

Homework 1

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Problem 1.0:

$$SSE = \sum_{i=1}^n e_i^2 = (Y - X\beta)'(Y - X\beta) \quad (1.1)$$

$$= Y'Y - Y'X\hat{\beta} - \hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta} \quad (1.2)$$

$$(1.3)$$

Substitute $\hat{\beta} = (X'X)^{-1}X'Y$ to the last term

$$SSE = Y'Y - Y'X\hat{\beta} - \hat{\beta}'X'Y + \hat{\beta}'X'X((X'X)^{-1}X'Y) \quad (1.4)$$

$$= Y'Y - Y'X\hat{\beta} - \hat{\beta}'X'Y + \hat{\beta}'X'Y \quad (1.5)$$

$$\boxed{SSE = Y'Y - \hat{\beta}'X'Y} \quad (1.6)$$

$$SSE = Y'Y - Y'X\hat{\beta} - \hat{\beta}'X'Y + \hat{\beta}'X'X((X'X)^{-1}X'Y) \quad (1.7)$$

$$= Y'Y - Y'X\hat{\beta} - \hat{\beta}'X'Y + \hat{\beta}'X'HY \quad (1.8)$$

$$\boxed{SSE = Y'Y - \hat{\beta}'X'\hat{Y}} \quad (1.9)$$

$$SSE = Y'Y - Y'X\hat{\beta} - \hat{\beta}'X'Y + \hat{\beta}'X'X((X'X)^{-1}X'Y) \quad (1.10)$$

$$= Y'Y - Y'X\hat{\beta} - \hat{\beta}'X'Y + \hat{\beta}'X'Y \quad (1.11)$$

$$SSE = Y'Y - \hat{\beta}'X'Y \quad (1.12)$$

Substitute normal equations $X'Y = X'X\hat{\beta}$

$$SSE = Y'Y - Y'X\hat{\beta} - \hat{\beta}'X'Y + \hat{\beta}'X'X((X'X)^{-1}X'Y) \quad (1.13)$$

$$= Y'Y - Y'X\hat{\beta} - \hat{\beta}'X'Y + \hat{\beta}'X'Y \quad (1.14)$$

$$\boxed{SSE = Y'Y - \hat{\beta}'X'X\hat{\beta}} \quad (1.15)$$

Problem 2.0:

$$X = \begin{pmatrix} 1 & x_{11} & x_{12} & \cdots & .001x_{1i} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & .001x_{2i} & \cdots & x_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ 1 & x_{i1} & x_{i2} & \cdots & .001x_{ii} & \cdots & x_{ik} \\ \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & .001x_{ni} & \cdots & x_{nk} \end{pmatrix} \quad (2.16)$$

$$X'X = \begin{pmatrix} n & \sum x_{i1} & \sum x_{i2} & \cdots & .001 \sum \sum x_{ji} & \cdots & \sum x_{ik} \\ \sum x_{i1} & \sum x_{i1}^2 & \sum x_{i1}x_{i2} & \cdots & .001 \sum x_{i1} \sum x_{ji} & \cdots & \sum x_{i1}x_{ik} \\ \sum x_{i2} & \sum x_{i1}x_{i2} & \sum x_{i2}^2 & \cdots & .001 \sum x_{i2} \sum x_{ji} & & \sum x_{i2}x_{ik} \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ .001 \sum \sum x_{ji} & .001 \sum x_{i1} \sum x_{ji} & .001 \sum x_{i2} \sum x_{ji} & \cdots & (.001)^2 \sum x_{ii}^2 & \cdots & .001 \sum x_{ik}x_{ji} \\ \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\ \sum x_{ik} & \sum x_{i1}x_{ik} & \sum x_{i1}x_{ik} & \cdots & .001 \sum x_{ik}x_{ji} & \cdots & \sum x_{ik}^2 \end{pmatrix} \quad (2.17)$$

$$\hat{\beta}_i = (.001)^2 X'_{ii} X_{ii}^{-1} (.001 X'_{ii}) Y \quad (2.18)$$

$$\hat{\beta}_i = 1000 (X'_{ii} X_{ii})^{-1} X'_{ii} Y \quad (2.19)$$

$$= 1000 \hat{\beta}_i \quad (2.20)$$

Problem 3.0:

```

library(MASS)
a <- read.table("http://www.stat.ucla.edu/~nchristo/statistics100c/restaurant.txt",
               header = T)
x <- matrix(c(rep(1,100),a$food,a$decor, a$ser), nrow=100, ncol=4)
x_transpose_x <- (t(x) %*% x) ;x_transpose_x

##      [,1] [,2] [,3] [,4]
## [1,] 100 1943 1609 1764
## [2,] 1943 38445 31764 34722
## [3,] 1609 31764 27671 29108
## [4,] 1764 34722 29108 31748

y <- matrix(a$cost)
beta_hat = ginv(x_transpose_x) %*% t(x) %*% y;beta_hat

##      [,1]
## [1,] -35.3098383
## [2,]  1.2838518
## [3,]  1.7773844
## [4,]  0.2804128

lm(cost ~ ., data= a)

##
## Call:
## lm(formula = cost ~ ., data = a)
##
## Coefficients:
## (Intercept)      food      decor      ser
##   -35.3098    1.2839    1.7774    0.2804

H <- x %*% ginv(x_transpose_x) %*% t(x)
H[1:5,1:5]

##      [,1] [,2] [,3] [,4] [,5]
## [1,] 0.02333445 0.017090141 0.010247706 0.030288494 0.008920920
## [2,] 0.01709014 0.030828287 0.009747574 0.007121384 0.012518985
## [3,] 0.01024771 0.009747574 0.013348148 0.008131717 0.015062149
## [4,] 0.03028849 0.007121384 0.008131717 0.053772247 0.001936585
## [5,] 0.00892092 0.012518985 0.015062149 0.001936585 0.018606756

```

Problem 4.0:

$$(Y - X\hat{\beta} + X\hat{\beta} - X\beta)'(Y - X\hat{\beta} + X\hat{\beta} - X\beta) \quad (4.21)$$

$$((Y - X\hat{\beta}) + X(\hat{\beta} - \beta))'((Y - X\hat{\beta}) + X(\hat{\beta} - \beta)) \quad (4.22)$$

$$(Y - X\hat{\beta})'(Y - X\hat{\beta}) + (Y - X\hat{\beta})'X(\hat{\beta} - \beta) + (\hat{\beta} - \beta)'X'(Y - X\hat{\beta}) + (\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta) \quad (4.23)$$

$$(Y - X\hat{\beta})'(Y - X\hat{\beta}) + (Y - \hat{Y})'X(\hat{\beta} - \beta) + (\hat{\beta} - \beta)'X'(Y - \hat{Y}) + (\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta) \quad (4.24)$$

$$(Y - X\hat{\beta})'(Y - X\hat{\beta}) + (\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta) \quad (4.25)$$

Problem 5.0:

$$E[\hat{Y}'\hat{Y}] = E[(HY)'HY] = E[Y'H'HY] \quad (5.26)$$

$$= E[\text{tr}(Y'HY)] = \text{tr}(HE[Y'Y]) \quad (5.27)$$

$$= \text{tr}(H(\sigma^2 I + X\beta\beta'X')) = \text{tr}(\sigma^2 H + X\beta\beta'X') \quad (5.28)$$

$$= \text{tr}(\sigma^2 H) + \text{tr}(X\beta\beta'X') \quad (5.29)$$

$$E[\hat{Y}'\hat{Y}] = \sigma(K + 1) + \beta'X'X\beta \quad (5.30)$$

Problem 6.0:

$$Q = \begin{pmatrix} \hat{\beta}_1 - 2\hat{\beta}_2 + 3\hat{\beta}_4 \\ \hat{\beta}_0 + \hat{\beta}_4 + 3\hat{\beta}_5 \end{pmatrix} \quad (6.31)$$

$$= \begin{pmatrix} 0 & 1 & -2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \\ \hat{\beta}_5 \end{pmatrix} \quad (6.32)$$

$$Q = c\hat{\beta} \quad (6.33)$$

We know that $\hat{\beta} \sim N_6(\beta, \sigma(X'X)^{-1})$ then,

$$E[Q] = E[c\hat{\beta}] = cE[\hat{\beta}] \quad (6.34)$$

$$E[Q] = c\beta \quad (6.35)$$

$$\text{Var}(Q) = \text{Var}(c\hat{\beta}) \quad (6.36)$$

$$\text{Var}(Q) = c(\sigma(X'X)^{-1})c' \quad (6.37)$$

$$\boxed{Q \sim N_2(c\beta, \sigma c(X'X)^{-1}c')} \quad (6.38)$$

Problem 7.0:**a**

$$\hat{\beta}_1 = (X_1' X_1)^{-1} X_1' Y_1 \quad (7.39)$$

$$\hat{\beta}_2 = (X_2' X_2)^{-1} X_2' Y_2 \quad (7.40)$$

b

$$\hat{\beta} = \left(\begin{pmatrix} X_1' & X_2' \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right)^{-1} \begin{pmatrix} X_1' & X_2' \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \quad (7.41)$$

$$= (X_1' X_1 + X_2' X_2)^{-1} \begin{pmatrix} X_1' & X_2' \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \quad (7.42)$$

$$\hat{\beta} = (X_1' X_1 + X_2' X_2)^{-1} (X_1' Y_1 + X_2' Y_2) \quad (7.43)$$

c

$$\hat{\beta} = (2X_1' X_1)^{-1} (X_1' Y_1 + X_2' Y_2) \quad (7.44)$$

$$= \frac{1}{2} (X_1' X_1)^{-1} (X_1' Y_1 + X_2' Y_2) \quad (7.45)$$

$$= \frac{1}{2} [(X_1' X_1)^{-1} X_1' Y_1 + (X_1' X_1)^{-1} X_2' Y_2] \quad (7.46)$$

$$= \frac{1}{2} [(X_1' X_1)^{-1} X_1' Y_1 + (X_2' X_2)^{-1} X_2' Y_2] \quad (7.47)$$

$$\hat{\beta} = \frac{1}{2} (\hat{\beta}_1 + \hat{\beta}_2) \quad (7.48)$$