Math With Sean

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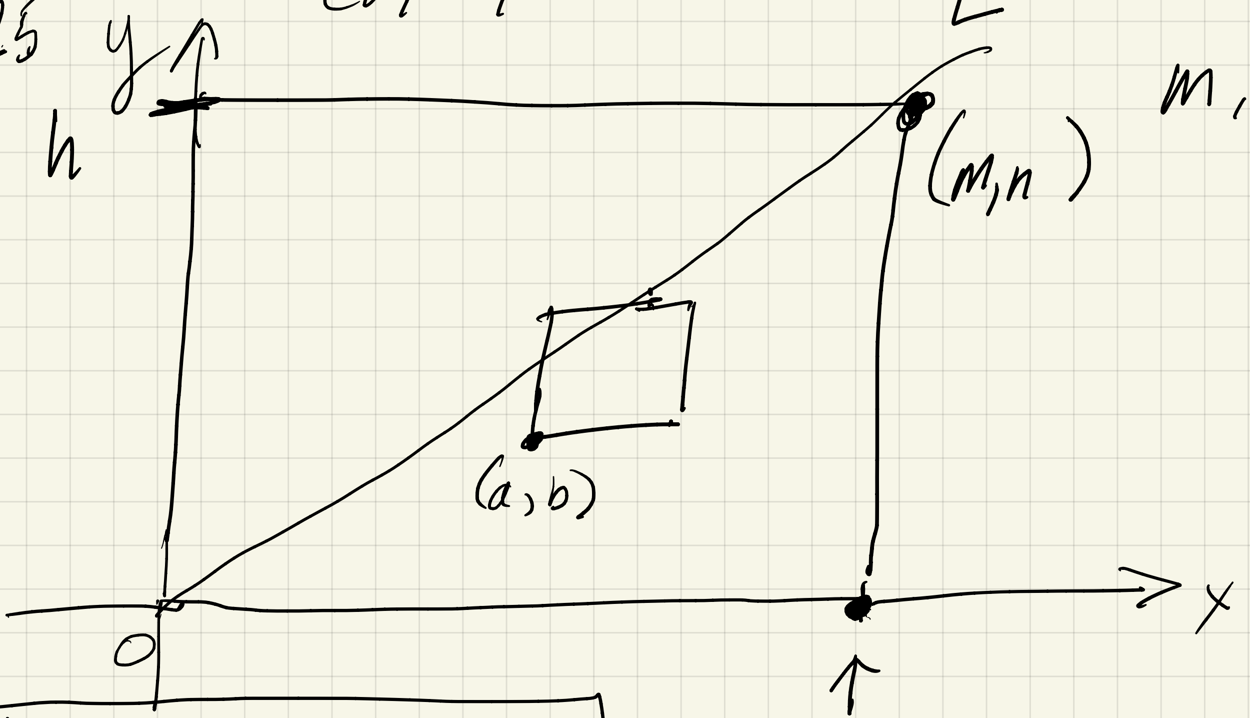
2021-09-18

#25 Cayley 2009



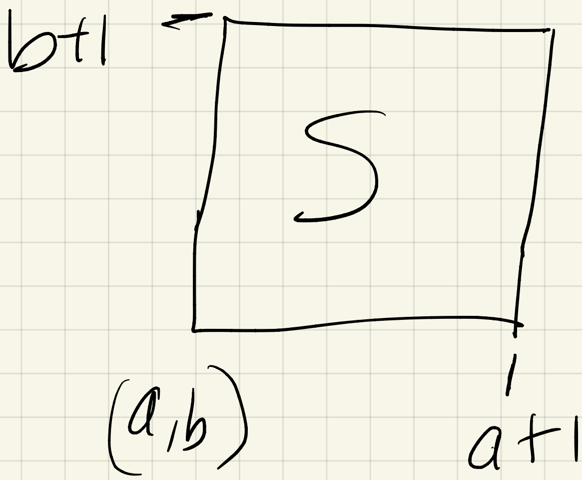
m, n

$$\boxed{L: \; y = \frac{n}{m} x}$$

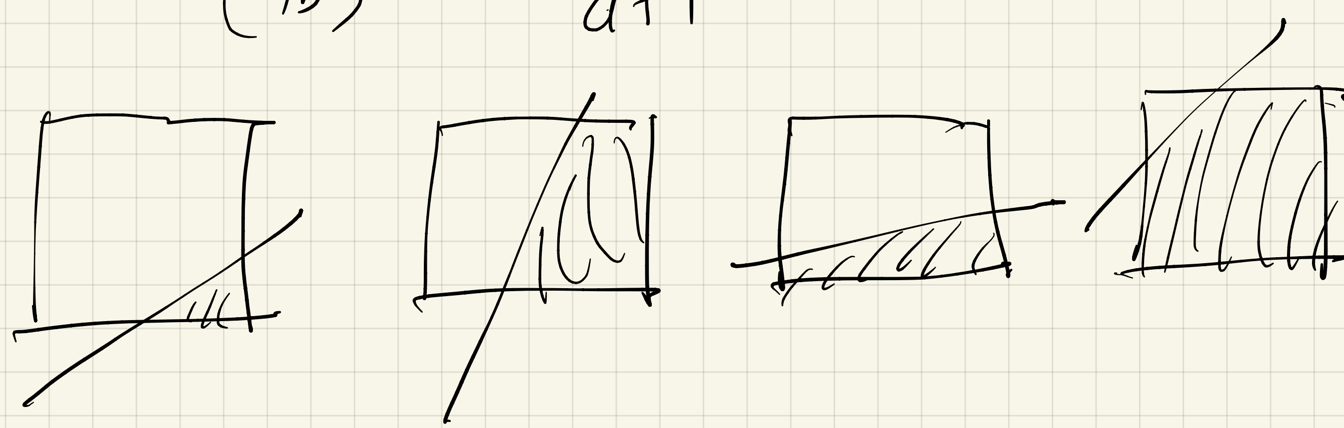
$0 < m$

$0 < n$

$$\boxed{\begin{array}{l} 0 \le a < m \\ 0 \le b < n \end{array}}$$



a, b integers



slope $s = \frac{n}{m}$

$2n < m < 3n$

$2 < 1/s < 3$

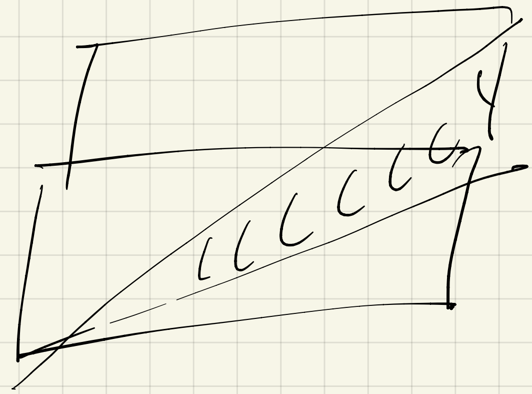
$$2 < \frac{1}{s} < 3$$

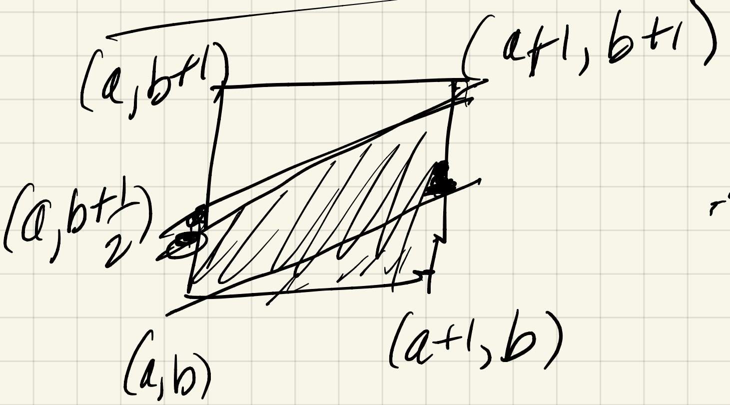
$2 < \frac{1}{s}$ $\qquad$ $\frac{1}{s} < 3$

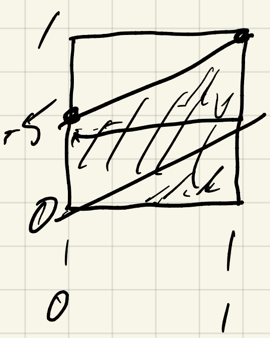
$2s < 1$ $\qquad$ $1 < 3s$

$s < \frac{1}{2}$ $\qquad$ $\frac{1}{3} < s$

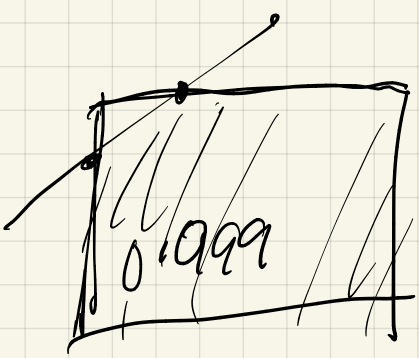
$$\frac{1}{3} < s < \frac{1}{2} < 1$$

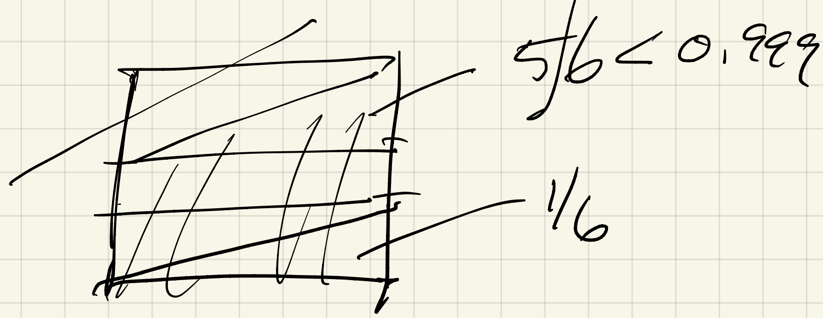




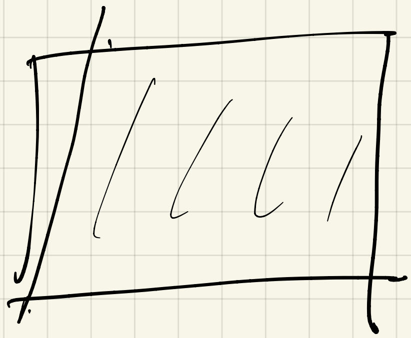


Shaded area
= 0.75
= 3/4





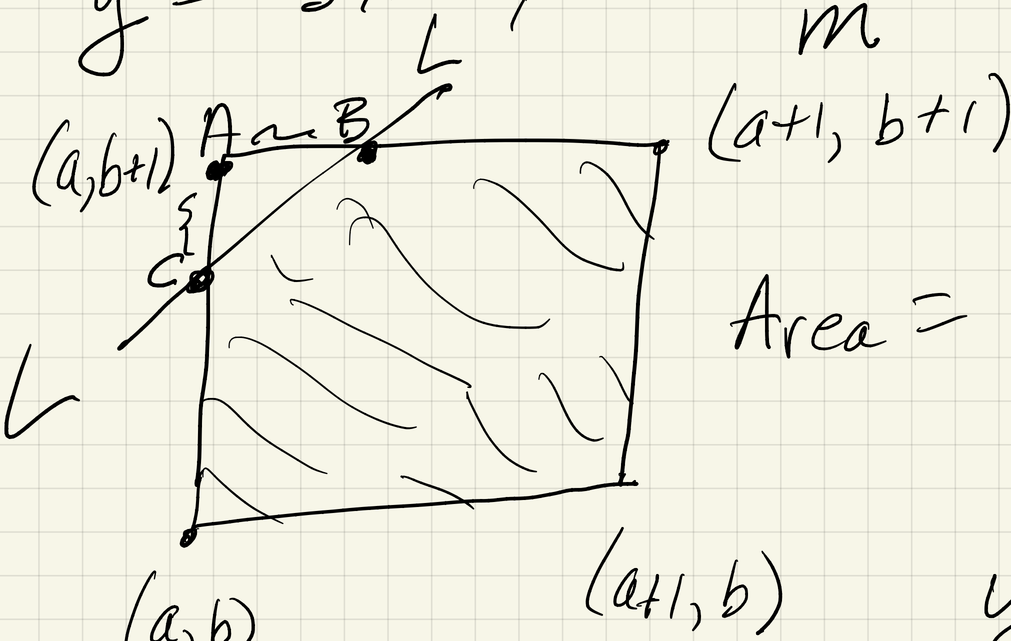
Only need to consider lines that go through the left & top of the unit square.



impossible!
because $s < 1/2$

Derive formula for shaded area when L goes through left and top of unit square whose lower left corner is at (a,b)

$L: \; y = sx \;, \quad s = \dfrac{n}{m}$



$\text{Area} = 1 - \triangle ABC$

$y = sx$

$$\left[\begin{array}{l} A = (a, b+1) \\ B = (x_1, b+1) \\ C = (a, y_1) \end{array}\right.$$

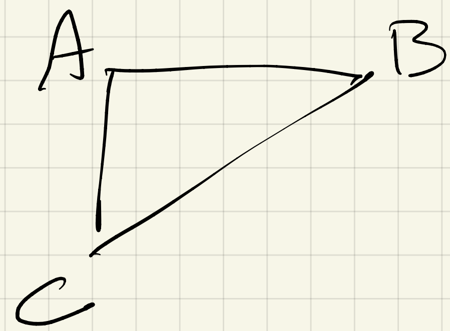
$b+1 = s x_1$

$y_1 = s a$

$AB = x_1 - a$

$AC = b+1-y_1$

$s = \frac{n}{m}$ $\quad y = sx$



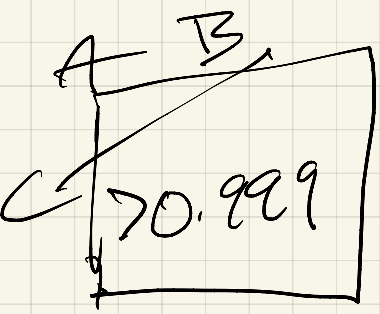
$AB = x_1 - a$

$b+1 = s x_1$ $\quad AC = b+1-y_1$

$y_1 = s a$

$$\Delta ABC = \frac{AB \times AC}{2}$$

$$= \frac{(x_1 - a)(b+1-y_1)}{2}$$



$$= \frac{\left(\frac{b+1}{s} - a\right)(b+1-sa)}{2}$$

$$= \frac{\left(\frac{b+1-sa}{s}\right)(b+1-sa)}{2}$$

$$\boxed{= \frac{(b+1-sa)^2}{2} < 0.001}$$

Solve

$$0 < \frac{(b+1-sa)^2}{a} < 0.001 \qquad s = \frac{n}{m}$$

Find mn such there exists a unit square at (a,b) that satisfies the above inequality.

$$\left.\begin{array}{l} 0 < 2n \leq m < 3n \\ 0 \leq a < m, \quad 0 \leq b < n \end{array}\right\} \text{integer}$$

$$(b+1-sa)^2 < 0.002$$

$$|b+1-sa| < \sqrt{0.002}$$

$$b+1 \approx sa \qquad sa = b+1 \pm \epsilon$$

$$s = \frac{n}{m} \quad 0 \leq a < m \qquad \epsilon < \sqrt{0.002}$$

$$sa = \frac{n}{m} \cdot a \quad 0 < sa < sm = n$$

$$0 < sa < n$$

Test (A) $496 \le mn \le 500$

$2n < m < 3n$

$2n^2 < mn < 3n^2$

$496 \le mn < 3n^2$

$496 < 3n^2$

$2n^2 < mn \le 500$

$2n^2 \le 500$

$n^2 \le 250$

$n \le \sqrt{250} =$

$n < \sqrt{250} = 15.8\ldots$

$n \le 15$

$496 < 3n^2$

$\frac{496}{3} < n^2$

$165.3 < n^2$

$13 \le n$

$\sqrt{165.3} < n$

$12.8 < n$

(A) $13 \le n \le 15$

$n = 13, 14, 15$

$2n < m < 3n$

$n = 13 \quad 26 < m < 39$

Try $n = 13 \quad m = 27$

Find a, b $\qquad s = \dfrac{n}{m} = \boxed{\dfrac{13}{27}}$

$$\boxed{|b+1 - sa| < \sqrt{0.002}}$$

$0 \le a < m \;, \quad 0 \le b < n$

$0 \le a < 27 \qquad 0 \le b < 13$

$$\left| b+1 - \frac{13}{27} \cdot a \right| < \sqrt{0.002}$$

$\dfrac{13}{27} \cdot a \quad \dfrac{13}{27} \cdot$