# INTEGERS

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ABSTRACT. This article contains Z Notation definitions for concepts related to the integers, $\mathbb{Z}$. It has been type checked by $f$UZZ.

## CONTENTS

## 1. INTRODUCTION

The integers $\mathbb{Z}$ are built-in to Z Notation. This article provides definitions for some related objects so that they can be used and type checked in formal Z specifications.

## 2. EXPONENTIATION

Let $x \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then $x$ raised to the exponent $n$ is the product of $x$ multiplied by itself $n$ times, with the convention that for $n = 0$ the result is 1. Let $exp(x, n)$ denote the result.

$$
\begin{array}{|l}
exp : \mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{Z} \\
\hline
\forall x : \mathbb{Z} \bullet exp(x, 0) = 1 \\
\forall x : \mathbb{Z};\ n : \mathbb{N}_1 \bullet exp(x, n) = x * exp(x, n - 1)
\end{array}
$$

**Example.**

$exp(5, 2) = 25$

**Remark.**

$\forall x : \mathbb{Z} \bullet$
$\qquad exp(x, 0) = 1$

**Remark.**

$\forall\, x : \mathbb{Z} \;\bullet$
$\qquad exp(x, 1) = x$

**Remark.**

$\forall\, x : \mathbb{Z};\, n, m : \mathbb{N} \;\bullet$
$\qquad exp(x, n + m) = exp(x, n) * exp(x, m)$

**Remark.**

$\forall\, x, y : \mathbb{Z};\, n : \mathbb{N} \;\bullet$
$\qquad exp(x * y, n) = exp(x, n) * exp(y, n)$

Exponentiation is normally denoted $x^n$ but Z Notation cannot reproduce that. Instead we use the infix operator notation that is common to programming languages such as FORTRAN and Python. Define the notation $x ** n = exp(x, n)$.

$(\_ ** \_) == exp$

**Example.**

$5 ** 2 = 25$

## 3. Divisibility

This section defines *divisibility* of integers.

Given integers $x$ and $y$ we say that $x$ *divides* $y$ if there is some integer $q$ such that $qx = y$.

┌─ *Divides* ─────────────────────────
│ $x, y, q : \mathbb{Z}$
├─────────────────
│ $q * x = y$
└──────────────────────────────────────

Let *divides* denote the divisibility relation between integers where $(x, y) \in divides$ means that $x$ divides $y$.

$divides == \{\, Divides \bullet x \mapsto y \,\}$

**Remark.**

$divides \in \mathbb{Z} \leftrightarrow \mathbb{Z}$

We introduce the usual infix notation $x \mid y$ to denote that $(x, y) \in divides$.

$(\_ \mid \_) == divides$

**Example.** *The integer* $7$ *divides* $42$ *because* $6 * 7 = 42$.

$7 \mid 42$

**Remark.** *Every integer* $x$ *divides* $0$ *because* $0 * x = 0$.

$\forall\, x : \mathbb{Z} \bullet x \mid 0$

## 4. Divisors

Let $x$ be a nonzero integer that divides the integer $y$. We say that $x$ is a *divisor* of $y$.

```
__ Divisor _____
  Divides
  _____
  x ≠ 0
```

Let the relation $(x, y) \in \mathit{is\_divisor\_of}$ denote that $x$ is a divisor of $y$.

$\mathit{is\_divisor\_of} == \{\, \mathit{Divisor} \bullet x \mapsto y \,\}$

Let the set $\mathit{divisors}(y)$ denote the set of all divisors of the integer $y$.

$\mathit{divisors} == (\lambda\, y : \mathbb{Z} \bullet \{\, x : \mathbb{Z} \mid (x, y) \in \mathit{is\_divisor\_of} \,\})$

**Remark.**

$\mathit{divisors} \in \mathbb{Z} \longrightarrow \mathbb{P}\,\mathbb{Z}$

**Example.** *The integer* $6$ *has the following divisors.*

$\mathit{divisors}(6) = \{-6, -3, -2, -1, 1, 2, 3, 6\}$

Let the set $\mathit{positive\_divisors}(y)$ denote the set of all positive divisors of the integer $y$.

$\mathit{positive\_divisors} == (\lambda\, y : \mathbb{Z} \bullet \mathit{divisors}(y) \cap \mathbb{N}_1)$

**Remark.**

$\mathit{positive\_divisors} \in \mathbb{Z} \longrightarrow \mathbb{P}\,\mathbb{N}_1$

**Example.** *The integer* $6$ *has the following positive divisors.*

$\mathit{positive\_divisors}(6) = \{1, 2, 3, 6\}$

## 5. Prime Numbers

An integer $p$ is *prime* if it is greater than one and only has one and itself as positive divisors.

```
__ Prime _____
  p : ℤ
  _____
  p > 1
  positive_divisors(p) = {1, p}
```

- $p$ is greater than 1.

- 1 and $p$ are the only positive divisors of $p$.

**Example.** *The integer 2 is prime.*

**let** $p == 2 \bullet$
    $Prime$

Let *primes* denote the set of all primes.

$primes == \{ \, Prime \bullet p \, \}$

**Remark.**

$primes \subset \mathbb{N}_1$

**Example.** *The natural numbers* $2, 3, 5,$ *and* $7$ *are primes.*

$\{2, 3, 5, 7\} \subseteq primes$

## 6. Addition of Integer Sequences

Let $l$ be a natural number and let $x$ and $y$ be two integer sequences of length $l$. Their sum $z = x + y$ is the integer sequence of length $l$ defined by pointwise addition of the terms in $x$ and $y$.

―――― $AddIntegerSequences$ ――――――――――――――
$l : \mathbb{N}$
$x, y, z : \text{seq}\, \mathbb{Z}$
――――――――――
$l = \#x = \#y$
$z = (\lambda\, i : 1 \,..\, l \bullet x\,i + y\,i)$
―――――――――――――――――――――――――――――――

- The sequence $z$ is defined by pointwise addition of the sequences $x$ and $y$.

Let the function $add\_int\_seq(x, y) = z$ be the sum of two equal-length integer sequences.

$add\_int\_seq == \{ \, AddIntegerSequences \bullet (x, y) \mapsto z \, \}$

**Remark.** *Addition is a partial function on the set of all pairs of integer sequences.*

$add\_int\_seq \in \text{seq}\, \mathbb{Z} \times \text{seq}\, \mathbb{Z} \nrightarrow \text{seq}\, \mathbb{Z}$

We introduce the usual infix notation $x + y = add\_int\_seq(x, y)$.

$(\_ + \_) == add\_int\_seq$

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