Strict inverse systems of locally convex vector bundles and Lie groupoids

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January 6, 2026

Abstract

First we study inverse systems of manifolds and vector bundles modeled on locally convex spaces using Bastiani calculus. A notion of strictness is employed, characterized by the compatibility of local charts with inverse-limit constructions, to obtain a chart on the inverse limit. Under this condition, the inverse limit is shown to inherit a manifold structure modeled on a the inverse limit of the modeling spaces. We then study strict inverse systems of vector bundles where the inverse limit is also a vector bundle. For generality, we do not refer to any topology on the spaces of continuous linear maps. We then prove several useful results related to pullback bundles and direct sums. We finish the first part by considering compatible submersions between strict inverse systems which define a submersion between the inverse limits. In the second part, we recall the construction of inverse systems of topological groupoids as a precursor to our main results concerning Lie groupoids. Several examples are used to emphasize that some properties fail to be inherited by the inverse limit. We then investigate inverse systems of Lie groupoids modeled on locally convex spaces and establish sufficient conditions for the existence of inverse limits. This relies heavily on the results developed on the first part for vector bundles and submersions. In particular, we prove that the inverse limit of a strict system of Lie groupoids carries the structure of a Lie groupoid and we construct the associated limit Lie algebroid. Finally, we discuss inverse systems of topological groupoids where the inverse limit is a Lie groupoid. As an example, we study the multiphase diffeomorphism groupoid arising in the study of the multiphase Euler equations.

Keywords. Lie groupoids; Lie algebroids; inverse limits; infinite-dimensional geometry; multiphase-diffeomorphism groupoids

1 Introduction

Lie groupoids, introduced by Ehresmann [4], have become a fundamental tool in modern differential geometry and mathematical physics. They provide a natural language for singular spaces, symmetries of PDEs, Poisson geometry, and geometric quantization [9, 12, 15, 17].

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While the finite-dimensional theory is well established, the study of infinite-dimensional Lie groupoids has only recently begun to receive systematic attention [1, 16].

A key insight in infinite-dimensional geometry is Omori's realization that the diffeomorphism group of a compact manifold, though naturally a Fréchet Lie group, can be understood as an inverse limit of Banach–Lie groups of finite regularity [14]. This idea of inverse limit Banach (ILB) and inverse limit Hilbert (ILH) structures has since been extended to various geometric objects, but the literature has largely focused on Banach manifolds and on groups rather than groupoids [2, 5].

In this paper, we develop a more general framework by considering inverse systems of manifolds and vector bundles modeled on arbitrary locally convex spaces. This setting naturally includes the classical ILB/ILH case, where the limit is Fréchet, but also accommodates broader classes of locally convex models. As an application, we consider the multiphase diffeomorphism groupoid which was introduced to extend Arnold's geometric description of the Euler equation to multiphase inviscid flows [8]. We show that our notion of strict inverse limits of Lie groupoids provides the precise geometric foundation needed to treat such groupoids and their associated Lie algebroids rigorously.

Notation and conventions. All topological spaces are assumed to be Hausdorff unless otherwise stated. We set $\mathbb{N}_d := \{n \in \mathbb{Z} : n \geq d\}$ for $d \geq 0$ and $\mathbb{N}_d^\infty := \mathbb{N}_d \cup \{\infty\}$.

We work with manifolds and vector bundles modeled on arbitrary locally convex spaces. In this setting, we employ the notion Bastiani calculus (also known as Keller's C_c^k-theory) for calculus in locally convex spaces [6, 11, 16]. Let E and F be locally convex spaces and $U \subseteq \mathsf{E}$ open. A continuous map $f\colon U \to \mathsf{F}$ is of class C^1 if the directional derivative $df(x,u) := \lim_{t\to 0} t^{-1}(f(x+tu) - f(x))$ exists for all $(x,u) \in U \times \mathsf{E}$, and the induced map $df\colon U \times \mathsf{E} \to \mathsf{F}$ is continuous. Inductively, f is C^k if it is C^1 and df is C^{k-1}; it is C^∞ if it is C^k for all $k \in \mathbb{N}$.

A chart about $x \in M$ is a homeomorphism $\phi\colon U_\phi \to V_\phi$ where $U_\phi \ni x$ is an open neighborhood and V_ϕ is an open subset of a locally convex space M_ϕ. We shall always adapt this notation for charts. A C^r-atlas is a collection of charts covering M whose transition maps are C^r in the Bastiani sense. A C^r-manifold is a Hausdorff space equipped with a maximal C^r-atlas. For connected C^1-manifolds, all modeling spaces are topologically isomorphic, so we may speak of a single model space up to isomorphism, but we opt for stressing the modeling space of each chart for generality.

Following [6], a subset $S \subseteq M$ is a *submanifold* if it is locally modeled on a closed subspace of the modeling space: for every $x \in S$, there is a chart ϕ such that $\phi(U_\phi \cap S) = V_\phi \cap \mathsf{S}_\phi$ for some closed subspace $\mathsf{S}_\phi \subseteq \mathsf{M}_\phi$. If each S_ϕ is complemented in M_ϕ, then S is a *split submanifold*.

A C^r-map $f\colon M \to N$ is a *submersion* if, for every $x \in M$, there exist charts ϕ about x and ψ about $f(x)$ such that $\psi \circ f \circ \phi^{-1}$ restricts to a projection $p\colon \mathsf{M}_\phi \simeq \mathsf{N}_\psi \times \mathsf{C} \to \mathsf{N}_\psi$. Similarly, f is an *immersion* if it locally looks like the inclusion of a complemented subspace. An immersion that is also a topological embedding is called a C^r*-embedding*.

Structure of the manuscript. Section 2 forms the backbone of the exposition. Here we introduce strict inverse systems of manifolds, submanifolds, vector bundles, and subbundles. Our treatment of inverse systems of vector bundles is general in that it avoids the use of any topology on spaces of continuous linear maps between locally convex spaces. We conclude this section with some facts about submersions. In particular, we are interested in a sufficient condition for a compatible family of submersions between strict inverse systems of manifolds to define a submersion between the inverse limits. Specific examples are stated to justify this requirement. In Section 3.1, we recall the construction of inverse systems of topological groupoids and the associated inverse systems of topological spaces. The category of topological groupoids is particularly well-behaved in that inverse limits always exist. In particular, we endow orbit spaces and isotropy groups with the subspace topologies and study their inverse systems. In Section 3, we introduce strict inverse systems of Lie groupoids and show that the associated Lie algebroids assemble into strict inverse systems as well where the inverse limit is naturally a Lie algebroid. A number of elementary examples are considered. Finally, in Section 3.3, we show that the multiphase diffeomorphism groupoid arises as an inverse limit of an inverse system of topological groupoids with certain properties.

2 Strict inverse systems of manifolds and bundles

Throughout our presentation, inverse systems are considered over an arbitrary directed set I.

Definition 2.1. Let $k \in \mathbb{N}_0^\infty$. Let $(\{\mathsf{E}\}_i, \{\epsilon_{ij}\}_{i\leq j})$ and $(\{\mathsf{F}\}_i, \{\nu_{ij}\}_{i\leq j})$ be inverse systems of locally convex spaces. A family of C^k maps $f_i\colon U_i \to \mathsf{F}_i$, where $U_i \subseteq \mathsf{E}_i$ are open sets, is *strict C^k-morphism* from $\{U_i\}_i$ to $\{\mathsf{F}_i\}_i$ if the following conditions hold:

1. $\epsilon_{ij}(U_j) \subseteq U_i$;
2. $f_i \circ \epsilon_{ij} = \nu_{ij} \circ f_j$;
3. $\varprojlim U_i$ is open in $\varprojlim \mathsf{E}_i$.

That is, if $\{f_i\}_i$ is a morphism of inverse systems of topological spaces and the limit $\varprojlim f_i$ has an open domain. [1]

Lemma 2.2. *Let $\{f_i\colon U_i \to \mathsf{F}_i\}_i$ be a strict C^k-morphism of inverse system. Then $\varprojlim f_i\colon \varprojlim U_i \to \varprojlim \mathsf{F}_i$ is a well-defined C^k-map. Furthermore, if $\{g_i\colon V_i \to \mathsf{H}_i\}$ is another strict C^k-morphism from $\{V_i \subseteq \mathsf{F}_i\}_i$ to $\{\mathsf{H}_i\}_i$ with $f_i(U_i) \subseteq V_i$ for all $i \in I$, then $\varprojlim g_i \circ \varprojlim f_i = \varprojlim (g_i \circ f_i)$.*

Proof. First recall that $\varprojlim f_i$ is continuous. It is enough to prove the result for C^1-maps and then proceed inductively. For all $((x_i)_i, (v_i)_i) \in \varprojlim U_i \times \varprojlim \mathsf{E}_i$, the limit

$$\begin{aligned} d(\varprojlim f_i)((x_i)_i, (v_i)_i) &= \lim_{t\to 0} \frac{\varprojlim f_i((x_i)_i + t(v_i)_i) - \varprojlim f_i((x_i)_i)}{t} \\ &= \lim_{t\to 0} \left(\frac{f_i(x_i + tv_i) - f_i(x_i)}{t} \right)_i = (df_i(x_i, v_i))_i \end{aligned}$$

converges. For $i \leq j$, using the chain rule, we have

$$\begin{aligned} df_i(x_i, v_i) &= df_i(\epsilon_{ij}(x_j), \epsilon_{ij}(v_j)) = d(f_i \circ \epsilon_{ij})(x_j, v_j) \\ &= d(\nu_{ij} \circ f_j)(x_j, v_j) = \nu_{ij}(df_j(x_j, v_j)). \end{aligned}$$

Moreover,

[1] One can instead require that $\varprojlim U_i$ has dense interior.

1. $(\epsilon_{ij} \times \epsilon_{ij})(U_j \times \mathsf{E}_j) \subseteq U_i \times \mathsf{E}_i$;
2. $\nu_{ij} \circ \varprojlim df_j = (\varprojlim df_i) \circ (\epsilon_{ij} \times \epsilon_{ij})$;
3. $\varprojlim (U_i \times \mathsf{E}_i)$ is open in $\varprojlim (\mathsf{E}_i \times \mathsf{E}_i)$.

Hence $\varprojlim df_i\colon \varprojlim U_i \times \mathsf{E}_i \to \varprojlim \mathsf{F}_i$ is a well-defined C^0-map. Using the linear homeomorphism $\varprojlim U_i \times \varprojlim \mathsf{E}_i \simeq \varprojlim (U_i \times \mathsf{E}_i)$, we see that $d \varprojlim f_i$ is continuous. The final statement is straightforward. $\square$

2.1 Inverse systems of manifolds and submanifolds

While inverse limits of inverse systems of locally convex spaces always exist, strong conditions are required for the existence of inverse limits in the category of manifolds modeled on locally convex spaces.

Definition 2.3. Let $k \in \mathbb{N}_0^\infty$ and let I be a directed set. A inverse system of C^k-manifolds $(\{M_i\}_i, \{g_{ij}\}_{i \leq j})$ is *strict* if for all $(x_i)_i \in \varprojlim M_i$, there exists a family of charts $\phi_i\colon M_i \supseteq U_{\phi_i} \to V_{\phi_i} \subseteq \mathsf{M}_{\phi_i}$, about each x_i, where the model spaces assemble into an inverse system of locally convex spaces $(\{\mathsf{M}_{\phi_i}\}_i, \{\epsilon^{\phi}_{ij}\}_{i \leq j})$ such that

1. $\{\phi_i\}$ is morphism of inverse systems of topological spaces from $(\{U_{\phi_i}\}_i, \{g_{ij}|_{U_{\phi_j}}\}_{i \leq j})$ to $(\{V_{\phi_i}\}_i, \{\epsilon^{\phi}_{ij}|_{V_{\phi_j}}\}_{i \leq j})$;
2. $\varprojlim V_{\phi_i}$ is an open neighborhood of $\{\phi_i(x_i)\}_i$ in $\varprojlim \mathsf{M}_{\phi_i}$.

The two conditions ensure that $\varprojlim \phi_i$ defines a chart about $(x_i)_i$ taking values in the locally convex space $\varprojlim \mathsf{M}_{\phi_i}$. We call such family of charts, a *strict inverse system of charts* about $(x_i)_i$. Obviously, $(\varprojlim \phi_i)^{-1} = \varprojlim \phi_i^{-1}$. Let $\{\psi_i\}_i$ be another strict inverse system of charts about $(x_i)_i$. Then

$$\epsilon^{\phi}_{ij}(\phi_j(U_{\phi_j} \cap U_{\psi_j})) = \phi_i(g_{ij}(U_{\phi_j} \cap U_{\psi_j})) \subseteq \phi_i(g_{ij}(U_{\phi_j}) \cap g_{ij}(U_{\psi_j})) \subseteq \phi_i(U_{\phi_i} \cap U_{\psi_i}),$$

and

$$\epsilon^{\psi}_{ij} \circ (\psi_j \circ \phi_j^{-1}) = \psi_i \circ (g_{ij} \circ \phi_j^{-1}) = \psi_i \circ (\phi_i^{-1} \circ \epsilon^{\phi}_{ij}) = (\psi_i \circ \phi_i^{-1}) \circ \epsilon^{\phi}_{ij}$$

over $\phi_j(U_{\phi_j} \cap U_{\psi_j})$. Moreover, $\varprojlim \phi_i(U_{\phi_i} \cap U_{\psi_i}) = \varprojlim \phi_i(\varprojlim U_{\phi_i} \cap \varprojlim U_{\psi_i})$ is open. Hence $\varprojlim \psi_i \circ \left(\varprojlim \phi_i\right)^{-1}$ is well-defined C^k-maps by Lemma 2.2. Using an atlas of such charts, $\varprojlim M_i$ can be endowed with the structure of a C^k-manifold with respect to which the *limit maps* $g_j\colon \varprojlim M_i \to M_j$, $(x_i)_i \mapsto x_j$ are C^k. In particular, the atlas topology on $\varprojlim M_i$ is Hausdorff which can be finer than the inverse limit topology.

Consider a C^k-manifold P with a family of C^k-maps $\{h_i\colon P \to M_i\}_i$ satisfying $g_{ij} \circ h_j = h_i$ whenever $i \leq j$. Then $h\colon P \to \varprojlim M_i, z \mapsto (h_i(z))_i$ is a well-defined C^k-map. The uniqueness of h is evident by construction. Hence strict inverse systems in the category of C^k-manifolds have the expected "universal property".

$$\begin{tikzcd}
& & M_k \arrow[dd, "g_{jk}"] \\
P \arrow[r, dashed, "h"] \arrow[urr, bend left, "h_k"] \arrow[drr, bend right, "h_j"'] & \varprojlim M_i \arrow[ur, "g_k"] \arrow[dr, "g_j"'] & \\
& & M_j
\end{tikzcd}$$

Fortunately, the strictness condition 2 is not hard to realize as we shall see later in the examples, these include many nontrivial inverse systems. One case is when $\{V_{\phi_i}\}_i$ becomes stable, that is, when there exists a cofinal set $J \subseteq I$ and an index $i_0 \in J$ such that for all $j \in J$ with $j \geq i_0$, the following equality holds:

$$V_{\phi_j} = \epsilon_{i_0 j}^{-1}(V_{\phi_{i_0}}).$$

We call $\{\phi_i\}_i$ satisfying the latter condition an *saturated*. It follows then that $\varprojlim V_{\phi_i}$ is open since

$$\varprojlim V_{\phi_i} = \bigcap_{i \in I} (\epsilon_i^{\phi})^{-1}(V_{\phi_i}) = (\epsilon_{i_0}^{\phi})^{-1}(V_{\phi_{i_0}}). \tag{1}$$

Obviously, in this case, $\varprojlim U_{\phi_i} = g_{i_0}^{-1}(U_{\phi_{i_0}})$ so it is open in the inverse limit topology. We call an inverse system of manifolds *saturated* if it can be covered by such systems of charts. In this case, the atlas topology agrees with the inverse limit topology.

For $k \geq 1$, $T \varprojlim M_i$ is a C^{k-1}-manifold where the change of charts is given by

$$\begin{aligned}
T \varprojlim \psi_i \circ \left(T \varprojlim \phi_i\right)^{-1} & ((y_i)_i, (v_i)_i) \\
&= \left(\varprojlim (\psi_i \circ \phi_i^{-1})((y_i)_i), \varprojlim d(\psi_i \circ \phi_i^{-1})((y_i)_i, (v_i)_i)\right), \\
&= \left(((\psi_i \circ \phi_i^{-1})(y_i))_i, (d(\psi_i \circ \phi_i^{-1})(y_i, v_i))_i\right),
\end{aligned}$$

for $(y_i)_i \in \varprojlim \phi_i \left(\varprojlim U_{\phi_i} \cap \varprojlim U_{\psi_i}\right)$ and $(v_i)_i \in \varprojlim \mathsf{M}_{\phi_i}$.

Associated with $(\{M_i\}_i, \{g_{ij}\}_{i \leq j \in I})$, we have a strict C^{k-1}-inverse system of manifolds $(\{TM_i\}_i, \{Tg_{ij}\}_{i \leq j})$ for which $\varprojlim TM_i$ is diffeomorphic to $T \varprojlim M_i$. We note that for each strict inverse system of charts $\{\phi_i\}_i$, $\{T\phi_i\}_i$ is a strict inverse system of charts for $\{TM_i\}_i$. Moreover, since $\varprojlim (V_{\phi_i} \times \mathsf{M}_{\phi_i})$ is linearly homeomorphic to $\varprojlim V_{\phi_i} \times \varprojlim \mathsf{M}_{\phi_i}$, $\varprojlim TU_{\phi_i}$ is diffeomorphic to $T \varprojlim U_{\phi_i}$. Now we construct this diffeomorphism globally.

The universal property gives us a unique C^{k-1}-map $\Xi \colon T \varprojlim M_i \to \varprojlim TM_i$ which is explicitly given by

$$\Xi \colon T \varprojlim M_i \ni [\gamma] \mapsto ([g_i \circ \gamma])_i \in \varprojlim TM_i.$$

On the other hand, suppose that $([\gamma_i])_i \in \varprojlim TM_i$ and let $\{\phi_i\}_i$ be a strict inverse system of charts at $(\gamma_i(0))_i$. By continuity and linearity, there exists an open interval $J \subseteq \mathbb{R}$ such that $(\phi_i(\gamma_i(0)) + t(\phi_i \circ \gamma_i)'(0))_i$ takes values in $\varprojlim V_{\phi_i}$. Then

$$\Xi^{-1} \colon ([\gamma_i])_i \mapsto [t \mapsto (\phi_i^{-1}(\phi_i(\gamma_i(0)) + t(\phi_i \circ \gamma_i)'(0)))_i]$$

is a well-defined C^{k-1}-inverse. Indeed, that Ξ^{-1} is a right-inverse is clear. On the other hand, using Lemma 2.2 we have

$$\Xi^{-1}(([g_i \circ \gamma])_i) = [t \mapsto (\varprojlim \phi_i)^{-1}((\varprojlim \phi_i)((g_i \circ \gamma)_i(0) + t\left((\varprojlim \phi_i) \circ (g_i \circ \gamma)_i\right)(0)))] = [\gamma].$$

Of course locally in the charts induced by $\{\phi_i\}_i$, Ξ is given by $\varprojlim V_{\phi_i} \times \varprojlim \mathsf{M}_{\phi_i} \ni ((y_i)_i, (v_i)_i) \mapsto (y_i, v_i) \in \varprojlim V_{\phi_i} \times \mathsf{M}_{\phi_i}$.

Let $(\{N_i\}_i, \{h_{ij}\}_{i \leq j})$ be a C^k-inverse system of manifolds. For $l \leq k$, a family of C^l-maps $\{f_i \colon M_i \to N_i\}_i$, is a *$C^l$-morphism of inverse systems of manifolds* if $h_{ij} \circ f_j = f_i \circ g_{ij}$ for all $i \leq j$. In particular, if $\{M_i\}_i$ and $\{N_i\}$ are strict, then $\varprojlim f_i \colon \varprojlim M_i \to \varprojlim N_i$ is a C^l-map. Note that $\{\pi_{TM_i}\}_i$ forms an inverse system of C^{k-1}-maps as $g_{ij} \circ \pi_{TM_j} = \pi_{TM_i} \circ Tg_{ij}$.

We call a family $X_i \in \Gamma_{C^{k-1}}(TM_i)$ *compatible* if they assemble into a morphism of inverse systems. That is, for $i \leq j$, $Tg_{ij} \circ X_j = X_i \circ g_{ij}$, that is, if they are related by g_{ij}. Now $\Xi^{-1} \circ \varprojlim X \in \Gamma_{C^{k-1}}(T \varprojlim M_i)$. For another compatible family of vector fields $\{Y_i\}_i$, one can show that

$$[\Xi^{-1} \circ \varprojlim X_i, \Xi^{-1} \circ \varprojlim Y_i]_{T \varprojlim M_i} = \Xi^{-1} \circ \varprojlim [X_i, Y_i]_{TM_i}.$$

Let $X_i^\phi, Y_i^\phi \in C^{k-1}(U_{\phi_i}, \mathsf{M}_{\phi_i})$ be the local representations of X_i and Y_i, respectively. Then we have

$$\begin{aligned}
d(\varprojlim X_i^\phi)((x_i)_i, (\varprojlim Y_i^\phi)((x_i)_i)) - d(\varprojlim Y_i^\phi)((x_i)_i, (\varprojlim X_i^\phi)((x_i)_i)) \\
&= (\varprojlim dX_i^\phi)((x_i, Y_i^\phi(x_i))_i) - (\varprojlim dY_i^\phi)((x_i, X_i^\phi(x_i))_i) \\
&= (dX_i^\phi(x_i, Y_i^\phi(x_i)))_i - (dY_i^\phi(x_i, X_i^\phi(x_i)))_i \\
&= ([X_i, Y_i]^\phi(x_i))_i.
\end{aligned}$$

We shall suppress Ξ^{-1} from now on.

The discussion above is summarized in the following result.

Theorem 2.4. *Let $k \in \mathbb{N}_0^\infty$ and let $(\{M_i\}_{i \in I}, \{g_{ij}\}_{i \leq j \in I})$ be a strict inverse system of C^k-manifolds. Then $\varprojlim M_i$ is a C^k-manifold. Moreover, for $k \geq 1$, $(\{TM_i\}_{i \in I}, \{Tg_{ij}\}_{i \leq j \in I})$ is a strict inverse system of C^{k-1}-manifolds for which*

$$\varprojlim TM_i \simeq T \varprojlim M_i.$$

Finally, for any compatible families of vector fields $X_i, Y_i \in \Gamma_{C^{k-1}}(TM_i)$

$$[\varprojlim X_i, \varprojlim Y_i]_{T \varprojlim M_i} = \varprojlim [X_i, Y_i]_{TM_i}.$$

Remark 2.5 (The atlas topology can be strictly finer than the inverse limit topology)**.** Let $I = \mathbb{N}_1$, let $M_n = \mathbb{R}^n$, and let $\epsilon_{nm} = g_{nm} \colon \mathbb{R}^m \to \mathbb{R}^n$ be the projection onto the first n-coordinates. Define $f \colon (-1, 1) \to \mathbb{R}$ by $f(t) = t/(1 - t^2)$. Since f is a C^∞-diffeomorphism, we can use it to define a chart $\phi_n \colon (-1, 1)^n \to \mathbb{R}^n$ by $\phi_n(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$ for all $n \in \mathbb{N}$. Obviously, $\epsilon_{nm} \circ \phi_m = \phi_n \circ g_{nm}$ for all $n \leq m$. Now $\varprojlim V_{\phi_n} = \varprojlim \mathbb{R}^n \simeq \mathbb{R}^{\mathbb{N}}$ which is open in the product topology on $\mathbb{R}^{\mathbb{N}}$ (as the inverse limit of the model spaces) however $\varprojlim U_{\phi_n} = (-1, 1)^{\mathbb{N}}$ which is not open in the product topology on $\mathbb{R}^{\mathbb{N}}$ (as the inverse limit of the manifolds).

Example 2.6. From the definition, we have the following obvious cases:

1. Every inverse system of locally convex spaces is a saturated inverse system of C^∞-manifolds;
2. If I is finite, then $(\{M_i\}_i, \{f_{ij}\}_{i\leq j})$ is strict;
3. Strict (or strong) ILH/ILB manifolds are inverse limits of strict inverse systems of Hilbert/Bananch manifolds.

Example 2.7. Let M be a smooth manifold and F be locally convex spaces. For $m \in \mathbb{N}_0$, $C^m(M,\mathsf{F})$ is a locally convex space. For all $n \leq m$, let $\iota_{nm}\colon C^m(M,\mathsf{F}) \to C^n(M,\mathsf{F})$ denote the canonical inclusion. Then $(\{C^m(M,\mathsf{F})\}_m, \{\iota_{nm}\}_{n\leq m})$ is a saturated C^∞-inverse system of manifolds. The inverse limit is realized by the locally convex space $C^\infty(M,\mathsf{F}) = \cap_{n\geq 0} C^n(M,\mathsf{F})$.

Example 2.8. Let $l \in \mathbb{N}_0^\infty$, let M be a C^l-manifold, and let $(\{\mathsf{E}_i\}_i, \{\epsilon_{ij}\}_{i\leq j})$ be an inverse system of locally convex spaces. Then, for $k \leq l$, $(\{C^k(M,\mathsf{E}_i)\}_i, \{C^k(M,\epsilon_{ij})\}_{i\leq j})$ is a saturated C^k-inverse system of manifolds for which $\varprojlim C^k(M,\mathsf{E}_i) \simeq C^k(M, \varprojlim \mathsf{E}_i)$.

Example 2.9. Let $k \in \mathbb{N}_1^\infty$. Let M be a smooth compact manifold and N be a possibly infinite-dimensional smooth manifold with local addition $\Sigma\colon TN \supseteq U \to N$ defining a diffeomorphism $(\pi_{TN}, \Sigma)\colon U \to U'$ onto an open subset U' of $N \times N$. We recall that $C^k(M,N)$ can be endowed with a smooth manifold structure with charts constructed as follows: for any $f \in C^k(M,N)$, there exists a chart $\psi_{k,f}\colon U_{k,f} \to V_{k,f}$ with

$$U_{k,f} := \{g \in C^k(M,N) : (f,g) \in C^k(M,U')\}$$

and

$$V_{k,f} := \{Y \in C^k(M,U) : \pi_{TN} \circ Y = f\}.$$

Here V_f is an open set of the locally convex space

$$\mathsf{C}^k_f := \{X \in C^k(M,TN) : \pi_{TN} \circ X = f\}.$$

Thus, for any $f \in C^\infty(M,N)$, we have an inverse system of locally convex spaces $\{\mathsf{C}^k_f\}_k$ where the connecting morphisms $\iota_{jk}\colon \mathsf{C}^k_f \to \mathsf{C}^j_f$ induced by the inclusions $C^k(M,TN) \to C^j(M,TN)$, for $j \leq k$. Furthermore, $V_{\infty,f} = \cap_{k\geq 0} V_{k,f}$ is open in the inverse limit $\mathsf{C}^\infty_f = \cap_{k\geq 0}\mathsf{C}^k_f$. Indeed, trivially we have, $\iota_{0j}^{-1}(V_{0,f}) = V_{j,f}$ for all $j \in \mathbb{N}_1$. Thus $\{C^k(M,N)\}_{k\in\mathbb{N}_1}$ is a saturated inverse system of Banach manifolds and $C^\infty(M,N)$ is the inverse limit.

In the same fashion, we can realize $C^\infty(M,N)$, where M is a smooth compact d-dimensional manifold and (N,g) is a finite-dimensional Riemannian manifold, as the the inverse limit of the saturated inverse system $\{H^s(M,N)\}_{s>d/2}$. Here $\mathsf{C}^\infty_f = \cap_{s>d/2}\mathsf{H}^s_f$ is the inverse limit of an inverse system of Hilbert spaces $\mathsf{H}^s_f := \{X \in H^s(M,TN) : \pi_{TN} \circ X = f\}$, where the charts $\phi_{s,f}\colon H^s(M,N) \supseteq U_{s,f} \to V_{s,f} \subseteq \mathsf{H}^s_f$ are defined using the Riemann exponential map $\exp_g$ of the metric g. See the discussion in [3]. In particular, $\mathcal{D}(M)$ is the inverse limit of $\{\mathcal{D}^s(M)\}_{s>d/2}$.

Example 2.10. Let $l \in \mathbb{N}_0^\infty$. Consider a C^l-compact manifold M and a C^∞-strict inverse system of manifolds $(\{N_i\}_i, \{g_{ij}\}_{i\leq j})$ with a family of local additions $\Sigma_i \colon TN_i \supseteq U_i \to N_i$ such that the diagram

$$\begin{tikzcd} U_j \arrow[r, "\Sigma_j"] \arrow[d, "Tg_{ij}"'] & N_j \arrow[d, "g_{ij}"] \\ U_i \arrow[r, "\Sigma_i"] & N_i \end{tikzcd}$$

commutes and $\varprojlim U_i$ is open. It follows then that $\varprojlim \Sigma_i$ defines a local addition on $\varprojlim N_i$ noting that $\{(\pi_{TM_i}, \Sigma_i)\}_i$ is a compatible family of diffeomorphisms since the following diagram commutes:

$$\begin{tikzcd} U_j \arrow[r, "(\pi_{TM_j}, \Sigma_j)"] \arrow[d, "Tg_{ij}"'] & U_j' \arrow[d, "g_{ij}\times g_{ij}"] \\ U_i \arrow[r, "(\pi_{TM_i}, \Sigma_i)"] & U_i' \end{tikzcd}$$

Now for any $k \leq l$ and all $i \in I$, the manifold structure on $C^k(M, N_i)$ is canonical. Then for $i \leq j$, the pushforward maps $C^k(M, g_{ij}) \colon C^k(M, N_j) \to C^k(M, N_i)$ are C^k. Let $(f_i)_i \in \varprojlim C^k(M, N_i)$, set $\mathsf{C}^k_{f_i} := \{X \in C^k(M, TN_i) : \pi_{TN_i} \circ X = f_i\}$, and, using the local additions, construct a family of charts $\phi_{f_i} \colon U_{k,f_i} \to V_{k,f_i}$ with $U_{k,f_i} := \{g \in C^k(M, N_i) : (f_i, g) \in C^k(M, U_i')\}$ and $V_{k,f_i} = \{X \in C^k(M, U_i) : \pi_{TN_i} \circ X = f_i\}$. Then $(\{\mathsf{C}^k_{f_i}\}, \{C^k(M, Tg_{ij})$ is an inverse system of locally convex spaces. Note that the restrictions $C^k(M, Tg_{ij})|_{\mathsf{C}^k_{f_j}}$ are linear and well-defined as

$$\pi_{TN_i} \circ (Tg_{ij} \circ X_j) = g_{ij} \circ (\pi_{TN_j} \circ X_j) = g_{ij} \circ f_j = f_i.$$

Moreover, the diagram

$$\begin{tikzcd} U_{k,f_j} \arrow[r, "\phi_{k,f_j}"] \arrow[d, "C^k(M, g_{ij}\times g_{ij})"'] & V_{k,f_j} \arrow[d, "C^k(M, Tg_{ij})"] \\ U_{k,f_i} \arrow[r, "\phi_{k,f_i}"] & V_{k,f_i} \end{tikzcd}$$

commutes. Finally, we note that $\varprojlim V_{k,f_i} = \varprojlim \mathsf{C}^k_{f_i} \cap C^k(M, \varprojlim U_i)$.[2] Thus, $(\{C^k(M, N_i)\}_i, \{C^k(M, g_{ij})\}_{i\leq j})$ is a strict C^k-inverse system of manifolds. Using the canonical manifold structure on $C^k(M, \varprojlim N_i)$ defined by $\varprojlim \Sigma_i$, we get a diffeomorphism

$$\varprojlim C^k(M, N_i) \simeq C^k(M, \varprojlim N_i),$$

generalizing Example 2.8.

Products of strict inverse systems of manifolds behave as expected.

[2]In fact, if $(\{N_i\}_i, \{g_{ij}\}_{i\leq j})$ is saturated so is $(\{C^k(M, N_i)\}_i, \{C^k(M, g_{ij})\}_{i\leq j})$.

Proposition 2.11. *Let $k \in \mathbb{N}_0^\infty$ and let $(\{M_i\}_i, \{f_{ij}\}_{i\leq j})$ and $(\{N_i\}_i, \{g_{ij}\}_{i\leq j})$ be strict (resp. saturated, eventually saturated) C^k-inverse systems of manifolds. Then $(\{M_i \times N_i\}_i, \{f_{ij} \times g_{ij}\}_{i\leq j})$ is a strict (resp. saturated, eventually saturated) C^k-inverse system of manifolds. Moreover, $\varprojlim M_i \times N_i$ is diffeomorphic to $\varprojlim M_i \times \varprojlim N_i$.*

Remark 2.12. We can center a chart $\varprojlim \phi_i$ about $(x_i)_i \in \varprojlim U_{\phi_i}$ by centering each ϕ_i about x_i, for all i, simultaneously. In fact, we can show that for $(x_i)_i \in \varprojlim U_{\phi_i}$, we have $\tau_{-(\phi_i(x_i))_i} = \varprojlim \tau_{-\phi_i(x_i)}$, where $\tau_{z_i}\colon y_i \mapsto y_i + z_i$ is the shift by z_i. This follows since $\epsilon^\phi_{ij} \circ \tau_{y_j} = \tau_{y_i} \circ \epsilon^\phi_{ij}$ for all $(y_i)_i \in \varprojlim \mathsf{M}_{\phi_i}$. Thus we have a diffeomorphism $\varprojlim \tau_{y_i}\colon \varprojlim \mathsf{M}_{\phi_i} \to \varprojlim \mathsf{M}_{\phi_i}$ such that $\varprojlim \tau_{y_i} = \tau_{(y_i)_i}$.

Definition 2.13. A *strict (resp. saturated) C^k-inverse system of submanifolds* of a strict C^k-inverse system of manifolds $(\{M_i\}_i, \{g_{ij}\}_{i\leq j})$ is a family of submanifolds $\{S_i\}_i$ such that

1. $S_i \subseteq M_i$ is a submanifold;
2. $g_{ij}|_{S_j}$ takes values in S_i for $i \leq j$;
3. for all $(x_i)_i \in \varprojlim S_i$, there exists a strict (resp. saturated) inverse system of submanifold charts $\{\phi_i\}_i$ that is, for all $i \in I$, $\phi_i(S_i \cap U_{\phi_i}) = V_{\phi_i} \cap \mathsf{S}_{\phi_i}$, where $\mathsf{S}_{\phi_i} \subseteq \mathsf{M}_{\phi_i}$ is a closed subspace.

Note that $(\{S_i\}_i, \{g_{ij}|_{S_j}\}_{i\leq j})$ is a strict (resp. saturated) inverse system of manifolds as the family $\{\phi_i\}_i$ defines a strict (resp. saturated) inverse system of charts for $\{S_i\}_i$, and $\varprojlim \phi_i$ is a submanifold chart around $(x_i)_i$. Consider a stict inverse system of submanifold charts $\{\phi_i\}_i$ centered at $(x_i)_i \in \varprojlim S_i$. Then, for $i \leq j$,

$$\epsilon^\phi_{ij}(V_{\phi_j} \cap \mathsf{S}_{\phi_j}) = \epsilon^\phi_{ij}(\phi_j(U_{\phi_j} \cap S_j)) = \phi_i(g_{ij}(U_{\phi_j} \cap S_j)) \subseteq \phi_i(U_{\phi_i} \cap S_i) = V_{\phi_i} \cap \mathsf{S}_{\phi_i}.$$

Since V_{ϕ_j} is absorbing, it follows that $\epsilon^\phi_{ij}|_{\mathsf{S}_{\phi_j}}$ takes values in S_{ϕ_i}. Hence $(\{\mathsf{S}_{\phi_i}\}_i, \{\epsilon^\phi_{ij}|_{\mathsf{S}_{\phi_j}}\}_{i\leq j})$ is an inverse system of locally convex spaces. Thus

$$\varprojlim \phi_i(\varprojlim U_{\phi_i} \cap \varprojlim S_i) = \varprojlim V_{\phi_i} \cap \varprojlim \mathsf{S}_{\phi_i},$$

noting that $\varprojlim \mathsf{S}_{\phi_i}$ is a closed subspace of $\varprojlim \mathsf{M}_{\phi_i}$.

In the same spirit, we define *strict (resp. saturated) C^k-inverse systems of split-submanifolds* by replacing the third condition by a stronger one:

3'. for all $(x_i)_i \in \varprojlim S_i$, there exists a strict (resp. saturated) inverse system of split-submanifold charts $\{\phi_i\}_i$ such that, for all $i \in I$, $\phi_i(S_i \cap U_{\phi_i}) = V_{\phi_i} \cap \mathsf{S}_{\phi_i}$, where $\mathsf{S}_{\phi_i} \subseteq \mathsf{M}_{\phi_i}$ is complemented and if $p^\phi_i\colon \mathsf{M}_{\phi_i} \to \mathsf{S}_{\phi_i}$ denote the projection then $p^\phi_i \circ \epsilon^\phi_{ij} = \epsilon^\phi_{ij} \circ p^\phi_j$.

In Condition 3', what we require is that $\{\mathsf{S}_{\phi_i}\}_i$ is complemented in $\{\mathsf{M}_{\phi_i}\}_i$ (see Definition 2.44). This ensures that $\varprojlim \mathsf{S}_{\phi_i}$ is complemented in $\varprojlim \mathsf{M}_{\phi_i}$ as one can show using Lemma 2.43.

From the above discussion, we have the following result.

Proposition 2.14. *Let $(\{M_i\}_i, \{g_{ij}\}_{i\leq j})$ be a strict C^k-inverse system of manifolds and let $\{S_i\}_i$ be a strict C^k-inverse system of (split-)submanifolds. Then $\varprojlim S_i$ is a C^k-(split-)submanifold of $\varprojlim M_i$.*

Remark 2.15 (An inverse sysetem of split-submanifolds for which the limit is not a (sub)manifold)**.** Let $I = \mathbb{N}_0$, let $M_n = \mathbb{R}$, $g_{nm} = \epsilon_{nm} = \mathrm{id}_{\mathbb{R}}$, let $S_0 = (0,1)$, and let S_n be constructed from S_{n-1} by removing the middle third from each component. Obviously, $g_{nm}(S_m) \subseteq S_n$ and the identity chart $\phi_n = \mathrm{id}_{\mathbb{R}}|_{S_n}$ defines a compatible family of split-submanifold charts where $\mathsf{S}_{\phi_n} = \mathbb{R}$ for all $n \in \mathbb{N}_0$. Passing to the limit $\varprojlim \mathsf{S}_{\phi_n} \simeq \mathbb{R}$ and $\varprojlim S_n = C_e$, the Cantor set without end points, which is neither a 0-dimensional nor a 1-dimensional (sub)manifold. This example also strengthens the requirement of Condition 2 in Definition 2.3.

Example 2.16. Let $(\{M_i\}_{i\in I}, \{g_{ij}\}_{i\leq j\in I})$ be a strict (resp. saturated) inverse system of manifolds. By Proposition 2.11, $(\{M_i \times M_i\}_{i\in I}, \{g_{ij} \times g_{ij}\}_{i\leq j\in I})$ is also a strict (resp. saturated) inverse system of manifolds. Recall that, for all $i \in I$, Δ_{M_i} is a split-submanifold of $M_i \times M_i$. It is easy to see that $(\{\Delta_{M_i}\}_i, \{(g_{ij} \times g_{ij})|_{\Delta_{M_i}}\}_{i\leq j})$ is a strict (resp. saturated) inverse system of split-submanifolds. Moreover, $\varprojlim \Delta_{M_i} \simeq \Delta_{\varprojlim M_i}$. Indeed, for any strict inverse system of charts $\{\phi_i\}_i$, the modeling space for Δ_{M_i} can be easily shown to be $\Delta_{\mathsf{M}_{\phi_i}}$ which is complemented by $\mathsf{M}_{\phi_i} \times \{0\}$, where we use the projection $p_i^\phi\colon \mathsf{M}_{\phi_i} \times \mathsf{M}_{\phi_i} \to \Delta_{\mathsf{M}_{\phi_i}}$, $(x_i, y_i) \mapsto (y_i, y_i)$. Obviously, $(\epsilon_{ij}^\phi \times \epsilon_{ij}^\phi) \circ p_j^\phi = p_i^\phi \circ (\epsilon_{ij}^\phi \times \epsilon_{ij}^\phi)$. Finally, $\Delta_{\varprojlim M_i}$ is the image of $\varprojlim \Delta_{M_i}$ under the diffeomorphism $\varprojlim M_i \times \varprojlim M_i \to \varprojlim M_i \times M_i$.

2.2 Inverse systems of vector bundles

Let $k \in \mathbb{N} \cup \{\infty\}$, let M be a C^k-manifold, and let E be a locally convex space. Recall that we call the product $M \times \mathsf{E}$ a trivial C^k-vector bundle over M and the projection $\mathrm{pr}_1\colon M \times \mathsf{E} \to M$ the bundle projection which is obviously a surjective C^k-submersion. Let $N \times \mathsf{F}$ be another trivial C^k-vector bundle. A vector bundle morphism is a C^k-map $F\colon M \times \mathsf{E} \to N \times \mathsf{F}$ which is defined using a pair of C^k-maps $f\colon M \to N$ and $\widehat{F}\colon M \times \mathsf{E} \to \mathsf{F}$ such that $F(x,u) = (f(x), \widehat{F}(x,u))$ and $F_x := \widehat{F}(x,\cdot) \in \mathrm{L}(\mathsf{E},\mathsf{F})$ for all $(x,u) \in M \times \mathsf{E}$. We say that F is *simple* if it is a product morphism, that is, $F = f \times \widehat{F}$ with $\widehat{F} \in \mathrm{L}(\mathsf{E},\mathsf{F})$. The composition of two vector bundle morphisms is defined in the obvious way; let $L\colon N \times \mathsf{F} \to P \times \mathsf{F}'$ be another vector bundle morphism taking values in a trivial C^k-vector bundle $P \times \mathsf{F}'$. The composition is given by

$$(L \circ F)(x,u) = ((l \circ f)(x), \widehat{L}(f(x), \widehat{F}(x,u)))$$

for all $(x,u) \in M \times \mathsf{E}$. That is, $\widehat{(L \circ F)} = \widehat{L} \circ ((f \circ \mathrm{pr}_1), \widehat{F})$ which is C^k and linear in the second argument (equivalently, $(L \circ F)_x = L_{f(x)} \circ F_x \in \mathrm{L}(\mathsf{E},\mathsf{F}')$ for all $x \in M$). We say that F is an (trivial vector bundle) isomorphism if there exists a trivial C^k-vector bundle morphism $F^{-1}\colon N \times \mathsf{F} \to M \times \mathsf{E}$ such that $F \circ F^{-1} = \mathrm{id}_{N\times\mathsf{F}}$ and $F^{-1} \circ F = \mathrm{id}_{M\times\mathsf{E}}$. In particular, f is a C^k-diffeomorphism, $\widehat{F^{-1}}(f(x), \widehat{F}(x,u)) = u$ for all $(x,u) \in M \times \mathsf{E}$, and $\widehat{F}(f^{-1}(y), \widehat{F^{-1}}(y,w)) = w$ for all $(y,w) \in N \times \mathsf{F}$. That is, F^{-1} is can be written as

$$F^{-1}(y,w) = (f^{-1}(y), (F_{f^{-1}(y)})^{-1}(w)) \quad \forall (y,w) \in N \times \mathsf{F}.$$

Definition 2.17. A C^k*-vector bundle* is a C^k-manifold E together with a C^k-surjective map $p\colon E \to M$ onto a C^k-manifold M such that, for each $x \in M$, the fiber $E_x := p^{-1}(x)$

is a locally convex space, and such that there exists a family of C^k-diffeomorphisms (*local trivializations*) $\{\phi\colon p^{-1}(U_\phi) \to U_\phi \times \mathsf{F}_\phi\}$ such that $\{\mathsf{F}_\phi\}$ are locally convex spaces and $\{U_\phi\}$ is an open cover of M, satisfying the following conditions:

1. $\phi|_{p^{-1}(x)}\colon p^{-1}(x) \to \mathsf{F}_\phi$ is a topological isomorphism for all $x \in U_\phi$;
2. $\mathrm{pr}_1 \circ \phi = p|_{p^{-1}(U_\phi)}$, where $\mathrm{pr}_1\colon U_\phi \times \mathsf{F}_\phi \to U_\phi$ is the projection onto U_ϕ;
3. for any two local trivialization ϕ and ψ, the *transition map* $\psi \circ \phi^{-1}\colon (U_\phi \cap U_\psi) \times \mathsf{F}_\phi \to (U_\phi \cap U_\psi) \times \mathsf{F}_\psi$ is a trivial C^k-vector bundle isomorphism (hence $\mathsf{F}_\psi \simeq \mathsf{F}_\phi$).

Similar to the finite-dimensional case, the local triviality condition ensures that p is a C^k-submersion, the model spaces F_ϕ are (topologically) isomorphic on connected components, and the set of transition maps forms a Čech cocycle. One can alternatively define a vector bundle as a set and use the manifold structure defined by the system of local trivializations.

Remark 2.18. For C^k-Banach vector bundles, the above definition reduces to the standard one. We only need to note that Condition 3 is equivalent to the existence of a C^k-map $A\colon U_\phi \cap U_\psi \to \mathrm{Iso}(\mathsf{F}_\phi, \mathsf{F}_\psi)$ such that $(\psi \circ \phi^{-1})(x, v) = (x, A(x)v)$ for all $(x, v) \in (U_\phi \cap U_\psi) \times \mathsf{F}_\phi$, where $\mathrm{Iso}(\mathsf{F}_\phi, \mathsf{F}_\psi)$ is the open set of isomorphisms in the Banach space $\mathrm{L}(\mathsf{F}_\phi, \mathsf{F}_\psi)$. If $\psi \circ \phi^{-1}$ is a trivial C^k-vector bundle isomorphism, then we can take $A\colon x \mapsto \widehat{\psi \circ \phi^{-1}}(x, \cdot)$, which takes values in $\mathrm{Iso}(\mathsf{F}_\phi, \mathsf{F}_\psi)$. On the other hand, $(\psi \circ \phi^{-1})(x, u) = \mathrm{ev}(A(x), u)$ defines a trivial C^k-vector bundle isomorphism on account of the evaluation map $\mathrm{ev}\colon \mathrm{L}(\mathsf{F}_\phi, \mathsf{F}_\psi) \times \mathsf{F}_\phi \to \mathsf{F}_\psi$ being smooth.

Vector bundles morphisms are defined in the usual way.

Definition 2.19. Let $p_i\colon E_i \to M_i$, $i = 1, 2$, be vector bundles. A *C^k-vector bundle morphism* is a pair (F, f) of C^k-maps $F\colon E_1 \to E_2$ and $f\colon M_1 \to M_2$ such that $p_2 \circ F = f \circ p_1$ and the restriction $F|_{E_{1,x}}\colon E_{1,x} \to E_{2,f(x)}$ is linear.

Composition of C^k-vector bundle morphisms is a C^k-vector bundle morphism. The base map f is fully determined by F as $f(x) = p_2(F(0_x))$ for $x \in M_1$. So, if f is not relevant, we shall denote the vector bundle morphism by F.

Definition 2.20. A *strict C^k-inverse system of vector bundles* is a pair $(\{p_i\colon E_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ such that

1. For all $i \in I$, $p_i\colon E_i \to M_i$ is a C^k-vector bundle and, for all $i \leq j$, (F_{ij}, f_{ij}) is a C^k-vector bundle morphism;
2. $(\{M_i\}_i, \{f_{ij}\}_{i\leq j})$ and $(\{E_i\}_i, \{F_{ij}\}_{i\leq j})$ are strict C^k-inverse systems of manifolds, and $\{p_i\}_i$ is a C^k-morphism of inverse system of manifolds;
3. for all $(x_i)_i \in \varprojlim M_i$, there exists a *strict inverse system of local trivializations* $\{\psi_i\colon p_i^{-1}(U_{\psi_i}) \to U_{\psi_i} \times \mathsf{F}_{\psi_i}\}_i$. That is, $\varprojlim U_{\psi_i} \ni (x_i)$ is open and $(\{\mathsf{F}_{\psi_i}\}_i, \{\nu^\psi_{ij}\}_{i\leq j})$ is an inverse system of locally convex spaces such that, for all $i \leq j$, the diagram

$$\begin{tikzcd} p_j^{-1}(U_{\psi_j}) \arrow[r, "\psi_j"] \arrow[d, "F_{ij}"'] & U_{\psi_j} \times \mathsf{F}_{\psi_j} \arrow[d, "f_{ij} \times \nu^\psi_{ij}"] \\ p_i^{-1}(U_{\psi_i}) \arrow[r, "\psi_i"] & U_{\psi_i} \times \mathsf{F}_{\psi_i} \end{tikzcd}$$

commutes.

It is clear that $\varprojlim p_i$ is surjective and C^k. Note that, for a strict inverse system of local trivialization $\{\phi_i\}$, the diffeomorphism $\varprojlim \psi_i \colon \varprojlim(p_i^{-1}(U_{\psi_i})) \to \varprojlim U_{\psi_i} \times \mathsf{F}_{\psi_i}$ defines a local trivialization for $\varprojlim E_i$ on account of $(\varprojlim p_i)^{-1}(\varprojlim U_i) = \varprojlim(p_i^{-1}(U_i))$ and of the connecting morphisms being simple ensuring that $\varprojlim(U_{\psi_i} \times \mathsf{F}_{\psi_i}) \simeq \varprojlim U_{\psi_i} \times \varprojlim \mathsf{F}_{\psi_i}$. To check the overlap condition, we consider another strict inverse system of local trivializations $\{\phi_i\}_i$. Then, for all $i \leq j$, the following diagram commutes:

$$\begin{tikzcd}
(U_{\psi_j} \cap U_{\phi_j}) \times \mathsf{F}_{\phi_j} \arrow[r, "\psi_j \circ \phi_j^{-1}"] \arrow[d, "f_{ij} \times \nu_{ij}^{\phi}"'] & (U_{\psi_j} \cap U_{\phi_j}) \times \mathsf{F}_{\psi_j} \arrow[d, "f_{ij} \times \nu_{ij}^{\psi}"] \\
(U_{\psi_i} \cap U_{\phi_i}) \times \mathsf{F}_{\phi_i} \arrow[r, "\psi_i \circ \phi_i^{-1}"] & (U_{\psi_i} \cap U_{\phi_i}) \times \mathsf{F}_{\psi_i}
\end{tikzcd}$$

Indeed, immediately from the definition, we see that

$$(\psi_i \circ \phi_i^{-1}) \circ (f_{ij} \times \nu_{ij}^{\phi}) = (\psi_i \circ \phi_i^{-1}) \circ (\phi_i \circ F_{ij} \circ \phi_j^{-1}) = \psi_i \circ F_{ij} \circ \phi_j^{-1}$$

and similarly

$$(f_{ij} \times \nu_{ij}^{\psi}) \circ (\psi_j \circ \phi_j^{-1}) = (\psi_i \circ F_{ij} \circ \psi_j^{-1}) \circ (\psi_j \circ \phi_j^{-1}) = \psi_i \circ F_{ij} \circ \phi_j^{-1}.$$

Also from the above diagram, it follows that

$$\widehat{\varprojlim \psi_i \circ (\varprojlim \phi_i)^{-1}} = \varprojlim(\widehat{\psi_i \circ \phi_i^{-1}}).$$

Let $(e_i)_i \in \varprojlim E_i$ and let $(x_i)_i = (p_i(e_i))_i$. About $(x_i)_i$, we let $\{\kappa_i\}_i$ be strict inverse system of charts and $\{\psi_i\}_i$ be a strict inverse system of local trivialization. Then

$$\{p_i^{-1}(U_{\kappa_i} \cap U_{\psi_i}) \xrightarrow{\psi_i} (U_{\kappa_i} \cap U_{\psi_i}) \times \mathsf{F}_{\psi_i} \xrightarrow{\kappa_i \times \mathrm{id}_{\mathsf{F}_{\psi_i}}} \kappa_i(U_{\kappa_i} \cap U_{\psi_i}) \times \mathsf{F}_{\psi_i}\}_i \tag{2}$$

is a strict inverse system of charts at $(e_i)_i$ which along $\{\phi_i\}_i$ define C^k-submersion charts for $\varprojlim p_i$. Thus, $\varprojlim p_i$ is a C^k-submersion.

We have proven the following.

Proposition 2.21. *Let $(\{p_i \colon E_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i \leq j})$ be a strict C^k-inverse system of vector bundles. Then $\varprojlim p_i \colon \varprojlim E_i \to \varprojlim M_i$ is a C^k-vector bundle.*

One can define saturated inverse systems of local trivializations so that 1 holds. In particular, if $\{E_i\}_i$ admits such system, then it is a strict inverse system of manifolds.

Example 2.22. Let $(\{M_i\}_i, \{f_{ij}\}_{i \leq j})$ be a strict C^k-inverse system of smooth manifolds and let $(\{\mathsf{E}_i\}_i, \{\epsilon_{ij}\}_{i \leq j})$ be an inverse system of locally convex spaces. Then $(\{\mathrm{pr}_1^i \colon M_i \times \mathsf{E}_i \to M_i\}, \{f_{ij} \times \epsilon_{ij}\}_{i \leq j})$ is a strict C^k-inverse system of vector bundles.

Example 2.23. Let $(\{M_i\}_i, \{f_{ij}\}_{i \leq j})$ be a strict C^k-inverse system of smooth manifolds. Then $(\{\pi_{TM_i} \colon TM_i \to M_i\}, \{(Tf_{ij}, f_{ij})\}_{i \leq j})$ is a strict C^{k-1}-inverse system of vector bundles. Let $\{\phi_i\}_i$ and $\{\psi_i\}_i$ be strict inverse system of charts and denote $\tilde{\phi}_i := (\phi_i^{-1} \times \mathrm{id}_{\mathsf{M}_{\phi_i}}) \circ T\phi_i$ and $\tilde{\psi}_i := (\psi_i^{-1} \times \mathrm{id}_{\mathsf{M}_{\psi_i}}) \circ T\psi_i$, for $i \in I$. Then the following diagram commutes.

$$\begin{array}{ccc}
(U_{\phi_j} \cap U_{\psi_j}) \times \mathsf{M}_{\phi_j} & \xrightarrow{\tilde{\psi}_j \circ \tilde{\phi}_j^{-1}} & (U_{\phi_j} \cap U_{\psi_j}) \times \mathsf{M}_{\psi_j} \\
{\scriptstyle f_{ij} \times \epsilon_{ij}^{\phi}} \big\downarrow & & \big\downarrow {\scriptstyle f_{ij} \times \epsilon_{ij}^{\psi}} \\
(U_{\phi_i} \cap U_{\psi_i}) \times \mathsf{M}_{\phi_i} & \xrightarrow{\tilde{\psi}_i \circ \tilde{\phi}_i^{-1}} & (U_{\phi_i} \cap U_{\psi_i}) \times \mathsf{M}_{\psi_i}
\end{array}$$

To define morphisms of inverse systems of vector bundles, we need compatibility on the total spaces and the base spaces. We record this explicitly in the following definition and lemma.

Definition 2.24. Let $(\{p_i\colon E_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ and $(\{\tilde{p}_i\colon \tilde{E}_i \to \tilde{M}_i\}_i, \{(\tilde{F}_{ij}, \tilde{f}_{ij})\}_{i\leq j})$ be C^k-inverse systems of vector bundles. For $m \leq k$, a *C^m-morphism of inverse systems of vector bundles* is a family

$$\{(\Theta_i\colon E_i \to \tilde{E}_i, \theta_i\colon M_i \to \tilde{M}_i)\}_i$$

of C^m-vector bundle morphisms such that the diagrams

$$\begin{array}{ccc}
E_j & \xrightarrow{\Theta_j} & \tilde{E}_j \\
{\scriptstyle F_{ij}} \big\downarrow & & \big\downarrow {\scriptstyle \tilde{F}_{ij}} \\
E_i & \xrightarrow{\Theta_i} & \tilde{E}_i
\end{array}
\qquad
\begin{array}{ccc}
M_j & \xrightarrow{\theta_j} & \tilde{M}_j \\
{\scriptstyle f_{ij}} \big\downarrow & & \big\downarrow {\scriptstyle \tilde{f}_{ij}} \\
M_i & \xrightarrow{\theta_i} & \tilde{M}_i
\end{array}$$

commute. If each (Θ_i, θ_i) is a C^m-vector bundle isomorphism then $\{(\Theta_i, \theta_i)\}_i$ is a *C^m-isomorphism of inverse systems of vector bundles*.

Lemma 2.25. *In Definition 2.24, if $(\{p_i\colon E_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ and $(\{\tilde{p}_i\colon \tilde{E}_i \to \tilde{M}_i\}_i, \{(\tilde{F}_{ij}, \tilde{f}_{ij})\}_{i\leq j})$ are strict, then $\varprojlim(\Theta_i, \theta_i) = (\varprojlim \Theta_i, \varprojlim \theta_i)$ is a C^m-vector bundle morphism. If $\{(\Theta_i, \theta_i)\}$ is a C^m-isomorphism of inverse systems of vector bundles, then $\varprojlim(\Theta_i, \theta_i)$ is a C^m-isomorphism.*

An immediate trivial example is the family $\{\mathrm{id}_{E_i}, \mathrm{id}_{M_i}\}_i$ where $\varprojlim(\mathrm{id}_{E_i}, \mathrm{id}_{M_i}) = (\mathrm{id}_{\varprojlim E_i}, \mathrm{id}_{\varprojlim M_i})$.

Definition 2.26. Let $p\colon E \to M$ be a C^k-vector bundle. A *subbundle* of E is a subset $L \subseteq E$ such that for every $x \in M$, the intersection $L_x := L \cap E_x$ is a closed linear subspace and the restriction $p|_L\colon L \to M$ defines a vector bundle over M. Equivalently, for every $x \in M$, there exists a local trivialization $\phi\colon p^{-1}(U_\phi) \to U_\phi \times \mathsf{F}_\phi$ and closed linear subspace $\mathsf{F}'_\phi \subseteq \mathsf{F}_\phi$ such that the restriction $\phi|_{p^{-1}(U_\phi)\cap L}\colon p^{-1}(U_\phi) \cap L \to U_\phi \times \mathsf{F}'_\phi$ is a C^k-diffeomorphism, thus defining a local trivialization for L about x. In particular L is a C^k-embedded submanifold of E.

Similar to the definitions of submanifolds and split-submanifolds, we define subbundles and split-subbundles.

Definition 2.27. Let $(\{p_i\colon E_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ be a strict C^k-inverse system of vector bundles. A family of subbundles $\{L_i\}_i$ is a *strict inverse system of subbundles* if it satisfies the following conditions:

1. $L_i \subseteq E_i$ is a subbundle for $i \in I$;

2. $F_{ij}|_{L_j}$ takes values in L_i for $i \leq j$;
3. for every $(x_i)_i \in \varprojlim M_i$, there exists a strict inverse system of local trivializations $\phi_i \colon p_i^{-1}(U_{\phi_i}) \to U_{\phi_i} \times \mathsf{F}_{\phi_i}$ such that $\phi_i(p_i^{-1}(U_{\phi_i}) \cap L_i) = U_{\phi_i} \times \mathsf{F}'_{\phi_i}$, where F'_{ϕ_i} is a closed linear subspace of F_{ϕ_i}.

Similarly, we define a *strict inverse systems of split-subbundles* by replacing Condition 3 by the following stronger condition:

3'. for every $(x_i)_i \in \varprojlim M_i$, there exists a strict inverse system of local trivializations $\phi_i \colon p_i^{-1}(U_{\phi_i}) \to U_{\phi_i} \times \mathsf{F}_{\phi_i}$ such that $\phi_i(p_i^{-1}(U_{\phi_i}) \cap L_i) = U_{\phi_i} \times \mathsf{F}'_{\phi_i}$, where F'_{ϕ_i} is a complemented subspace of F_{ϕ_i} such that for all $i \leq j$, the projections $p_i^{\phi} \colon \mathsf{F}_{\phi_i} \to \mathsf{F}'_{\phi_i}$ satisfies $p_i^{\phi} \circ \nu_{ij}^{\phi} = \nu_{ij}^{\phi} \circ p_j^{\phi}$.

To see that $\varprojlim L_i$ is a well-defined subbundle of $\varprojlim E_i$, we note that, for all $i \leq j$,

$$\begin{aligned}(f_{ij} \times \nu_{ij}^{\phi})(U_{\phi_j} \times \mathsf{F}'_{\phi_j}) &= (f_{ij} \times \nu_{ij}^{\phi})(\phi_j(p_j^{-1}(U_{\phi_j}) \cap L_j)) \\ &= \phi_i(F_{ij}(p_j^{-1}(U_{\phi_j}) \cap L_j)) \subseteq \phi_i(p_i^{-1}(U_{\phi_i}) \cap L_i) = U_{\phi_i} \times \mathsf{F}'_{\phi_i}.\end{aligned}$$

In particular, $(\{\mathsf{F}'_{\phi_i}\}_i, \{\nu_{ij}^{\phi}|_{\mathsf{F}_{\phi_j}}\}_{i \leq j})$ is an inverse system of locally convex space and the following diagram commutes.

$$\begin{array}{ccc} p_j^{-1}(U_{\phi_j}) \cap L_j & \xrightarrow{\phi_j} & U_{\phi_j} \times \mathsf{F}_{\phi_j} \\ \Big\downarrow{\scriptstyle F_{ij}} & & \Big\downarrow{\scriptstyle f_{ij} \times \nu_{ij}^{\phi}} \\ p_i^{-1}(U_{\phi_i}) \cap L_i & \xrightarrow{\phi_i} & U_{\phi_i} \times \mathsf{F}_{\phi_i} \end{array}$$

Thus,

$$\varprojlim \phi_i(\varprojlim p_j^{-1}(U_{\phi_j}) \cap \varprojlim L_j) \simeq \varprojlim U_{\phi_i} \times \varprojlim \mathsf{F}'_{\phi_i}$$

defines a local trivialization for $\varprojlim L_i$. Moreover if $\{L_i\}$ is a strict inverse system of split-subbundles, then $\varprojlim p_i^{\phi} \colon \varprojlim \mathsf{F}_{\phi_i} \to \varprojlim \mathsf{F}'_{\phi_i}$ is a projection onto $\varprojlim \mathsf{F}'_{\phi_i}$ showing that $\varprojlim L_i$ is split-subbundle.

Remark 2.28. Definition 2.27 can be generalized to subbundles over submanifolds, that is, to a strict C^k-inverse systems of (split-)subbundles $\{p_i|_{E_{N_i}} \colon E_{N_i} \to N_i\}_i$ where $\{N_i\}_i$ is a strict inverse system of (split-)submanifolds, by requiring that for all $(x_i) \in \varprojlim N_i$, there exists a strict inverse system of local trivializations $\{\phi_i\}_i$ such that

$$\phi_i \colon p_i^{-1}(U_{\phi_i} \cap N_i) \cap E_{N_i} \to (U_{\phi_i} \cap N_i) \times \tilde{\mathsf{F}}_{\phi_i},$$

is C^k-diffeomorphism, where $\tilde{\mathsf{F}}_{\phi_i} \subseteq \mathsf{F}_{\phi_i}$ is a (complemented) closed subspace. In fact, this is consistent with the definition above, by realizing this as a strict C^k-inverse system of (split-)subbundles of the strict C^k-inverse system of pullback bundles $(\{\iota_i^* p_i \colon \iota_i^* E_i \to N_i\}_i, \{f_{ij}|_{N_j} \times F_{ij}\}_{i \leq j})$, where $\iota_i \colon N_i \to M_i$ is the canonical inclusion.

Proposition 2.29. *Let* $(\{p_i\colon E_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\le j})$ *be a strict* C^k*-inverse system of vector bundles and let* $\{L_i\}_i$ *be a strict inverse system of (split-)subbundles. Then* $\varprojlim L_i$ *is a (split-)subbundle of* $\varprojlim E_i$*. In particular,* $(\{p_i|_{L_i}\colon L_i \to M_i\}_i, \{(F_{ij}|_{L_j}, f_{ij})\}_{i\le j})$ *is strict* C^k*-inverse system of vector bundles.*

Example 2.30. Let $(\{M_i\}_i, \{f_{ij}\}_{i\le j})$ be a strict C^k-inverse system of smooth manifolds, let $(\{\mathsf{E}_i\}_i, \{\epsilon_{ij}\}_{i\le j})$ be an inverse system of locally convex spaces, and let $\{\mathsf{F}_i\}_i$ be a family of closed linear subspaces such that $\epsilon_{ij}(\mathsf{F}_j) \subseteq \mathsf{F}_i$ for $i \le j$. Then $\{M_i \times \mathsf{F}_i\}_i$ is a strict inverse system of subbundles of $(\{\mathrm{pr}_1^i\colon M_i \times \mathsf{E}_i \to M_i\}, \{f_{ij} \times \epsilon_{ij}\}_{i\le j})$. Moreover, if each F_i is complemented such that the projections $\pi_i\colon \mathsf{E}_i \to \mathsf{F}_i$ satisfy $\pi_i \circ \epsilon_{ij} = \epsilon_{ij} \circ \pi_j$ whenever $i \le j$, then $\{M_i \times \mathsf{F}_i\}_i$ is a strict inverse system of split-subbundles. In particular, $\varprojlim M_i \times \varprojlim \mathsf{F}_i$ is a (split-)subbundle of $\varprojlim M_i \times \varprojlim \mathsf{E}_i$.

Example 2.31. Let $(\{M_i\}_{i\in I}, \{f_{ij}\}_{i\le j\in I})$ be a strict C^k-inverse system of manifolds. Let $\{S_i\}_i$ be a strict inverse system of (split-)submanifolds. Then $\{TS_i\}_i$ is a strict inverse system of (split-)subbundles of $(\{\pi_{TM_i}\colon TM_i \to M_i\}, \{(Tf_{ij}, f_{ij})\}_{i\le j})$ with $T\varprojlim S_i = \varprojlim TS_i$.

Recall that we have a canonical embedding $Z_{M_i}\colon M_i \to E_i$ defined by $Z_{M_i}(x_i) = 0_{x_i}$. Then $\{Z_{M_i}\}_i$ is a strict C^k-embedding of inverse systems satisfying $\varprojlim Z_{M_i} = Z_{\varprojlim M_i}$.

Proposition 2.32. *Let* $(\{p_i\colon E_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\le j})$ *be a strict* C^k*-inverse system of vector bundles, let* $(\{N_i\}_i, \{g_{ij}\}_{i\le j})$ *be a strict* C^k*-inverse system of manifolds, and let* $\{h_i\colon N_i \to M_i\}_i$ *be a* C^k*-morphism of inverse systems of manifolds. Then* $(\{h_i^*p_i\colon h_i^*E_i \to N_i\}_i, \{(g_{ij} \times F_{ij}, g_{ij})\}_{i\le j})$ *is a strict* C^k*-inverse system of vector bundles such that its limit is isomorphic to the pullback bundle*

$$(\varprojlim h_i)^* \varprojlim p_i\colon (\varprojlim h_i)^* \varprojlim E_i \to \varprojlim N_i.$$

Proof. By Theorem 2.48, $(\{h_i^*E_i\}_i, \{(g_{ij} \times F_{ij})|_{h_j^*E_j}\}_{i\le j})$ is a strict C^k-inverse system of manifolds. For any $(y_i)_i \in \varprojlim N_i$, there exists a strict inverse system of charts $\{\phi_i\}_i$ about $(y_i)_i$ and strict inverse system of local trivialization $\{\psi_i\}_i$ about $(h_i(y_i))_i$ such that (possibly after shrinking) $h_i(U_{\phi_i}) \subseteq U_{\psi_i}$. Then

$$\Theta_{\phi_i,\psi_i} := \mathrm{id}_{U_{\phi_i}} \times (\mathrm{pr}_2 \circ \psi_i)\colon (h_i^*E_i) \cap (U_{\phi_i} \times p_i^{-1}(U_{\psi_i})) \to U_{\phi_i} \times \mathsf{F}_{\psi_i}$$

is a local trivialization for $h_i^*E_i$ noting that

$$(h_i^*p_i)^{-1}(U_{\phi_i}) = (U_{\phi_i} \times p_i^{-1}(U_{\psi_i})).$$

The inverse $\Theta^{-1}_{\phi_i,\psi_i}$ is the C^k-map

$$\Theta^{-1}_{\phi_i,\psi_i}\colon U_{\phi_i} \times \mathsf{F}_{\psi_i} \ni (z_i, w_i) \mapsto (z_i, \psi_i^{-1}(h_i(z_i), w_i)) \in (h_i^*E_i) \cap (U_{\phi_i} \times p_i^{-1}(U_{\psi_i})).$$

Let $\{\tilde\phi_i\}_i$ be another strict inverse of charts about $(y_i)_i$ and $\{\tilde\psi_i\}_i$ be another strict inverse system of local trivializations about $(h_i(y_i))_i$. Then the transition map is given by

$$(U_{\phi_i} \cap U_{\tilde\phi_i}) \times \mathsf{F}_{\psi_i} \ni (z_i, w_i) \mapsto (z_i, \mathrm{pr}_2((\tilde\psi_i \circ \psi_i^{-1})(h_i(z_i), w_i))) \in (U_{\phi_i} \cap U_{\tilde\phi_i}) \times \mathsf{F}_{\tilde\psi_i}.$$

This defines a C^k-vector bundle structure over N_i on $h_i^*E_i$ for all $i \in I$. For any $i \leq j$ and $(y_j, e_j) \in h_j^*E_j$, we have

$$p_i(F_{ij}(e_j)) = f_{ij}(p_j(e_j)) = f_{ij}(h_j(y_j)) = h_i(g_{ij}(y_j)).$$

Hence $(g_{ij} \times F_{ij})|_{h_j^*E_j} \colon h_j^*E_j \to h_i^*E_i$ is a well-defined map. Obviously, $(g_{ij} \times F_{ij})|_{\{y_i\}\times E_{i,f_i(y_i)}}$ is linear and $(h_i^*p_i) \circ (g_{ij} \times F_{ij}) = g_{ij} \circ (h_j^*p_j)$. Hence it is a C^k-vector bundle morphism. Now consider the following commutative diagram

$$\begin{array}{ccc}
(h_j^*p_j)^{-1}(U_{\phi_j}) = (h_j^*E_j) \cap (U_{\phi_j} \times p_j^{-1}(U_{\psi_j})) & \xrightarrow{\Theta_{\phi_j,\psi_j}} & U_{\phi_j} \times \mathsf{F}_{\psi_j} \\
{\scriptstyle g_{ij}\times F_{ij}}\big\downarrow & & \big\downarrow{\scriptstyle f_{ij}\times \nu_{ij}^{\psi}} \\
(h_i^*p_i)^{-1}(U_{\phi_i}) = (h_i^*E_i) \cap (U_{\phi_i} \times p_i^{-1}(U_{\psi_i})) & \xrightarrow{\Theta_{\phi_i,\psi_i}} & U_{\phi_i} \times \mathsf{F}_{\psi_i}
\end{array}$$

This shows that $(\{h_i^*p_i \colon h_i^*E_i \to N_i\}_i, \{(g_{ij} \times F_{ij}, g_{ij})\}_{i\leq j})$ is a strict C^k-inverse system of vector bundles. By Proposition 2.21, $\varprojlim h_i^*p_i \colon \varprojlim h_i^*E_i \to N_i$ is a C^k-vector bundle. The final part follows by constructing a C^k-vector bundle isomorphism $\Upsilon \colon \varprojlim h_i^*E_i \to (\varprojlim h_i)^* \varprojlim E_i$, $(y_i, e_i)_i \mapsto ((y_i)_i, (e_i)_i)$. Clearly Υ is a bijection, linear on the fibers, and $(\varprojlim h_i)^*(\varprojlim p_i) \circ \Upsilon = \varprojlim h_i^*p_i$. To show that Υ is C^k, we pick a strict system of charts $\{\phi_i\}_i$ about $(y_i)_i$ and a strict system of local trivializations about $(h_i(y_i))_i$ as above. Then it is enough to note that the composition $\Theta_{\varprojlim \phi_i, \varprojlim \psi_i} \circ \Upsilon \circ (\varprojlim \Theta_{\phi_i,\psi_i})^{-1}$, given by

$$\varprojlim U_{\phi_i} \times \mathsf{F}_{\psi_i} \ni (y_i, u_i)_i \mapsto ((y_i)_i, (u_i)_i) \in \varprojlim U_{\phi_i} \times \varprojlim \mathsf{F}_{\psi_i},$$

is C^k, but this is trivial. This completes the proof. $\square$

Remark 2.33. We note that the transition map can be verified to be a trivial vector bundle C^k-morphism explicitly by showing that the following diagram commutes.

$$\begin{array}{ccc}
(U_{\phi_j} \cap U_{\tilde{\phi}_j}) \times \mathsf{F}_{\psi_j} & \xrightarrow{\Theta_{\tilde{\phi}_j,\tilde{\psi}_j} \circ \Theta_{\phi_j,\psi_j}^{-1}} & (U_{\phi_j} \cap U_{\tilde{\phi}_j}) \times \mathsf{F}_{\tilde{\psi}_j} \\
{\scriptstyle g_{ij}\times \nu_{ij}^{\psi}}\big\downarrow & & \big\downarrow{\scriptstyle g_{ij}\times \nu_{ij}^{\tilde{\psi}}} \\
(U_{\phi_i} \cap U_{\tilde{\phi}_i}) \times \mathsf{F}_{\psi_i} & \xrightarrow{\Theta_{\tilde{\phi}_i,\tilde{\psi}_i} \circ \Theta_{\phi_i,\psi_i}^{-1}} & (U_{\phi_i} \cap U_{\tilde{\phi}_i}) \times \mathsf{F}_{\tilde{\psi}_i}
\end{array}$$

To prove this, let $(z_j, w_j) \in (U_{\phi_j} \cap U_{\tilde{\phi}_j}) \times \mathsf{F}_{\psi_j}$ then compute

$$((g_{ij} \times \nu_{ij}^{\tilde{\psi}}) \circ (\Theta_{\tilde{\phi}_j,\tilde{\psi}_j} \circ \Theta_{\phi_j,\psi_j}^{-1}))(z_j, w_j) = (g_{ij} \times \nu_{ij}^{\tilde{\psi}})(z_j, \mathrm{pr}_2((\tilde{\psi}_j \circ \psi_j^{-1})(h_j(z_j), w_j)))$$

and

$$\begin{aligned}
&((\Theta_{\tilde{\phi}_i,\tilde{\psi}_i} \circ \Theta_{\phi_i,\psi_i}^{-1}) \circ (g_{ij} \times \nu_{ij}^{\psi}))(z_j, w_j) \\
&\qquad = (g_{ij}(z_j), \mathrm{pr}_2((\tilde{\psi}_i \circ \psi_i^{-1})(h_i(g_{ij}(z_j)), \nu_{ij}^{\psi}(w_j)))) \\
&\qquad = (g_{ij}(z_j), \mathrm{pr}_2(((\tilde{\psi}_i \circ \psi_i^{-1}) \circ (f_{ij} \times \nu_{ij}^{\psi}))(h_j(z_j), w_j))) \\
&\qquad = (g_{ij}(z_j), \mathrm{pr}_2(((f_{ij} \times \nu_{ij}^{\tilde{\psi}}) \circ (\tilde{\psi}_j \circ \psi_j^{-1}))(h_j(z_j), w_j))).
\end{aligned}$$

Lemma 2.34. *Let* $(\{p_i\colon E_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ *and* $(\{\tilde{p}_i\colon \tilde{E}_i \to M_i\}_i, \{(\tilde{F}_{ij}, f_{ij})\}_{i\leq j})$ *be strict* C^k*-inverse systems of vector bundles. Denote* $q_i := p_i \circ \mathrm{pr}_1 |_{E_i\oplus \tilde{E}_i}$ *for all* i*. Then* $(\{q_i\colon E_i \oplus \tilde{E}_i \to M_i\}_i, \{(F_{ij} \times \tilde{F}_{ij}, f_{ij})\}_{i\leq j})$ *is a strict* C^k*-inverse system of vector bundles. Moreover, we have a* C^k*-verctor bundle isomorphism over* $\varprojlim M_i$

$$\varprojlim(E_i \oplus \tilde{E}_i) \simeq \varprojlim E_i \oplus \varprojlim \tilde{E}_i.$$

Proof. It is enough to construct a strict inverse system of local trivializations for $\{E_i\oplus\tilde{E}_i\}_i$. Let $(x_i)_i \in \varprojlim M_i$ and let $\{\phi_i\}_i$ and $\{\tilde{\phi}_i\}_i$ be strict inverse systems of local trivializations about (x_i) such that (possibly after shrinking) $U_{\phi_i} = U_{\tilde{\phi}_i}$ for all i. Define $\theta_i\colon q_i^{-1}(U_{\phi_i}) \to U_{\phi_i} \times \mathsf{F}_{\phi_i} \times \mathsf{F}_{\tilde{\phi}_i}$ by

$$\theta_i(e_i, \tilde{e}_i) = (q_i(e_i, \tilde{e}_i), \mathrm{pr}_2 \circ\phi_i(e_i), \mathrm{pr}_2 \circ\tilde{\phi}_i(\tilde{e}_i)) \quad \forall (e_i, \tilde{e}_i) \in q_i^{-1}(U_{\phi_i}), \tag{3}$$

noting that $q_i^{-1}(U_{\phi_i}) = (p_i^{-1}(U_{\phi_i}) \times \tilde{p}_i^{-1}(U_{\tilde{\phi}_i})) \cap (E_i \oplus \tilde{E}_i)$ is open. Then θ_i is clearly a C^k-diffeomorphism where its inverse is given by

$$\theta_i^{-1}(x_i, u_i, \tilde{u}_i) = (\phi_i^{-1}(x_i, u_i), \tilde{\phi}_i^{-1}(x_i, \tilde{u}_i)) \quad \forall (x_i, u_i, \tilde{u}_i) \in U_{\phi_i} \times \mathsf{F}_{\phi_i} \times \mathsf{F}_{\tilde{\phi}_i}.$$

Moreover, for any strict inverse systems of local trivializations $\{\psi_i\}_i$ and $\{\tilde{\psi}_i\}$ for $\{E_i\}_i$ and $\{\tilde{E}_i\}_i$ about (x_i) defining a C^k-diffeomorphism in the same way as θ_i, using (3), the transition map

$$\theta_i \circ \tau_i^{-1}\colon (U_{\phi_i} \cap U_{\psi_i}) \times \mathsf{F}_{\psi_i} \times \mathsf{F}_{\tilde{\psi}_i} \to (U_{\phi_i} \cap U_{\psi_i}) \times \mathsf{F}_{\phi_i} \times \mathsf{F}_{\tilde{\phi}_i}$$

is C^k and is given by

$$\theta_i \circ \tau_i^{-1}(x_i, w_i, \tilde{w}_i) = (x_i, (\phi_i \circ \psi_i^{-1})(x_i, w_i), (\tilde{\phi}_i \circ \tilde{\psi}_i^{-1})(x_i, \tilde{w}_i)).$$

Finally, note that

$$\begin{array}{ccc} q_j^{-1}(U_{\phi_j}) & \xrightarrow{\theta_j} & U_{\phi_j} \times \mathsf{F}_{\phi_j} \times \mathsf{F}_{\tilde{\phi}_j} \\ {\scriptstyle F_{ij}\times\tilde{F}_{ij}}\Big\downarrow & & \Big\downarrow{\scriptstyle f_{ij}\times\nu_{ij}^{\phi}\times\nu_{ij}^{\tilde{\phi}}} \\ q_i^{-1}(U_{\phi_i}) & \xrightarrow{\theta_i} & U_{\phi_i} \times \mathsf{F}_{\phi_i} \times \mathsf{F}_{\tilde{\phi}_i} \end{array}$$

is commutative and $\varprojlim q_i^{-1}(U_{\phi_i}) = (\varprojlim p_i^{-1}(U_{\phi_i}) \times \varprojlim \tilde{p}_i^{-1}(U_{\tilde{U}_{\tilde{\phi}_i}})) \cap \varprojlim E_i \oplus \tilde{E}_i$ is open. Thus, $\{\theta_i\}_i$ is a strict inverse system of local trivializations. $\square$

Globally complemented subbundles and the decomposition of $T\varprojlim E_i$

Let $(\{p_i\colon E_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ be a strict C^k-inverse system of vector bundles. Note that a strict inverse system of split-subbundles $\{L_i\}_i$ is only locally complemented. In fact, $\{L_i\}_i$ is globally complemented if there exists a stict C^k-inverse system of split-subbundles $\{\tilde{L}_i\}_i$ such that

1. for all $i \in I$, the exact sequence (of vector bundles)

$$0 \to L_i \xrightarrow{\iota_i} E_i \xrightarrow{\pi_i} \tilde{L}_i \to 0$$

   splits. That is, $E_i = L_i \oplus \tilde{L}_i$;

2. $\{\pi_i\}_i$ is a C^k-morphism of inverse systems of vector bundles.

In turn, the exact sequence

$$0 \to \varprojlim L_i \xrightarrow{\iota} \varprojlim E_i \xrightarrow{\pi} \varprojlim \tilde{L}_i \to 0$$

C^k-splits, where $\iota = \varprojlim \iota_i$ and $\pi = \varprojlim \pi_i$.[3] In particular, $\varprojlim E_i = \varprojlim L_i \oplus \varprojlim \tilde{L}_i$.

A particular case of interest is when $\{h_i \colon M_i \to N_i\}_i$ is a strict C^k-submersion. Then $\{\ker(Th_i)\}_i$ is a strict C^{k-1}-inverse systems of split-distributions of $\{TM_i\}_i$. Moreover, $\{\ker(Th_i)\}_i$ is globally complemented if there exists a C^{k-1}-morphism of strict inverse vector bundles $\{\sigma_i \colon h_i^*TN_i \to TM_i\}_i$ such that

$$0 \to \ker(Th_i) \xrightarrow{\iota_i} TM_i \xrightarrow{(\pi_{TM_i}, Th_i)} h_i^*TN_i \to 0$$

splits. On the other hand, if there exists a distribution L_i such that $TM_i = \ker(Th_i) \oplus L_i$, hence $(L_i)_{x_i} \simeq (h_i^*TN_i)_{x_i}$, then $\sigma_i \colon h_i^*TN_i \to TM_i$ is defined by $\sigma_i(v_i) = (T_{x_i}h_i|_{L_i})^{-1}(v_i)$ $\forall v_i \in (h_i^*TN_i)_{x_i}$.

Alternatively, this correspondence can be easily seen locally. Let $(x_i)_i \in \varprojlim M_i$ and let $\{\phi_i\}_i$ and $\{\psi_i\}$ be strict submersion charts for $\{h_i\}_i$ at (x_i) and $(h_i(x_i))_i$ such that $h_i(U_{\phi_i}) \subseteq U_{\psi_i}$. We define a C^{k-1}-chart $\theta_i \colon U_{\theta_i} \to V_{\theta_i}$ for $M_i \times TN_i$ by setting $U_{\theta_i} = U_{\phi_i} \times TU_{\psi_i}$ and

$$\theta_i(x_i, v_{y_i}) := (\phi_i(x_i), \psi_i(y_i) - \psi_i(h_i(x_i)), \mathrm{pr}_2 \circ T\psi_i(v_{y_i})) \quad \forall (x_i, v_{y_i}) \in U_{\phi_i} \times TU_{\psi_i}.$$

Now

$$V_{\theta_i} = \{(a_i, b_i, c_i) \in V_\phi \times \mathsf{N}_{\psi_i} \times \mathsf{N}_{\psi_i} : b_i + \psi_i(h_i(\phi_i^{-1}(a_i))) \in V_{\psi_i}\}$$

is open in $\mathsf{M}_{\phi_i} \times \mathsf{N}_{\psi_i} \times \mathsf{N}_{\psi_i}$ as

$$V_{\theta_i} = \kappa_i^{-1}(V_{\phi_i} \times V_{\psi_i} \times \mathsf{N}_{\psi_i}),$$

where the C^k-map $\kappa_i \colon V_{\phi_i} \times \mathsf{N}_{\psi_i} \times \mathsf{N}_{\psi_i} \to V_{\phi_i} \times \mathsf{N}_{\psi_i} \times \mathsf{N}_{\psi_i}$ is defined by $(a_i, b_i, c_i) \mapsto (a_i, b_i + \psi_i(h_i(\phi_i^{-1}(a_i))), c_i)$. The inverse $\theta_i^{-1} \colon V_{\theta_i} \to U_{\theta_i}$ is given by

$$\theta_i^{-1}(a_i, b_i, c_i) = (\phi_i^{-1}(a_i), (T\psi_i)^{-1}(b_i + \psi_i(h_i(\phi_i^{-1}(a_i))), c_i)).$$

Now note that for all $a_j \in V_{\phi_j}$,

$$\begin{aligned}\psi_i(h_i(\phi_i^{-1}(\epsilon_{ij}^\phi(a_j)))) &= \psi_i(h_i(f_{ij}(\phi_j^{-1}(a_j))) \\ &= \psi_i(g_{ij}(h_j(\phi_j^{-1}(a_j))) = \epsilon_{ij}^\psi(\psi_j(h_j(\phi_j^{-1}(a_j)))).\end{aligned}$$

Hence, the diagram

$$\begin{array}{ccc} V_{\phi_j} \times \mathsf{N}_{\psi_j} \times \mathsf{N}_{\psi_j} & \xrightarrow{\kappa_j} & V_{\phi_j} \times \mathsf{N}_{\psi_j} \times \mathsf{N}_{\psi_j} \\ {\scriptstyle \epsilon_{ij}^\phi \times \epsilon_{ij}^\psi \times \epsilon_{ij}^\psi}\downarrow & & \downarrow{\scriptstyle \epsilon_{ij}^\phi \times \epsilon_{ij}^\psi \times \epsilon_{ij}^\psi} \\ V_{\phi_i} \times \mathsf{N}_{\psi_i} \times \mathsf{N}_{\psi_i} & \xrightarrow{\kappa_i} & V_{\phi_i} \times \mathsf{N}_{\psi_i} \times \mathsf{N}_{\psi_i} \end{array}$$

[3] Recall that $\varprojlim$ is only left exact. The strong compatibility conditions assumed here makes $\varprojlim \pi_i$ a projection.

commutes. As a result,

$$\varprojlim V_{\theta_i} = (\varprojlim \kappa_i)^{-1}(\varprojlim V_{\phi_i} \times \varprojlim V_{\psi_i} \times \varprojlim \mathsf{N}_{\psi_i})$$

is open. Finally, using a similar argument, the diagram

$$\begin{array}{ccc} U_{\phi_j} \times TU_{\psi_j} & \xrightarrow{\theta_j} & V_{\phi_j} \times \mathsf{N}_{\psi_j} \times \mathsf{N}_{\psi_j} \\ {\scriptstyle f_{ij}\times Tg_{ij}}\downarrow & & \downarrow{\scriptstyle \epsilon^{\phi}_{ij}\times\epsilon^{\psi}_{ij}\times\epsilon^{\psi}_{ij}} \\ U_{\phi_i} \times TU_{\psi_i} & \xrightarrow{\theta_i} & V_{\phi_i} \times \mathsf{N}_{\psi_i} \times \mathsf{N}_{\psi_i} \end{array}$$

commutes. Thus $\{\theta_i\}_i$ is a strict chart. Since for all $i \in I$, $\theta_i(h_i^*TN_i \cap U_{\theta_i}) = V_{\phi_i} \times \{0\} \times \mathsf{N}_{\psi_i}$, it is easy to see that $\{\theta_i\}_i$ is a strict split-submanifolds chart for $\{h_i^*TN_i\}_i$. In fact, they define a system of local trivialization over $\{M_i\}_i$. Now note that $\{T\phi_i\}_i$ is a strict inverse system of subbundle charts for $\{\ker(Th_i)\}_i$. Indeed, $T\phi_i(\ker(Th_i) \cap TU_{\phi_i}) = V_{\phi_i} \times \{0\} \times \mathsf{C}_i$, where C_i is such that $\mathsf{M}_{\phi_i} \simeq \mathsf{N}_{\psi_i} \times \mathsf{C}_i$. In particular, if $\{L_i\}_i$ is a strict inverse system of split-subbundles such that $TM_i \simeq \ker(Th_i) \oplus L_i$. Then for all $i \in I$

$$T\phi_i(L_i \cap TU_{\phi_i}) = V_{\phi_i} \times \mathsf{N}_{\psi_i} \times \{0\} \simeq V_{\phi_i} \times \{0\} \times \mathsf{N}_{\psi_i} = T\theta_i(h_i^*TN_i \cap (U_{\phi_i} \times TU_{\psi_i})).$$

In other words, $\{h_i^*TN_i\}_i$ locally complements $\{\ker(Th_i)\}_i$.

The existence of a strict C^k-inverse system of (necessarily split) subbundles HE_i such that $HE_i \simeq p^*TM_i$ is equivalent to the existence of a C^{k-1}-morphism of inverse system of vector bundles $\{\omega_i\colon TE_i \to VE_i\}_i$ such that $HE_i = \ker(\omega_i)$. The map $\sigma_i\colon p_i^*TM_i \to HE_i$ constructed as above is then the horizontal lift associated with ω_i. In particular, for any $e_i \in E_i$ and any $u_{e_i} \in T_{e_i}E_i$, we can write

$$\omega_i(u_{e_i}) = u_{e_i} - \sigma_i(e_i, Tp_i(u_{e_i})).$$

For any $(u_{e_i})_i \in T_{(e_i)_i} \varprojlim E_i$, we have

$$\varprojlim \omega_i((u_{e_i})_i) = (u_{e_i})_i - (\varprojlim \sigma_i)((e_i)_i, T \varprojlim p_i((u_{e_i})_i)) = (u_{e_i} - \sigma_i(e_i, Tp_i(u_{e_i})))_i.$$

Now note that $\{p_i^*p_i\colon p_i^*E_i \to E_i\}$ is a strict C^k-inverse system of vector bundles. Recall that we have C^{k-1}-isomorphisms of vector bundles $\mathrm{vl}_{E_i}\colon p_i^*E_i \to VE_i$ and $\mathrm{vl}_{\varprojlim E_i}\colon (\varprojlim p_i)^*(\varprojlim E_i) \to V \varprojlim E_i$ obtained by considering the vertical lifts as vector bundle morphisms over E_i and $\varprojlim E_i$, respectively. Moreover, $(\{\pi_{TE_i}|_{VE_i}\colon VE_i \mapsto E_i\}_i, \{(TF_{ij}, F_{ij})\}_{i\leq j})$ is an strict C^{k-1}-inverse system of vector bundles. To see this, note that $TF_{ij}|_{VE_j}\colon VE_j \to VE_i$ is well-defined vector bundle morphism. It is enough to note that $TF_{ij}([t \mapsto e_j + tu_j]) = [t \mapsto F_{ij}(e_j) + tF_{ij}(u_j)] \in VE_i$ on account of

$$p_i(F_{ij}(e_j)) = f_{ij}(p_j(e_j)) = f_{ij}(p_j(u_j)) = p_i(F_{ij}(u_j)),$$

and that for any strict inverse system of local trivialization $\{\phi_i\}_i$ for $\{E_i\}$ defining a strict inverse of charts $\{\tilde{\phi} := \phi_i \times \mathrm{id}_{F_{\phi_i}}\}_i$, we have

$$T\tilde{\phi}_i(T(p_i^{-1}(U_{\phi_i})) \cap VE_i) = (V_{\phi_i} \times \mathsf{F}_{\phi_i}) \times (\{0\} \times \mathsf{F}_{\phi_i}).$$

Thus, $\{VE_i\}_i$ is a strict inverse system of subbundles of $(\{\pi_{TE_i}\colon TE_i \to E_i\}_i, \{(TF_{ij}, F_{ij})\}_{i\leq j})$.

Now we show that

$$V \varprojlim E_i \simeq \varprojlim VE_i. \tag{4}$$

is a C^{k-1}-isomorphism of vector bundles over $\varprojlim E_i$ Indeed, $\{(\mathrm{vl}_{E_i}, \mathrm{id}_{E_i})\}_i$ is a C^{k-1}-isomorphism of inverse system of vector bundles. It is enough to show that the diagram

$$\begin{tikzcd} p_j^*E_j \arrow[r, "\mathrm{vl}_{E_j}"] \arrow[d, "F_{ij}\times F_{ij}"'] & VE_j \arrow[d, "TF_{ij}"] \\ p_i^*E_i \arrow[r, "\mathrm{vl}_{E_i}"] & VE_i \end{tikzcd}$$

commutes. But this is straightforward since for any $(e_j, u_j) \in p_j^*E_j$,

$$\begin{aligned} TF_{ij}(\mathrm{vl}_{E_j}(u_j, e_j)) &= TF_{ij}([t \mapsto e_j + tu_j]) = [t \mapsto F_{ij}(e_j + tu_j)] \\ &= [t \mapsto F_{ij}(e_j) + tF_{ij}(u_j)] = \mathrm{vl}_{E_i}((F_{ij} \times F_{ij})(e_j, u_j)). \end{aligned}$$

Alternatively, we can consider the strict inverse system of pullback bundles $(\{Z_{M_i}^*\pi_{TE_i}\colon Z_{M_i}^*TE_i \to M_i\}_i, \{(f_{ij} \times TF_{ij}, f_{ij})\}_{i\leq j})$ which is isomorphic to $(\{p \circ \pi_{TE_i}|_{VE_i}\colon VE_i \to M_i\}_i, \{(TF_{ij}|_{VE_j}, f_{ij})\}_{i\leq j})$ via the canonical vertical lifts over $\{M_i\}_i$. The obtained isomorphism in (4) is then over $\varprojlim M_i$.

Given a connection as above $H \varprojlim E_i \simeq \varprojlim HE_i$.

Definition 2.35. A C^k-inverse system of anchored vector bundles $(\{p_i\colon E_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ is *anchored* if the anchor morphisms $\{\rho_i\colon E_i \to TM_i\}_i$ forms a C^k-morphism of inverse systems of vector bundles.

Lemma 2.36. *Let $(\{p_i\colon E_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ be an anchored strict C^k-inverse system of vector bundles with anchors $\{\rho_i\colon E_i \to M_i\}_i$. Then $\varprojlim p_i\colon \varprojlim E_i \to \varprojlim M_i$ is anchored with $\varprojlim \rho_i$ as the anchor morphism.*

Inverse systems of Lie algebroids

In this section, we work exclusively in the smooth category.

Recall that a Lie algebroid over M if a vector bundle $A \to M$ with anchor $\mathbf{a}$ equipped with a Lie bracket on its space of section $[\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ satisfying the Leibniz rule (that is a bi-derivation)

$$[\xi, f\eta]_A = (\mathbf{a}(\xi).f)\eta + f[\xi, \eta]$$

for all $\xi, \eta \in \Gamma(A)$ and $f \in C^\infty(M)$. So that, $\mathbf{a}\colon \Gamma(A) \to \Gamma(TM)$ is a Lie-algebra homomorphism. A stronger notion is that of localized Lie algebroid. For any C^∞-manifold M, $TM \Rightarrow M$ is a localized Lie algebroid. This transfers to Lie algebroids associated with Lie groupoids.

Definition 2.37. A Lie algebroid $A \Rightarrow M$ is *localized* if

1. the Lie bracket induces a morphism of sheaves of Lie algebras. That is, for all $U \subseteq M$ open, $[\cdot,\cdot]_U \colon \Gamma(A|U) \times \Gamma(A|U) \to \Gamma(A|U)$ is Lie bracket that is compatible with restrictions;
2. for all $U \subseteq M$ open, $[\,\cdot\,,\cdot\,]_U$ is a bi-derivation, that is,

$$[\xi, f\eta]_U = f[\xi,\eta] + (\mathbf{a}(\xi).f)\eta, \quad \forall f \in C^\infty(U)\ \forall \xi,\eta \in \Gamma(A|U).$$

A *Lie algebroid morphism* from $\tilde{A} \Rightarrow \tilde{M}$ to $A \Rightarrow M$ is a vector bundle map (F, f) such that $\tilde{\mathbf{a}} \circ F = Tf \circ \mathbf{a}$ and and, for all $\xi_j \in \Gamma(A)$ and $\eta_j \in \Gamma(\tilde{A})$ such that $\eta_j \circ f = F \circ \xi_j$, $j = 1, 2$,

$$F \circ [\xi_1, \xi_2]_A = [\eta_1, \eta_2]_{\tilde{A}} \circ f.$$

Definition 2.38. A *strict inverse system of Lie algebroid* is a pair $(\{A_i \Rightarrow M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\le j})$ such that

1. $(\{A_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\le j})$ is a (strict) inverse system of anchored vector bundles;
2. for all $i \le j$, (F_{ij}, f_{ij}) is a morphism of Lie algebroids.

The inverse limit of a strict inverse system of Lie algebroids is an anchored vector bundle with a Lie bracket that is defined on compatible families of local sections which does not necessarily extend to a Lie algebroid. For this, we require the existence of Lie algebroid structure on the inverse limit that is compatible with the Lie brackets.

Definition 2.39. A Lie algebroid $A \Rightarrow M$ is *strict pro-Lie algebroid* if there exists a strict inverse system of Lie algebroids $(\{A_i \Rightarrow M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\le j})$ such that $A \to M$ is the inverse limit of $(\{A_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\le j})$ and

$$[\varprojlim \xi_i, \varprojlim \eta_i]_A = \varprojlim [\xi_i, \eta_i]_{A_i},$$

for all compatible families of local sections $\{\xi\}_i$ and $\{\eta_i\}_i$.

Example 2.40. Let $\{M_i\}_i$ be a strict inverse system of manifolds. Then $\{TM_i\}_i$ is a strict inverse system of Lie algebroids for which $\varprojlim TM_i \Rightarrow \varprojlim M_i$ is a pro-Lie algebroid.

Inverse systems of fiber and affine bundles

One can generalize the notion of strict inverse system of vector bundles in Definition 2.20 to affine bundles and more generally to fiber bundles in a similar fashion. This is done by requiring that the local trivialization assemble into a morphism of inverse systems taking value in a strict inverse system of trivial affine (resp. fiber) bundle. We omit the details. To bridge the gaps, we recall the definition of trivial fiber and affine bundles.

A trivial fiber bundle is simply a surjective submersion $p \colon M \times B \to M$, where M and B are C^k-manifolds. A C^k-morphisms of trivial fiber bundle from $p_1 \colon M_1 \times B_1$ to $M_2 \to B_2$ is a C^k-map $F \colon M_1 \times B_1 \to M_2 \times B_2$ such that $F(x, b) = (f(x), \widehat{F}(x, b))$ for all $(x, b) \in M_1 \times B_1$, where $f \colon M_1 \to M_2$ and $\widehat{F} \colon M_1 \times B_1 \to B_2$ are C^k-maps. In the affine case, we require B_1 and B_2 to be affine spaces and additionally require $\widehat{F}(x, \cdot) \colon B_1 \to B_2$ to be an affine map. Definition 2.20 can be adapted directly to affine bundles by requiring $(\{\mathsf{F}_{\psi_i}\}_i, \{\nu^{\psi}_{ij}\}_{i\le j})$ to be an inverse system of affine spaces. Note that in this case $\{\ker(Tp_i)\}_i$ is the strict inverse system of model vector bundles. For fiber bundles, we replace this by a strict inverse systems of manifolds $(\{B_{\psi_i}\}_i, \{h_{ij}\}_{i\le j})$ to get a local trivialization where the fiber is the manifolds $\varprojlim B_{\psi_i}$.

Lemma 2.41. *Let $(\{M_i\}_i, \{g_{ij}\}_{i\le j})$ be an inverse system of C^k-manifolds. Suppose that for each $(x_i)_i \in \varprojlim M_i$, there exists an index $i_0 \in I$ and open neighborhood U_{i_0} of x_{i_0} such that $(\{g_{i_0j}\colon g_{i_0j}^{-1}(U_{i_0}) \to U_{i_0}\}_{j\in I_{i_0}}, \{(g_{jk}, \mathrm{id}_{U_{i_0}}\}_{j\le k})$ is an inverse system of trivializable fiber bundles such that the fibers are locally convex spaces, where $I_{i_0} := \{j \in I : j \ge i_0\}$. Then $\varprojlim M_i$ is a C^k-manifold.*

Proof. Let $\{\tau_i\}$ be a compatible family of trivializations and let ϕ_{i_0} be a chart at x_{i_0} such that (possibly after shrinking) $U_{\phi_{i_0}} = U_{i_0}$. For all $j \in I_{i_0}$, the compositions $\phi_j := (\phi_{i_0} \times \mathrm{id}_{\mathsf{F}_{\tau_j}}) \circ \tau_j$ define a saturated family of charts about (x_i). This shows that $\{g_{i_0j}\colon g_{i_0j}^{-1}(U_{i_0}) \to U_{i_0}\}$ is saturated. Endow $\varprojlim M_i$ with its inverse limit topology. We define a chart ϕ for $\varprojlim M$ about (x_i) by

$$\phi\colon g_{i_0}^{-1}(U_{i_0}) \ni (y_i)_i \mapsto (y_{i_0}, ((\mathrm{pr}_2 \circ \phi_j)(y_j))_{j\ge i_0}) \in V_{\phi_{i_0}} \times \varprojlim \mathsf{F}_{\tau_i}.$$

The map ϕ is a homeomorphism where the continuous inverse is constructed as follows. First, for any $i \notin I_{i_0}$, we fix $\hat{i} \in I_{i_0}$ such that $i \le \hat{i}$. Now for any $(y_{i_0}, (e_i)_{i\in I_{i_0}}) \in V_{\phi_{i_0}} \times \varprojlim_{i\in I_{i_0}} \mathsf{F}_{\tau_i}$, we set $y_j = \phi_j^{-1}(y_{i_0}, e_j)$ for $j \ge i_0$ and $y_i = g_{i\hat{i}}(y_{\hat{i}})$ for $i \notin I_{i_0}$. □

Corollary 2.42. $(\{M_n\}_{n\in\mathbb{N}}, \{g_{nm}\}_{n\le m})$ *be an inverse system of C^k-manifolds such that for all $(x_n)_n \in \varprojlim M_n$, there exists an open neighborhood U_0 of x_0 such that $(\{g_{0n}\colon g_{0n}^{-1}(U_0) \to U_0\}_n, \{(g_{0n}, \mathrm{id}_{U_0}\}_n)$ is an inverse system of trivializable fiber bundles where the fibers are locally convex spaces. Then $\{M_n\}$ is saturated.*

This shows that jet groupids assemble into a strict inverse system of manifolds. We shall see an explicit construction for this in Example 3.20.

2.3 Strict morphisms between inverse systems of manifolds

We extend some standard results about submersions, immersions, embeddings to inverse systems.

Let $k \in \mathbb{N} \cup \{0, \infty\}$.

Lemma 2.43. *Let $(\{\mathsf{E}_i\}_{i\in I}, \{\epsilon_{ij}\}_{i\le j})$ be an inverse system of locally convex spaces. Suppose that each $\mathsf{F}_i \subseteq \mathsf{E}_i$ is a complemented subspace such that $\mathsf{F}_i = p_i(\mathsf{E}_i)$ where the continuous projections $\{p_i\colon \mathsf{E}_i \to \mathsf{E}_i\}$ assemble into an endomorphism of inverse systems of locally convex spaces. Then $\varprojlim \mathsf{F}_i$ is complemented in $\varprojlim \mathsf{E}_i$ with the projection $\varprojlim p_i\colon \varprojlim \mathsf{E}_i \to \varprojlim \mathsf{F}_i$.*

Denote $\mathsf{C}_i := \ker(p_i)$ for all i. We then have $p_i \circ \epsilon_{ij}|_{\mathsf{C}_j} = \epsilon_{ij} \circ p_j|_{\mathsf{C}_j} = 0$ for all $i \le j$. In turn $(\{\mathsf{C}_i\}_i, \{\epsilon_{ij}|_{\mathsf{C}_j}\})$ is an inverse system of locally convex spaces for which $\varprojlim \mathsf{E}_i \simeq \varprojlim \mathsf{F}_i \times \varprojlim \mathsf{C}_i$. That is, the topological isomorphisms $\mathsf{E}_i \simeq \mathsf{F}_i \times \mathsf{C}_i$ assemble into an isomorphism of inverse systems of locally convex spaces.

Definition 2.44. Let $(\{\mathsf{E}_i\}_i, \{\epsilon_{ij}\}_{i\leq j})$ and $(\{\mathsf{F}_i\}_i, \{\nu_{ij}\}_{i\leq j})$ be inverse system of locally convex spaces. We say that $\{\mathsf{F}_i\}_i$ is *complemented* in $\{\mathsf{E}_i\}_i$ if there exists an inverse system of locally convex spaces $(\{\mathsf{C}_i\}_i, \{\kappa_{ij}\})$ such that we have a topological isomorphism of inverse systems of locally convex spaces

$$(\{\mathsf{E}_i\}_i, \{\epsilon_{ij}\}_{ij}) \simeq (\{\mathsf{F}_i \times \mathsf{C}_i\}_i, \{\nu_{ij} \times \kappa_{ij}\}_{i\leq j}).$$

In particular, $\varprojlim \mathsf{E}_i \simeq \varprojlim \mathsf{F}_i \times \varprojlim \mathsf{C}_i$.

Example 2.45. 1. (*The inverse limit of submersions is not a submersion.*) Set $M_n = C^\infty(\mathbb{R})$ and $N_n = \mathbb{R}^{n+1}$ for all $n \in \mathbb{N}$. For $n \leq m$ in $\mathbb{N}$, set $g_{nm} = \mathrm{id}_{C^\infty(\mathbb{R})}$ and $h_{nm} = \mathrm{pr}^m_n \colon \mathbb{R}^m \ni (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then, using the usual order on $\mathbb{N}$, $(\{M_n\}_n, \{g_{nm}\}_{n\leq m})$ and $(\{N_n\}_n, \{h_{nm}\}_{n\leq m})$ are inverse systems of smooth manifolds, where the inverse limits are $C^\infty(\mathbb{R})$ and $\mathbb{R}^{\mathbb{N}}$, respectively. Define $J_n \colon M_n \to N_n$ by

$$J_n \colon f \mapsto (f(0), \ldots, f^{(n)}(0)).$$

Then J_n is a smooth submersion. Indeed, we establish a topological isomorphism $C^\infty(\mathbb{R}) \simeq N_n \oplus \mathsf{C}_n$, where

$$\mathsf{C}_n := \ker(J_n) = \{f \in C^\infty(\mathbb{R}) : f^{(j)}(0) = 0 \text{ for } 0 \leq j \leq n\}.$$

Define a smooth section $c_n \colon N_n \to C^\infty(\mathbb{R})$ by

$$c_n \colon (a_0, \ldots, a_n) \mapsto \left(x \mapsto \sum_{j=0}^{n} a_j x^j \chi(x) \right),$$

for some smooth bump function χ such that $\chi \equiv 1$ about 0. Clearly, $J_n \circ c_n = \mathrm{id}_{N_n}$ hence the splitting. Now the space of flat functions

$$\mathsf{C}_\infty = \varprojlim \mathsf{C}_n = \{f \in C^\infty(M) : f^{(j)}(0) = 0, j \geq 0\}$$

is not complemented in $C^\infty(M)$ [7]. Thus, $J_\infty \colon C^\infty(\mathbb{R}) \to \mathbb{R}^{\mathbb{N}}$ is not a submersion even though it is surjective by Borel's Lemma.

Let $p_n \colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ denote the projection onto C_n. That is, $p_n(f) = f - c_n(J_n(f))$ for all $f \in C^\infty(\mathbb{R})$. Obviously, $p_n \neq p_m$. Hence $\{\mathsf{C}_n\}_n$ is an inverse system of locally convex spaces where the connecting morphisms are continuous inclusions $\mathsf{C}_m \to \mathsf{C}_n$, for $n \leq m$ but fails to complement $\{N_n\}_n$ in $\{M_n\}_n$.

2. (*The inverse limit of submersion is a submersion but the complements are not an inverse system.*) Set $I = \mathbb{N}$ with its standard order, set $M_n = \mathbb{R}^2$ and $N_n = \mathbb{R}$. Define a smooth map $g_{nm} \colon M_m \to M_n$ by

$$g_{nm}(x_m, y_m) = (x_m + (m-n)y_m, y_m),$$

and set $h_{nm} = \mathrm{id}_{\mathbb{R}}$, for all $n \leq m$. Then

$$\varprojlim M_n = \left\{(x + (n-1)y, y)_n \in (\mathbb{R}^2)^{\mathbb{N}} : x, y \in \mathbb{R}\right\}$$

and $\varprojlim N_n = \{(x, \ldots, x, \ldots) \in \mathbb{R}^{\mathbb{N}} : x \in \mathbb{R}\}$ which is just $\mathbb{R}$. Let $f_i = \mathrm{pr}_2$. Obviously f_i is a submersion and $\varprojlim f_i \colon (x + (n-1)y, y)_n \mapsto (y, \ldots, y, \ldots)$ is also a submersion. Trivially, $\mathsf{C}_n = \{((x, 0), \ldots, (x, 0), \ldots) \in (\mathbb{R}^2)^{\mathbb{N}} : x \in \mathbb{R}\}$ is a complement of $\{((0, x), \ldots, (0, x), \ldots) \in (\mathbb{R}^2)^{\mathbb{N}} : x \in \mathbb{R}\} \simeq N_n$ for each n. However, $\{\mathsf{C}_n\}_n$ does not complement $\{N_n\}_n$ in $\{M_n\}_n$. Indeed, let $p_n \colon M_n \to M_n$ be the projection $(x, y) \mapsto (x, 0)$ then $p_n \circ g_{nm} \neq g_{nm} \circ p_m$, for $n \leq m$.

3. Finally, we consider a trivial case where the inverse limit of non-submersion smooth maps is a submersion. Let E and F be non-zero locally convex spaces. Then with the identity map and the zero map as connecting morphisms for $\{\mathsf{E} \times \mathsf{F}\}_i$ and $\{\mathsf{F}\}_i$, respectively, the zero map $z_i \colon \mathsf{E} \times \mathsf{F} \to \mathsf{F}$, $(x, y) \mapsto 0$ is not a submersion, but $\varprojlim z_i$ is on account of the inverse limit of $\{\mathsf{F}\}$ being null.

Definition 2.46. Let $(\{M_i\}_{i \in I}, \{g_{ij}\}_{i \leq j \in I})$ and $(\{N_i\}_{i \in I}, \{h_{ij}\}_{i \leq j \in I})$ be strict C^k-inverse systems of manifolds. Let $\{f_i \colon M_i \to N_i\}_{i \in I}$ be C^k-morphism of inverse systems of manifolds such that f_i is a submersion for all $i \in I$. The family $\{f_i\}_i$ is called a *strict family of C^k-submersions* if for all $(x_i)_i \in M$ there exists strict inverse systems of submersion charts $\{\phi_i\}_i$ about $(x_i)_i$ and $\{\psi_i\}_i$ about $(f_i(x_i))_i$ such that $(\{\mathsf{N}_{\psi_i}\}_i, \{\epsilon^{\psi}_{ij}\}_{i \leq j})$ is complemented in $(\{\mathsf{M}_{\phi_i}\}_i, \{\epsilon^{\phi}_{ij}\}_{i \leq j})$.

Proposition 2.47. *Let $(\{M_i\}_{i \in I}, \{g_{ij}\}_{i \leq j \in I})$ and $(\{N_i\}_{i \in I}, \{h_{ij}\}_{i \leq j \in I})$ be strict C^k-inverse systems of manifolds. Let $\{f_i \colon M_i \to N_i\}_i$ be a strict family of C^k-submersions. Then $\varprojlim f_i$ is a C^k-submersion. Moreover, if $\{S_i\}_i$ is a strict inverse system of (split) submanifolds of $\{N_i\}_i$, then $\{f_i^{-1}(S_i)\}_i$ is a strict inverse system of (split-)submanifolds and the following hold:*

1. $(\varprojlim f_i)^{-1}(\varprojlim S_i) = \varprojlim f_i^{-1}(S_i)$;
2. $(\varprojlim f_i)|_{\varprojlim f_i^{-1}(S_i)} \colon \varprojlim f_i^{-1}(S_i) \to \varprojlim S_i$ *is a submersion;*
3. $T_{(x_i)_i}\left((\varprojlim f_i)^{-1}(\varprojlim S_i)\right) = \left(T_{(x_i)_i} \varprojlim f_i\right)^{-1}\left(T_{(f_i(x_i))_i} \varprojlim S_i\right)$.

Proof. Immediately from definition, at any $(x_i)_i \in \varprojlim M_i$ we have strict inverse systems of charts $\{\phi_i\}_i$ and $\{\psi_i\}_i$ about $(x_i)_i$ and $(f_i(x_i))_i$, respectively, such that $\varprojlim f_i$ is locally given by the projection $\varprojlim \mathsf{M}_{\phi_i} \to \varprojlim \mathsf{N}_{\psi_i}$. Hence $\varprojlim f_i$ is a C^k-submersion.

Let $(x_i)_i \in (\varprojlim f_i)^{-1}(\varprojlim S_i)$. There exists strict inverse systems of submersion charts $\{\phi_i\}_i$ about $(x_i)_i$ and $\{\psi_i\}_i$ about $(f_i(x_i))_i$ such that, possibly after shrinking, $f_i(U_{\phi_i}) \subseteq U_{\psi_i}$, $\psi_i \circ f_i \circ \phi_i^{-1} = p_i|_{V_{\phi_i}}$, where $p_i \colon \mathsf{M}_{\phi_i} \simeq \mathsf{N}_{\psi_i} \times \mathsf{C}_i \to \mathsf{N}_{\psi_i}$ is a continuous linear projection, and $\psi_i(S_i \cap U_{\psi_i}) = \mathsf{S}_{\psi_i} \cap V_{\psi_i}$. Hence $\phi_i(f_i^{-1}(S_i) \cap U_{\phi_i}) = V_{\phi_i} \cap (\mathsf{S}_{\psi_i} \times \mathsf{C}_i)$. In particular, $V_{\phi} \cap (\mathsf{S}_{\psi_i} \times \mathsf{C}_i)$ is an open subset of the closed (complemented) subspace $\mathsf{S}_{\psi_i} \times \mathsf{C}_i$ of $\mathsf{N}_{\psi_i} \times \mathsf{C}_i$. In case $\{S_i\}_i$ is split, the compatibility of the projections $\mathsf{M}_{\phi} \simeq \mathsf{N}_{\psi_i} \times \mathsf{C}_i \to \mathsf{S}_{\psi_i} \times \mathsf{C}_i$ is evident by noting that $\epsilon^{\psi}_{ij}|_{\mathsf{S}_{\psi_j}}$ takes values in S_{ψ_i}. Hence $\{\phi_i\}_i$ is a strict inverse system of split-subamnifold charts of $\{f_i^{-1}(S_i)\}_i$ and thus

$$\varprojlim \phi_i(\varprojlim f_i^{-1}(S_i) \cap \varprojlim U_{\phi_i}) = \varprojlim V_{\phi_i} \cap (\varprojlim \mathsf{S}_{\psi_i} \times \varprojlim \mathsf{C}_i). \tag{5}$$

That $(\varprojlim f_i)^{-1}(\varprojlim S_i) = \varprojlim f_i^{-1}(S_i)$ is clear.

1 is clear.

For 2, it is enough to note that $f_i|_{f_i^{-1}(S_i)}\colon f_i^{-1}(S_i) \to S_i$ are given locally by $V_{\phi_i} \cap (\mathsf{S}_{\psi_i} \times \mathsf{C}_i) \to \mathsf{S}_{\psi_i} \cap V_{\psi_i}$, the restrictions p_i. This is also obvious using $\varprojlim f_i^{-1}(S_i) = (\varprojlim f_i)^{-1}(\varprojlim S_i)$.

For 3, pick up charts as above, then locally both sides are equal to $\varprojlim \mathsf{S}_{\psi_i} \times \varprojlim \mathsf{C}_i$. □

Note that $\varprojlim f_i$ is open as each f_i is a submersion, however, if each f_i is surjective, $\varprojlim f_i$ is not necessarily surjective.

Theorem 2.48 (Strict inverse system of fibred product)**.** *Let* $(\{M_i\}_{i\in I}, \{g_{ij}\}_{i\leq j\in I})$, $(\{\tilde{M}_i\}, \{\tilde{g}_{ij}\}_{i\leq j\in I})$, *and* $(\{N_i\}_{i\in I}, \{h_{ij}\}_{i\leq j\in I})$ *be strict C^k-inverse systems of manifolds. Suppose that* $\{f_i\colon M_i \to N_i\}_i$ *and* $\{\tilde{f}_i\colon \tilde{M}_i \to N_i\}_i$ *are C^k-morphisms of inverse systems of manifolds such that $\{f_i\}_i$ is a strict family of C^k-submersions. Then $\{M_i \times_{N_i} \tilde{M}_i\}_i$ is a strict inverse systems of split-submanifolds of* $(\{M_i \times \tilde{M}_i\}_i, \{g_{ij} \times \tilde{g}_{ij}\})$. *Moreover, the projections* $p_i\colon M_i \times_{N_i} \tilde{M}_i \to \tilde{M}_i$ *form a strict family of C^k-submersions. Finally, we have a C^k-diffeomorphism*

$$\varprojlim M_i \times_{\varprojlim N_i} \varprojlim \tilde{M}_i \simeq \varprojlim M_i \times_{N_i} \tilde{M}_i.$$

Proof. Recall that $M_i \times_{N_i} \tilde{M}_i$ is a split-submanifold of $M_i \times_{N_i} \tilde{M}_i$. Indeed, let $(x_i, \tilde{x}_i) \in M_i \times_{N_i} \tilde{M}_i$. Then there exists charts: ϕ_i about x_i, $\tilde{\phi}_i$ about $\tilde{x}_i$, and ψ_i about $f_i(x_i)$ such that $f_i(U_{\phi_i}) \subseteq U_{\psi_i}$, $\tilde{f}_i(U_{\tilde{\phi}_i}) \subseteq U_{\psi_i}$, and such that ϕ_i and ψ_i are submersion charts for f_i so that $\mathsf{M}_{\phi_i} \simeq \mathsf{N}_{\psi_i} \times \mathsf{C}_i$ for a locally convex space C_i. Define $\theta_i\colon U_{\phi_i} \times U_{\tilde{\phi}_i} \to \mathsf{N}_{\psi_i} \times \mathsf{C}_i \times \tilde{\mathsf{M}}_{\tilde{\phi}_i}$ by

$$\theta_i\colon (x_i, \tilde{x}_i) \mapsto (\psi_i(f_i(x_i)) - \psi_i(\tilde{f}_i(\tilde{x}_i)), \mathrm{pr}_{\mathsf{C}_i}(\phi_i(x_i)), \tilde{\phi}_i(\tilde{x}_i)).$$

Denote $U_{\theta_i} = U_{\phi_i} \times U_{\tilde{\phi}_i}$ and $V_{\theta_i} = \theta_i(U_{\theta_i})$. Then V_{θ_i} is open in $\mathsf{N}_{\psi_i} \times \mathsf{C}_i \times \tilde{\mathsf{M}}_{\tilde{\phi}_i}$ as it is the preimage of $V_{\phi_i} \times V_{\tilde{\phi}_i}$ under the continuous map

$$\alpha_i\colon \mathsf{N}_{\psi_i} \times \mathsf{C}_i \times V_{\tilde{\phi}_i} \ni (a_i, b_i, c_i) \mapsto (a_i + \psi_i(\tilde{f}_i(\tilde{\phi}_i^{-1}(c_i))), b_i, c_i) \in \mathsf{N}_{\psi_i} \times \mathsf{C}_i \times V_{\tilde{\phi}_i}$$

Note that $\theta_i\colon U_{\theta_i} \to V_{\theta_i}$ is an invertible C^k-map and the inverse

$$\theta_i^{-1}\colon (a_i, b_i, c_i) \mapsto (\phi_i^{-1}(a_i + \psi_i(\tilde{f}_i(\tilde{\phi}_i^{-1}(c_i))), b_i), \tilde{\phi}_i^{-1}(c_i))$$

is C^k. This defines a chart for $M_i \times \tilde{M}_i$. In fact, it is a split-submanifold chart for $M_i \times_{N_i} M_i$ where the model space is $\{0\} \times \mathsf{C}_i \times \tilde{M}_{\tilde{\phi}_i}$ which is complemented in $\mathsf{N}_{\psi_i} \times \mathsf{C}_i \times \tilde{M}_{\tilde{\phi}_i}$. Now we show that $(\{M_i \times_{N_i} \tilde{M}_i\}_i, \{g_{ij} \times \tilde{g}_{ij}\}_{i\leq j})$ is a strict inverse system of split-submanifolds of $\{M_i \times \tilde{M}_i\}_i$. Note that $(g_{ij} \times \tilde{g}_{ij})(x_j, \tilde{x}_j) \in M_i \times_{N_i} \tilde{M}_i$ for all $(x_j, \tilde{x}_j) \in M_j \times_{N_j} \tilde{M}_j$. Now let $(x_i, \tilde{x}_i)_i \in \varprojlim M_i \times_{N_i} \tilde{M}_i$ and let $\{\phi_i\}_i$, $\{\tilde{\phi}_i\}_i$, and $\{\psi_i\}_i$ be strict inverse systems of charts for $(x_i)_i$, $(\tilde{x}_i)_i$, and $(f_i(x_i))_i$ such that $\{\phi_i\}_i$ and $\{\psi_i\}_i$ are strict inverse system of submersion charts making $\{f_i\}_i$ a strict family of C^k-submersions. Suppose that $f_i(U_{\phi_i})$ and $\tilde{f}_i(U_{\tilde{\phi}_i})$ for all i. Define a family of charts $\{\theta_i\}_i$ as above. We claim that $\{\theta_i\}_i$ is a strict family of split-submanifold charts for $(x_i, \tilde{x}_i)_i$. Note that the diagram

$$\begin{tikzcd}
U_{\phi_j} \times U_{\tilde{\phi}_j} \arrow[r, "\theta_j"] \arrow[d, "g_{ij} \times \tilde{g}_{ij}"'] & V_{\theta_j} \arrow[d, "\epsilon_{ij}^{\psi} \times \nu_{ij} \times \epsilon_{ij}^{\tilde{\phi}}"] \\
U_{\phi_j} \times U_{\tilde{\phi}_j} \arrow[r, "\theta_i"] & V_{\theta_i}
\end{tikzcd}$$

commutes since

$$
\begin{aligned}
&(\epsilon^{\psi}_{ij}\times\nu_{ij}\times\epsilon^{\tilde{\phi}}_{ij})(\theta_j(y_j,\tilde{y}_j))\\
&\qquad\qquad = (\epsilon^{\psi}_{ij}\times\nu_{ij}\times\epsilon^{\tilde{\phi}}_{ij})(\psi_j(f_j(y_j))-\psi_j(\tilde{f}_j(\tilde{y}_j)),\mathrm{pr}_{\mathsf{C}_j}(\phi_j(y_j)),\tilde{\phi}_j(\tilde{y}_j))))\\
&\qquad\qquad = (\epsilon^{\psi}_{ij}(\psi_j(f_j(y_j))-\psi_j(\tilde{f}_j(\tilde{y}_j))),\nu_{ij}(\mathrm{pr}_{\mathsf{C}_j}(\phi_j(y_j))),\epsilon^{\tilde{\phi}}_{ij}(\tilde{\phi}_j(\tilde{y}_j)))\\
&\qquad\qquad = (\epsilon^{\psi}_{ij}(\psi_j(f_j(y_j)))-\epsilon^{\psi}_{ij}(\psi_j(\tilde{f}_j(\tilde{y}_j))),\mathrm{pr}_{\mathsf{C}_i}(\epsilon^{\phi}_{ij}(\phi_j(y_j))),\tilde{\phi}_i(\tilde{g}_{ij}(\tilde{y}_j)))\\
&\qquad\qquad = (\psi_i(f_i(g_{ij}(y_j)))-\psi_i(\tilde{f}_i(\tilde{g}_{ij}(\tilde{y}_j))),\mathrm{pr}_{\mathsf{C}_i}(\phi_i(g_{ij}(y_j))),\tilde{\phi}_i(\tilde{g}_{ij}(\tilde{y}_j)))\\
&\qquad\qquad = \theta_i((g_{ij}\times\tilde{g}_{ij})(y_j,\tilde{y}_j)),
\end{aligned}
$$

for all $(y_j,\tilde{y}_j)\in U_{\phi_j}\times U_{\tilde{\phi}_j}$. Similarly we see that the diagram

$$
\begin{array}{ccc}
\mathsf{N}_{\psi_j}\times\mathsf{C}_j\times V_{\tilde{\phi}_j} & \xrightarrow{\alpha_j} & \mathsf{N}_{\psi_j}\times\mathsf{C}_j\times V_{\tilde{\phi}_j}\\
{\scriptstyle \epsilon^{\psi}_{ij}\times\nu_{ij}\times\epsilon^{\tilde{\phi}}_{ij}}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon^{\psi}_{ij}\times\nu_{ij}\times\epsilon^{\tilde{\phi}}_{ij}}\\
\mathsf{N}_{\psi_i}\times\mathsf{C}_i\times V_{\tilde{\phi}_i} & \xrightarrow{\alpha_i} & \mathsf{N}_{\psi_i}\times\mathsf{C}_i\times V_{\tilde{\phi}_i}
\end{array}
$$

commutes. Moreover, $\varprojlim V_{\tilde{\phi}_i}$ is open. Hence $\{\alpha_i\}_i$ is strict and the map

$$\varprojlim\alpha_i\colon \varprojlim\mathsf{N}_{\psi_i}\times\varprojlim\mathsf{C}_i\times\varprojlim V_{\tilde{\phi}_i}\to\varprojlim\mathsf{N}_{\psi_i}\times\varprojlim\mathsf{C}_i\times\varprojlim V_{\tilde{\phi}_i}\simeq\varprojlim\mathsf{M}_{\phi_i}\times\varprojlim V_{\tilde{\phi}_i}$$

is C^k. In particular, $\varprojlim V_{\theta_i}=(\varprojlim\alpha_i)^{-1}(\varprojlim V_{\phi_i}\times\varprojlim V_{\tilde{\phi}_i})$ is open. Thus $\{M_i\times_{N_i}\tilde{M}_i\}_i$ is a stict inverse system of split-submanifolds of $(\{M_i\times\tilde{M}_i\}_i,\{g_{ij}\times\tilde{g}_{ij}\}_{i\le j})$.

Next we show that $\{\theta_i\}_i$ and $\{\tilde{\phi}_i\}$ are submersion charts for the maps $p_i\colon M_i\times_{N_i}\tilde{M}_i\to\tilde{M}_i$, $(x_i,\tilde{x}_i)\mapsto\tilde{x}_i$ such that $\{p_i\}_i$ is a strict family of C^k-submersions. As a result, $\varprojlim p_i\colon\varprojlim M_i\times_{N_i}\tilde{M}_i\to\varprojlim\tilde{M}_i$ is a C^k-submersion. Obviously, $\tilde{g}_{ij}\circ p_j=p_i\circ(g_{ij}\times\tilde{g}_{ij})|_{M_j\times N_j\tilde{M}_j}$. Hence $\{p_i\}_i$ is indeed a morphism of inverse systems of manifolds. Now it is enough to note that $\tilde{\phi}_i\circ p_i\circ\theta_i^{-1}$ is the restriction of the projection $\mathsf{C}_i\times\tilde{\mathsf{M}}_{\tilde{\phi}_i}\to\tilde{\mathsf{M}}_{\tilde{\phi}_i}$ to V_{θ_i} which shows that $\{p_i\}_i$ is a C^k-submersion.

Now that $\varprojlim p_i$ is a submersion, $\varprojlim M_i\times_{\varprojlim N_i}\varprojlim\tilde{M}_i$ is a split-submanifold of $\varprojlim M_i\times\varprojlim\tilde{M}_i$. The diffeomorphism is clearly given by $((x_i)_i,(\tilde{x}_i)_i)\mapsto(x_i,\tilde{x}_i)$ which is C^k as one can show using the charts constructed above or simply by using Proposition 2.11. $\square$

Definition 2.49. Let $(\{M_i\}_{i\in I},\{g_{ij}\}_{i\le j\in I})$ and $(\{N_i\}_{i\in I},\{h_{ij}\}_{i\le j\in I})$ be strict C^k-inverse systems of manifolds. Let $\{f_i\colon M_i\to N_i\}_{i\in I}$ be C^k-morphism of inverse systems of manifolds such that f_i is an immersion (resp. embedding) for all $i\in I$. The family $\{f_i\}_i$ is called a *strict family of C^k-immersions* (resp. *embeddings*) if for all $(x_i)_i\in M$ there exists strict inverse systems of immersion charts $\{\phi_i\}_i$ about $(x_i)_i$ and $\{\psi_i\}_i$ about $(f_i(x_i))_i$ such that $(\{\mathsf{M}_{\phi_i}\}_i,\{\epsilon^{\phi}_{ij}\}_{i\le j})$ is complemented in $(\{\mathsf{N}_{\psi_i}\}_i,\{\epsilon^{\psi}_{ij}\}_{i\le j})$.

Lemma 2.50. *Let $\{f_i\colon M_i\to N_i\}_{i\in I}$ be a strict family of C^k-immersions. Then $\varprojlim f_i$ is an immersion. Moreover, if each f_i is an embedding, then $\varprojlim f_i$ is an embedding.*

Unlike surjectivity, if each f_i is injective, then $\varprojlim f_i$ is injective.

Suppose that f_i is injective and set $S_i:=f_i(M_i)$ for all $i\in I$. We say that $\{S_i\}_i$ is a *strict inverse system of immersed submanifolds of* $\{M_i\}_i$ if $\{f_i\}_i$ is a strict inverse system of injective immersions.

Corollary 2.51. *Let $\{S_i\}_i$ be a strict inverse system of immersed submanifolds of $\{M_i\}_i$. Then $\varprojlim S_i$ is a C^k-immersed submanifold of $\varprojlim M_i$.*

3 Inverse systems of topological and Lie groupoids

We start this section by recall the construction inverse limits of inverse systems of topological groupoids. Afterwards, we consider several examples showing that some features are not inherited by the inverse limits. In particular, the inverse limit groupoid is not necessarily open nor locally transitive. This is in contrast to strict inverse limits in the Lie case.

3.1 Inverse systems of topological groupoids

We start by the definition of topological groupoids.

Definition 3.1. A *topological groupoid* $G \rightrightarrows M$ consists of topological spaces G (arrows) and M (objects), together with continuous structure maps $(\mathbf{s}, \mathbf{t}, \mathbf{m}, \mathbf{1}, \mathbf{i})$

1. the source and target maps $\mathbf{s}, \mathbf{t} : G \to M$,
2. a partial multiplication map $\mathbf{m} : G^{(2)} \to G$, $(g, h) \mapsto gh$,
3. the unit map $\mathbf{1} : M \to G$, $x \mapsto 1_x$,
4. the inverse map $\mathbf{i} \colon G \to G$, $g \mapsto g^{-1}$,

where $G^{(2)} := G \times_{\mathbf{s},\mathbf{t}} G = \{(g, h) \in G \times G : \mathbf{s}(g) = \mathbf{t}(h)\}$ is endowed with the induced topology, such that

1. $(g_1 g_2)g_3 = g_1(g_2 g_3)$ whenever defined;
2. $1_{\mathbf{t}(g)} g = g = g 1_{\mathbf{s}(g)}$ for all $g \in G$;
3. $gg^{-1} = 1_{\mathbf{t}(g)}$ and $g^{-1}g = 1_{\mathbf{s}(g)}$. for all $g \in G$.

A topological groupoid is *open* if $\mathbf{s}$ is open.

The *source fiber at $x \in M$* of $G \rightrightarrows M$ is the set $G|_x := \mathbf{s}^{-1}(x)$. The *target fiber at y* is the set $G|^y := \mathbf{t}^{-1}(y)$. These are endowed with their subspace topologies. The set of arrows with source at x and target at y is denoted by $G|_x^y := G|_x \cap G|^y$. In particular, the *vertex group at x* is the set $G|_x^x = G|_x \cap G|^x$ which becomes a topological group with the subspace topology. The *orbit* through $x \in M$ is the set $O_x := \mathbf{t}(G|_x)$. We say that $G \rightrightarrows M$ is *transitive* if $(\mathbf{s}, \mathbf{t}) \colon G \to M \times M$ is surjective. In this case, all the orbits are equal to M. Denote $G|_U := \mathbf{s}^{-1}(U)$, $G|^V = \mathbf{t}^{-1}(V)$, and $G|_U^V = \mathbf{s}^{-1} \cap \mathbf{t}^{-1}(V)$ for any $U, V \subseteq M$ open. We say that $G \rightrightarrows M$ is *locally transitive at $x \in M$* if there exists an open neighborhood of x such that $(\mathbf{s}, \mathbf{t})|_{G|_U^U} \colon G|_U^U \to U \times U$ is surjective.

From the definition, it follows that $\mathbf{s}$ and $\mathbf{t}$ are surjective maps, $\mathbf{1}$ is a topological embedding, and $\mathbf{i}$ is a homeomophism. In particular, since $\mathbf{s} = \mathbf{t} \circ \mathbf{i}$, $\mathbf{s}$ is open if and only if $\mathbf{t}$ is open.

Example 3.2 (The continuous action groupoid)**.** Let G be a topological group acting continuously on a topological space M. Then $G \ltimes M \rightrightarrows M$ is a topological groupoid where the structure is defined as follows:

1. $\mathbf{s}(g,x) = x$ and $\mathbf{t}(g,x) = g.x$ for all $(g,x) \in G \times X$;
2. $(G \ltimes M)^{(2)} = \{((g_2,x_2),(g_1,x_1)) \in (G \times M)^2 : x_2 = g_1.x_1\}$;
3. $\mathbf{m}((g_2, g_1.x),(g_1,x)) = (g_2g_1, x)$ for all $x \in M$ and $g_1, g_2 \in G$;
4. $\mathbf{1}(x) = (e,x)$ for all x;
5. $\mathbf{i}(g,x) = (g^{-1}, g.x)$ for all $(g,x) \in G \times M$.

Definition 3.3. Let $G \rightrightarrows M$ and $H \rightrightarrows N$ be topological groupoids. A *topological groupoid morphism* $(F,f)\colon (G \rightrightarrows M) \to (H \rightrightarrows N)$ is a pair of continuous map $F\colon G \to H$ and $f\colon M \to N$ such that

$$\mathbf{s}_H \circ F = f \circ \mathbf{s}_G, \quad \mathbf{t}_H \circ F = f \circ \mathbf{t}_G, \quad \mathbf{m}_H \circ (F \times F)|_{G^{(2)}} = F \circ \mathbf{m}_G, \quad \mathbf{1}_H \circ f = F \circ \mathbf{1}_G.$$

It follows that $F \circ \mathbf{i}_G = \mathbf{i}_H \circ F$.

Definition 3.4. Let $(I, \leq)$ be a directed set. An *inverse system of topological groupoids* is a pair $(\{G_i \rightrightarrows M_i\}_{i\in I}, \{(F_{ij}, f_{ij})\}_{i\leq j})$ where $\{G_i \rightrightarrows M_i\}_{i\in I}$ is a family of topological groupoids, with structure maps $(\mathbf{s}_i, \mathbf{t}_i, \mathbf{m}_i, \mathbf{1}_i, \mathbf{i}_i)$, and $\{(F_{ij}, f_{ij})\}_{i\leq j}$ is a family of morphisms of topological groupoids $(F_{ij}, f_{ij})\colon (G_j \rightrightarrows M_j) \to (G_i \rightrightarrows M_i)$ whenever $i \leq j$, such that $(\{G_i\}_{i\in I}, \{F_{ij}\}_{i\leq j})$ and $(\{M_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ are inverse systems of topological spaces.

Explicitly, whenever $i \leq j$, we have the following commutative diagrams:

$$\begin{tikzcd} G_j \arrow[r, "F_{ij}"] \arrow[d, "\mathbf{s}_j"'] & G_i \arrow[d, "\mathbf{s}_i"] \\ M_j \arrow[r, "f_{ij}"'] & M_i \end{tikzcd} \quad \begin{tikzcd} G_j \arrow[r, "F_{ij}"] \arrow[d, "\mathbf{t}_j"'] & G_i \arrow[d, "\mathbf{t}_i"] \\ M_j \arrow[r, "f_{ij}"'] & M_i \end{tikzcd} \quad \begin{tikzcd} G_j^{(2)} \arrow[r, "F_{ij} \times F_{ij}"] \arrow[d, "\mathbf{m}_j"'] & G_i^{(2)} \arrow[d, "\mathbf{m}_i"] \\ G_j \arrow[r, "F_{ij}"'] & G_i \end{tikzcd} \quad \begin{tikzcd} M_j \arrow[r, "f_{ij}"] \arrow[d, "\mathbf{1}_j"'] & M_i \arrow[d, "\mathbf{1}_i"] \\ G_j \arrow[r, "F_{ij}"'] & G_i \end{tikzcd} \quad \begin{tikzcd} G_j \arrow[r, "F_{ij}"] \arrow[d, "\mathbf{i}_j"'] & G_i \arrow[d, "\mathbf{i}_i"] \\ G_j \arrow[r, "F_{ij}"'] & G_i \end{tikzcd}$$

In other words the families $\{\mathbf{s}_i\}_i$, $\{\mathbf{t}_i\}_i$, $\{\mathbf{1}_i\}_i$, $\{\mathbf{i}_i\}_i$, and $\{\mathbf{m}_i\}_i$ assemble into morphisms of inverse systems of topological spaces.

Lemma 3.5. *Let $(\{G_i \rightrightarrows M_i\}_{i\in I}, \{(F_{ij}, f_{ij})\}_{i\leq j})$ be an inverse system of topological groupoids. Then the following are inverse system of topological spaces:*

1. $(\{G_i \times_{\mathbf{s}_i,\mathbf{s}_i} G_i\}_i, \{F_{ij} \times F_{ij}\}_{i\leq j})$*;*
2. $(\{G_i \times_{\mathbf{t}_i,\mathbf{t}_i} G_i\}_i, \{F_{ij} \times F_{ij}\}_{i\leq j})$*;*
3. $(\{G_i \times_{\mathbf{s}_i,\mathbf{t}_i} G_i\}_i, \{F_{ij} \times F_{ij}\}_{i\leq j})$*;*
4. $(\{G_i|_{x_i}\}_i, \{F_{ij}\}_{i\leq j})$ *for* $(x_i)_i \in \varprojlim M_i$*;*
5. $(\{G_i|^{y_i}\}_i, \{F_{ij}\}_{i\leq j})$ *for* $(y_i)_i \in \varprojlim M_i$*;*
6. $(\{G_i|_{x_i}^{y_i}\}_i, \{F_{ij}\}_{i\leq j})$ *for* $(x_i)_i \in \varprojlim M_i$ *and* $(y_i)_i \in O_{(x_i)_i}$*.*

Proof. We only show 3; the rest follows similarly. It is enough to note that for any $(g_j, h_j) \in G_j \times_{\mathbf{s}_j, \mathbf{t}_j} G_j$

$$(\mathbf{s}_i \times \mathbf{t}_i)((F_{ij} \times F_{ij})(g_j, h_j)) = (\mathbf{s}_i(F_{ij}(g_j)), \mathbf{t}_i(F_{ij}(h_j))) = (f_{ij}(\mathbf{s}_j(g_j)), f_{ij}(\mathbf{t}_j(h_j))).$$

Hence $(F_{ij} \times F_{ij})(g_j, h_j) \in G_i \times_{\mathbf{s}_i, \mathbf{t}_i} G_i$ for $i \leq j$. $\square$

The inverse limit for an inverse system of topological groupoids always exists.

Theorem 3.6. *Let $(\{G_i \rightrightarrows M_i\}_{i \in I}, \{(F_{ij}, f_{ij})\}_{i \leq j \in I})$ be an inverse system of topological groupoids. Then $\varprojlim G_i \rightrightarrows \varprojlim M_i$ is a topological groupoid where the structure maps are defined by $(\varprojlim \mathbf{s}_i, \varprojlim \mathbf{t}_i, \varprojlim \mathbf{m}_i, \varprojlim \mathbf{1}_i, \varprojlim \mathbf{i}_i)$.*

Proof. The maps $\varprojlim \mathbf{s}_i$, $\varprojlim \mathbf{t}_i$, $\varprojlim \mathbf{1}_i$, and $\varprojlim \mathbf{i}_i$ are clearly continuous. We show that $(\varprojlim G_i)^{(2)}$ is homeomorphic to $\varprojlim {G_i}^{(2)}$. We define θ by

$$\theta \colon (\varprojlim G_i)^{(2)} \ni ((g_i)_i, (h_i)_i) \mapsto (g_i, h_i)_i \in \varprojlim {G_i}^{(2)}.$$

Now let $((g_i)_i, (h_i)_i) \in (\varprojlim G_i)^{(2)}$. Then

$$(\mathbf{s}_i(g_i))_i = (\varprojlim \mathbf{s}_i)((g_i)_i) = (\varprojlim \mathbf{t}_i)((h_i)_i) = (\mathbf{t}_i(h_i))_i.$$

By compatibility, $(F_{ij} \times F_{ij})|_{{G_j}^{(2)}}(g_j, h_j) = (g_i, h_i)$ for all $i \leq j$. Hence $(g_i, h_i)_i \in \varprojlim {G_i}^{(2)}$. This shows that θ is well-defined. In a similar manner, the inverse $\theta^{-1} \colon (g_i, h_i) \mapsto ((g_i)_i, (h_i)_i)$ is a well-defined map. Recall that the topologies on $(\varprojlim G_i)^{(2)}$ and $\varprojlim {G_i}^{(2)}$ are the induced topologies from $\varprojlim G_i \times \varprojlim G_i$ and $\varprojlim (G_i \times G_i)$. These are initial topologies with respect to the families

$$\{p_j \colon (\varprojlim G_i)^{(2)} \to G_j \times G_j, ((g_i)_i, (h_i)_i) \mapsto (g_j, h_j)\}_{j \in I}$$

and

$$\{q_j \colon \varprojlim {G_i}^{(2)} \to G_j \times G_j, (g_i, h_i)_i \mapsto (g_j, h_j)\}_{j \in I},$$

respectively. In particular,

$$\theta^{-1}(q_j^{-1}(U_j \times V_j)) = p_j^{-1}(U_j \times V_j)$$

for all open sets $U_j, V_j \subseteq G_j$, for all $j \in I$. Thus, θ is a homeomorphism. The map $\mathbf{m}_{\varprojlim G_i} := \varprojlim \mathbf{m}_i \circ \theta$ defines a partial multiplication on $\varprojlim G_i$. Given the definition of θ, we use $\varprojlim \mathbf{m}_i$ to denote the multiplication structure $\varprojlim G_i$ directly. The associativity, unit, and inverse axioms of groupoids are easy to verify. $\square$

Equivalently, the homeomorphism $(\varprojlim G_i)^{(2)} \simeq \varprojlim {G_i}^{(2)}$ can be obtained directly from the commutative diagram

$$\begin{tikzcd}
(\varprojlim G_i)^{(2)} \arrow[r, hook] \arrow[d, "\theta"] & \varprojlim G_i \times \varprojlim G_i \arrow[r, "\mathrm{pr}_{G_j} \times \mathrm{pr}_{G_j}"] \arrow[d, "\wr"] & G_j \times G_j \arrow[d, equal] \\
\varprojlim G_i^{(2)} \arrow[r, hook] & \varprojlim (G_i \times G_i) \arrow[r, "\mathrm{pr}_{G_j \times G_j}"] & G_j \times G_j
\end{tikzcd}$$

where the homeomorphism $\varprojlim G_i \times \varprojlim G_i \simeq \varprojlim (G_i \times G_i)$ follows from the universal property of inverse systems of topological spaces.

In the same fashion, one can construct bijections which, by an argument involving the initial topologies, are seen to be homeomorphisms, Lemma 3.7. As an example, we illustrate the homeomorphism $(\varprojlim G_i)\big|_{(x_i)_i}^{(y_i)_i} \simeq \varprojlim (G_i|_{x_i}^{y_i})$ using the following commutative diagram.

$$\begin{tikzcd}
(\varprojlim G_i)\big|_{(x_i)_i}^{(y_i)_i} \arrow[r, hook] \arrow[d, "\kappa"] & \varprojlim G_i \arrow[r, "F_j"] & G_j \arrow[d, equal] \\
\varprojlim \left(G_i|_{x_i}^{y_i}\right) \arrow[r, "F_j"] & G_j|_{x_j}^{y_j} \arrow[r, hook] & G_j
\end{tikzcd}$$

Clearly $\kappa\colon (g_i)_i \mapsto (g_i)_i$ is a bijection and the diagram shows that the initial topologies makes κ a homeomorphism.

Lemma 3.7. *For the inverse systems in Lemma 3.5, we have the following homeomorphisms:*

1. $\varprojlim G_i \times_{\varprojlim \mathbf{s}_i, \varprojlim \mathbf{s}_i} \varprojlim G_i \simeq \varprojlim (G_i \times_{\mathbf{s}_i, \mathbf{s}_i} G_i)$,
2. $\varprojlim G_i \times_{\varprojlim \mathbf{t}_i, \varprojlim \mathbf{t}_i} \varprojlim G_i \simeq \varprojlim (G_i \times_{\mathbf{t}_i, \mathbf{t}_i} G_i)$,
3. $\varprojlim G_i \times_{\varprojlim \mathbf{s}_i, \varprojlim \mathbf{t}_i} \varprojlim G_i \simeq \varprojlim (G_i \times_{\mathbf{s}_i, \mathbf{t}_i} G_i)$,
4. $(\varprojlim G_i)\big|_{(x_i)_i} \simeq \varprojlim (G_i|_{x_i})$,
5. $(\varprojlim G_i)\big|^{(y_i)_i} \simeq \varprojlim (G_i|^{y_i})$,
6. $(\varprojlim G_i)\big|_{(x_i)_i}^{(y_i)_i} \simeq \varprojlim (G_i|_{x_i}^{y_i})$.

Corollary 3.8. *Let $(\{G_i \rightrightarrows M_i\}_{i\in I}, \{(F_{ij}, f_{ij})\}_{i\leq j})$ be an inverse system of topological groupoids. Then for all $(x_i)_i \in \varprojlim M_i$, $(\{G_i|_{x_i}^{x_i}\}_i, \{F_{ij}\}_{i\leq j})$ is an inverse system of topological groups. Moreover, $\varprojlim (G_i|_{x_i}^{x_i}) \simeq (\varprojlim G_i)\big|_{(x_i)}^{(x_i)_i}$.*

Remark 3.9. The orbit through $(x_i)_i \in \varprojlim M_i$ is the set $O_{(x_i)_i} := (\varprojlim \mathbf{t}_i)\left((\varprojlim G_i)|_{(x_i)_i}\right)$. With the topology induced from $\varprojlim M_i$, we note that $O_{(x_i)_i}$ is not necessarily homeomorphic to $\varprojlim O_{x_i}$, where $O_{x_i} = \mathbf{t}_i(G_i|_{x_i})$ is endowed with the topology induced from M_i, for all $i \in I$. First note that $(\{O_{x_i}\}_i, \{f_{ij}\}_{i\leq j})$ forms an inverse system of topological spaces. Indeed, let $z_j \in O_{x_j}$ and let $g_j \in G|_{x_j}^{z_j}$. Then

$$f_{ij}(z_j) = f_{ij}(\mathbf{t}_j(g_j)) = \mathbf{t}_i(F_{ij}(g_i)) \in O_{x_i},$$

since $F_{ij}(G|_{x_j}) \subseteq G|_{x_i}$. Let $(y_i)_i \in \varprojlim O_{x_i}$. For each $i \in I$, pick $g_i \in G_i|_{x_i}^{y_i}$. Then in general $(g_i)_i \notin \varprojlim G_i$.

Thus, in general $O_{(x_i)_i} \subseteq \varprojlim O_{x_i}$. The inclusion can be strict as we shall see in the examples below.

Example 3.10 (The inverse limit is not necessarily open)**.** For all $n \in \mathbb{N}_1$, set $M_n := [-1, 1]$ with the subspace topology and $G_n = M_n \times \mathbb{R}$ with the product topology. Then the continuous group bundle $\mathrm{pr}_1\colon G_n \to M_n$ defines a topological groupoid. Indeed, the groupoid structure is defined as follows:

1. $\mathbf{s}_n = \mathbf{t}_n = \mathrm{pr}_1$;
2. $(x, y_1).(x, y_2) = (x, y_1 + y_2)$;
3. $(x, y)^{-1} = (x, -y)$;
4. $1_x = (x, 0)$.

Define the connecting morphisms by $f_{n,n+1} := \mathrm{id}_{[-1,1]}$ and $F_{n,n+1}(x, y) := (x, \theta_n(x).y)$ for all $(x, y) \in G_{n+1}$, where $\theta_n : [-1, 1] \to [-1, 1]$ is a continuous cut-off function such that $\theta_n(0) = 1$ and $\theta_n(x) = 0$ whenever $|x| \geq 1/n$. It is easy to see that $(\{G_n \rightrightarrows M_n\}_n, \{(f_{nm}, F_{nm})\}_{n\leq m})$ is an inverse system of topological groupoids. Moreover, we can identify $\varprojlim M_n$ with $[-1, 1]$ and $\varprojlim G_n$ with $(\{0\} \times \mathbb{R}) \cup ((0, 1] \times \{0\}) \cup ([-1, 0) \times \{0\})$. But then $\varprojlim \mathbf{s}_n$ is just pr_1 restricted to this union. Now for any $0 < a < b \in \mathbb{R}$, $V = \{0\} \times (a, b)$ is open in the limit, but $\mathrm{pr}_1(V) = \{0\}$ which is not open in $[-1, 1]$.

Example 3.11. Let $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ be an inverse system of topological spaces. Then $(\{X_i \times X_i \rightrightarrows X_i\}_i, \{(f_{ij} \times f_{ij}, f_{ij})\}_{i\leq j}$ is an inverse system of topological groupoids. The inverse limit is the pair groupoid $\varprojlim X_i \times \varprojlim X_i \rightrightarrows \varprojlim X_i$. Pair groupoids are transitive and open. In fact $O_{(x_i)_i} = \varprojlim O_{x_i} = \varprojlim X_i$ for all $(x_i)_i \in \varprojlim X_i$.

Lemma 3.12 (Inverse limits of continuous action groupoids)**.** *Let $(\{G_i\}_i, \{F_{ij}\}_{i\leq j})$ be an inverse system of topological groups and $(\{M_i\}_i, \{f_{ij}\}_{i\leq j})$ be an inverse system of topological spaces together with a family of continuous group actions $\{A_i : G_i \times M_i \to M_i\}_i$ that is a morphism of inverse systems. Then $(\{G_i \ltimes M_i \rightrightarrows M_i\}_i, \{F_{ij} \times f_{ij}\}_{i\leq j})$ is an inverse system of topological groupoids where the limit $\varprojlim G_i \ltimes \varprojlim M_i \rightrightarrows \varprojlim M_i$ has $\varprojlim A_i$ as the continuous action on $\varprojlim M_i$.*

Proof. It is enough to show that $F_{ij} \times f_{ij}$ is a topological groupoid morphism for all $i \leq j$, but this is straightforward. □

Example 3.13. For any $n \in \mathbb{N}$, define $A_n : \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ by $A_n(k, [l]_n) = [l + k]_n$. Obviously A_n is a well-defined continuous map where $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ have their discrete topologies. In particular, it defines a continuous action. Hence $\mathbb{Z} \ltimes \mathbb{Z}/n\mathbb{Z} \rightrightarrows \mathbb{Z}/n\mathbb{Z}$ is a continuous action groupoid. Now let $m \in \mathbb{N}$ such that $n|m$ and define

$$f_{nm} : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}, \quad [l]_m \mapsto [l]_n,$$

and

$$F_{nm} : \mathbb{Z} \ltimes \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z} \ltimes \mathbb{Z}/n\mathbb{Z}, \quad (k, [l]_m) \mapsto (k, [l]_n).$$

These well-define continuous maps define a topological groupoid morphism (F_{nm}, f_{nm}). Define a direction $\preceq$ on $\mathbb{N}$ so that $a \preceq b$ iff $a \mid b$. Then, with the directed set $(\mathbb{N}, \preceq)$, $(\{\mathbb{Z}/n\mathbb{Z}\}_n, \{f_{nm}\}_{n\leq m})$ and $(\{\mathbb{Z}\ltimes\mathbb{Z}/n\mathbb{Z}\}_n, \{F_{nm}\}_{n\leq m})$ are inverse systems of topological spaces. Thus, $(\{\mathbb{Z} \ltimes \mathbb{Z}/n\mathbb{Z} \rightrightarrows \mathbb{Z}/n\mathbb{Z}\}_n, \{(F_{nm}, f_{nm})\}_{n\leq m})$ is an inverse of (open) topological groupoids. Moreover,

$$f_{nm}(A_m(k, [l]_m)) = f_{nm}([l + k]_m) = [l + k]_n = A_n(F_{nm}(k, [l]_m)).$$

Now $\varprojlim \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}}$ (the set of profinite integers) and $\varprojlim(\mathbb{Z} \ltimes \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z} \ltimes \widehat{\mathbb{Z}}$. For $([0]_n)_n \in \varprojlim \mathbb{Z}/n\mathbb{Z}$, $O_{[0]_n} = \mathbb{Z}/n\mathbb{Z}$, hence $\varprojlim O_{[0]_n} = \widehat{\mathbb{Z}}$, but

$$O_{([0]_n)_n} = \{([k]_n)_n : k \in \mathbb{Z}\} \simeq \mathbb{Z}.$$

Thus, $O_{([0]_n)_n} \subsetneq \varprojlim O_{[0]_n}$.

Finally, we note that $\mathbb{Z} \ltimes \mathbb{Z}/n\mathbb{Z} \rightrightarrows \mathbb{Z}/n\mathbb{Z}$ is transitive for all $n \in \mathbb{N}$. However, $\mathbb{Z} \ltimes \widehat{\mathbb{Z}} \rightrightarrows \widehat{\mathbb{Z}}$ is not locally transitive. We claim that $O_{([0]_n)_n} \simeq \mathbb{Z}$ is not an open neighborhood of $([0]_n)_n$. It is enough to note that for every $m \in \mathbb{N}$,

$$f_m^{-1}([k]_m) = \{([l_n]_n)_n \in \widehat{\mathbb{Z}} : [l_m]_m = [k]_m\}$$

is uncountable.

3.2 Strict inverse system of Lie groupoids and their algebroids

Definition 3.14. A *strict inverse system of Lie groupoids* is a pair $(\{G_i \rightrightarrows M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ such that

1. $(\{G_i\}_i, \{F_{ij}\}_{i\leq j})$ and $(\{M_i\}_i, \{f_{ij}\}_{i\leq j})$ are strict inverse systems of manifolds;
2. $G_i \rightrightarrows M_i$ is a Lie groupoid for all $i \in I$ and (F_{ij}, f_{ij}) is a Lie groupoid morphism for $i \leq j$;
3. $\{\mathbf{s}_i\}_i$ and $\{\mathbf{t}_i\}_i$ are strict families of smooth submersions.

That is, an inverse system of Lie groupoids is strict if the arrows and units inverse systems are strict.

Lemma 3.15. *Let $(\{G_i \rightrightarrows M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ be a strict inverse system of Lie groupoids. Then $\varprojlim G_i \rightrightarrows \varprojlim M_i$ is a Lie groupoid where the structure maps are defined by $(\varprojlim \mathbf{s}_i, \varprojlim \mathbf{t}_i, \varprojlim \mathbf{m}_i, \varprojlim \mathbf{1}_i, \varprojlim \mathbf{i}_i)$.*

Proof. Since $\mathbf{t}_i = \mathbf{s}_i \circ \mathbf{i}_i$, the family $\{\mathbf{t}_i\}_i$ becomes a strict family of submersions. By Proposition 2.47 $\varprojlim \mathbf{s}_i$ and $\varprojlim \mathbf{t}_i$ are submersions. It follows then that $(\{{G_i}^{(2)}\}_i, \{F_{ij} \times F_{ij}\})$ is a strict inverse system of manifolds either by using Theorem 2.48 or by showing directly noting that $\mathbf{s}_i \times \mathbf{t}_i$ is a submersion hence ${G_i}^{(2)} = (\mathbf{s}_i \times \mathbf{t}_i)^{-1}(\Delta_{M_i})$ is a split-submanifold by Proposition 2.47. Thus, $\varprojlim \mathbf{m}_i$ defines a smooth multiplication on $\varprojlim G_i$. □

Lemma 3.16. *An inverse system of topological groupoids $(\{G_i \rightrightarrows M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ is a strict inverse system of Lie groupoids if and only if*

1. *$(\{G_i\}_i, \{F_{ij}\}_{i\leq j})$ and $(\{M_i\}_i, \{f_{ij}\}_{i\leq j})$ are strict inverse systems of manifolds;*
2. *$\{\mathbf{s}_i\}_i$ is a strict family of smooth submersions;*
3. *$\{\mathbf{i}_i\}_i$ is a smooth diffeomorphism of inverse systems of manifolds;*
4. *$\{\mathbf{m}_i\}_i$ is smooth morphism of inverse systems of manifolds.*

Let $(\{G_i \rightrightarrows M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ be a strict inverse system of Lie groupoids. Then $(\{TG_i \rightrightarrows TM_i\}_{i\in I}, \{(TF_{ij}, Tf_{ij})\}_{i\leq j\in I})$ is a strict inverse system of Lie groupoids. Obviously, $(\{TG_i\}_i, \{TF_{ij}\}_{i\leq j})$ and $(\{TM_i\}_i, \{Tf_{ij}\}_{i\leq j})$ are strict inverse systems of manifolds, and $\{T\mathbf{s}_i\}_i$ is a strict family of submersions. We note that there exists a diffeomorphism between strict inverse systems of submanifolds $\{T({G_i}^{(2)})\}_i \simeq \{(TG_i)^{(2)}\}_i$ that is realized as the restriction of the canonical diffeomorphism $(\{T(G_i \times G_i)\}_i, \{T(F_{ij} \times TF_{ij})\}_{i\leq j}) \simeq (\{TG_i \times TG_i\}, \{TF_{ij} \times TF_{ij}\}_{i\leq j})$, by locally identifying $T({G_i}^{(2)})$ with $(TG_i)^{(2)}$ for all $i \in I$. Thus, T is also well-behaved with respect to inverse limits of Lie groupoids.

Lemma 3.17. *Let $(\{G_i \rightrightarrows M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ be a strict inverse system of Lie groupoids. Then*

1. $(\{G_i \times_{\mathbf{p}_i,\mathbf{q}_i} G_i\}_i, \{F_{ij} \times F_{ij}\}_{i\leq j})$*, for* $\mathbf{p}_i, \mathbf{q}_i \in \{\mathbf{s}_i, \mathbf{t}_i\}$,
2. $(\{G_i|_{x_i}\}_i, \{F_{ij}\}_{i\leq j})$ *for any* $(x_i)_i \in \varprojlim M_i$,
3. $(\{G_i|^{y_i}\}_i, \{F_{ij}\}_{i\leq j})$ *for any* $(y_i)_i \in \varprojlim M_i$,

are strict inverse systems of manifolds for which we have the following identifications for their inverse limits:

1. $\varprojlim G_i \times_{\varprojlim \mathbf{p}_i, \varprojlim \mathbf{q}_i} \varprojlim G_i \simeq \varprojlim (G_i \times_{\mathbf{p}_i,\mathbf{q}_i} G_i)$*, for* $\mathbf{p}_i, \mathbf{q}_i \in \{\mathbf{s}_i, \mathbf{t}_i\}$,
2. $(\varprojlim G_i)\big|_{(x_i)_i} \simeq \varprojlim (G_i|_{x_i})$,
3. $(\varprojlim G_i)\big|^{(y_i)_i} \simeq \varprojlim (G_i|^{y_i})$.

Remark 3.18. There is no general result regarding smooth structures on vertex groups and orbits in the locally convex case, although we obtain strict inverse limits if the vertex groups are embedded. Consequently, we cannot replicate our remarks regarding strict inverse systems of vertex groups and orbits as found in the topological groupoid case. In the finite-dimensional case, for $x_i \in G_i$, $y_i \in O_{x_i}$, $G_i|_{x_i}^{y_i}$ is an embedded submanifold of G_i, so it holds in this case. In the case of a Banach Lie groupoid $G \rightrightarrows M$, for $x \in M$ and $y \in O_x$, it was shown in [1] that for the inclusion $\iota_{x,y} \colon G|_x^y \hookrightarrow G$, the map $T\iota_{x,y}$ is injective, and both $T_g\iota_{x,y}(T_g G|_x^y)$ and $\iota_{x,y}(G|_x^y)$ are closed. This demonstrates that $G|_x^y$ is a closed immersed submanifold in the authors' terminology (noting that this is weaker than the definition of submersion we employ here). This was established by arguing that the distribution $\Delta \subseteq TG|_x$, where the fiber at $g \in G|_x$ is the subspace $\ker(T_g\mathbf{s}) \cap \ker(T_g\mathbf{t})$, is integrable. Furthermore, the authors show that for any $x \in M$, the inclusion $O_x \hookrightarrow M$ has an injective derivative (which is termed a *weak immersion* in their terminology).

Inverse systems of Lie algerboids and the functor L

Let $(\{G_i \rightrightarrows M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ be a strict inverse system of Lie groupoids. Denote $SG_i := \ker(T\mathbf{s}_i)$. Then $\{SG_i\}_i$ is a strict inverse system of split-subbundles of $(\{\pi_{TG_i} \colon TG_i \to G_i\}_i, \{(TF_{ij}, F_{ij})\}_{i\leq j})$. Moreover, $S\varprojlim G \simeq \varprojlim SG_i$ seen as a restriction of the diffeomorphism $T\varprojlim G_i \simeq \varprojlim TG_i$. This define a strict inverse system of pullback vector bundles $\{\mathbf{1}_i^* SG_i \to M_i\}_i$. We set $\mathbf{L}(G)_i = \mathbf{1}_i^* SG_i$ and define a vector bundle map $\mathbf{a}_i \colon \mathbf{L}(G)_i \to TM_i$

by the composition $\mathbf{L}(G)_i \to SG_i \hookrightarrow TG_i \xrightarrow{T\mathbf{t}_i} TM_i$. Similarly, we define $\mathbf{a}_{\varprojlim G_i}$. Then $\{\mathbf{a}_i\}_i$ is a morphism of inverse systems of vector bundles making $\{\mathbf{L}G_i\}_i$ a strict inverse system of anchored bundles. In fact, $\varprojlim \mathbf{a}_i$ is identified with $\mathbf{a}_{\varprojlim G_i}$ by virtue of the following commutative diagram:

$$\begin{tikzcd}
\mathbf{L}\varprojlim G_i \arrow[r] \arrow[d, "\wr"] & S\varprojlim G_i \arrow[r, hook] \arrow[d, "\wr"] & T\varprojlim G_i \arrow[r, "T\varprojlim \mathbf{t}_i"] \arrow[d, "\wr"] & T\varprojlim M_i \arrow[d, "\wr"] \\
\varprojlim \mathbf{L}G_i \arrow[r] & \varprojlim SG_i \arrow[r, hook] & \varprojlim TG_i \arrow[r, "\varprojlim T\mathbf{t}_i"] & \varprojlim TM_i
\end{tikzcd}$$

Now for $i \leq j$, it is clear that $\mathbf{a}_i \circ (f_{ij} \times TF_{ij})|_{\mathbf{L}G_j} = Tf_{ij} \circ \mathbf{a}_j$. Let $\xi_i, \eta_i \in \Gamma(\mathbf{L}G_i)$ and $\xi_j, \eta_j \in \Gamma(\mathbf{L}G_j)$ be such that $(f_{ij} \times TF_{ij}) \circ \xi_j = \xi_i \circ f_{ij}$ and $(f_{ij} \times TF_{ij}) \circ \eta_j = \eta_i \circ f_{ij}$. Define a smooth source-tangent vector field $\hat{\xi}_i$ by

$$\hat{\xi}_i(g_i) = T\mathbf{m}_i(\xi_i(\mathbf{t}_i(g_i)), Z_i(g_i)) \quad (\forall g_i \in G_i),$$

where $Z_i \colon G_i \to TG_i$ is the zero section. By construction, $\hat{\xi}_i$ is right-invariant. For all $i \leq j$, we have

$$\begin{aligned}
TF_{ij} \circ \hat{\xi}_j &= TF_{ij} \circ (T\mathbf{m}_j \circ (\xi_j \circ \mathbf{t}_j, Z_j)) \\
&= T\mathbf{m}_i \circ (TF_{ij} \circ (\xi_j \circ \mathbf{t}_j), TF_{ij} \circ Z_j) \\
&= T\mathbf{m}_i \circ ((\xi_i \circ \mathbf{t}_i) \circ F_{ij}, Z_i \circ F_{ij}) \\
&= \hat{\xi}_i \circ F_{ij}.
\end{aligned} \tag{6}$$

In turn, $TF_{ij} \circ [\hat{\xi}_j, \hat{\eta}_j]_{TG_j} = [\hat{\xi}_i, \hat{\eta}_i]_{TG_i} \circ F_{ij}$, where $\hat{\eta}_i$ is defined similarly. Now as customary, the Lie bracket on each $\mathbf{L}(G)_i$ is defined by $[\xi_i, \eta_i]_{\mathbf{L}(G)_i} = [\hat{\xi}_i, \hat{\eta}_i]_{TG_i} \circ \mathbf{1}_i$. Using 6, we see that

$$(f_{ij} \times TF_{ij}) \circ [\xi_j, \eta_j]_{\mathbf{L}G_j} = [\xi_i, \eta_i]_{\mathbf{L}G_i} \circ f_{ij}.$$

This shows that $(\{\mathbf{L}(G)_i\}_i, \{f_{ij} \times TF_{ij}\}_{i \leq j})$ is a strict inverse system of Lie algebroid.

We can define a Lie bracket $[\cdot, \cdot]_{\mathbf{L}\varprojlim G_i}$ on $\mathbf{L}\varprojlim G_i$ similarly. Now consider two families of sections $\xi_i, \eta_i \in \Gamma(\mathbf{L}G_i)$ such that $\{\xi_i\}_i$ and $\{\eta_i\}_i$ are morphisms of inverse systems of smooth manifolds. Then $\varprojlim \hat{\xi}_i$ and $\varprojlim \hat{\eta}_i$ are right-invariant and source-tangent. Indeed, $(\varprojlim \hat{\xi}_i)((g_i)_i) = (\varprojlim T\mathbf{m}_i)(\varprojlim \xi_i(\varprojlim \mathbf{t}_i((g_i)_i)))$. Hence $[\varprojlim \hat{\xi}_i, \varprojlim \hat{\eta}_i]_{T\varprojlim G_i} = \varprojlim [\hat{\xi}_i, \hat{\eta}_i]_{TG_i}$. It follows that

$$\begin{aligned}
[\varprojlim \hat{\xi}_i, \varprojlim \hat{\eta}_i]_{T\varprojlim G_i} \circ \varprojlim \mathbf{1}_i &= \varprojlim [\hat{\xi}_i, \hat{\eta}_i]_{TG_i} \circ \varprojlim \mathbf{1}_i \\
&= \varprojlim \left([\hat{\xi}_i, \hat{\eta}_i]_{TG_i} \circ \mathbf{1}_i \right) \\
&= \varprojlim [\xi_i, \eta_i]_{\mathbf{L}G_i}.
\end{aligned}$$

Thus, $\mathbf{L}\varprojlim G_i$ is a pro-Lie algebroid.

Example 3.19. Let $(\{M_i\}_i, \{g_{ij}\}_{i \leq j})$ be a strict inverse system of manifolds and let $(\{G_i\}_i, \{h_{ij}\}_{i \leq j})$ be a strict inverse system of Lie groupoids.

1. $(\{M_i \times M_i \rightrightarrows M_i\}_i, \{(g_{ij} \times g_{ij}, g_{ij})\}_{i\le j})$ is strict inverse system of Lie groupoids, where each $M_i \times M_i \rightrightarrows M_i$ is the pair groupoid associated with M_i. Obviously, the limit is the pair groupoid $\varprojlim M_i \times \varprojlim M_i \rightrightarrows \varprojlim M_i$.
2. More generally, let $\{f_i\colon M_i \to N_i\}_i$ be a strict family of submersions taking values in a strict inverse system of manifolds $(\{N_i\}_i, \{h_{ij}\}_{i\le j})$. Then $(\{M_i \times_{f_i,f_i} M_i \rightrightarrows M_i\}_i, (g_{ij} \times g_{ij}, g_{ij})_{i\le j})$ is a strict inverse system of Lie groupoids where the inverse limit is $\varprojlim M_i \times_{\varprojlim f_i, \varprojlim f_i} M_i \rightrightarrows \varprojlim M_i$ by Theorem 2.48. It is easy to see that $\mathbf{L}(M_i \times_{f_i,f_i} M_i)$ is just $\ker(Tf_i)$ and hence $\mathbf{L}\varprojlim(M_i \times_{f_i,f_i} M_i)$ can be identified with $\ker(T\varprojlim f_i)$.
3. Recall that $M_i \times G_i \times M_i \rightrightarrows M_i$ has the structure of a Lie groupoid with the structure defined by $\mathbf{s}_i\colon (y_i, g_i, x_i) = x_i$, $\mathbf{t}_i(y_i, g_i, x_i) = y_i$, $(z_i, h_i, y_i).(y_i, g_i, x_i) = (z_i, h_i g_i, x_i)$, $1_{x_i} = (x_i, e_i, x_i)$, and $(y_i, g_i, x_i)^{-1} = (x_i, g_i^{-1}, y_i)$. Then $(\{M_i \times G_i \times M_i \rightrightarrows M_i\}_i, \{g_{ij} \times h_{ij} \times g_{ij}\}_{i\le j})$ is a strict inverses system of trivial groupoids where the limit is the trivial groupoid $\varprojlim M_i \times \varprojlim G_i \times \varprojlim M_i \rightrightarrows \varprojlim M_i$. We note that $\mathbf{L}(M_i \times G_i \times M_i) = TM_i \times \mathbf{L}G_i$, where we use the negative of the Lie bracket on $\mathbf{L}G_i$. The Lie algebroid of the inverse limit groupoid is then $T\varprojlim M_i \times \varprojlim \mathbf{L}G_i$.
4. Suppose that we have a family of smooth group actions $\{A_i\colon G_i \times M_i \to M_i\}_i$ that is a morphism of inverse systems of manifolds. Then $(\{G_i \ltimes M_i \rightrightarrows M_i\}_i, \{h_{ij} \times g_{ij}\}_{i\le j})$ is a strict inverse system of groupoids and the limit $\varprojlim G_i \times \varprojlim M_i \rightrightarrows \varprojlim M_i$ is an action groupoid with the action $\varprojlim A_i$. We then have $\varprojlim(\mathbf{L}G_i \ltimes M_i) \simeq \mathbf{L}\varprojlim G_i \ltimes \varprojlim M_i$.

Example 3.20. Let $G \rightrightarrows M$ be a finite-dimensional Lie groupoid, where $\dim M = n$ and $\dim G = n + d$. Set $J^0G = G$. For $k \in \mathbb{N}_1$, let J^kG denote the set of k-jets of local bisections of G. First, recall that the product of two local bisections σ and τ is defined by $(\sigma \circ \tau)(x) := \mathbf{m}(\sigma(\mathbf{t}(\tau(x))), \tau(x))$ whenever it makes sense. The following construction of the k-groupoid prolongation is standard [13]. For $k \in \mathbb{N}_0$, $J^kG \rightrightarrows M$ is a Lie groupoid with the maps $(\mathbf{s}_k, \mathbf{t}_k, \mathbf{m}_k, \mathbf{1}_k, \mathbf{i}_k)$ defined by:

1. $\mathbf{s}_k(j^k_x\sigma) = \mathbf{s}(\sigma(x)) = x$ and $\mathbf{t}_k(j^k_x\sigma) = \mathbf{t}(\sigma(x))$ for any local bisection σ;
2. $\mathbf{m}_k(j^k_y\sigma, j^k_x\tau) = j^k_x(\sigma \circ \tau)$ for any composable $j^k_y\sigma, j^k_x\tau \in J^kG$ (that is, $y = \mathbf{t}(\tau(x))$);
3. $\mathbf{1}_k\colon x \mapsto j^k_x\mathbf{1}$;
4. $\mathbf{i}_k\colon j^k_x\sigma \mapsto j^k_{\mathbf{t}(\sigma(x))}\tilde{\sigma}$, where $\tilde{\sigma} = \mathbf{i} \circ \sigma \circ (\mathbf{t} \circ \sigma)^{-1}$.

First we recall the smooth manifold structure on J^kG. This is done by showing that J^kG is an open set of $J^k\mathbf{s}$, the set of k-jets of sections of $\mathbf{s}$ which is a submanifold of $J^k(M, G)$. Define $\mathcal{S}\colon J^k(M, G) \to J^k(M, M)$ by $\mathcal{S}(j^k_x\sigma) = j^k_x(\mathbf{s} \circ \sigma)$. In local coordinates, one can show that $\mathcal{S}$ is a submersion and that $J^k\mathbf{s}$ is the preimage of $\{j^k_x \operatorname{id}_M\colon x \in M\}$ which is a closed submanifold of $J^k(M, M)$. For $k \ge 1$, let $\pi^k_{k-1}\colon J^k\mathbf{s} \to J^{k-1}\mathbf{s}$ denote the projection $j^k\sigma \mapsto j^{k-1}\sigma$. Recall that π^k_{k-1} has the structure of an affine bundle [10]. Define $\mathcal{T}\colon J^1\mathbf{s} \to J^1(M, M)$ by $\mathcal{T}(j^1_x\sigma) = j^1_x(\mathbf{t} \circ \sigma)$. By computing in local coordinates, one can see that $\mathcal{T}$ is smooth. Let $U = \{j^1_x f \in J^1(M, M) : \det d_x f \neq 0\}$. This is an open set. It remains to note that $J^kG = (\mathcal{T} \circ \pi^k_1)^{-1}(U)$. We note that $\mathbf{s}_k = \pi_k\colon J^kG \to M, j^k_x\sigma \mapsto x$ and $\mathbf{t}_k = \mathbf{t} \circ \pi^k_0$ hence both are smooth surjective submersions. The unit map $\mathbf{1}_k$ is obviously smooth. One can argue using the chain rule for jets (Faà Di Bruno formula in charts) that $\mathbf{m}_k$ and $\mathbf{i}_k$ are smooth.

We claim that $(\{J^kG \rightrightarrows M\}_{k\in\mathbb{N}_0}, \{(\pi^k_{k-1}, \mathrm{id}_M)\}_{k\in\mathbb{N}})$ is a strict inverse system of Lie groupoids. That $\{J^kG \rightrightarrows M\}_k$ is an inverse system of Lie groupoid is obvious. It follows from Corollary 2.42 that it is saturated. Let us consider the associated charts explicitly. Let $(j^k_x\sigma)_{k\in\mathbb{N}_0} \in J^\infty G$ and let ϕ be a chart about x and let $\phi_0\colon U_0 \to V_\phi \times W \subseteq \mathbb{R}^m \times \mathbb{R}^d$ be an adapted chart about $\sigma(x)$. For $k \in \mathbb{N}$, a chart $\phi_k\colon (\pi^k_0)^{-1}(U_0) \to V_\phi \times W \times \bigoplus_{j=1}^k \mathbb{R}^d \otimes S^j(\mathbb{R}^m)^*$ about $j^k_x\sigma$ is constructed in the natural manner by sending $j^k_x\tau$ to the base point and the derivatives of the coordinate representation of σ. In local coordinates, $\{\mathbf{s}_k\}_k$ are all projections onto V_ϕ. This is obviously a strict family of submersions. Thus, $\{\mathbf{L}J^kG\}$ is a strict inverse system of Lie algebroid whose inverse limit is also a Lie algebroid. Utilizing the Lie algebroid isomorphism $J^k\mathbf{L}G \simeq \mathbf{L}J^kG$, one obtains: $J^\infty\mathbf{L}G \simeq \mathbf{L}J^\infty G$.

Example 3.21 (Strict inverse systems of Gauge groupoids)**.** We show that associated with a strict inverse system of principle bundles, there is a strict inverse system of gauge groupoid.

1. A strict inverse system of Banach principal bundles is given by a strict inverse family of Banach fiber bundle $(\{\pi_i\colon P_i \to M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\le j})$ and a strict inverse system of Banach-Lie groups $(\{G_i\}_i, \{h_{ij}\}_{i\le j})$ such that $\pi_i\colon P_i \to M_i$ is a principal G_i-bundle and the the group actions are compatible. That is, $F_{ij}(p_j.g_j) = F_{ij}(p_j).h_{ij}(g_j)$ for all $p_j \in P_j$ and $g_j \in G_j$ and such that for all $(x_i)_i \in \varprojlim M_i$ there exists a strict family of local trivializations $\{\tau_i\colon \pi_i^{-1}(U_i) \to U_i \times G_i\}_i$. Denote $\hat\tau_i := \mathrm{pr}_2 \circ \tau_i\colon \pi_i^{-1}(U_i) \to G_i$ which is G_i-equivariant: $\hat\tau_i(p_i.g_i) = \hat\tau_i(p_i).g_i$. For another bundle trivialization λ_i, the overlap map $g_{\lambda_i\tau_i}\colon U_{\lambda_i} \cap U_{\tau_i} \to G_i$ is a smooth map satisfying $\hat\lambda_i(p_i) = g_{\lambda_i\tau_i}(\pi_i(p_i))\hat\tau_i(p_i)$. Obviously, $h_{ij}(\hat\tau_j(p_j.g_j)) = \hat\tau_i(F_{ij}(p_j).h_{ij}(g_j))$ and
$$\hat\lambda_i(F_{ij}(p_j)) = h_{ij}(\hat\lambda_j(p_j)) = h_{ij}(g_{\lambda_j\tau_j}(\pi_j(p_j))\hat\tau_j(p_j)) = h_{ij}(g_{\lambda_j\tau_j}(\pi_j(p_j))\hat\tau_i(F_{ij}(p_j)).$$
Hence $g_{\lambda_i\tau_i} \circ f_{ij} = h_{ij} \circ g_{\lambda_j\tau_j}$. Let $\sigma_{\tau_i}\colon U_{\tau_i} \to P_i$ be a section of π_i such that $\tau_i(\sigma_{\tau_i}(x_i)) = (x_i, e_i)$ and note that for all $x_j \in U_{\tau_j}$:
$$\begin{aligned}\tau_i((F_{ij} \circ \sigma_{\tau_j}))(x_j)) &= ((\tau_i \circ F_{ij}) \circ \sigma_{\tau_i})(x_j) = (((f_{ij} \times h_{ij}) \circ \tau_j) \circ \sigma_{\tau_i})(x_j)\\ &= (f_{ij} \times h_{ij})(x_j, e_j) = (f_{ij}(x_j), h_{ij}(e_j))\\ &= (f_{ij}(x_j), e_i) = \tau_i((\sigma_{\tau_i} \circ f_{ij})(x_j)).\end{aligned}$$

2. Set $Q_i := (P_i \times P_i)/G_i$. We can endow Q_i with a smooth manifold structure in the following way: let $[p_i, q_i] \in Q_i$, let τ_i be a trivialization about $\pi_i(p_i)$, and let λ_i be a trivialization about $\pi_i(q_i)$. Set $W_{\tau_i\lambda_i} := (\mathbf{t}_i, \mathbf{s}_i)^{-1}(U_{\tau_i} \times U_{\lambda_i})$ and define
$$\Theta_{\tau_i\lambda_i}\colon W_{\tau_i\lambda_i} \to U_{\tau_i} \times U_{\lambda_i} \times G_i, \quad [p_i, q_i] \mapsto (\pi_i(p_i), \pi(q_i), \hat\tau_i(p_i)\hat\lambda_i(q_i)^{-1}).$$
Note that W_{τ_i} is just $(\pi_i^{-1}(U_{\tau_i}) \times \pi_i^{-1}(U_{\tau_i}))/G_i$. The inverse is given by
$$\Theta^{-1}_{\tau_i\lambda_i}\colon (x_i, y_i, g_i) \mapsto [\sigma_{\tau_i}(x_i).g_i, \sigma_{\lambda_i}(y_i)]$$
which is well-defined. Let $\tilde\tau_i$ and $\tilde\lambda_i$ be local trivializations about $\pi_i(p_i)$ and $\pi_i(q_i)$, respectively. Then the overlap map is given by
$$(\Theta_{\tilde\tau_i\tilde\lambda_i} \circ \Theta^{-1}_{\tau_i\lambda_i})(x_i, y_i, g_i) = (x_i, y_i, g_{\tilde\tau_i\tau_i}(x_i)g_ig_{\tilde\lambda_i\lambda_i}(y_i)^{-1}).$$

We can construct a smooth map $\chi_i\colon P_i \times_{\pi_i,\pi_i} P_i \to G_i$ such that $(p_i, q_i) \mapsto g_i$ such that $p_i.g_i = q_i$. This maps is given in the trivializations τ_i and λ_i about $\pi_i(p_i) = \pi_(q_i)$ by $\chi_{\tau_i\lambda_i}\colon (p_i, q_i) \mapsto \hat{\tau}_i(p_i)^{-1}.\hat{\lambda}_i(q_i)$, hence it is smooth.

3. Let us now recall the construction of the groupoid structure on $Q_i \rightrightarrows M_i$. Define $\mathbf{s}_i, \mathbf{t}_i\colon Q_i \to M_i$ by $\mathbf{s}_i([p_i, q_i]) = \pi_i(q_i)$ and $\mathbf{t}_i([p_i, q_i]) = \pi_i(p_i)$. We define multiplication by $\mathbf{m}_i([p_i, q_i], [p_i', q_i']) = [p_i, q_i'.\chi_i(q_i, p_i')^{-1}]$. The inverse is the map $[p_i, q_i] \mapsto [q_i, p_i]$ and the unit map are given by $\mathbf{1}_i\colon x_i \mapsto \sigma_{\tau_i}(x_i)$ for any local trivialization about x_i. This is easily verified to be well-defined.
4. For $i \leq j$, define $q_{ij}\colon Q_j \to Q_i$ by $[p_j, q_j] \mapsto [F_{ij}(p_j), F_{ij}(q_j)]$. Note that for any $g_j \in G_j$:

$$q_{ij}([p_j.g_j, q_j.g_j]) = [F_{ij}(p_j.g_j), F_{ij}(q_j.g_j)] = \\ [F_{ij}(p_j).h_{ij}(g_j), F_{ij}(q_j).h_{ij}(g_j)] = [F_{ij}(p_j), F_{ij}(q_j)].$$

Hence q_{ij} is well-defined. Note that $\pi_{Q_i}\colon P_i \times P_i \to Q_i$ is a surjective submersion. Indeed, with τ_i, λ_i being local trivializations about $p_i, q_i \in P_i$, respectively, π_{Q_i} is given by

$$U_{\tau_i} \times G_i \times U_{\lambda_i} \times G_i \ni ((x_i, g_i), (y_i, h_i)) \mapsto (x_i, y_i, g_i h_i^{-1}) \in U_{\tau_i} \times U_{\lambda_i} \times G_i.$$

Now $q_{ij} \circ \pi_{Q_j} = \pi_{Q_i} \circ (F_{ij} \times F_{ij})$ is smooth which implies that q_{ij} is smooth. Let us compute q_{ij} locally:

$$\begin{aligned}(\Theta_{\tau_i\lambda_i} \circ q_{ij} \circ \Theta^{-1}_{\tau_j\lambda_j})(x_j, y_j, g_j) &= (\Theta_{\tau_i\lambda_i} \circ q_{ij})([\sigma_{\tau_j}(x_j).g_j, \sigma_{\lambda_j}(y_j)]) \\ &= \Theta_{\tau_i\lambda_i}([F_{ij}(\sigma_{\tau_j}(x_j).g_j), F_{ij}(\sigma_{\lambda_j}(y_j))]) \\ &= \Theta_{\tau_i\lambda_i}([\sigma_{\tau_i}(f_{ij}(x_j)).h_{ij}(g_j), \sigma_{\lambda_i}(f_{ij}(y_j))]) \\ &= (f_{ij}(x_j), f_{ij}(y_j), \hat{\tau}_i(\sigma_{\tau_i}(f_{ij}(x_j)).h_{ij}(g_j))\hat{\lambda}_i(\sigma_{\lambda_i}(f_{ij}(y_j))) \\ &= (f_{ij}(x_j), f_{ij}(y_j), h_{ij}(g_j)).\end{aligned}$$

This computation shows, in particular, that the following diagram commutes

$$\begin{array}{ccc} W_{\tau_j\lambda_j} & \xrightarrow{\Theta_{\tau_j\lambda_j}} & U_{\tau_j} \times U_{\lambda_j} \times G_j \\ {\scriptstyle q_{ij}}\big\downarrow & & \big\downarrow{\scriptstyle f_{ij} \times f_{ij} \times h_{ij}} \\ W_{\tau_i\lambda_i} & \xrightarrow{\Theta_{\tau_i\lambda_i}} & U_{\tau_i} \times U_{\lambda_i} \times G_i \end{array}$$

It easily follows that $(\{Q_i\}_i, \{q_{ij}\}_{i\leq j})$ is an inverse system of smooth manifolds. It remains to show that it is strict. But this follows immediately by construction since $\varprojlim U_{\tau_j} \times \varprojlim U_{\lambda_j} \times \varprojlim G_j$ is open in $\varprojlim M_i \times \varprojlim M_i \times \varprojlim G_i$. Let $([p_j, q_j], [p_j', q_j']) \in {Q_j}^{(2)}$ and compute

$$\begin{aligned}(\mathbf{m}_i \circ (q_{ij} \times q_{ij}))([p_j, q_j], [p_j', q_j']) &= \mathbf{m}_i([F_{ij}(p_j), F_{ij}(q_j)], [F_{ij}(p_j'), F_{ij}(q_j')]) \\ &= [F_{ij}(p_j), F_{ij}(q_j').\chi_i(F_{ij}(q_j), F_{ij}(p_j'))^{-1}] \\ &= [F_{ij}(p_j), F_{ij}(q_j').h_{ij}(\chi_j(p_j, q_j')^{-1})] \\ &= [F_{ij}(p_j), F_{ij}(q_j'.\chi_j(p_j, q_j')^{-1})] \\ &= (q_{ij} \circ \mathbf{m}_j)([p_j, q_j], [p_j', q_j']).\end{aligned}$$

This shows that $\{\mathbf{m}_i\}_i$ is a compatible family of smooth maps. Compatibility of the remaining groupoid strucutre maps are straightforward. This shows that (q_{ij}, f_{ij}) is a Lie groupoid morphism for all $i \leq j$. Thus, $(\{Q_i \rightrightarrows M_i\}_i, \{(q_{ij}, f_{ij})\}_{i\leq j})$ is a strict inverse system of Lie groupoids. This shows that $\varprojlim Q_i \rightrightarrows \varprojlim M_i$ is a Lie groupoid. However, in general it will not be a principal $\varprojlim G_i$-bundle.

3.3 The multiphase diffeomorphism groupoid

Similar to half-Lie groups, there are situations where the multiplication defined on a groupoid has different regularities for left and right multiplication. This applies to the multiphase diffeomorphism groupoid introduced in [8]. Unlike the characterization given for strict ILH/ILB groups in [14], it is not possible to obtain a nice characterization of when a topological groupoid is a strict inverse limit of topological groupoid where each component has a different regularity for the multiplication and/or inverse maps. We record the following trivial definition to account for such case.

Definition 3.22. An inverse system of topological groupoids $(\{G_i \rightrightarrows M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ is *strict* if

1. $(\{G_i\}_i, \{F_{ij}\}_{i\leq j})$ and $(\{M_i\}_i, \{f_{ij}\}_{i\leq j})$ are strict inverse systems of manifolds;
2. $\{\mathbf{s}_i\}_i$ and $\{\mathbf{t}_i\}_i$ are strict families of smooth submersions;
3. $\mathbf{1}_i$ is a smooth embedding, for all i;
4. $\varprojlim \mathbf{m}_i$ and $\varprojlim \mathbf{i}_i$ are smooth.

The following lemma is immediate.

Lemma 3.23. *Let $(\{G_i \rightrightarrows M_i\}_i, \{(F_{ij}, f_{ij})\}_{i\leq j})$ be a strict inverse system of topological groupoids. Then $\varprojlim G_i \rightrightarrows \varprojlim M_i$ is a Lie groupoid.*

Let (M, g) be a smooth compact d-dimensional Riemannian manifold with a Riemannian metric g defining a volume form μ. For simplicity we suppose that M has no boundary. Let $m \in \mathbb{N}$ represents the number of phases of inviscid fluid occuping M. An m-tuple of d-forms on M will be denoted by $\underline{\nu} := (\nu_1, \dots, \nu_m)$. Similarly, an m-tuple of diffeomorphisms will be denoted by $\underline{\phi} := (\phi_1, \dots, \phi_m)$ with $\underline{\phi}^{-1} := (\phi_1^{-1}, \dots, \phi_m^{-1})$. We denote $\underline{\phi}^*\underline{\nu} := (\phi_1^*\nu_1, \dots, \phi_m^*\nu_m)$. We consider the case where the total volume of each phase is fixed. Therefore, we fix $\mathrm{vol}_a \in \mathbb{R}_{>0}$ such that $\sum_a \mathrm{vol}_a = \int_M \mu$. Set $\mu_a = \frac{\mathrm{vol}_a}{\int_M \mu}\mu$ and $\underline{\hat{\mu}} := (\mu_1, \dots, \mu_m)$. and, for $s > d/2 + 2$, denote

$$\mathcal{V}_a^s := \{\nu \in \Omega^d_{H^s}(M) : \nu > 0, \int_M \nu = \mathrm{vol}_a\},$$

$$\mathcal{D}^s := \{\psi \in H^s(M, M) : \psi \text{ is a bijection and } \psi^{-1} \in H^s(M, M)\},$$

and

$$\mathcal{D}_a^s := \{\phi \in \mathcal{D}^s : \phi^*\mu_a = \mu_a\}.$$

Note that $\mathcal{V}_a^s$ is an open subset of $\mu_a + \mathrm{d}\,\Omega_{H^s}^{d-1}(M)$ which is an affine subspace of $\Omega_{H^s}^d(M)$. In fact (by Moser's trick [3, Lemma 5.2]) there exists a continuous map $\chi^s\colon \prod_a \mathcal{V}_a^s \to \prod_a \mathcal{D}^s$ such that $\chi^s(\underline{\nu})^*\underline{\hat{\mu}} = \underline{\nu}$. Now we define a topological groupoid $\mathcal{MD}^s \rightrightarrows \mathcal{MV}^s$ by setting:

$$\mathcal{MV}^s := \{\underline{\nu} \in \prod_a \mathcal{V}_a^s : \sum_a \nu_a = \mu\}$$

and

$$\mathcal{MD}^s := \{(\underline{\phi}, \underline{\nu}, \underline{\tilde{\nu}}) \in (\mathcal{D}^s)^m \times \mathcal{MV}^s \times \mathcal{MV}^s : \underline{\phi}^*\underline{\tilde{\nu}} = \underline{\nu}\}.$$

We remark that the construction given in [8] is done by using the pushforward. By construction, $\mathcal{MV}^s$ is a Hilbert manifold where $\Omega_{H^s}^d(M)^m$ is the modeling space. We put a manifold structure on $\mathcal{MD}^s$ by constructing a bijection

$$\Psi^s\colon \mathcal{MD}^s \to \mathcal{MV}^s \times \left(\prod_a \mathcal{D}_a^s\right) \times \mathcal{MV}^s,\ (\underline{\phi}, \underline{\nu}, \underline{\tilde{\nu}}) \mapsto (\underline{\nu}, \chi^s(\underline{\tilde{\nu}}) \circ \underline{\phi} \circ \chi^s(\underline{\nu})^{-1}, \underline{\tilde{\nu}}),$$

with the inverse being defined in the obvious way. We use Ψ^s (as a global chart) to put a Hilbert manifold structure on $\mathcal{MD}^s$. The following proposition is then obvious.

Proposition 3.24. *Let $s > d/2 + 2$ and let $I = \mathbb{N}_s$. Then $\{\mathcal{MV}^s\}_s$ and $\{\mathcal{MD}^s\}_s$ are strict inverse systems of Hilbert manifolds (with the dense embeddings as the connecting morphisms). Moreover, with the following groupoid structure maps:*

1. *smooth source and target maps given by $\mathbf{s}_s\colon (\underline{\phi}, \underline{\nu}, \underline{\nu}') \mapsto \underline{\nu}$ and $\mathbf{t}_s\colon (\underline{\phi}, \underline{\nu}, \underline{\nu}') \mapsto \underline{\nu}'$,*
2. *a smooth unit map $\mathbf{1}_s\colon \underline{\nu} \mapsto (\mathrm{id}, \underline{\nu}, \underline{\nu})$,*
3. *a continuous partial multiplication map given by*

$$\mathbf{m}_s\colon ((\underline{\tilde{\phi}}, \underline{\nu}', \underline{\nu}''), (\underline{\phi}, \underline{\nu}, \underline{\nu}')) \mapsto (\underline{\tilde{\phi}} \circ \underline{\phi}, \underline{\nu}, \underline{\nu}''),$$

4. *a continuous inverse $\mathbf{i}_s\colon (\underline{\phi}, \underline{\nu}, \underline{\nu}') \mapsto (\underline{\phi}^{-1}, \underline{\nu}', \underline{\nu})$,*

$\{\mathcal{MD}^s \rightrightarrows \mathcal{MV}^s\}_s$ is a strict inverse system of transitive topological groupoids.

This makes $\mathcal{MD} \rightrightarrows \mathcal{MV}$ a transitive Fréchet Lie groupoid by leveraging the Lie group structures on $\mathcal{D}_a$. In fact, passing to the limit, the comdomain of Ψ^s is an trivial ILH-groupoid.

References

[1] D. Beltiţă, T. Goliński, G. Jakimowicz, and F. Pelletier. Banach–Lie groupoids and generalized inversion. *Journal of Functional Analysis*, 276(5):1528–1574, 2019.

[2] P. Cabau and F. Pelletier. *Direct and projective limits of geometric Banach structures.* CRC Press, 2023.

[3] D. G. Ebin and J. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Annals of Mathematics*, 92(1):102–163, 1970.

[4] C. Ehresmann. *Catégories topologiques et catégories différentiables*. Librairie universitaire, 1959.

[5] G. N. Galanis. Projective limits of Banach vector bundles. *Portugaliae Mathematica*, 55 (1):11–24, 1998.

[6] H. Glöckner. Fundamentals of submersions and immersions between infinite-dimensional manifolds. *arXiv preprint arXiv:1502.05795*, 2015.

[7] A. Goncharov and Z. Ural. Mityagin's extension problem: Progress report. *Journal of Mathematical Analysis and Applications*, 448(1):357–375, 2017.

[8] A. Izosimov and B. Khesin. Geometry of generalized fluid flows. *Calculus of Variations and Partial Differential Equations*, 63(1):3, 2024.

[9] K. Landsman. Lie groupoids and Lie algebroids in physics and noncommutative geometry. *Journal of Geometry and Physics*, 56(1):24–54, 2006.

[10] A. D. Lewis. The affine structure of jet bundles. MATH 949 Course Notes, 2005.

[11] A. D. Lewis. Infinite-dimensional analysis and geometry. Unpublished course notes, 2025.

[12] K. C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 2005.

[13] E. Meinrenken. Lie groupoids and lie algebroids lecture notes, fall 2017. *Last accessed*, 16, 2017.

[14] H. Omori. *Infinite dimensional Lie transformation groups*, volume 427 of *Lecture Notes in Mathematics*. Springer, 1974. Reprinted 2006.

[15] M. A. Salazar. *Pfaffian groupoids*. PhD thesis, Utrecht University, 2013.

[16] A. Schmeding. *An introduction to infinite-dimensional differential geometry*, volume 202 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2022.

[17] A. Schmeding and C. Wockel. The Lie group of bisections of a Lie groupoid. *Annals of Global Analysis and Geometry*, 48(1):87–123, 2015.