Math with Sean

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Q25, Pascal 2009

$A(m,n) = (n,m)$

$B(m,n) = (m+3n, \quad n)$

$C(m,n) = (m-2n, \quad n)$

Start with $(0,1)$

$AA\ (m,n) = (m,n)$

$A^2 = I$

B

A) $(2009, 1016) \xleftarrow{A} (1016, 2009)$

$\xleftarrow{B}$ (

$B \Rightarrow 2009 = m' + 3n' = m' + 3 \times 1016$

$1016 = n'$

$m' = -1039$

$$\begin{array}{r} 3048 \\ -2009 \\ \hline 1039 \end{array}$$

$(-1039, 1016)$

$B ; C$

$B : (m, n) \mapsto (m + 3n, n)$

$C : (m+3n, n) \mapsto (m + 3n - 2n, n)$

$= (m+n, n)$

$$(B;C)^{2009} = (0,1) \to (2009, 1)$$

$$\begin{tikzcd}
 & & & & (0,1) \arrow[dlll, "A"'] \arrow[d, "B"] \arrow[drrr] & & & & \\
 & (1,0) \arrow[dl, "A"'] \arrow[d, "B"] \arrow[dr, bend left, "C"] & & & (3,1) \arrow[dl, "A"'] \arrow[d, "B"] \arrow[dr, "C"] & & & (-2,1) \arrow[dl, "A"'] \arrow[d, "B"] \arrow[dr, "C"] & \\
(0,1) & (1,0) & (1,0) & (1,3) & (6,1) & (1,1) & (1,-2) & (1,1) & (-4,1)
\end{tikzcd}$$

$B;C : (m,n) \mapsto (m+n, n)$

$(B;C)^3 = (m,n) \mapsto (m+3n, n) = B$

$$B;\underbrace{C;B;C;B;}_{I}C \qquad \underbrace{A;A}_{I}$$

$C ; B = I$

$BBACB$

$(0,1) \xrightarrow{B} (3,1) \xrightarrow{B} (6,1) \xrightarrow{A} (1,6) \xrightarrow{C} (-11,6) \xrightarrow{B} (7,6)$

$I : (m,n) \mapsto (m,n)$

$BC : (m,n) \mapsto (m+n, n)$

$CB : (m,n) \mapsto (m+n, n) = BC$

$A^2 = I$

Given a word composed of B's & C's we can put it in canonical form since $BC = CB$. Put all the B's first followed by C.

$W = B^\alpha C^\beta : (m,n) \mapsto (m + (3\alpha - 2\beta) n)$

3 & 2 are coprime, therefor Every integer is expressible as $3\alpha - 2\beta$ so $W : (m,n) \mapsto (m + kn, n)$

$A^2 = I$. Therefore every sequence of machines W can be expressed as $W_{k_0} \, A \, W_{k_1} \, A \, W_{k_2} \cdots A \, W_{k_L}$

where $L \geq 0$

and $k_i \in \mathbb{Z}$

the case $L = 0$ is just W_{k_0}

but $(0,1) W_{k_0} = (k_0, 1)$ which is NOT one of the answers.

Therefore $L > 0$

So. $L = 1 \Rightarrow$

$(0,1) W_{k_i} \equiv (0,1) k_i = (k_i, 1)$

$L = 0$

Word	(m, n)	
ϵ	$(0, 1)$	$=$
W_{k_0}	$(k_0, 1)$	$= (m_0, n_0)$
A_0	$(1, k_0)$	$= (n_0, m_0)$
W_{k_1}	$(1 + k_0 k_1, k_0)$	$= (m_1, n_1)$
A_1	$(k_0, 1 + k_0 k_1)$	
W_{k_2}	$(k_0 + (1 + k_0 k_1) k_2, 1 + k_0 k_1)$	
A_2	$(1 + k_0 k_1, k_0 + (1 + k_0 k_1) k_2)$	
W_{k_3}	$(1 + k_0 k_1 + (k_0 + (1 + k_0 k_1) k_2) k_3$	
A_i	(n_i, m_i)	
$W_{k_{i+1}}$	$(n_i + k_i \cdot m_i, m_i)$	
	$= (m_{i+1}, n_{i+1})$	

$$g = k_0 \cdot A \cdot k_1 \cdot A \cdot \cdots k_L$$

$L > 0$

$k_0 \in \mathbb{Z}$, $k_L \in \mathbb{Z}$

$k_1, \cdots, k_{L-1} \neq 0$

answers $= [(2009, 1016), (2009, 1004) \ldots$

$\{(2009, 1000 + x) \mid x \in \{16, 4, 2, 8, 32\}\}$

$k_0 : (0,1) \mapsto (k_0, 1) = (m_0, n_0) \quad n_0 = 1$

~~so~~ $L > 0$

$+A : (m_0, n_0) \longmapsto (n_0, m_0) = (1, k_0)$

$+k_1 : (1, k_0) \mapsto (1 + k_0 \cdot k_1, k_0)$

$= (2009, 1000 + 2^{\alpha}) \quad \alpha = 1, 2, 3, 4, 5.$

$1 + k_0 k_1 = 2009$

$k_0 k_1 = 2008 = 2 \times 1004 \Rightarrow k_0 = 1004$

$= 2 \times 2 \times 502 \quad k_1 = 2$

$= 2 \times 2 \times 2 \times 251 = 2^3 \cdot 251$

$$\begin{array}{r} 8 \\ 3\overline{)251} \\ 24 \\ \hline 11 \end{array} \quad \begin{array}{r} 3 \\ 7\overline{)251} \\ 21 \\ \hline 41 \end{array} \quad \begin{array}{r} 2 \\ 11\overline{)251} \\ 22 \\ \hline 31 \end{array} \quad \begin{array}{r} 1 \\ 13\overline{)251} \\ 13 \\ \hline 121 \end{array} \quad \begin{array}{r} 1 \\ 17\overline{)251} \\ 17 \\ \hline 81 \end{array}$$

so $(2009, 1004)$ is possible

$k_0 = 1004 \quad k_1 = 2$

$G = W_{1004}; A; W_2$

$(0,1) \to (1004, 1) \to (1, 1004) \to (1 + 2 \cdot 1004, 1004)$
$= (2009, 1004)$

$k_0; A; k_1 = (1 + k_0 k_1, k_0) = (m_1, n_1)$

$; A = (n_1, m_1) = (2009, 1000 + 2^\alpha)$

$n_1 = k_0 = 2009$

$m_1 = 1 + k_0 k_1 = 1 + 2009 \cdot k_1 = 1000 + 2^\alpha$

so $k_1 > 0$ since $999 + 2^\alpha > 0$

no solution since $2009 \cdot k_1 > 1000 + 2^\alpha$
for $k_1 > 0$ $\forall \alpha \in 1..5$

Try $k_2 \in \mathbb{Z}$

$k_0; A; k_1; A = (n_1, m_1) = (k_0, 1 + k_0 k_1)$
where $k_1 \neq 0$

Try $k_2 \in \mathbb{Z}$

$$k_0 ; A ; k_1 ; A = (n_1, m_1) = (k_0, 1 + k_0 k_1)$$

where $k_1 \neq 0$

$$; k_2 \quad (m_2, n_2) = (n_1 + k_2 m_1, m_1)$$

$$= (2009, 1000 + 2^\alpha)$$

$$m_1 = 1000 + 2^\alpha \qquad > 0$$

$$n_1 + k_2 m_1 = 2009 \qquad k_2 \neq 0$$

There are 4 values for m_1 : $1000 + \{2, 8, 16, 32\}$

$$n_1 + k_2 1002 = 2009$$

$$n_1 = 2009 - k_2 1002 = k_0$$

$$m_1 = 1 + k_0 k_1 = 1002$$

$$k_0 k_1 = 1001$$

what are the prime factors of 1001, etc.?

$$\begin{array}{r} 33 \\ 3 \overline{)1001} \\ 9 \\ \hline 10 \\ 9 \\ \hline 11 \end{array} \qquad \begin{array}{r} 143 \\ 7 \overline{)1001} \\ 7 \\ \hline 30 \\ 28 \\ \hline 21 \end{array}$$

$143 = 13 \times 11$ so $1001 = 7 \times 11 \times 13$

$$2009 - k_2 1002 = k_0$$

$$k_0 k_1 = 1001 = 7 \times 11 \times 13$$

So $-1001 \le k_0 \le 1001$

$$2009 - k_0 = k_2 1002$$

$$2009 - k_0 > 0 \Rightarrow k_2 > 0$$

$$2009 - 1001 = 1008$$

$$1008 \le 2009 - k_0 \le 3006$$

$$2009 - k_0 = k_2 1002 \le 3006$$
$$1 \le k_2 \le 3$$

$$2009 - k_0 \in 1002, 2004, 3006$$

$$k_0 \in \underbrace{2009 - 1002}_{1007}, \cancel{5}, \cancel{-997}$$

$$\begin{array}{r} 13 \\ 7 \\ \hline 91 \end{array}$$

So $m_1 = 1002$ is impossible

2020-05-17 4:21pm

Use the fact that $A, B, \& C$ are linear transformations of the plane $\mathbb{R}^2$. We are asked to compute the orbit of the point $(0,1)$ under the action of the group generated by $\{A, B, C\}$

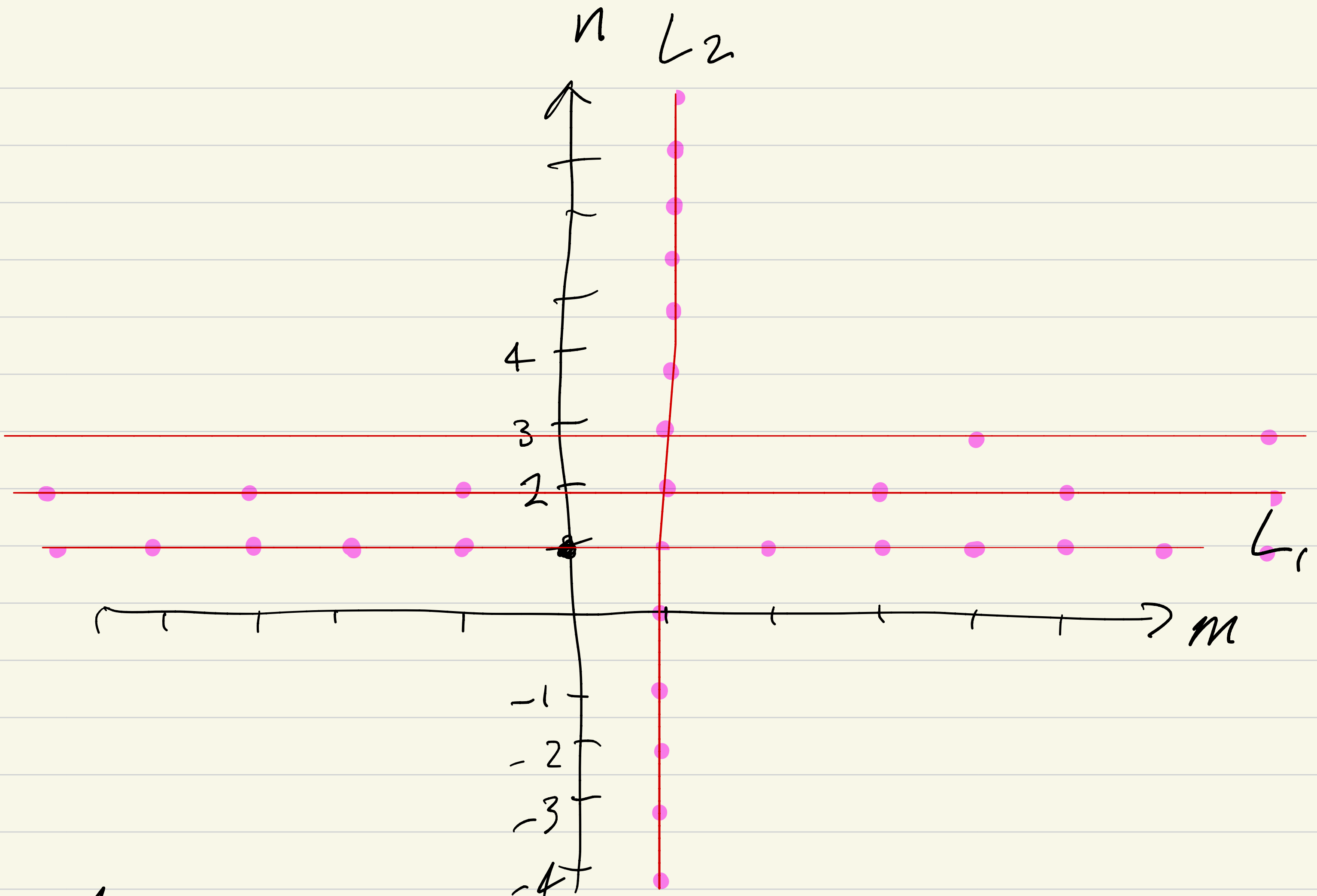
$A(m,n) = (n,m)$ $B(m,n) = (m+3n, n)$
$C(m,n) = (m-2n, n)$

interpret the product XY as do X then do Y, ie. forward fuction composition.

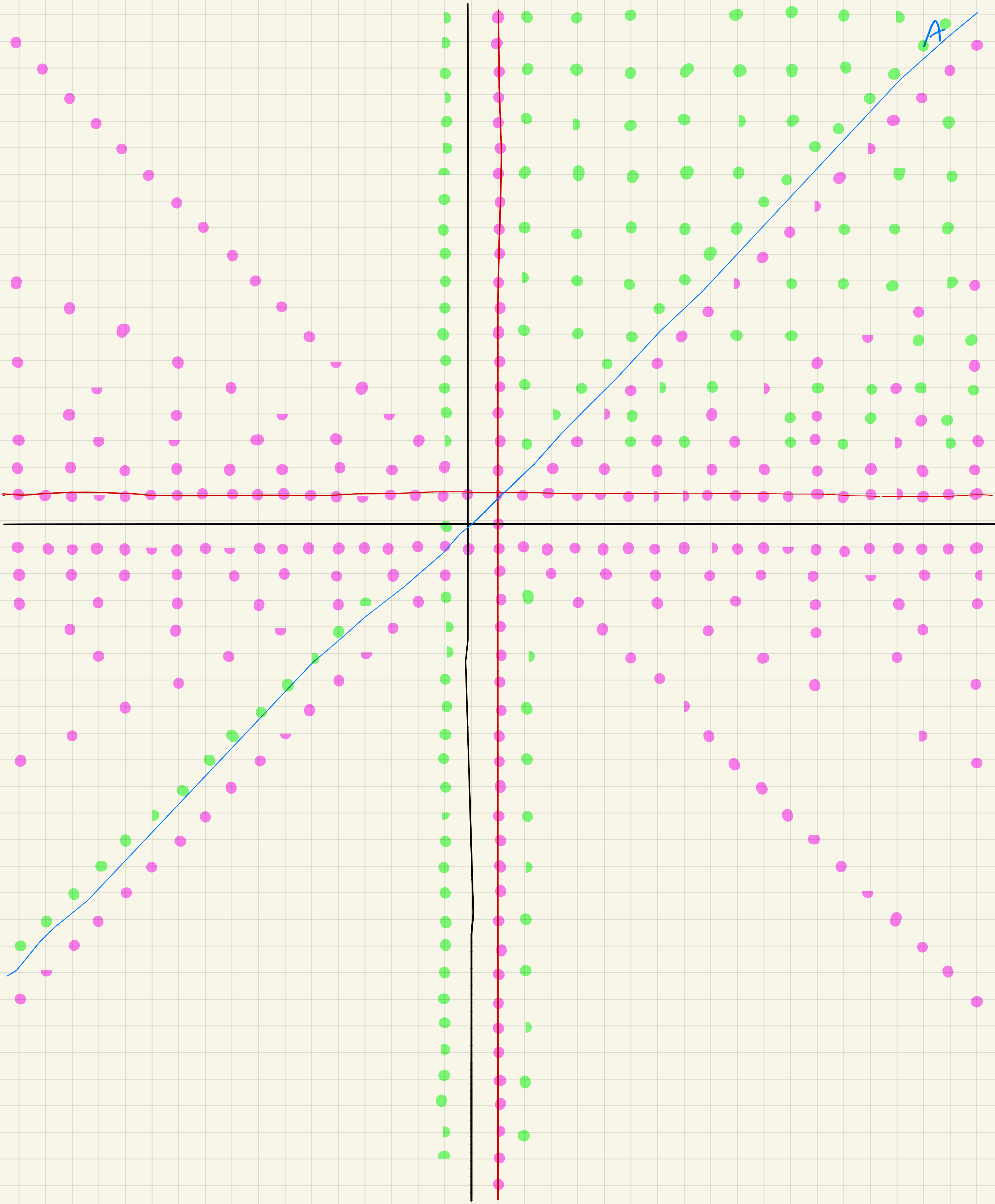
$$(BC)(m,n) = (m+n, n)$$

$(BC)^k(m,n) = (m+kn, n)$ which is a horizontal line through (m,n) which contains all integer multiples of $(n,0)$ ie. $L = \{(m,n) + k(n,0)$

Therefore the orbit contains $(0,1) + k(1,0)$ ie all integer points



A is reflection in the main diagonal, so it sends L_1 to L_2


A

define $S(k)(m,n) = (m+kn, n)$

so $B = S(3)$ $C = S(-2)$

$$BC = S(1)$$

$$S(k) = S(1)^k$$ generates an Abelian subgroup $\cong \mathbb{Z}$.

$A\,S(k)\,(m,n)$

$= S(k)(n,m)$

$= (n+km, m)$

$A \cdot S(k) \cdot A\ (m,n) = (m, n+km)$

$p_0 = (0,1)$ $A(p_0) = (1,0)$

$$S(k)(m,0) = (m+k\cdot 0, 0) = (m, 0)$$

so $(0,1) * A * S(k) * A = (0,1)$

Therefore consider words that start with $S(k)$ $\forall$ $k \neq 0$

$k_1 \in \mathbb{Z}$

$$\begin{tikzcd}
 & & (1+k_2k_1, k_1) \\
(k_1, 1) \arrow[r, "A"] & (1, k_1) \arrow[ur, "S(k_2)"] & \\
(0,1) \arrow[u, "S(k_1)"] \arrow[d, "A"] & & \\
(1,0) \arrow[loop below, "S(k_1)"] & &
\end{tikzcd}$$

A) $(2009, 1016) = (1+k_1k_2, k_1)$

$$k_1 = 1016$$

$$1+k_1k_2 = 2009$$

$$k_1k_2 = 2008$$

$$k_2 = \frac{2008}{1016} \notin \mathbb{Z}$$

but $k_1 = 1004$ $\quad k_2 = 2$

B) $(2009, 1004)$ is reachable.

$$S(1004)\,A\,S(2)$$

apply A to $(1+k_1k_2, k_1) = (k_1, 1+k_1k_2)$

Try A) $(2009, 1016) = (k_1, 1+k_1k_2)$

$k_1 = 2009$

$1 + k_1k_2 = 1016$ (or $1002, 1008, 1032$)

$k_1k_2 = 1015$

$2009\, k_2 = 1015$ No Solutions.

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apply $S(k_3)$ to $(k_1, 1+k_1k_2)$

$= (k_1 + k_3(1+k_1k_2), 1+k_1k_2)$

A) $(2009, 1016) = (k_1 + k_3(1+k_1k_2), 1+k_1k_2)$

$k_1 + k_3(1+k_1k_2) = 2009$

$1 + k_1k_2 = 1016$

$k_1k_2 = 1015 \Rightarrow -1015 \le k_1 \le 1015$

$1015 = 5 \times 203 = 5 \times 7 \times 29$

$$\begin{array}{r} 29 \\ 7\overline{)203} \\ 14 \\ \hline 63 \end{array}$$

$1016\, k_3 = 2009 - k_1 > 0$

$$\begin{array}{r} 2009 \\ -1015 \\ \hline 984 \end{array} \qquad 3024$$

$984 \le 1016\, k_3 \le 3024$

The only possible values for k_3 are $1, 2$

if $(2009, 1016)$ is reachable then so is

$$\begin{array}{r} 2009 \\ -1016 \\ \hline 983 \end{array} \quad (983, 1016)$$

so $(1016, 983)$ is reachable.

so $\begin{array}{r} 1016 \\ -983 \\ \hline 33 \end{array} \quad (33, 983)$

so $(983, 33)$ is reachable.

$$\begin{array}{r} 29 \\ 33\overline{)983} \\ 66 \\ \hline 323 \end{array}$$

$$(m,n) \to (m+an, n) \to (n, m+an)$$

$$\to (n+b(m+an), m+an)$$

$$= (bm+an, m+an)$$

suppose $\gcd(m,n) = c$

then $\exists\, a, b$ s.t. $am+bn = c$

2020-05-18 10:44 am

I see the "trick". The key observation is that the machines are linear transformations. After applying an arbitrary combination of machines A, B, + C, the result can be expressed as

$$(m,n) \longrightarrow (am+bn,\ cm+dn)$$

for integers (a,b,c,d). Similarly, the inverse of the combination can also be expressed in that form, namely an integer combination.

We are asked which given (m,n) are possible when starting from $(0,1)$

If (m,n) is possible, then there are integers a,b,c,d

$$am+bn = 0 \qquad (1)$$
$$cm+dn = 1 \qquad (2)$$

But (2) has solutions exactly when $\gcd(m,n) = 1$. Therefore compute $\gcd(m,n)$ for each answer.

(A) $(2009, 1016)$

$$\begin{array}{r} 1 \\ 1016\overline{)2009} \\ \underline{1016} \\ 983 \end{array} \qquad \begin{array}{r} 1 \\ 983\overline{)1016} \\ \underline{983} \\ 33 \end{array} \qquad \begin{array}{r} 29 \\ 33\overline{)983} \\ \underline{66} \\ 323 \\ \underline{297} \\ 26 \end{array}$$

$$\begin{array}{r} 1 \\ 26\overline{)33} \\ \underline{26} \\ 7 \end{array} \qquad \begin{array}{r} 3 \\ 7\overline{)26} \\ \underline{21} \\ 5 \end{array} \qquad \begin{array}{r} 1 \\ 5\overline{)7} \\ \underline{5} \\ 2 \end{array}$$

$$\begin{array}{r} 2 \\ 2\overline{)5} \\ \underline{4} \\ 1 \end{array}$$

so $\gcd(2009, 1016) = 1$ ✓

B) $(2009, 1004)$ observe

$$2 \times 1004 = 2008$$

so $2 \times 1004 + 1 = 2009$

$$2009 - 2 \times 1004 = 1$$

$$\Rightarrow \gcd(2009, 1004) = 1$$

C) $(2009, 1002)$

$$\begin{array}{r} 2 \\ 1002 \overline{)2009} \\ 2004 \\ \hline 5 \end{array} \qquad \begin{array}{r} 200 \\ 5 \overline{)1002} \\ 1000 \\ \hline 2 \end{array} \qquad \begin{array}{r} 2 \\ 2 \overline{)5} \\ 4 \\ \hline 1 \end{array} \Rightarrow \gcd(2009, 1002) = 1$$

D) $(2009, 1008)$

$$\begin{array}{r} 1 \\ 1008 \overline{)2009} \\ 1008 \\ \hline 1001 \end{array} \qquad \begin{array}{r} 1 \\ 1001 \overline{)1008} \\ 1001 \\ \hline 7 \end{array}$$

$$\begin{array}{r} 143 \\ 7 \overline{)1001} \\ 7 \\ \hline 30 \\ 28 \\ \hline 21 \\ 21 \\ \hline 0 \end{array}$$!

so $\gcd(2009, 1008) = 7$! X

$$\begin{array}{r} 287 \\ 7 \overline{)2009} \\ 14 \\ \hline 60 \\ 56 \\ \hline 49 \\ 49 \\ \hline 0 \end{array} \qquad \begin{array}{r} 144 \\ 7 \overline{)1008} \\ 7 \\ \hline 30 \\ 28 \\ \hline 28 \\ 28 \\ \hline 0 \end{array}$$

The answer is D

Double check E

(E) (2009, 1032)

$$\begin{array}{r} 1 \\ 1032\overline{)2009} \\ 1032 \\ \hline 977 \end{array} \qquad \begin{array}{r} 1 \\ 977\overline{)1032} \\ 977 \\ \hline 55 \end{array} \qquad \begin{array}{r} 17 \\ 55\overline{)977} \\ 55 \\ \hline 427 \\ 385 \\ \hline 42 \end{array} \qquad \begin{array}{r} 1 \\ 42\overline{)55} \\ 42 \\ \hline 17 \end{array}$$

$$\begin{array}{r} 2 \\ 17\overline{)42} \\ 34 \\ \hline 8 \end{array} \qquad \begin{array}{r} 2 \\ 8\overline{)17} \\ 16 \\ \hline 1 \end{array} \qquad \gcd(2009, 1032) = \underline{1} \checkmark$$

The only impossible pair is (D).