Complex Numbers

Arthur Ryman, arthur.ryman@gmail.com

January 15, 2022

Abstract

This article contains Z Notation type declarations for the complex numbers, $\mathbb{C}$, and some related objects. It has been type checked by *f*UZZ.

1 Introduction

The complex numbers, $\mathbb{C}$, are foundational to many mathematical objects such as vector spaces and manifolds, but are not built-in to Z Notation. This article provides type declarations for $\mathbb{C}$ and related objects so that they can be used and type checked in formal Z specifications.

No attempt has been made to provide complete, axiomatic definitions of all these objects since that would only be of use for proof checking. Although proof checking is highly desirable, it is beyond the scope of this article. The type declarations given here are intended to provide a basis for future axiomatization.

2 Complex Numbers

Z notation does not predefine the set of complex numbers, so we define it here.

2.1 $\mathbb{C}$ `\C`

Let $\mathbb{C}$ denote the set of complex numbers. We define it to be simply the set of pairs of real numbers. Note that it would be more correct to define complex numbers as a new free type based on pairs of real numbers since otherwise we have no way to prevent accidentally conflating plain pairs of real numbers with complex numbers. We'll take this risk for the moment. We'll add further axioms as needed below.

$$COMPLEX ::= complex\langle\!\langle \mathbb{R} \times \mathbb{R} \rangle\!\rangle$$

We introduce the usual notation for the set of complex numbers.

$$\mathbb{C} == \mathit{COMPLEX}$$

Given real numbers x and y, we can construct the complex number $z = \mathit{complex}(x, y)$. The real numbers x and y are referred to as the *real* and *imaginary* parts of z. It's useful to introduce schema *Complex* that relates the complex number z to its real and imaginary parts x and y.

Complex

$$z : \mathbb{C}$$
$$x, y : \mathbb{R}$$

$$z = \mathit{complex}(x, y)$$

Let $\mathit{real\_complex}(z)$ denote the real part of z.

$$\mathit{real\_complex} == \{\, \mathit{Complex} \bullet z \mapsto x \,\}$$

We introduce the usual notation $x = \mathrm{Re}(z)$ for the real part of z.

$$\mathrm{Re} == \mathit{real\_complex}$$

Let $\mathit{imag\_complex}(z)$ denote the imaginary part of z.

$$\mathit{imag\_complex} == \{\, \mathit{Complex} \bullet z \mapsto y \,\}$$

We introduce the usual notation $y = \mathrm{Im}(z)$ for the imaginary part of z.

$$\mathrm{Im} == \mathit{imag\_complex}$$

2.2 $+$ `\addC`, 0 `\zeroC`, - `\negC`, and $-$ `\subC`

We can *add* the complex numbers z_1 and z_2 to give the complex number z'. Let the schema *AddComplex* denote this situation.

AddComplex

$$\mathit{Complex}_1$$
$$\mathit{Complex}_2$$
$$\mathit{Complex}'$$

$$x' = x_1 + x_2$$
$$y' = y_1 + y_2$$

Let $\mathit{add\_complex}(z_1, z_2)$ denote the result of adding z_1 and z_2.

$$add\_complex == \{ AddComplex \bullet (z_1, z_2) \mapsto z' \}$$

We introduce the usual notation $z' = z_1 + z_2$ for addition in $\mathbb{C}$.

$$(\_ + \_) == add\_complex$$

Let *zero_complex* denote the *zero* of $\mathbb{C}$.

$$zero\_complex == complex(0, 0)$$

We introduce the usual notation $0 \in \mathbb{C}$ for the zero of $\mathbb{C}$.

$$0 == zero\_complex$$

We can *negate* the complex number z to give its negative z'. Let the schema *NegComplex* denote this situation.

$$\begin{array}{|l}
\hline
\textit{NegComplex} \\
\textit{Complex} \\
\textit{Complex}' \\
\hline
x' = \text{-}\, x \\
y' = \text{-}\, y \\
\hline
\end{array}$$

Let $neg\_complex(z)$ denote the negative of z.

$$neg\_complex == \{ NegComplex \bullet z \mapsto z' \}$$

We introduce the usual notation $z' = \text{-}\, z$ for the negative of z.

$$\text{-} == neg\_complex$$

The complex numbers form an Abelian group under addition.

$$(\_ + \_) \in \mathrm{abgroup}\, \mathbb{C}$$

$$0 = identity\_element(\_ + \_)$$

$$\text{-} = inverse\_operation(\_ + \_)$$

We can *subtract* the complex number z_1 from z_2 to give the difference z'. Let the schema *SubComplex* denote this situation.

$$\begin{array}{|l}
\multicolumn{1}{l}{\underline{\quad SubComplex \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}} \\
Complex_1 \\
Complex_2 \\
Complex' \\
\hline
x' = x_1 - x_2 \\
y' = y_1 - y_2 \\
\hline
\end{array}$$

Let $sub\_complex(z_1, z_2)$ denote z_1 subtract z_2.

$$sub\_complex == \{ SubComplex \bullet (z_1, z_2) \mapsto z' \}$$

We introduce the usual notation $z' = z_1 - z_2$ for subtraction in $\mathbb{C}$.

$$(\_ - \_) == sub\_complex$$

Although these complex number objects are displayed using the same symbols as the corresponding real number objects, they represent distinct mathematical objects. This distinction is apparent to the *f*UZZ type-checker and should not cause confusion to the human reader because the underlying types of the objects will, as a rule, be clear from the context. Visually distinct symbols will be used in cases where confusion is possible.

2.3 $\mathbb{C}_*$ `\Cnz`

Let *nonzero_complex* denote the set of nonzero complex numbers.

$$nonzero\_complex == \mathbb{C} \setminus \{0\}$$

We introduce the usual notation $\mathbb{C}_* \subseteq \mathbb{C}$ to denote the set of nonzero complex numbers, also referred to as the *punctured complex number plane*.

$$\mathbb{C}_* == nonzero\_complex$$

2.4 $*$ `\mulC`

We can *multiply* the complex numbers z_1 times z_2 to give the product z'. Let the schema *MulComplex* denote this situation.

$$\begin{array}{|l}
\multicolumn{1}{l}{\underline{\quad MulComplex \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}} \\
Complex_1 \\
Complex_2 \\
Complex' \\
\hline
x' = x_1 * x_2 - y_1 * y_2 \\
y' = x_1 * y_2 + y_1 * x_2 \\
\hline
\end{array}$$

Let $mul\_complex(z_1, z_2)$ denote z_1 multiplied by z_2.

$$mul\_complex == \{ MulComplex \bullet (z_1, z_2) \mapsto z' \}$$

We introduce the usual notation $z' = z_1 * z_2$ for multiplication in $\mathbb{C}$.

$$(\_ * \_) == mul\_complex$$

2.5 $*$ \mulCnz, 1 \oneC, $^{-1}$ \invCnz, and / \divC

We can restrict multiplication in $\mathbb{C}$ to $\mathbb{C}_*$. Let the schema *MulNonzeroComplex* denote this situation.

$$\begin{array}{|l}
\textit{MulNonzeroComplex} \\
\hline
\textit{MulComplex} \\
\hline
z_1 \in \mathbb{C}_* \\
z_2 \in \mathbb{C}_* \\
\hline
\end{array}$$

Let $mul\_nonzero\_complex(z_1, z_2)$ denote the product of nonzero complex numbers.

$$mul\_nonzero\_complex == \{ MulNonzeroComplex \bullet (z_1, z_2) \mapsto z' \}$$

We introduce the usual notation $z' = z_1 * z_2$ to denote the product.

$$(\_ * \_) == mul\_nonzero\_complex$$

Let $one\_complex$ denote the multiplicative unit in $\mathbb{C}$.

$$one\_complex == complex(1, 0)$$

We introduce the usual notation $1 \in \mathbb{C}$ for the unit of $\mathbb{C}$.

$$1 == one\_complex$$

We can *invert* the nonzero complex number z to get its inverse or reciprocal z'. Let the schema *InvNonzeroComplex* denote this situation.

$$\begin{array}{|l}
\textit{InvNonzeroComplex} \\
\hline
z, z' : \mathbb{C}_* \\
\hline
z * z' = 1 \\
\hline
\end{array}$$

Let $z' = inv\_nonzero\_complex(z)$ denote the inverse of z.

$$inv\_nonzero\_complex == \{ InvNonzeroComplex \bullet z \mapsto z' \}$$

We introduce the usual notation $z' = z^{-1}$.

$$(\_^{-1}) == inv\_nonzero\_complex$$

We can *divide* the nonzero complex numbers z_1 by z_2 to get their quotient z'. Let the schema *DivNonzeroComplex* denote this situation.

$$\begin{array}{|l}
\textit{DivNonzeroComplex} \\
\hline
z_1, z_2, z' : \mathbb{C}_* \\
\hline
z_1 = z' * z_2 \\
\hline
\end{array}$$

Let $z' = div\_nonzero\_complex(z_1, z_2)$ denote z_1 divided by z_2.

$$div\_nonzero\_complex == \{ DivNonzeroComplex \bullet (z_1, z_2) \mapsto z' \}$$

We introduce the usual notation $z_1 \mathbin{/} z_2$ to denote division.

$$(\_ \mathbin{/} \_) == div\_nonzero\_complex$$

The nonzero complex numbers form an Abelian group under multiplication.

$$(\_ * \_) \in \mathrm{abgroup}\, \mathbb{C}_*$$

$$1 = identity\_element(\_ * \_)$$

$$(\_^{-1}) = inverse\_operation(\_ * \_)$$

2.6 norm `\normC`

The *norm* of a complex number z is a non-negative real number r. Let the schema *NormComplex* denote this situation.

$$\begin{array}{|l}
\textit{NormComplex} \\
\hline
\textit{Complex} \\
r : \mathbb{R} \\
\hline
r = \mathrm{sqrt}(x * x + y * y) \\
\hline
\end{array}$$