Homework 1

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Problem 1.0:

a

$L_{Shuffle}(L_1, L_2)$ **not regular**
Let $L_1 = L_2 = \{0^n 1^n\}$ then

$$L_{Shuffle}(L_1, L_2) = \{0^n 0^n 1^n 1^n\} \tag{1.1}$$
$$L_{Shuffle}(L_1, L_2) = \{0^{2n} 1^{2n}\} = \{0^m 1^m | m = 2n\} \tag{1.2}$$

We know that $\{0^m 1^m\}$ is not regular so $L_{Shuffle}(L_1, L_2)$ is not regular.

$L_{Shuffle}(L_1, \bar{L_2})$ **not regular**
Let $L_2^{'} = \{2^n 3^n\} \subseteq \bar{L_2}$ then

$$L_{Shuffle}(L_1, \bar{L_2}) = \{\{02\}^n \{13\}^n\} \tag{1.3}$$

$\{\{02\}^n \{13\}^n\}$ is not regular and therefore $L_{Shuffle}(L_1, \bar{L_2})$ is also not regular.

b

Claim: For $x \in L_1$ and $y \in L_2$, s.t. $|x| = |y| = n$, $\exists$ machines N_1 and N_2 s.t $N_1(x) : Accepts$ and $N_2(y) : Accepts \iff \exists$ M that recognizes $w \in L_{shuffle}(x, y)$, $M(w) : Accepts$.

Proof.
Let M be a DFA recognizing $L_{shuffle}$:

- $Q = Q_1 \cup Q_2$
- $q_0^{''} = q_0 \in Q_1$
- $F^{''} = F^{'}$
- $\delta^{'}(s \in Q, w_i) = \begin{cases} \delta_1(q_0, w_i), \ s = q_0 \\ \delta_2(s^{'}, w_i), \ s \in Q_1, \ s^{'} \in Q_2 \\ \delta_1(s^{'}, w_i), \ s \in Q_2, \ s^{'} \in Q_1 \end{cases}$

($\Rightarrow$) L_1 and L_2 be regular $\implies$ $\exists$ DFAs N_1 and N_2 that accept x and y respectively. Now N_1 recognizes x as follows. $\exists$ states $a_1, a_2, \dots, a_n$ s.t. $\delta_1(a_i, x_i) = \{a_j \in Q_1\}$ and $a_n \in F$. N_2 recognizes y since $\exists$ states $b_1, b_2, \dots, b_n$ s.t $\delta_2(b_i, y_i) = \{b_j \in Q_2\}$. Now the states $a_1 b_1 a_2 b_2 \dots a_n b_n$ is concatenation of paths on N_1 and N_2 from $q_0 \in Q_1$ to $q_n \in Q_2$ for for $L_{shuffle}(x, y)$.
($\Leftarrow$) Now M recognizes $L_{Shuffle}$ as follows. Let w=$w_1 w_2 \dots w_m \in L_{shuffle}$, then $\exists$ states $s_1, ... s_{2n}$ s.t $s_1 = a_1$ and $s_m = b_n$. Therefore a concatenation of their paths $a_1 b_1 a_2 b_2 \dots a_n b_n$ is an accpeting compuation path on M for w.

□

Problem 2.0:

a

Proof. Assume $L_2 = b^* \cup L_1$ is regular. Then by Pumping Lemma $\exists$ $p \in \mathbb{N}$ s.t $w = b^p = xyz$, $|xy| \leq p$ and $|y| \geq 1$. Now by Pumping Lemma we can pump so that $ab^ic \in L_2$ since $b*$ is the set off all possible strings created by concatenating any number i of b it will produce an $ab^ic \in L_2$. □

b

Proof. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing L and $n = k + p + m$ be the number of states in M. Let $s = s_1 s_2 \dots s_n$ be a string in L s.t a partion $\exists$ where $s = xyz$ as such:
1) $x = s_1 \dots s_k$
2) $y = s_{1'} \dots s_l$ s.t $|y| \geq p$
3) $z = s_{1''} \dots s_m$
Let abc be strings s.t $y = abc$. Observe $r_{1'} \dots r_{l+1}$ the sequence of states M enters while processing Y so $r_{i+1} = \delta(r_i, s_{i'})$ for $1' \leq i \leq l$. The sequence has length $l + 1$ which is at least $p + 1$. By the pigeonhole principle $\exists$ i, j s.t $r_i = r_j$, $1' \leq i < j \leq l$.

Let $a = r_{1'} \dots r_{i-1}$, $b = r_i \dots r_{j-1}$, $c = r_j \dots r_l$ Since $i \neq j \implies |b| > 0$ and since x takes M r_1 to $r_{1'}$, a takes M from $r_{'}$ *to* r_i, b takes M from r_i *to* r_i, c takes M from r_j *to* r_l, and z takes M from $r_{1''}$ *to* r_m then $xab^icz \in L$ □

c

Proof.
Assume for contradiction L_2 is regular

- By Pumping Lemma $\exists\, p$
- Consider $\alpha^i\beta^n = xy$, where $y = \beta^n$ and $n > p$.
- By pumping lemma $\beta^n = abc$, $|b| > 0$
- Let $|b| = q$ s.t $y = \beta^q\beta^{n-q}$
- Let $i = n+1 \implies \beta^{q(n+1)}\beta^{n-q} = \beta^{n(q+1)}$
- If $q = 1$ then $2n$ is even or iff $q > 1$ then $q+1|n(q+1)$ therefore
- $\alpha\beta^{n(q+1)} \notin L_2$ so L_2 is not regular.

$\square$

Problem 3.0:
Given $0^*21^* \in L$ and L is regular. By inspection we note the following pattern:

L	$L_{\frac{1}{3}-\frac{1}{3}}(L)$
0$\underline{\mathbf{2}}$1	01
00$\underline{\mathbf{21}}$11	0011
000$\underline{\mathbf{211}}$111	000111
0000$\underline{\mathbf{2111}}$1111	00001111
$\vdots$	$\vdots$
$0^n21^{\lvert xyz\rvert-(n+1)}$	0^n1^n

Let $L_1 = \{0^n21^{|xyz|-(n+1)}\} \in L$. Regular languages are closed under intersection so then $L_{\frac{1}{3}-\frac{1}{3}}(L_1) \cap L = \{0^n1^n\}$, but we know that $\{0^n1^n\}$ not regular.