SETS

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Abstract. This article contains Z Notation definitions for concepts related to sets. It has been type checked by *f*UZZ.

Contents

1. Introduction

Typed set theory forms the mathematical foundation of Z Notation[1]. Many set theory concepts are defined in its built-in mathematical toolkit. This article augments the toolkit with some additional concepts. It has been type checked by *f*UZZ[2].

2. Binary Digits

Let *bit* denote the set of *binary digits*, namely the set $\{0, 1\} \subseteq \mathbb{Z}$.

$$bit == \{0, 1\}$$

Let the notation $\mathbb{B}$ denote the set of bits.

$$\mathbb{B} == bit$$

Example. *Zero and one are bits, but two isn't.*

$$0 \in \mathbb{B}$$

$$1 \in \mathbb{B}$$

$$2 \notin \mathbb{B}$$

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3. Families of Subsets

Let t be a set. A *family* of subsets of t is a set of subsets of t. Let $\mathit{Fam}[\mathsf{t}]$ denote the set of all families of subsets of t.

$$\mathit{Fam}[\mathsf{t}] == \mathbb{P}(\mathbb{P}\,\mathsf{t})$$

Example. *The set consisting of the empty set and* X *is a family of subsets of* X.

$$\{\varnothing, \mathsf{X}\} \in \mathit{Fam}[\mathsf{X}]$$

Let the notation $\mathcal{F}\,\mathsf{t}$ denote the family of subsets of t .

$$\mathcal{F}\,\mathsf{t} == \mathit{Fam}[\mathsf{t}]$$

4. Functions

4.1. **Binary Functions.** Let t be a set. A function that maps t to $\mathbb{B}$ is called a *binary function* on t.

$$\mathit{binary\_function}[\mathsf{t}] == \mathsf{t} \longrightarrow \mathbb{B}$$

Example. *The function that maps the set* T *to* 0 *is a binary function.*

$$(\lambda\, x : \mathsf{T} \bullet 0) \in \mathit{binary\_function}[\mathsf{T}]$$

4.2. **Constant Functions.** Let t and u be sets and let c be some given element of u. The mapping f that sends every element x of t to c is called the *constant function* on t with value c.

$$
\begin{array}{|l}
\multicolumn{1}{l}{\mathit{ConstantFunction}[\mathsf{t}, \mathsf{u}]} \\ \hline
c : \mathsf{u} \\
f : \mathsf{t} \longrightarrow \mathsf{u} \\ \hline
f = (\lambda\, x : \mathsf{t} \bullet c) \\ \hline
\end{array}
$$

Let *constant_function* c denote the constant function f on t with value c.

$$\mathit{constant\_function}[\mathsf{t}, \mathsf{u}] == \{\, \mathit{ConstantFunction}[\mathsf{t}, \mathsf{u}] \bullet c \mapsto f \,\}$$

Remark. *The mapping constant_function maps each element* $c \in \mathsf{U}$ *to a function* $\mathsf{T} \longrightarrow \mathsf{U}$.

$$\mathit{constant\_function}[\mathsf{T}, \mathsf{U}] \in \mathsf{U} \longrightarrow (\mathsf{T} \longrightarrow \mathsf{U})$$

Let the notation const c denote the constant function with value c.

$$\mathrm{const}[\mathsf{t}, \mathsf{u}] == \mathit{constant\_function}[\mathsf{t}, \mathsf{u}]$$

Remark.

$$\forall\, c : \mathsf{U};\ x : \mathsf{T} \bullet \mathrm{const}[\mathsf{T}, \mathsf{U}]\, c\, x = c$$

4.3. **Delta Functions.** Let t be a set and let $x, y \in \mathsf{t}$. Define the *equality indicator bit* z of (x, y) to be 1 if $x = y$ and 0 otherwise.

$$
\begin{array}{|l}
\hline
\multicolumn{1}{l}{\mathit{EqualityIndicator}[\mathsf{t}]} \\
x, y : \mathsf{t} \\
z : \mathbb{B} \\
\hline
z = \textbf{if } x = y \textbf{ then } 1 \textbf{ else } 0 \\
\hline
\end{array}
$$

Define the delta function $\mathit{delta}(x, y)$ to be the equality indication bit of (x, y).

$\mathit{delta}[\mathsf{t}] == \{ \mathit{EqualityIndicator}[\mathsf{t}] \bullet (x, y) \mapsto z \}$

Remark.

$\mathit{delta}[\mathsf{X}] \in \mathsf{X} \times \mathsf{X} \longrightarrow \mathbb{B}$

Example.

$\mathit{delta}(0, 0) = 1$

$\mathit{delta}(0, 1) = 0$

Remark.

$$
\begin{array}{l}
\forall x : \mathsf{X} \bullet \\
\qquad \mathit{delta}(x, x) = 1
\end{array}
$$

Let the notation $\delta\,\mathsf{t}$ denote the delta function $\mathit{delta}[\mathsf{t}]$.

$\delta\,\mathsf{t} == \mathit{delta}[\mathsf{t}]$

Example.

$(\delta\,\mathbb{Z})(0, 1) = 0$

4.4. **Function Restriction.** Let t and u be sets, let $f : \mathsf{t} \longrightarrow \mathsf{u}$, and let $T \subseteq \mathsf{t}$ be a subset. Let g denote the restriction of f to T.

$$
\begin{array}{|l}
\hline
\multicolumn{1}{l}{\mathit{FunctionRestriction}[\mathsf{t}, \mathsf{u}]} \\
f : \mathsf{t} \longrightarrow \mathsf{u} \\
T : \mathbb{P}\,\mathsf{t} \\
g : \mathsf{t} \mathrel{+\!\!\!\rightarrow} \mathsf{u} \\
\hline
g = T \lhd f \\
\hline
\end{array}
$$

Let $\mathit{restriction}(f, T)$ denote the restriction of f to T.

$\mathit{restriction}[\mathsf{t}, \mathsf{u}] == \{ \mathit{FunctionRestriction}[\mathsf{t}, \mathsf{u}] \bullet (f, T) \mapsto g \}$

Remark.

$\mathit{restriction}[\mathsf{T}, \mathsf{U}] \in (\mathsf{T} \longrightarrow \mathsf{U}) \times \mathbb{P}\,\mathsf{T} \longrightarrow (\mathsf{T} \mathrel{+\!\!\!\rightarrow} \mathsf{U})$

Let the notation $f \mid T$ denote the restriction of f to T.

$(\_ \mid \_)[\mathsf{t}, \mathsf{u}] == \mathit{restriction}[\mathsf{t}, \mathsf{u}]$

Remark. *Function restriction is domain restriction with arguments reversed.*

$\forall\, \mathit{FunctionRestriction}[\mathsf{T}, \mathsf{U}] \bullet$
$\quad f \mid T = T \lhd f$

4.5. **Indicator Functions.** Let t be a set and let $X \subseteq \mathsf{t}$ be a subset. The *indicator function* f of X maps each element $a \in \mathsf{t}$ to 1 if $a \in X$ and 0 otherwise. The indicator function is also referred to as the *characteristic function* of X.

$$
\begin{array}{|l}
\mathit{IndicatorFunction}[\mathsf{t}] \\
\hline
X : \mathbb{P}\,\mathsf{t} \\
f : \mathsf{t} \longrightarrow \mathbb{B} \\
\hline
f = (\lambda\, a : \mathsf{t} \bullet \textbf{if } a \in X \textbf{ then } 1 \textbf{ else } 0) \\
\hline
\end{array}
$$

Let *indicator_function* X denote the indicator function of X.

$\mathit{indicator\_function}[\mathsf{t}] == \{\, \mathit{IndicatorFunction}[\mathsf{t}] \bullet X \mapsto f \,\}$

Remark. *For each subset $X \subseteq \mathsf{T}$, the indicator function of X is a binary function on T.*

$\mathit{indicator\_function}[\mathsf{T}] \in \mathbb{P}\,\mathsf{T} \longrightarrow \mathsf{T} \longrightarrow \mathbb{B}$

We introduce the prefix generic symbol $(\mathbf{1}\,\_)$ where $(\mathbf{1}\,\mathsf{t})X = \mathit{indicator\_function}[\mathsf{t}]X$.

$\mathbf{1}\,\mathsf{t} == \mathit{indicator\_function}[\mathsf{t}]$

Remark. *The domain of the range restriction of the indicator function of a set X to the range $\{1\}$ is X.*

$\forall\, X : \mathbb{P}\,\mathsf{T} \bullet$
$\quad \operatorname{dom}((\mathbf{1}\,\mathsf{T})X \rhd \{1\}) = X$

5. The Support of a Function

Let t be a set and let f be an integer-valued function on t. The *support* S of f is the set of elements x in t that take nonzero values.

$$
\begin{array}{|l}
\mathit{Support}[\mathsf{t}] \\
\hline
f : \mathsf{t} \longrightarrow \mathbb{Z} \\
S : \mathbb{P}\,\mathsf{t} \\
\hline
S = \{\, x : \mathsf{t} \mid f\,x \neq 0 \,\} \\
\hline
\end{array}
$$

Let *support* f denote the support S of f.

$\mathit{support}[\mathsf{t}] == \{\, \mathit{Support}[\mathsf{t}] \bullet f \mapsto S \,\}$

Example. *The support of the indicator function of a set X is X.*

$\forall\, X : \mathbb{P}\,\mathsf{T} \bullet$
$\quad \mathit{support}((\mathbf{1}\,\mathsf{T})X) = X$

An integer-valued function is said to have *finite support* if its support is a finite set.

$$\begin{array}{|l}
\hline
\multicolumn{1}{l}{\mathit{FiniteSupport}[\mathsf{t}]} \\
\mathit{Support}[\mathsf{t}] \\
\hline
S \in \mathbb{F}\,\mathsf{t} \\
\hline
\end{array}$$

Let *finite_support*[t] denote the set of all integer-valued functions on `t` that have finite support.

$\mathit{finite\_support}[\mathsf{t}] == \{\, \mathit{FiniteSupport}[\mathsf{t}] \bullet f \,\}$

Remark.

$\mathit{finite\_support}[\mathsf{T}] \subseteq \mathsf{T} \longrightarrow \mathbb{Z}$

References

[1] J. M. Spivey. *The Z Notation*. Second Edition. Prentice Hall International, 1992. URL: https://spivey.oriel.ox.ac.uk/wiki/files/zrm/zrm.pdf.

[2] Mike Spivey. *The fUZZ Manual*. Second Edition. The Spivey Partnership, 2000. URL: https://github.com/Spivoxity/fuzz/blob/59313f201af2d536f5381e65741ee6d98db54a70/doc/fuzzman-pub.pdf.

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