COMPLEX NUMBERS

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Abstract. This article contains Z Notation definitions for the complex numbers, $\mathbb{C}$, and some related objects. It has been type checked by fUZZ.

Contents

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1. Introduction

The complex numbers, $\mathbb{C}$, are foundational to many mathematical objects such as vector spaces and manifolds, but are not built into Z Notation. This article provides type declarations for $\mathbb{C}$ and related objects so that they can be used and type checked in formal Z specifications.

No attempt has been made to provide complete, axiomatic definitions of all these objects since that would only be of use for proof checking. Although proof checking is highly desirable, it is beyond the scope of this article. The type declarations given here are intended to provide a basis for future axiomatization.

1.1. The Set of Complex Numbers.

Z Notation does not predefine the set of complex numbers, so we define them and operations on them here.

Although complex number operations are displayed using the same symbols as the analogous real number operations, they are distinct mathematical objects. This distinction is apparent to the fUZZ type-checker and should not cause confusion to the human reader because the underlying types of the objects will, as a rule, be clear from the context. Visually distinct symbols will be used in cases where confusion is possible.

1.1.1. *COMPLEX*. A complex number can be thought of a pair of real numbers. However, it is not correct to view any pair of real numbers as a complex number. Therefore to denote a complex number we map a pair of real numbers into the free type *COMPLEX*.

$COMPLEX ::= complex\langle\!\langle \mathbb{R}^2 \rangle\!\rangle$

Here the function *complex* is a constructor that constructs a complex number from a pair of real numbers.

1.1.2. $\mathbb{C}$ `\C`. We introduce the usual notation $\mathbb{C} = COMPLEX$ for the set of complex numbers.

$\mathbb{C} == COMPLEX$

1.2. Real and Imaginary Parts.

1.2.1. *Complex*. Given real numbers x and y, we can construct the complex number $z = complex(x, y)$. The real numbers x and y are referred to as the *real* and *imaginary* parts of z. It's useful to introduce the schema *Complex* that relates the complex number z to its real and imaginary parts x and y.

Complex

$z : \mathbb{C}$
$x, y : \mathbb{R}$

$z = complex(x, y)$

1.2.2. *real_of_complex*. Let $real\_of\_complex(z)$ denote the real part x of z.

$real\_of\_complex : \mathbb{C} \longrightarrow \mathbb{R}$

$real\_of\_complex = \{ Complex \bullet z \mapsto x \}$

1.2.3. Re `\realC`. We introduce the usual notation $x = \mathrm{Re}(z)$ for the real part of z.

$\mathrm{Re} == real\_of\_complex$

1.2.4. *imag_of_complex*. Let $imag\_of\_complex(z)$ denote the imaginary part y of z.

$imag\_of\_complex : \mathbb{C} \longrightarrow \mathbb{R}$

$imag\_of\_complex = \{ Complex \bullet z \mapsto y \}$

1.2.5. Im `\imagC`. We introduce the usual notation $y = \mathrm{Im}(z)$ for the imaginary part of z.

$\mathrm{Im} == imag\_of\_complex$

1.3. **Real Numbers as Complex Numbers.** The complex numbers contain a natural copy of the real numbers, namely the set of complex numbers with vanishing imaginary part.

1.3.1. *real_as_complex*. Let the function $real\_as\_complex(x) = z$ map the real number x to its corresponding complex number z which has x as its real part and 0 as its imaginary part.

$real\_as\_complex : \mathbb{R} \longrightarrow \mathbb{C}$

$real\_as\_complex = (\lambda\, x : \mathbb{R} \bullet complex(x, 0))$

1.3.2. $\mathsf{complex}$ `\asRC`. We introduce the notation $\mathsf{complex}\, x = real\_as\_complex(x)$.

$\mathsf{complex} == real\_as\_complex$

1.4. **Conjugation.**

1.4.1. *ComplexConjugate*. Let $z' = z^*$ be the *complex conjugate* of the complex number z. Let the schema *ComplexConjugate* denote this situation.

$$\begin{array}{|l}
\hline
\textit{ComplexConjugate} \\
\textit{Complex} \\
\textit{Complex}' \\
\hline
x' = x \\
y' = \text{-}\, y \\
\hline
\end{array}$$

- The complex conjugate z' of a complex number z has the same real part x and the negative imaginary part $-y$.

1.4.2. *complex_conjugate*. Let the function $complex\_conjugate(z) = z^*$ map the complex number z to its complex conjugate z^*.

$$\begin{array}{|l}
complex_conjugate : \mathbb{C} \longrightarrow \mathbb{C} \\
\hline
complex_conjugate = \{ ComplexConjugate \bullet z \mapsto z' \}
\end{array}$$

1.4.3. * `\conjC`. We introduce the usual postfix operator notation $z^* = complex\_conjugate(z)$.

$(\_^*) == complex\_conjugate$

2. Some Important Complex Numbers

We next define some important complex numbers.

2.1. **Zero.**

2.1.1. *zero_complex*. Let *zero_complex* denote the *zero* of $\mathbb{C}$.

$$\begin{array}{|l}
zero_complex : \mathbb{C} \\
\hline
zero_complex = \mathsf{complex}\, 0
\end{array}$$

2.1.2. 0 `\zeroC`. We introduce the usual notation $0 \in \mathbb{C}$ for the zero of $\mathbb{C}$.

$0 == zero\_complex$

2.2. **One.**

2.2.1. *one_complex*. Let *one_complex* denote the multiplicative unit in $\mathbb{C}$.

$$\begin{array}{|l}
one_complex : \mathbb{C} \\
\hline
one_complex = \mathsf{complex}\, 1
\end{array}$$

2.2.2. 1 `\oneC`. We introduce the usual notation $1 \in \mathbb{C}$ for the unit of $\mathbb{C}$.

$1 == one\_complex$

2.3. **The Square Root of -1.**

2.3.1. *i_complex*. Let *i_complex* denote the usual square root of -1 in $\mathbb{C}$.

$$\begin{array}{|l}
i_complex : \mathbb{C} \\
\hline
i_complex = complex(0, 1)
\end{array}$$

2.3.2. i `\iC`. We introduce the usual notation $i = i\_complex$.

$i == i\_complex$

3. Arithmetic

3.1. **Addition.**

3.1.1. *AddComplex.* We can *add* the complex numbers z_1 and z_2 to give their *sum* $z' = z_1 + z_2$. Let the schema *AddComplex* denote this situation.

AddComplex
$Complex_1$
$Complex_2$
$Complex'$

$x' = x_1 + x_2$
$y' = y_1 + y_2$

3.1.2. *add_complex.* Let $add\_complex(z_1, z_2)$ denote the result of adding z_1 and z_2.

$add\_complex : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$

$add\_complex = \{ AddComplex \bullet (z_1, z_2) \mapsto z' \}$

3.1.3. $+$ `\addC`. We introduce the usual notation $z' = z_1 + z_2$ for addition in $\mathbb{C}$.

$(\_ + \_) == add\_complex$

3.2. **Negation.**

3.2.1. *NegComplex.* We can *negate* the complex number z to give its *negative* $z' = -z$. Let the schema *NegComplex* denote this situation.

NegComplex
$Complex$
$Complex'$

$x' = \text{-}\, x$
$y' = \text{-}\, y$

3.2.2. $neg\_complex(z)$. Let $neg\_complex(z)$ denote the negative of z.

$neg\_complex : \mathbb{C} \longrightarrow \mathbb{C}$

$neg\_complex = \{ NegComplex \bullet z \mapsto z' \}$

3.2.3. `-` `\negC`. We introduce the usual notation $z' = \text{-}\, z$ for the negative of z.

$\text{-} == neg\_complex$

3.3. **Subtraction.**

3.3.1. *SubComplex.* We can *subtract* the complex number z_2 from z_1 to give their *difference* $z' = z_1 - z_2$. Let the schema *SubComplex* denote this situation.

$$
\begin{array}{|l}
\textit{SubComplex} \\
\hline
\textit{Complex}_1 \\
\textit{Complex}_2 \\
\textit{Complex}' \\
\hline
x' = x_1 - x_2 \\
y' = y_1 - y_2 \\
\hline
\end{array}
$$

3.3.2. *sub_complex.* Let $\mathit{sub\_complex}(z_1, z_2)$ denote the difference $z_1 - z_2$.

$$
\begin{array}{|l}
\mathit{sub\_complex} : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C} \\
\hline
\mathit{sub\_complex} = \{\, \mathit{SubComplex} \bullet (z_1, z_2) \mapsto z' \,\}
\end{array}
$$

3.3.3. $-$ `\subC`. We introduce the usual notation $z' = z_1 - z_2$ for subtraction in $\mathbb{C}$.

$$(\_ - \_) == \mathit{sub\_complex}$$

3.4. **Multiplication.**

3.4.1. *MulComplex.* We can *multiply* the complex numbers z_1 times z_2 to give their *product* $z' = z_1 z_2$. Let the schema *MulComplex* denote this situation.

$$
\begin{array}{|l}
\textit{MulComplex} \\
\hline
\textit{Complex}_1 \\
\textit{Complex}_2 \\
\textit{Complex}' \\
\hline
x' = x_1 * x_2 - y_1 * y_2 \\
y' = x_1 * y_2 + y_1 * x_2 \\
\hline
\end{array}
$$

3.4.2. *mul_complex.* Let $\mathit{mul\_complex}(z_1, z_2)$ denote the product $z_1 z_2$.

$$
\begin{array}{|l}
\mathit{mul\_complex} : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C} \\
\hline
\mathit{mul\_complex} = \{\, \mathit{MulComplex} \bullet (z_1, z_2) \mapsto z' \,\}
\end{array}
$$

3.4.3. $*$ `\mulC`. We introduce the usual notation $z' = z_1 * z_2$ for multiplication in $\mathbb{C}$.

$$(\_ * \_) == \mathit{mul\_complex}$$

3.4.4. *mul_real_complex.* Let the function $\mathit{mul\_real\_complex}(x, z) = xz$ denote the product of the real number x and the complex number z.

$$
\begin{array}{|l}
\mathit{mul\_real\_complex} : \mathbb{R} \times \mathbb{C} \longrightarrow \mathbb{C} \\
\hline
\mathit{mul\_real\_complex} = (\lambda\, x : \mathbb{R};\, z : \mathbb{C} \bullet (\mathsf{complex}\, x) * z)
\end{array}
$$

3.4.5. $*$ `\mulRC`. We introduce the notation $x * z = \mathit{mul\_real\_complex}(x, z)$.

$$(\_ * \_) == \mathit{mul\_real\_complex}$$

3.4.6. *mul_complex_real*. Similarly, let the function $mul\_complex\_real(z, x) = zx$ denote the product of the complex number z and the real number x.

$$
\begin{array}{|l}
mul\_complex\_real : \mathbb{C} \times \mathbb{R} \longrightarrow \mathbb{C} \\
\hline
mul\_complex\_real = (\lambda\, z : \mathbb{C};\, x : \mathbb{R} \bullet z * (\mathsf{complex}\, x))
\end{array}
$$

3.4.7. $*$ `\mulCR`. We introduce the notation $z * z = mul\_complex\_real(z, x)$.

$(\_ * \_) == mul\_complex\_real$

3.5. **Nonzero Complex Numbers.**

3.5.1. *nonzero_complex*. Let *nonzero_complex* denote the set of nonzero complex numbers.

$$
\begin{array}{|l}
nonzero\_complex : \mathbb{P}\, \mathbb{C} \\
\hline
nonzero\_complex = \mathbb{C} \setminus \{0\}
\end{array}
$$

3.5.2. $\mathbb{C}_*$ `\Cnz`. We introduce the usual notation $\mathbb{C}_*$ to denote the set of nonzero complex numbers, also referred to as the *punctured complex number plane*.

$\mathbb{C}_* == nonzero\_complex$

3.6. **Nonzero Multiplication.**

3.6.1. *MulNonzeroComplex*. We can restrict multiplication in $\mathbb{C}$ to $\mathbb{C}_*$. Let the schema *MulNonzeroComplex* denote this situation.

$$
\begin{array}{|l}
\hline
\textit{MulNonzeroComplex} \\
\hline
\textit{MulComplex} \\
\hline
z_1 \in \mathbb{C}_* \\
z_2 \in \mathbb{C}_* \\
\hline
\end{array}
$$

3.6.2. *mul_nonzero_complex*. Let $mul\_nonzero\_complex(z_1, z_2)$ denote the product of nonzero complex numbers.

$$
\begin{array}{|l}
mul\_nonzero\_complex : \mathbb{C}_* \times \mathbb{C}_* \longrightarrow \mathbb{C}_* \\
\hline
mul\_nonzero\_complex = \{\, MulNonzeroComplex \bullet (z_1, z_2) \mapsto z' \,\}
\end{array}
$$

3.6.3. $*$ `\mulCnz`. We introduce the usual notation $z' = z_1 * z_2$ to denote the product.

$(\_ * \_) == mul\_nonzero\_complex$

3.7. **Inversion.**

3.7.1. *InvNonzeroComplex*. We can *invert* the nonzero complex number z to get its *inverse* or *reciprocal* $z' = z^{-1}$. Let the schema *InvNonzeroComplex* denote this situation.

$$\begin{array}{|l}
\hline
\textit{InvNonzeroComplex} \\
z, z' : \mathbb{C}_* \\
\hline
z * z' = 1 \\
\hline
\end{array}$$

3.7.2. *inv_nonzero_complex*. Let $z' = \mathit{inv\_nonzero\_complex}(z)$ denote the inverse of z.

$$\begin{array}{|l}
\mathit{inv_nonzero_complex} : \mathbb{C}_* \longrightarrow \mathbb{C}_* \\
\hline
\mathit{inv_nonzero_complex} = \{\, \mathit{InvNonzeroComplex} \bullet z \mapsto z' \,\}
\end{array}$$

3.7.3. $^{-1}$ `\invCnz`. We introduce the usual notation $z' = z^{-1}$ for the inverse.

$$(\_^{-1}) == \mathit{inv\_nonzero\_complex}$$

3.8. **Division.**

3.8.1. *DivNonzeroComplex*. We can *divide* the complex number z_1 by the nonzero complex number z_2 to get their *quotient* $z' = z_1/z_2$. Let the schema *DivNonzeroComplex* denote this situation.

$$\begin{array}{|l}
\hline
\textit{DivNonzeroComplex} \\
z_1, z' : \mathbb{C} \\
z_2 : \mathbb{C}_* \\
\hline
z_1 = z' * z_2 \\
\hline
\end{array}$$

3.8.2. *div_nonzero_complex*. Let $z' = \mathit{div\_nonzero\_complex}(z_1, z_2)$ denote z_1 divided by z_2.

$$\begin{array}{|l}
\mathit{div_nonzero_complex} : \mathbb{C} \times \mathbb{C}_* \longrightarrow \mathbb{C} \\
\hline
\mathit{div_nonzero_complex} = \{\, \mathit{DivNonzeroComplex} \bullet (z_1, z_2) \mapsto z' \,\}
\end{array}$$

3.8.3. / `\divCnz`. We introduce the usual notation $z_1 \,/\, z_2$ to denote division.

$$(\_ \,/\, \_) == \mathit{div\_nonzero\_complex}$$

3.9. **Power.**

3.9.1. *power_complex_nat*. Let the function $\mathit{power\_complex\_nat}(z, n) = z^n$ denote the result of raising the complex number z to the power n where n is a natural number.

$$\begin{array}{|l}
\mathit{power_complex_nat} : \mathbb{C} \times \mathbb{N} \longrightarrow \mathbb{C} \\
\hline
\forall\, z : \mathbb{C} \bullet \\
\qquad \mathit{power_complex_nat}(z, 0) = 1 \\
\forall\, z : \mathbb{C};\, n : \mathbb{N}_1 \bullet \\
\qquad \mathit{power_complex_nat}(z, n) = z * \mathit{power_complex_nat}(z, n - 1)
\end{array}$$

Remark. *The expression* 0^0 *is problematic, but here we define it to be* 1 *since this is the most convenient for complex polynomials.*

$power\_complex\_nat(0, 0) = 1$

3.9.2. $**$ `\powCN`. We introduce the notation $z ** n = power\_complex\_nat(z, n)$.

$(\_ ** \_) == power\_complex\_nat$

3.10. **Norm.**

3.10.1. *NormComplex*. The *norm* of a complex number z is a non-negative real number r equal to the Euclidean length of its underlying pair of real numbers regarded as a vector in the Euclidean plane. Let the schema *NormComplex* denote this situation.

NormComplex

$Complex$
$r : \mathbb{R}$

$r = \mathrm{sqrt}(x * x + y * y)$

3.10.2. *norm_complex*. Let $r = norm\_complex(z)$ be the norm of z.

$norm\_complex : \mathbb{C} \longrightarrow \mathbb{R}$

$norm\_complex = \{ NormComplex \bullet z \mapsto r \}$

3.10.3. norm `\normC`. We introduce the notation $r = \mathrm{norm}(z)$ to denote the norm of z.

$\mathrm{norm} == norm\_complex$

4. Some Important Functions

We declare the usual exponential and trigonometric functions here.

4.1. **Exponential.**

4.1.1. *exp_complex*. Let $exp\_complex(z) = e^z$.

$exp\_complex : \mathbb{C} \longrightarrow \mathbb{C}$

4.1.2. exp `\expC`. We introduce the notation $\exp z = exp\_complex(z)$.

$\exp == exp\_complex$

4.2. **Logarithm.**

4.2.1. *log_complex*. Let $log\_complex(z) = \log z$.

$log\_complex : \mathbb{C}_* \longrightarrow \mathbb{C}$

4.2.2. log `\logC`. We introduce the notation $\log z = log\_complex(z)$.

$\log == log\_complex$

4.3. **Sine.**

4.3.1. *sin_complex*. Let $sin\_complex(z) = \sin z$.

| $sin\_complex : \mathbb{C} \longrightarrow \mathbb{C}$

4.3.2. sin `\sinC`. We introduce the notation $\sin z = sin\_complex(z)$.

$\sin == sin\_complex$

4.4. **Cosine.**

4.4.1. *cos_complex*. Let $cos\_complex(z) = \cos z$.

| $cos\_complex : \mathbb{C} \longrightarrow \mathbb{C}$

4.4.2. cos `\cosC`. We introduce the notation $\cos z = cos\_complex(z)$.

$\cos == cos\_complex$

4.5. **Tangent.**

4.5.1. *tan_complex*. Let $tan\_complex(z) = \tan z$.

| $tan\_complex : \mathbb{C} \nrightarrow \mathbb{C}$

4.5.2. tan `\tanC`. We introduce the notation $\tan z = tan\_complex(z)$.

$\tan == tan\_complex$

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