VECTOR SPACES

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ABSTRACT. This article contains Z Notation type declarations for vector spaces and some related objects. It has been type checked by *f*UZZ.

CONTENTS

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1. Real Vector Spaces

Real vector spaces are multidimensional generalizations of real numbers. They are the objects studied in linear algebra and are foundational to differential geometry.

In the following let `t` denote a set of elements which we'll refer to as *vectors* and let A denote an Abelian group over the vectors in which the binary operation is denoted as addition. Let v and w denote vectors and and let x and y denote real numbers.

1.1. **Notation for Vector Addition, Zero, and Negative:** $+$ `\addV`, $\mathbf{0}$ `\zeroV`, **and** - `\negV`. Let $v + w$ denote vector addition, let $\mathbf{0}$ denote the zero vector, and let - v denote the negative vector.

1.2. **Real Scalar Multiplication:** $*$ `\mulS`, $\times$ `\timesS`, **and** *RealScalarMultiplication*. A *real scalar multiplication* operation on the vectors is an operation *smul* that maps the pair (x, v) to another vector, typically denoted $x * v$ or $x \times y$, such that multiplication by 0 maps all vectors to the group identity element, multiplication by 1 maps each vector to itself, multiplication preserves group addition, and multiplication distributes over both real and group addition.

Let *RealScalarMultiplication* denote this situation.

RealScalarMultiplication[t]

A : abgroup `t`
$smul : \mathbb{R} \times \mathtt{t} \longrightarrow \mathtt{t}$

let $(\_ + \_) == A$;
$\mathbf{0} == \mathit{identity\_element}\ A$;
$(\_ * \_) == \mathit{smul}$ •
$\forall x, y : \mathbb{R};\ v, w : \mathtt{t}$ •
$0 * v = \mathbf{0} \land$
$1 * v = v \land$
$(x * y) * v = x * (y * v) \land$
$(x + y) * v = x * v + y * v \land$
$x * (v + w) = x * v + x * w$

- Multiplying by 0 gives the zero vector.
- Multiplying by 1 gives the same vector.
- Scalar multiplication is associative.
- Scalar addition distributes over scalar multiplication.
- Vector addition distributes over scalar multiplication.

1.3. **The Set of All Real Vector Spaces:** $\mathrm{vec}_{\mathbb{R}}$ `\vecR`. A *real vector space* is a pair (A, smul) where A is an Abelian group and *smul* is a real scalar multiplication on the elements of A. The elements of A are referred to as *vectors*.

Let $\mathrm{vec}_{\mathbb{R}}\ \mathtt{t}$ denote the set of all real vector spaces over `t`,

$\mathrm{vec}_{\mathbb{R}}\ \mathtt{t} == \{\ \mathit{RealScalarMultiplication}[\mathtt{t}] \bullet (A, \mathit{smul})\ \}$

1.4. **Real Linear Transformations:** *RealLinearTransformation*. Let V_1 and V_2 be real vector spaces and let f be a homomorphism of the underlying Abelian groups. The map f is said to be a *linear transformation* if f maps scalar multiples of vectors to the scalar multiple of the mapped vectors.

Let *RealLinearTransformation* denote this situation.

$$
\begin{array}{|l}
\multicolumn{1}{l}{\textit{RealLinearTransformation}[\mathsf{t}, \mathsf{u}]} \\
f : \mathsf{t} \longrightarrow \mathsf{u} \\
V_1 : \mathrm{vec}_{\mathbb{R}}\, \mathsf{t} \\
V_2 : \mathrm{vec}_{\mathbb{R}}\, \mathsf{u} \\
\hline
\mathbf{let}\ A_1 == \mathit{first}\ V_1;\ (\_ * \_) == \mathit{second}\ V_1; \\
\qquad A_2 == \mathit{first}\ V_2;\ (\_ \times \_) == \mathit{second}\ V_2 \bullet \\
\qquad\quad f \in \mathrm{hom}_{\mathrm{grp}}(A_1, A_2) \land \\
\qquad\quad (\forall x : \mathbb{R};\ v : \mathsf{t} \bullet \\
\qquad\qquad f(x * v) = x \times (f\ v)) \\
\hline
\end{array}
$$

- The vector space V_1 has Abelian group A_1 and scalar multiplication $(\_*\_)$.
- The vector space V_2 has Abelian group A_2 and scalar multiplication $(\_\times\_)$.
- The map f is a homomorphism of the underlying Abelian groups.
- The map f maps scalar multiples of vectors in t to scalar multiples of the mapped vectors in u.

1.5. **The Set of All Real Linear Transformations:** $\mathrm{L}_{\mathbb{R}}$ `\homVecR`. Let V_1 and V_2 be real vector spaces. Let $\mathrm{L}_{\mathbb{R}}(V_1, V_2)$ denote the set of all linear transformations from V_1 to V_2. A linear transformation is also referred to as a *homomorphism* of vector spaces.

$$
\begin{array}{l}
\mathrm{L}_{\mathbb{R}}[\mathsf{t}, \mathsf{u}] == \\
\quad (\lambda\, V_1 : \mathrm{vec}_{\mathbb{R}}\, \mathsf{t};\ V_2 : \mathrm{vec}_{\mathbb{R}}\, \mathsf{u} \bullet \\
\qquad \{ f : \mathsf{t} \longrightarrow \mathsf{u} \mid \\
\qquad\quad \textit{RealLinearTransformation}[\mathsf{t}, \mathsf{u}] \})
\end{array}
$$

2. Real n-tuples

The preceding section described real vector spaces abstractly. In this section we define a family of finite-dimensional real vector spaces whose elements are finite sequences of real numbers, also referred to as *real tuples*.

2.1. **The Set of All Finite Sequences of Real Numbers:** $\mathbb{R}^\infty$ `\Rinf`. Let n be a natural number. A finite sequence of n real numbers is called a *real n-tuple*. Let $\mathbb{R}^\infty$ denote the set of all real n-tuples for any n.

$$\mathbb{R}^\infty == \mathrm{seq}\, \mathbb{R}$$

2.2. **The Component Projection Function:** π `\piRinf`. The real numbers that comprise an n-tuple are called its *components*. Let v be a real n-tuple and let i be an integer where $1 \leq i \leq n$. The real number $v(i)$ is the i-th component of v. Let $\pi(i)$ be the projection function that maps an n-tuple v to its i-th component $v(i)$.

$$
\begin{array}{|l}
\pi : \mathbb{N}_1 \longrightarrow \mathbb{R}^\infty \nrightarrow \mathbb{R} \\
\hline
\forall i : \mathbb{N}_1 \bullet \\
\qquad \pi(i) = (\lambda\, v : \mathbb{R}^\infty \mid i \in \mathrm{dom}\, v \bullet v(i))
\end{array}
$$

2.3. **The Set of All Well-Dimensioned Subsets of $\mathbb{R}^\infty$: $\Delta_\mathbb{R}$ `\DeltaRinf`.** A non-empty subset of $\mathbb{R}^\infty$ is said to be *well-dimensioned* if each of its elements has the same number of components. Let $\Delta_\mathbb{R}$ denote the family of all well-dimensioned subsets of $\mathbb{R}^\infty$.

$$\Delta_\mathbb{R} : \mathcal{F}\,\mathbb{R}^\infty$$
$$\Delta_\mathbb{R} = \{\, S : \mathbb{P}_1\,\mathbb{R}^\infty \mid \forall\, v, w : S \bullet \#v = \#w \,\}$$

2.4. **The Dimension of a Well-Dimensioned Set of Tuples:** dim **`\dimRinf`.** Let $S \in \Delta_\mathbb{R}$ be a well-dimensioned set of tuples. The number of components of each tuple in S is called its *dimension*. Let $\dim(S)$ denote the dimension of S.

$$\dim : \Delta_\mathbb{R} \longrightarrow \mathbb{N}$$
$$\forall\, S : \Delta_\mathbb{R} \bullet \quad \dim S = (\mu\, v : S \bullet \#v)$$

2.5. **The Set of All Compatible Pairs of Tuples: $\mathbb{R}^\Delta$ `\RinfDelta`.** The pair of real tuples (v, w) is said to be *compatible* if each member has the same number of components. Let $\mathbb{R}^\Delta$ denote the set of all compatible pairs of real tuples. If the pair (v, w) is compatible then v and w are said to be compatible with each other.

$$\mathbb{R}^\Delta : \mathbb{R}^\infty \leftrightarrow \mathbb{R}^\infty$$
$$\mathbb{R}^\Delta = \{\, v, w : \mathbb{R}^\infty \mid \#v = \#w \,\}$$

2.6. **Addition of Compatible Tuples: $+$ `\addRinf`.** Let v and w be n-tuples. Vector addition of v and w is the n-tuple $v+w$ defined by component-wise addition.

$$\_ + \_ : \mathbb{R}^\Delta \longrightarrow \mathbb{R}^\infty$$
$$\langle\rangle + \langle\rangle = \langle\rangle$$
$$\forall\, n : \mathbb{N}_1;\ v, w : \mathbb{R}^\infty \mid n = \#v = \#w \bullet \quad v + w = (\lambda\, i : 1 \,..\, n \bullet v\, i + w\, i)$$

2.7. **Subtraction of Compatible Tuples: $-$ `\subRinf`.** Vector subtraction is defined similarly.

$$\_ - \_ : \mathbb{R}^\Delta \longrightarrow \mathbb{R}^\infty$$
$$\langle\rangle - \langle\rangle = \langle\rangle$$
$$\forall\, n : \mathbb{N}_1;\ v, w : \mathbb{R}^\infty \mid n = \#v = \#w \bullet \quad v - w = (\lambda\, i : 1 \,..\, n \bullet v\, i - w\, i)$$

2.8. **The Negative of a Tuple: - `\negRinf`.** Let - v denote the negative of v.

$$\text{-} : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$$
$$\text{-}\langle\rangle = \langle\rangle$$
$$\forall\, n : \mathbb{N}_1;\ v : \mathbb{R}^\infty \mid n = \#v \bullet \quad \text{-}\, v = (\lambda\, i : 1 \,..\, n \bullet \text{-}(v\, i))$$

2.9. **Scalar Multiplication of a Tuple:** $*$ `\smulRinf`. Let v be an n-tuple and let c be a real number. Scalar multiplication of v by c is the n-tuple $c * v$ defined by component-wise multiplication.

$$
\begin{array}{|l}
\_ * \_ : \mathbb{R} \times \mathbb{R}^{\infty} \longrightarrow \mathbb{R}^{\infty} \\
\hline
\forall\, c : \mathbb{R} \bullet \\
\qquad c * \langle\rangle = \langle\rangle \\
\forall\, c : \mathbb{R};\ n : \mathbb{N}_1;\ v : \mathbb{R}^{\infty} \mid n = \# v \bullet \\
\qquad c * v = (\lambda\, i : 1 \ldots n \bullet c * (v\ i))
\end{array}
$$

Remark. *Scalar multiplication is associative in the sense that* $(a * b) * v = a * (b * v)$

$$
\begin{array}{l}
\forall\, a, b : \mathbb{R};\ v : \mathbb{R}^{\infty} \bullet \\
\qquad (a * b) * v = a * (b * v)
\end{array}
$$

2.10. **The Set of All Real n-tuples:** $\mathbb{R}$ `\Rtup`. Let $\mathbb{R}(n)$ denote $\mathbb{R}^n$, the set of all n-tuples for some given n.

$$
\begin{array}{|l}
\mathbb{R} : \mathbb{N} \longrightarrow \mathbb{P}\, \mathbb{R}^{\infty} \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathbb{R}(n) = \{\, v : \mathbb{R}^{\infty} \mid \# v = n \,\}
\end{array}
$$

Remark.

$$\mathbb{R}^{\infty} = \bigcup \{\, n : \mathbb{N} \bullet \mathbb{R}(n) \,\}$$

Remark. *The subset* $\mathbb{R}(n)$ *is well-dimensioned.*

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathbb{R}(n) \in \Delta_{\mathbb{R}}
\end{array}
$$

Remark. *The dimension of* $\mathbb{R}(n)$ *is* n.

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\qquad \dim(\mathbb{R}(n)) = n
\end{array}
$$

2.11. **Addition of n-tuples:** *addRtup*. Let $addRtup(n)$ denote the restriction of addition to $\mathbb{R}(n)$.

$$
\begin{array}{l}
addRtup == \\
\qquad (\lambda\, n : \mathbb{N} \bullet \\
\qquad\qquad (\lambda\, v, w : \mathbb{R}(n) \bullet v + w))
\end{array}
$$

Example. *The binary operation* $addRtup(n)$ *defines an Abelian group over* $\mathbb{R}(n)$.

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\qquad addRtup(n) \in \mathrm{abgroup}(\mathbb{R}(n))
\end{array}
$$

2.12. **Subtraction of n-tuples:** *subRtup*. Let $subRtup(n)$ denote the restriction of subtraction to $\mathbb{R}(n)$.

$$
\begin{array}{l}
subRtup == \\
\qquad (\lambda\, n : \mathbb{N} \bullet \\
\qquad\qquad (\lambda\, v, w : \mathbb{R}(n) \bullet v - w))
\end{array}
$$

2.13. **The Negative of an n-tuple:** *negRtup***.** Let $negRtup(n)$ denote the restriction of the negative operation to $\mathbb{R}(n)$.

$$
\begin{aligned}
&negRtup == \\
&\quad (\lambda\, n : \mathbb{N} \bullet \\
&\qquad (\lambda\, v : \mathbb{R}(n) \bullet \text{-}\, v))
\end{aligned}
$$

Remark. *The operation $negRtup(n)$ is the inverse operation of the Abelian group $addRtup(n)$.*

$$
\begin{aligned}
&\forall\, n : \mathbb{N} \bullet \\
&\quad negRtup(n) = inverse\_operation(addRtup(n))
\end{aligned}
$$

2.14. **The Zero Real n-tuple: $\mathbf{0}$ `\zeroRtup`.** Let $\mathbf{0}(n)$ denote the n-tuple consisting of all zeroes.

$$
\begin{array}{|l}
\mathbf{0} : \mathbb{N} \longrightarrow \mathbb{R}^{\infty} \\
\hline
\mathbf{0}(0) = \langle\rangle \\
\\
\forall\, n : \mathbb{N}_1 \bullet \\
\quad \mathbf{0}(n) = (\lambda\, i : 1 \mathrel{..} n \bullet 0)
\end{array}
$$

Remark. *Every component of $\mathbf{0}(n)$ is 0.*

$$
\begin{aligned}
&\forall\, n : \mathbb{N} \bullet \\
&\quad \forall\, i : 1 \mathrel{..} n \bullet \\
&\qquad (\pi\ i)(\mathbf{0}\ n) = 0
\end{aligned}
$$

Remark. *The tuple $\mathbf{0}(n)$ is in $\mathbb{R}(n)$.*

$$
\begin{aligned}
&\forall\, n : \mathbb{N} \bullet \\
&\quad \mathbf{0}(n) \in \mathbb{R}(n)
\end{aligned}
$$

Remark. *The tuple $\mathbf{0}(n)$ is the identity element of the Abelian group $addRtup(n)$.*

$$
\begin{aligned}
&\forall\, n : \mathbb{N} \bullet \\
&\quad \mathbf{0}(n) = identity\_element(addRtup(n))
\end{aligned}
$$

2.15. **Scalar Multiplication of an n-tuple:** *smulRtup***.** Let $smulRtup(n)$ denote scalar multiplication restricted to $\mathbb{R}(n)$.

$$
\begin{aligned}
&smulRtup == \\
&\quad (\lambda\, n : \mathbb{N} \bullet \\
&\qquad (\lambda\, c : \mathbb{R};\ v : \mathbb{R}(n) \bullet c * v))
\end{aligned}
$$

2.16. **The Real Vector Space of n-tuples:** *vecRtup***.** Let $vecRtup(n)$ denote the real vector space of n-tuples.

$$
\begin{aligned}
&vecRtup == \\
&\quad (\lambda\, n : \mathbb{N} \bullet (addRtup(n), smulRtup(n)))
\end{aligned}
$$

Remark. *The pair $vecRtup(n)$ defines a vector space over $\mathbb{R}(n)$.*

$$
\begin{aligned}
&\forall\, n : \mathbb{N} \bullet \\
&\quad vecRtup(n) \in \mathrm{vec}_{\mathbb{R}}(\mathbb{R}(n))
\end{aligned}
$$

2.17. **Linear Transformations of n-tuples:** $\mathrm{L}_{\mathbb{R}}$ `\linRtup`. Define $\mathrm{L}_{\mathbb{R}}(n, m)$ to be the set of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$\mathrm{L}_{\mathbb{R}} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^{\infty} \nrightarrow \mathbb{R}^{\infty})$$

$$\forall\, n, m : \mathbb{N} \bullet \\ \quad \mathrm{L}_{\mathbb{R}}(n, m) = \mathrm{L}_{\mathbb{R}}(\mathit{vecRtup}(n), \mathit{vecRtup}(m))$$

2.18. **The Identity Transformation of n-tuples:** I `\idRtup`. Let $\mathrm{I}(n)$ denote the identity function on $\mathbb{R}(n)$.

$$\mathrm{I} : \mathbb{N} \longrightarrow \mathbb{R}^{\infty} \nrightarrow \mathbb{R}^{\infty}$$

$$\forall\, n : \mathbb{N} \bullet \\ \quad \mathrm{I}(n) = \mathrm{id}(\mathbb{R}(n))$$

Remark. *The function* $\mathrm{I}(n)$ *is a linear transformation.*

$$\forall\, n : \mathbb{N} \bullet \\ \quad \mathrm{I}(n) \in \mathrm{L}_{\mathbb{R}}(n, n)$$

3. The Metric Topology on Real n-tuples

3.1. **The Dot Product of Tuples:** $\cdot$ `\dotRinf`. The *inner* or *dot* product of n-tuples v and w is the real number $v \cdot w$ defined by the sum of the component-wise products.

$$\_ \cdot \_ : \mathbb{R}^{\Delta} \longrightarrow \mathbb{R}$$

$$\langle\rangle \cdot \langle\rangle = 0$$

$$\forall\, x, y : \mathbb{R};\ v, w : \mathbb{R}^{\infty} \mid \#v = \#w \bullet \\ \quad (\langle x \rangle \frown v) \cdot (\langle y \rangle \frown w) = x * y + v \cdot w$$

Each $\mathbb{R}(n)$ is a real inner product space under the operation of dot product defined above.

3.2. **The Norm of a Tuple:** norm `\normRinf`. The norm $\|v\|$ of the n-tuple v is the positive square root of its dot product with itself.

$$\|v\| = \sqrt{v \cdot v}$$

Define $\mathrm{norm}(v)$ to be $\|v\|$.

$$\mathrm{norm} : \mathbb{R}^{\infty} \longrightarrow \mathbb{R}$$

$$\forall\, v : \mathbb{R}^{\infty} \bullet \\ \quad \mathrm{norm}(v) = \mathrm{sqrt}(v \cdot v)$$

The concepts of continuity, limits, and differentiability extend to functions between normed vector spaces such as $\mathbb{R}^n$.

3.3. **The Open Ball at a Tuple:** ball `\ballRinf`. Let $\mathrm{ball}(v, r)$ denote the *open ball* in $\mathbb{R}(n)$ of radius $r \in \mathbb{R}_+$ centred at $v \in \mathbb{R}(n)$.

$$
\begin{array}{|l}
\mathrm{ball} : \mathbb{R}^\infty \times \mathbb{R}_+ \longrightarrow \mathbb{P}\,\mathbb{R}^\infty \\
\hline
\forall\, v : \mathbb{R}^\infty;\ r : \mathbb{R}_+ \bullet \\
\quad \mathbf{let}\ n == \#v \bullet \\
\quad\quad \mathrm{ball}(v, r) = \{\, w : \mathbb{R}(n) \mid \mathrm{norm}(v - w) < r \,\}
\end{array}
$$

3.4. **The Set of All Open Balls at an n-tuple:** balls `\ballsRtup`. Let $\mathrm{balls}(n)$ denote the family of all open balls in $\mathbb{R}(n)$.

$$
\begin{array}{|l}
\mathrm{balls} : \mathbb{N} \longrightarrow \mathcal{F}\,\mathbb{R}^\infty \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\quad \mathrm{balls}(n) = \{\, v : \mathbb{R}(n);\ r : \mathbb{R}_+ \bullet \mathrm{ball}(v, r) \,\}
\end{array}
$$

Remark. *The set of all open balls in* $\mathbb{R}(n)$ *is a family of sets in* $\mathbb{R}(n)$.

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\quad \mathrm{balls}(n) \in \mathcal{F}(\mathbb{R}(n))
\end{array}
$$

3.5. **The Usual Topology on n-tuples:** $\tau_{\mathbb{R}}$ `\tauRtup`. The *usual topology* on $\mathbb{R}(n)$ is the topology generated by the open balls in $\mathbb{R}(n)$. Let $\tau_{\mathbb{R}}(n)$ denote the usual topology on $\mathbb{R}(n)$.

$$
\begin{array}{|l}
\tau_{\mathbb{R}} : \mathbb{N} \longrightarrow \mathcal{F}\,\mathbb{R}^\infty \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\quad \tau_{\mathbb{R}}(n) = \mathit{topGen}[\mathbb{R}(n)](\mathrm{balls}(n))
\end{array}
$$

Remark. *If* $n \in \mathbb{N}$ *then* $\tau_{\mathbb{R}}(n)$ *is a topology on* $\mathbb{R}(n)$.

$$\forall\, n : \mathbb{N} \bullet \tau_{\mathbb{R}}(n) \in \mathit{top}[\mathbb{R}(n)]$$

3.6. **The Set of All Neighbourhoods of a Tuple:** neigh `\neighRinf`. Let $v \in \mathbb{R}(n)$. An open set U in the usual topology $\tau_{\mathbb{R}}(n)$ that contains v is called a *neighbourhood* of v. Let $\mathrm{neigh}(v)$ denote the set of all neighbourhoods of x.

$$
\begin{array}{|l}
\mathrm{neigh} : \mathbb{R}^\infty \longrightarrow \mathcal{F}\,\mathbb{R}^\infty \\
\hline
\forall\, n : \mathbb{N};\ v : \mathbb{R}^\infty \mid n = \#v \bullet \\
\quad \mathrm{neigh}(v) = \{\, U : \tau_{\mathbb{R}}(n) \mid v \in U \,\}
\end{array}
$$

Remark. *The set of all neighbourhoods of* $v \in \mathbb{R}(n)$ *is a family of sets in* $\mathbb{R}(n)$.

$$
\begin{array}{l}
\forall\, n : \mathbb{N};\ v : \mathbb{R}^\infty \mid n = \#v \bullet \\
\quad \mathrm{neigh}(v) \in \mathcal{F}(\mathbb{R}(n))
\end{array}
$$

3.7. **The Topological Space of n-tuples:** $\mathbb{R}_\tau$ `\tsRtup`. Let $\mathbb{R}_\tau(n)$ denote the topological space defined by the usual topology on $\mathbb{R}(n)$.

$$
\begin{array}{|l}
\mathbb{R}_\tau : \mathbb{N} \longrightarrow \mathit{topSpaces}[\mathbb{R}^\infty] \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\quad \mathbb{R}_\tau(n) = (\mathbb{R}(n), \tau_{\mathbb{R}}(n))
\end{array}
$$

4. Continuity

4.1. **Real-Valued Functions That Are Continuous on the Set of All n-tuples:** C^0 **`\CzeroRtup`.** A function $f \in \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be *continuous* if it is continuous with respect to the usual topologies on $\mathbb{R}^n$ and $\mathbb{R}$. Let $\mathrm{C}^0(n)$ denote the set of these continuous functions.

$$\begin{array}{|l}
\mathrm{C}^0 : \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \rightarrow\!\!\!\!\!+\, \mathbb{R}) \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathrm{C}^0(n) = \mathrm{C}^0(\mathbb{R}_\tau(n), \mathbb{R}_\tau)
\end{array}$$

4.2. **Real-Valued Functions That Are Continuous on a Subset of n-tuples:** C^0 **`\CzeroSubsetRtup`.** Let U be a subset of $\mathbb{R}^n$. A function $f \in U \longrightarrow \mathbb{R}$ is said to be *continuous on U* if it is continuous with respect to the topology induced on U. Let $\mathrm{C}^0(U)$ denote the set of these continuous functions.

$$\begin{array}{|l}
\mathrm{C}^0 : \Delta_\mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \rightarrow\!\!\!\!\!+\, \mathbb{R}) \\
\hline
\forall\, U : \Delta_\mathbb{R} \bullet \\
\qquad \mathbf{let}\ n == \dim U \bullet \\
\qquad\qquad \mathrm{C}^0(U) = \mathrm{C}^0(\mathbb{R}_\tau(n) \mid_{\mathsf{top}} U, \mathbb{R}_\tau)
\end{array}$$

4.3. **Real-Valued Functions That Are Continuous at an n-tuple:** C^0 **`\CzeroPointRtup`.** A partial function f from $\mathbb{R}^n$ to $\mathbb{R}$ is said to be *continuous at* $x \in \mathbb{R}^n$ if its domain contains a neighbourhood U of x such that its restriction to U is continuous on U. Let $\mathrm{C}^0(x)$ denote the set of such functions.

$$\begin{array}{|l}
\mathrm{C}^0 : \mathbb{R}^\infty \longrightarrow \mathbb{P}(\mathbb{R}^\infty \rightarrow\!\!\!\!\!+\, \mathbb{R}) \\
\hline
\forall\, x : \mathbb{R}^\infty \bullet \\
\qquad \mathbf{let}\ n == \#x \bullet \\
\qquad\qquad \mathrm{C}^0(x) = \{\, f : \mathbb{R}(n) \rightarrow\!\!\!\!\!+\, \mathbb{R} \mid \exists\, U : \mathrm{neigh}(x) \mid U \subseteq \operatorname{dom} f \bullet U \lhd f \in \mathrm{C}^0(U) \,\}
\end{array}$$

4.4. **m-tuple-Valued Functions That Are Continuous on the Set of All n-tuples:** C^0 **`\CzeroRtupRtup`.** A mapping f from $\mathbb{R}(n)$ to $\mathbb{R}(m)$ is said to be continuous if it is continuous with respect to the usual topologies. Let $\mathrm{C}^0(n, m)$ denote the set of these continuous mappings.

$$\begin{array}{|l}
\mathrm{C}^0 : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \rightarrow\!\!\!\!\!+\, \mathbb{R}^\infty) \\
\hline
\forall\, n, m : \mathbb{N} \bullet \\
\qquad \mathrm{C}^0(n, m) = \mathrm{C}^0(\mathbb{R}_\tau(n), \mathbb{R}_\tau(m))
\end{array}$$

Example. *The function* $\mathrm{I}(n)$ *is continuous.*

$$\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathrm{I}(n) \in \mathrm{C}^0(n, n)
\end{array}$$

Theorem 1. *Linear functions are continuous.*

$$\begin{array}{l}
\forall\, n, m : \mathbb{N} \bullet \\
\qquad \mathrm{L}_\mathbb{R}(n, m) \subseteq \mathrm{C}^0(n, m)
\end{array}$$

4.5. **m-tuple-Valued Functions That Are Continuous on a Subset of n-tuples:** C^0 `\CzeroSubsetRtupRtup`**.** Let U be any subset of $\mathbb{R}(n)$. Let $\mathrm{C}^0(U, m)$ denote the set of continuous mappings from the topology induced by $\mathbb{R}_\tau(n)$ on U to $\mathbb{R}_\tau(m)$.

$$
\begin{array}{|l}
\mathrm{C}^0 : \Delta_{\mathbb{R}} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \mathrel{+\!\!\!\rightarrow} \mathbb{R}^\infty) \\
\hline
\forall\, n, m : \mathbb{N} \bullet \\
\quad \forall\, U : \Delta_{\mathbb{R}} \mid \dim(U) = n \bullet \\
\quad\quad \mathrm{C}^0(U, m) = \mathrm{C}^0(\mathbb{R}_\tau(n) \mid_{\mathsf{top}} U, \mathbb{R}_\tau(m))
\end{array}
$$

Remark.

$$
\begin{array}{l}
\forall\, n, m : \mathbb{N} \bullet \\
\quad \mathrm{C}^0(\mathbb{R}(n), m) = \mathrm{C}^0(n, m)
\end{array}
$$

4.6. **m-tuple-Valued Functions That Are Continuous at an n-tuple:** *VectorConinuous*, C^0 `\CzeroPointRtupRtup`**.** Let $x \in \mathbb{R}(n)$ and let f be a partial function from $\mathbb{R}(n)$ to $\mathbb{R}(m)$ whose domain includes some neighbourhood U of x such that f restricted to U is continuous. In this case f is said to be *continuous at* x.

$$
\begin{array}{|l}
\hline
\textit{VectorContinuous} \\
n, m : \mathbb{N} \\
f : \mathbb{R}^\infty \mathrel{+\!\!\!\rightarrow} \mathbb{R}^\infty \\
x : \mathbb{R}^\infty \\
\hline
f \in \mathbb{R}(n) \mathrel{+\!\!\!\rightarrow} \mathbb{R}(m) \\
\exists\, U : \mathrm{neigh}(x) \mid \\
\quad U \subseteq \mathrm{dom}\, f \bullet \\
\quad\quad U \lhd f \in \mathrm{C}^0(U, m) \\
\hline
\end{array}
$$

Let $\mathrm{C}^0(x, m)$ denote the set of all partial functions f from $\mathbb{R}(n)$ to $\mathbb{R}(m)$ that are continuous at x.

$$
\begin{array}{|l}
\mathrm{C}^0 : \mathbb{R}^\infty \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \mathrel{+\!\!\!\rightarrow} \mathbb{R}^\infty) \\
\hline
\forall\, n, m : \mathbb{N} \bullet \forall\, x : \mathbb{R}(n) \bullet \\
\quad \mathrm{C}^0(x, m) = \\
\quad\quad \{\, f : \mathbb{R}(n) \mathrel{+\!\!\!\rightarrow} \mathbb{R}(m) \mid \textit{VectorContinuous} \,\}
\end{array}
$$

Example. *The function* $\mathrm{I}(n)$ *is continuous at every point* $x \in \mathbb{R}(n)$.

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \forall\, x : \mathbb{R}(n) \bullet \\
\quad \mathrm{I}(n) \in \mathrm{C}^0(x, n)
\end{array}
$$

Theorem 2. *Linear functions are continuous everywhere.*

$$
\begin{array}{l}
\forall\, n, m : \mathbb{N} \bullet \\
\quad \forall\, x : \mathbb{R}(n);\ L : \mathrm{L}_{\mathbb{R}}(n, m) \bullet \\
\quad\quad L \in \mathrm{C}^0(x, m)
\end{array}
$$

5. Differentiability

Let $x \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \nrightarrow \mathbb{R}^m$ be continuous at x. Then f is said to be *differentiable at x* if there exists a linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that $f(x+h) - f(x)$ is approximately linear in h for very small h.

$$f(x+h) - f(x) \approx L(h) + O(h^2) \quad \text{when} \quad \|h\| \approx 0$$

This condition can be written as a limit.

$$\lim_{h \to 0} \frac{\big\|f(x+h) - f(x) - L(h)\big\|}{\|h\|} = 0$$

5.1. The Difference Quotient: *DifferenceQuotient* and *diffQuot*.

The limit exists when the following difference quotient function $q : \mathbb{R}^n \nrightarrow \mathbb{R}$ is continuous at 0.

$$q(h) = \begin{cases} \frac{\|f(x+h)-f(x)-L(h)\|}{\|h\|} & \text{if } h \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Given a function f that is continuous at x, and a linear transformation L, we can define the difference quotient q. Clearly q is uniquely determined by f, x, and L. Let *DifferenceQuotient* denote this situation.

$$\begin{array}{|l}
\textit{DifferenceQuotient} \\
\hline
\textit{VectorContinuous} \\
L : \mathbb{R}^\infty \nrightarrow \mathbb{R}^\infty \\
q : \mathbb{R}^\infty \nrightarrow \mathbb{R} \\
\hline
L \in \mathrm{L}_{\mathbb{R}}(n, m) \\
\operatorname{dom} q = \{\, h : \mathbb{R}(n) \mid x + h \in \operatorname{dom} f \,\} \\
\forall\, h : \operatorname{dom} q \mid h \neq \mathbf{0}(n) \bullet \\
\qquad q(h) = \operatorname{norm}(f(x+h) - f(x) - L(h)) \,/\, \operatorname{norm}(h) \\
q(\mathbf{0}(n)) = 0 \\
\hline
\end{array}$$

- L is a linear transformation from $\mathbb{R}(n)$ to $\mathbb{R}(m)$.
- The difference quotient q is defined on a subset of $\mathbb{R}(n)$ that contains $\mathbf{0}(n)$.
- $q(h)$ is defined as the quotient when h is non-zero.
- $q(0)$ is defined as zero.

Let *diffQuot*(f, x, L) denote the difference quotient q.

$$\textit{diffQuot} == \{\, \textit{DifferenceQuotient} \bullet (f, x, L) \mapsto q \,\}$$

5.2. The Derivative of a Continuous m-tuple-Valued Function: *VectorDifferentiable*.

The continuous function f is *differentiable at x* when there exists a linear transformation L such that the difference quotient q is continuous at 0. In this case L is unique and is referred to as the *derivative at x*.

$$
\begin{array}{|l}
\hline
\textit{VectorDifferentiable} \\
\textit{VectorContinuous} \\
L : \mathbb{R}^\infty \twoheadrightarrow \mathbb{R}^\infty \\
\hline
\textbf{let } q == \textit{diffQuot}(f, x, L) \bullet \\
\qquad q \in \mathrm{C}^0(\mathbf{0}(n)) \\
\hline
\end{array}
$$

- The continuous function f is differentiable at x with derivative L if the resulting difference quotient q is continuous at $\mathbf{0}(n)$.

Remark. *If L exists then it is unique.*

Let $\mathrm{C}^\infty(x, m)$ denote the set of all functions $f \in \mathbb{R}(n) \twoheadrightarrow \mathbb{R}(m)$ that are smooth at $x \in \mathbb{R}(n)$.

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