VECTOR SPACES

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ABSTRACT. This article contains Z Notation type declarations for vector spaces and some related objects. It has been type checked by fUZZ.

CONTENTS

1. REAL VECTOR SPACES

Real vector spaces are multidimensional generalizations of real numbers. They are the objects studied in linear algebra and are foundational to differential geometry.

In the following let `t` denote a set of elements which we'll refer to as *vectors* and let A denote an Abelian group over the vectors in which the binary operation is denoted as addition. Let v and w denote vectors and and let x and y denote real numbers.

1.1. **Notation for Vector Addition, Zero, and Negative: $+$ `\addV`, $\mathbf{0}$ `\zeroV`, and - `\negV`.** Let $v + w$ denote vector addition, let $\mathbf{0}$ denote the zero vector, and let - v denote the negative vector.

1.2. **Real Scalar Multiplication: $*$ `\mulS`, $\times$ `\timesS`, and *RealScalarMultiplication*.** A *real scalar multiplication* operation on the vectors is an operation *smul* that maps the pair (x, v) to another vector, typically denoted $x * v$ or $x \times y$, such that multiplication by 0 maps all vectors to the group identity element, multiplication by 1 maps each vector to itself, multiplication preserves group addition, and multiplication distributes over both real and group addition.

Let *RealScalarMultiplication* denote this situation.

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$$
\begin{array}{|l}
\hline
\textit{RealScalarMultiplication}[\mathsf{t}] \\
A : \text{abgroup}\,\mathsf{t} \\
\mathit{smul} : \mathbb{R} \times \mathsf{t} \longrightarrow \mathsf{t} \\
\hline
\textbf{let } (\_ + \_) == \mathit{second}\ A; \\
\quad\quad \mathbf{0} == \mathit{identity\_element}\ A; \\
\quad\quad (\_ * \_) == \mathit{smul} \bullet \\
\quad\quad\quad \forall\, x, y : \mathbb{R};\ v, w : \mathsf{t} \bullet \\
\quad\quad\quad\quad 0 * v = \mathbf{0} \land \\
\quad\quad\quad\quad 1 * v = v \land \\
\quad\quad\quad\quad (x * y) * v = x * (y * v) \land \\
\quad\quad\quad\quad (x + y) * v = x * v + y * v \land \\
\quad\quad\quad\quad x * (v + w) = x * v + x * w \\
\hline
\end{array}
$$

- Multiplying by 0 gives the zero vector.
- Multiplying by 1 gives the same vector.
- Scalar multiplication is associative.
- Scalar addition distributes over scalar multiplication.
- Vector addition distributes over scalar multiplication.

1.3. **The Set of All Real Vector Spaces:** $\mathrm{vec}_{\mathbb{R}}$ **`\vecR`.** A *real vector space* is a pair (A, smul) where A is an Abelian group and *smul* is a real scalar multiplication on the elements of A. The elements of A are referred to as *vectors*.

Let $\mathrm{vec}_{\mathbb{R}}\,\mathsf{t}$ denote the set of all real vector spaces over t.

TODO: Do not assume that the set of vectors coincides with t. In general, it will be a subset of t.

$\mathrm{vec}_{\mathbb{R}}\,\mathsf{t} == \{\, \mathit{RealScalarMultiplication}[\mathsf{t}] \bullet (A, \mathit{smul}) \,\}$

1.4. **Real Linear Transformations:** ***RealLinearTransformation*.** Let V_1 and V_2 be real vector spaces and let f be a homomorphism of the underlying Abelian groups. The map f is said to be a *linear transformation* if f maps scalar multiples of vectors to the scalar multiple of the mapped vectors.

Let *RealLinearTransformation* denote this situation.

$$
\begin{array}{|l}
\hline
\textit{RealLinearTransformation}[\mathsf{t}, \mathsf{u}] \\
f : \mathsf{t} \longrightarrow \mathsf{u} \\
V_1 : \mathrm{vec}_{\mathbb{R}}\,\mathsf{t} \\
V_2 : \mathrm{vec}_{\mathbb{R}}\,\mathsf{u} \\
\hline
\textbf{let } A_1 == \mathit{first}\ V_1;\ (\_ * \_) == \mathit{second}\ V_1; \\
\quad\quad A_2 == \mathit{first}\ V_2;\ (\_ \times \_) == \mathit{second}\ V_2 \bullet \\
\quad\quad\quad f \in \mathrm{hom}_{\mathrm{grp}}(A_1, A_2) \land \\
\quad\quad\quad (\forall\, x : \mathbb{R};\ v : \mathsf{t} \bullet \\
\quad\quad\quad\quad f(x * v) = x \times (f\ v)) \\
\hline
\end{array}
$$

- The vector space V_1 has Abelian group A_1 and scalar multiplication $(\_ * \_)$.

- The vector space V_2 has Abelian group A_2 and scalar multiplication $(\_ \times \_)$.
- The map f is a homomorphism of the underlying Abelian groups.
- The map f maps scalar multiples of vectors in t to scalar multiples of the mapped vectors in u.

1.5. **The Set of All Real Linear Transformations:** $\mathrm{L}_{\mathbb{R}}$ `\homVecR`. Let V_1 and V_2 be real vector spaces. Let $\mathrm{L}_{\mathbb{R}}(V_1, V_2)$ denote the set of all linear transformations from V_1 to V_2. A linear transformation is also referred to as a *homomorphism* of vector spaces.

$$
\begin{array}{l}
\mathrm{L}_{\mathbb{R}}[\mathsf{t}, \mathsf{u}] == \\
\quad (\lambda\, V_1 : \mathrm{vec}_{\mathbb{R}}\, \mathsf{t};\ V_2 : \mathrm{vec}_{\mathbb{R}}\, \mathsf{u} \bullet \\
\qquad \{\, f : \mathsf{t} \longrightarrow \mathsf{u} \mid \\
\qquad\quad \mathit{RealLinearTransformation}[\mathsf{t}, \mathsf{u}] \,\})
\end{array}
$$

2. Real n-tuples

The preceding section described real vector spaces abstractly. In this section we define a family of finite-dimensional real vector spaces whose elements are finite sequences of real numbers, also referred to as *real tuples*.

2.1. **The Set of All Finite Sequences of Real Numbers:** $\mathbb{R}^{\infty}$ `\Rinf`. Let n be a natural number. A finite sequence of n real numbers is called a *real n-tuple*. Let $\mathbb{R}^{\infty}$ denote the set of all real n-tuples for any n.

$$\mathbb{R}^{\infty} == \mathrm{seq}\, \mathbb{R}$$

2.2. **The Component Projection Function:** π `\piRinf`. The real numbers that comprise an n-tuple are called its *components*. Let v be a real n-tuple and let i be an integer where $1 \leq i \leq n$. The real number $v(i)$ is the i-th component of v. Let $\pi(i)$ be the projection function that maps an n-tuple v to its i-th component $v(i)$.

$$
\begin{array}{|l}
\pi : \mathbb{N}_1 \longrightarrow \mathbb{R}^{\infty} \mathrel{+\!\!\!\!\!\rightarrow} \mathbb{R} \\
\hline
\forall\, i : \mathbb{N}_1 \bullet \\
\quad \pi(i) = (\lambda\, v : \mathbb{R}^{\infty} \mid i \in \mathrm{dom}\, v \bullet v(i))
\end{array}
$$

2.3. **The Set of All Well-Dimensioned Subsets of $\mathbb{R}^{\infty}$:** $\Delta_{\mathbb{R}}$ `\DeltaRinf`. A non-empty subset of $\mathbb{R}^{\infty}$ is said to be *well-dimensioned* if each of its elements has the same number of components. Let $\Delta_{\mathbb{R}}$ denote the family of all well-dimensioned subsets of $\mathbb{R}^{\infty}$.

$$
\begin{array}{|l}
\Delta_{\mathbb{R}} : \mathcal{F}\, \mathbb{R}^{\infty} \\
\hline
\Delta_{\mathbb{R}} = \{\, S : \mathbb{P}_1\, \mathbb{R}^{\infty} \mid \forall\, v, w : S \bullet \#v = \#w \,\}
\end{array}
$$

2.4. **The Dimension of a Well-Dimensioned Set of Tuples:** dim `\dimRinf`. Let $S \in \Delta_{\mathbb{R}}$ be a well-dimensioned set of tuples. The number of components of each tuple in S is called its *dimension*. Let $\dim(S)$ denote the dimension of S.

$$
\begin{array}{|l}
\dim : \Delta_{\mathbb{R}} \longrightarrow \mathbb{N} \\
\hline
\forall S : \Delta_{\mathbb{R}} \bullet \\
\quad \dim S = (\mu\, v : S \bullet \# v)
\end{array}
$$

2.5. **The Set of All Compatible Pairs of Tuples: $\mathbb{R}^{\Delta}$ `\RinfDelta`.** The pair of real tuples (v, w) is said to be *compatible* if each member has the same number of components. Let $\mathbb{R}^{\Delta}$ denote the set of all compatible pairs of real tuples. If the pair (v, w) is compatible then v and w are said to be compatible with each other.

$$
\begin{array}{|l}
\mathbb{R}^{\Delta} : \mathbb{R}^{\infty} \leftrightarrow \mathbb{R}^{\infty} \\
\hline
\mathbb{R}^{\Delta} = \{\, v, w : \mathbb{R}^{\infty} \mid \# v = \# w \,\}
\end{array}
$$

2.6. **Addition of Compatible Tuples: $+$ `\addRinf`.** Let v and w be n-tuples. Vector addition of v and w is the n-tuple $v+w$ defined by component-wise addition.

$$
\begin{array}{|l}
\_ + \_ : \mathbb{R}^{\Delta} \longrightarrow \mathbb{R}^{\infty} \\
\hline
\langle\rangle + \langle\rangle = \langle\rangle \\
\\
\forall n : \mathbb{N}_1;\ v, w : \mathbb{R}^{\infty} \mid n = \# v = \# w \bullet \\
\quad v + w = (\lambda\, i : 1 \,..\, n \bullet v\, i + w\, i)
\end{array}
$$

2.7. **Subtraction of Compatible Tuples: $-$ `\subRinf`.** Vector subtraction is defined similarly.

$$
\begin{array}{|l}
\_ - \_ : \mathbb{R}^{\Delta} \longrightarrow \mathbb{R}^{\infty} \\
\hline
\langle\rangle - \langle\rangle = \langle\rangle \\
\\
\forall n : \mathbb{N}_1;\ v, w : \mathbb{R}^{\infty} \mid n = \# v = \# w \bullet \\
\quad v - w = (\lambda\, i : 1 \,..\, n \bullet v\, i - w\, i)
\end{array}
$$

2.8. **The Negative of a Tuple: - `\negRinf`.** Let - v denote the negative of v.

$$
\begin{array}{|l}
\text{-} : \mathbb{R}^{\infty} \longrightarrow \mathbb{R}^{\infty} \\
\hline
\text{-}\langle\rangle = \langle\rangle \\
\\
\forall n : \mathbb{N}_1;\ v : \mathbb{R}^{\infty} \mid n = \# v \bullet \\
\quad \text{-}\, v = (\lambda\, i : 1 \,..\, n \bullet \text{-}(v\, i))
\end{array}
$$

2.9. **Scalar Multiplication of a Tuple: $*$ `\smulRinf`.** Let v be an n-tuple and let c be a real number. Scalar multiplication of v by c is the n-tuple $c * v$ defined by component-wise multiplication.

$$
\begin{array}{|l}
\_ * \_ : \mathbb{R} \times \mathbb{R}^{\infty} \longrightarrow \mathbb{R}^{\infty} \\
\hline
\forall c : \mathbb{R} \bullet \\
\quad c * \langle\rangle = \langle\rangle \\
\\
\forall c : \mathbb{R};\ n : \mathbb{N}_1;\ v : \mathbb{R}^{\infty} \mid n = \# v \bullet \\
\quad c * v = (\lambda\, i : 1 \,..\, n \bullet c * (v\, i))
\end{array}
$$

Remark. *Scalar multiplication is associative in the sense that* $(a*b)*v = a*(b*v)$

$$\forall\, a, b : \mathbb{R};\ v : \mathbb{R}^\infty \bullet$$
$$\quad (a * b) * v = a * (b * v)$$

2.10. **The Set of All Real n-tuples:** $\mathbb{R}$ **\Rtup.** Let $\mathbb{R}(n)$ denote $\mathbb{R}^n$, the set of all n-tuples for some given n.

$$\mathbb{R} : \mathbb{N} \longrightarrow \mathbb{P}\,\mathbb{R}^\infty$$
$$\forall\, n : \mathbb{N} \bullet$$
$$\quad \mathbb{R}(n) = \{\, v : \mathbb{R}^\infty \mid \# v = n \,\}$$

Remark.

$$\mathbb{R}^\infty = \bigcup \{\, n : \mathbb{N} \bullet \mathbb{R}(n) \,\}$$

Remark. *The subset $\mathbb{R}(n)$ is well-dimensioned.*

$$\forall\, n : \mathbb{N} \bullet$$
$$\quad \mathbb{R}(n) \in \Delta_{\mathbb{R}}$$

Remark. *The dimension of $\mathbb{R}(n)$ is n.*

$$\forall\, n : \mathbb{N} \bullet$$
$$\quad \dim(\mathbb{R}(n)) = n$$

2.11. **Addition of n-tuples:** *addRtup***.** Let *addRtup*(n) denote the restriction of addition to $\mathbb{R}(n)$.

$$\mathit{addRtup} ==$$
$$\quad (\lambda\, n : \mathbb{N} \bullet$$
$$\qquad (\lambda\, v, w : \mathbb{R}(n) \bullet v + w))$$

Example. *The binary operation addRtup(n) defines an Abelian group over $\mathbb{R}(n)$.*

$$\forall\, n : \mathbb{N} \bullet$$
$$\quad (\mathbb{R}(n), \mathit{addRtup}(n)) \in \mathrm{abgroup}(\mathbb{R}(n))$$

2.12. **Subtraction of n-tuples:** *subRtup***.** Let *subRtup*(n) denote the restriction of subtraction to $\mathbb{R}(n)$.

$$\mathit{subRtup} ==$$
$$\quad (\lambda\, n : \mathbb{N} \bullet$$
$$\qquad (\lambda\, v, w : \mathbb{R}(n) \bullet v - w))$$

2.13. **The Negative of an n-tuple:** *negRtup***.** Let *negRtup*(n) denote the restriction of the negative operation to $\mathbb{R}(n)$.

$$\mathit{negRtup} ==$$
$$\quad (\lambda\, n : \mathbb{N} \bullet$$
$$\qquad (\lambda\, v : \mathbb{R}(n) \bullet \text{-}\, v))$$

Remark. *The operation negRtup(n) is the inverse operation of the Abelian group $(\mathbb{R}(n), \mathit{addRtup}(n))$.*

$$\forall\, n : \mathbb{N} \bullet$$
$$\quad \mathit{negRtup}(n) = \mathit{inverse\_operation}(\mathbb{R}(n), \mathit{addRtup}(n))$$

2.14. **The Zero Real n-tuple: $\mathbf{0}$ `\zeroRtup`.** Let $\mathbf{0}(n)$ denote the n-tuple consisting of all zeroes.

$$
\begin{array}{|l}
\mathbf{0} : \mathbb{N} \longrightarrow \mathbb{R}^{\infty} \\
\hline
\mathbf{0}(0) = \langle\rangle \\
\forall\, n : \mathbb{N}_1 \bullet \\
\quad \mathbf{0}(n) = (\lambda\, i : 1 \mathinner{..} n \bullet 0)
\end{array}
$$

Remark. *Every component of* $\mathbf{0}(n)$ *is* 0.

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\quad \forall\, i : 1 \mathinner{..} n \bullet \\
\quad\quad (\pi\, i)(\mathbf{0}\, n) = 0
\end{array}
$$

Remark. *The tuple* $\mathbf{0}(n)$ *is in* $\mathbb{R}(n)$.

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\quad \mathbf{0}(n) \in \mathbb{R}(n)
\end{array}
$$

Remark. *The tuple* $\mathbf{0}(n)$ *is the identity element of the Abelian group* $(\mathbb{R}(n), \mathit{addRtup}(n))$.

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\quad \mathbf{0}(n) = \mathit{identity\_element}(\mathbb{R}(n), \mathit{addRtup}(n))
\end{array}
$$

2.15. **Scalar Multiplication of an n-tuple: *smulRtup*.** Let $\mathit{smulRtup}(n)$ denote scalar multiplication restricted to $\mathbb{R}(n)$.

$$
\begin{array}{l}
\mathit{smulRtup} == \\
\quad (\lambda\, n : \mathbb{N} \bullet \\
\quad\quad (\lambda\, c : \mathbb{R};\ v : \mathbb{R}(n) \bullet c * v))
\end{array}
$$

2.16. **The Real Vector Space of n-tuples: *vecRtup*.** Let $\mathit{vecRtup}(n)$ denote the real vector space of n-tuples.

$$
\begin{array}{l}
\mathit{vecRtup} == \\
\quad (\lambda\, n : \mathbb{N} \bullet ((\mathbb{R}(n), \mathit{addRtup}(n)), \mathit{smulRtup}(n)))
\end{array}
$$

Remark. *The pair* $\mathit{vecRtup}(n)$ *defines a vector space over* $\mathbb{R}(n)$.

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\quad \mathit{vecRtup}(n) \in \mathrm{vec}_{\mathbb{R}}(\mathbb{R}(n))
\end{array}
$$

2.17. **Linear Transformations of n-tuples: $\mathrm{L}_{\mathbb{R}}$ `\linRtup`.** Define $\mathrm{L}_{\mathbb{R}}(n, m)$ to be the set of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$
\begin{array}{|l}
\mathrm{L}_{\mathbb{R}} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^{\infty} \nrightarrow \mathbb{R}^{\infty}) \\
\hline
\forall\, n, m : \mathbb{N} \bullet \\
\quad \mathrm{L}_{\mathbb{R}}(n, m) = \mathrm{L}_{\mathbb{R}}(\mathit{vecRtup}(n), \mathit{vecRtup}(m))
\end{array}
$$

2.18. **The Identity Transformation of n-tuples:** I `\idRtup`. Let $\mathrm{I}(n)$ denote the identity function on $\mathbb{R}(n)$.

$$\begin{array}{|l}
\mathrm{I} : \mathbb{N} \longrightarrow \mathbb{R}^\infty \rightarrowtail \mathbb{R}^\infty \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathrm{I}(n) = \mathrm{id}(\mathbb{R}(n))
\end{array}$$

Remark. *The function* $\mathrm{I}(n)$ *is a linear transformation.*

$$\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathrm{I}(n) \in \mathrm{L}_{\mathbb{R}}(n, n)
\end{array}$$

3. The Metric Topology on Real n-tuples

3.1. **The Dot Product of Tuples:** $\cdot$ `\dotRinf`. The *inner* or *dot* product of n-tuples v and w is the real number $v \cdot w$ defined by the sum of the component-wise products.

$$\begin{array}{|l}
_ \cdot _ : \mathbb{R}^\Delta \longrightarrow \mathbb{R} \\
\hline
\langle\rangle \cdot \langle\rangle = 0 \\
\\
\forall\, x, y : \mathbb{R};\ v, w : \mathbb{R}^\infty \mid \#v = \#w \bullet \\
\qquad (\langle x \rangle \frown v) \cdot (\langle y \rangle \frown w) = x * y + v \cdot w
\end{array}$$

Each $\mathbb{R}(n)$ is a real inner product space under the operation of dot product defined above.

3.2. **The Norm of a Tuple:** norm `\normRinf`. The norm $\|v\|$ of the n-tuple v is the positive square root of its dot product with itself.

$$\|v\| = \sqrt{v \cdot v}$$

Define $\mathrm{norm}(v)$ to be $\|v\|$.

$$\begin{array}{|l}
\mathrm{norm} : \mathbb{R}^\infty \longrightarrow \mathbb{R} \\
\hline
\forall\, v : \mathbb{R}^\infty \bullet \\
\qquad \mathrm{norm}(v) = \mathrm{sqrt}(v \cdot v)
\end{array}$$

The concepts of continuity, limits, and differentiability extend to functions between normed vector spaces such as $\mathbb{R}^n$.

3.3. **The Open Ball at a Tuple:** ball `\ballRinf`. Let $\mathrm{ball}(v, r)$ denote the *open ball* in $\mathbb{R}(n)$ of radius $r \in \mathbb{R}_+$ centred at $v \in \mathbb{R}(n)$.

$$\begin{array}{|l}
\mathrm{ball} : \mathbb{R}^\infty \times \mathbb{R}_+ \longrightarrow \mathbb{P}\, \mathbb{R}^\infty \\
\hline
\forall\, v : \mathbb{R}^\infty;\ r : \mathbb{R}_+ \bullet \\
\qquad \mathbf{let}\ n == \#v \bullet \\
\qquad\qquad \mathrm{ball}(v, r) = \{\, w : \mathbb{R}(n) \mid \mathrm{norm}(v - w) < r \,\}
\end{array}$$

3.4. **The Set of All Open Balls at an n-tuple:** balls `\ballsRtup`**.** Let $\mathrm{balls}(n)$ denote the family of all open balls in $\mathbb{R}(n)$.

$$\mathrm{balls} : \mathbb{N} \longrightarrow \mathcal{F}\,\mathbb{R}^\infty$$
$$\forall\, n : \mathbb{N} \bullet \quad \mathrm{balls}(n) = \{\, v : \mathbb{R}(n);\ r : \mathbb{R}_+ \bullet \mathrm{ball}(v, r) \,\}$$

Remark. *The set of all open balls in $\mathbb{R}(n)$ is a family of sets in $\mathbb{R}(n)$.*

$$\forall\, n : \mathbb{N} \bullet \quad \mathrm{balls}(n) \in \mathcal{F}(\mathbb{R}(n))$$

3.5. **The Usual Topology on n-tuples:** $\tau_{\mathbb{R}}$ `\tauRtup`**.** The *usual topology* on $\mathbb{R}(n)$ is the topology generated by the open balls in $\mathbb{R}(n)$. Let $\tau_{\mathbb{R}}(n)$ denote the usual topology on $\mathbb{R}(n)$.

$$\tau_{\mathbb{R}} : \mathbb{N} \longrightarrow \mathcal{F}\,\mathbb{R}^\infty$$
$$\forall\, n : \mathbb{N} \bullet \quad \tau_{\mathbb{R}}(n) = \mathit{topGen}[\mathbb{R}(n)](\mathrm{balls}(n))$$

Remark. *If $n \in \mathbb{N}$ then $\tau_{\mathbb{R}}(n)$ is a topology on $\mathbb{R}(n)$.*

$$\forall\, n : \mathbb{N} \bullet \tau_{\mathbb{R}}(n) \in \mathit{top}[\mathbb{R}(n)]$$

3.6. **The Set of All Neighbourhoods of a Tuple:** neigh `\neighRinf`**.** Let $v \in \mathbb{R}(n)$. An open set U in the usual topology $\tau_{\mathbb{R}}(n)$ that contains v is called a *neighbourhood* of v. Let $\mathrm{neigh}(v)$ denote the set of all neighbourhoods of x.

$$\mathrm{neigh} : \mathbb{R}^\infty \longrightarrow \mathcal{F}\,\mathbb{R}^\infty$$
$$\forall\, n : \mathbb{N};\ v : \mathbb{R}^\infty \mid n = \#v \bullet \quad \mathrm{neigh}(v) = \{\, U : \tau_{\mathbb{R}}(n) \mid v \in U \,\}$$

Remark. *The set of all neighbourhoods of $v \in \mathbb{R}(n)$ is a family of sets in $\mathbb{R}(n)$.*

$$\forall\, n : \mathbb{N};\ v : \mathbb{R}^\infty \mid n = \#v \bullet \quad \mathrm{neigh}(v) \in \mathcal{F}(\mathbb{R}(n))$$

3.7. **The Topological Space of n-tuples:** $\mathbb{R}_\tau$ `\tsRtup`**.** Let $\mathbb{R}_\tau(n)$ denote the topological space defined by the usual topology on $\mathbb{R}(n)$.

$$\mathbb{R}_\tau : \mathbb{N} \longrightarrow \mathit{topSpaces}[\mathbb{R}^\infty]$$
$$\forall\, n : \mathbb{N} \bullet \quad \mathbb{R}_\tau(n) = (\mathbb{R}(n), \tau_{\mathbb{R}}(n))$$

4. Continuity

4.1. **Real-Valued Functions That Are Continuous on the Set of All n-tuples:** C^0 `\CzeroRtup`**.** A function $f \in \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be *continuous* if it is continuous with respect to the usual topologies on $\mathbb{R}^n$ and $\mathbb{R}$. Let $\mathrm{C}^0(n)$ denote the set of these continuous functions.

$$
\begin{array}{|l}
\mathrm{C}^0 : \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \mathrel{+\!\!\!\to} \mathbb{R}) \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathrm{C}^0(n) = \mathrm{C}^0(\mathbb{R}_\tau(n), \mathbb{R}_\tau)
\end{array}
$$

4.2. **Real-Valued Functions That Are Continuous on a Subset of n-tuples:** C^0 `\CzeroSubsetRtup`. Let U be a subset of $\mathbb{R}^n$. A function $f \in U \longrightarrow \mathbb{R}$ is said to be *continuous on* U if it is continuous with respect to the topology induced on U. Let $\mathrm{C}^0(U)$ denote the set of these continuous functions.

$$
\begin{array}{|l}
\mathrm{C}^0 : \Delta_{\mathbb{R}} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \mathrel{+\!\!\!\to} \mathbb{R}) \\
\hline
\forall\, U : \Delta_{\mathbb{R}} \bullet \\
\qquad \mathbf{let}\ n == \dim U \bullet \\
\qquad\qquad \mathrm{C}^0(U) = \mathrm{C}^0(\mathbb{R}_\tau(n) \mid_{\mathsf{top}} U, \mathbb{R}_\tau)
\end{array}
$$

4.3. **Real-Valued Functions That Are Continuous at an n-tuple:** C^0 `\CzeroPointRtup`. A partial function f from $\mathbb{R}^n$ to $\mathbb{R}$ is said to be *continuous at* $x \in \mathbb{R}^n$ if its domain contains a neighbourhood U of x such that its restriction to U is continuous on U. Let $\mathrm{C}^0(x)$ denote the set of such functions.

$$
\begin{array}{|l}
\mathrm{C}^0 : \mathbb{R}^\infty \longrightarrow \mathbb{P}(\mathbb{R}^\infty \mathrel{+\!\!\!\to} \mathbb{R}) \\
\hline
\forall\, x : \mathbb{R}^\infty \bullet \\
\qquad \mathbf{let}\ n == \#x \bullet \\
\qquad\qquad \mathrm{C}^0(x) = \{\, f : \mathbb{R}(n) \mathrel{+\!\!\!\to} \mathbb{R} \mid \exists\, U : \mathrm{neigh}(x) \mid U \subseteq \mathrm{dom}\, f \bullet U \lhd f \in \mathrm{C}^0(U) \,\}
\end{array}
$$

4.4. **m-tuple-Valued Functions That Are Continuous on the Set of All n-tuples:** C^0 `\CzeroRtupRtup`. A mapping f from $\mathbb{R}(n)$ to $\mathbb{R}(m)$ is said to be continuous if it is continuous with respect to the usual topologies. Let $\mathrm{C}^0(n, m)$ denote the set of these continuous mappings.

$$
\begin{array}{|l}
\mathrm{C}^0 : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \mathrel{+\!\!\!\to} \mathbb{R}^\infty) \\
\hline
\forall\, n, m : \mathbb{N} \bullet \\
\qquad \mathrm{C}^0(n, m) = \mathrm{C}^0(\mathbb{R}_\tau(n), \mathbb{R}_\tau(m))
\end{array}
$$

Example. *The function* $\mathrm{I}(n)$ *is continuous.*

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathrm{I}(n) \in \mathrm{C}^0(n, n)
\end{array}
$$

Theorem 1. *Linear functions are continuous.*

$$
\begin{array}{l}
\forall\, n, m : \mathbb{N} \bullet \\
\qquad \mathrm{L}_{\mathbb{R}}(n, m) \subseteq \mathrm{C}^0(n, m)
\end{array}
$$

4.5. **m-tuple-Valued Functions That Are Continuous on a Subset of n-tuples:** C^0 `\CzeroSubsetRtupRtup`. Let U be any subset of $\mathbb{R}(n)$. Let $\mathrm{C}^0(U, m)$ denote the set of continuous mappings from the topology induced by $\mathbb{R}_\tau(n)$ on U to $\mathbb{R}_\tau(m)$.

$$
\begin{array}{|l}
\mathrm{C}^0 : \Delta_{\mathbb{R}} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \rightarrowtail \mathbb{R}^\infty) \\
\hline
\forall\, n, m : \mathbb{N} \bullet \\
\quad \forall\, U : \Delta_{\mathbb{R}} \mid \dim(U) = n \bullet \\
\quad\quad \mathrm{C}^0(U, m) = \mathrm{C}^0(\mathbb{R}_\tau(n) \mid_{\mathsf{top}} U, \mathbb{R}_\tau(m))
\end{array}
$$

Remark.

$$
\begin{array}{l}
\forall\, n, m : \mathbb{N} \bullet \\
\quad \mathrm{C}^0(\mathbb{R}(n), m) = \mathrm{C}^0(n, m)
\end{array}
$$

4.6. **m-tuple-Valued Functions That Are Continuous at an n-tuple:** *VectorConinuous*, C^0 `\CzeroPointRtupRtup`. Let $x \in \mathbb{R}(n)$ and let f be a partial function from $\mathbb{R}(n)$ to $\mathbb{R}(m)$ whose domain includes some neighbourhood U of x such that f restricted to U is continuous. In this case f is said to be *continuous at* x.

$$
\begin{array}{|l}
\multicolumn{1}{l}{\mathit{VectorContinuous}} \\
\hline
n, m : \mathbb{N} \\
f : \mathbb{R}^\infty \rightarrowtail \mathbb{R}^\infty \\
x : \mathbb{R}^\infty \\
\hline
f \in \mathbb{R}(n) \rightarrowtail \mathbb{R}(m) \\
\exists\, U : \mathrm{neigh}(x) \mid \\
\quad U \subseteq \operatorname{dom} f \bullet \\
\quad\quad U \lhd f \in \mathrm{C}^0(U, m) \\
\hline
\end{array}
$$

Let $\mathrm{C}^0(x, m)$ denote the set of all partial functions f from $\mathbb{R}(n)$ to $\mathbb{R}(m)$ that are continuous at x.

$$
\begin{array}{|l}
\mathrm{C}^0 : \mathbb{R}^\infty \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \rightarrowtail \mathbb{R}^\infty) \\
\hline
\forall\, n, m : \mathbb{N} \bullet \forall\, x : \mathbb{R}(n) \bullet \\
\quad \mathrm{C}^0(x, m) = \\
\quad\quad \{\, f : \mathbb{R}(n) \rightarrowtail \mathbb{R}(m) \mid \mathit{VectorContinuous} \,\}
\end{array}
$$

Example. *The function* $\mathrm{I}(n)$ *is continuous at every point* $x \in \mathbb{R}(n)$.

$$
\begin{array}{l}
\forall\, n : \mathbb{N} \bullet \forall\, x : \mathbb{R}(n) \bullet \\
\quad \mathrm{I}(n) \in \mathrm{C}^0(x, n)
\end{array}
$$

Theorem 2. *Linear functions are continuous everywhere.*

$$
\begin{array}{l}
\forall\, n, m : \mathbb{N} \bullet \\
\quad \forall\, x : \mathbb{R}(n); L : \mathrm{L}_{\mathbb{R}}(n, m) \bullet \\
\quad\quad L \in \mathrm{C}^0(x, m)
\end{array}
$$

5. Differentiability

Let $x \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \rightarrowtail \mathbb{R}^m$ be continuous at x. Then f is said to be *differentiable at* x if there exists a linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that $f(x + h) - f(x)$ is approximately linear in h for very small h.

$$f(x + h) - f(x) \approx L(h) + O(h^2) \quad \text{when} \quad \|h\| \approx 0$$

This condition can be written as a limit.

$$\lim_{h \to 0} \frac{\left\| f(x+h) - f(x) - L(h) \right\|}{\|h\|} = 0$$

5.1. **The Difference Quotient:** *DifferenceQuotient* **and** *diffQuot***.** The limit exists when the following difference quotient function $q : \mathbb{R}^n \nrightarrow \mathbb{R}$ is continuous at 0.

$$q(h) = \begin{cases} \frac{\left\| f(x+h) - f(x) - L(h) \right\|}{\|h\|} & \text{if } h \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Given a function f that is continuous at x, and a linear transformation L, we can define the difference quotient q. Clearly q is uniquely determined by f, x, and L. Let *DifferenceQuotient* denote this situation.

DifferenceQuotient

$$\begin{array}{l} \mathit{VectorContinuous} \\ L : \mathbb{R}^\infty \nrightarrow \mathbb{R}^\infty \\ q : \mathbb{R}^\infty \nrightarrow \mathbb{R} \\ \hline L \in \mathrm{L}_{\mathbb{R}}(n, m) \\ \mathrm{dom}\, q = \{ h : \mathbb{R}(n) \mid x + h \in \mathrm{dom}\, f \} \\ \forall h : \mathrm{dom}\, q \mid h \neq \mathbf{0}(n) \bullet \\ \quad q(h) = \mathrm{norm}(f(x+h) - f(x) - L(h)) \,/\, \mathrm{norm}(h) \\ q(\mathbf{0}(n)) = 0 \end{array}$$

- L is a linear transformation from $\mathbb{R}(n)$ to $\mathbb{R}(m)$.
- The difference quotient q is defined on a subset of $\mathbb{R}(n)$ that contains $\mathbf{0}(n)$.
- $q(h)$ is defined as the quotient when h is non-zero.
- $q(0)$ is defined as zero.

Let $\mathit{diffQuot}(f, x, L)$ denote the difference quotient q.

$\mathit{diffQuot} == \{ \mathit{DifferenceQuotient} \bullet (f, x, L) \mapsto q \}$

5.2. **The Derivative of a Continuous m-tuple-Valued Function:** *VectorDifferentiable***.** The continuous function f is *differentiable at* x when there exists a linear transformation L such that the difference quotient q is continuous at 0. In this case L is unique and is referred to as the *derivative at* x.

VectorDifferentiable

$$\begin{array}{l} \mathit{VectorContinuous} \\ L : \mathbb{R}^\infty \nrightarrow \mathbb{R}^\infty \\ \hline \textbf{let } q == \mathit{diffQuot}(f, x, L) \bullet \\ \quad q \in \mathrm{C}^0(\mathbf{0}(n)) \end{array}$$

- The continuous function f is differentiable at x with derivative L if the resulting difference quotient q is continuous at $\mathbf{0}(n)$.

Remark. *If L exists then it is unique.*

Let $\mathrm{C}^\infty(x, m)$ denote the set of all functions $f \in \mathbb{R}(n) \nrightarrow \mathbb{R}(m)$ that are smooth at $x \in \mathbb{R}(n)$.

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