Vector Spaces

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Abstract

This article contains Z Notation type declarations for vector spaces and some related objects. It has been type checked by *f*UZZ.

1 Introduction

Real vector spaces are multidimensional generalizations of real numbers. They are the objects studied in linear algebra and are foundational to differential geometry.

2 Real n-tuples

2.1 $\mathbb{R}^\infty$ `\Rinf`

Let n be a natural number. A finite sequence of n real numbers is called a *real n-tuple*. Let $\mathbb{R}^\infty$ denote the set of all real n-tuples for any n.

$$\mathbb{R}^\infty == \operatorname{seq} \mathbb{R}$$

2.2 $\mathbb{R}$ `\Rtuples`

Let $\mathbb{R}(n)$ denote $\mathbb{R}^n$, the set of all n-tuples for some given n.

$$\mathbb{R} : \mathbb{N} \longrightarrow \mathbb{P}\, \mathbb{R}^\infty$$

$$\forall\, n : \mathbb{N} \bullet \quad \mathbb{R}(n) = \{\, v : \mathbb{R}^\infty \mid \# v = n \,\}$$

Remark.

$$\mathbb{R}^\infty = \bigcup \{\, n : \mathbb{N} \bullet \mathbb{R}(n) \,\}$$

2.3 $\Delta_{\mathbb{R}}$ `\DeltaR`

Let $\Delta_{\mathbb{R}}$ denote the family of subsets of $\mathbb{R}^{\infty}$ such that all tuples in each subset have the same number of components. Such as subset is said to be *well-dimensioned*.

$$
\begin{array}{|l}
\Delta_{\mathbb{R}} : \mathcal{F}\,\mathbb{R}^{\infty} \\
\hline
\Delta_{\mathbb{R}} = \bigcup \{\, n : \mathbb{N} \bullet \mathbb{P}(\mathbb{R}(n)) \,\}
\end{array}
$$

Example. *The subset* $\mathbb{R}(n)$ *is well-dimensioned.*

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathbb{R}(n) \in \Delta_{\mathbb{R}}
\end{array}
$$

2.4 dim `\dimR`

Let $\dim(U)$ denote the number of components of the tuples in $U \in \Delta_{\mathbb{R}}$.

$$
\begin{array}{|l}
\dim : \Delta_{\mathbb{R}} \longrightarrow \mathbb{N} \\
\hline
\forall\, n : \mathbb{N} \bullet \forall\, U : \mathbb{P}(\mathbb{R}(n)) \bullet \\
\qquad \dim(U) = n
\end{array}
$$

Example. *The dimension of* $\mathbb{R}(n)$ *is* n*.*

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\qquad \dim(\mathbb{R}(n)) = n
\end{array}
$$

2.5 $\mathbf{0}$ `\zeroRn`

Let $\mathbf{0}(n)$ denote the n-tuple consisting of all zeroes.

$$
\begin{array}{|l}
\mathbf{0} : \mathbb{N} \longrightarrow \mathbb{R}^{\infty} \\
\hline
\mathbf{0}(0) = \langle\rangle \\
\\
\forall\, n : \mathbb{N}_1 \bullet \\
\qquad \mathbf{0}(n) = (\lambda\, i : 1 \,..\, n \bullet 0)
\end{array}
$$

Remark. *The tuple* $\mathbf{0}(n)$ *is in* $\mathbb{R}(n)$*.*

$$
\forall\, n : \mathbb{N} \bullet \mathbf{0}(n) \in \mathbb{R}(n)
$$

2.6 π `\piR`

The real numbers that comprise an n-tuple are called its components. The real number $v(i)$ is the i-th component of the n-tuple v where $1 \leq i \leq n$. Let $\pi(i)$ be the projection function that maps an n-tuple v to its i-th component $v(i)$.

$$
\begin{array}{|l}
\pi : \mathbb{N}_1 \longrightarrow \mathbb{R}^\infty \mathrel{+\!\!\!\!\rightarrow} \mathbb{R} \\
\hline
\forall\, i : \mathbb{N}_1 \bullet \\
\qquad \pi(i) = (\lambda\, v : \mathbb{R}^\infty \mid i \in \operatorname{dom} v \bullet v(i))
\end{array}
$$

Remark. *Every component of* $\mathbf{0}(n)$ *is* 0.

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \forall\, i : 1 \mathinner{..} n \bullet \\
\qquad \pi(i)(\mathbf{0}(n)) = 0
\end{array}
$$

3 Scalar Multiplication

3.1 $*$ `\smulR`

Let v be an n-tuple and let c be a real number. Scalar multiplication of v by c is the n-tuple $c * v$ defined by component-wise multiplication.

$$
\begin{array}{|l}
\_ * \_ : \mathbb{R} \times \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty \\
\hline
\forall\, c : \mathbb{R} \bullet \\
\qquad c * \langle\rangle = \langle\rangle \\
\forall\, c : \mathbb{R};\ n : \mathbb{N}_1 \bullet \\
\qquad \forall\, v : \mathbb{R}(n);\ i : 1 \mathinner{..} n \bullet \\
\qquad\qquad (c * v)(i) = c * v(i)
\end{array}
$$

Remark. *Scalar multiplication is associative in the sense that* $(a * b) * v = a * (b * v)$

$$
\begin{array}{l}
\forall\, a, b : \mathbb{R};\ v : \mathbb{R}^\infty \bullet \\
\qquad (a * b) * v = a * (b * v)
\end{array}
$$

4 Vector Addition and Subtraction

4.1 $\mathbb{R}^\Delta$ `\Rdelta`

Let $\mathbb{R}^\Delta$ denote the set of all pairs of tuples that have the same number of components.

$$
\begin{array}{|l}
\mathbb{R}^\Delta : \mathbb{R}^\infty \longleftrightarrow \mathbb{R}^\infty \\
\hline
\mathbb{R}^\Delta = \{\, v, w : \mathbb{R}^\infty \mid \#v = \#w \,\}
\end{array}
$$

4.2 $+$ `\vaddR`

Let v and w be n-tuples. Vector addition of v and w is the n-tuple $v + w$ defined by component-wise addition.

$$
\begin{array}{|l}
\_ + \_ : \mathbb{R}^{\Delta} \longrightarrow \mathbb{R}^{\infty} \\
\hline
\langle\rangle + \langle\rangle = \langle\rangle \\
\forall\, n : \mathbb{N}_1 \bullet \\
\quad \forall\, v, w : \mathbb{R}(n);\ i : 1 \mathinner{..} n \bullet \\
\qquad (v + w)(i) = v(i) + w(i)
\end{array}
$$

4.3 $-$ `\vsubR`

Vector subtraction is defined similarly.

$$
\begin{array}{|l}
\_ - \_ : \mathbb{R}^{\Delta} \longrightarrow \mathbb{R}^{\infty} \\
\hline
\langle\rangle - \langle\rangle = \langle\rangle \\
\forall\, n : \mathbb{N}_1 \bullet \\
\quad \forall\, v, w : \mathbb{R}(n);\ i : 1 \mathinner{..} n \bullet \\
\qquad (v - w)(i) = v(i) - w(i)
\end{array}
$$

Each $\mathbb{R}(n)$ is a real vector space under the operations of scalar multiplication and vector addition defined above.

5 Vector Spaces

The sets $\mathbb{R}^n$ with the operations of scalar multiplication and vector addition form vector spaces. In general, a vector space is a set of vectors endowed with scalar multiplication and vector addition operations that follow rules analogous to those for $\mathbb{R}^n$.

5.1 *VectorSpace*

Let V be a set and let $\mathit{VectorSpace}[V]$ denote the set of all vector spaces whose vectors are V.

$$
\begin{array}{|l}
\multicolumn{1}{l}{\_\, \mathit{VectorSpace}[V] \,\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\
\mathbf{0} : V \\
\_ + \_ : V \times V \longrightarrow V \\
\_ * \_ : \mathbb{R} \times V \longrightarrow V \\
\hline
\forall\, v : V \bullet \\
\qquad \mathbf{0} + v = v = v + \mathbf{0} \\
\forall\, v, w : V \bullet \\
\qquad v + w = w + v \\
\forall\, u, v, w : V \bullet \\
\qquad u + (v + w) = (u + v) + w \\
\forall\, v : V \bullet \\
\qquad 0 * v = \mathbf{0} \\
\forall\, v : V \bullet \\
\qquad 1 * v = v \\
\forall\, a, b : \mathbb{R};\ v : V \bullet \\
\qquad (a + b) * v = (a * v) + (b * v) \\
\forall\, a, b : \mathbb{R};\ v : V \bullet \\
\qquad (a * b) * v = a * (b * v) \\
\forall\, a : \mathbb{R};\ v, w : V \bullet \\
\qquad a * (v + w) = (a * v) + (a * w) \\
\hline
\end{array}
$$

- the zero vector $\mathbf{0}$ is the identity element for vector addition
- vector addition is commutative
- vector addition is associative
- scalar multiplication by 0 gives the zero vector
- scalar multiplication by 1 leaves any vector unchanged
- real addition distributes over scalar multiplication
- real multiplication associates over scalar multiplication
- scalar multiplication distributes over vector addition

5.2 *vectorSpace*

Let *vectorSpace*$[V]$ the set of all triples consisting of a zero vector, a vector addition operation, and a scalar multiplication operation that define a vector space whose vectors are V,

6 Linear Transformations

6.1 Linear

Let n and m be natural numbers. A mapping L from $\mathbb{R}^n$ to $\mathbb{R}^m$ is said to be a *linear transformation* if it preserves scalar multiplication and vector addition.

Linear

$$n, m : \mathbb{N}$$
$$L : \mathbb{R}^\infty \rightarrowtail\!\!\!\!\!+ \, \mathbb{R}^\infty$$

$$L \in \mathbb{R}(n) \longrightarrow \mathbb{R}(m)$$

$$\forall\, c : \mathbb{R};\ v : \mathbb{R}(n) \bullet$$
$$\quad L(c * v) = c * L(v)$$

$$\forall\, v, w : \mathbb{R}(n) \bullet$$
$$\quad L(v + w) = L(v) + L(w)$$

6.2 lin `\linR`

Define $\mathrm{lin}(n, m)$ to be the set of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$\mathrm{lin} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \rightarrowtail\!\!\!\!\!+ \, \mathbb{R}^\infty)$$

$$\forall\, n, m : \mathbb{N} \bullet$$
$$\quad \mathrm{lin}(n, m) = \{\, L : \mathbb{R}^\infty \rightarrowtail\!\!\!\!\!+ \, \mathbb{R}^\infty \mid \mathit{Linear} \,\}$$

6.3 I `\In`

Let $\mathrm{I}(n)$ denote the identity function on $\mathbb{R}(n)$.

$$\mathrm{I} : \mathbb{N} \longrightarrow \mathbb{R}^\infty \rightarrowtail\!\!\!\!\!+ \, \mathbb{R}^\infty$$

$$\forall\, n : \mathbb{N} \bullet$$
$$\quad \mathrm{I}(n) = \mathrm{id}(\mathbb{R}(n))$$

Remark. *The function* $\mathrm{I}(n)$ *is a linear transformation.*

$$\forall\, n : \mathbb{N} \bullet$$
$$\quad \mathrm{I}(n) \in \mathrm{lin}(n, n)$$

7 The Dot Product

7.1 $\cdot$ \dotR

The *inner* or *dot* product of n-tuples v and w is the real number $v \cdot w$ defined by the sum of the component-wise products.

$$
\begin{array}{|l}
\_ \cdot \_ : \mathbb{R}^{\Delta} \longrightarrow \mathbb{R} \\
\hline
\langle\rangle \cdot \langle\rangle = 0 \\
\forall x, y : \mathbb{R};\ v, w : \mathbb{R}^{\infty} \mid \#v = \#w \bullet \\
\qquad (\langle x \rangle \frown v) \cdot (\langle y \rangle \frown w) = x * y + v \cdot w
\end{array}
$$

Each $\mathbb{R}(n)$ is a real inner product space under the operation of dot product defined above.

8 The Norm

8.1 norm \normR

The norm $\|v\|$ of the n-tuple v is the positive square root of its dot product with itself.

$$\|v\| = \sqrt{v \cdot v}$$

Define $\mathrm{norm}(v)$ to be $\|v\|$.

$$
\begin{array}{|l}
\mathrm{norm} : \mathbb{R}^{\infty} \longrightarrow \mathbb{R} \\
\hline
\forall v : \mathbb{R}^{\infty} \bullet \\
\qquad \mathrm{norm}(v) = \mathrm{sqrt}(v \cdot v)
\end{array}
$$

The concepts of continuity, limits, and differentiability extend to functions between normed vector spaces such as $\mathbb{R}^n$.

8.2 ball \ballRn

Let $\mathrm{ball}(v, r)$ denote the open ball in $\mathbb{R}(n)$ of radius $r \in \mathbb{R}$ centred at $v \in \mathbb{R}(n)$.

$$
\begin{array}{|l}
\mathrm{ball} : \mathbb{R}^{\infty} \times \mathbb{R} \longrightarrow \mathbb{P}\, \mathbb{R}^{\infty} \\
\hline
\forall v : \mathbb{R}^{\infty};\ r : \mathbb{R} \bullet \mathbf{let}\ n == \#v \bullet \\
\qquad \mathrm{ball}(v, r) = \{\, w : \mathbb{R}(n) \mid \mathrm{norm}(v - w) < r \,\}
\end{array}
$$

8.3 balls `\ballsRn`

Let balls(n) denote the family of all open balls in $\mathbb{R}(n)$.

$$\begin{array}{|l}
\text{balls} : \mathbb{N} \longrightarrow \mathcal{F}\,\mathbb{R}^\infty \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\qquad \text{balls}(n) = \{\, v : \mathbb{R}(n);\ r : \mathbb{R} \bullet \text{ball}(v, r) \,\}
\end{array}$$

8.4 $\tau_{\mathbb{R}}$ `\tauRn`

The usual topology on $\mathbb{R}(n)$ is the topology generated by the open balls in $\mathbb{R}(n)$. Let $\tau_{\mathbb{R}}(n)$ denote the usual topology on $\mathbb{R}(n)$.

$$\begin{array}{|l}
\tau_{\mathbb{R}} : \mathbb{N} \longrightarrow \mathcal{F}\,\mathbb{R}^\infty \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\qquad \tau_{\mathbb{R}}(n) = \mathit{topGen}[\mathbb{R}(n)](\text{balls}(n))
\end{array}$$

Remark. *If $n \in \mathbb{N}$ then $\tau_{\mathbb{R}}(n)$ is a topology on $\mathbb{R}(n)$.*

$$\forall\, n : \mathbb{N} \bullet \tau_{\mathbb{R}}(n) \in \mathit{top}[\mathbb{R}(n)]$$

8.5 neigh `\neighRn`

Let $x \in \mathbb{R}(n)$. Let neigh(x) denote the set of all open sets U in the usual topology $\tau_{\mathbb{R}}(n)$ that contain x. Such a set U is called a neighbourhood of x.

$$\begin{array}{|l}
\text{neigh} : \mathbb{R}^\infty \longrightarrow \mathcal{F}\,\mathbb{R}^\infty \\
\hline
\forall\, x : \mathbb{R}^\infty \bullet \textbf{let } n == \#x \bullet \\
\qquad \text{neigh}(x) = \{\, U : \tau_{\mathbb{R}}(n) \mid x \in U \,\}
\end{array}$$

Remark.

$$\forall\, v : \mathbb{R}^\infty \bullet \textbf{let } n == \#v \bullet \text{neigh}(v) \in \mathcal{F}(\mathbb{R}(n))$$

8.6 $\mathbb{R}_\tau$ `\RtauN`

Let $\mathbb{R}_\tau(n)$ denote the topological space defined by the usual topology on $\mathbb{R}(n)$.

$$\begin{array}{|l}
\mathbb{R}_\tau : \mathbb{N} \longrightarrow \mathit{topSpaces}[\mathbb{R}^\infty] \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathbb{R}_\tau(n) = (\mathbb{R}(n), \tau_{\mathbb{R}}(n))
\end{array}$$

9 Continuity

9.1 C^0 \CzeroN

A function f from $\mathbb{R}^n$ to $\mathbb{R}$ is said to be continuous if it is continuous with respect to the usual topologies. Let $\mathrm{C}^0(n)$ denote the set of these continuous mappings.

$$
\begin{array}{|l}
\mathrm{C}^0 : \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \mathrel{+\!\!\!\to} \mathbb{R}) \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathrm{C}^0(n) = \mathrm{C}^0(\mathbb{R}_\tau(n), \mathbb{R}_\tau)
\end{array}
$$

9.2 C^0 \CzeroPRn

Let U be a subset of $\mathbb{R}^n$. A function $f \in U \longrightarrow \mathbb{R}$ is said to be continuous if it is continuous with respect to the topology induced on U. Let $\mathrm{C}^0(U)$ denote the set of these continuous functions.

$$
\begin{array}{|l}
\mathrm{C}^0 : \Delta_{\mathbb{R}} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \mathrel{+\!\!\!\to} \mathbb{R}) \\
\hline
\forall\, U : \Delta_{\mathbb{R}} \bullet \\
\qquad \textbf{let } n == \dim U \bullet \\
\qquad\qquad \mathrm{C}^0(U) = \mathrm{C}^0(\mathbb{R}_\tau(n) \mid_{\mathsf{top}} U, \mathbb{R}_\tau)
\end{array}
$$

9.3 C^0 \CzeroRn

A partial function f from $\mathbb{R}^n$ to $\mathbb{R}$ is said to be continuous at $x \in \mathbb{R}^n$ if its domain contains a neighbourhood U of x such that its restriction to U is continuous on U. Let $\mathrm{C}^0(x)$ denote the set of such functions.

$$
\begin{array}{|l}
\mathrm{C}^0 : \mathbb{R}^\infty \longrightarrow \mathbb{P}(\mathbb{R}^\infty \mathrel{+\!\!\!\to} \mathbb{R}) \\
\hline
\forall\, x : \mathbb{R}^\infty \bullet \\
\qquad \textbf{let } n == \#x \bullet \\
\qquad\qquad \mathrm{C}^0(x) = \{\, f : \mathbb{R}(n) \mathrel{+\!\!\!\to} \mathbb{R} \mid \exists\, U : \mathrm{neigh}(x) \mid U \subseteq \mathrm{dom}\, f \bullet U \lhd f \in \mathrm{C}^0(U) \,\}
\end{array}
$$

9.4 C^0 \CzeroNN

A mapping f from $\mathbb{R}(n)$ to $\mathbb{R}(m)$ is said to be continuous if it is continuous with respect to the usual topologies. Let $\mathrm{C}^0(n, m)$ denote the set of these continuous mappings.

$$
\begin{array}{|l}
\mathrm{C}^0 : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \mathrel{+\!\!\!\to} \mathbb{R}^\infty) \\
\hline
\forall\, n, m : \mathbb{N} \bullet \\
\qquad \mathrm{C}^0(n, m) = \mathrm{C}^0(\mathbb{R}_\tau(n), \mathbb{R}_\tau(m))
\end{array}
$$

Example. *The function* $\mathrm{I}(n)$ *is continuous.*

$$\forall\, n : \mathbb{N} \bullet \\ \quad \mathrm{I}(n) \in \mathrm{C}^0(n, n)$$

Theorem 1. *Linear functions are continuous.*

$$\forall\, n, m : \mathbb{N} \bullet \\ \quad \mathrm{lin}(n, m) \subseteq \mathrm{C}^0(n, m)$$

9.5 C^0 `\CzeroPRnN`

Let U be any subset of $\mathbb{R}(n)$. Let $\mathrm{C}^0(U, m)$ denote the set of continuous mappings from the topology induced by $\mathbb{R}_\tau(n)$ on U to $\mathbb{R}_\tau(m)$.

$$\mathrm{C}^0 : \Delta_{\mathbb{R}} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \rightarrow\!\!\!\!\!\!+\; \mathbb{R}^\infty)$$
$$\forall\, n, m : \mathbb{N} \bullet \\ \quad \forall\, U : \Delta_{\mathbb{R}} \mid \dim(U) = n \bullet \\ \quad\quad \mathrm{C}^0(U, m) = \mathrm{C}^0(\mathbb{R}_\tau(n) \mid_{\mathsf{top}} U, \mathbb{R}_\tau(m))$$

Remark.

$$\forall\, n, m : \mathbb{N} \bullet \\ \quad \mathrm{C}^0(\mathbb{R}(n), m) = \mathrm{C}^0(n, m)$$

9.6 C^0 `\CzeroRnN`

Let $x \in \mathbb{R}(n)$ and let f be a partial function from $\mathbb{R}(n)$ to $\mathbb{R}(m)$ whose domain includes some neighbourhood U of x such that f restricted to U is continuous. In this case f is said to be *continuous at* x.

VectorContinuous

$$n, m : \mathbb{N} \\ f : \mathbb{R}^\infty \rightarrow\!\!\!\!\!\!+\; \mathbb{R}^\infty \\ x : \mathbb{R}^\infty$$
$$f \in \mathbb{R}(n) \rightarrow\!\!\!\!\!\!+\; \mathbb{R}(m)$$
$$\exists\, U : \mathrm{neigh}(x) \mid \\ \quad U \subseteq \mathrm{dom}\, f \bullet \\ \quad\quad U \lhd f \in \mathrm{C}^0(U, m)$$

Let $\mathrm{C}^0(x, m)$ denote the set of all partial functions f from $\mathbb{R}(n)$ to $\mathbb{R}(m)$ that are continuous at x.

$$
\begin{array}{|l}
\mathrm{C}^0 : \mathbb{R}^\infty \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \mathrel{+\!\!\!\!\rightarrow} \mathbb{R}^\infty) \\
\hline
\forall\, n, m : \mathbb{N} \bullet \forall\, x : \mathbb{R}(n) \bullet \\
\quad \mathrm{C}^0(x, m) = \\
\quad\quad \{\, f : \mathbb{R}(n) \mathrel{+\!\!\!\!\rightarrow} \mathbb{R}(m) \mid \mathit{VectorContinuous} \,\}
\end{array}
$$

Example. *The function* $\mathrm{I}(n)$ *is continuous at every point* $x \in \mathbb{R}(n)$.

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \forall\, x : \mathbb{R}(n) \bullet \\
\quad \mathrm{I}(n) \in \mathrm{C}^0(x, n)
\end{array}
$$

Theorem 2. *Linear functions are continuous everywhere.*

$$
\begin{array}{l}
\forall\, n, m : \mathbb{N} \bullet \\
\quad \forall\, x : \mathbb{R}(n);\ L : \mathrm{lin}(n, m) \bullet \\
\quad\quad L \in \mathrm{C}^0(x, m)
\end{array}
$$

10 Differentiability

Let $x \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \mathrel{+\!\!\!\!\rightarrow} \mathbb{R}^m$ be continuous at x. Then f is said to be *differentiable at x* if there exists a linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that $f(x+h) - f(x)$ is approximately linear in h for very small h.

$$f(x+h) - f(x) \approx L(h) + O(h^2) \quad \text{when} \quad \|h\| \approx 0$$

This condition can be written as a limit.

$$\lim_{h \to 0} \frac{\big\| f(x+h) - f(x) - L(h) \big\|}{\|h\|} = 0$$

10.1 *diffQuot*

The limit exists when the following difference quotient function $q : \mathbb{R}^n \mathrel{+\!\!\!\!\rightarrow} \mathbb{R}$ is continuous at 0.

$$q(h) = \begin{cases} \frac{\|f(x+h) - f(x) - L(h)\|}{\|h\|} & \text{if } h \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$
\begin{array}{|l}
\hline
\textit{DifferenceQuotient} \\
\textit{VectorContinuous} \\
L : \mathbb{R}^\infty \rightarrowtail\!\!\!\!\!\!+ \ \mathbb{R}^\infty \\
q : \mathbb{R}^\infty \rightarrowtail\!\!\!\!\!\!+ \ \mathbb{R} \\
\hline
L \in \mathrm{lin}(n, m) \\
\mathrm{dom}\, q = \{\, h : \mathbb{R}(n) \mid x + h \in \mathrm{dom}\, f \,\} \\
\forall\, h : \mathrm{dom}\, q \mid h \neq \mathbf{0}(n) \bullet \\
\qquad q(h) = \mathrm{norm}(f(x + h) - f(x) - L(h)) \,/\, \mathrm{norm}(h) \\
q(\mathbf{0}(n)) = 0 \\
\hline
\end{array}
$$

The function f is differentiable at x when there exists a linear transformation L such that the difference quotient q is continuous at 0.

$$
\begin{array}{|l}
\hline
\textit{VectorDifferentiable} \\
\textit{DifferenceQuotient} \\
\hline
q \in \mathrm{C}^0(\mathbf{0}(n)) \\
\hline
\end{array}
$$

Clearly q is uniquely determined by f, x, and L. Let $\textit{diffQuot}(f, x, L)$ denote the difference quotient.

$$
\begin{array}{|l}
\textit{diffQuot} : (\mathbb{R}^\infty \rightarrowtail\!\!\!\!\!\!+ \ \mathbb{R}^\infty) \times \mathbb{R}^\infty \times (\mathbb{R}^\infty \rightarrowtail\!\!\!\!\!\!+ \ \mathbb{R}^\infty) \rightarrowtail\!\!\!\!\!\!+ \ (\mathbb{R}^\infty \rightarrowtail\!\!\!\!\!\!+ \ \mathbb{R}) \\
\hline
\textit{diffQuot} = \{\, \textit{VectorDifferentiable} \bullet (f, x, L) \mapsto q \,\}
\end{array}
$$

Let $\mathrm{C}^\infty(x, m)$ denote the set of all functions $f \in \mathbb{R}(n) \rightarrowtail\!\!\!\!\!\!+ \ \mathbb{R}(m)$ that are smooth at $x \in \mathbb{R}(n)$.