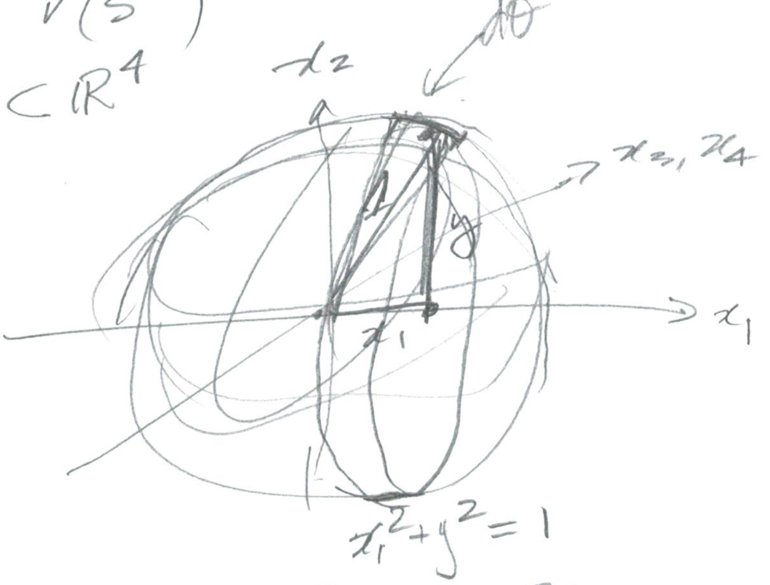
Compute $v(S^3)$ 2017-03-19

$S^3 \subset \mathbb{R}^4$



$x_1 = \cos\theta$

$y = \sin\theta = r$

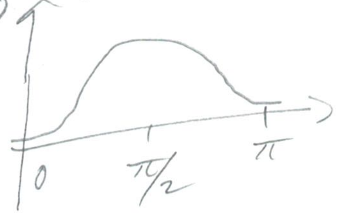
$r = y$

$dl = \sin\theta \, d\theta$

$= r \, d\theta$

$\mathrm{vol}(S^n_r) = r^n \mathrm{vol}(S^n)$

$$v(S^3) = \int_0^\pi d\theta \cdot v(S^2_r) \quad r = \sin\theta$$



so $\mathrm{vol}(S^3) = \int_0^\pi d\theta \; \mathrm{vol}(S^2_{\sin\theta})$

in general $\mathrm{vol}(S^{n+1}) = \int_0^\pi d\theta \; \mathrm{vol}(S^n_{\sin\theta})$

$$\mathrm{vol}(S^1) = \int_0^\pi d\theta \; \mathrm{vol}(S^0_{\sin\theta})$$

$$= \int_0^\pi d\theta \cdot 2$$

$$= 2\pi$$

$$\mathrm{vol}(S^2) = \int_0^\pi d\theta \cdot \mathrm{vol}(S^1_{\sin\theta})$$

$$= \int_0^\pi d\theta \cdot (\sin\theta)^1 \cdot S^1$$

$$= 2\pi \int_0^\pi d\theta \cdot \sin\theta$$

$$= 2\pi \left| \cos\theta \right|_0^\pi$$

$$= 2\pi \left(\cos 0 - \cos(\pi) \right)$$

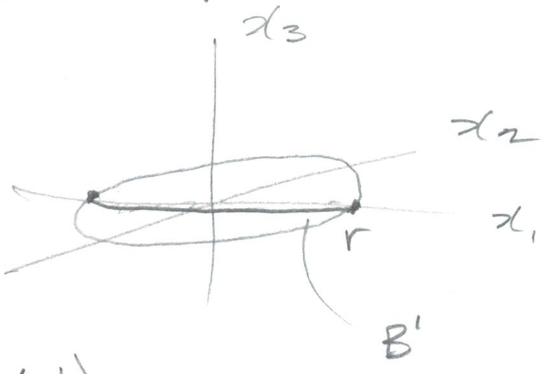
$$= 2\pi (1 - -1)$$

$$= 4\pi$$

S^n - n-sphere $\quad S^n_r$
B^n - n-ball $\quad B^n_r$ } r = radius

$$S^n = S^n_1$$
$$B^n = B^n_1$$

$$\partial B^n_r = S^{n-1}_r \quad n \geq 1 \qquad S^0_r = \{r, -r\} \subset \mathbb{R}^1$$



ball $B^n_r = \{x : \mathbb{R}^n \mid \|x\| \leq r\}$
Cube $C^n_r = \{x : \mathbb{R}^n \mid |x_i| \leq r\}$
sphere $S^n_r = \{x : \mathbb{R}^{n+1} \mid \|x\| = r\}$

$$v(B^1) = 2r$$

$$v(B^{n+1}_r) = \int_0^r v(S^n_t)\,dt \qquad S^n_r \subseteq B^{n+1}_r \subseteq C^{n+1}_r$$

$$\frac{\partial}{\partial r} v(B^{n+1}_r) = v(S^n_r)$$

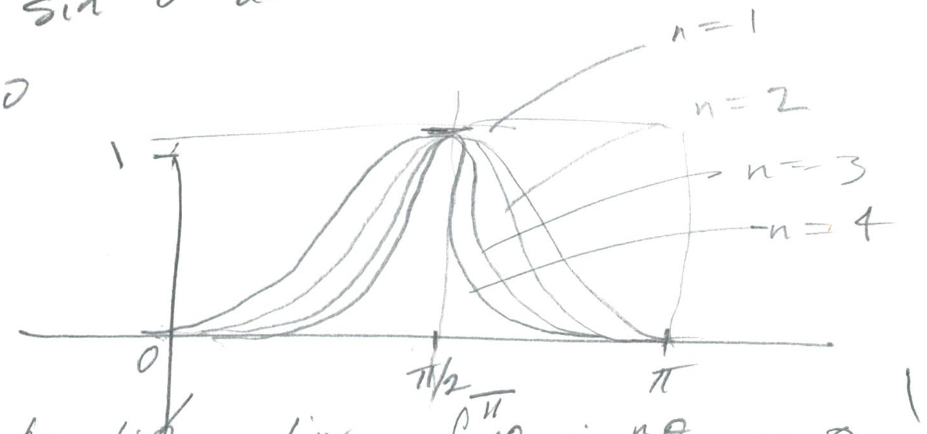
n	$v(S^n_r)$	$v(B^n_r)$	$v(C^n_r) = (2r)^n$	$v(B^n_r)/v(C^n_r)$		
0	2	1	1	$1/1 =$	1	1
1	$2\pi r$	$2r$	$2r$	$2r/2r =$	1	1
2	$4\pi r^2$	πr^2	$4r^2$	$\pi r^2/4r^2 =$	$\frac{\pi}{4}$	$\sim \frac{3}{4}$
3	$2\pi^2 r^3$	$\frac{4\pi}{3} r^3$	$8r^3$	$\frac{4}{3}\pi r^3/8r^3 =$	$\frac{\pi}{6}$	$\sim \frac{1}{2}$
4		$\frac{\pi^2}{2} r^4$ ✓	$16r^4$	$\frac{\pi^2}{2}/16 =$	$\frac{\pi^2}{32}$	$\sim \frac{1}{3}$

Since the volume of the ball is a decreasing proportion of the cube Then

$$\lim_{n \to \infty} \frac{B^n}{C^n} = 0$$

so we need a more efficient way to compute random samples on S^n.
i.e a The probability that a random point in C^n lies in B^n goes to zero

$$\int_0^{\pi} \sin^n\theta \, d\theta$$



Looks like $\lim_{n\to\infty} \int_0^{\pi} d\theta \cdot \sin^n\theta = 0$

Defin $I_n = \int_0^{\pi} d\theta \cdot \sin^n\theta$

$v(S^0) = 2$

$v(S^1) = v(S^0) \cdot I_0 = 2\pi \qquad I_0 = \pi$

$v(S^2) = v(S^1) \cdot I_1 = 4\pi = 2\pi \cdot I_1 \qquad I_1 = 2 < \pi = I_0$

$v(S^3) = \frac{\pi^2}{2} = v(S^2) \cdot I_2 = 4\pi \cdot I_2 \qquad I_2 = \frac{\pi^2}{2 \cdot 4\pi} = \frac{\pi}{8}$

$= v(S^1) \cdot I_1 \cdot I_2$

$= v(S^0) \cdot I_0 \cdot I_1 \cdot I_2$

$= 2 \cdot I_0 \cdot I_1 \cdot I_2$

$$v(S^n) = 2 \cdot \prod_{i=0}^{n} I_i$$

$I_4 = \int_0^{\pi} \sin^4\theta \cdot d\theta$

we can probably express $\sin^4\theta$ as a polynomial in z & $\frac{1}{z}$ of degree 4

$$vol(S^3) = \int_0^\pi d\theta \; vol(S^2_{\sin\theta})$$

$$= \int_0^\pi d\theta \; vol(S^2) \cdot (\sin\theta)^2$$

$$= 4\pi \int_0^\pi d\theta \cdot \sin^2\theta = 4\pi \cdot \frac{\pi}{2} = 2\pi^2$$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$$e^{2\theta i} = (e^{i\theta})^2$$

$$= (\sin\theta + i\cos\theta)^2$$

$$= \sin^2\theta - \cos^2\theta + 2i(\sin\theta \cdot \cos\theta)$$

$$e^{-2i\theta} = (\sin\theta - i\cos\theta)^2$$

$$= \sin^2\theta - \cos^2\theta - 2i\sin\theta\cos\theta$$

$$e^{2i\theta} = \sin^2\theta - [1 - \sin^2\theta] + i \ldots$$

$$= 2\sin^2\theta - 1 + i \ldots$$

$$e^{-2i\theta} = 2\sin^2\theta - 1 - i \ldots$$

$$e^{2i\theta} + e^{-2i\theta} = 4\sin^2\theta - 2$$

$$\frac{e^{2i\theta} + e^{-2i\theta} + 2}{4} = \sin^2\theta$$

$$I = \int_0^\pi \sin^2\theta \, d\theta = \frac{1}{4}\int_0^\pi d\theta \left[e^{2i\theta} + e^{-2i\theta} + 2 \right]$$

$$= \frac{1}{4} \left[\frac{e^{2i\theta}}{2i} + \frac{e^{-2i\theta}}{-2i} + 2\theta \right]\Bigg|_0^\pi$$

$$= \frac{1}{4}\left[\left(\frac{e^{2\pi i}}{2i} - \frac{e^{-2\pi i}}{2i} + 2\pi \right) - \left(\frac{1}{2i} - \frac{1}{2i} + 0 \right) \right]$$

$$= \frac{1}{4}\left[\frac{1}{2i} - \frac{1}{2i} + 2\pi \right]$$

$$= \frac{2\pi}{4}$$

$$\boxed{= \frac{\pi}{2}}$$

2017-03-19

Devise the distribution of X_1 for S^2

Establish a stable naming convention for distributions.

ρ_n is the uniform distribution on S^n.

$X_n : S^n \longrightarrow \mathbb{R}$ is the coordinate function.

What is its pdf called?

we could define standard projection mappings.

$$\pi_i^n : S^n \to \mathbb{R}$$

ρ_n: ~~$\mathbb{P}\, S^n \to [0,1]$~~

use pdf.

The pdf for uniform distributions is a constant

Let $M \xrightarrow{f} M'$ be a morphism of measurable spaces.

Let (M, ρ) be a probability space defined be ρ a pdf. Then f induces a morphism

$$(M,\rho) \xrightarrow{f^*} (M', \rho')$$

where $\rho'(X) = \rho(\{m \in M \mid f(m) \in X\})$

$$\begin{tikzcd}
(M,\rho) \arrow[r, "f^*"] \arrow[d] & (M', f^*\rho) = f^*(\rho) \arrow[d] \\
M \arrow[r, "f"] & M'
\end{tikzcd}$$