

# VECTOR SPACES

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ABSTRACT. This article contains Z Notation type declarations for vector spaces and some related objects. It has been type checked by *fUZZ*.

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## 1. REAL VECTOR SPACES

Real vector spaces are multidimensional generalizations of real numbers. They are the objects studied in linear algebra and are foundational to differential geometry.

In the following let  $\mathbf{t}$  denote a set of elements which we'll refer to as *vectors* and let  $A$  denote an Abelian group over the vectors in which the binary operation is denoted as addition. Let  $v$  and  $w$  denote vectors and let  $x$  and  $y$  denote real numbers.

**1.1. Notation for Vector Addition, Zero, and Negative:**  $+$  `\addV`,  $\mathbf{0}$  `\zeroV`, and  $-$  `\negV`. Let  $v + w$  denote vector addition, let  $\mathbf{0}$  denote the zero vector, and let  $-v$  denote the negative vector.

**1.2. Real Scalar Multiplication:**  $*$  `\mulS`,  $\times$  `\timesS`, and *RealScalarMultiplication*.

A *real scalar multiplication* operation on the vectors is an operation *smul* that maps the pair  $(x, v)$  to another vector, typically denoted  $x * v$  or  $x \times y$ , such that multiplication by 0 maps all vectors to the group identity element, multiplication by 1 maps each vector to itself, multiplication preserves group addition, and multiplication distributes over both real and group addition.

Let *RealScalarMultiplication* denote this situation.

$RealScalarMultiplication[t]$
$A : abgroup[t]$ $smul : \mathbb{R} \times t \rightarrow t$
$let$ $(- + -) == second\ A;$ $\mathbf{0} == identity\_element\ A;$ $(- * -) == smul \bullet$ $\forall x, y : \mathbb{R}; v, w : t \bullet$ $\quad 0 * v = \mathbf{0} \wedge$ $\quad 1 * v = v \wedge$ $\quad (x * y) * v = x * (y * v) \wedge$ $\quad (x + y) * v = x * v + y * v \wedge$ $\quad x * (v + w) = x * v + x * w$

- Multiplying by 0 gives the zero vector.
- Multiplying by 1 gives the same vector.
- Scalar multiplication is associative.
- Scalar addition distributes over scalar multiplication.
- Vector addition distributes over scalar multiplication.

**1.3. The Set of All Real Vector Spaces:**  $vec_{\mathbb{R}} \setminus vecR$ . A *real vector space* is a pair  $(A, smul)$  where  $A$  is an Abelian group and  $smul$  is a real scalar multiplication on the elements of  $A$ . The elements of  $A$  are referred to as *vectors*.

Let  $vec_{\mathbb{R}} t$  denote the set of all real vector spaces over  $t$ .

TODO: Do not assume that the set of vectors coincides with  $t$ . In general, it will be a subset of  $t$ .

$$vec_{\mathbb{R}} t == \{ RealScalarMultiplication[t] \bullet (A, smul) \}$$

**1.4. Real Linear Transformations:** *RealLinearTransformation*. Let  $V_1$  and  $V_2$  be real vector spaces and let  $f$  be a homomorphism of the underlying Abelian groups. The map  $f$  is said to be a *linear transformation* if  $f$  maps scalar multiples of vectors to the scalar multiple of the mapped vectors.

Let *RealLinearTransformation* denote this situation.

$RealLinearTransformation[t, u]$
$f : t \rightarrow u$ $V_1 : vec_{\mathbb{R}} t$ $V_2 : vec_{\mathbb{R}} u$
$let$ $A_1 == first\ V_1; (- * -) == second\ V_1;$ $A_2 == first\ V_2; (- \times -) == second\ V_2 \bullet$ $(A_1, A_2) \mapsto f \in group\_hom(A_1, A_2) \wedge$ $(\forall x : \mathbb{R}; v : t \bullet$ $\quad f(x * v) = x \times (f\ v))$

- The vector space  $V_1$  has Abelian group  $A_1$  and scalar multiplication  $(- * -)$ .

- The vector space  $V_2$  has Abelian group  $A_2$  and scalar multiplication  $(\cdot \times \cdot)$ .
- The map  $f$  is a homomorphism of the underlying Abelian groups.
- The map  $f$  maps scalar multiples of vectors in  $\mathbf{t}$  to scalar multiples of the mapped vectors in  $\mathbf{u}$ .

**1.5. The Set of All Real Linear Transformations:**  $L_{\mathbb{R}} \setminus \text{homVecR}$ . Let  $V_1$  and  $V_2$  be real vector spaces. Let  $L_{\mathbb{R}}(V_1, V_2)$  denote the set of all linear transformations from  $V_1$  to  $V_2$ . A linear transformation is also referred to as a *homomorphism* of vector spaces.

$$L_{\mathbb{R}}[\mathbf{t}, \mathbf{u}] == (\lambda V_1 : \text{vec}_{\mathbb{R}} \mathbf{t}; V_2 : \text{vec}_{\mathbb{R}} \mathbf{u} \bullet \{ f : \mathbf{t} \rightarrow \mathbf{u} \mid \text{RealLinearTransformation}[\mathbf{t}, \mathbf{u}] \})$$

## 2. REAL $n$ -TUPLES

The preceding section described real vector spaces abstractly. In this section we define a family of finite-dimensional real vector spaces whose elements are finite sequences of real numbers, also referred to as *real tuples*.

**2.1. The Set of All Finite Sequences of Real Numbers:**  $\mathbb{R}^{\infty} \setminus \text{Rinf}$ . Let  $n$  be a natural number. A finite sequence of  $n$  real numbers is called a *real  $n$ -tuple*. Let  $\mathbb{R}^{\infty}$  denote the set of all real  $n$ -tuples for any  $n$ .

$$\mathbb{R}^{\infty} == \text{seq } \mathbb{R}$$

**2.2. The Component Projection Function:**  $\pi \setminus \text{piRinf}$ . The real numbers that comprise an  $n$ -tuple are called its *components*. Let  $v$  be a real  $n$ -tuple and let  $i$  be an integer where  $1 \leq i \leq n$ . The real number  $v(i)$  is the  $i$ -th component of  $v$ . Let  $\pi(i)$  be the projection function that maps an  $n$ -tuple  $v$  to its  $i$ -th component  $v(i)$ .

$$\begin{array}{|l} \pi : \mathbb{N}_1 \rightarrow \mathbb{R}^{\infty} \rightarrow \mathbb{R} \\ \hline \forall i : \mathbb{N}_1 \bullet \\ \pi(i) = (\lambda v : \mathbb{R}^{\infty} \mid i \in \text{dom } v \bullet v(i)) \end{array}$$

**2.3. The Set of All Well-Dimensioned Subsets of  $\mathbb{R}^{\infty}$ :**  $\Delta_{\mathbb{R}} \setminus \text{DeltaRinf}$ . A non-empty subset of  $\mathbb{R}^{\infty}$  is said to be *well-dimensioned* if each of its elements has the same number of components. Let  $\Delta_{\mathbb{R}}$  denote the family of all well-dimensioned subsets of  $\mathbb{R}^{\infty}$ .

$$\begin{array}{|l} \Delta_{\mathbb{R}} : \mathcal{F} \mathbb{R}^{\infty} \\ \hline \Delta_{\mathbb{R}} = \{ S : \mathbb{P}_1 \mathbb{R}^{\infty} \mid \forall v, w : S \bullet \#v = \#w \} \end{array}$$

**2.4. The Dimension of a Well-Dimensioned Set of Tuples:**  $\text{dim} \setminus \text{dimRinf}$ . Let  $S \in \Delta_{\mathbb{R}}$  be a well-dimensioned set of tuples. The number of components of each tuple in  $S$  is called its *dimension*. Let  $\text{dim}(S)$  denote the dimension of  $S$ .

$$\begin{array}{|l} \dim : \Delta_{\mathbb{R}} \longrightarrow \mathbb{N} \\ \hline \forall S : \Delta_{\mathbb{R}} \bullet \\ \dim S = (\mu v : S \bullet \#v) \end{array}$$

**2.5. The Set of All Compatible Pairs of Tuples:**  $\mathbb{R}^{\Delta} \setminus \text{RinfDelta}$ . The pair of real tuples  $(v, w)$  is said to be *compatible* if each member has the same number of components. Let  $\mathbb{R}^{\Delta}$  denote the set of all compatible pairs of real tuples. If the pair  $(v, w)$  is compatible then  $v$  and  $w$  are said to be compatible with each other.

$$\begin{array}{|l} \mathbb{R}^{\Delta} : \mathbb{R}^{\infty} \longleftrightarrow \mathbb{R}^{\infty} \\ \hline \mathbb{R}^{\Delta} = \{ v, w : \mathbb{R}^{\infty} \mid \#v = \#w \} \end{array}$$

**2.6. Addition of Compatible Tuples:**  $+$   $\setminus \text{addRinf}$ . Let  $v$  and  $w$  be  $n$ -tuples. Vector addition of  $v$  and  $w$  is the  $n$ -tuple  $v + w$  defined by component-wise addition.

$$\begin{array}{|l} \_ + \_ : \mathbb{R}^{\Delta} \longrightarrow \mathbb{R}^{\infty} \\ \hline \langle \rangle + \langle \rangle = \langle \rangle \\ \forall n : \mathbb{N}_1; v, w : \mathbb{R}^{\infty} \mid n = \#v = \#w \bullet \\ v + w = (\lambda i : 1..n \bullet v\ i + w\ i) \end{array}$$

**2.7. Subtraction of Compatible Tuples:**  $-$   $\setminus \text{subRinf}$ . Vector subtraction is defined similarly.

$$\begin{array}{|l} \_ - \_ : \mathbb{R}^{\Delta} \longrightarrow \mathbb{R}^{\infty} \\ \hline \langle \rangle - \langle \rangle = \langle \rangle \\ \forall n : \mathbb{N}_1; v, w : \mathbb{R}^{\infty} \mid n = \#v = \#w \bullet \\ v - w = (\lambda i : 1..n \bullet v\ i - w\ i) \end{array}$$

**2.8. The Negative of a Tuple:**  $-$   $\setminus \text{negRinf}$ . Let  $-v$  denote the negative of  $v$ .

$$\begin{array}{|l} - : \mathbb{R}^{\infty} \longrightarrow \mathbb{R}^{\infty} \\ \hline -\langle \rangle = \langle \rangle \\ \forall n : \mathbb{N}_1; v : \mathbb{R}^{\infty} \mid n = \#v \bullet \\ -v = (\lambda i : 1..n \bullet -(v\ i)) \end{array}$$

**2.9. Scalar Multiplication of a Tuple:**  $*$   $\setminus \text{smulRinf}$ . Let  $v$  be an  $n$ -tuple and let  $c$  be a real number. Scalar multiplication of  $v$  by  $c$  is the  $n$ -tuple  $c * v$  defined by component-wise multiplication.

$$\begin{array}{|l} \_ * \_ : \mathbb{R} \times \mathbb{R}^{\infty} \longrightarrow \mathbb{R}^{\infty} \\ \hline \forall c : \mathbb{R} \bullet \\ c * \langle \rangle = \langle \rangle \\ \forall c : \mathbb{R}; n : \mathbb{N}_1; v : \mathbb{R}^{\infty} \mid n = \#v \bullet \\ c * v = (\lambda i : 1..n \bullet c * (v\ i)) \end{array}$$

**Remark.** *Scalar multiplication is associative in the sense that  $(a * b) * v = a * (b * v)$*

$$\forall a, b : \mathbb{R}; v : \mathbb{R}^\infty \bullet \\ (a * b) * v = a * (b * v)$$

**2.10. The Set of All Real  $n$ -tuples:**  $\mathbb{R} \setminus \text{Rtup}$ . Let  $\mathbb{R}(n)$  denote  $\mathbb{R}^n$ , the set of all  $n$ -tuples for some given  $n$ .

$$\begin{array}{|l} \mathbb{R} : \mathbb{N} \longrightarrow \mathbb{P} \mathbb{R}^\infty \\ \hline \forall n : \mathbb{N} \bullet \\ \mathbb{R}(n) = \{ v : \mathbb{R}^\infty \mid \#v = n \} \end{array}$$

**Remark.**

$$\mathbb{R}^\infty = \bigcup \{ n : \mathbb{N} \bullet \mathbb{R}(n) \}$$

**Remark.** *The subset  $\mathbb{R}(n)$  is well-dimensioned.*

$$\forall n : \mathbb{N} \bullet \\ \mathbb{R}(n) \in \Delta_{\mathbb{R}}$$

**Remark.** *The dimension of  $\mathbb{R}(n)$  is  $n$ .*

$$\forall n : \mathbb{N} \bullet \\ \dim(\mathbb{R}(n)) = n$$

**2.11. Addition of  $n$ -tuples:**  $\text{addRtup}$ . Let  $\text{addRtup}(n)$  denote the restriction of addition to  $\mathbb{R}(n)$ .

$$\begin{array}{l} \text{addRtup} == \\ (\lambda n : \mathbb{N} \bullet \\ (\lambda v, w : \mathbb{R}(n) \bullet v + w)) \end{array}$$

**Example.** *The binary operation  $\text{addRtup}(n)$  defines an Abelian group over  $\mathbb{R}(n)$ .*

$$\forall n : \mathbb{N} \bullet \\ (\mathbb{R}(n), \text{addRtup}(n)) \in \text{abgroup}[\mathbb{R}(n)]$$

**2.12. Subtraction of  $n$ -tuples:**  $\text{subRtup}$ . Let  $\text{subRtup}(n)$  denote the restriction of subtraction to  $\mathbb{R}(n)$ .

$$\begin{array}{l} \text{subRtup} == \\ (\lambda n : \mathbb{N} \bullet \\ (\lambda v, w : \mathbb{R}(n) \bullet v - w)) \end{array}$$

**2.13. The Negative of an  $n$ -tuple:**  $\text{negRtup}$ . Let  $\text{negRtup}(n)$  denote the restriction of the negative operation to  $\mathbb{R}(n)$ .

$$\begin{array}{l} \text{negRtup} == \\ (\lambda n : \mathbb{N} \bullet \\ (\lambda v : \mathbb{R}(n) \bullet -v)) \end{array}$$

**Remark.** *The operation  $\text{negRtup}(n)$  is the inverse operation of the Abelian group  $(\mathbb{R}(n), \text{addRtup}(n))$ .*

$$\forall n : \mathbb{N} \bullet \\ \text{negRtup}(n) = \text{inverse\_operation}(\mathbb{R}(n), \text{addRtup}(n))$$

**2.14. The Zero Real  $n$ -tuple:  $\mathbf{0} \setminus \text{zeroRtup}$ .** Let  $\mathbf{0}(n)$  denote the  $n$ -tuple consisting of all zeroes.

$$\begin{array}{|l} \mathbf{0} : \mathbb{N} \rightarrow \mathbb{R}^\infty \\ \hline \mathbf{0}(0) = \langle \rangle \\ \forall n : \mathbb{N}_1 \bullet \\ \mathbf{0}(n) = (\lambda i : 1 \dots n \bullet 0) \end{array}$$

**Remark.** Every component of  $\mathbf{0}(n)$  is 0.

$$\begin{array}{l} \forall n : \mathbb{N} \bullet \\ \quad \forall i : 1 \dots n \bullet \\ \quad (\pi i)(\mathbf{0} n) = 0 \end{array}$$

**Remark.** The tuple  $\mathbf{0}(n)$  is in  $\mathbb{R}(n)$ .

$$\begin{array}{l} \forall n : \mathbb{N} \bullet \\ \mathbf{0}(n) \in \mathbb{R}(n) \end{array}$$

**Remark.** The tuple  $\mathbf{0}(n)$  is the identity element of the Abelian group  $(\mathbb{R}(n), \text{addRtup}(n))$ .

$$\begin{array}{l} \forall n : \mathbb{N} \bullet \\ \mathbf{0}(n) = \text{identity\_element}(\mathbb{R}(n), \text{addRtup}(n)) \end{array}$$

**2.15. Scalar Multiplication of an  $n$ -tuple:  $\text{smulRtup}$ .** Let  $\text{smulRtup}(n)$  denote scalar multiplication restricted to  $\mathbb{R}(n)$ .

$$\begin{array}{l} \text{smulRtup} == \\ (\lambda n : \mathbb{N} \bullet \\ (\lambda c : \mathbb{R}; v : \mathbb{R}(n) \bullet c * v)) \end{array}$$

**2.16. The Real Vector Space of  $n$ -tuples:  $\text{vecRtup}$ .** Let  $\text{vecRtup}(n)$  denote the real vector space of  $n$ -tuples.

$$\begin{array}{l} \text{vecRtup} == \\ (\lambda n : \mathbb{N} \bullet ((\mathbb{R}(n), \text{addRtup}(n)), \text{smulRtup}(n))) \end{array}$$

**Remark.** The pair  $\text{vecRtup}(n)$  defines a vector space over  $\mathbb{R}(n)$ .

$$\begin{array}{l} \forall n : \mathbb{N} \bullet \\ \text{vecRtup}(n) \in \text{vec}_{\mathbb{R}}(\mathbb{R}(n)) \end{array}$$

**2.17. Linear Transformations of  $n$ -tuples:  $\mathbb{L}_{\mathbb{R}} \setminus \text{linRtup}$ .** Define  $\mathbb{L}_{\mathbb{R}}(n, m)$  to be the set of all linear transformations from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ .

$$\begin{array}{|l} \mathbb{L}_{\mathbb{R}} : \mathbb{N} \times \mathbb{N} \rightarrow \mathcal{P}(\mathbb{R}^\infty \rightarrow \mathbb{R}^\infty) \\ \hline \forall n, m : \mathbb{N} \bullet \\ \mathbb{L}_{\mathbb{R}}(n, m) = \mathbb{L}_{\mathbb{R}}(\text{vecRtup}(n), \text{vecRtup}(m)) \end{array}$$

**2.18. The Identity Transformation of  $n$ -tuples:**  $\text{I} \setminus \text{idRtup}$ . Let  $\text{I}(n)$  denote the identity function on  $\mathbb{R}(n)$ .

$$\begin{array}{|l} \text{I} : \mathbb{N} \rightarrow \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty \\ \hline \forall n : \mathbb{N} \bullet \\ \quad \text{I}(n) = \text{id}(\mathbb{R}(n)) \end{array}$$

**Remark.** *The function  $\text{I}(n)$  is a linear transformation.*

$$\begin{array}{|l} \forall n : \mathbb{N} \bullet \\ \quad \text{I}(n) \in \text{L}_{\mathbb{R}}(n, n) \end{array}$$

### 3. THE METRIC TOPOLOGY ON REAL $n$ -TUPLES

**3.1. The Dot Product of Tuples:**  $\cdot \setminus \text{dotRinf}$ . The *inner* or *dot* product of  $n$ -tuples  $v$  and  $w$  is the real number  $v \cdot w$  defined by the sum of the component-wise products.

$$\begin{array}{|l} \cdot : \mathbb{R}^\Delta \rightarrow \mathbb{R} \\ \hline \langle \rangle \cdot \langle \rangle = 0 \\ \forall x, y : \mathbb{R}; v, w : \mathbb{R}^\infty \mid \#v = \#w \bullet \\ \quad (\langle x \rangle \frown v) \cdot (\langle y \rangle \frown w) = x * y + v \cdot w \end{array}$$

Each  $\mathbb{R}(n)$  is a real inner product space under the operation of dot product defined above.

**3.2. The Norm of a Tuple:**  $\text{norm} \setminus \text{normRinf}$ . The norm  $\|v\|$  of the  $n$ -tuple  $v$  is the positive square root of its dot product with itself.

$$\|v\| = \sqrt{v \cdot v}$$

Define  $\text{norm}(v)$  to be  $\|v\|$ .

$$\begin{array}{|l} \text{norm} : \mathbb{R}^\infty \rightarrow \mathbb{R} \\ \hline \forall v : \mathbb{R}^\infty \bullet \\ \quad \text{norm}(v) = \text{sqrt}(v \cdot v) \end{array}$$

The concepts of continuity, limits, and differentiability extend to functions between normed vector spaces such as  $\mathbb{R}^n$ .

**3.3. The Open Ball at a Tuple:**  $\text{ball} \setminus \text{ballRinf}$ . Let  $\text{ball}(v, r)$  denote the *open ball* in  $\mathbb{R}(n)$  of radius  $r \in \mathbb{R}_+$  centred at  $v \in \mathbb{R}(n)$ .

$$\begin{array}{|l} \text{ball} : \mathbb{R}^\infty \times \mathbb{R}_+ \rightarrow \mathcal{P} \mathbb{R}^\infty \\ \hline \forall v : \mathbb{R}^\infty; r : \mathbb{R}_+ \bullet \\ \quad \text{let } n == \#v \bullet \\ \quad \quad \text{ball}(v, r) = \{ w : \mathbb{R}(n) \mid \text{norm}(v - w) < r \} \end{array}$$

**3.4. The Set of All Open Balls at an  $n$ -tuple:**  $\text{balls} \setminus \text{ballsRtup}$ . Let  $\text{balls}(n)$  denote the family of all open balls in  $\mathbb{R}(n)$ .

$$\begin{array}{|l} \text{balls} : \mathbb{N} \longrightarrow \mathcal{F}\mathbb{R}^\infty \\ \hline \forall n : \mathbb{N} \bullet \\ \text{balls}(n) = \{ v : \mathbb{R}(n); r : \mathbb{R}_+ \bullet \text{ball}(v, r) \} \end{array}$$

**Remark.** *The set of all open balls in  $\mathbb{R}(n)$  is a family of sets in  $\mathbb{R}(n)$ .*

$$\begin{array}{|l} \forall n : \mathbb{N} \bullet \\ \text{balls}(n) \in \mathcal{F}(\mathbb{R}(n)) \end{array}$$

**3.5. The Usual Topology on  $n$ -tuples:**  $\tau_{\mathbb{R}} \setminus \text{tauRtup}$ . The *usual topology* on  $\mathbb{R}(n)$  is the topology generated by the open balls in  $\mathbb{R}(n)$ . Let  $\tau_{\mathbb{R}}(n)$  denote the usual topology on  $\mathbb{R}(n)$ .

$$\begin{array}{|l} \tau_{\mathbb{R}} : \mathbb{N} \longrightarrow \mathcal{F}\mathbb{R}^\infty \\ \hline \forall n : \mathbb{N} \bullet \\ \tau_{\mathbb{R}}(n) = \text{topGen}[\mathbb{R}(n)](\text{balls}(n)) \end{array}$$

**Remark.** *If  $n \in \mathbb{N}$  then  $\tau_{\mathbb{R}}(n)$  is a topology on  $\mathbb{R}(n)$ .*

$$\forall n : \mathbb{N} \bullet \tau_{\mathbb{R}}(n) \in \text{top}[\mathbb{R}(n)]$$

**3.6. The Set of All Neighbourhoods of a Tuple:**  $\text{neigh} \setminus \text{neighRinf}$ . Let  $v \in \mathbb{R}(n)$ . An open set  $U$  in the usual topology  $\tau_{\mathbb{R}}(n)$  that contains  $v$  is called a *neighbourhood* of  $v$ . Let  $\text{neigh}(v)$  denote the set of all neighbourhoods of  $v$ .

$$\begin{array}{|l} \text{neigh} : \mathbb{R}^\infty \longrightarrow \mathcal{F}\mathbb{R}^\infty \\ \hline \forall n : \mathbb{N}; v : \mathbb{R}^\infty \mid n = \#v \bullet \\ \text{neigh}(v) = \{ U : \tau_{\mathbb{R}}(n) \mid v \in U \} \end{array}$$

**Remark.** *The set of all neighbourhoods of  $v \in \mathbb{R}(n)$  is a family of sets in  $\mathbb{R}(n)$ .*

$$\begin{array}{|l} \forall n : \mathbb{N}; v : \mathbb{R}^\infty \mid n = \#v \bullet \\ \text{neigh}(v) \in \mathcal{F}(\mathbb{R}(n)) \end{array}$$

**3.7. The Topological Space of  $n$ -tuples:**  $\mathbb{R}_\tau \setminus \text{tsRtup}$ . Let  $\mathbb{R}_\tau(n)$  denote the topological space defined by the usual topology on  $\mathbb{R}(n)$ .

$$\begin{array}{|l} \mathbb{R}_\tau : \mathbb{N} \longrightarrow \text{topSpaces}[\mathbb{R}^\infty] \\ \hline \forall n : \mathbb{N} \bullet \\ \mathbb{R}_\tau(n) = (\mathbb{R}(n), \tau_{\mathbb{R}}(n)) \end{array}$$

## 4. CONTINUITY

**4.1. Real-Valued Functions That Are Continuous on the Set of All  $n$ -tuples:**  $\text{C}^0 \setminus \text{CzeroRtup}$ . A function  $f \in \mathbb{R}^n \longrightarrow \mathbb{R}$  is said to be *continuous* if it is continuous with respect to the usual topologies on  $\mathbb{R}^n$  and  $\mathbb{R}$ . Let  $\text{C}^0(n)$  denote the set of these continuous functions.



$$\begin{array}{|l} C^0 : \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \multimap \mathbb{R}) \\ \hline \forall n : \mathbb{N} \bullet \\ \quad C^0(n) = C^0(\mathbb{R}_\tau(n), \mathbb{R}_\tau) \end{array}$$

**4.2. Real-Valued Functions That Are Continuous on a Subset of  $n$ -tuples:**  $C^0 \setminus \text{CzeroSubsetRtup}$ . Let  $U$  be a subset of  $\mathbb{R}^n$ . A function  $f \in U \longrightarrow \mathbb{R}$  is said to be *continuous on  $U$*  if it is continuous with respect to the topology induced on  $U$ . Let  $C^0(U)$  denote the set of these continuous functions.

$$\begin{array}{|l} C^0 : \Delta_{\mathbb{R}} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \multimap \mathbb{R}) \\ \hline \forall U : \Delta_{\mathbb{R}} \bullet \\ \quad \text{let } n == \dim U \bullet \\ \quad \quad C^0(U) = C^0(\mathbb{R}_\tau(n) \mid_{\text{top}} U, \mathbb{R}_\tau) \end{array}$$

**4.3. Real-Valued Functions That Are Continuous at an  $n$ -tuple:**  $C^0 \setminus \text{CzeroPointRtup}$ . A partial function  $f$  from  $\mathbb{R}^n$  to  $\mathbb{R}$  is said to be *continuous at  $x \in \mathbb{R}^n$*  if its domain contains a neighbourhood  $U$  of  $x$  such that its restriction to  $U$  is continuous on  $U$ . Let  $C^0(x)$  denote the set of such functions.

$$\begin{array}{|l} C^0 : \mathbb{R}^\infty \longrightarrow \mathbb{P}(\mathbb{R}^\infty \multimap \mathbb{R}) \\ \hline \forall x : \mathbb{R}^\infty \bullet \\ \quad \text{let } n == \#x \bullet \\ \quad \quad C^0(x) = \{ f : \mathbb{R}(n) \multimap \mathbb{R} \mid \exists U : \text{neigh}(x) \mid U \subseteq \text{dom } f \bullet U \triangleleft f \in C^0(U) \} \end{array}$$

**4.4.  $m$ -tuple-Valued Functions That Are Continuous on the Set of All  $n$ -tuples:**  $C^0 \setminus \text{CzeroRtupRtup}$ . A mapping  $f$  from  $\mathbb{R}(n)$  to  $\mathbb{R}(m)$  is said to be *continuous* if it is continuous with respect to the usual topologies. Let  $C^0(n, m)$  denote the set of these continuous mappings.

$$\begin{array}{|l} C^0 : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \multimap \mathbb{R}^\infty) \\ \hline \forall n, m : \mathbb{N} \bullet \\ \quad C^0(n, m) = C^0(\mathbb{R}_\tau(n), \mathbb{R}_\tau(m)) \end{array}$$

**Example.** The function  $I(n)$  is continuous.

$$\begin{array}{|l} \forall n : \mathbb{N} \bullet \\ \quad I(n) \in C^0(n, n) \end{array}$$

**Theorem 1.** Linear functions are continuous.

$$\begin{array}{|l} \forall n, m : \mathbb{N} \bullet \\ \quad L_{\mathbb{R}}(n, m) \subseteq C^0(n, m) \end{array}$$

**4.5.  $m$ -tuple-Valued Functions That Are Continuous on a Subset of  $n$ -tuples:**  $C^0 \setminus \text{CzeroSubsetRtupRtup}$ . Let  $U$  be any subset of  $\mathbb{R}(n)$ . Let  $C^0(U, m)$  denote the set of continuous mappings from the topology induced by  $\mathbb{R}_\tau(n)$  on  $U$  to  $\mathbb{R}_\tau(m)$ .

$$\begin{array}{|l}
C^0 : \Delta_{\mathbb{R}} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^{\infty} \rightharpoonup \mathbb{R}^{\infty}) \\
\hline
\forall n, m : \mathbb{N} \bullet \\
\quad \forall U : \Delta_{\mathbb{R}} \mid \dim(U) = n \bullet \\
\quad \quad C^0(U, m) = C^0(\mathbb{R}_{\tau}(n) \mid_{\text{top}} U, \mathbb{R}_{\tau}(m))
\end{array}$$

**Remark.**

$$\begin{array}{|l}
\forall n, m : \mathbb{N} \bullet \\
\quad C^0(\mathbb{R}(n), m) = C^0(n, m)
\end{array}$$

**4.6.  $m$ -tuple-Valued Functions That Are Continuous at an  $n$ -tuple:** *VectorContinuous*,  $C^0 \setminus \text{CzeroPointRtupRtup}$ . Let  $x \in \mathbb{R}(n)$  and let  $f$  be a partial function from  $\mathbb{R}(n)$  to  $\mathbb{R}(m)$  whose domain includes some neighbourhood  $U$  of  $x$  such that  $f$  restricted to  $U$  is continuous. In this case  $f$  is said to be *continuous at  $x$* .

$$\begin{array}{|l}
\text{VectorContinuous} \\
\hline
n, m : \mathbb{N} \\
f : \mathbb{R}^{\infty} \rightharpoonup \mathbb{R}^{\infty} \\
x : \mathbb{R}^{\infty} \\
\hline
f \in \mathbb{R}(n) \rightharpoonup \mathbb{R}(m) \\
\exists U : \text{neigh}(x) \mid \\
\quad U \subseteq \text{dom } f \bullet \\
\quad \quad U \triangleleft f \in C^0(U, m)
\end{array}$$

Let  $C^0(x, m)$  denote the set of all partial functions  $f$  from  $\mathbb{R}(n)$  to  $\mathbb{R}(m)$  that are continuous at  $x$ .

$$\begin{array}{|l}
C^0 : \mathbb{R}^{\infty} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^{\infty} \rightharpoonup \mathbb{R}^{\infty}) \\
\hline
\forall n, m : \mathbb{N} \bullet \forall x : \mathbb{R}(n) \bullet \\
\quad C^0(x, m) = \\
\quad \quad \{ f : \mathbb{R}(n) \rightharpoonup \mathbb{R}(m) \mid \text{VectorContinuous} \}
\end{array}$$

**Example.** The function  $I(n)$  is continuous at every point  $x \in \mathbb{R}(n)$ .

$$\begin{array}{|l}
\forall n : \mathbb{N} \bullet \forall x : \mathbb{R}(n) \bullet \\
\quad I(n) \in C^0(x, n)
\end{array}$$

**Theorem 2.** Linear functions are continuous everywhere.

$$\begin{array}{|l}
\forall n, m : \mathbb{N} \bullet \\
\quad \forall x : \mathbb{R}(n); L : L_{\mathbb{R}}(n, m) \bullet \\
\quad \quad L \in C^0(x, m)
\end{array}$$

## 5. DIFFERENTIABILITY

Let  $x \in \mathbb{R}^n$  and let  $f : \mathbb{R}^n \rightharpoonup \mathbb{R}^m$  be continuous at  $x$ . Then  $f$  is said to be *differentiable at  $x$*  if there exists a linear transformation  $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$  such that  $f(x + h) - f(x)$  is approximately linear in  $h$  for very small  $h$ .

$$f(x + h) - f(x) \approx L(h) + O(h^2) \quad \text{when} \quad \|h\| \approx 0$$

This condition can be written as a limit.

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - L(h)\|}{\|h\|} = 0$$

**5.1. The Difference Quotient:** *DifferenceQuotient* and *diffQuot*. The limit exists when the following difference quotient function  $q : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous at 0.

$$q(h) = \begin{cases} \frac{\|f(x+h) - f(x) - L(h)\|}{\|h\|} & \text{if } h \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Given a function  $f$  that is continuous at  $x$ , and a linear transformation  $L$ , we can define the difference quotient  $q$ . Clearly  $q$  is uniquely determined by  $f$ ,  $x$ , and  $L$ . Let *DifferenceQuotient* denote this situation.

<i>DifferenceQuotient</i>
<i>VectorContinuous</i>
$L : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$
$q : \mathbb{R}^\infty \rightarrow \mathbb{R}$
$L \in \mathbb{L}_{\mathbb{R}}(n, m)$
$\text{dom } q = \{ h : \mathbb{R}(n) \mid x + h \in \text{dom } f \}$
$\forall h : \text{dom } q \mid h \neq \mathbf{0}(n) \bullet$
$q(h) = \text{norm}(f(x+h) - f(x) - L(h)) / \text{norm}(h)$
$q(\mathbf{0}(n)) = 0$

- $L$  is a linear transformation from  $\mathbb{R}(n)$  to  $\mathbb{R}(m)$ .
- The difference quotient  $q$  is defined on a subset of  $\mathbb{R}(n)$  that contains  $\mathbf{0}(n)$ .
- $q(h)$  is defined as the quotient when  $h$  is non-zero.
- $q(0)$  is defined as zero.

Let  $\text{diffQuot}(f, x, L)$  denote the difference quotient  $q$ .

$$\text{diffQuot} == \{ \text{DifferenceQuotient} \bullet (f, x, L) \mapsto q \}$$

**5.2. The Derivative of a Continuous  $m$ -tuple-Valued Function:** *VectorDifferentiable*.

The continuous function  $f$  is *differentiable at  $x$*  when there exists a linear transformation  $L$  such that the difference quotient  $q$  is continuous at 0. In this case  $L$  is unique and is referred to as the *derivative at  $x$* .

<i>VectorDifferentiable</i>
<i>VectorContinuous</i>
$L : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$
<b>let</b> $q == \text{diffQuot}(f, x, L) \bullet$
$q \in C^0(\mathbf{0}(n))$

- The continuous function  $f$  is differentiable at  $x$  with derivative  $L$  if the resulting difference quotient  $q$  is continuous at  $\mathbf{0}(n)$ .

**Remark.** *If  $L$  exists then it is unique.*

Let  $C^\infty(x, m)$  denote the set of all functions  $f \in \mathbb{R}(n) \rightarrow \mathbb{R}(m)$  that are smooth at  $x \in \mathbb{R}(n)$ .

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