Homework 2

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Problem 1.0:

$$Q = \begin{pmatrix} \hat{\beta} \\ \hat{Y} \end{pmatrix} = \begin{pmatrix} (X'X)^{-1}X'Y \\ (I-H)Y \end{pmatrix} \tag{1.1}$$

$$Q = LY \tag{1.2}$$

$$Var(Q) = Var(LY) = LVar(Y)L' \tag{1.3}$$

$$= \sigma^2 LL' \tag{1.4}$$

$$= \sigma^2 \begin{pmatrix} (X'X)^{-1}X' \\ (I-H) \end{pmatrix} \begin{pmatrix} (X'X)^{-1}X' & (I-H) \end{pmatrix} \tag{1.5}$$

$$= \sigma^2 \begin{pmatrix} (X'X)^{-1} & (X'X)^{-1}X'(I-H) \\ (I-H)X(X'X)^{-1} & (I-H) \end{pmatrix} \tag{1.6}$$

$$Var(Q) = \sigma^2 \begin{pmatrix} (X'X)^{-1} & 0 \\ 0 & (I-H) \end{pmatrix} \tag{1.7}$$

$\boxed{\text{Therefore } \hat{Y} \text{ and } e \text{ are independent.}}$

Problem 2.0:

$$\hat{\beta}_c = \hat{\beta} - (X'X)^{-1}c'[c(X'X)^{-1}c']^{-1}(c\hat{\beta} - \gamma) \tag{2.8}$$

$$\tag{2.9}$$

let $A = (X'X)^{-1}c'[c(X'X)^{-1}c']^{-1}$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - Var(A(c\hat{\beta} - \gamma)) \tag{2.10}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - AVar(c\hat{\beta} - \gamma)A' \tag{2.11}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - A[cVar(\hat{\beta})c' - Var(\gamma)]A' \tag{2.12}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - AcVar(\hat{\beta})c'A' + AVar(\gamma)A' \tag{2.13}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - AcVar(\hat{\beta})c'A' + AVar(c\beta)A' \tag{2.14}$$

$$Var(\hat{\beta}_c) = \sigma^2(X'X)^{-1} - Ac[\sigma^2(X'X)^{-1}]c'A' \tag{2.15}$$

$$\boxed{Var(\hat{\beta}_c) = \sigma^2(X'X)^{-1} - \sigma^2(X'X)^{-1}c'[c'(X'X)^{-1}c]^{-1}c(X'X)^{-1}} \tag{2.16}$$

Problem 2.0 continued on next page...

Problem 3.0:

We have that $S^2 = \frac{\sum_{i=1}^{n}(y_i-\bar{y})^2}{n-1}$ and $S_e^2 = \frac{e'e}{n-k-1} = \frac{\sum_{i=1}^{n}(y_i-\bar{y})^2 - \sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2}{n-k-1}$

$$R_a^2 = \frac{S^2 - s_e^2}{S^2} \tag{3.17}$$

$$= \frac{n-1}{\sum_{i=1}^{n}(y_i-\bar{y})^2}\left[\frac{\sum_{i=1}^{n}(y_i-\bar{y})^2}{n-1} - \frac{\sum_{i=1}^{n}(y_i-\bar{y})^2}{n-k-1} + \frac{\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2}{n-k-1}\right] \tag{3.18}$$

$$= \frac{n-1}{\sum_{i=1}^{n}(y_i-\bar{y})^2}\left[\frac{(n-k-1)\sum_{i=1}^{n}(y_i-\bar{y})^2 - (n-1)\sum_{i=1}^{n}(y_i-\bar{y})^2 + (n-1)\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2}{(n-1)(n-k-1)}\right] \tag{3.19}$$

$$= \left[\frac{(n-k-1)-(n-1)+(n-1)\frac{\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2}{\sum_{i=1}^{n}(y_i-\bar{y})^2}}{n-k-1}\right] \tag{3.20}$$

$$= 1 - \frac{(n-1)-(n-1)R^2}{n-k-1} \tag{3.21}$$

$$\boxed{R_a^2 = 1 - \frac{n-1}{n-k-1}(1-R^2)} \tag{3.22}$$

Problem 4.0:

a

$$var\left(\frac{(n-k-1)S_e^2}{\sigma^2}\right) = 2(n-k-1) \tag{4.23}$$

$$\frac{(n-k-1)^2}{\sigma^4}Var(S_e^2) = 2(n-k-1) \tag{4.24}$$

$$\frac{n-k-1}{\sigma^4}Var(S_e^2) = 2 \tag{4.25}$$

$$Var(S_e^2) = \frac{2\sigma^4}{n-k-1} \tag{4.26}$$

Problem 4.0 continued on next page. . .

b

c

Problem 5.0:

$$(Y - Xc)^{'}(Y - Xc) - (Y - X\hat{\beta})^{'}(Y - X\hat{\beta}) \tag{5.27}$$
$$Y^{'}Y - Y^{'}Xc - c^{'}X^{'}Y + c^{'}X^{'}Xc - Y^{'}Y - Y^{'}Y + Y^{'}X\hat{\beta} + \hat{\beta}^{'}X^{'}Y - \hat{\beta}^{'}X^{'}X\hat{\beta} \tag{5.28}$$
$$\tag{5.29}$$

We use the normal equations to substitute for $Y^{'}X$ and $X^{'}Y$.

$$-\hat{\beta}^{'}X^{'}Xc - c^{'}X^{'}X\hat{\beta} + c^{'}X^{'}Xc + \hat{\beta}X^{'}X\hat{\beta} + \hat{\beta}^{'}X^{'}X\hat{\beta} - \hat{\beta}X^{'}X\hat{\beta} \tag{5.30}$$
$$c^{'}X^{'}Xc - \hat{\beta}^{'}X^{'}Xc - c^{'}X^{'}X\hat{\beta} + \hat{\beta}^{'}X^{'}X\hat{\beta} \tag{5.31}$$
$$(c^{'}X^{'}X - \hat{\beta}^{'}X^{'}X)(c - \hat{\beta}) \tag{5.32}$$
$$(c^{'} - \hat{\beta}^{'})X^{'}X(c - \hat{\beta}) \tag{5.33}$$
$$\boxed{(c - \hat{\beta})^{'}X^{'}X(c - \hat{\beta})} \tag{5.34}$$

Problem 6.0:

a

1

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = (X^{'}X)^{-1}X^{'}Y \tag{6.35}$$
$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \tag{6.36}$$
$$\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} 8 \\ -11 \\ 7 \end{pmatrix} \tag{6.37}$$

2

$$\sigma^2(X^{'}X) - 1 = \begin{pmatrix} 2\sigma^2 & 0 & 0 \\ 0 & 3\sigma^2 & -\sigma^2 \\ 0 & -\sigma^2 & \sigma^2 \end{pmatrix} \tag{6.38}$$

Problem 6.0 continued on next page...

$\boxed{\text{Therefore } Cov(\hat{\beta}_1, \hat{\beta}_2) = 0}$

b

$$X = \begin{pmatrix} 1 & 3 & 1 & -1 & 1 \\ 1 & 4 & 1 & 1 & -1 \\ 1 & 5 & -1 & 1 & 1 \\ 1 & 6 & 0.5 & 0.2 & 0.3 \\ 1 & 8 & 0.8 & 0.1 & 0.1 \\ 1 & 9 & 0.3 & 0.5 & 0.2 \\ 1 & 10 & 0.2 & 0.3 & 0.5 \\ 1 & 13 & 0.1 & 0.6 & 0.3 \end{pmatrix} \tag{6.39}$$

$x_2 + x_3 + x_4 = 1 \implies$ matrix is not full rank and we cannot invert $X'X$.

Problem 7.0:

a

$E[\hat{y}_{n+1}]$**:**

$$E[\hat{y}_{n+1}] = E[x'_{n+1}\hat{\beta}] \tag{7.40}$$
$$= x'_{n+1}E[\hat{\beta}] \tag{7.41}$$
$$\boxed{E[\hat{y}_{n+1}] = x'_{n+1}\beta} \tag{7.42}$$

$E[y_{n+1}]$**:**

$$\boxed{E[y_{n+1}] = E[x'_{n+1}\beta + \varepsilon_{n+1}] = x'_{n+1}\beta} \tag{7.43}$$

b

$$E(\tilde{y}_{n+1} - y_{n+1}) = 0 \tag{7.44}$$
$$E[\tilde{y}_{n+1}] - E[y_{n+1}] = 0 \tag{7.45}$$
$$a'E[Y] - x'_{n+1}\beta = 0 \tag{7.46}$$
$$a'X\beta - x'_{n+1}\beta = 0 \tag{7.47}$$
$$\boxed{(a'X - x'_{n+1})\beta = 0 \implies a'X = x'_{n+1}} \tag{7.48}$$

Problem 7.0 continued on next page...

c

$$
\begin{aligned}
Var(\hat{y}_{n+1}) &= Var(x_{n+1}^{'}\hat{\beta}) && (7.49)\\
&= x_{n+1}^{'}Var(\hat{\beta})x_{n+1} && (7.50)\\
&= x_{n+1}^{'}[\sigma^2(X^{'}X)^{-1}]x_{n+1} && (7.51)\\
\boxed{Var(\hat{y}_{n+1}) = \sigma^2 x_{n+1}^{'}(X^{'}X)^{-1}x_{n+1}} & && (7.52)
\end{aligned}
$$

$$
\boxed{Var(\tilde{y}_{n+1}) = Var(a^{'}Y) = \sigma^2 a^{'}a} \qquad (7.53)
$$

d

$$
\begin{aligned}
Var(\tilde{y}_{n+1}) - Var(\hat{y}_{n+1}) &= 0 && (7.54)\\
\sigma^2 a^{'}a - \sigma^2 x_{n+1}^{'}(X^{'}X)^{-1}x_{n+1} &= \sigma^2 a^{'}a - \sigma^2 a^{'}x(X^{'}X)^{-1}x^{'}a && (7.55)\\
\sigma^2 a^{'}(I - X(X^{'}X)^{-1}X^{'})a &= \sigma^2 a^{'}(I-H)a && (7.56)\\
\sigma^2 a^{'}(I-H)(I-H)^{'}a &= \sigma^2((I-H)^{'}a)^{'}(I-H)^{'}a && (7.57)
\end{aligned}
$$

We notice that $((I-H)^{'}a)^{'}(I-H)^{'}a$ is of the form $(M^{'}l)^{'}(M^{'}l)$ which implies that $(I-H)$ is at least postive semidefinite so then

$$
\sigma^2((I-H)^{'}a)^{'}(I-H)^{'}a \geq 0 \qquad (7.58)
$$

which implies that

$$
\begin{aligned}
Var(\tilde{y}_{n+1}) - Var(\hat{y}_{n+1}) &\geq 0 && (7.59)\\
Var(\tilde{y}_{n+1}) &\geq Var(\hat{y}_{n+1}) && (7.60)
\end{aligned}
$$

Problem 8.0:

a

$$
\begin{aligned}
e^{'}e &= ((I-H)Y)^{'}((I-H)Y) && (8.61)\\
&= Y^{'}(I-H)^{'}(I-H)Y && (8.62)\\
\boxed{e^{'}e = Y^{'}(I-H)Y} & && (8.63)
\end{aligned}
$$

b

$$tr[(I-H)E[Y^{'}Y]] = tr[(I-H)(\sigma^2 I + X\beta\beta^{'}X^{'})] \tag{8.64}$$
$$= tr[\sigma^2 I + X\beta\beta^{'}X^{'} - \sigma^2 H - HX\beta\beta^{'}X^{'}] \tag{8.65}$$
$$= \sigma^2 tr(I) - \sigma^2 tr(H) \tag{8.66}$$
$$\boxed{tr[(I-H)E[Y^{'}Y]] = \sigma^2(n-k-1)} \tag{8.67}$$

c

$$Var(e) = Var((I-H)Y) \tag{8.68}$$
$$= \sigma^2(I-H)(I-H)^{'} \tag{8.69}$$
$$= \sigma^2(I-H) \tag{8.70}$$
$$Var(e) = \sigma^2(I - X(X^{'}X)^{-1}X^{'}) \tag{8.71}$$

Therefore $Var(e_i) = \sigma^2(1 - x_i^{'}(X^{'}X)^{-1}x_i)$

d

$$Var(\hat{Y}) = Var(HY) \tag{8.72}$$
$$= HVar(Y)H^{'} \tag{8.73}$$
$$= H(\sigma^2 I)H^{'} \tag{8.74}$$
$$= \sigma^2 H \tag{8.75}$$
$$Var(\hat{Y}) = \sigma^2 X(X^{'}X)^{-1}X^{'} \tag{8.76}$$

Therefore $Var(e_i) = \sigma^2 - Var(\hat{Y})$