Homework 2

April XX, 2017

Earle Aguilar (804501476)

Problem 1.0:

$$Q = \begin{pmatrix} \hat{\beta} \\ \hat{Y} \end{pmatrix} = \begin{pmatrix} (X'X)^{-1}X'Y \\ (I-H)Y \end{pmatrix} \tag{1.1}$$

$$Q = LY \tag{1.2}$$

$$Var(Q) = Var(LY) = LVar(Y)L' \tag{1.3}$$

$$= \sigma^2 LL' \tag{1.4}$$

$$= \sigma^2 \begin{pmatrix} (X'X)^{-1}X' \\ (I-H) \end{pmatrix} \begin{pmatrix} (X'X)^{-1}X' & (I-H) \end{pmatrix} \tag{1.5}$$

$$= \sigma^2 \begin{pmatrix} (X'X)^{-1} & (X'X)^{-1}X'(I-H) \\ (I-H)X(X'X)^{-1} & (I-H) \end{pmatrix} \tag{1.6}$$

$$Var(Q) = \sigma^2 \begin{pmatrix} (X'X)^{-1} & 0 \\ 0 & (I-H) \end{pmatrix} \tag{1.7}$$

Therefore $\hat{Y}$ and e are independent.

Problem 2.0:

$$\hat{\beta}_c = \hat{\beta} - (X'X)^{-1}c'[c(X'X)^{-1}c']^{-1}(c\hat{\beta} - \gamma) \tag{2.8}$$

$$\tag{2.9}$$

let $A = (X'X)^{-1}c'[c(X'X)^{-1}c']^{-1}$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - Var(A(c\hat{\beta} - \gamma)) \tag{2.10}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - AVar(c\hat{\beta} - \gamma)A' \tag{2.11}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - A[cVar(\hat{\beta})c' - Var(\gamma)]A' \tag{2.12}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - AcVar(\hat{\beta})c'A' + AVar(\gamma)A' \tag{2.13}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - AcVar(\hat{\beta})c'A' + AVar(c\beta)A' \tag{2.14}$$

$$Var(\hat{\beta}_c) = \sigma^2(X'X)^{-1} - Ac[\sigma^2(X'X)^{-1}]c'A' \tag{2.15}$$

$$Var(\hat{\beta}_c) = \sigma^2(X'X)^{-1} - \sigma^2(X'X)^{-1}c'[c'(X'X)^{-1}c]^{-1}c(X'X)^{-1} \tag{2.16}$$

Problem 2.0 continued on next page...

Problem 3.0:

We have that $S^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1}$ and $S_e^2 = \frac{e^{'}e}{n-k-1} = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2 - \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{n-k-1}$

$$R_a^2 = \frac{S^2 - s_e^2}{S^2} \tag{3.17}$$

$$= \frac{n-1}{\sum_{i=1}^{n}(y_i - \bar{y})^2}\left[\frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1} - \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-k-1} + \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{n-k-1}\right] \tag{3.18}$$

$$= \frac{n-1}{\sum_{i=1}^{n}(y_i - \bar{y})^2}\left[\frac{(n-k-1)\sum_{i=1}^{n}(y_i - \bar{y})^2 - (n-1)\sum_{i=1}^{n}(y_i - \bar{y})^2 + (n-1)\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{(n-1)(n-k-1)}\right] \tag{3.19}$$

$$= \left[\frac{(n-k-1) - (n-1) + (n-1)\frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}}{n-k-1}\right] \tag{3.20}$$

$$= 1 - \frac{(n-1) - (n-1)R^2}{n-k-1} \tag{3.21}$$

$$\boxed{R_a^2 = 1 - \frac{n-1}{n-k-1}(1 - R^2)} \tag{3.22}$$

Problem 4.0:

a

$$var\left(\frac{(n-k-1)S_e^2}{\sigma^2}\right) = 2(n-k-1) \tag{4.23}$$

$$\frac{(n-k-1)^2}{\sigma^4}Var(S_e^2) = 2(n-k-1) \tag{4.24}$$

$$\frac{n-k-1}{\sigma^4}Var(S_e^2) = 2 \tag{4.25}$$

$$Var(S_e^2) = \frac{2\sigma^4}{n-k-1} \tag{4.26}$$

Problem 4.0 continued on next page...

b

c

Problem 5.0:

$$\begin{aligned}(Y - Xc)^{'}(Y - Xc) - (Y - X\hat{\beta})^{'}(Y - X\hat{\beta}) && (5.27)\\ Y^{'}Y - Y^{'}Xc - c^{'}X^{'}Y + c^{'}X^{'}Xc - Y^{'}Y - Y^{'}Y + Y^{'}X\hat{\beta} + \hat{\beta}^{'}X^{'}Y - \hat{\beta}^{'}X^{'}X\hat{\beta} && (5.28)\\ && (5.29)\end{aligned}$$

We use the normal equations to substitute for $Y^{'}X$ and $X^{'}Y$.

$$\begin{aligned}-\hat{\beta}^{'}X^{'}Xc - c^{'}X^{'}X\hat{\beta} + c^{'}X^{'}Xc + \hat{\beta}X^{'}X\hat{\beta} + \hat{\beta}^{'}X^{'}X\hat{\beta} - \hat{\beta}X^{'}X\hat{\beta} && (5.30)\\ c^{'}X^{'}Xc - \hat{\beta}^{'}X^{'}Xc - c^{'}X^{'}X\hat{\beta} + \hat{\beta}^{'}X^{'}X\hat{\beta} && (5.31)\\ (c^{'}X^{'}X - \hat{\beta}^{'}X^{'}X)(c - \hat{\beta}) && (5.32)\\ (c^{'} - \hat{\beta}^{'})X^{'}X(c - \hat{\beta}) && (5.33)\\ \boxed{(c - \hat{\beta})^{'}X^{'}X(c - \hat{\beta})} && (5.34)\end{aligned}$$

Problem 6.0:

a

$$\hat{\beta} = \begin{pmatrix}\hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\end{pmatrix} = (X^{'}X)^{-1}X^{'}Y \qquad (6.35)$$

$$= \begin{pmatrix}2 & 0 & 0\\ 0 & 3 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}4\\ -2\\ 5\end{pmatrix} \qquad (6.36)$$

$$\begin{pmatrix}\hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\end{pmatrix} = \begin{pmatrix}8\\ -11\\ 7\end{pmatrix} \qquad (6.37)$$

b

$$X = \begin{pmatrix} 1 & 3 & 1 & -1 & 1 \\ 1 & 4 & 1 & 1 & -1 \\ 1 & 5 & -1 & 1 & 1 \\ 1 & 6 & 0.5 & 0.2 & 0.3 \\ 1 & 8 & 0.8 & 0.1 & 0.1 \\ 1 & 9 & 0.3 & 0.5 & 0.2 \\ 1 & 10 & 0.2 & 0.3 & 0.5 \\ 1 & 13 & 0.1 & 0.6 & 0.3 \end{pmatrix} \tag{6.38}$$

$x_2 + x_3 + x_4 = 1 \implies$ matrix is not full rank and we cannot invert $X'X$.
Problem 7.0:

a

$E[\hat{Y}_{n+1}]$**:**

$$E[\hat{Y}_{n+1}] = E[X'_{n+1}\hat{\beta}] \tag{7.39}$$

$$= X'_{n+1}E[\hat{\beta}] \tag{7.40}$$

$$E[\hat{Y}_{n+1}] = X'_{n+1}\beta \tag{7.41}$$

$E[Y_{n+1}]$**:**

$$E[Y_{n+1}] = E[X'_{n+1}\beta + \varepsilon_{n+1}] = X'_{n+1}\beta \tag{7.42}$$

b

$$E(\tilde{Y}_{n+1} - Y_{n+1}) = 0 \tag{7.43}$$

$$E[\tilde{Y}_{n+1}] - E[Y_{n+1}] = 0 \tag{7.44}$$

$$a'E[Y_{n+1}] - X'_{n+1}\beta = 0 \tag{7.45}$$

$$a'X\beta - X'_{n+1}\beta = 0 \tag{7.46}$$

$$(a'X - X'_{n+1})\beta = 0 \implies a'X = X'_{n+1} \tag{7.47}$$

Problem 7.0 continued on next page. . .

c

$$
\begin{aligned}
Var(\hat{Y}_{n+1}) &= Var(X_{n+1}^{'}\hat{\beta}) && (7.48)\\
&= X_{n+1}^{'}Var(\hat{\beta})X_{n+1} && (7.49)\\
&= X_{n+1}^{'}[\sigma^2(X^{'}X)^{-1}]X_{n+1} && (7.50)\\
Var(\hat{Y}_{n+1}) &= \sigma^2 X_{n+1}^{'}(X^{'}X)^{-1}X_{n+1} && (7.51)
\end{aligned}
$$

$$
\begin{aligned}
Var(\tilde{Y}_{n+1}) &= Var(a^{'}Y_{n+1}) = \sigma^2 a^{'}a && (7.52)\\
&= \sigma^2(X_{n+1}^{'}X^{-1})(X^{'})^{-1}X_{n+1} && (7.53)\\
&= \sigma^2 X_{n+1}^{'}(XX^{'})^{-1}X_{n+1} && (7.54)
\end{aligned}
$$

d

$$ \tag{7.55} $$

Problem 8.0:

a

$$
\begin{aligned}
e^{'}e &= ((I-H)Y)^{'}((I-H)Y) && (8.56)\\
&= Y^{'}(I-H)^{'}(I-H)Y && (8.57)\\
e^{'}e &= Y^{'}(I-H)Y && (8.58)
\end{aligned}
$$

b

$$
\begin{aligned}
tr[(I-H)E[Y^{'}Y]] &= tr[(I-H)(\sigma^2 I + X\beta\beta^{'}X^{'})] && (8.59)\\
&= tr[\sigma^2 I + X\beta\beta^{'}X^{'} - \sigma^2 H - HX\beta\beta^{'}X^{'}] && (8.60)\\
&= \sigma^2 tr(I) - \sigma^2 tr(H) && (8.61)\\
tr[(I-H)E[Y^{'}Y]] &= \sigma^2(n-k-1) && (8.62)
\end{aligned}
$$

c

$$ Var(e) = Var((I-H)Y) \tag{8.63} $$

d