Integers

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Abstract

This article contains Z Notation type declarations for the integers, $\mathbb{Z}$, and some related objects. It has been type checked by *f*UZZ.

Contents

1 Introduction

The integers, $\mathbb{Z}$, are built-in to Z Notation. This article provides type declarations for some related objects so that they can be used and type checked in formal Z specifications.

2 Divisibility and Prime Numbers

2.1 Divisibility

This section specifies divisibility of integers.

2.1.1 *Divides*

Given integers x and y we say the x *divides* y if there is some integer q such that $qx = y$. Let the schema *Divides* denote this situation.

$$
\begin{array}{|l}
\textit{Divides} \\ \hline
x, y, q : \mathbb{Z} \\ \hline
q * x = y \\ \hline
\end{array}
$$

- y is a multiple of x.

2.1.2 *divides*

Let *divides* denote the divisibility relation between integers where $(x, y) \in \mathit{divides}$ means that x divides y.

$$
\begin{array}{|l}
\mathit{divides} : \mathbb{Z} \leftrightarrow \mathbb{Z} \\ \hline
\mathit{divides} = \{ \mathit{Divides} \bullet x \mapsto y \}
\end{array}
$$

2.1.3 $\mid$ `\divides`

We introduce the usual notation $x \mid y$ to denote that $(x, y) \in \mathit{divides}$.

$$(\_ \mid \_) == \mathit{divides}$$

Example. *The integer* 7 *divides* 42 *because* $6 * 7 = 42$.

$$7 \mid 42$$

Remark. *Every integer* x *divides* 0 *because* $0 * x = 0$.

$$\forall x : \mathbb{Z} \bullet x \mid 0$$

2.1.4 *Divisor*

Let x be a nonzero integer that divides the integer y. We say that x is a *divisor* of y. Let the schema *Divisor* denote this situation.

$$
\begin{array}{|l}
\hline
\textit{Divisor} \\
x, y : \mathbb{Z} \\
\hline
x \neq 0 \\
x \mid y \\
\hline
\end{array}
$$

- x is nonzero.
- x divides y.

2.1.5 *divisors*

Let the set $\textit{divisors}(y)$ denote the set of all divisors of the integer y.

$$
\begin{array}{|l}
\textit{divisors} : \mathbb{Z} \longrightarrow \mathbb{P}\,\mathbb{Z} \\
\hline
\forall\, y : \mathbb{Z} \bullet \\
\qquad \textit{divisors}(y) = \{\, x : \mathbb{Z} \mid \textit{Divisor} \,\}
\end{array}
$$

Example. *The integer* 6 *has the following divisors.*

$$\textit{divisors}(6) = \{-6, -3, -2, -1, 1, 2, 3, 6\}$$

2.1.6 *positive_divisors*

Let the set $\textit{positive\_divisors}(y)$ denote the set of all positive divisors of the integer y.

$$
\begin{array}{|l}
\textit{positive\_divisors} : \mathbb{Z} \longrightarrow \mathbb{P}\,\mathbb{N}_1 \\
\hline
\forall\, y : \mathbb{Z} \bullet \\
\qquad \textit{positive\_divisors}(y) = \textit{divisors}(y) \cap \mathbb{N}_1
\end{array}
$$

Example. *The integer* 6 *has the following positive divisors.*

$$\textit{positive\_divisors}(6) = \{1, 2, 3, 6\}$$

2.2 Prime Numbers

2.2.1 *Prime*

An integer p is *prime* if it is greater than one and only has one and itself as positive divisors. Let the schema *Prime* denote this situation.

$$
\begin{array}{|l}
\textit{Prime} \\
\hline
p : \mathbb{N} \\
\hline
p > 1 \\
\textit{positive\_divisors}(p) = \{1, p\} \\
\hline
\end{array}
$$

- p is greater than 1.
- 1 and p are the only positive divisors of p.

Example. *The integer* 2 *is prime.*

$$\textbf{let } p == 2 \bullet \textit{Prime}$$

2.2.2 *primes*

Let *primes* denote the set of all primes.

$$
\begin{array}{|l}
\textit{primes} : \mathbb{P}\,\mathbb{N}_1 \\
\hline
\textit{primes} = \{\, \textit{Prime} \bullet p \,\}
\end{array}
$$

Example. *The natural numbers* $2, 3, 5,$ *and* 7 *are primes.*

$$\{2, 3, 5, 7\} \subseteq \textit{primes}$$

3 Integer Sequences

3.1 Addition of Integer Sequences

3.1.1 *AddIntegerSequences*

Let l be a natural number and let x and y be two integer sequences of length l. Their sum $z = x + y$ is the integer sequence of length l defined by pointwise addition of the terms in x and y. Let the schema *AddIntegerSequences* denote this situation.

AddIntegerSequences

$$
\begin{array}{|l}
l : \mathbb{N} \\
x, y, z : \mathrm{seq}\,\mathbb{Z} \\
\hline
l = \#x = \#y \\
z = (\lambda\, i : 1 \mathrel{..} l \bullet x\, i + y\, i)
\end{array}
$$

- The sequence z is defined by pointwise addition of the sequences x and y.

3.1.2 *add_int_seq*

Let the function $add\_int\_seq(x, y) = z$ be the sum of two equal-length integer sequences.

$$
\begin{array}{|l}
add\_int\_seq : \mathrm{seq}\,\mathbb{Z} \times \mathrm{seq}\,\mathbb{Z} \pfun \mathrm{seq}\,\mathbb{Z} \\
\hline
add\_int\_seq = \{\, AddIntegerSequences \bullet (x, y) \mapsto z \,\}
\end{array}
$$

3.1.3 $+$ \addSeqZ

We introduce the notation $x + y = add\_int\_seq(x, y)$.

$$(\_ + \_) == add\_int\_seq$$