Groups

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Abstract

This article contains Z Notation type declarations for groups and some related objects. It has been type checked by fUZZ.

1 Introduction

Groups are ubiquitous throughout mathematics and physics.

2 Groups

A *group* is a set G of elements $x, y, \ldots$ endowed with an associative multiplication operation $x * y$ that has an identity element $\mathbf{1}$ such that all elements have inverses $x^{-1}, y^{-1}, \ldots$.

2.1 *Group*

Let $Group[G]$ denote the set of all group structures defined on some given set G of elements.

$$
\begin{array}{|l}
\hline
Group[G] \\
\_ * \_ : G \times G \longrightarrow G \\
\mathbf{1} : G \\
\_^{-1} : G \longrightarrow G \\
\hline
\forall x, y, z : G \bullet \\
\quad (x * y) * z = x * (y * z) \\
\forall x : G \bullet \\
\quad x * \mathbf{1} = x = \mathbf{1} * x \\
\forall x : G \bullet \\
\quad x * x^{-1} = \mathbf{1} = x^{-1} * x \\
\hline
\end{array}
$$

- The group multiplication operation is associative.

- The element $\mathbf{1}$ is both a left and right identity element under group multiplication.
- Every group element has an inverse element that is both a left and right inverse under group multiplication.

Remark. *The group multiplication operation uniquely determines the set of elements.*

$$\begin{array}{l} \forall\ \mathit{Group}[\mathsf{X}] \bullet \\ \quad \mathsf{X} = \mathrm{ran}(\_ * \_) \end{array}$$

Remark. *The group multiplication operation uniquely determines the identity operation.*

$$\begin{array}{l} \forall\ \mathit{Group}[\mathsf{X}] \bullet \\ \quad \forall\ x, y : \mathsf{X} \bullet \\ \quad\quad x * y = x \Rightarrow y = \mathbf{1} \end{array}$$

Remark. *The group multiplication uniquely determines the inverses.*

$$\begin{array}{l} \forall\ \mathit{Group}[\mathsf{X}] \bullet \\ \quad \forall\ x, y : \mathsf{X} \bullet \\ \quad\quad x * y = \mathbf{1} \Rightarrow x = y^{-1} \land y = x^{-1} \end{array}$$

2.2 *group*

Let $\mathit{group}[G]$ denote the set of all group multiplication operations on G.

$$\begin{array}{|l} [G] \\ \hline \mathit{group} : \mathbb{P}(G \times G \longrightarrow G) \\ \hline \mathit{group} = \{\ \mathit{Group}[G] \bullet (\_ * \_)\ \} \\ \hline \end{array}$$

2.3 *bij*

Let X be a set and let $\mathit{bij}[X]$ denote the set of a bijections $X \rightarrowtail\!\!\!\!\twoheadrightarrow X$ from X to itself.

$$\begin{array}{|l} [X] \\ \hline \mathit{bij} : \mathbb{P}(X \longrightarrow X) \\ \hline \mathit{bij} = X \rightarrowtail\!\!\!\!\twoheadrightarrow X \\ \hline \end{array}$$

Remark. *The composition of bijections is a bijection.*

$$\begin{array}{l} \forall\ f, g : \mathit{bij}[\mathsf{X}] \bullet \\ \quad f \circ g \in \mathit{bij}[\mathsf{X}] \end{array}$$

Remark. *Composition is associative.*

$$\forall f, g, h : bij[\mathsf{X}] \bullet \\ \quad f \circ (g \circ h) = (f \circ g) \circ h$$

Remark. *The identity function* $\mathrm{id}\, X$ *acts as a left and right identity element under composition.*

$$\forall f : bij[\mathsf{X}] \bullet \\ \quad \mathrm{id}\, \mathsf{X} \circ f = f = f \circ \mathrm{id}\, \mathsf{X}$$

Remark. *The inverse* $f^{\sim}$ *of a bijection* f *is its left and right inverse under composition.*

$$\forall f : bij[\mathsf{X}] \bullet \\ \quad f \circ f^{\sim} = \mathrm{id}\, \mathsf{X} = f^{\sim} \circ f$$

2.4 *Bij*

The preceding remarks show that set $bij[X]$ under the operation of composition has the structure of a group. Let $Bij[X]$ denote this group.

$$[X]$$
$$Bij : bij[X] \times bij[X] \longrightarrow bij[X]$$
$$Bij = (\lambda f, g : bij[X] \bullet f \circ g)$$

Theorem 1. *The multiplication operation* $Bij[X]$ *is a group.*

$$Bij[\mathsf{X}] \in group[bij[\mathsf{X}]]$$

3 Abelian Groups

An *abelian group* is a group in which the multiplication is commutative, i.e. $x * y = y * x$. In this case, the group multiplication is denoted as addition $x + y$, the identity element is denoted as a zero $\mathbf{0}$, and the inverse of an element is denoted as its negative $-x$.

3.1 *AbelianGroup*

Let $AbelianGroup[G]$ denote the abelian group structures on G.

$$
\begin{array}{|l}
\hline
\multicolumn{1}{l}{\textit{AbelianGroup}[G]} \\
\textit{Group}[G] \\
\_ + \_ : G \times G \longrightarrow G \\
\mathbf{0} : G \\
- : G \longrightarrow G \\
\hline
(\_ + \_) = (\_ * \_) \\
\mathbf{0} = \mathbf{1} \\
- = (\_^{-1}) \\
\forall\, x, y : G \bullet \\
\qquad x + y = y + x \\
\hline
\end{array}
$$

- The group multiplication is denoted as addition.
- The group identify element is denoted as the zero element.
- The inverse of a group element is denoted as the negative of the element.
- The group addition is commutative.

3.2 *abelianGroup*

Let $\mathit{abelianGroup}[G]$ denote the subset of abelian groups in $\mathit{group}[G]$.

$$
\begin{array}{|l}
\hline\hline
\multicolumn{1}{l}{[G]} \\
\mathit{abelianGroup} : \mathbb{P}(G \times G \longrightarrow G) \\
\hline
\mathit{abelianGroup} = \{\, \mathit{AbelianGroup}[G] \bullet (\_ + \_) \,\} \\
\hline
\end{array}
$$

Remark. *The set* $\mathit{abelianGroup}[G]$ *is a subset of* $\mathit{group}[G]$*.*

$$\mathit{abelianGroup}[\mathsf{X}] \subseteq \mathit{group}[\mathsf{X}]$$

Example. *The integer addition is an abelian group.*

$$(\_ + \_) \in \mathit{abelianGroup}[\mathbb{Z}]$$