cayley-1997-problem-21

July 26, 2020

1 Cayley 1997 Problem 21

21. If $\dfrac{\left(\frac{a}{c}+\frac{a}{b}+1\right)}{\left(\frac{b}{a}+\frac{b}{c}+1\right)}=11$, where a, b, and c are positive integers, the number of different ordered triples (a,b,c) such that $a+2b+c\le 40$ is

(A) 33 **(B)** 37 **(C)** 40 **(D)** 42 **(E)** 45

The problem involves integers but divides them so that the result is a rational number. If we use Python `float` numbers then we'll get round-off errors. Instead, let's use SymPy which can do exact calculations involving rational numbers.

First, let's define some SymPy variables for the positive integers a, b, and c.

[1]:
```
from sympy import symbols

a, b, c = symbols('a:c', integer=True, positive=True)
```

[2]:
```
a, b, c
```

[2]:
```
(a, b, c)
```

Next, lets define the equation that constrains the variables.

[3]:
```
from sympy import Eq
eq21 = Eq((a/c + a/b + 1)/(b/a + b/c + 1), 11)
eq21
```

[3]:
$$\frac{\frac{a}{c}+\frac{a}{b}+1}{\frac{b}{c}+1+\frac{b}{a}}=11$$

[4]:
```
eq21.subs(a, 1).subs(b, 1).subs(c, 1)
```

[4]:
```
False
```

[5]:
```
def is_solution(va, vb, vc):
    if va + 2 * vb + vc <= 40:
```

```
        return eq21.subs(a, va).subs(b, vb).subs(c, vc)
    return False

is_solution(1, 1, 1)
```

[5]:
```
False
```

[6]:
```
def find_all_solutions():
    return [(va, vb, vc)
            for va in range(1, 40)
            for vb in range(1, 20)
            for vc in range(1, 40)
            if is_solution(va, vb, vc)]

solutions = find_all_solutions()
print('Number of solutions is', len(solutions))
```

```
Number of solutions is 42
```

The solutions are:

[7]:
```
for s in solutions:
    print(s)
```

```
(11, 1, 1)
(11, 1, 2)
(11, 1, 3)
(11, 1, 4)
(11, 1, 5)
(11, 1, 6)
(11, 1, 7)
(11, 1, 8)
(11, 1, 9)
(11, 1, 10)
(11, 1, 11)
(11, 1, 12)
(11, 1, 13)
(11, 1, 14)
(11, 1, 15)
(11, 1, 16)
(11, 1, 17)
(11, 1, 18)
(11, 1, 19)
(11, 1, 20)
(11, 1, 21)
(11, 1, 22)
(11, 1, 23)
(11, 1, 24)
(11, 1, 25)
```

```
(11, 1, 26)
(11, 1, 27)
(22, 2, 1)
(22, 2, 2)
(22, 2, 3)
(22, 2, 4)
(22, 2, 5)
(22, 2, 6)
(22, 2, 7)
(22, 2, 8)
(22, 2, 9)
(22, 2, 10)
(22, 2, 11)
(22, 2, 12)
(22, 2, 13)
(22, 2, 14)
(33, 3, 1)
```