Topological Spaces

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Abstract

This article defines topological spaces and related concepts.

1 Topological Spaces

1.1 *Topology*

A *topology* τ on X is a family of subsets of X, referred to as the *open* subsets of X, that satisfy the following axioms.

Topology[X]

$\tau : \mathcal{F} X$

$\varnothing \in \tau$

$X \in \tau$

$\forall F : \mathbb{F}\,\tau \bullet \bigcap F \in \tau$

$\forall F : \mathbb{P}\,\tau \bullet \bigcup F \in \tau$

- The empty set is open.
- The whole set is open.
- The intersection of a finite family of open sets is open.
- The union of any family of open sets is open.

1.2 *top* and *tops*

Let $top[X]$ denote the set of all topologies on X.

$$
\begin{array}{l}
[X] \\
\hline
top : \mathbb{P}(\mathcal{F} X) \\
\hline
top = \{\, Topology[X] \bullet \tau \,\}
\end{array}
$$

Let $tops[X]$ denote the set of all topologies on subsets $U \subseteq X$.

$$
\begin{array}{l}
[X] \\
\hline
tops : \mathbb{P}(\mathcal{F} X) \\
\hline
tops = \bigcup\{\, U : \mathbb{P}\, X \bullet top[U] \,\}
\end{array}
$$

1.3 *discrete* and *indiscrete*

The *discrete* topology on X consists of all subsets of X. The *indiscrete* topology on X consists of just X and $\varnothing$. Let $discrete[X]$ and $indiscrete[X]$ denote the discrete and indiscrete topologies on X.

$$
\begin{array}{l}
[X] \\
\hline
discrete, indiscrete : \mathcal{F} X \\
\hline
discrete = \mathbb{P}\, X \\
indiscrete = \{\varnothing, X\}
\end{array}
$$

Example. *Let* X *be an arbitrary set. Then* $discrete[\mathsf{X}]$ *and* $indiscrete[\mathsf{X}]$ *are topologies on* X.

$$discrete[\mathsf{X}] \in top[\mathsf{X}]$$

$$indiscrete[\mathsf{X}] \in top[\mathsf{X}]$$

1.4 *topGen*

Remark. *The intersection of a set of topologies on* X *is also a topology on* X.

Given a family B of subsets of X, the topology *generated by* B is the intersection of all topologies that contain B. The set B is referred to as a *basis* for the topology it generates. Let $topGen[X]\, B$ denote the topology on X generated by the basis B.

$$
\begin{array}{l}
[X] \\
\hline
topGen : \mathcal{F} X \longrightarrow top[X] \\
\hline
\forall\, B : \mathcal{F} X \bullet \\
\qquad topGen\, B = \bigcap\{\, \tau : top[X] \mid B \subseteq \tau \,\}
\end{array}
$$

Example. *Let* X *be an arbitrary set.*

$$topGen[\mathsf{X}]\varnothing = indiscrete[\mathsf{X}]$$

$$topGen[\mathsf{X}]\{\varnothing\} = indiscrete[\mathsf{X}]$$

$$topGen[\mathsf{X}]\{\mathsf{X}\} = indiscrete[\mathsf{X}]$$

1.5 *topSpace*

Let X be a set. A *topological space* is a pair (X, τ) where τ is a topology on X. Let $topSpace[X]$ denote the set of all topological spaces (X, τ).

$$topSpace[X] == \{\, \tau : top[X] \bullet (X, \tau) \,\}$$

Example. *Let* X *be an arbitrary set.*

$$(\mathsf{X}, indiscrete[\mathsf{X}]) \in topSpace[\mathsf{X}]$$

$$(\mathsf{X}, discrete[\mathsf{X}]) \in topSpace[\mathsf{X}]$$

1.6 *topSpaces*

Let $topSpaces[t]$ dentote the set of all topological spaces (X, τ) where X is a subset of t.

$$[t]$$

$$topSpaces : \mathbb{P}\, t \leftrightarrow \mathcal{F} t$$

$$topSpaces = \{\, X : \mathbb{P}\, t;\ \tau : \mathcal{F} t \mid \tau \in top[X] \,\}$$

Remark.

$$topSpace[\mathsf{X}] \subseteq topSpaces[\mathsf{X}]$$

2 Continuous Mappings

Let (X, τ) and (Y, σ) be topological spaces.

2.1 *Continuous*

A mapping $f \in X \longrightarrow Y$ is said to be *continuous* if the inverse image of every open set is open.

$$
\begin{array}{|l}
\hline
\textit{Continuous}[X, Y] \\
f : X \longrightarrow Y \\
\tau : \textit{top}[X] \\
\sigma : \textit{top}[Y] \\
\hline
\forall\, U : \sigma \bullet \\
\quad f^{\sim}(\!| U |\!) \in \tau \\
\hline
\end{array}
$$

2.2 C^0 `\CzeroTT`

Let A and B be topological spaces, and let $\mathrm{C}^0(A, B)$ denote the set of continuous mappings from A to B.

$$
\begin{array}{|l}
\hline\hline
[X, Y] \\
\mathrm{C}^0 : \textit{topSpace}[X] \times \textit{topSpace}[Y] \longrightarrow \mathbb{P}(X \longrightarrow Y) \\
\hline
\forall\, \tau : \textit{top}[X]; \sigma : \textit{top}[Y] \bullet \\
\quad \mathbf{let}\ A == (X, \tau); B == (Y, \sigma) \bullet \\
\qquad \mathrm{C}^0(A, B) = \{ f : X \longrightarrow Y \mid \textit{Continuous}[X, Y] \} \\
\hline
\end{array}
$$

2.3 The Identity Mapping

Remark. *The identity mapping is continuous.*

$$
\begin{array}{l}
\forall\, \tau : \textit{top}[\mathsf{X}] \bullet \\
\quad \mathbf{let}\ A == (\mathsf{X}, \tau) \bullet \\
\qquad \mathrm{id}\,\mathsf{X} \in \mathrm{C}^0(A, A)
\end{array}
$$

Remark. *The constant mapping is continuous.*

$$
\begin{array}{l}
\forall\, \tau : \textit{top}[\mathsf{X}]; \sigma : \textit{top}[\mathsf{Y}]; c : \mathsf{Y} \bullet \\
\quad \mathbf{let}\ A == (\mathsf{X}, \tau); B == (\mathsf{Y}, \sigma) \bullet \\
\qquad \mathrm{const}[\mathsf{X}, \mathsf{Y}]c \in \mathrm{C}^0(A, B)
\end{array}
$$

2.4 Composition of Continuous Mapping

Remark. *Let* X, Y, *and* Z *be arbitrary sets. The composition of continuous mappings is a continuous mapping.*

$$
\begin{array}{l}
\forall\, A : \textit{topSpace}[\mathsf{X}]; B : \textit{topSpace}[\mathsf{Y}]; C : \textit{topSpace}[\mathsf{Z}] \bullet \\
\quad \forall\, f : \mathrm{C}^0(A, B); g : \mathrm{C}^0(B, C) \bullet \\
\qquad g \circ f \in \mathrm{C}^0(A, C)
\end{array}
$$

3 Induced Topology

Let $A = (X, \tau)$ be a topological space and let $U \subseteq X$ be a subset. The topology on X *induces* a topology on U. This topology is variously referred to as the *induced*, *relative*, or *subspace* topology on U.

3.1 $|_{\mathcal{F}}$ \inducedFam

Let ϕ be a family of subsets of X and let U be a subset of X. The family of subsets of U *induced* by ϕ is the set of intersections of the members of ϕ with U. Let $\phi \,|_{\mathcal{F}}\, U$ denote the family on U induced by ϕ.

$$
\begin{array}{l}
[X] \\
\hline
\_ \,|_{\mathcal{F}}\, \_ : \mathcal{F}X \times \mathbb{P}\, X \longrightarrow \mathcal{F}X \\
\hline
\forall\, \phi : \mathcal{F}X;\ U : \mathbb{P}\, X \bullet \\
\qquad \phi \,|_{\mathcal{F}}\, U = \{\ Y : \phi \bullet Y \cap U\ \}
\end{array}
$$

Remark. *If τ is a topology on X then $\tau \,|_{\mathcal{F}}\, U$ is a topology on U.*

$$
\begin{array}{l}
\forall\, \tau : \mathit{top}[\mathsf{X}];\ U : \mathbb{P}\, \mathsf{X} \bullet \\
\qquad \tau \,|_{\mathcal{F}}\, U \in \mathit{top}[U]
\end{array}
$$

3.2 $|_{\mathsf{top}}$ \inducedTopSp

Let $(X, \tau) \,|_{\mathsf{top}}\, U$ denote the corresponding induced topological space.

$$
\begin{array}{l}
[X] \\
\hline
\_ \,|_{\mathsf{top}}\, \_ : \mathit{topSpace}[X] \times \mathbb{P}\, X \longrightarrow \mathit{topSpaces}[X] \\
\hline
\forall\, \tau : \mathit{top}[X];\ U : \mathbb{P}\, X \bullet \\
\qquad (X, \tau) \,|_{\mathsf{top}}\, U = (U, \tau \,|_{\mathcal{F}}\, U)
\end{array}
$$

4 Product Topology

Let (X, τ) and (Y, σ) be topological spaces. There is a natural topology on $X \times Y$ generated by the products of the sets in τ and σ.

4.1 $\times_{\mathcal{F}}$ \prodFam

Let X and Y be sets and let ϕ and ψ be families on them. The product of these families is the family that consists of the products of the sets in them and is a family on $X \times Y$. Let $\phi \times_{\mathcal{F}} \psi$ denote the product of the families.

$$
\begin{array}{l}
[X, Y] \\
\hline
\_ \times_{\mathcal{F}} \_ : \mathcal{F}X \times \mathcal{F}Y \longrightarrow \mathcal{F}(X \times Y) \\
\hline
\forall \phi : \mathcal{F}\,X; \psi : \mathcal{F}\,Y \bullet \\
\quad \phi \times_{\mathcal{F}} \psi = \{ U : \phi;\ V : \psi \bullet U \times V \}
\end{array}
$$

Remark. *If* τ *and sigma are topologies then* $\tau \times_{\mathcal{F}} \sigma$ *is not, in general, a topology. However, we can use it to generate a topology.*

4.2 $\times_{\mathsf{top}}$ `\prodTop`

Let $\tau \times_{\mathsf{top}} \sigma$ denote the topology generated by $\tau \times_{\mathcal{F}} \sigma$.

$$
\begin{array}{l}
[X, Y] \\
\hline
\_ \times_{\mathsf{top}} \_ : \mathit{top}[X] \times \mathit{top}[Y] \longrightarrow \mathit{top}[X \times Y] \\
\hline
\forall \tau : \mathit{top}[X]; \sigma : \mathit{top}[Y] \bullet \\
\quad \tau \times_{\mathsf{top}} \sigma = \mathit{topGen}(\tau \times_{\mathcal{F}} \sigma)
\end{array}
$$

4.3 $\times_{\mathsf{top}}$ `\prodTopSp`

Let $(X, \tau) \times_{\mathsf{top}} (Y, \sigma)$ denote the product topological space.

$$
\begin{array}{l}
[X, Y] \\
\hline
\_ \times_{\mathsf{top}} \_ : \mathit{topSpace}[X] \times \mathit{topSpace}[Y] \longrightarrow \mathit{topSpace}[X \times Y] \\
\hline
\forall \tau : \mathit{top}[X]; \sigma : \mathit{top}[Y] \bullet \\
\quad (X, \tau) \times_{\mathsf{top}} (Y, \sigma) = (X \times Y, \tau \times_{\mathsf{top}} \sigma)
\end{array}
$$