Manifolds

Arthur Ryman, `arthur.ryman@gmail.com`

June 25, 2022

Abstract

This article contains Z Notation type declarations for manifolds and some related objects. It has been type checked by fUZZ.

Contents

1 Introduction **1**

2 Charts **1**

2.1 Transition Functions 1

2.2 Compatible Charts 2

3 Atlases **2**

4 Smooth Mappings **2**

5 Tangent Vectors **2**

6 Tangent Bundles **3**

1 Introduction

Manifolds can be defined in several ways. The way I prefer to think about them is that, first of all, they are based on topological spaces. A manifold is therefore a topological space with some additional structure. This additional structure allows one to regard a manifold as, locally, being like an open subset of $\mathbb{R}^n$ for some natural number n referred to as the dimension of the manifold. In the following, let M be a topological space of dimension n.

2 Charts

A chart ϕ on M is a continuous injection of some open subset $U \subseteq M$ into $\mathbb{R}^n$. A chart gives every point $p \in U$ in its domain of definition a tuple of n real number coordinates.

$$\phi : U \rightarrowtail \mathbb{R}^n \tag{1}$$

2.1 Transition Functions

Let U, V, W be open subsets of M with $W = U \cap V$. Let $\phi : U \rightarrowtail \mathbb{R}^n$ and $\psi : V \rightarrowtail \mathbb{R}^n$ be charts. Every point $p \in W$ is therefore given two, typically distinct, tuples of coordinates. The mapping from one coordinate tuple to the other is called the transition function defined by the pair of charts. Let $t_{\phi,\psi}$ denote that transition function that maps the ϕ coordinates to the ψ coordinates.

$$\forall\, x \in \phi(W) \bullet t_{\phi,\psi}(x) = \psi(\phi^{-1}(x)) \tag{2}$$

2.2 Compatible Charts

Let $\mathcal{F}$ be some family of partial injections from $\mathbb{R}^n$ to $\mathbb{R}^n$, e.g. continuous, differentiable, smooth, defined on the open subsets.

$$\mathcal{F} \subseteq \mathbb{R}^n \rightarrowtail\mkern-14mu\rightarrow \mathbb{R}^n \tag{3}$$

A pair of charts are said to be compatible with respect to $\mathcal{F}$ when their transition functions belong to $\mathcal{F}$.

3 Atlases

A set of pairwise compatible charts that cover M is called an atlas for M. An atlas gives M a manifold structure. If the charts are only required to be continuous then M is called a topological manifold. If the charts are required to be differentiable then the atlas is called a differential or differentiable structure and M is called a differentiable manifold. Infinitely differentiable charts are called smooth charts. We are only concerned with smooth charts and manifolds.

In general, we normally consider an atlas to be a maximal set of charts. A given set of mutually compatible charts belongs to a unique maximal atlas. The given set is said to generate the maximal atlas.

4 Smooth Mappings

Mappings from one smooth manifold to another are called smooth when they are smooth when expressed in their coordinate charts. A smooth mapping that has a smooth inverse is called a diffeomorphism.

5 Tangent Vectors

A tangent vector X at the point $p \in M$ is a mapping from the set of smooth functions at p to $\mathbb{R}$ that satisfies the following for all $c \in \mathbb{R}$ and $f, g \in C^\infty(M, p)$

$$X(cf) = cX(f) \tag{4}$$

$$X(f + g) = X(f) + X(g) \tag{5}$$

$$X(fg) = g(p)X(f) + f(p)X(g) \tag{6}$$

A smooth curve $\gamma : \mathbb{R} \longrightarrow M$ defines a tangent vector X at $p = \gamma(0)$ by

$$X(f) = \left. \frac{df(\gamma(t))}{dt} \right|_{t=0} \tag{7}$$

6 Tangent Bundles

The set of all tangent vectors at p is denoted M_p or $T_p(M)$. It is an n-dimensional vector space and is called the tangent space at p. The set of all tangent spaces is called the tangent bundle and is denoted $T(M)$

$$T(M) = \{ (p, X) \mid p \in M, X \in M_p \} \tag{8}$$

The tangent bundle $T(M)$ is a smooth vector bundle over M under the natural projection $\pi : T(M) \longrightarrow M, \pi(p, X) = p$.