Complex Numbers

Arthur Ryman, arthur.ryman@gmail.com

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Abstract

This article contains Z Notation type declarations for the complex numbers, $\mathbb{C}$, and some related objects. It has been type checked by *f*UZZ.

1 Introduction

The complex numbers, $\mathbb{C}$, are foundational to many mathematical objects such as vector spaces and manifolds, but are not built-in to Z Notation. This article provides type declarations for $\mathbb{C}$ and related objects so that they can be used and type checked in formal Z specifications.

No attempt has been made to provide complete, axiomatic definitions of all these objects since that would only be of use for proof checking. Although proof checking is highly desirable, it is beyond the scope of this article. The type declarations given here are intended to provide a basis for future axiomatization.

2 Complex Numbers

Z Notation does not predefine the set of complex numbers, so we define them and operations on them here.

Although complex number operations are displayed using the same symbols as the analogous real number operations, they are distinct mathematical objects. This distinction is apparent to the *f*UZZ type-checker and should not cause confusion to the human reader because the underlying types of the objects will, as a rule, be clear from the context. Visually distinct symbols will be used in cases where confusion is possible.

2.1 *COMPLEX*

A complex number can be thought of a pair of real numbers. However, it is not correct to view any pair of real numbers as a complex number. Therefore to denote a complex number we map a pair of real numbers into the free type $COMPLEX$.

$$COMPLEX ::= complex \langle\!\langle \mathbb{R} \times \mathbb{R} \rangle\!\rangle$$

Here the function *complex* is a constructor that constructs a complex number from a pair of real numbers.

2.2 $\mathbb{C}$ `\C`

We introduce the usual notation $\mathbb{C} = COMPLEX$ for the set of complex numbers.

$$\mathbb{C} == COMPLEX$$

2.3 *Complex*

Given real numbers x and y, we can construct the complex number $z = complex(x, y)$. The real numbers x and y are referred to as the *real* and *imaginary* parts of z. It's useful to introduce the schema *Complex* that relates the complex number z to its real and imaginary parts x and y.

```
┌─ Complex ──────────────────────
│ z : ℂ
│ x, y : ℝ
├──────────
│ z = complex(x, y)
└────────────────────────────────
```

2.4 *real_complex*

Let $real\_complex(z)$ denote the real part x of z.

$$real\_complex == \{ Complex \bullet z \mapsto x \}$$

2.5 Re `\realC`

We introduce the usual notation $x = \mathrm{Re}(z)$ for the real part of z.

$$\mathrm{Re} == real\_complex$$

2.6 *imag_complex*

Let $imag\_complex(z)$ denote the imaginary part y of z.

$$imag\_complex == \{ Complex \bullet z \mapsto y \}$$

2.7 Im \imagC

We introduce the usual notation $y = \mathrm{Im}(z)$ for the imaginary part of z.

$$\mathrm{Im} == \mathit{imag\_complex}$$

2.8 *AddComplex*

We can *add* the complex numbers z_1 and z_2 to give their sum z'. Let the schema *AddComplex* denote this situation.

$$\begin{array}{|l}
\multicolumn{1}{l}{\mathit{AddComplex}} \\
\hline
\mathit{Complex}_1 \\
\mathit{Complex}_2 \\
\mathit{Complex}' \\
\hline
x' = x_1 + x_2 \\
y' = y_1 + y_2 \\
\hline
\end{array}$$

2.9 *add_complex*

Let $\mathit{add\_complex}(z_1, z_2)$ denote the result of adding z_1 and z_2.

$$\mathit{add\_complex} == \{\, \mathit{AddComplex} \bullet (z_1, z_2) \mapsto z' \,\}$$

2.10 + \addC

We introduce the usual notation $z' = z_1 + z_2$ for addition in $\mathbb{C}$.

$$(\_ + \_) == \mathit{add\_complex}$$

2.11 *zero_complex*

Let *zero_complex* denote the *zero* of $\mathbb{C}$.

$$\mathit{zero\_complex} == \mathit{complex}(0, 0)$$

2.12 0 \zeroC

We introduce the usual notation $0 \in \mathbb{C}$ for the zero of $\mathbb{C}$.

$$0 == \mathit{zero\_complex}$$

2.13 *NegComplex*

We can *negate* the complex number z to give its negative z'. Let the schema *NegComplex* denote this situation.

$$\begin{array}{|l}
\hline
\textit{NegComplex} \\
\textit{Complex} \\
\textit{Complex}' \\
\hline
x' = -x \\
y' = -y \\
\hline
\end{array}$$

2.14 $neg\_complex(z)$

Let $neg\_complex(z)$ denote the negative of z.

$$neg\_complex == \{ NegComplex \bullet z \mapsto z' \}$$

2.15 - \negC

We introduce the usual notation $z' = -z$ for the negative of z.

$$- == neg\_complex$$

2.16 The Additive Abelian Group $\mathbb{C}$

Theorem 1. *The complex numbers $\mathbb{C}$ form an Abelian group under addition.*

$$(\_ + \_) \in \text{abgroup}\,\mathbb{C}$$
$$0 = identity\_element(\_ + \_)$$
$$- = inverse\_operation(\_ + \_)$$

2.17 *SubComplex*

We can *subtract* the complex number z_1 from z_2 to give their difference z'. Let the schema *SubComplex* denote this situation.

$$\begin{array}{|l}
\hline
\textit{SubComplex} \\
\textit{Complex}_1 \\
\textit{Complex}_2 \\
\textit{Complex}' \\
\hline
x' = x_1 - x_2 \\
y' = y_1 - y_2 \\
\hline
\end{array}$$

2.18 $sub\_complex$

Let $sub\_complex(z_1, z_2)$ denote the difference of z_1 and z_2.

$$sub\_complex == \{ SubComplex \bullet (z_1, z_2) \mapsto z' \}$$

2.19 $-$ `\subC`

We introduce the usual notation $z' = z_1 - z_2$ for subtraction in $\mathbb{C}$.

$$(\_ - \_) == sub\_complex$$

2.20 $nonzero\_complex$

Let $nonzero\_complex$ denote the set of nonzero complex numbers.

$$nonzero\_complex == \mathbb{C} \setminus \{0\}$$

2.21 $\mathbb{C}_*$ `\Cnz`

We introduce the usual notation $\mathbb{C}_* \subseteq \mathbb{C}$ to denote the set of nonzero complex numbers, also referred to as the *punctured complex number plane*.

$$\mathbb{C}_* == nonzero\_complex$$

2.22 $MulComplex$

We can *multiply* the complex numbers z_1 times z_2 to give their product z'. Let the schema $MulComplex$ denote this situation.

$$\begin{array}{|l}
\multicolumn{1}{l}{MulComplex} \\
\hline
Complex_1 \\
Complex_2 \\
Complex' \\
\hline
x' = x_1 * x_2 - y_1 * y_2 \\
y' = x_1 * y_2 + y_1 * x_2 \\
\hline
\end{array}$$

2.23 $mul\_complex$

Let $mul\_complex(z_1, z_2)$ denote z_1 multiplied by z_2.

$$mul\_complex == \{ MulComplex \bullet (z_1, z_2) \mapsto z' \}$$

2.24 $*$ `\mulC`

We introduce the usual notation $z' = z_1 * z_2$ for multiplication in $\mathbb{C}$.

$$(\_ * \_) == mul\_complex$$

2.25 *MulNonzeroComplex*

We can restrict multiplication in $\mathbb{C}$ to $\mathbb{C}_*$. Let the schema *MulNonzeroComplex* denote this situation.

$$\begin{array}{|l}
\multicolumn{1}{l}{\textit{MulNonzeroComplex}} \\
\hline
\textit{MulComplex} \\
\hline
z_1 \in \mathbb{C}_* \\
z_2 \in \mathbb{C}_* \\
\hline
\end{array}$$

2.26 *mul_nonzero_complex*

Let *mul_nonzero_complex*(z_1, z_2) denote the product of nonzero complex numbers.

$$mul\_nonzero\_complex == \{ \, MulNonzeroComplex \bullet (z_1, z_2) \mapsto z' \, \}$$

2.27 $*$ `\mulCnz`

We introduce the usual notation $z' = z_1 * z_2$ to denote the product.

$$(\_ * \_) == mul\_nonzero\_complex$$

2.28 *one_complex*

Let *one_complex* denote the multiplicative unit in $\mathbb{C}$.

$$one\_complex == complex(1, 0)$$

2.29 1 `\oneC`

We introduce the usual notation $1 \in \mathbb{C}$ for the unit of $\mathbb{C}$.

$$1 == one\_complex$$

2.30 *InvNonzeroComplex*

We can *invert* the nonzero complex number z to get its inverse or reciprocal z'. Let the schema *InvNonzeroComplex* denote this situation.

$$
\begin{array}{|l}
\hline
\textit{InvNonzeroComplex} \\
z, z' : \mathbb{C}_* \\
\hline
z * z' = 1 \\
\hline
\end{array}
$$

2.31 *inv_nonzero_complex*

Let $z' = \mathit{inv\_nonzero\_complex}(z)$ denote the inverse of z.

$$\mathit{inv\_nonzero\_complex} == \{ \mathit{InvNonzeroComplex} \bullet z \mapsto z' \}$$

2.32 $^{-1}$ `\invCnz`

We introduce the usual notation $z' = z^{-1}$ for the inverse.

$$(\_^{-1}) == \mathit{inv\_nonzero\_complex}$$

2.33 *DivNonzeroComplex*

We can *divide* the nonzero complex numbers z_1 by z_2 to get their quotient z'. Let the schema *DivNonzeroComplex* denote this situation.

$$
\begin{array}{|l}
\hline
\textit{DivNonzeroComplex} \\
z_1, z_2, z' : \mathbb{C}_* \\
\hline
z_1 = z' * z_2 \\
\hline
\end{array}
$$

2.34 *div_nonzero_complex*

Let $z' = \mathit{div\_nonzero\_complex}(z_1, z_2)$ denote z_1 divided by z_2.

$$\mathit{div\_nonzero\_complex} == \{ \mathit{DivNonzeroComplex} \bullet (z_1, z_2) \mapsto z' \}$$

2.35 / `\divCnz`

We introduce the usual notation $z_1 \,/\, z_2$ to denote division.

$$(\_ \,/\, \_) == \mathit{div\_nonzero\_complex}$$

2.36 The Multiplicative Abelian Group $\mathbb{C}_*$

Theorem 2. *The nonzero complex numbers $\mathbb{C}_*$ form an Abelian group under multiplication.*

$$(\_ * \_) \in \mathrm{abgroup}\, \mathbb{C}_*$$

$$1 = \mathit{identity\_element}(\_ * \_)$$

$$(\_^{-1}) = \mathit{inverse\_operation}(\_ * \_)$$

2.37 *NormComplex*

The *norm* of a complex number z is a non-negative real number r equal to the Euclidean length of its underlying pair of real numbers regarded as a vector in the Euclidean plane. Let the schema *NormComplex* denote this situation.

$$\begin{array}{|l}
\hline \mathit{NormComplex} \\
\mathit{Complex} \\
r : \mathbb{R} \\
\hline
r = \mathrm{sqrt}(x * x + y * y) \\
\hline
\end{array}$$

2.38 *norm_complex*

Let $r = \mathit{norm\_complex}(z)$ be the norm of z.

$$\mathit{norm\_complex} == \{\, \mathit{NormComplex} \bullet z \mapsto r \,\}$$

2.39 norm `\normC`

We introduce the usual notation $r = \mathrm{norm}(z)$ to denote the norm of z.

$$\mathrm{norm} == \mathit{norm\_complex}$$