Math With Sean

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#24 $S(n)$ is the smallest number divisible by $1, 2, 3, \cdots, n$

Example: $S(5) = 60$

$$S(n) = LCM(1, 2, \cdots, n)$$

$$S(5) = LCM(1, 2, 3, 4, 5)$$

$$1 = 2^0\ 3^0\ 5^0$$

$$2 = 2^1\ 3^0\ 5^0$$

$$3 = 2^0\ 3^1\ 5^0$$

$$4 = 2^2\ 3^0\ 5^0$$

$$5 = 2^0\ 3^0\ 5^1$$

$$LCM = 2^2 \cdot 3^1 \cdot 5^1$$

$$= 4 \times 3 \times 5$$

$$= 12 \times 5$$

$$= 60$$

$S(n) = S(n+4)$

$1, 2, 3, \dots, n$ divide $S(n)$ (Def)

if $S(n) = S(n+4)$ then

$n+1, n+2, n+3, n+4$ also

divide $S(n)$

Can 5 divide $S(4)$? NO.

$\Rightarrow n+1, n+2, n+3, n+4$ are NOT prime

Look at primes ≤ 100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104						

$\boxed{C}$ $\boxed{11 \text{ solutions}}$

Look for n such that $n+1, n+2, n+3, n+4$ are NOT prime.

$S(23) = LCM(1, 2, 3, \ldots, 23)$

$2 \quad 4 \quad 8 \quad 16 \quad$ (32 too big.)

			$S(24)$	$S(25)$
2	$\to 2^4 = 16$	2^4		
3	$\to 3^2 = 9$	3^2		
5	$\to 5^1$	5^1		5^2
7	7^1	$\vdots$		
11	11^1			
13	13^1			
17	17^1			
19	19^1			
23	23^1			

$S(23) = S(24) \neq S(25)$

$\Rightarrow$ 23 is NOT a solution.

24 is NOT a solution

...

	~~31~~	32	33	34	35	36
2	4	5	5	5	5	5
3		3	✓	✓	✓	✓
5		2				
7		1				
11		1				
13		1				
17		1				
19		1				
23		1				
29		1				
31						

32 is a solution!

47 48 49 50 51 52

7 1 1 2

47, 48 are not solutions.

53 54 55 56 57 58

2 $32 \to 64$ ✓

3 $27 \to 81$ ✓

~~5 $25 \to 125$~~ ✓

~~7 $49 \to$ big.~~ ✓

~~11 $11 \to 121$~~ ✓

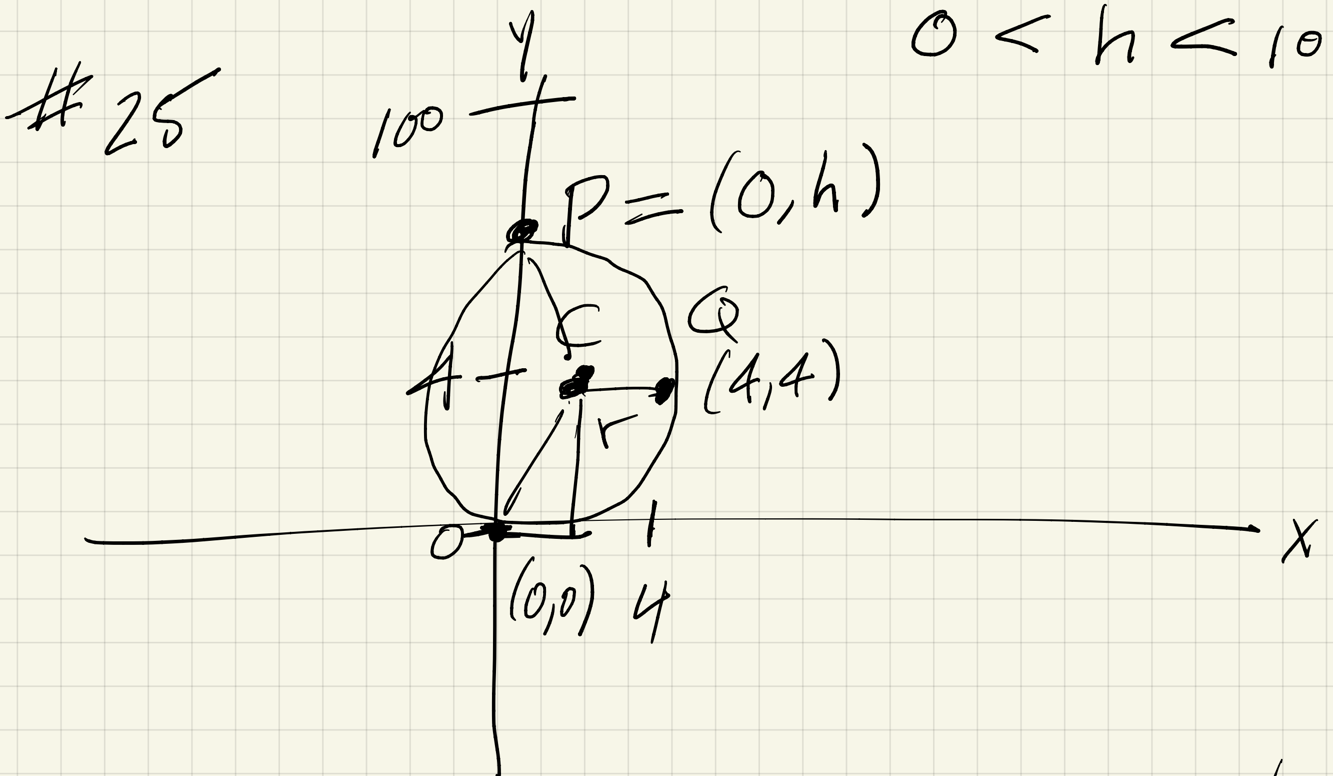
13 $13 \to 169$.

53 & 54

are solutions!

#25

$0 < h < 100$



$C = (x, y)$ centre of the circle

$O = (0, 0)$

$Q = (4, 4)$

$P = (0, h)$

$$\begin{cases} r^2 = x^2 + y^2 \\ r^2 = (x-4)^2 + (y-4)^2 \\ \underline{r}^2 = (x)^2 + (y-h)^2 \end{cases}$$

$$= \underline{x^2 + y^2} - 2yh + h^2$$

$$\underline{r}^2 = x^2 - 8x + 16 + y^2 - 8y + 16$$

$$= \underline{x^2 + y^2} - 8(x+y) + 32$$

$32 = 8(x+y)$

$$0 = -8(x+y) + 32$$

$4 = x+y$

$$r^2 = (x)^2 + (y-h)^2$$

$$= \underline{x^2 + y^2} - 2yh + h^2$$

$$h^2 = +2yh \qquad h > 0$$

$$h = 2y$$

$$4 = x + y$$

$$y = \frac{h}{2} \qquad x = 4 - y$$

$$= 4 - \frac{h}{2}$$

$$r^2 = x^2 + y^2$$

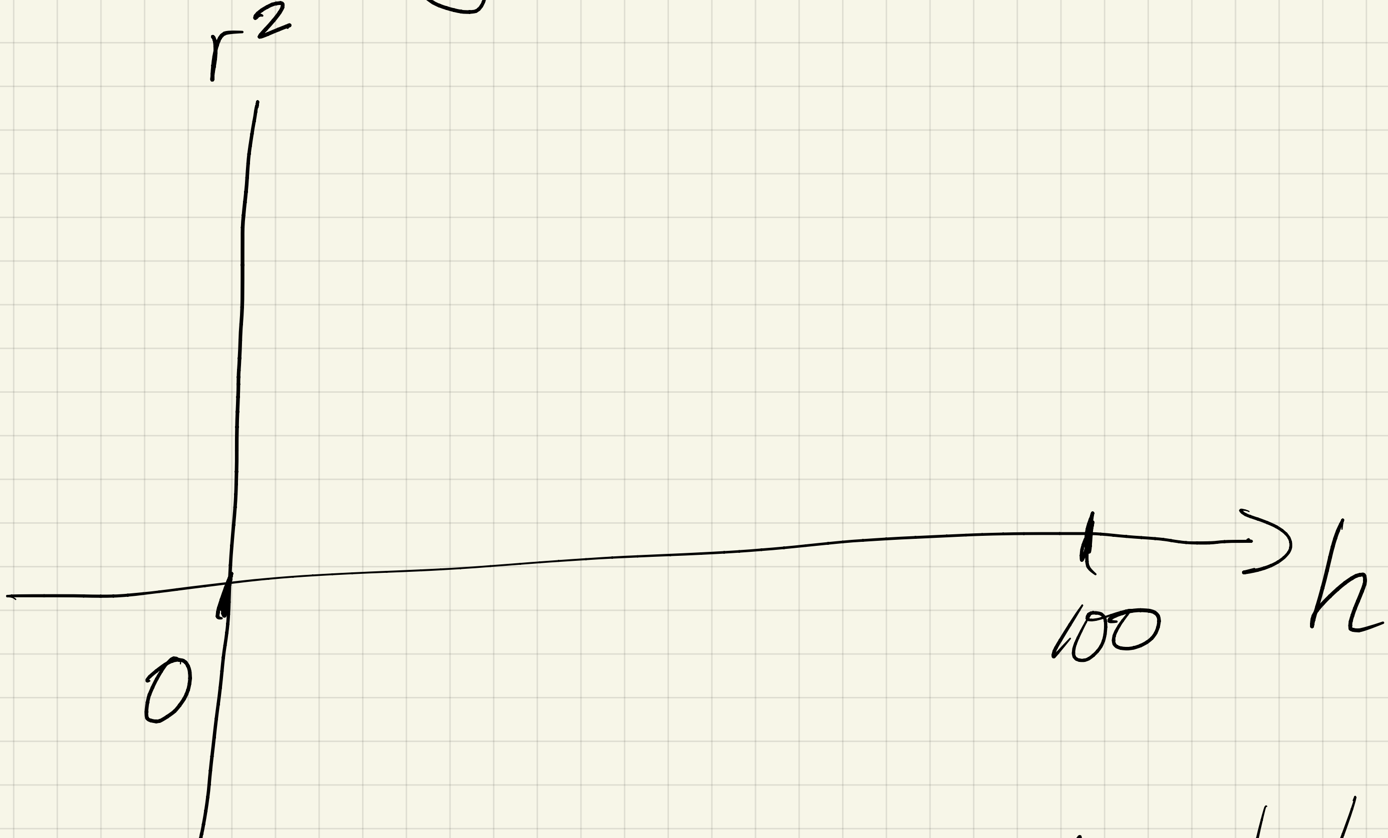
$$= \left(4 - \frac{h}{2}\right)^2 + \left(\frac{h}{2}\right)^2$$

$$= 16 - 4h + \frac{h^2}{4} + \frac{h^2}{4}$$

$$= 16 - 4h + \frac{h^2}{2}$$

$$r^2 = 16 - 4h + \frac{h^2}{2}$$ $0 < h < 100$

How many values of h make r an integer?



$$f(x) = ax^2 + bx + c$$ Parabola

$$y = \frac{h^2}{2} - 4h + 16$$

$$h = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

roots $y = 0$ on x-axis.

$$y = ax^2 + bx + c$$

roots $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{h^2}{2} - 4h + 16$$

$a = \frac{1}{2}$ $\quad b = -4 \quad c = 16$

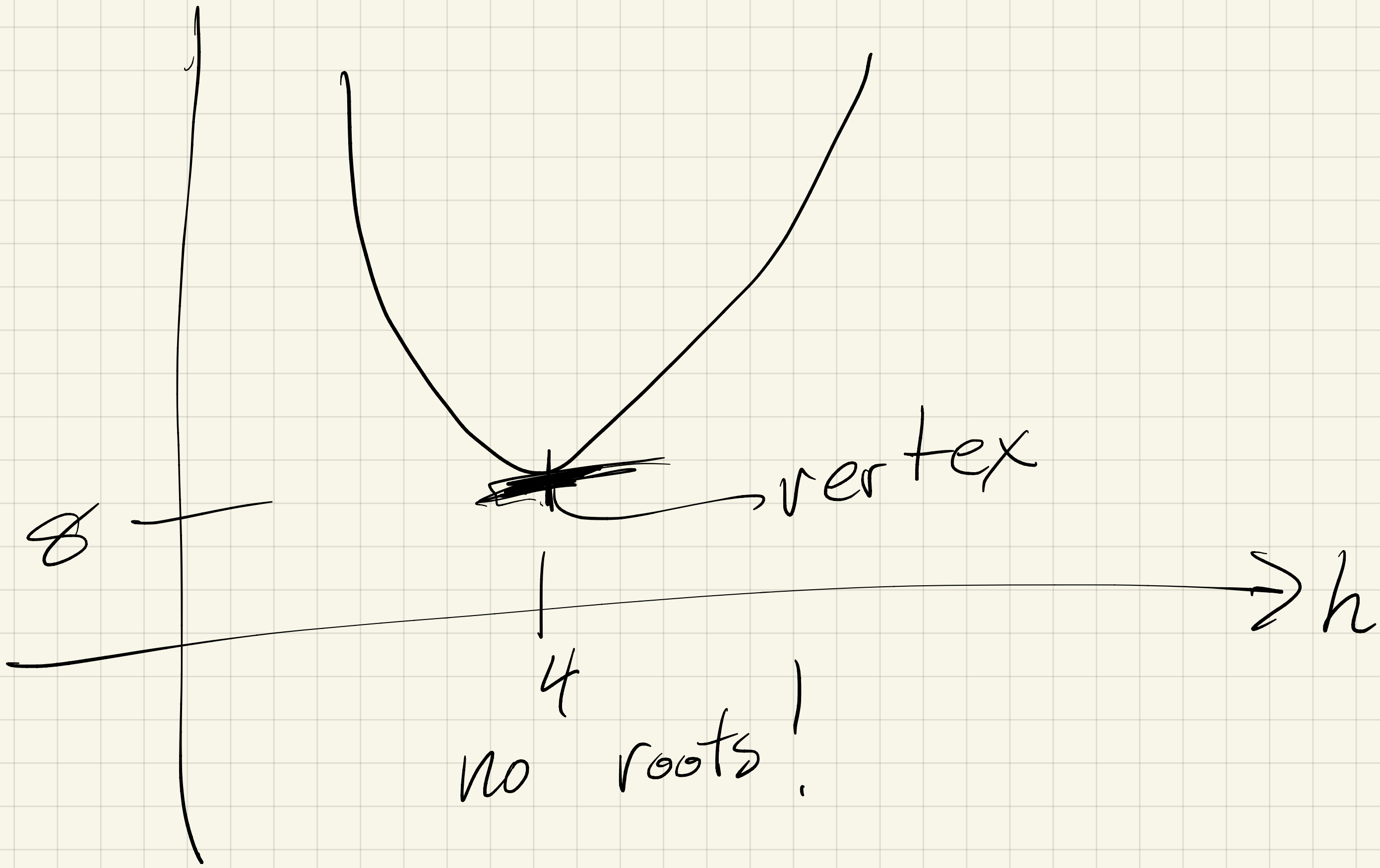
$$b^2 - 4ac = (-4)^2 - 4\left(\frac{1}{2}\right)(16)$$

$$= 16 - 2 \times 16$$

$$= -16$$

$\sqrt{-16}$ = imaginary number.

NOT a real number



$$y = ax^2 + bx + c$$

$$y' = 2ax + b \quad = 0 \text{ at vertex}$$

$$x = -\frac{b}{2a} \quad \text{at vertex}$$

$$a = \frac{1}{2} \qquad b = -4$$

$$x = -\frac{(-4)}{\left(2 \cdot \frac{1}{2}\right)}$$

$$= 4$$

at vertex $x = 4 = h$

$$r^2 = y = \frac{h^2}{2} - 4h + 16$$

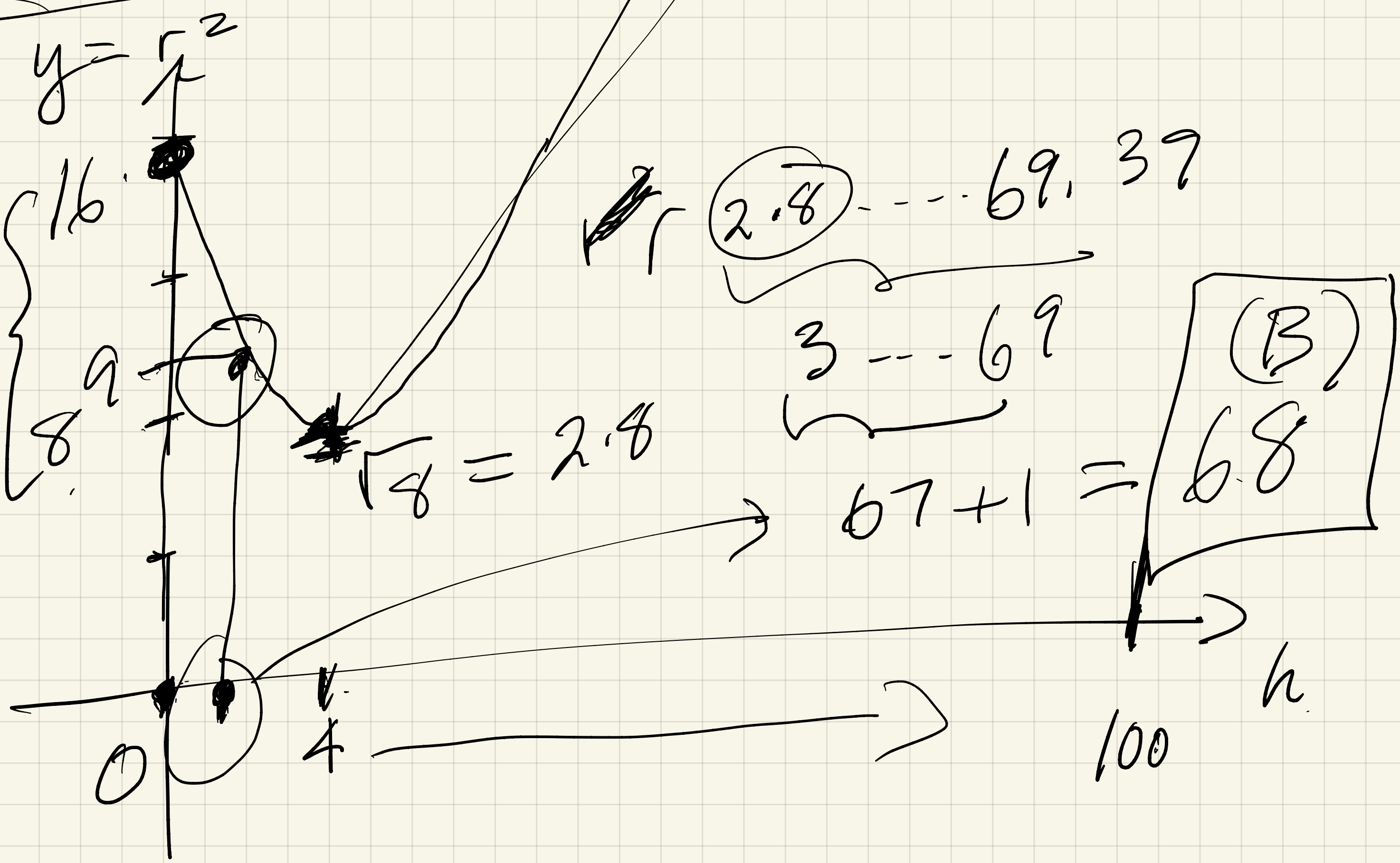
$$= \frac{4^2}{2} - 4 \cdot 4 + 16$$

$$= \frac{16}{2} - 16 + 16$$

$$= 8$$

$(h, y) = (4, 8)$ at vertex.

$r^2 = y = \frac{h^2}{2} - 4h + 16$ vertex $(4, 8)$



$0 < h < 100$

max $y = r^2$ for $h > 4$ $h < 100$

$= \frac{100^2}{2} - 2 \cdot 100 + 16$

$= \frac{10000}{2} - 200 + 16$

$r^2 = 5000 - 200 + 16$

$r^2 = 4816$ $\sqrt{4816} = 69.39$