# Homework 2

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**Problem 1.0:**

$$Q = \begin{pmatrix} \hat{\beta} \\ \hat{Y} \end{pmatrix} = \begin{pmatrix} (X'X)^{-1}X'Y \\ (I-H)Y \end{pmatrix} \tag{1.1}$$

$$Q = LY \tag{1.2}$$

$$Var(Q) = Var(LY) = LVar(Y)L' \tag{1.3}$$

$$= \sigma^2 LL' \tag{1.4}$$

$$= \sigma^2 \begin{pmatrix} (X'X)^{-1}X' \\ (I-H) \end{pmatrix} \begin{pmatrix} (X'X)^{-1}X' & (I-H) \end{pmatrix} \tag{1.5}$$

$$= \sigma^2 \begin{pmatrix} (X'X)^{-1} & (X'X)^{-1}X'(I-H) \\ (I-H)X(X'X)^{-1} & (I-H) \end{pmatrix} \tag{1.6}$$

$$Var(Q) = \sigma^2 \begin{pmatrix} (X'X)^{-1} & 0 \\ 0 & (I-H) \end{pmatrix} \tag{1.7}$$

Therefore $\hat{Y}$ and $e$ are independent.

**Problem 2.0:**

$$\hat{\beta}_c = \hat{\beta} - (X'X)^{-1}c'[c(X'X)^{-1}c']^{-1}(c\hat{\beta} - \gamma) \tag{2.8}$$

$$\tag{2.9}$$

let $A = (X'X)^{-1}c'[c(X'X)^{-1}c']^{-1}$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - Var(A(c\hat{\beta} - \gamma)) \tag{2.10}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - AVar(c\hat{\beta} - \gamma)A' \tag{2.11}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - A[cVar(\hat{\beta})c' - Var(\gamma)]A' \tag{2.12}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - AcVar(\hat{\beta})c'A' + AVar(\gamma)A' \tag{2.13}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - AcVar(\hat{\beta})c'A' + AVar(c\beta)A' \tag{2.14}$$

$$Var(\hat{\beta}_c) = \sigma^2(X'X)^{-1} - Ac[\sigma^2(X'X)^{-1}]c'A' \tag{2.15}$$

$$Var(\hat{\beta}_c) = \sigma^2(X'X)^{-1} - \sigma^2(X'X)^{-1}c'[c'(X'X)^{-1}c]^{-1}c(X'X)^{-1} \tag{2.16}$$

Problem 2.0 continued on next page. . .

**Problem 3.0:**

We have that $S^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1}$ and $S_e^2 = \frac{e^{'}e}{n-k-1} = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2 - \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{n-k-1}$

$$R_a^2 = \frac{S^2 - s_e^2}{S^2} \tag{3.17}$$

$$= \frac{n-1}{\sum_{i=1}^{n}(y_i - \bar{y})^2}\left[\frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1} - \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-k-1} + \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{n-k-1}\right] \tag{3.18}$$

$$= \frac{n-1}{\sum_{i=1}^{n}(y_i - \bar{y})^2}\left[\frac{(n-k-1)\sum_{i=1}^{n}(y_i - \bar{y})^2 - (n-1)\sum_{i=1}^{n}(y_i - \bar{y})^2 + (n-1)\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{(n-1)(n-k-1)}\right] \tag{3.19}$$

$$= \left[\frac{(n-k-1) - (n-1) + (n-1)\frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}}{n-k-1}\right] \tag{3.20}$$

$$= 1 - \frac{(n-1) - (n-1)R^2}{n-k-1} \tag{3.21}$$

$$\boxed{R_a^2 = 1 - \frac{n-1}{n-k-1}(1 - R^2)} \tag{3.22}$$

**Problem 4.0:**

## a

$$var\left(\frac{(n-k-1)S_e^2}{\sigma^2}\right) = 2(n-k-1) \tag{4.23}$$

$$\frac{(n-k-1)^2}{\sigma^4}Var(S_e^2) = 2(n-k-1) \tag{4.24}$$

$$\frac{n-k-1}{\sigma^4}Var(S_e^2) = 2 \tag{4.25}$$

$$Var(S_e^2) = \frac{2\sigma^4}{n-k-1} \tag{4.26}$$

Problem 4.0 continued on next page. . .

### b

### c

### Problem 5.0:

$$(Y - Xc)^{'}(Y - Xc) - (Y - X\hat{\beta})^{'}(Y - X\hat{\beta}) \tag{5.27}$$

$$Y^{'}Y - Y^{'}Xc - c^{'}X^{'}Y + c^{'}X^{'}Xc - Y^{'}Y - Y^{'}Y + Y^{'}X\hat{\beta} + \hat{\beta}^{'}X^{'}Y - \hat{\beta}^{'}X^{'}X\hat{\beta} \tag{5.28}$$

$$\tag{5.29}$$

We use the normal equations to substitute for $Y^{'}X$ and $X^{'}Y$.

$$-\hat{\beta}^{'}X^{'}Xc - c^{'}X^{'}X\hat{\beta} + c^{'}X^{'}Xc + \hat{\beta}X^{'}X\hat{\beta} + \hat{\beta}^{'}X^{'}X\hat{\beta} - \hat{\beta}X^{'}X\hat{\beta} \tag{5.30}$$

$$c^{'}X^{'}Xc - \hat{\beta}^{'}X^{'}Xc - c^{'}X^{'}X\hat{\beta} + \hat{\beta}^{'}X^{'}X\hat{\beta} \tag{5.31}$$

$$(c^{'}X^{'}X - \hat{\beta}^{'}X^{'}X)(c - \hat{\beta}) \tag{5.32}$$

$$(c^{'} - \hat{\beta}^{'})X^{'}X(c - \hat{\beta}) \tag{5.33}$$

$$\boxed{(c - \hat{\beta})^{'}X^{'}X(c - \hat{\beta})} \tag{5.34}$$

### Problem 6.0:

### a

### 1

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = (X^{'}X)^{-1}X^{'}Y \tag{6.35}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \tag{6.36}$$

$$\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} 8 \\ -11 \\ 7 \end{pmatrix} \tag{6.37}$$

### 2

$$\sigma^2(X^{'}X){-}1 = \begin{pmatrix} 2\sigma^2 & 0 & 0 \\ 0 & 3\sigma^2 & -\sigma^2 \\ 0 & -\sigma^2 & \sigma^2 \end{pmatrix} \tag{6.38}$$

Therefore $Cov(\hat{\beta}_1, \hat{\beta}_2) = 0$

## b

$$X = \begin{pmatrix} 1 & 3 & 1 & -1 & 1 \\ 1 & 4 & 1 & 1 & -1 \\ 1 & 5 & -1 & 1 & 1 \\ 1 & 6 & 0.5 & 0.2 & 0.3 \\ 1 & 8 & 0.8 & 0.1 & 0.1 \\ 1 & 9 & 0.3 & 0.5 & 0.2 \\ 1 & 10 & 0.2 & 0.3 & 0.5 \\ 1 & 13 & 0.1 & 0.6 & 0.3 \end{pmatrix} \tag{6.39}$$

$x_2 + x_3 + x_4 = 1 \implies$ matrix is not full rank and we cannot invert $X'X$.

**Problem 7.0:**

## a

$E[\hat{y}_{n+1}]$**:**

$$E[\hat{y}_{n+1}] = E[x_{n+1}^{'}\hat{\beta}] \tag{7.40}$$
$$= x_{n+1}^{'}E[\hat{\beta}] \tag{7.41}$$
$$E[\hat{y}_{n+1}] = x_{n+1}^{'}\beta \tag{7.42}$$

$E[y_{n+1}]$**:**

$$E[y_{n+1}] = E[x_{n+1}^{'}\beta + \varepsilon_{n+1}] = x_{n+1}^{'}\beta \tag{7.43}$$

## b

$$E(\tilde{y}_{n+1} - y_{n+1}) = 0 \tag{7.44}$$
$$E[\tilde{y}_{n+1}] - E[y_{n+1}] = 0 \tag{7.45}$$
$$a^{'}E[Y] - x_{n+1}^{'}\beta = 0 \tag{7.46}$$
$$a^{'}X\beta - x_{n+1}^{'}\beta = 0 \tag{7.47}$$
$$(a^{'}X - x_{n+1}^{'})\beta = 0 \implies a^{'}X = x_{n+1}^{'} \tag{7.48}$$

Problem 7.0 continued on next page...

### c

$$
\begin{aligned}
Var(\hat{y}_{n+1}) &= Var(x^{'}_{n+1}\hat{\beta}) && (7.49)\\
&= x^{'}_{n+1}Var(\hat{\beta})x_{n+1} && (7.50)\\
&= x^{'}_{n+1}[\sigma^2(X^{'}X)^{-1}]x_{n+1} && (7.51)\\
Var(\hat{y}_{n+1}) &= \sigma^2 x^{'}_{n+1}(X^{'}X)^{-1}x_{n+1} && (7.52)
\end{aligned}
$$

$$
Var(\tilde{y}_{n+1}) = Var(a^{'}Y) = \sigma^2 a^{'}a \quad (7.53)
$$

### d

$$
\begin{aligned}
Var(\tilde{y}_{n+1}) - Var(\hat{y}_{n+1}) &= 0 && (7.54)\\
\sigma^2 a^{'}a - \sigma^2 x^{'}_{n+1}(X^{'}X)^{-1}x_{n+1} &= 0 && (7.55)\\
\sigma^2 a^{'}a - \sigma^2 a^{'}x(X^{'}X)^{-1}x^{'}a &= 0 && (7.56)\\
\sigma^2 a^{'}(I - X(X^{'}X)^{-1}X^{'})a &= 0 && (7.57)\\
\sigma^2 a^{'}(I - H)a &= 0 && (7.58)
\end{aligned}
$$

## Problem 8.0:

### a

$$
\begin{aligned}
e^{'}e &= ((I-H)Y)^{'}((I-H)Y) && (8.59)\\
&= Y^{'}(I-H)^{'}(I-H)Y && (8.60)\\
e^{'}e &= Y^{'}(I-H)Y && (8.61)
\end{aligned}
$$

### b

$$
\begin{aligned}
tr[(I-H)E[Y^{'}Y]] &= tr[(I-H)(\sigma^2 I + X\beta\beta^{'}X^{'})] && (8.62)\\
&= tr[\sigma^2 I + X\beta\beta^{'}X^{'} - \sigma^2 H - HX\beta\beta^{'}X^{'}] && (8.63)\\
&= \sigma^2 tr(I) - \sigma^2 tr(H) && (8.64)\\
tr[(I-H)E[Y^{'}Y]] &= \sigma^2(n-k-1) && (8.65)
\end{aligned}
$$

Problem 8.0 continued on next page. . . 

## c

$$
\begin{aligned}
Var(e) &= Var((I-H)Y) && (8.66)\\
&= \sigma^2(I-H)(I-H)^{'} && (8.67)\\
&= \sigma^2(I-H) && (8.68)\\
Var(e) &= \sigma^2(I - X(X^{'}X)^{-1}X^{'}) && (8.69)
\end{aligned}
$$

Therefore $Var(e_i) = \sigma^2(1 - x_i^{'}(X^{'}X)^{-1}x_i)$

## d

$$
\begin{aligned}
Var(\hat{Y}) &= Var(HY) && (8.70)\\
&= HVar(Y)H^{'} && (8.71)\\
&= H(\sigma^2 I)H^{'} && (8.72)\\
&= \sigma^2 H && (8.73)\\
Var(\hat{Y}) &= \sigma^2 X(X^{'}X)^{-1}X^{'} && (8.74)
\end{aligned}
$$

Therefore $Var(e_i) = \sigma^2 - Var(\hat{Y})$