2017-03-01 (1)

Suppose we have N particles of mass m. Suppose the total kinetic energy is $E = \frac{m}{2}\sum_{i=1}^{N} v_i^2 = \frac{M}{2}V^2$ where $V_i \in \mathbb{R}^3$.

Suppose the particles are in a small region of space so we don't care where they are. We only care about their velocity distribution. The total velocity state is given by

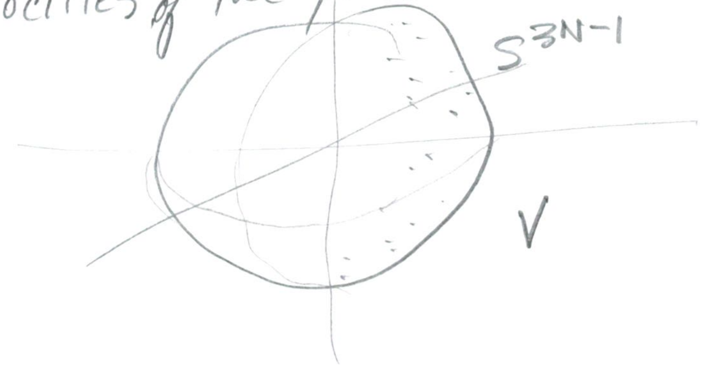
$$V \in \mathbb{R}^{3N}$$

In fact V lies on a sphere of radius $R = \sqrt{\frac{2E}{m}}$

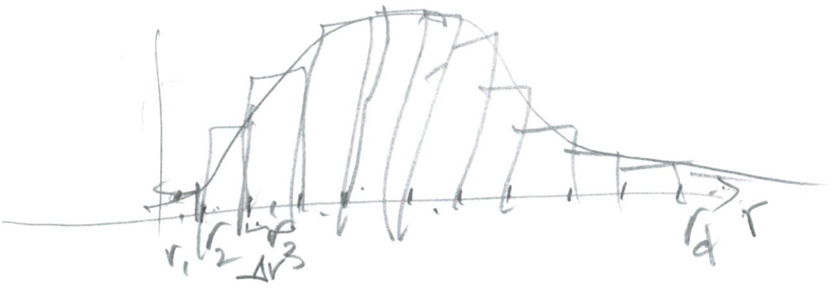
$$R^2 = \frac{2E}{m}$$

$$\frac{mV^2}{2} = E \qquad v^2 = \frac{2E}{m} = R^2$$

Consider the distribution of the individual velocities of the particles.



$$\Big\downarrow f$$



histogram

$$f(v)(r_j) = \#\text{ of particles that have } \big|\, \|v_i\| - r_j \,\big| \le \Delta v$$

Try this for a 1-dimension configuration space.

$$V \in R^N$$

$$f: \mathbb{R}^N \longrightarrow (1..d \longrightarrow [0,1])$$

a probability distribution after normalization by N.

$$S^{N-1} \longrightarrow \mathrm{Prob}(\mathbb{R}^+)$$

microscopic state — macroscopic state.

Given the total energy E there is a maximum velocity that a single particle can have.

$$\frac{m}{2} V_{max}^2 = E \qquad V_{max} = \sqrt{\frac{2E}{m}} > 0$$

We can divide the single particle velocity space into d cells for some $d > 1$.

$$V_{max} : \mathbb{R}^+$$

Use spherical coordinates. $\psi : \mathbb{R}^3 \cong S^2 \times \mathbb{R}^+$

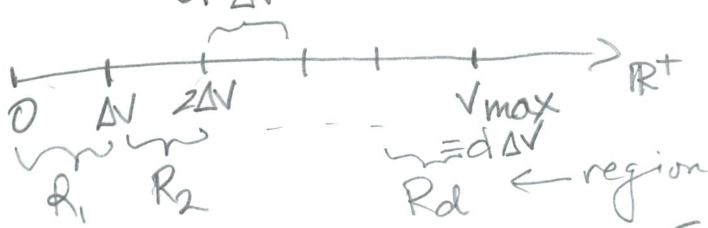
$$\psi(x,y,z) = (\Omega, r)$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$dx\,dy\,dz = r^2 dr \cdot d^2\Omega$$

$$\int_{S^2} d^2\Omega = 4\pi$$

define $\Delta V = \dfrac{V_{max}}{d}$



← regions in velocity space.

Given a $v \in \mathbb{R}^+$ define $\chi_X(v) = \begin{cases} 1 & \text{if } v \in X \\ 0 & \text{if } v \notin X \end{cases}$ $\quad X \subseteq \mathbb{R}^+$, $v \in \mathbb{R}^+$, $v \leq V_{max}$

The microstates $= \left\{ (v_1, v_2, \ldots, v_N) \in (\mathbb{R}^3)^N \;\middle|\; \sum_{i=1}^{N} \|v_i\|^2 = E \right\}$

Define the occupancy number $n : \mathbb{N}_d \longrightarrow \mathbb{N}$

$$n_v(j) = \sum_{i=1}^{N} \chi_{R_j}(\|v_i\|)$$

Express $n_v(j)$ in terms of a density function ρ

assume ρ_v is smooth. $\quad \rho_v : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$

$$n_v(j) \cong \rho_v(j\Delta v) \cdot \Delta v$$

$$= \rho_v\left(\frac{j \cdot V_{max}}{d}\right) \frac{V_{max}}{d}$$

~~Improve notation~~.

A microstate is a point $V \in \mathbb{R}^{3N}$ that has total kinetic energy $E > 0$. Let $N \in \mathbb{N}_1$ be the number of particles.
Let $P = \{1, 2, \ldots, N\} = 1..N$. P is the set of particle indices.
Regard $\mathbb{R}^{3N}$ as $P \to \mathbb{R}^3$ = the set of microstates $= A$.
$P == 1..N$

$\mathbb{R}^{3N} \cong A == P \to \mathbb{R}^3$ or $seq(\mathbb{R}^3) \mid \# V = N$.
$N : \mathbb{N}_1$ denote $A(i) = V_i$

Let $m : \mathbb{R}^+$ be a mass.

Define the single particle kinetic energy

$$T_1 : \mathbb{R}^3 \to \mathbb{R}$$
$$T_1(x,y,z) = \frac{m}{2}(x^2+y^2+z^2) = \frac{m}{2}\|(x,y,z)\|^2$$

The N-particle kinetic energy is T

$$T : A \to \mathbb{R}$$
$$T(V) = \sum_{i \in P} T_1(A(i)) = \sum_{i \in P} T_1(V_i)$$

Let $E : \mathbb{R}^+$ be a given total kinetic energy.

Define the energy sphere in A

$S : \mathbb{R}^+ \to \mathbb{P} A$ $\quad S_E = \{V : A \mid T(V) = E\}$

Observe that $S_E \cong S^{3N+1}$ as a smooth manifold.

$$\dim S_E = 3N-1.$$

Let $D : \mathbb{N}_1$ be the number of cells that we divide the single particle space into.

Since $T(V) = \sum_i T_1(V_i) = E$

we have $T_1(V_i) \leq E$ for all $i \in P$

Define $B_E \subseteq \mathbb{R}^3$ to be the E-energy ball in the single particle space

$$A = \Phi_{micro} = \Phi_N = \Phi^N$$
$$\mathbb{R}^3 = \Phi_1 = \Phi$$

$$B_E = \{ V : \mathbb{R}^3 \mid T_1(V) \le E \}$$

Given $V : A$ let $E_i(V) = T_1(V_i)$

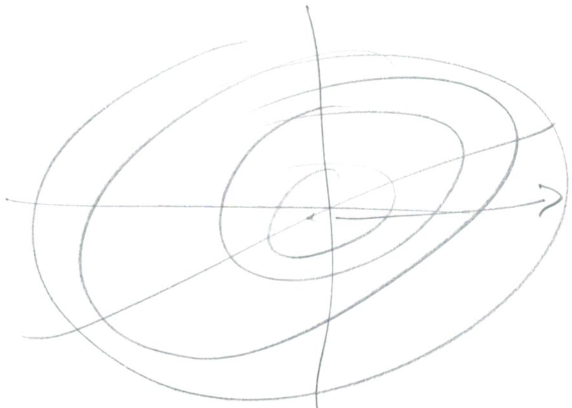
Then $T(V) = \sum_i E_i(V)$

To compute entropy, define the macro state as follows. Let D be some number of cells that we'll divide the single particle ball B_E into.

$$B_E = \bigcup_j B_E(D,j)$$

$$B_E(D,j) = \left\{ V : B_E \;\middle|\; \left(\frac{j-1}{D}\right) \le \frac{\|V\|}{V_{max}} \le \left(\frac{j}{D}\right) \right\}$$

$j \in 1 .. D$



where $\frac{m}{2} V_{max}^2 = E$ so $V_{max} = \sqrt{\frac{2E}{m}}$

The velocity space volume of B_E is

$$\text{volume}(B_E) = \cancel{4\pi V_{max}^2 = \frac{4\pi 2E}{m}}$$

$$\frac{4}{3}\pi r^3 = \frac{4}{3}\pi V_{max}^3$$

$$= \cancel{\frac{8\pi E}{m}}$$

The radius of $B_E(D,j)$ in velocity space is

$$\text{radius}(B_E(D,j)) \cong \frac{j}{D} V_{max}$$

The volume of $B_E(D,j)$ is approximately

$$4\pi r^2 \underbrace{dr}_{\substack{\text{thickness} \\ = \frac{V_{max}}{D}}} = 4\pi \left(\frac{j}{D} V_{max}\right)^2 \frac{V_{max}}{D}$$

The macrostate is the distribution of occupancies of the $B_E(D,j)$ for $j = 1, .. D$ for $V \in S_E$

$$n_V(j) = \sum_{i=1}^{N} \chi_{B_E(D,j)}(V_i)$$

In order to approximate the situation with smooth functions we need to take two limits.

- First suppose that instead of having a large number N of particles, suppose we have a smooth matter distribution. This actually loses a lot of information since we lose the identity of the particles. Can we recover that?
- Second, we need to treat space as continuous.

For the occupancy number, if d is very large we can approximate it using a smooth density function.

$$n_V(j) = \rho_V(r(j)) \cdot \Delta r$$

$$r(j) = \frac{j}{D} V_{max} \qquad \Delta r = \frac{V_{max}}{D}$$

So $n_V(j) = \rho_V\left(\frac{jV_{max}}{D}\right)\frac{V_{max}}{D}$

9:30 — 12:00pm PST } BayTech
12:30pm – 3:00pm EST
1:00pm – 2:00pm EST Assante