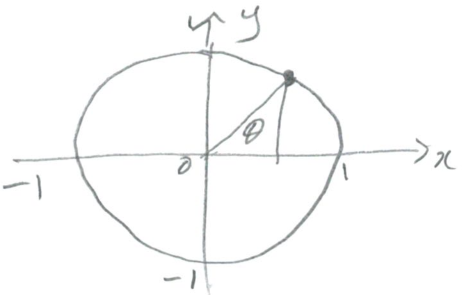
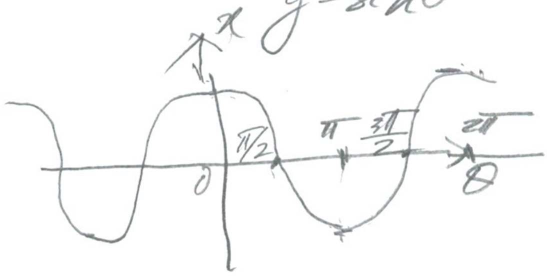
Let $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

Regard S^1 as a state space.

The volume measure on S^1 is $d\theta$ where $x=\cos\theta$, $y=\sin\theta$





$$d\theta = dx\left(\frac{d\theta}{dx}\right) = \frac{dx}{dx/d\theta} = \frac{dx}{-\sin\theta}$$

$$\frac{dx}{d\theta} = -\sin\theta$$

Define a uniform probability measure on S^1 by the density function $\rho(\theta) = \frac{1}{2\pi}$ $\quad 0 \le \theta < 2\pi$

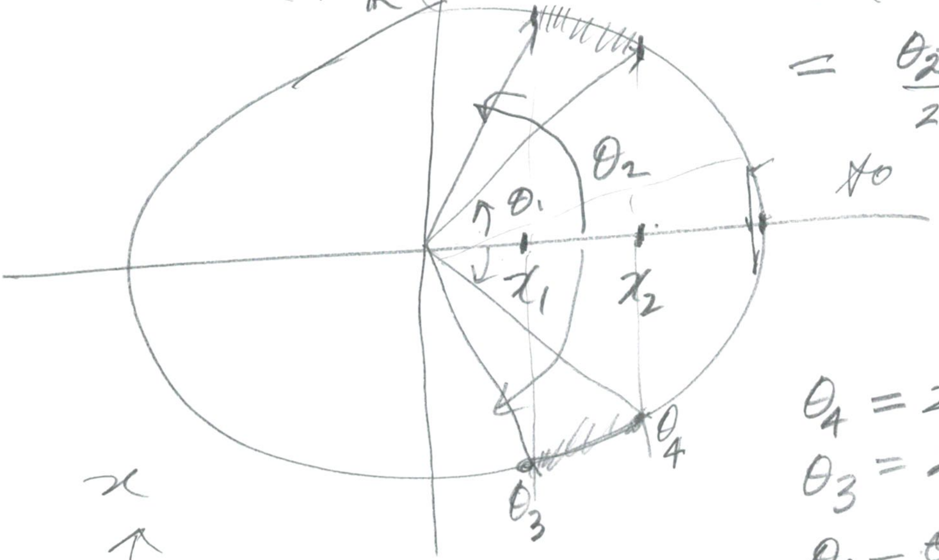
<u>Problem</u> What is the probability distribution of x regarded as a function $x : S^1 \to \mathbb{R}$?

$x(\theta) = \cos\theta$.

<u>Solution</u> On S^1 we have $P_{S^1}([\theta_1,\theta_2]) = \frac{\theta_2-\theta_1}{2\pi}$

where $0 \le \theta_1 \le \theta_2 < 2\pi$.

$$(x_* P_{\mathbb{R}})([x_1,x_2]) = P_{S^1}(\{\theta \mid x_1 \le x(\theta) \le x_2\})$$



$$= \frac{\theta_2-\theta_1}{2\pi} + \frac{\theta_4-\theta_3}{2\pi}$$

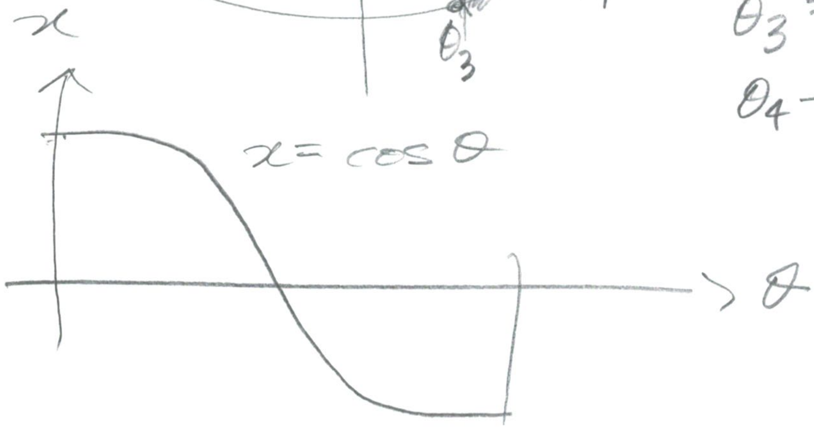
$\neq 0$ $\quad \sim \frac{d\theta}{2\pi} + \frac{d\theta}{2\pi}$

$$\sim \frac{d\theta}{\pi}$$

$\theta_4 = 2\pi - \theta_1$

$\theta_3 = 2\pi - \theta_2$

$$\theta_4 - \theta_3 = (2\pi-\theta_1) - (2\pi-\theta_2)$$
$$= 2\pi - \theta_1 - 2\pi + \theta_2$$
$$= \theta_2 - \theta_1$$



for small $d\theta$, $\theta_2 = \theta_1 + d\theta$

$$dx = -\sin\theta \, d\theta$$

$$x_2 = x_1 + dx$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + x^2 = 1$$

$$\sin^2\theta = 1 - x^2$$

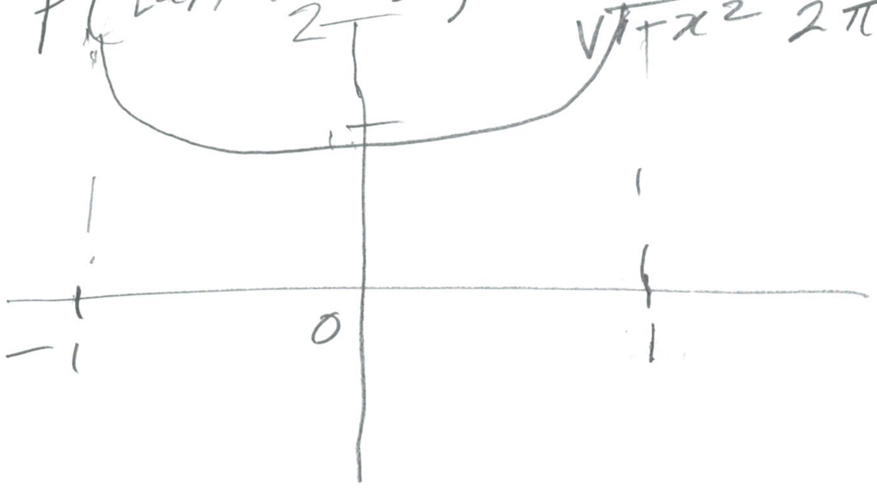
$$\sin\theta = \sqrt{1-x^2}$$

$$dx = -\sqrt{1-x^2}\, d\theta$$

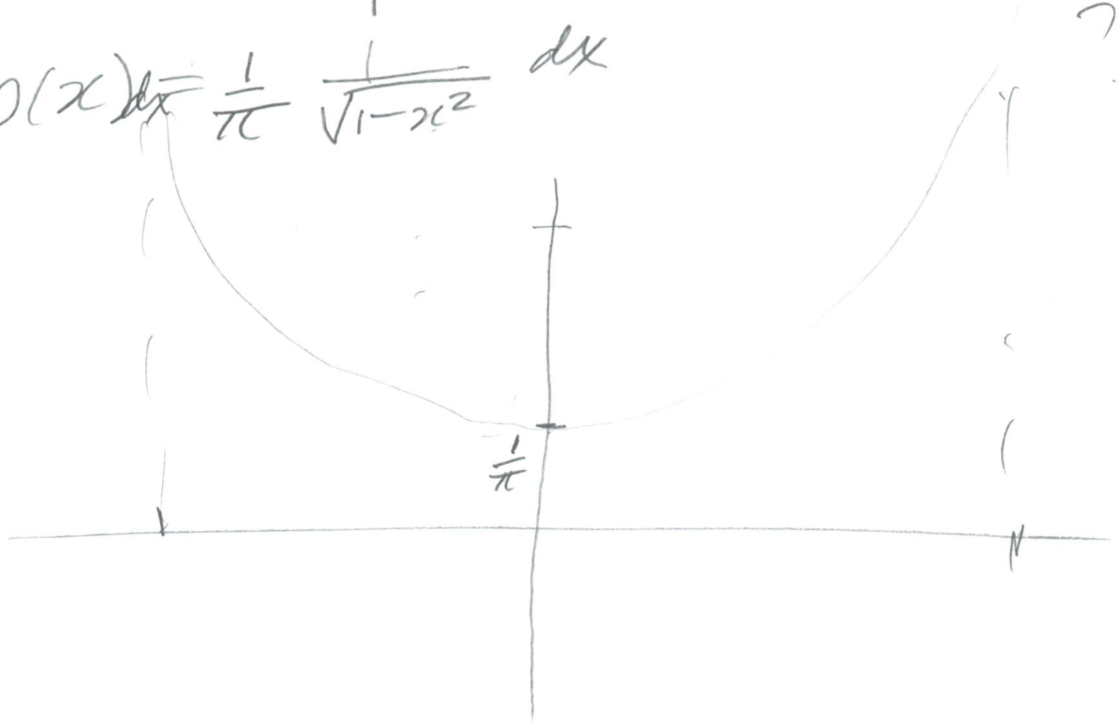
$$d\theta = -\frac{dx}{\sqrt{1-x^2}} = \rho(x)\, dx$$

$$\rho(x) = \frac{1}{\sqrt{1-x^2}}$$

$$P([x_1, x_1+dx]) = \frac{2}{\sqrt{1-x^2}} \frac{dx}{2\pi} \quad ? \qquad |x| \le 1$$



$$\rho(x)dx = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} dx$$



$x_1 = \cos\theta_1 \qquad \theta_1 = \arccos x_1$

$x_2 = \cos\theta_2 \qquad \theta_2 = \arccos(x_1 + dx)$

$x_2 = x_1 + dx$

$$\theta_2 = \theta_1 + d\theta$$

$$\cos(\theta_1 + d\theta) = \cos(\theta_1)\cos(d\theta) - \sin(\theta_1)\sin(d\theta)$$

for $|d\theta| << 1 \qquad \cos(d\theta) \approx 1 \quad + o(d\theta)^2$

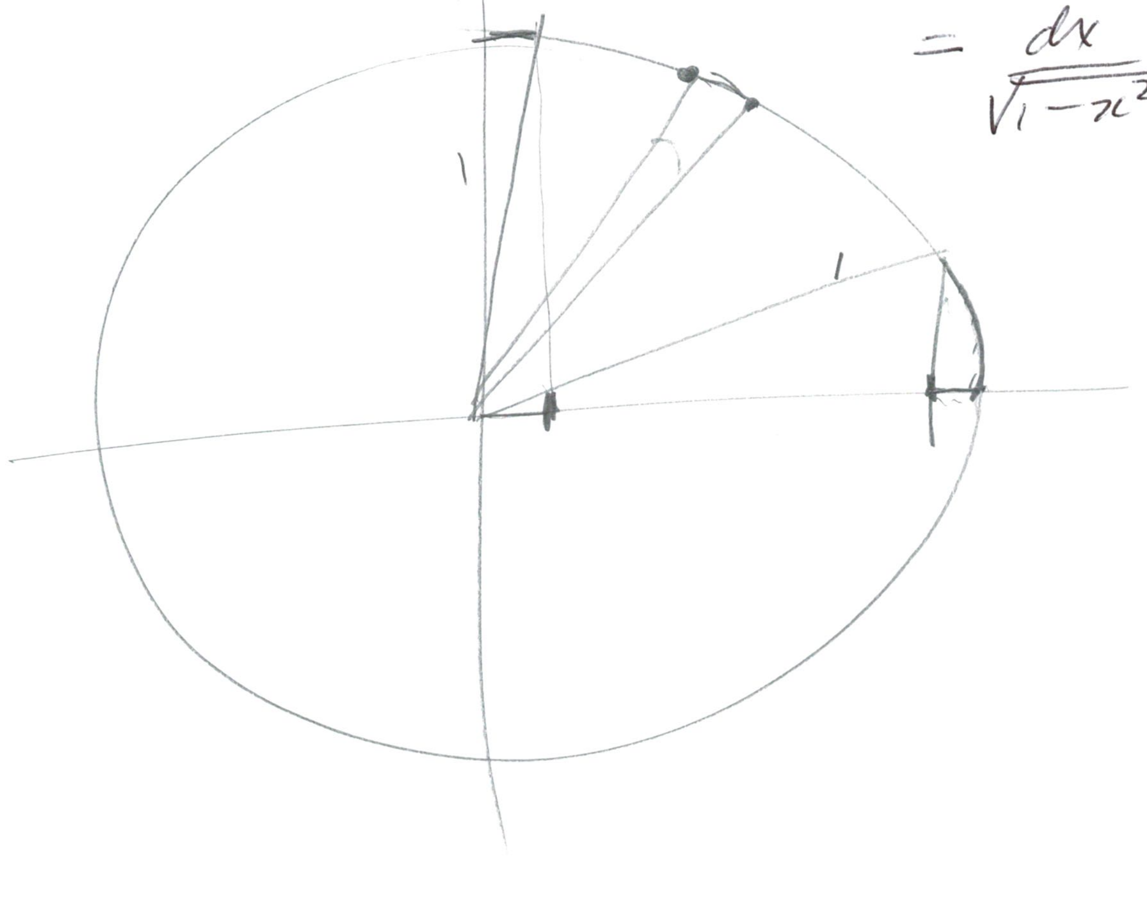
$\qquad\qquad\qquad \sin(d\theta) \approx d\theta$

so $\cos\theta_2 \approx \cos(\theta_1) - \sin(\theta_1)\, d\theta$

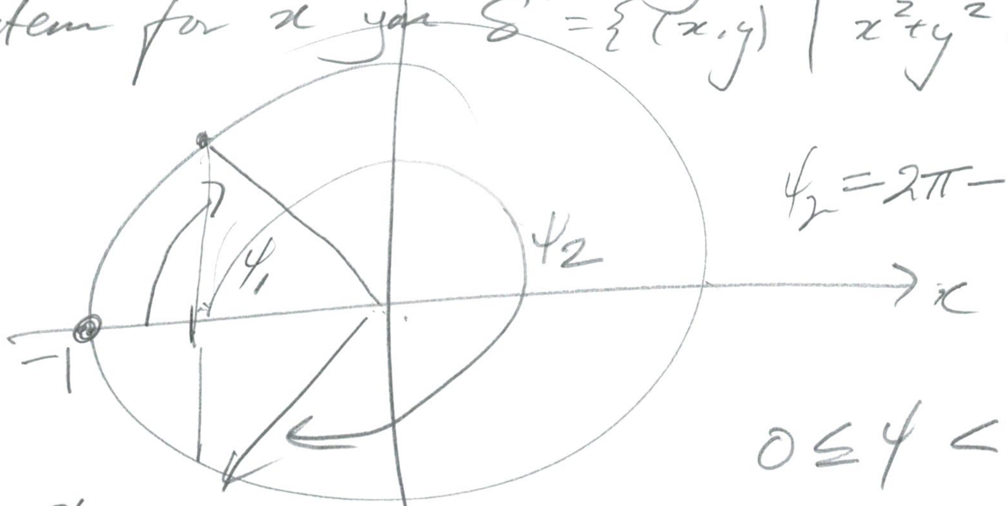
$$x_2 - x_1 = dx = \cos(\theta_2) - \cos(\theta_1)$$
$$= -\sin(\theta_1)\, d\theta$$

$$d\theta = \frac{-dx}{\sin(\theta_1)}$$

$$P([x_1, x_2]) = |2(\theta_2 - \theta_1)| = \frac{dx}{\sin(\theta_1)}$$
$$= \frac{dx}{\sqrt{1 - x^2}}$$

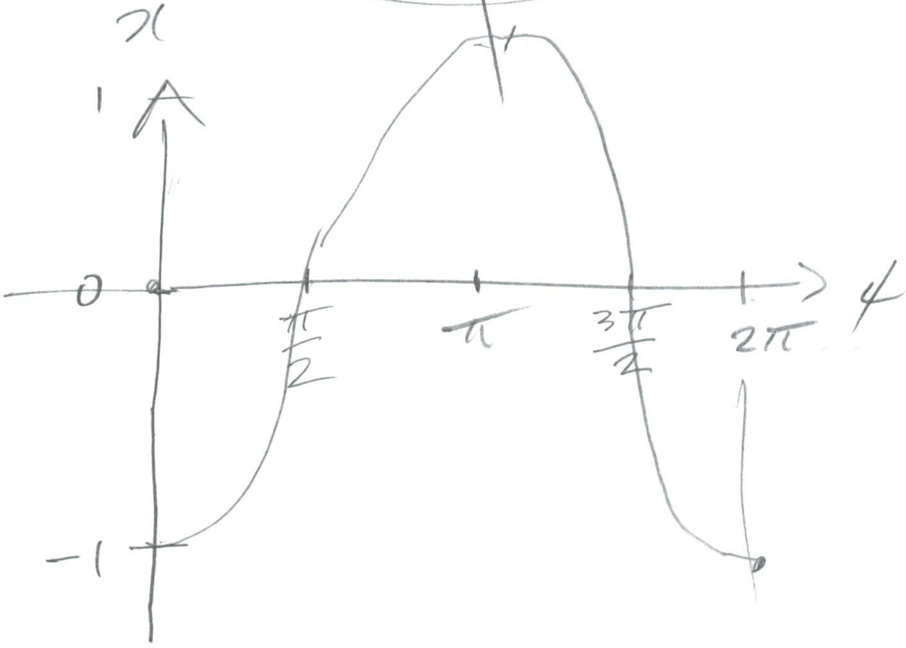


use a monotonic increasing coordinate system for x you $S^1 = \{(x,y) \mid x^2+y^2=1\}$



$\psi_2 = 2\pi - \psi_1$

$0 \le \psi < 2\pi$



$$x = -\cos\psi$$
$$dx = \sin\psi \, d\psi$$

$$\rho(\psi) = \frac{1}{2\pi}$$
$$\rho(\psi)d\psi = \frac{d\psi}{2\pi}$$

$f_*(\omega)$

$$M \xrightarrow{f} N \quad \omega$$

$f_*\omega \in \Omega^1(M)$ $\quad$ $\omega \in \Omega^1(N)$

$\Uparrow$

S^1 uniform dist

$$[-1,1] \xrightarrow[x \mapsto \psi]{f} \omega(\psi) = \frac{d\psi}{2\pi}$$

$$\psi = \arccos(-x)$$

$$\rho(\psi)d\psi = \frac{d\psi}{2\pi} = \frac{dx}{2\pi\sin\psi} \text{ on } 0 \le \psi < \pi$$

$$+ \frac{dx}{2\pi\sin\psi} \text{ on } \pi \le \psi < 2\pi$$

The distribution of x for S^1 is anomalous because volume$(S^0)=2$ independent of the radius of the sphere.

define $S^n_R = \{ v \in \mathbb{R}^n \mid |v| = R \}$

R is the radius of the n-sphere.

hypercube of length 2R

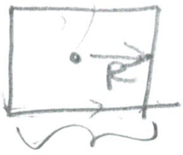
$(2R)^0 \neq 1$ $\quad (2R)^0 \cdot 2$

$(2R)^1 \times 4$

$\text{Volume}(S^0_R) = 2$

$\text{Volume}(S^1_R) = 2\pi R$

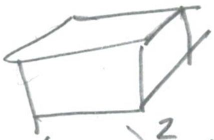
$2\pi R$ enclose πR^2 $\quad 2 \cdot 4R$



$(2R)^1 \cdot 4$

$\underbrace{\quad}_{2R}$

$\text{Volume}(S^2_R) = 4\pi R^2$ encloses $\frac{4\pi}{3} R^3$



$(2R)^2 \times 8$

$\text{Volume}(S^3_R) =$

$\underbrace{(2R)^n}_{\substack{\text{size} \\ \text{of a} \\ \text{"face"}}} \cdot \underbrace{2^{n+1}}_{\substack{\#\text{ of} \\ \text{"faces"}}}$

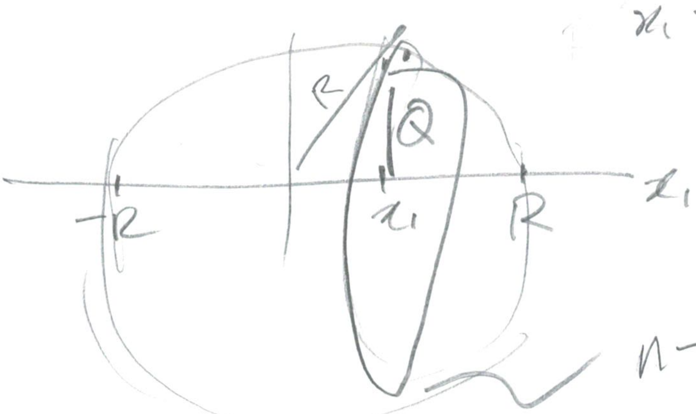
$x, y, z,$

(x_1, x_2, x_3, x_4)

$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$

$x_1^2 + Q^2 = R^2$

$Q = \sqrt{R^2 - x_1^2}$



n-1 sphere of radius Q.

$\text{Volume}(S^n_R) = \int_{-R}^{R} dx_1 \cdot \text{Volume}(S^{n-1}_Q) \qquad Q = \sqrt{R^2 - x_1^2}$

$= 2 \int_0^R dx \, \text{Vol}(S^{n-1}_Q)$

$$\cancel{Vol(S^1_R) = 2\int_0^R dx\ vol(S^0_Q)}$$

$n > 1$

$$=$$

$$Vol(S^2_R) = 2\int_0^R dx\ Vol(S^1_Q)$$

$$Vol(S^1_Q) = 2\pi Q$$

~~define coordinates in $\mathbb{R}^4$ - spherical polar.~~

Consider the positive hemisphere $H^n_R \subseteq S^n_R$

$$H^3_R = \{(x_1, x_2, x_3, x_4) \in S^3_R \mid x_1 \geq 0\}$$

By symmetry volume $(H^n_R) = \frac{1}{2}$ volume(S^3_R)

$$x_1 = -R\cos\varphi$$

$$x_1 = R\cos\theta_1$$
$$x_2 = R\sin\theta_1\cos\theta_2$$
$$x_3 = R\sin\theta_1\sin\theta_2\cos\theta_3$$
$$x_4 = R\sin\theta_1\sin\theta_2\sin\theta_3$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2$$
$$= R^2\big(\cos^2\theta_1$$
$$+ \sin^2\theta_1\cos^2\theta_2$$
$$+ \sin^2\theta_1$$