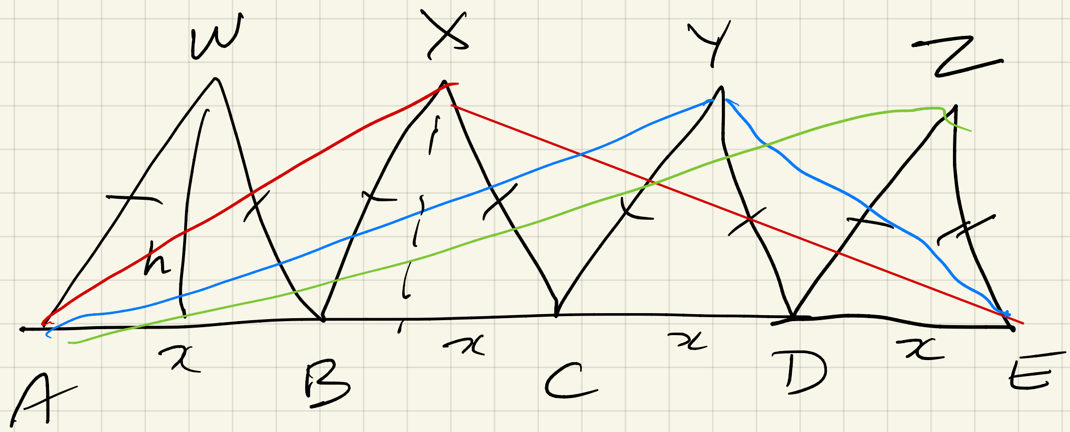
Math with Sean

Arthur Ryman

2020-12-26

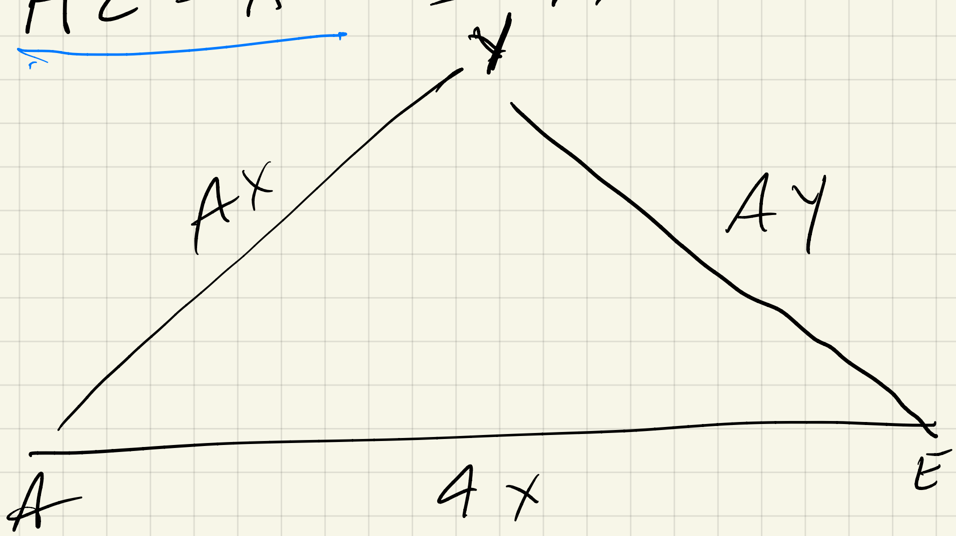
2004 Cayley #24

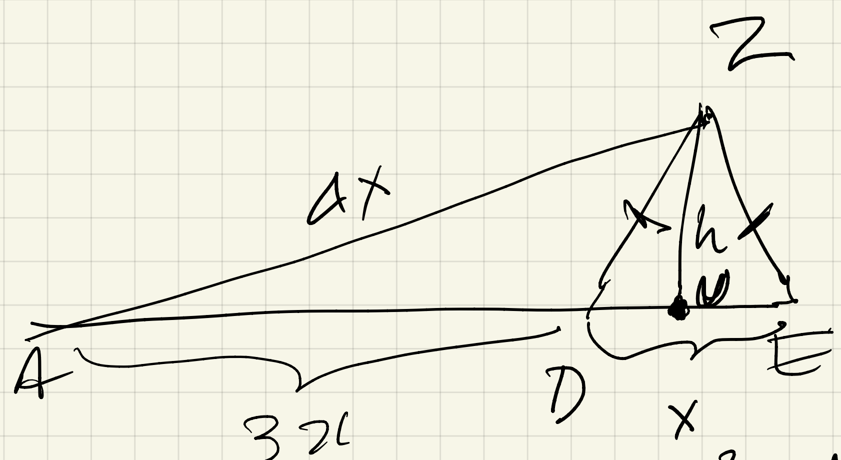


new triangle

sides are AX, AY, AZ

$AZ = AE = 4x$





$AE = 4x$

$AZ = AE$

$AU = AE - UE$

$= 4x - \frac{1}{2}x$

$= \frac{7}{2}x$

$\Delta AUZ: \; AU^2 + UZ^2 = AZ^2$

$\left(\frac{7}{2}x\right)^2 + h^2 = (4x)^2$

1

$$\left(\frac{7}{2}x\right)^2 + h^2 = (4x)^2$$

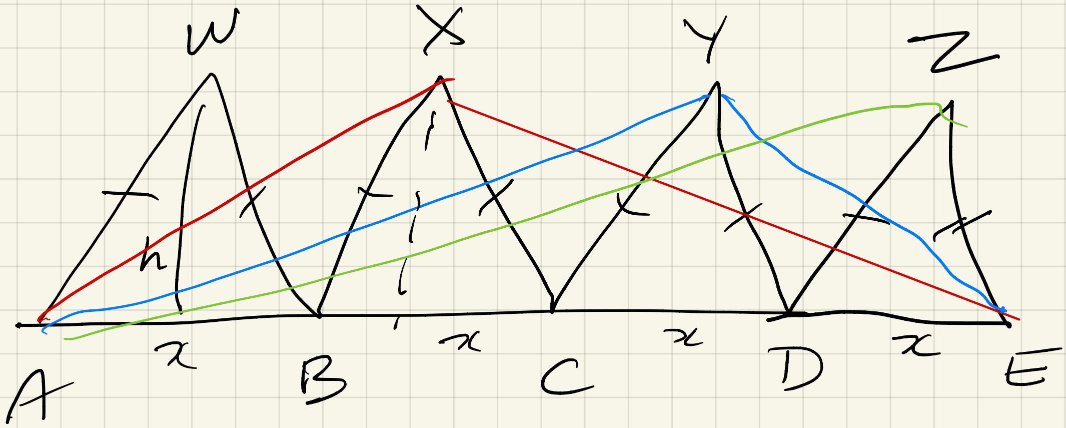
$$\frac{49}{4}x^2 + h^2 = 16x^2$$

$$h^2 = 16x^2 - \frac{49}{4}x^2$$

$$= \frac{64-49}{4}x^2$$

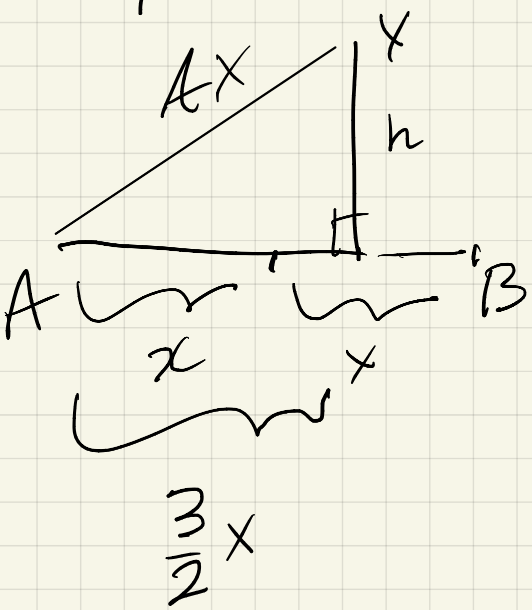
$$= \frac{15}{4}x^2$$

$$h = \sqrt{\frac{15}{4}x^2} = \frac{\sqrt{15}}{2}x$$



$$h = \frac{\sqrt{15}}{2}x$$

Compute AX, AY, $AZ = AE = 4x$



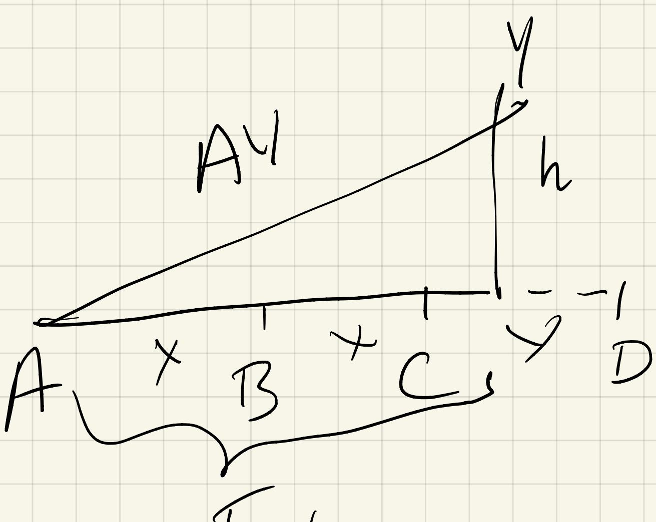
$$\left(\frac{3}{2}x\right)^2 + h^2 = AX^2$$

$$\frac{9}{4}x^2 + \frac{15}{4}x^2 = AX^2$$

$$\frac{24}{4}x^2 = AX^2$$

$$6x^2 = AX^2$$

$$AX = \sqrt{6}\,x$$



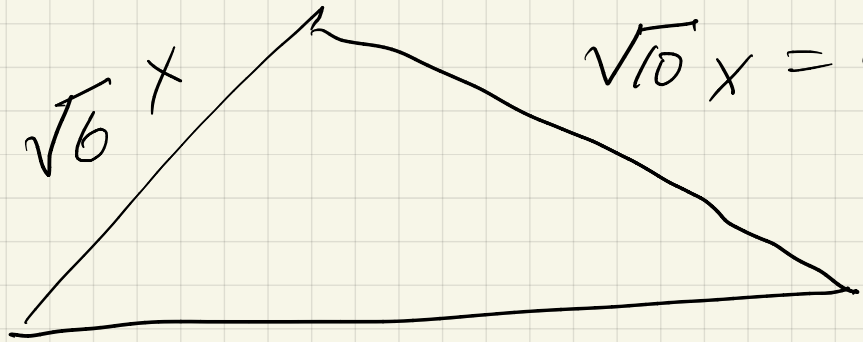
$$AY^2 = \left(\frac{5}{2}x\right)^2 + h^2$$

$$= \frac{25}{4}x^2 + \frac{15}{4}x^2$$

$$= \frac{40}{4}x^2$$

$$= 10x^2$$

$$AY = \sqrt{10}\,x$$



Heron's Formula $p = \dfrac{a+b+c}{2}$

$$\text{Area} = \sqrt{p(p-a)(p-b)(p-c)}$$

$$p = \frac{4x + \sqrt{6}\,x + \sqrt{10}\,x}{2}$$

$$= \underbrace{\frac{4+\sqrt{6}+\sqrt{10}}{2}}_{4.806}\,x$$

$$p - a = (4.806 - 4)x$$
$$= 0.806\,x$$

$$p - b = (4.806 - \sqrt{6})x$$
$$2.357\,x$$

$$p - c = (4.806 - \sqrt{10})\,x$$
$$1.644\,x$$

$$\text{Area} = \sqrt{(4.806 \times 0.806 \times 2.357 \times 1.644)}\; x^2$$

$$6.847 x^2 \leq 2004$$

$$x^2 \leq \frac{2004}{6.847} = 292.68$$

$$x \leq \sqrt{292.68}$$

$$= 17.1$$

$x = 17$ is the largest possible integer.

#25 Number of positive integers $x \le 60$ s.t.

$\frac{7x+1}{2}, \ldots, \frac{7x+300}{301}$ are all in lowest terms.

$$\gcd(7x+k,\ 1+k) = 1$$

for all $1 \le k \le 300$

x	$7x+1$	
1	8	$8/2 = 4$

$\frac{15}{2}$ ✓ $\frac{16}{3}$ ✓ $\frac{17}{4}$ ✓

if x is odd then $7x+1$ is even so $\frac{7x+1}{2}$ is NOT in lowest terms

$\boxed{\text{rule out all odd } x}$

$\gcd(7x+k,\ 1+k) \geq 1$

x is even $\quad x = 2, 4, \ldots, 60$

if $7x+1 = 2d \qquad \frac{7x+1}{2} = d$

This is impossible since x is even.

if $7x+2 = 3d$

$x+2 = 0 \mod 3$

$\boxed{x = 1 \mod 3}$

Rule out all $x = 1 \mod 3$

$x = 2,\ 4,\ 6,\ 8,\ 10,\ \ldots\ 60$

$x \bmod 3 = \underbrace{2 \quad 1 \quad 0}\ \underbrace{2 \quad 1 \quad \ldots}\ \underbrace{0}$

$\frac{7 \times 4 + 2}{3} = \frac{28+2}{3} = \frac{30}{3} = 10$

$\frac{7 \times 10 + 2}{3} = \frac{70+2}{3} = \frac{72}{3} = 24$

20 x's left $\quad 2, 6, 8, 12, \ldots\ 60$

$$\frac{7x+3}{4}$$

$7x+3 = 4d$ then rule out x

$$7x+3 = 0 \mod 4$$

$$3x+3 = 0 \mod 4$$

$7x+3 = 2d$ then $\frac{7x+3}{4} = \frac{d}{2}$

$$\boxed{7x+3 = 0 \mod 2}$$

$$x+1 = 0$$

$$x = 1 \mod 2$$

$\Rightarrow$ x is odd

rule out x odd

But we already ruled out x odd.

$$\frac{7x+4}{5}$$

Rule out x if $7x+4 = 5d$

$$7x+4 = 0 \mod 5$$
$$2x+4 = 0 \mod 5$$
$$x+2 = 0 \mod 5$$
$$x = 3 \mod 5$$

2, 6, ~~8~~, 12, 14, ~~18~~, 20, 24, 26,
30, 32, 36, ~~38~~, 42, 44, ~~48~~,
50, 54, 56, 60

$$\frac{7 \cdot 8+4}{5} = \frac{60}{5} = 12 \qquad \frac{7 \cdot 18+4}{5} = \frac{126+4}{5} = \frac{130}{5}$$

$$\frac{7x+5}{6} = \begin{cases} 2d \\ 3d \end{cases} \qquad \begin{matrix} 7x+5 = 2d \\ 7x+5 = 3d \end{matrix}$$

$7x+5 = 0 \mod 2 \Rightarrow x+1=0$ x odd

$7x+5 = 0 \mod 3 \Rightarrow x+2=0$
$\Rightarrow x = 1 \mod 3$

$$\frac{7x+6}{7} \Rightarrow 7x+6 = 7d$$

$$7x+6 = 0 \mod 7$$

$$\Rightarrow \quad 6 = 0 \mod 7$$

never happens.

$$\frac{7x+7}{8} \qquad 7x+7 = 2d$$

$$7x+7 = 0 \mod 2$$

$$x+1 = 0 \qquad x = \text{odd}.$$

$$\frac{7x+8}{9} \qquad 7x+8 \equiv 0 \mod 3$$

$$x+2 = 0$$

$$x = 1 \mod 3$$

$$\frac{7x+9}{10} \qquad 7x+9 \equiv 0 \mod 2$$

$$x+1 = 0 \Rightarrow x \text{ is odd}$$

$$7x+9 \equiv 0 \mod 5$$

$$2x+4 \equiv 0 \mod 5$$

already ruled out.

11

$$\frac{7x+10}{11}$$

$7x+10=0 \mod 11$

$7x+10=0 \mod 11$

2, 6, ~~8~~, 12, 14, ~~18~~, 20, 24, 26,
~~30~~, 32, 36, ~~38~~, 42, 44, ~~48~~,
50, 54, 56, 60

$7x+10=0 \mod 11$

$7x=-10 \mod 11$

$7x=1 \mod 11$

2, 6, 1, 3, 9, 2, 4,
~~8~~, 10, 3, 9, 0, 6, 10,
1, 5

$12 - 2\times2\times3$

13

4:30pm