MANIFOLDS

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ABSTRACT. This article contains Z Notation type declarations for manifolds and some related objects. It has been type checked by *f*UZZ.

CONTENTS

1. INTRODUCTION

Manifolds can be defined in several ways. The way I prefer to think about them is that, first of all, they are based on topological spaces. A manifold is therefore a topological space with some additional structure. This additional structure allows one to regard a manifold as, locally, being like an open subset of $\mathbb{R}^n$ for some natural number n referred to as the dimension of the manifold. In the following, let M be a topological space of dimension n.

2. CHARTS

A chart ϕ on M is a continuous injection of some open subset $U \subseteq M$ into $\mathbb{R}^n$. A chart gives every point $p \in U$ in its domain of definition a tuple of n real number coordinates.

$$\phi : U \rightarrowtail \mathbb{R}^n \tag{1}$$

2.1. **Transition Functions.** Let U, V, W be open subsets of M with $W = U \cap V$. Let $\phi : U \rightarrowtail \mathbb{R}^n$ and $\psi : V \rightarrowtail \mathbb{R}^n$ be charts. Every point $p \in W$ is therefore given two, typically distinct, tuples of coordinates. The mapping from one coordinate tuple to the other is called the transition function defined by the pair of charts. Let $t_{\phi,\psi}$ denote that transition function that maps the ϕ coordinates to the ψ coordinates.

$$\forall\, x \in \phi(W) \bullet t_{\phi,\psi}(x) = \psi(\phi^{-1}(x)) \tag{2}$$

2.2. **Compatible Charts.** Let $\mathcal{F}$ be some family of partial injections from $\mathbb{R}^n$ to $\mathbb{R}^n$, e.g. continuous, differentiable, smooth, defined on the open subsets.

$$\mathcal{F} \subseteq \mathbb{R}^n \rightarrowtail\!\!\!\!\!\!\!+\; \mathbb{R}^n \tag{3}$$

A pair of charts are said to be compatible with respect to $\mathcal{F}$ when their transition functions belong to $\mathcal{F}$.

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3. Atlases

A set of pairwise compatible charts that cover M is called an atlas for M. An atlas gives M a manifold structure. If the charts are only required to be continuous then M is called a topological manifold. If the charts are required to be differentiable then the atlas is called a differential or differentiable structure and M is called a differentiable manifold. Infinitely differentiable charts are called smooth charts. We are only concerned with smooth charts and manifolds.

In general, we normally consider an atlas to be a maximal set of charts. A given set of mutually compatible charts belongs to a unique maximal atlas. The given set is said to generate the maximal atlas.

4. Smooth Mappings

Mappings from one smooth manifold to another are called smooth when they are smooth when expressed in their coordinate charts. A smooth mapping that has a smooth inverse is called a diffeomorphism.

5. Tangent Vectors

A tangent vector X at the point $p \in M$ is a mapping from the set of smooth functions at p to $\mathbb{R}$ that satisfies the following for all $c \in \mathbb{R}$ and $f, g \in C^\infty(M, p)$

$$X(cf) = cX(f) \tag{4}$$

$$X(f + g) = X(f) + X(g) \tag{5}$$

$$X(fg) = g(p)X(f) + f(p)X(g) \tag{6}$$

A smooth curve $\gamma : \mathbb{R} \longrightarrow M$ defines a tangent vector X at $p = \gamma(0)$ by

$$X(f) = \left.\frac{df(\gamma(t))}{dt}\right|_{t=0} \tag{7}$$

6. Tangent Bundles

The set of all tangent vectors at p is denoted M_p or $T_p(M)$. It is an n-dimensional vector space and is called the tangent space at p. The set of all tangent spaces is called the tangent bundle and is denoted $T(M)$

$$T(M) = \{\, (p, X) \mid p \in M, X \in M_p \,\} \tag{8}$$

The tangent bundle $T(M)$ is a smooth vector bundle over M under the natural projection $\pi : T(M) \longrightarrow M, \pi(p, X) = p$.

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