Math with Sean

2020-07-25

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Cayley 1998

#17

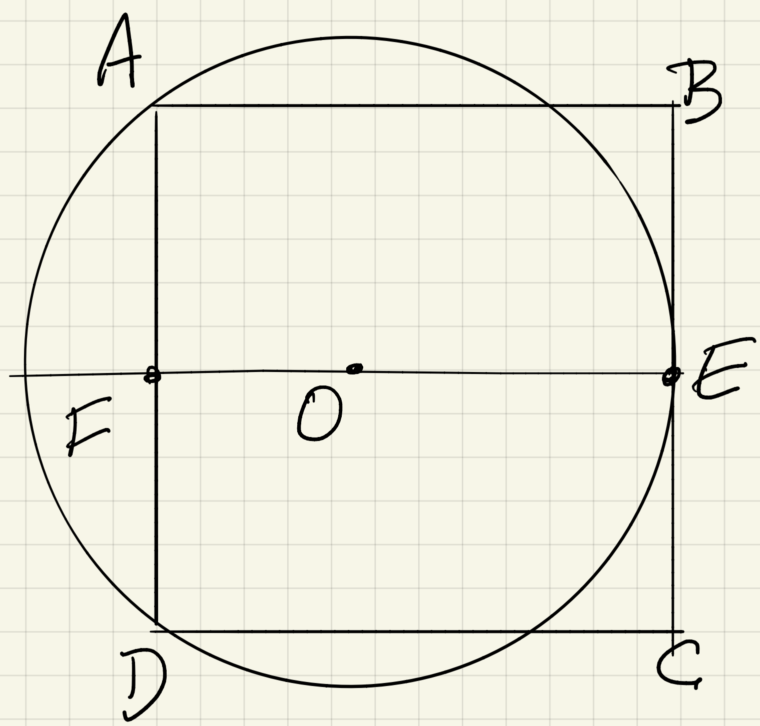
$N = (7^{p+4})(5^q)(2^3)$ is a perfect cube where p and q are positive integers. Find the smallest possible value of $p+q$.

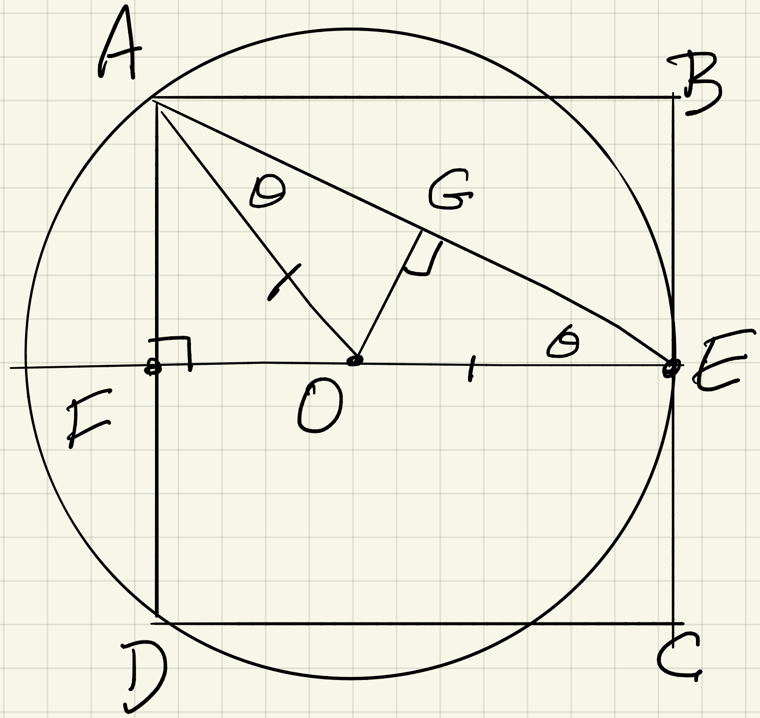
<u>Solution</u>: 7, 5, and 2 are primes so if N is a perfect cube then $p+4$ and q must be multiples of 3. Therefore, the smallest $p=2$ and $q=3$. So $p+q=5$

The answer is (A).

20. ABCD is a square of side 8. A circle through A and D is tangent to BC.
"Tangent" means that the circle just touches BC – it does not cross BC.
What is the radius of the circle?

Solution Label the point where the circle touches BC as E. Label the centre of the circle as O (for origin).
Draw the diameter from E through O and label where it crosses AB as F.





Draw lines AE and AO. Notice that $\Delta\, OAE$ is isoceles. Draw the perpendicular bisector from O to AE and label the point where it crosses AE as G.

Observe that $AF = 4$ and $FE = 8$ since the square has length 8. By Pythagoras' Theorem

$$AE^2 = AF^2 + FE^2 = 4^2 + 8^2 = 16 + 64 = 80$$

Therefore $AE = \sqrt{80} = 4\sqrt{5}$

But $\Delta\, AFE$ is similar to $\Delta\, OGE$, $GE = \frac{AE}{2} = 2\sqrt{5}$

So $\frac{AE}{FE} = \frac{OE}{GE} \Rightarrow \frac{4\sqrt{5}}{8} = \frac{OE}{2\sqrt{5}}$

$\Rightarrow$ radius $= OE = \frac{4\sqrt{5} \times 2\sqrt{5}}{8}$

$= 5$

The answer is (B) 5

21. $f(x) = ax^3 - 2x + c$

$f(1) = a - 2 + c = -5$

$f(4) = a4^3 - 2 \cdot 4 + c$

$= 64a - 8 + c = 52$

$f(4) - f(1) = 52 - (-5) = 57$

$= 64a - 8 + c$

$- (a - 2 + c)$

$= 63a - 8 + 2$

$= 63a - 6$

so $63a - 6 = 57$

$63a = 63$

$a = 1$

$f(1) = -5 = a - 2 + c$

$= 1 - 2 + c$

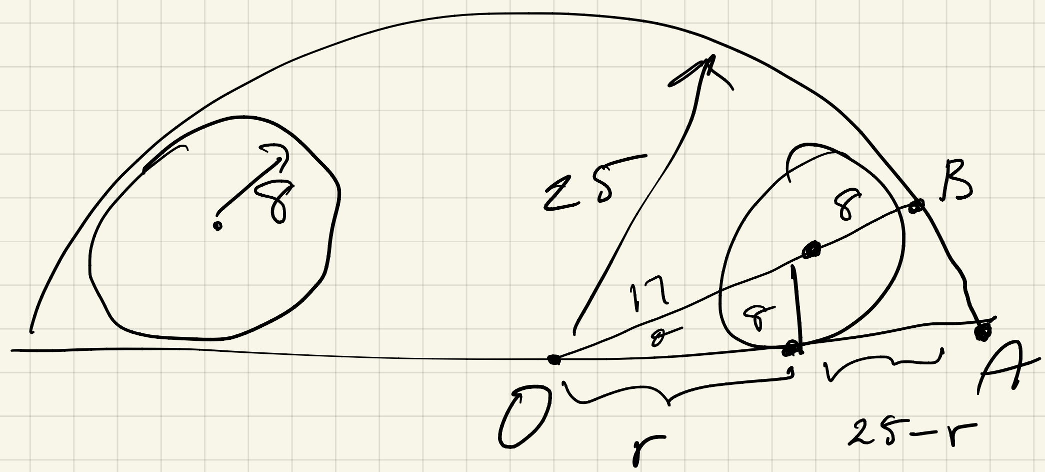
$= -1 + c$

$c = -4$

$f(x) = x^3 - 2x - 4$

(A) $f(2) = 8 - 4 - 4 = 0$

22.



17
17
119
17
289

$$r^2 + 8^2 = 17^2 = 289$$

$$r^2 = 289 - 64 = 225$$

$$r = \sqrt{225}$$
$$= 15$$

15
15
75
15
225

The length that cannot be touched $= 2(25-r)$

$$= 2(25-15)$$
$$= 2 \cdot 10$$
$$= 20 \quad (E)$$

23. a, b, c, N are unequal positive integers.

$$N = 5a + 3b + 5c$$

$$N = 4a + 5b + 4c$$

$(131) \le N \le 150$. Find $a+b+c$.

define $t = a+b+c$

$$N = 5t - 2b$$

$$N = 4t + b$$

$$N + 2N = 3N = 5t - 2b + 8t + 2b$$
$$= 13t$$

$3N = 13t$ so N is a multiple of 13

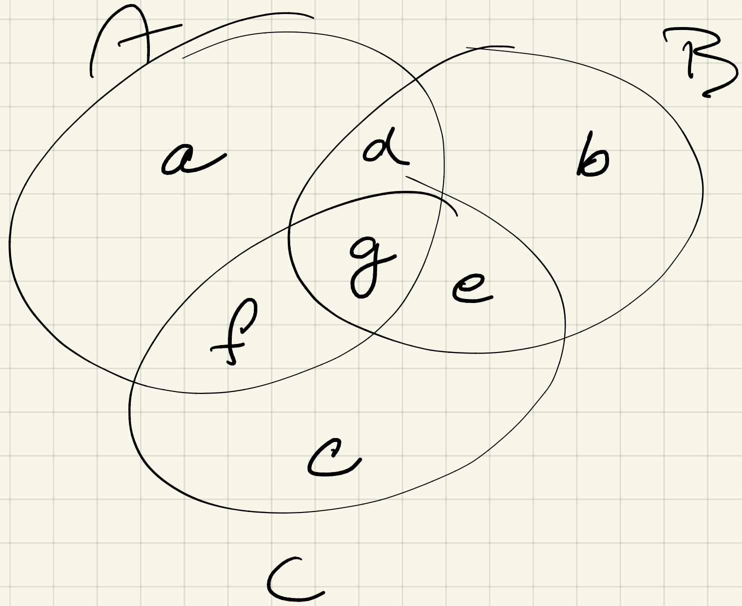
$130, 143, 156 \Rightarrow N = 143$
$= 13 \times 11$

t is a multiple of 3

$3 \times 13 \times 11 = 13t$ so $t = 33$

(D)

24.



$$A + B + C = 200$$

$$A = a + d + f + g$$

$$B = b + d + e + g$$

$$C = c + e + f + g$$

$$a + b + c + d + e + f + g = 140$$

$$d + e + f = 24$$

$$x_1 = a + b + c$$

$$x_2 = d + e + f = 24$$

$$x_3 = g$$

$$200 = a + b + c + 2(d + e + f) + 3g$$

$$= x_1 + 2x_2 + 3x_3 = x_1 + 3x_3 + 48$$

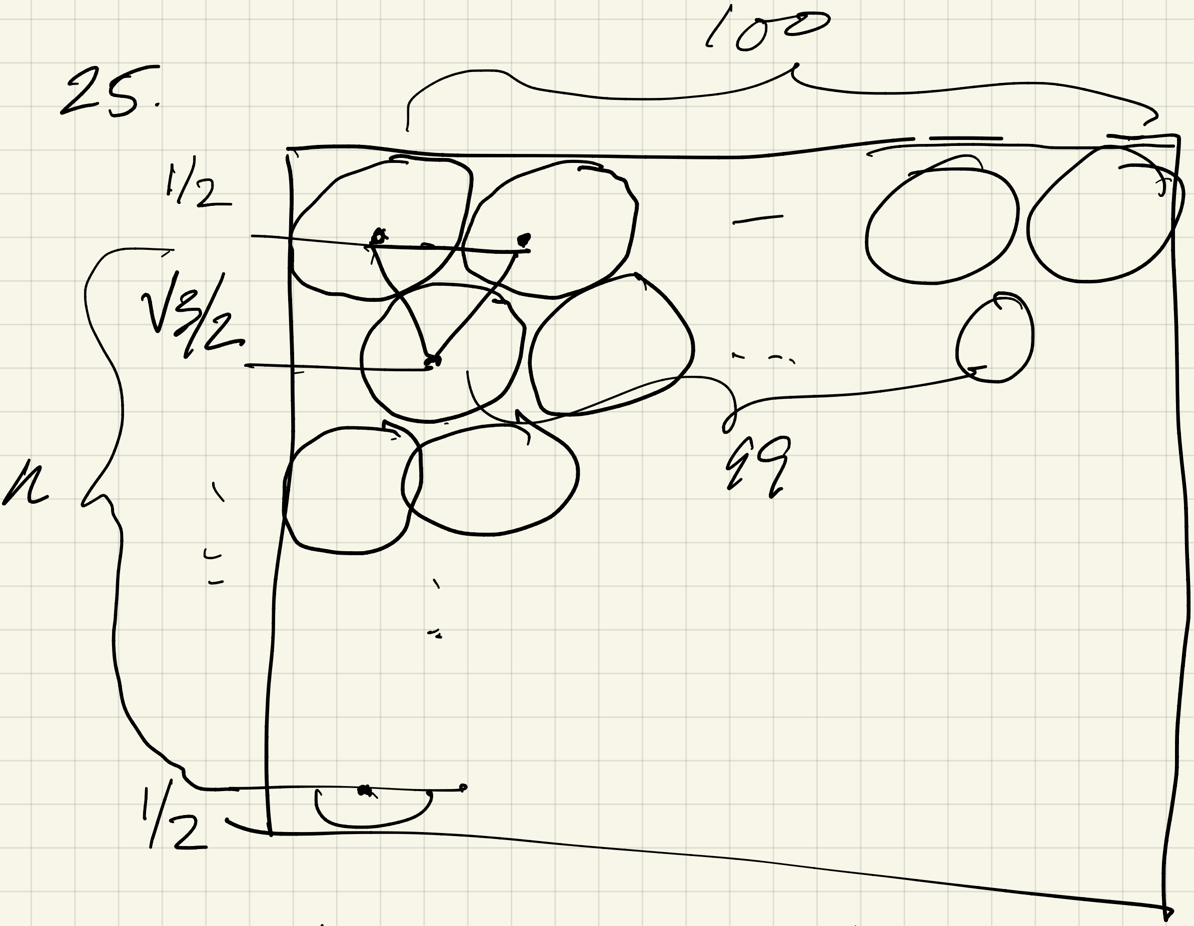
$$140 = x_1 + x_2 + x_3 = x_1 + x_2 + 24$$

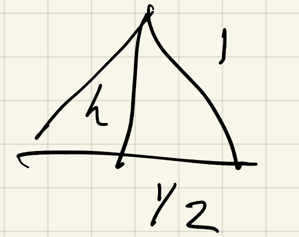
$$200 = a+b+c+2(d+e+f)+3g$$
$$= x_1 + 2x_2 + 3x_3 = x_1 + 3x_3 + 48$$
$$140 = x_1 + x_2 + x_3 = x_1 + x_2 + 24$$

$$x_1 + 3x_3 = 200 - 48 = 152$$
$$x_1 + x_3 = 140 - 24 = 116$$

$$2x_3 = 152 - 116 = 36$$
$$x_3 = 18 \quad (B)$$

25.





$$h^2 = 1^2 - \left(\frac{1}{2}\right)^2$$
$$= 1 - \frac{1}{4}$$
$$= \frac{3}{4}$$
$$h = \frac{\sqrt{3}}{2}$$

n circles take $x = 1 + (n-1)\frac{\sqrt{3}}{2}$ length.

$$1 + (n-1)\frac{\sqrt{3}}{2} \leq 100$$

$$(n-1)\frac{\sqrt{3}}{2} \leq 99$$

$$n-1 \leq \frac{99 \cdot 2}{\sqrt{3}} = 114.315\ldots$$

$$n-1 \leq 114$$

$n = 115$ an odd number

58 rows 1, 3, 5, ... ,115 have 100 circles

57 rows 2, 4, 6, ... 114 have 99 circles

$\#$ of circles $= 58 \times 100 + 57 \times 99$

$= 5800 + 5643$

$= 11443$

increase is 1443 (D)

$$\begin{array}{r} 99 \\ 57 \\ \hline 693 \\ 495 \\ \hline 5643 \end{array}$$