Math with Sean

2020-07-11

Arthur Ryman

1997 Cayley Contest

#15

$$20^{50} = 2^{50} \times 10^{50}$$

5

$$\boxed{(xy)^n = x^n \cdot y^n}$$

$$(xy)^n = \underbrace{xy \cdot xy \cdot \cdots \cdot xy}_{n}$$

$$\boxed{x^n \cdot x^m = x^{n+m}}$$

$$= \underbrace{x \cdot x \cdot \cdots x}_{n} \cdot \underbrace{y \cdot y \cdots y}_{n}$$

$$= x^n \cdot y^n$$

$$50^{20} = (5 \times 10)^{20} = 5^{20} \cdot 10^{20}$$

$$20^{50} \cdot 50^{20} = 2^{50} \cdot 10^{50} \cdot 5^{20} \cdot 10^{20}$$

$$= 2^{50} \cdot 5^{20} \cdot 10^{50} \cdot 10^{20}$$

$$= 2^{30} \cdot 2^{20} \cdot 5^{20} \cdot 10^{70}$$

$$= 2^{30} \cdot 10^{20} \cdot 10^{70}$$

$$\boxed{= 2^{30} \cdot 10^{90}}$$

(C) = 90

$$7^5 \cdot 2^5 = (7 \cdot 2)^5$$
$$= 14^5$$

#18 $x > 0$, $y > 0$, $z > 0$ integers

$$\frac{30}{7} = x + \frac{1}{y + \frac{1}{z}}$$

$$0 < \frac{1}{z} \leq 1$$

$$\frac{30}{7} = 4\frac{2}{7}$$

$$y + \frac{1}{z} > 1$$

$$0 < \frac{1}{y + \frac{1}{z}} < 1$$

$$30 = 28 + 2$$

$$\frac{30}{7} = \frac{28+2}{7} = 4 + \frac{2}{7}$$

$$x = 4$$

$$\frac{1}{y + \frac{1}{z}} = \frac{2}{7} \qquad y + \frac{1}{z} = \frac{7}{2} = 3\frac{1}{2}$$

$$y =$$

$x, y, z > 0$ integers.

$x \geq 1, y \geq 1, z \geq 1$

$$0 < \frac{1}{z} \leq 1$$

$$y + \frac{1}{z} > 1$$

$$0 < \frac{1}{y + \frac{1}{z}} < 1$$

$$\boxed{\frac{30}{7} = 4\frac{2}{7}} = x + \frac{1}{y + \frac{1}{z}}$$

$$x = 4 \quad \boxed{\frac{1}{y + \frac{1}{z}} = \frac{2}{7}}$$

$$y + \frac{1}{z} = \frac{7}{2} = 3\frac{1}{2}$$

$$y = 3 \quad z = 2$$

$$x + y + z = 4 + 3 + 2 = 9$$

(B)

#19

$$x^2 y z^3 = 7^4$$

$$x y^2 = 7^5$$

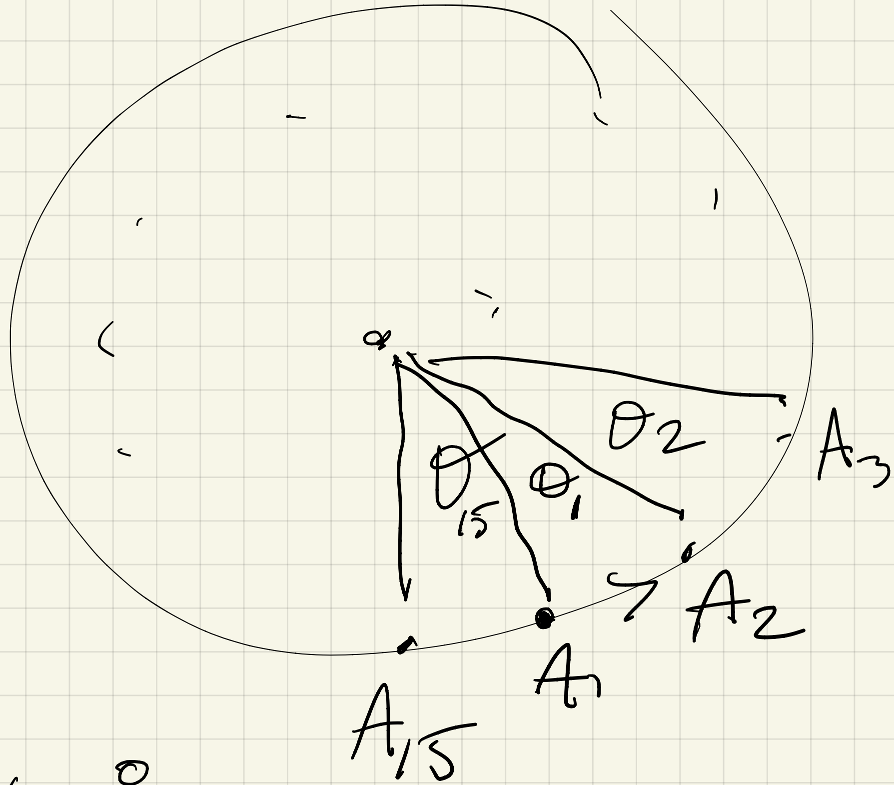
$$\underbrace{x^2 y z^3 \cdot x y^2} = 7^4 \cdot 7^5 = 7^9$$

$$= x^3 y^3 z^3 = 7^9 = (7^3)^3$$

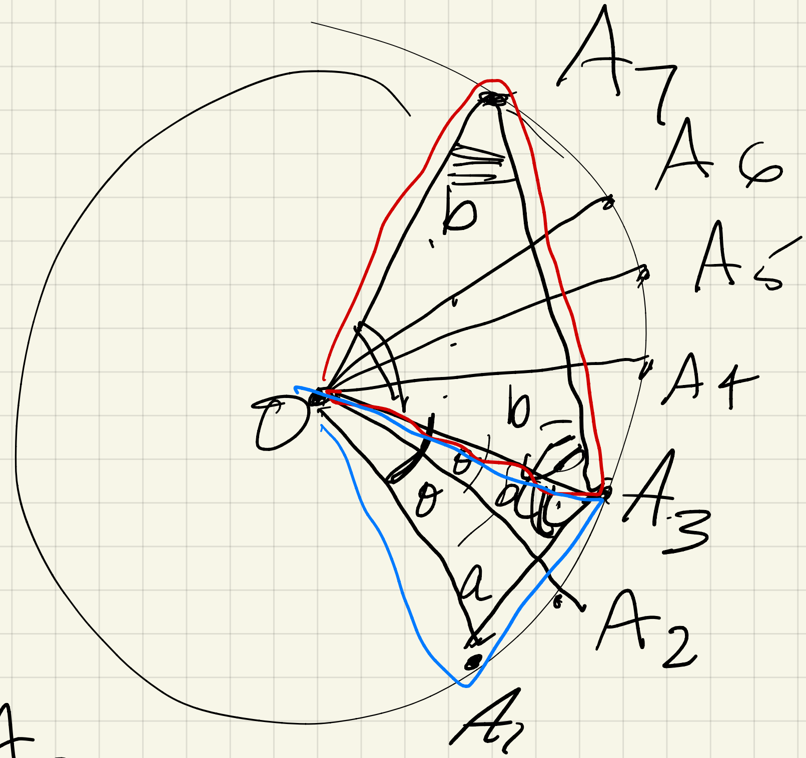
$$xyz = 7^3$$

(C)

20



$$\theta = \frac{360^\circ}{15}$$



$\triangle OA_1A_3$

$\angle A_1OA_3 = 2\theta$

$2\theta + 2a = 180$

$\triangle OA_3A_7$

$\angle A_3OA_7 = 4\theta$

$4\theta + 2b = 180$

$$\begin{cases} 2\theta + 2a = 180 \\ 4\theta + 2b = 180 \end{cases}$$

$$\theta = \frac{360^\circ}{15}$$

Solve for $\psi = a + b$

$$6\theta + 2\psi = 360$$

$$2\psi = 360 - 6\theta$$

$$\psi = 180 - 3\theta$$

$$= 180 - 3 \times \frac{360}{15}$$

$$= 180 - \frac{360}{5}$$

$$= 180 - 72$$

$$= 108$$

(D)

#21 a, b, c are positive integers.

$$\frac{\frac{a}{c} + \frac{a}{b} + 1}{\frac{b}{a} + \frac{b}{c} + 1} = 11$$

Find the number of (a, b, c) such that $\boxed{a + 2b + c \leq 40}$

$1 \leq a \quad 1 \leq b \quad 1 \leq c$

$$\frac{a}{c} + \frac{a}{b} + 1 = 11\left(\frac{b}{a} + \frac{b}{c} + 1\right)$$

$$\boxed{\frac{ab + ac + bc}{\cancel{c}b} = 11\left(\frac{bc + ab + ac}{a\cancel{c}}\right)}$$

$\frac{a}{c} = \frac{ab}{cb} \quad \frac{a}{b} = \frac{ac}{bc} \quad 1 = \frac{bc}{bc}$

$$\boxed{a(ab + ac + bc) = 11 \cdot b \cdot (bc + ab + ac)}$$

$a = 11b$

$$\boxed{a + 2b + c \le 40} \qquad 1 \le a,\ 1 \le b,\ 1 \le c$$

$$a = 11b$$

$$11b + 2b + c \le 40$$

$$13b + c \le 40$$

$$13b \le 40 - c$$

$$\Rightarrow b \le 3$$

$$b = 1, 2, 3$$

Case $b = 1$

$$c \le 40 - 13 \cdot b$$

$$c \le 40 - 13 \cdot 1$$

$$c \le 40 - 13$$

$$1 \le c \le 27 \quad \} \Rightarrow 27 \text{ sol's}$$

$b = 2$

$$c \le 40 - 26$$

$$c \le 14 \quad \Rightarrow 14 \text{ sol's}$$

$b = 3$

$$c \le 1 \quad \Rightarrow 1 \text{ sol}$$

$$\boxed{42}$$

(D)