INTEGERS

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ABSTRACT. This article contains Z Notation definitions for concepts related to the integers, $\mathbb{Z}$. It has been type checked by fUZZ.

CONTENTS

1. INTRODUCTION

The integers $\mathbb{Z}$ are built-in to Z Notation. This article provides definitions for some related objects so that they can be used and type checked in formal Z specifications.

2. EXPONENTIATION

Let $x \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then x raised to the exponent n is the product of x multiplied by itself n times, with the convention that for $n = 0$ the result is 1. Let $exp(x, n)$ denote the result.

$$exp : \mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{Z}$$

$$\forall x : \mathbb{Z} \bullet exp(x, 0) = 1$$

$$\forall x : \mathbb{Z}; n : \mathbb{N}_1 \bullet exp(x, n) = x * exp(x, n - 1)$$

Example.

$exp(5, 2) = 25$

Remark.

$\forall x : \mathbb{Z} \bullet$
$\quad exp(x, 0) = 1$

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Remark.

$\forall x : \mathbb{Z} \bullet$
$\quad exp(x, 1) = x$

Remark.

$\forall x : \mathbb{Z}; n, m : \mathbb{N} \bullet$
$\quad exp(x, n + m) = exp(x, n) * exp(x, m)$

Remark.

$\forall x, y : \mathbb{Z}; n : \mathbb{N} \bullet$
$\quad exp(x * y, n) = exp(x, n) * exp(y, n)$

Exponentiation is normally denoted x^n but Z Notation cannot reproduce that. Instead we use the infix operator notation that is common to programming languages such as FORTRAN and Python. Define the notation $x ** n = exp(x, n)$.

$(\_ ** \_) == exp$

Example.

$5 ** 2 = 25$

3. Divisibility

This section defines *divisibility* of integers.

Given integers x and y we say that x *divides* y if there is some integer q such that $qx = y$.

Divides

$x, y, q : \mathbb{Z}$

$q * x = y$

Let *divides* denote the divisibility relation between integers where $(x, y) \in divides$ means that x divides y.

$divides == \{ Divides \bullet x \mapsto y \}$

Remark.

$divides \in \mathbb{Z} \leftrightarrow \mathbb{Z}$

We introduce the usual infix notation $x \mid y$ to denote that $(x, y) \in divides$.

$(\_ \mid \_) == divides$

Example. *The integer* 7 *divides* 42 *because* $6 * 7 = 42$.

$7 \mid 42$

Remark. *Every integer* x *divides* 0 *because* $0 * x = 0$.

$\forall x : \mathbb{Z} \bullet x \mid 0$

4. Divisors

Let x be a nonzero integer that divides the integer y. We say that x is a *divisor* of y.

$$\begin{array}{|l}
\multicolumn{1}{l}{\textit{Divisor}} \\
\hline
\textit{Divides} \\
\hline
x \neq 0 \\
\hline
\end{array}$$

Let the relation $(x, y) \in \mathit{is\_divisor\_of}$ denote that x is a divisor of y.

$\mathit{is\_divisor\_of} == \{\, \mathit{Divisor} \bullet x \mapsto y \,\}$

Let the set $\mathit{divisors}(y)$ denote the set of all divisors of the integer y.

$\mathit{divisors} == (\lambda\, y : \mathbb{Z} \bullet \{\, x : \mathbb{Z} \mid (x, y) \in \mathit{is\_divisor\_of} \,\})$

Remark.

$\mathit{divisors} \in \mathbb{Z} \longrightarrow \mathbb{P}\,\mathbb{Z}$

Example. *The integer* 6 *has the following divisors.*

$\mathit{divisors}(6) = \{-6, -3, -2, -1, 1, 2, 3, 6\}$

Let the set $\mathit{positive\_divisors}(y)$ denote the set of all positive divisors of the integer y.

$\mathit{positive\_divisors} == (\lambda\, y : \mathbb{Z} \bullet \mathit{divisors}(y) \cap \mathbb{N}_1)$

Remark.

$\mathit{positive\_divisors} \in \mathbb{Z} \longrightarrow \mathbb{P}\,\mathbb{N}_1$

Example. *The integer* 6 *has the following positive divisors.*

$\mathit{positive\_divisors}(6) = \{1, 2, 3, 6\}$

5. Prime Numbers

An integer p is *prime* if it is greater than one and only has one and itself as positive divisors.

$$\begin{array}{|l}
\multicolumn{1}{l}{\textit{Prime}} \\
\hline
p : \mathbb{Z} \\
\hline
p > 1 \\
\mathit{positive_divisors}(p) = \{1, p\} \\
\hline
\end{array}$$

- p is greater than 1.
- 1 and p are the only positive divisors of p.

Example. *The integer* 2 *is prime.*

$$\mathbf{let}\ p == 2 \bullet \\ \quad \mathit{Prime}$$

Let *primes* denote the set of all primes.

$$\mathit{primes} == \{\, \mathit{Prime} \bullet p \,\}$$

Remark.

$$\mathit{primes} \subset \mathbb{N}_1$$

Example. *The natural numbers* $2, 3, 5,$ *and* 7 *are primes.*

$$\{2, 3, 5, 7\} \subseteq \mathit{primes}$$

6. Addition of Integer Sequences

Let l be a natural number and let x and y be two integer sequences of length l. Their sum $z = x+y$ is the integer sequence of length l defined by pointwise addition of the terms in x and y.

AddIntegerSequences

$$l : \mathbb{N} \\ x, y, z : \operatorname{seq} \mathbb{Z}$$

$$l = \#x = \#y$$

$$z = (\lambda\, i : 1 \mathinner{..} l \bullet x\ i + y\ i)$$

- The sequence z is defined by pointwise addition of the sequences x and y.

Let the function $\mathit{add\_int\_seq}(x, y) = z$ be the sum of two equal-length integer sequences.

$$\mathit{add\_int\_seq} == \{\, \mathit{AddIntegerSequences} \bullet (x, y) \mapsto z \,\}$$

Remark. *Addition is a partial function on the set of all pairs of integer sequences.*

$$\mathit{add\_int\_seq} \in \operatorname{seq} \mathbb{Z} \times \operatorname{seq} \mathbb{Z} \pfun \operatorname{seq} \mathbb{Z}$$

We introduce the usual infix notation $x + y = \mathit{add\_int\_seq}(x, y)$.

$$(\_ + \_) == \mathit{add\_int\_seq}$$

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