Collective motion in the nuclear shell model
II. The introduction of intrinsic wave-functions

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The wave functions for a number of particles in a degenerate oscillator level, classified in part I according to irreducible representations of the group U_3, are expressed as integrals of the Hill–Wheeler type over intrinsic states. The rotational band structure which appeared in the classification is now understood, since all states of a band are shown to involve the same intrinsic state in the integral. It is possible to use the quantum number K of the intrinsic states as an additional label for the final wave functions, thus distinguishing states which, in the classification of part I, had the same values for all other quantum numbers used. The integral form for the wave functions enables simple expressions to be obtained for the quadrupole moments which resemble those of the rotational model for a permanent deformation.

1. Introduction

In part I of this series (Elliott 1958) it was shown that the orbital wave functions in L–S coupling for a number of nucleons in a degenerate level of an oscillator potential could be classified according to irreducible representations of the group U_3. These representations were labelled by a pair of integers $(\lambda\mu)$. That the classification is physically important was demonstrated, at least for the nuclei of masses 18, 19 and 20, by comparing the wave functions so defined with those resulting from standard shell model calculations. There was found to be an overlap greater than 90 %.

Each representation $(\lambda\mu)$ contains a set of states with different values of the total orbital angular momentum L. It was found that these values are precisely those which one would obtain from a series of rotational bands cut off at some upper limit. In this paper it will be shown that the states belonging to a given representation $(\lambda\mu)$ do in fact resemble those of a rotational band by exhibiting them as integrals over functions which we shall call intrinsic functions. For example, in the $(\lambda 0)$ representation, *all* states may be written as integrals over a *single* intrinsic function. A quantum number K will be introduced in these intrinsic functions labelling the component of angular momentum along the intrinsic Z-axis. It will be seen, in general, that all states belonging to a representation $(\lambda\mu)$ may be expressed as integrals involving intrinsic functions with

$$K = \mu, \mu-2, \mu-4, \ldots, 0 \text{ or } 1.$$

An intrinsic function with a given value of K gives rise to states having just those values of L to be expected from a rotational K-band cut-off above $L = K+\lambda$. When $\mu > 1$ there will be more than one state in that representation for certain values of L as shown in I. The quantum number K may now be used to distinguish them, describing the intrinsic function from which they may be constructed.

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The integral form of the wave functions in the U_3 classification makes it possible to derive explicit formulae for the matrix elements of the quadrupole mass moment by transforming the operator into the intrinsic frame. The result shows that in the limit of large λ these matrix elements are related in the same way as are those in the strong-coupling rotational model.

Sections 2 and 3 describe the construction and properties of the eigenfunctions of the operator Q_0 introduced in I. This is done in preparation for § 4 where the integral form for the states is derived with these eigenfunctions as the intrinsic functions. Normalization coefficients which occur in the integral form are calculated in § 5, and in § 6 explicit formulae are derived for the matrix elements of the quadrupole mass moment.

2. The single-particle eigenfunctions of Q_0

The degenerate states of an oscillator level are usually labelled by quantum numbers l and K denoting the angular momentum and its component in the z-direction. In that scheme, the operators L^2 and L_0 (or L_z) are diagonal. We may equally well classify these degenerate states by the quantum numbers ϵ and K by making the operators Q_0 and L_0 diagonal. Here $\epsilon = \langle Q_0 \rangle$, where

$$Q_0 = \{(2z^2 - x^2 - y^2) + b^4(2p_z^2 - p_x^2 - p_y^2)\}/2b^2 \tag{1}$$

was introduced in I. Let us denote these states by $\chi(\epsilon K)$ to distinguish them from the $\phi(lK)$. In fact, the $\chi(\epsilon K)$ will not only diagonalize the operator Q_0 within a given oscillator shell but, because Q_0 is the product of one creation with one destruction operator of the oscillator quanta, the functions $\chi(\epsilon K)$ will actually be eigenfunctions of Q_0.

In making Q_0 diagonal, we are at the same time making n_z, the number of oscillator quanta in the z-direction, a good quantum number as well as $N = n_x + n_y + n_z$. In fact, it follows that

$$\epsilon = (2n_z - n_x - n_y) = 3n_z - N. \tag{2}$$

The symmetry of the quanta remaining in the x and y directions may be described in exactly the same way that the total spin S of a system of q particles of spin $\frac{1}{2}$ describes the symmetry of the function with respect to the two substates $m_S = \pm \frac{1}{2}$. In the totally symmetric state, for example, $S = \frac{1}{2}q$. We shall introduce a quantum number Λ in analogy with S. Since we are so far dealing with a single particle this is rather trivial because a single-particle function must necessarily be symmetric in the x and y quanta so that

$$\Lambda = \tfrac{1}{2}(n_x + n_y) = \tfrac{1}{6}(2N - \epsilon). \tag{3}$$

In the following section where we consider many-particle functions the quantum number Λ will not be trivial.

The introduction of Λ may be done rigorously as follows, by appealing to the eight infinitesimal operators Q_q, L_q of the group SU_3 discussed in I. It can be seen from I (8) that the three operators $Q_{\pm 2}$, L_0 all commute with Q_0 and satisfy the commutation relations

$$[Q_2, L_0] = -2Q_2, \quad [Q_{-2}, L_0] = 2Q_{-2}, \quad [Q_2, Q_{-2}] = 6L_0 \tag{4}$$

among themselves. This means that the operators $Q_{\pm 2}$, L_0, Q_0 describe a subgroup of U_3 and that this subgroup is a product of two groups described by the operators $Q_{\pm 2}$, L_0 and by Q_0. If we write

$$v_{\pm} = \mp\left(\frac{1}{2\sqrt{3}}\right) Q_{\pm 2}, \quad v_0 = \tfrac{1}{2}L_0, \tag{5}$$

the commutation relations (4) reduce to

$$[v_{\pm}, v_0] = \mp v_{\pm}, \quad [v_+, v_-] = -v_0, \tag{6}$$

which are recognized to be those of the three infinitesimal operators of a three-dimensional rotation group. It is clear then that we have carried out the reduction $SU_3 \to SU_2 \times U_1$ or $SU_3 \to R_3 \times U_1$, there being a well-known isomorphism between the three-dimensional rotation group and the two-dimensional unitary group. The group U_1 is described by the single operator Q_0, and its representations which are one-dimensional are labelled by a single quantum number ϵ. The representations of R_3 are labelled by an integral or half-integral number which we call Λ where, like the square of an angular momentum,

$$\begin{aligned}\Lambda(\Lambda+1) = \langle v^2 \rangle &= \langle v_0^2 - v_+ v_- - v_- v_+ \rangle \\ &= \tfrac{1}{4}\langle L_0^2 \rangle + \tfrac{1}{12}\langle Q_2 Q_{-2} + Q_{-2} Q_2 \rangle.\end{aligned} \tag{7}$$

The states belonging to each representation Λ will be labelled by

$$\langle v_0 \rangle = \Lambda, \Lambda - 1, \ldots, -\Lambda,$$

like angular momentum substates. However, since $v_0 = \frac{1}{2}L_0$ and $\langle L_0 \rangle = K$, it follows that the states belonging to a given representation Λ may be labelled by K, which may therefore take the values

$$K = 2\Lambda, 2\Lambda - 2, \ldots, -2\Lambda. \tag{8}$$

The classification of the single-particle states in any oscillator level N is now immediate. Given N, we may take $n_z = 0, 1, 2, \ldots, N$. For each value of n_z, the value of ϵ is fixed from (2) and, for a single particle, Λ is also fixed from (3) with K taking values $2\Lambda, 2\Lambda - 2, \ldots, -2\Lambda$. We list the single-particle states $\chi(\epsilon\Lambda K)$ for $N < 4$ in table 1. These results could no doubt have been obtained in a more direct way, but the analysis above will also be applicable to the many-particle wave functions discussed later.

TABLE 1. THE SINGLE-PARTICLE EIGENFUNCTIONS OF Q_0 FOR $N \leqslant 3$

N	Λ	K	ϵ
0	0	0	0
1	0	0	2
	$\frac{1}{2}$	± 1	-1
2	0	0	4
	$\frac{1}{2}$	± 1	1
	1	$0, \pm 2$	-2
3	0	0	6
	$\frac{1}{2}$	± 1	3
	1	$0, \pm 2$	0
	$\frac{3}{2}$	$\pm 1, \pm 3$	-3

The matrix elements of the operators $Q_{\pm 2}$ in our states $\chi(\epsilon\Lambda K)$ may be calculated like those of the angular momentum operators, because of the relations (5) and (6). One finds

$$\begin{aligned} Q_{\pm 2}\chi(\epsilon\Lambda K) &= \mp 2\sqrt{3}\, v_{\pm}\chi(\epsilon\Lambda K) \\ &= \tfrac{1}{2}\{6(2\Lambda \mp K)(2\Lambda \pm K + 2)\}^{\frac{1}{2}}\chi(\epsilon\Lambda K \pm 2). \end{aligned} \tag{9}$$

Here, the phases of $\chi(\epsilon\Lambda K \pm 2)$ relative to $\chi(\epsilon\Lambda K)$ have been chosen according to the convention used by Condon and Shortley for the angular momentum substates.

3. A CLASSIFICATION FOR THE MANY-PARTICLE EIGENFUNCTIONS OF Q_0

Let us now put a number k of particles into some oscillator level N. We shall be able to build up a great many totally antisymmetric states in a variety of ways. In fact, this will be the same complete set of states which we considered in I. However, instead of developing a classification in which L^2 and L_0 are diagonal as was done in I, we shall now develop a classification in which Q_0 and L_0 are diagonal. The operators Q_q and L_q are now defined as many-particle operators,

$$Q_q = \sum_{i=1}^{k} Q_q(i), \quad L_q = \sum_{i=1}^{k} L_q(i).$$

However, it may easily be seen that the relations (4) to (9) are true also for these many-particle operators. In the same way, $\epsilon = \langle Q_0 \rangle$ now denotes the eigenvalue of the many-particle operator Q_0 and (2) becomes

$$\epsilon = \sum_{i=1}^{k} (2n_z(i) - n_x(i) - n_y(i)) = 2N_z - N_x - N_y = 3N_z - kN, \tag{10}$$

where

$$N_x = \sum_{i=1}^{k} n_x, \text{ etc.}, \quad \text{and} \quad kN = N_x + N_y + N_z.$$

The classification of the many-particle eigenfunctions of Q_0 is then a generalization of the method used in § 2 for the single-particle functions. First, the states are labelled by the partition $[f]$ with its associated supermultiplet of T and S values in the usual way. Then, just as in I, the orbital states belonging to a given partition $[f]$ are classified according to the irreducible representations $(\lambda\mu)$ of the group SU_3. At this point we depart from the classification of I and instead follow that introduced in § 2. We choose a basis of states to spread out each representation $(\lambda\mu)$ in which Q_0 and L_0 are diagonal. As well as $K = \langle L_0 \rangle$, each state then has a label ϵ which, from (10), describes the excess (or deficiency) of quanta in the z-direction compared to those in the xy-plane. This will still leave some states undistinguished, but by specifying the symmetry of the remaining quanta in the xy-plane by Λ as indicated in § 2, a unique labelling of states may be obtained. The values of K occurring in each set labelled by Λ are given by (8). As an illustration of this classification, the many-particle eigenfunctions of Q_0 are listed in table 2 for the $N = 2$ oscillator shell with $k \leqslant 3$.

To determine which values of ϵ and Λ occur for a given representation $(\lambda\mu)$ we must study the reduction from SU_3 to $SU_2 \times U_1$. This is a standard process (Littlewood 1940, p. 94; Jahn 1950) and may be carried out using the Young tableau

(see I) which illustrate the representations of a unitary group. One finds the simple result that for a given representation $(\lambda\mu)$, ϵ may take the values

$$\epsilon = 2\lambda+\mu,\ 2\lambda+\mu-3,\ \ldots,\ -\lambda-2\mu, \tag{11}$$

while for each value of ϵ, Λ takes values

$$\Lambda = \tfrac{1}{6}\,|\,2\lambda-2\mu-\epsilon\,|,\ \tfrac{1}{6}\,|\,2\lambda-2\mu-\epsilon\,|+1,\ \ldots,\ \min\{\tfrac{1}{6}(2\lambda+4\mu-\epsilon),\ \tfrac{1}{6}(2\mu+4\lambda+\epsilon)\}. \tag{12}$$

The latter expression simplifies greatly for the two extreme values of ϵ from (11), giving unique values for Λ,

$$\left.\begin{aligned} &\Lambda = \tfrac{1}{2}\mu \quad \text{for} \quad \epsilon = \epsilon_{\max.} = 2\lambda+\mu, \\ \text{and} \quad &\Lambda = \tfrac{1}{2}\lambda \quad \text{for} \quad \epsilon = \epsilon_{\min.} = -\lambda-2\mu. \end{aligned}\right\} \tag{13}$$

It is clear from (12) why the quantum number Λ is superfluous for a single particle, as remarked in § 2. In that case, the wave function must be symmetric so that $\mu = 0$ and from (12) this leads to the single value

$$\Lambda = \tfrac{1}{6}(2\lambda-\epsilon)$$

for Λ, relating it to ϵ.

It seems appropriate at this stage to discuss the physical significance of these quantum numbers which we have introduced. To do this we must regard the wave functions for k-particles in the N-shell as tensors of rank kN in the creation operators of the oscillator well, acting on the 'vacuum state' in which all particles are in $(1s)$ states. The numbers λ and μ describe the symmetry of this tensor with respect to transformations among the three components of each single-creation operator. These are the transformations of the group U_3. Thus, for example, if $\mu = 0$, that function transforms like a symmetric tensor of rank λ. A function labelled by $\mu = 1$ will transform like a tensor of rank $\lambda+2$ which is symmetric in only $\lambda+1$ of its component vectors. In general, the functions labelled by $(\lambda\mu)$ will transform like tensors of rank $\lambda+2\mu$ whose symmetry is described by the partition $[\lambda+\mu, \mu]$. This symmetry is *not* the same as the particle permutation symmetry which is described by $[f]$. The symmetry described by λ and μ is in principle similar to that described by the angular momentum L. A function with $L = 1$ will transform like the vector x, y, z under rotation, although that function may be a very complicated one of many particles. To bring out the same point we note that the functions

$$z_1^4, \quad z_1^2 z_2^2, \quad z_1 z_2 z_3 z_4$$

have very different particle symmetry although they will all behave in the same way under some transformation of z. In the special case of the $(1p)$ shell, because each single-particle function is also the result of a *single*-creation operation, it follows that there is a one-to-one correspondence between the particle symmetry and the transformation symmetry under U_3. Thus, as pointed out in I, the quantum numbers $(\lambda\mu)$ and $[f]$ are identical in the $(1p)$ shell. In the general case, each single-particle function is the result of N-creation operations so that the particle symmetry and the creation operator symmetry are different.

A more direct physical interpretation of this symmetry in the creation operators is obtained in terms of the total numbers of quanta N_x, N_y, N_z in each direction. The values of λ and μ are related to the values of N_x, N_y and N_z which may occur in that representation $(\lambda\mu)$ by

$$\lambda+\mu = \max N_z - \min (N_x, N_y)_{\text{at max. } N_z}, \tag{14}$$

$$\mu = \max |N_x - N_y|_{\text{at max. } N_z}. \tag{15}$$

In the same way, for a given λ, μ and ϵ,

$$2\Lambda = \max |N_x - N_y|. \tag{16}$$

The relation $$\min (N_x, N_y)_{\text{at max. } N_z} = \tfrac{1}{3}(kN - \lambda - 2\mu),$$

together with (10) reduces (14) to $\epsilon_{\max.} = 2\lambda+\mu$, the result given in (11). In the same way, (15) and (16) lead to $\Lambda = \frac{1}{2}\mu$ for $N_z = \max N_z$, i.e. for $\epsilon = \epsilon_{\max.}$, as also given in (13).

The interpretation of λ and μ in (14) and (15) is of course equally valid for any interchange of the subscripts x, y and z because λ and μ are not related to any set of axes. The quantum numbers ϵ, Λ on the other hand, are related to the z-axis, along which the angular momentum of the state has a component K.

4. An integral form for the U_3 wave functions

It has been remarked earlier that the wave functions defined in § 3, which we shall call $\chi([f](\lambda\mu)\epsilon\Lambda K)$, provide a complete set of states, equivalent with those $\psi([f](\lambda\mu)\alpha LM)$ defined in I. The symbol α serves to distinguish, in a hitherto unspecified way, states having the same values for $[f]$, $(\lambda\mu)$, L and M. This was found necessary in I when both $\lambda > 1$ and $\mu > 1$. In this section we shall study the relation between these two complete sets and show how the χ may be interpreted as intrinsic wave functions, exhibiting rotational properties in the set ψ.

Because of the equivalence of these two complete sets we may expand

$$\chi([f](\lambda\mu)\epsilon\Lambda K) = \sum_{\alpha, L', M'} c(\lambda\mu\epsilon\Lambda K, \alpha L'M')\,\psi([f](\lambda\mu)\alpha L'M'), \tag{17}$$

where c is a numerical coefficient and the summation runs over all states of the representation $(\lambda\mu)$ in the scheme of I. It is clear that only terms with $M' = K$ can appear in this expansion. Consider a new set of axes rotated through an angle, which we denote symbolically by Ω, from the original laboratory frame. Now define χ_Ω to be the function χ discussed above but referred to the new, rotated, set of axes. If we define ψ_Ω in a similar way, then (17) is equally true for these rotated functions. Thus

$$\chi_\Omega([f](\lambda\mu)\epsilon\Lambda K) = \sum_{\alpha, L'} c(\lambda\mu\epsilon\Lambda K, \alpha L')\,\psi_\Omega([f](\lambda\mu)\alpha L'K), \tag{18}$$

where we have used the fact that c is zero unless $M' = K$.

It is well known that a function with definite angular momentum and referred to a rotated set of axes may be expressed as a linear combination of states of the same angular momentum but referred to the original set of axes. The coefficients in this

expansion are known functions of the angles of rotation of the axes which depend only on the angular momentum and its projections on the z-axes of the two frames. Thus we may write

$$\psi_\Omega([f](\lambda\mu)\alpha L'K) = \sum_{K'} (\mathscr{D}^{L'}_{K'K}(\Omega))^* \psi([f](\lambda\mu)\alpha L'K'), \tag{19}$$

where ψ is referred to the original laboratory frame.

If we now multiply both sides of (18) by $\mathscr{D}^L_{MK}(\Omega)$ and integrate over all Ω, substituting for ψ_Ω from (19) we find

$$\int \chi_\Omega([f](\lambda\mu)\epsilon\Lambda K)\,\mathscr{D}^L_{MK}(\Omega)\,\mathrm{d}\Omega = \sum_\alpha \frac{c(\lambda\mu\epsilon\Lambda K, \alpha L)}{(2L+1)}\,\psi([f](\lambda\mu)\alpha LM). \tag{20}$$

By $\int \mathrm{d}\Omega$ we mean explicitly

$$\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \mathrm{d}\theta_1,\ \sin\theta_2\ \mathrm{d}\theta_2\ \mathrm{d}\theta_3$$

where θ_1, θ_2, θ_3 are the three Euler angles defining the orientation of the new axes and represented symbolically by Ω.

Here we have made use of the orthogonality of $\mathscr{D}$-functions to carry out the integration on the right-hand side of the equation. In discussing (20), let us first consider the case when $\lambda < 2$ or $\mu < 2$ so that the label α is unnecessary. Then, (20) shows that each function ψ may be expressed as an integral of the Hill–Wheeler (1953) type over any state χ_Ω for which the coefficient $c(\lambda\mu\epsilon\Lambda K, \alpha L)$ is non-zero. This suggests the interpretation of the functions χ_Ω as intrinsic wave functions referred to the intrinsic axes oriented at an angle Ω relative to the laboratory frame. The value of such an interpretation depends on the number of intrinsic states χ needed to produce all the ψ. For example, if it were necessary to use a different intrinsic function for each state ψ then this use of intrinsic states would be rather meaningless. However, if we can represent a number of states ψ with different values of L as integrals of the form (20) over the *same* intrinsic state, then we shall have brought out a relationship between those states ψ similar to that between states of a rotational band. This is indeed the case and in the remainder of this section we shall prove the following results, first given in particular and then in general form.

If $\mu = 0$ then *all* states ψ belonging to that representation $(\lambda 0)$ may be expressed as integrals over a *single* intrinsic state with $K = 0$.

If $\mu = 1$ then *all* states ψ belonging to that representation $(\lambda 1)$ may be expressed as integrals over a *single* intrinsic state with $K = 1$.

If $\mu = 2$ then we need two intrinsic states, with $K = 0$ and $K = 2$ in order to construct all ψ of that representation $(\lambda 2)$. This is very pleasing because when $\mu > 1$ we have more than one state in that representation for certain values of L, as pointed out in I, and we introduced an unspecified symbol α to distinguish such states. In fact, we shall find that the number of intrinsic states needed is exactly the same as the number of values of α needed to distinguish all states of the representation. Furthermore, we may use the value of K in the intrinsic states to serve as the unspecified symbol α. For example, the two D-states occurring in a representation $(\lambda 2)$ may be distinguished by K, since one is formed as an integral over an

intrinsic function with $K = 0$ and another with $K = 2$. Even when $\mu < 2$ we may still include K among the labels of ψ although it will be unnecessary.

In general we may write

$$\psi([f](\lambda\mu)KLM) = \frac{(2L+1)}{c(\lambda\mu K, L)}\int \chi_\Omega([f](\lambda\mu)K)\,\mathscr{D}^L_{MK}(\Omega)\,\mathrm{d}\Omega, \tag{21}$$

where
$$K = \min(\lambda,\mu), \min(\lambda,\mu)-2, \ldots, 0 \text{ or } 1. \tag{22}$$

For each K, the ψ which result from the integral (21) have the following values of L,

$$L = K, K+1, \ldots, K+\max(\lambda,\mu), \tag{23}$$

except when $K = 0$, in which case

$$L = \max(\lambda,\mu), \max(\lambda,\mu)-2, \ldots, 0 \text{ or } 1.$$

These particular values of L are those for which $c \neq 0$ in the expansion of the intrinsic function into functions of definite angular momentum as in (17). It is clear from (23) that the L-values associated with a given K are just those of a rotational K-band cut-off at some upper limit $K+\max(\lambda,\mu)$.

We must now prove these results. We must specify precisely which intrinsic states are involved in (21) and prove that by such an integral over the comparatively few intrinsic states given in (22), we can produce *all* ψ of the representation $(\lambda\mu)$. The specification is that if $\lambda \geqslant \mu$ the intrinsic states needed in (21) are those with $\epsilon = \epsilon_{\max.}$, while if $\lambda < \mu$ they are those with $\epsilon = \epsilon_{\min.}$. For these values, Λ is uniquely determined by (13) and therefore need not be written in (21). When, as in (21), we omit the symbol ϵ from χ and c it means that ϵ has the value given above, viz. $\epsilon_{\max.}$ if $\lambda \geqslant \mu$ and $\epsilon_{\min.}$ if $\lambda < \mu$. Then (22) is an immediate consequence of (13) and (8). It is clearly only necessary to use the positive values of K in (21).

It remains to prove that all states ψ belonging to a given representation $(\lambda\mu)$ may be produced by the integral (21) involving only those intrinsic states with $\epsilon = \epsilon_{\max.}$ if $\lambda \geqslant \mu$ and $\epsilon = \epsilon_{\min.}$ if $\lambda < \mu$. In fact we shall show that irrespective of whether $\lambda \geqslant \mu$ or not, all states ψ may be produced either from intrinsic states with $\epsilon_{\max.}$ or from those with $\epsilon_{\min.}$. The reason why we choose to use the χ with $\epsilon_{\max.}$ when $\lambda \geqslant \mu$ and $\epsilon_{\min.}$ when $\lambda < \mu$ is to involve as few intrinsic states as possible. An inspection of the intrinsic states belonging to the representations (20) and (02) given in table 2 illustrates this point. The proof requires the following lemma:

In the expansions

$$\chi([f](\lambda\mu)K) = \sum_L c(\lambda\mu K, L)\,\psi([f](\lambda\mu)KLK) \tag{24}$$

of all χ with $\epsilon = \epsilon_{\max.}$ the states $\psi([f](\lambda\mu)KLM)$, where as usual $M = L, L-1, \ldots, -L$, span *all* states of that representation $(\lambda\mu)$.

We prove this lemma by a *reductio ad absurdum* argument. If the lemma is not true then there will exist some function $\psi'([f](\lambda\mu)LM)$ which is linearly independent of all the $\psi([f](\lambda\mu)KLM)$ produced by (24). Therefore

$$\int \overline{\psi}'([f](\lambda\mu)LK)\,\chi([f](\lambda\mu)K)\,\mathrm{d}\tau = 0 \tag{25}$$

TABLE 2. A CLASSIFICATION FOR THE MANY PARTICLE EIGENFUNCTIONS OF Q_0 FOR $N=2$ AND $k \leqslant 3$

k	$[f]$	$(\lambda\mu)$	ϵ	2Λ
1	[1]	(20)	4	0
			1	1
			−2	2
2	[2]	(40)	8	0
			5	1
			2	2
			−1	3
			−4	4
		(02)	2	2
			−1	1
			−4	0
	[11]	(21)	5	1
			2	0, 2
			−1	1, 3
			−4	2
3	[3]	(60)	12	0
			9	1
			6	2
			3	3
			0	4
			−3	5
			−6	6
		(22)	6	2
			3	1, 3
			0	0, 2, 4
			−3	1, 3
			−6	2
		(00)	0	0
	[21]	(41)	9	1
			6	0, 2
			3	1, 3
			0	2, 4
			−3	3, 5
			−6	4
		(22)	6	2
			3	1, 3
			0	0, 2, 4
			−3	1, 3
			−6	2
		(11)	3	1
			0	0, 2
			−3	1
	[111]	(30)	6	0
			3	1
			0	2
			−3	3
		(03)	3	3
			0	2
			−3	1
			−6	0

for all χ of $\epsilon_{\max.}$. The integration here runs over all co-ordinates of all particles, the usual region in which the orthogonality of the wave functions is defined.

We must now recall the group operators L_q and Q_q. From I (8),

$$[Q_0, Q_{\pm 1}] = \pm 3\sqrt{3}\, L_{\pm 1}, \quad [Q_0, L_{\pm 1}] = \pm \sqrt{3}\, Q_{\pm 1}. \tag{26}$$

Defining $$\xi_\pm = Q_{\pm 1} + \sqrt{3}\, L_{\pm 1}$$

and $$\eta_\pm = Q_{\pm 1} - \sqrt{3}\, L_{\pm 1},$$

it follows from (26) that

$$[Q_0, \xi_\pm] = \pm 3\xi_\pm \quad \text{and} \quad [Q_0, \eta_\pm] = \mp 3\eta_\pm. \tag{27}$$

From (27) and the fact that $\chi([f](\lambda\mu)\,\epsilon\Lambda K)$ is an eigenfunction of Q_0 with eigenvalue ϵ we have

$$\left.\begin{aligned} Q_0 \xi_\pm \chi([f](\lambda\mu)\,\epsilon\Lambda K) &= (\epsilon \pm 3)\,\xi_\pm \chi([f](\lambda\mu)\,\epsilon\Lambda K) \\ \text{and} \quad Q_0 \eta_\pm \chi([f](\lambda\mu)\,\epsilon\Lambda K) &= (\epsilon \mp 3)\,\eta_\pm \chi([f](\lambda\mu)\,\epsilon\Lambda K). \end{aligned}\right\} \tag{28}$$

This shows that the functions $\xi_\pm\chi$ and $\eta_\pm\chi$ are also eigenfunctions of Q_0 and that ξ_+ and η_- increase the eigenvalue by 3 while ξ_- and η_+ decrease it by 3. In fact, ξ and η are the operators which shift an oscillator quantum from the z-direction into the x-y plane and vice-versa. For the extreme values of ϵ within the representation $(\lambda\mu)$ it follows that

$$\left.\begin{aligned} \xi_+\chi([f](\lambda\mu)\,\epsilon_{\max.}\Lambda K) &= 0, \quad \eta_+\chi([f](\lambda\mu)\,\epsilon_{\min.}\Lambda K) = 0, \\ \eta_-\chi([f](\lambda\mu)\,\epsilon_{\max.}\Lambda K) &= 0, \quad \xi_-\chi([f](\lambda\mu)\,\epsilon_{\min.}\Lambda K) = 0. \end{aligned}\right\} \tag{29}$$

Returning to the proof of the lemma, let us consider for definiteness the case $\lambda \geqslant \mu$ when our convention implies that the χ in (21) and (24) have $\epsilon = \epsilon_{\max.}$. A general group operation P may be written $P = \sum_e \Pi^e$ as a sum of products Π^e of the eight infinitesimal group operators Q_q, L_q. We may equally well take the eight operators

$$Q_0, \quad Q_{\pm 2}, \quad L_0, \quad \xi_+, \quad \eta_- \quad \text{and} \quad L_{\pm 1} \tag{30}$$

as the basic infinitesimal operators. Then, making use of the commutation relations of these operators, we may write

$$P = \sum_e \Pi_1^e \Pi_2^e, \tag{31}$$

where Π_1^e is a product involving only the operators $L_{\pm 1}$, and Π_2^e is a product involving only the first six operators listed in (30).

Now consider the integral

$$I = \int \overline{\psi}'([f](\lambda\mu)\,LM)\, P\chi([f](\lambda\mu)\,K)\,\mathrm{d}\tau. \tag{32}$$

It is clear that Π_1^e can at most change the M-value of a state with definite angular momentum, i.e.

$$\Pi_1^e \psi'([f](\lambda\mu)\,LM) = \sum_{M'} a^e(M')\,\psi'([f](\lambda\mu)\,LM'),$$

where the $a^e(M')$ are numerical coefficients. In a similar way, making use of (29) and (9) the operators Π_2^ϵ can at most change the K-value of a state χ with $\epsilon = \epsilon_{\max.}$. Thus

$$\Pi_2^\epsilon \chi([f](\lambda\mu)K) = \sum_{K'} b^e(K')\chi([f](\lambda\mu)K')$$

and (32) becomes

$$I = \sum_e \sum_{M'} \sum_{K'} a^e(M') b^e(K') \int \overline{\psi}'([f](\lambda\mu)LM')\chi([f](\lambda\mu)K')\,d\tau.$$

Now, if $M' \neq K'$ this integral clearly vanishes and if $M' = K'$ and the lemma is untrue, then from (25) the integral also vanishes. Hence, if the lemma is untrue the integral (32) vanishes for all P. This would mean that $\psi'([f](\lambda\mu)LM)$ was orthogonal to all states which can be formed by any group operation on $\chi([f](\lambda\mu)K)$. Thus ψ' is orthogonal to all states of the representation and cannot therefore belong to that representation $(\lambda\mu)$. This contradicts the supposition that the lemma is untrue and therefore proves the lemma (24).

The lemma is equally true in the oriented frame so that the final result (21) may be derived from (24) in exactly the same way that (20) was derived from (18). When $\lambda < \mu$, and we therefore define the χ in (21) and (24) to have $\epsilon = \epsilon_{\min.}$, the proof proceeds in a similar fashion.

We now give a few examples to clarify this procedure. From I (§ 5) the (60) representation contains states with $L = 0, 2, 4, 6$. The result of this section is that all these states may be expressed in the form (21) with a single intrinsic state,

$$\chi_\Omega([f](60)\epsilon = 12, \Lambda = 0, K = 0).$$

From I (§ 5) the (41) representation contains states with $L = 1, 2, 3, 4, 5$. These may all be expressed in the form (21) with the single intrinsic state,

$$\chi_\Omega([f](41)\epsilon = 9, \Lambda = \tfrac{1}{2}, K = 1).$$

The (22) representation contains states with $L = 0, 2^2, 3, 4$, there being two states with $L = 2$. It is necessary to use two intrinsic states in this case,

$$\chi_\Omega([f](22)\epsilon = 6, \Lambda = 1, K = 0)$$

and

$$\chi_\Omega([f](22)\epsilon = 6, \Lambda = 1, K = 2).$$

The two D-states may be distinguished by forming one from each intrinsic state, thereby labelling one with $K = 0$ and one with $K = 2$.

Although this procedure defines two linearly independent D-states they are not orthogonal. This is a general feature of the use of K as a label when $\min(\lambda, \mu) > 1$. However, the departure from orthogonality is small unless the states concerned are near the top of the band, i.e. unless the value of L involved is close to the maximum L of that representation. For more than a few particles, the states near the top of a band will be so high in energy that they are not involved in any practical application of these results.

A further complication connected with these high states is that when $\min(\lambda, \mu) > 1$, some of them may be formed from more than one of the intrinsic states involved. This means that the assignment of K to these high states is somewhat arbitrary. In the example above, the G-state of the representation (22) may

be produced either from the $K = 0$ or $K = 2$ intrinsic function. It is not surprising that such effects occur near the top of the band, because in a true rotational picture the band extends up to infinity. The mere fact that the formalism of these papers involves a cut-off band implies a departure from such a picture.

The reason why so few intrinsic states are needed is clear from the above analysis. Using the form (31) for a general group operation it is seen that any intrinsic state χ' of a representation $(\lambda\mu)$ may be obtained from states of that representation which have $\epsilon = \epsilon_{\max.}$ by means of operators Π_1. But, since $L_{\pm 1}$ are the infinitesimal rotation operators, it follows that χ' may be obtained by rotation of states with $\epsilon = \epsilon_{\max.}$. Hence, in the integral (21) where we integrate over all orientations of the intrinsic state, χ' cannot produce any state ψ not produced by the χ having $\epsilon = \epsilon_{\max.}$. In this sense, we can speak of all χ with $\epsilon \neq \epsilon_{\max.}$ as redundant states when $\lambda \geqslant \mu$, and likewise for those with $\epsilon \neq \epsilon_{\min.}$ when $\lambda < \mu$. We stress that this is partly convention, in that such a state is only redundant with reference to the chosen intrinsic states. By choosing them in the way described in this section we have used as few intrinsic states as possible. (By mixing the intrinsic states with different K, which occur when $\min(\lambda, \mu) > 1$, it would be possible to reduce further the number of intrinsic states. We shall not develop such a scheme at this stage.)

5. The expansion coefficients $c(\lambda\mu K, L)$

To make use of (21) we must know the values of $c(\lambda\mu K, L)$. We may regard this number as a normalization coefficient in (21) or an expansion coefficient in (19). It would be ideal to have a general formula for these coefficients, but this has not yet been found possible in the general case. Nevertheless, they may be determined from a formula for a special case together with recursion formulae. The derivation is complicated by the fact that we need the coefficients only for the chosen intrinsic states so that we want to avoid involving the coefficients relating to the redundant states in any recursion formula.

Because the states $\chi(\lambda\mu K)$ and $\chi(\mu\lambda K)$ are conjugate in the sense of replacing particles by holes, they must have the same expansion into angular momentum states. Thus

$$c(\lambda\mu K, L) = c(\mu\lambda K, L). \tag{33}$$

We are free at this stage to make a phase convention. We therefore define the phase of our states $\psi([f](\lambda\mu)KLM)$ by making all $c(\lambda\mu KL)$ positive for non-negative values of K. This fixes the relative phase of states ψ of the same band. Their over-all phase depends on that of the intrinsic function χ. The relative phases of different χ with the same values of λ and μ are fixed from (9). For intrinsic states belonging to different representations, the phases will be specified in a later paper when fractional parentage coefficients are derived for the intrinsic functions.

If $\mu = 0$, we saw in § 3 that that function transforms like one with λ quanta in the z-direction. Thus, the desired expansion is obtained in terms of vector coupling coefficients,

$$\chi(200) = \sum_{L_2} (1100 \mid L_2 0)\, \psi((20) L_2 0) \quad \text{for} \quad \lambda = 2,$$

$$\chi(300) = \sum_{L_3} \sum_{L_2} (1100 \mid L_2 0)\, (L_2 100 \mid L_3 0)\, \psi((30) L_2 L_3 0) \quad \text{for} \quad \lambda = 3,$$

and in general

$$\chi(\lambda 00) = \sum_{L_\lambda} \sum_{L_{\lambda-1}} \dots \sum_{L_2} (1100 \mid L_2 0)\,(L_2 100 \mid L_3 0) \dots (L_{\lambda-1} 100 \mid L_\lambda 0) \times \psi((\lambda 0) L_2 L_3 \dots L_\lambda 0),$$

where L_2, L_3, ..., etc., describe the manner in which that particular state was constructed. The total intensity of all states with $L_\lambda = L$ must be the square of the expansion coefficient that we are looking for,

$$c^2(\lambda 00, L) = \sum_{L_{\lambda-1}} \sum_{L_{\lambda-2}} \dots \sum_{L_2} \{(1100 \mid L_2 0)\,(L_2 100 \mid L_3 0) \dots (L_{\lambda-1} 100 \mid L_\lambda 0)\}^2. \quad (34)$$

It is surely possible to reduce this summation in a direct way, but here we obtain a simple result for the sum in an indirect way. The expansion of z^λ into Legendre polynomials is well known (Whittaker & Watson 1927, p. 310), to be

$$z^\lambda = \sum_L \left\{ \frac{(2L+1)\, 2^L \lambda! (\frac{1}{2}\lambda + \frac{1}{2}L)!}{(\lambda + L + 1)!\, (\frac{1}{2}\lambda - \frac{1}{2}L)!} \right\} P_L(z). \quad (35)$$

However, successive applications of the general formula

$$Y_\beta^{(b)}\, Y_\gamma^{(c)} = \sum_a \left\{ \frac{(2b+1)\,(2c+1)}{4\pi(2a+1)} \right\}^{\frac{1}{2}} (bc00 \mid a0)\,(bc\beta\gamma \mid a\alpha)\, Y_\alpha^{(a)}$$

for combining spherical harmonics gives the result

$$z^\lambda = \sum_L \sum_{L_{\lambda-1}} \dots \sum_{L_2} \{(1100 \mid L_2 0)\,(L_2 100 \mid L_3 0) \dots (L_{\lambda-1} 100 \mid L_\lambda 0)\}^2 P_L(z). \quad (36)$$

Comparing (35) and (36) gives a very simple result for the multiple sum in (34), viz.

$$c^2(\lambda 00, L) = \left\{ \frac{(2L+1)\, 2^L \lambda!\, (\frac{1}{2}\lambda + \frac{1}{2}L)!}{(\lambda + L + 1)!\, (\frac{1}{2}\lambda - \frac{1}{2}L)!} \right\}. \quad (37)$$

Thus, $c(\lambda 00, L)$ is simply the square root of the corresponding coefficient in the expansion (35).

To determine these coefficients when $\mu \neq 0$ we must involve some of the redundant intrinsic states, in particular those with $\epsilon = \epsilon_{\max.}$ when $\lambda < \mu$ and those with $\epsilon = \epsilon_{\min.}$ when $\lambda \geqslant \mu$. Such states belong to a convention just opposite from that chosen in § 4 and will be denoted by $\tilde{\chi}$ with expansion coefficients $\tilde{c}$. The states $\tilde{\chi}(\mu 0K)$ will transform like functions with μ quanta in the xy-plane. Thus we have the following relation between $\tilde{c}(\mu 0K, L)$ and the $\tilde{c}(\mu - 10K', L')$, which describes the coupling of the last quantum,

$$\begin{aligned} \tilde{c}^2(\mu 0K, L) &= \sum_{L'} \left\{ \sum_{\pm} \tilde{c}(\mu - 10K', L') \left(\frac{\mu-1}{2}\,\frac{1}{2}\,\frac{K'}{2} \pm \frac{1}{2} \middle| \frac{\mu}{2}\,\frac{K}{2} \right) (L'1\,K' \pm 1 \mid LK) \right\}^2 \\ &= \sum_{L'} \left\{ \sum_{\pm} \tilde{c}(\mu - 10K \mp 1, L') \left(\frac{\mu \pm K}{2\mu} \right)^{\frac{1}{2}} (L'1\,K \pm 1 \mid LK) \right\}^2. \end{aligned} \quad (38)$$

We have used the formalism of § 2 here to describe the construction of states with definite $\Lambda = \frac{1}{2}\mu$. From (38) and the trivial relation $c(000L) = \delta(L, 0)$ we may calculate all $\tilde{c}(\mu 0K, L)$. Using (33) for these redundant states,

$$\tilde{c}(0\mu K, L) = \tilde{c}(\mu 0K, L),$$

we then know also the values of $\tilde{c}(0\mu K, L)$.

Finally, for states with $\epsilon = \epsilon_{\max.}$, we have the relation

$$c^2(\lambda+1\mu\epsilon_{\max.}K,L) = \sum_{L'}\{c(\lambda\mu\epsilon_{\max.}K,L')\,(L'1K0\,|\,LK)\}^2 \tag{39}$$

describing the coupling of the last quantum into the z-direction. Knowing $\tilde{c}(0\mu K,L)$ we may use (39) to deduce all $c(\lambda\mu\epsilon_{\max.}K,L)$. When $\lambda \geqslant \mu$ these coefficients refer to the chosen states and are therefore the ones required. From these and (33) we deduce the coefficients for the chosen states when $\lambda < \mu$.

Coefficients calculated in this way are given in table 3 for $\lambda+2\mu \leqslant 8$.

TABLE 3. THE EXPANSION, OR NORMALIZATION, COEFFICIENTS $c^2(\lambda\mu K, L)$

$(\lambda\mu)$	K \ L	0 S	1 P	2 D	3 F	4 G	5 H	6 I	7 K	8 L
(00)	0	1	—	—	—	—	—	—	—	—
(10)	0	—	1	—	—	—	—	—	—	—
(20)	0	1/3	—	2/3	—	—	—	—	—	—
(30)	0	—	3/5	—	2/5	—	—	—	—	—
(40)	0	1/5	—	4/7	—	8/35	—	—	—	—
(50)	0	—	3/7	—	4/9	—	8/63	—	—	—
(60)	0	1/7	—	10/21	—	24/77	—	16/231	—	—
(70)	0	—	1/3	—	14/33	—	8/39	—	48/1287	—
(80)	0	1/9	—	40/99	—	48/143	—	64/495	—	128/6435
(11)	1	—	1/2	1/2	—	—	—	—	—	—
(21)	1	—	2/5	1/3	4/15	—	—	—	—	—
(31)	1	—	3/10	5/14	1/5	1/7	—	—	—	—
(41)	1	—	9/35	2/7	4/15	4/35	8/105	—	—	—
(51)	1	—	3/14	5/18	2/9	2/11	4/63	4/99	—	—
(61)	1	—	4/21	5/21	8/33	12/77	32/273	8/231	64/3003	—
(22)	0	4/15	—	13/21	—	4/35	—	—	—	—
(22)	2	—	—	11/21	1/3	1/7	—	—	—	—
(32)	0	—	18/35	—	19/45	—	4/63	—	—	—
(32)	2	—	—	3/7	1/3	6/35	1/15	—	—	—
(42)	0	6/35	—	11/21	—	104/385	—	8/231	—	—
(42)	2	—	—	23/63	14/45	78/385	4/45	16/495	—	—

6. QUADRUPOLE MOMENTS

Having determined the coefficients involved in the integral form (21) we are now in a position to study the properties of our U_3 wave functions revealed by this form. Because of the similarity between the quadrupole moment operator and the group operators Q_q we shall first investigate the quadrupole moments and electric quadrupole matrix elements in the states $\psi([f]\,(\lambda\mu)\,KLM)$.

The quadrupole moment operator is defined (see Blatt & Weisskopf 1952) as a second degree tensor operator,

$$O_q^{(2)} = e\sum_p r_p^2\,Y_q^{(2)}(p) = \frac{e}{2}\sum_i (1-\tau_z(i))\,r_i^2\,Y_q^{(2)}(i), \tag{40}$$

as a sum over protons p, or over all particles i if we use the isotopic spin formalism. The quadrupole moment is then the expectation value of $\left(\frac{16\pi}{5}\right)^{\frac{1}{2}} O_0^{(2)}$ in the magnetic

substate with $M = J$. The lifetime for an E2 transition of energy E from a level J' to a level J is given by $\tau = \hbar/\Gamma$, where

$$\Gamma = \frac{4\pi E^5}{75\hbar^5 c^5} \frac{(J' \| \mathbf{O}^{(2)} \| J)}{(2J'+1)}$$

and the amplitude matrix element is defined by

$$(J'M' \mid O_q^{(2)} \mid JM) = (J2Mq \mid J'M')\,(J' \| \mathbf{O}^{(2)} \| J)/(2J'+1)^{\frac{1}{2}}. \tag{41}$$

Although the group operators Q_q defined in I (7) involve the particle momenta, the symmetry between the momenta and co-ordinates in an oscillator well means that, *within a given oscillator shell*,

$$Q_q \equiv \left(\frac{16\pi}{5}\right)^{\frac{1}{2}} \sum_i r_i^2\, Y_q^{(2)}(i)/b^2.$$

Thus $\mathbf{Q}$ is equivalent to a quadrupole mass moment operator within a shell, differing from (40) in that it is summed over all particles and not just protons. In a nucleus with $T = 0$, the term involving $\tau_z(i)$ in (40) gives zero contribution so that in this case,

$$O_q^{(2)}(T=0) \equiv \frac{eb^2}{8}\left(\frac{5}{\pi}\right)^{\frac{1}{2}} Q_q.$$

In general, it is not simple to deal with the isotopic spin dependence of (40) so we shall first study the matrix elements of Q_q. This will at least have direct physical meaning for the $T = 0$ nuclei.

Since Q_q is an operator of the group U_3 its matrix elements between states of different representations $(\lambda\mu)$ of U_3 must be zero. Furthermore, its matrix elements between states of a given representation will depend only on the quantum numbers $(\lambda\mu)$ labelling that representation and the quantum numbers KLM labelling the states of that representation. They will not depend on the label $[f]$ or on the intrinsic or isotopic spin quantum numbers. Because Q_q is a symmetric operator in the particles it cannot couple states belonging to different partitions $[f]$. Hence, in this section we omit the labels $[f]\,(\lambda\mu)$ from ψ, χ and c for brevity. We shall see, however, that Q_q can couple states belonging to different bands of the same representation, i.e. having different values of K, although this will be small when λ is large.

Because of the well-known relation (41) it is necessary to study the matrix elements of only one component of Q_q. We therefore study the operation of Q_0 on a general state $\psi([f]\,(\lambda\mu)\,KLM)$ of the U_3 classification for which we use the abbreviated notation $\psi(KLM)$. From (21),

$$Q_0\psi(KLM) = \frac{(2L+1)}{c(K,L)} \int Q_0 \chi_\Omega(K)\, \mathscr{D}^L_{MK}(\Omega)\, \mathrm{d}\Omega. \tag{42}$$

Now, transform Q_0 into the oriented frame in the usual way,

$$Q_0 = \sum_q Q_q(\Omega)\, \mathscr{D}^2_{0q}(\Omega),$$

where $Q_q(\Omega)$ is now referred to the oriented frame. If we substitute this expression for Q_0 into (42) and combine the two $\mathscr{D}$-functions of Ω we find

$$Q_0\psi(KLM) = \frac{(2L+1)}{c(K,L)} \sum_q \sum_{L'} (L2M0 \mid L'M) \times (L2Kq \mid L'K+q) \int Q_q(\Omega)\, \chi_\Omega(K)\, \mathscr{D}^{L'}_{MK+q}(\Omega)\, \mathrm{d}\Omega. \tag{43}$$

This has taken the problem into the intrinsic frame. We now have to find the result of the operations $Q_q(\Omega)\,\chi_\Omega(K)$ for all q (which is the same problem as studying $Q_q\chi(K)$) rather than the operation $Q_0\psi(KLM)$. However, we know from the definition of the intrinsic functions that

$$Q_0\chi(K) = \epsilon\chi(K)$$

and, by our choice of χ for use in (21), the value of ϵ involved is $(2\lambda+\mu)$ or $(-\lambda-2\mu)$ according as $\lambda \geqslant \mu$ or $\lambda < \mu$. In the same way, the effect of $Q_{\pm 2}$ on $\chi(K)$ is given by (9) where $2\Lambda = \min(\lambda, \mu)$. For the remaining two operators $Q_{\pm 1}$ we make use of (29), which shows the equivalences

$$Q_{\pm 1} \equiv \mp\sqrt{3}\, L_\pm \quad \text{for} \quad \lambda \geqslant \mu$$

and

$$Q_\pm \equiv \pm\sqrt{3}\, L_\pm \quad \text{for} \quad \lambda < \mu$$

when the operators act on the chosen intrinsic states. The effect of $L_{\pm 1}$ acting on $\chi_\Omega(K)$ may be found by expanding χ into states of definite angular momentum as in (24). Using these results, (43) reduces to

$$\begin{aligned} Q_0\psi(KLM) = \sum_{L'} \frac{(2L+1)}{c(K,L)} (L2M0 \mid L'M) \\ \times \Bigg[(L2K0 \mid L'K) \begin{Bmatrix} \mu+2\lambda \\ -\lambda-2\mu \end{Bmatrix} \frac{c(K,L')}{(2L'+1)} \psi(KL'M) + \sum_{\pm} (L2K \pm 2 \mid L'K \pm 2) \\ \times \left(\frac{3(2\Lambda \mp K)(2\Lambda \pm K + 2)}{2} \right)^{\frac{1}{2}} \frac{c(K \pm 2, L')}{(2L'+1)} \psi(K \pm 2L'M) \\ + \begin{Bmatrix} - \\ + \end{Bmatrix} (3L'(L'+1))^{\frac{1}{2}} \frac{c(KL')}{(2L'+1)} \psi(KL'M) \\ \times \sum_{\pm} \pm (L2K \pm 1 \mid L'K \pm 1)\, (L'1\, K \pm 1 \mid L'K \pm 1) \Bigg]. \end{aligned} \tag{44}$$

Here, the upper value in the curly bracket applies to the case $\lambda \geqslant \mu$ and the lower to the case $\lambda < \mu$. As explained above, $\Lambda = \frac{1}{2}\min(\lambda, \mu)$. The summation in the third term of (44) may be carried out to give finally,

$$\begin{aligned} Q_0\psi(KLM) = \sum_{L'} \left(\frac{2L+1}{2L'+1} \right) \frac{(L2M0 \mid L'M)}{c(K,L)} \Bigg[(L2K0 \mid L'K)\, c(K,L') \\ \times \begin{Bmatrix} \mu+2\lambda+\frac{1}{2}(L'(L'+1)+6-L(L+1)) \\ -\lambda-2\mu-\frac{1}{2}(L'(L'+1)+6-L(L+1)) \end{Bmatrix} \psi(KL'M) + \sum_{\pm} (L2K \pm 2 \mid L'K \pm 2) \\ \times \left(\frac{3(2\Lambda \mp K)(2\Lambda \pm K+2)}{2} \right)^{\frac{1}{2}} c(K \pm 2, L')\, \psi(K \pm 2L'M) \Bigg]. \end{aligned} \tag{45}$$

If $K<2$, then the second term of (45) involves a function ψ with negative K. Although $\psi(KLM)$ and $\psi(-KLM)$ are essentially the same function their phases may differ. However, we can define $\psi(-KLM)$ to be identical with $\psi(KLM)$ if the phase of the expansion coefficient $c(-K,L)$ is chosen appropriately. Although we have taken $c(K,L)$ positive for $k\geqslant 0$ we are free to introduce a phase if $k<0$. Using (9) it can be shown that the choice

$$c(-K,L) = (-1)^{L+\lambda+\mu}\, c(K,L)$$

ensures that $\psi(KLM)\equiv\psi(-KLM)$.

If the $\psi(KLM)$ were orthogonal for different values of K we could say that the first term of (45) gives the matrix element of Q_0 diagonal in K, while the second term gives the matrix element coupling K to $K\pm 2$. The non-orthogonality of states of the same representation distinguished by K, which was pointed out in § 4, prevents such a simple interpretation from being exact. However, the departure from orthogonality arises only when $\min(\lambda,\mu)\geqslant 2$ and is then appreciable only near the top of a band.

In the special case $K=1$, both terms of (45) contribute to the diagonal matrix element because $|K-2| = |K|$ for $K=1$.

To understand the significance of (45) let us look at some special cases. When $\mu=0$ (a case of some physical interest, see I, table 3), (45) simplifies considerably to give for the amplitude matrix of $\mathbf{Q}$,

$$(L' \,\|\, \mathbf{Q} \,\|\, L) = \frac{(2L+1)}{(2L'+1)^{\frac{1}{2}}}\frac{c(0L')}{c(0L)}(L200\,|\,L'0)\{2\lambda+3+\tfrac{1}{2}L'(L'+1)-\tfrac{1}{2}L(L+1)\}. \quad (46)$$

For the diagonal element, $L=L'$, the expansion coefficients cancel out to give the very simple result,

$$(L \,\|\, \mathbf{Q} \,\|\, L) = (2L+1)^{\frac{1}{2}}(L200\,|\,L0)\,(2\lambda+3),$$

i.e.
$$(LM=L\,|\,Q_0\,|\,LM=L) = -\left(\frac{L}{2L+3}\right)(2\lambda+3). \quad (47)$$

This result is identical in form to that obtained for the quadrupole moment of a state with $K=0$ in the strong coupling version of the collective model. (See Alder *et al.* 1957.) The constant $2\lambda+3$ takes the place of the intrinsic quadrupole moment of the collective model. The non-diagonal matrix elements of $\mathbf{Q}$ from (46) are not of quite the same form as those of the collective model. However, when $\lambda\gg L$, i.e. when the state of interest is not near the top of the band, the difference becomes small and vanishes in the limit $\lambda\to\infty$. This limit is approached as the number of particles is increased or the complexity of the shell, measured by N, is increased. Even with a few particles in the $N=2$ shell we find values quite close to the limiting ones. In this limit, the second term of (45) disappears, reducing that general expression to

$$(L' \,\|\, \mathbf{Q} \,\|\, L) = 2\lambda(2L+1)^{\frac{1}{2}}(L2K0\,|\,L'K),$$

giving the moment as

$$(LL \mid Q_0 \mid LL) = \frac{2\lambda(3K^2 - L(L+1))}{(L+1)(2L+3)}. \tag{48}$$

These are exactly the rotational forms for a K-band.

The examples quoted in I indicated that the states with highest values for λ were lowest in the spectrum when the usual short range two-body interaction was included. Thus, with 1, 2, 3 or 4 particles in the N-shell the lowest level will belong to representations $(N, 0)$, $(2N, 0)$, $(3N, 0)$ and $(4N, 0)$, respectively (see I, table 1, for example). Since (48) shows that, in this limit, the quadrupole (mass) moment is proportional to λ, it follows that as the first four particles are added to a closed shell, the moment increases as the number of particles. This is a very different result from that obtained in j-j coupling without configuration mixing, where the quadrupole moments are generally *less* than the single-particle value.

We stress that Q_0 is the quadrupole mass moment rather than the electric quadrupole moment. Calculation of the latter would involve the fractional parentage coefficients (f.p.c.). However, we may use the formalism of this paper to simplify the fractional parentage method considerably. We can derive f.p.c. for the functions ψ in terms of those for the intrinsic states χ, which are much simpler. This problem will be discussed in a later paper.

As we shall discuss in § 7, the quadrupole moment calculated in this section is only that of the particles outside the closed shell and does not take into account any excitation or deformation of the closed shell core. If one uses a model of weak coupling to surface oscillations to describe such core excitation, the quadrupole moment is enhanced by a term proportional to the *mass* moment of the outside particles (Bohr & Mottelson 1953). This term will therefore be given immediately by the matrix elements of **Q** discussed above.

7. Conclusion

The classification of states according to irreducible representations of the group SU_3 led, in part I, to the association of levels into sets resembling rotational bands, cut-off at some upper value of L. Here, we have seen that this is not just a chance occurrence and that we are justified in talking of such sets of states as bands. The reason for this is that it is possible its express all states of such a set as integrals of the Hill–Wheeler type over a *single* intrinsic state. The quantum number K of the intrinsic state (measuring the component of angular momentum along the intrinsic z-axis), may therefore be associated with all the states of the set, introducing a new quantum number into the classification in I. This is most satisfactory, because we saw in I that, in a representation $(\lambda\mu)$ with $\min(\lambda, \mu) \geqslant 2$, there occurred more than one state for certain values of L. The introduction of the quantum number K serves to distinguish all such states. This quantum number K is a rather unusual one because it does not correspond to any subgroup of U_3 containing R_3 as might have been expected. It is consequently a somewhat inexact quantum number in several ways. First, states of the same λ, μ and L but different K are not orthogonal, although nearly so. Secondly, states near the cut-off of the band do not have a unique value of K.

Because of the inevitable redundancy in the intrinsic states in the Hill–Wheeler integral there is a certain amount of freedom in choosing the intrinsic states for use in this integral. However, we want to ensure that *all* states of a given representation $(\lambda\mu)$ are produced by such integrals over *as few* intrinsic states as possible. We do this by making the convention that if $\lambda \geqslant \mu$ we take those intrinsic states with maximum value of ϵ, while for $\lambda < \mu$ we take those with minimum value of ϵ. It is proved that this convention generates all states of the representation. Examples indicate that states with $\lambda > \mu$ are lowest in the first half of each oscillator shell, while those with $\lambda < \mu$ are lowest in the second half. Thus we shall expect prolate intrinsic shapes, $\langle Q_0 \rangle > 0$ in the first half-shell and oblate intrinsic shapes in the second half. In the middle of an oscillator shell, $A = 10, 28$, etc., we can equally well use an intrinsic state of either shape. Further discussion of this point must await a study of the energy matrix in a following paper.

It is found possible to derive an explicit formula for the quadrupole moment, apart from its isotopic spin dependence. We find that it resembles that obtained in the rotational model for a permanent deformation and that it reduces exactly to that form in the limit of large max (λ, μ). The way in which the quadrupole moment increases as particles are added brings out the collective nature of the wave functions and contrasts with the seniority classification (Racah 1949; Flowers 1952) in a pure l or j shell, which gives less than the single-particle value.

In shell-model calculations on the nuclei ^{16}N, ^{17}O and ^{19}F (Elliott & Flowers 1955, 1957) it has been found necessary to supplement the particle contribution to the quadrupole moments and E2 transition strengths by a weak-coupling collective contribution arising from excitation of the closed shell core. In all these cases, as well as in ^{19}Ne (a recently reported E2 lifetime, Lehmann 1957, private communication), the data fit very well with a single parameter, the coupling strength. This effect is roughly equivalent to putting an extra charge of $\frac{1}{2}e$ on both neutrons and protons outside the closed shell. The collective behaviour discussed in this paper has nothing to do with this weak coupling effect. In the series so far we have discussed only the motion of the particles outside closed shells, so that even with the wave functions described here we must still include the core polarization effect to explain the large observed moments. Another way of appreciating this fact is by noting that the single-particle intrinsic wave functions χ are still eigenvalues of a *spherical* oscillator potential as well as being eigenfunctions of Q_0. However, because the operators Q_0 and $(2z^2 - x^2 - y^2)$ are equivalent within an oscillator shell, we may look upon the χ as *approximations* to the wave functions of a deformed oscillator potential obtained by neglecting the mixing of different oscillator shells N, which such a potential would cause. In this approximation, the wave functions do not depend on the degree of deformation. We might include the effect of core polarization by improving these intrinsic wave functions, allowing them to deform. This would result in a complicated mixing of excited configurations in the final wave functions ψ. Such an approach is being studied.

Having seen that the wave functions classified in I have simple collective properties we must now try to understand why the wave functions resulting from the energy matrix of the two-body nuclear interaction are so close to those defined in I.

Examples given in I show this to be true at least in the cases studied. We shall discuss this problem in detail in part III of the series.

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CORRECTION TO PART I OF THIS SERIES, *Proc. Roy. Soc.* A, **245**, 136

In table 1 the entry 1 in the row $k = 3$, $[f] = [3]$ should be removed from the column (06) to the column (60).

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