Homework 1

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Problem 1.0:

$$SSE = \sum_{i=1}^{n} e_i^2 = (Y - X\beta)^{'}(Y - X\beta) \tag{1.1}$$
$$= Y^{'}Y - Y^{'}X\hat{\beta} - \hat{\beta}^{'}X^{'}Y + \hat{\beta}^{'}X^{'}X\hat{\beta} \tag{1.2}$$
$$\tag{1.3}$$

Substitude $\hat{\beta} = (X^{'}X)^{-1}X^{'}Y$ to the last term

$$SSE = Y^{'}Y - Y^{'}X\hat{\beta} - \hat{\beta}^{'}X^{'}Y + \hat{\beta}^{'}X^{'}X((X^{'}X)^{-1}X^{'}Y) \tag{1.4}$$
$$= Y^{'}Y - Y^{'}X\hat{\beta} - \hat{\beta}^{'}X^{'}Y + \hat{\beta}^{'}X^{'}Y \tag{1.5}$$
$$\boxed{SSE = Y^{'}Y - \hat{\beta}^{'}X^{'}Y} \tag{1.6}$$

$$SSE = Y^{'}Y - Y^{'}X\hat{\beta} - \hat{\beta}^{'}X^{'}Y + \hat{\beta}^{'}X^{'}X((X^{'}X)^{-1}X^{'}Y) \tag{1.7}$$
$$= Y^{'}Y - Y^{'}X\hat{\beta} - \hat{\beta}^{'}X^{'}Y + \hat{\beta}^{'}X^{'}HY \tag{1.8}$$
$$\boxed{SSE = Y^{'}Y - \hat{\beta}^{'}X^{'}\hat{Y}} \tag{1.9}$$

$$SSE = Y^{'}Y - Y^{'}X\hat{\beta} - \hat{\beta}^{'}X^{'}Y + \hat{\beta}^{'}X^{'}X((X^{'}X)^{-1}X^{'}Y) \tag{1.10}$$
$$= Y^{'}Y - Y^{'}X\hat{\beta} - \hat{\beta}^{'}X^{'}Y + \hat{\beta}^{'}X^{'}Y \tag{1.11}$$
$$SSE = Y^{'}Y - \hat{\beta}^{'}X^{'}Y \tag{1.12}$$

Substitude normal equations $X^{'}Y = X^{'}X\hat{\beta}$

$$SSE = Y^{'}Y - Y^{'}X\hat{\beta} - \hat{\beta}^{'}X^{'}Y + \hat{\beta}^{'}X^{'}X((X^{'}X)^{-1}X^{'}Y) \tag{1.13}$$
$$= Y^{'}Y - Y^{'}X\hat{\beta} - \hat{\beta}^{'}X^{'}Y + \hat{\beta}^{'}X^{'}Y \tag{1.14}$$
$$\boxed{SSE = Y^{'}Y - \hat{\beta}^{'}X^{'}X\hat{\beta}} \tag{1.15}$$

Problem 2.0 continued on next page. . .

Problem 2.0:

$$X = \begin{pmatrix} 1 & x_{11} & x_{12} & \cdots & .001x_{1i} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & .001x_{2i} & \cdots & x_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ 1 & x_{i1} & x_{i2} & \cdots & .001x_{ii} & \cdots & x_{ik} \\ \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & .001x_{ni} & \cdots & x_{nk} \end{pmatrix} \tag{2.16}$$

$$X'X = \begin{pmatrix} n & \sum x_{i1} & \sum x_{i2} & \cdots & .001\sum\sum x_{ji} & \cdots & \sum x_{ik} \\ \sum x_{i1} & \sum x_{i1}^2 & \sum x_{i1}x_{i2} & \cdots & .001\sum x_{i1}\sum x_{ji} & \cdots & \sum x_{i1}x_{ik} \\ \sum x_{i2} & \sum x_{i1}x_{i2} & \sum x_{i2}^2 & \cdots & .001\sum x_{i2}\sum x_{ji} & & \sum x_{i2}x_{ik} \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ .001\sum\sum x_{ji} & .001\sum x_{i1}\sum x_{ji} & .001\sum x_{i2}\sum x_{ji} & \cdots & (.001)^2\sum x_{ii}^2 & \cdots & .001\sum x_{ik}x_{ji} \\ \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\ \sum x_{ik} & \sum x_{i1}x_{ik} & \sum x_{i1}x_{ik} & \cdots & .001\sum x_{ik}x_{ji} & \cdots & \sum x_{ik}^2 \end{pmatrix} \tag{2.17}$$

$$\hat{\beta}_{i_{new}} = ((.001)^2 X_{ii}' X_{ii})^{-1}(.001X_{ii}')Y \tag{2.18}$$

$$\hat{\beta}_{i_{new}} = 1000(X_{ii}' X_{ii})^{-1} X_{ii}' Y \tag{2.19}$$

$$\hat{\beta}_{i_{new}} = 1000\hat{\beta}_i \tag{2.20}$$

$$Var(\hat{\beta}_{i_{new}}) = Var(1000\hat{\beta}_i) \tag{2.21}$$

$$= (1000)^2(\sigma(X_{ii}' X_{ii})^{-1}) \tag{2.22}$$

Problem 3.0:

Problem 3.0 continued on next page...

```r
library(MASS)
a <- read.table("http://www.stat.ucla.edu/~nchristo/statistics100c/restaurant.txt",
                header = T)
x <- matrix(c(rep(1,100),a$food,a$decor, a$ser), nrow=100, ncol=4)
x_transpose_x <- (t(x) %*% x) ;x_transpose_x
```

```
##      [,1]  [,2]  [,3]  [,4]
## [1,]  100  1943  1609  1764
## [2,] 1943 38445 31764 34722
## [3,] 1609 31764 27671 29108
## [4,] 1764 34722 29108 31748
```

```r
y <- matrix(a$cost)
beta_hat = ginv(x_transpose_x) %*% t(x) %*% y;beta_hat
```

```
##             [,1]
## [1,] -35.3098383
## [2,]   1.2838518
## [3,]   1.7773844
## [4,]   0.2804128
```

```r
lm(cost ~ ., data= a)
```

```
##
## Call:
## lm(formula = cost ~ ., data = a)
##
## Coefficients:
## (Intercept)         food        decor          ser
##    -35.3098       1.2839       1.7774       0.2804
```

```r
H <- x %*% ginv(x_transpose_x) %*% t(x)
H[1:5,1:5]
```

```
##            [,1]        [,2]        [,3]        [,4]        [,5]
## [1,] 0.02333445 0.017090141 0.010247706 0.030288494 0.008920920
## [2,] 0.01709014 0.030828287 0.009747574 0.007121384 0.012518985
## [3,] 0.01024771 0.009747574 0.013348148 0.008131717 0.015062149
## [4,] 0.03028849 0.007121384 0.008131717 0.053772247 0.001936585
## [5,] 0.00892092 0.012518985 0.015062149 0.001936585 0.018606756
```

Problem 3.0 continued on next page. . .

Problem 4.0:

$$(Y - X\hat{\beta} + X\hat{\beta} - X\beta)^{'}(Y - X\hat{\beta} + X\hat{\beta} - X\beta) \tag{4.23}$$
$$((Y - X\hat{\beta}) + X(\hat{\beta} - \beta))^{'}((Y - X\hat{\beta}) + X(\hat{\beta} - \beta)) \tag{4.24}$$
$$(Y - X\hat{\beta})^{'}(Y - X\hat{\beta}) + (Y - X\hat{\beta})^{'}X(\hat{\beta} - \beta) + (\hat{\beta} - \beta)^{'}X^{'}(Y - X\hat{\beta}) + (\hat{\beta} - \beta)^{'}X^{'}X(\hat{\beta} - \beta) \tag{4.25}$$
$$(Y - X\hat{\beta})^{'}(Y - X\hat{\beta}) + (Y - \hat{Y})^{'}X(\hat{\beta} - \beta) + (\hat{\beta} - \beta)^{'}X^{'}(Y - \hat{Y}) + (\hat{\beta} - \beta)^{'}X^{'}X(\hat{\beta} - \beta) \tag{4.26}$$
$$(Y - X\hat{\beta})^{'}(Y - X\hat{\beta}) + (\hat{\beta} - \beta)^{'}X^{'}X(\hat{\beta} - \beta) \tag{4.27}$$

Problem 5.0:

$$\begin{aligned}
E[\hat{Y}^{'}\hat{Y}] &= E[(HY)^{'}HY] = E[Y^{'}H^{'}HY] && (5.28)\\
&= E[tr(Y^{'}HY)] = tr(HE[Y^{'}Y]) && (5.29)\\
&= tr(H(\sigma^2 I + X\beta\beta^{'}X^{'})) = tr(\sigma^2 H + X\beta\beta^{'}X^{'}) && (5.30)\\
&= tr(\sigma^2 H) + tr(X\beta\beta^{'}X^{'}) && (5.31)\\
E[\hat{Y}^{'}\hat{Y}] &= \sigma(K+1) + \beta^{'}X^{'}X\beta && (5.32)
\end{aligned}$$

Problem 6.0:

$$Q = \begin{pmatrix} \hat{\beta}_1 - 2\hat{\beta}_2 + 3\hat{\beta}_4 \\ \hat{\beta}_0 + \hat{\beta}_4 + 3\hat{\beta}_5 \end{pmatrix} \tag{6.33}$$

$$= \begin{pmatrix} 0 & 1 & -2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \\ \hat{\beta}_5 \end{pmatrix} \tag{6.34}$$

$$Q = c\hat{\beta} \tag{6.35}$$

We know that $\hat{\beta} \sim N_6(\beta, \sigma(X^{'}X)^{-1})$ then,

$$\begin{aligned}
E[Q] &= E[c\hat{\beta}] = cE[\hat{\beta}] && (6.36)\\
E[Q] &= c\beta && (6.37)\\
Var(Q) &= Var(c\hat{\beta}) && (6.38)\\
Var(Q) &= c(\sigma(X^{'}X)^{-1})c^{'} && (6.39)
\end{aligned}$$

$$\boxed{Q \sim N_2(c\beta, \sigma c(X^{'}X)^{-1}c^{'})} \tag{6.40}$$

Problem 7.0:

a

$$\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'Y_1 \tag{7.41}$$
$$\hat{\beta}_2 = (X_2'X_2)^{-1}X_2'Y_2 \tag{7.42}$$

b

$$\hat{\beta} = \left(\begin{pmatrix} X_1' & X_2' \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right)^{-1} \begin{pmatrix} X_1' & X_2' \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \tag{7.43}$$
$$= \left(X_1'X_1 + X_2'X_2 \right)^{-1} \begin{pmatrix} X_1' & X_2' \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \tag{7.44}$$
$$\hat{\beta} = \left(X_1'X_1 + X_2'X_2 \right)^{-1} \left(X_1'Y_1 + X_2'Y_2 \right) \tag{7.45}$$

c

$$\hat{\beta} = (2X_1'X_1)^{-1}(X_1'Y_1 + X_2'Y_2) \tag{7.46}$$
$$= \frac{1}{2}(X_1'X_1)^{-1}(X_1'Y_1 + X_2'Y_2) \tag{7.47}$$
$$= \frac{1}{2}[(X_1'X_1)^{-1}X_1'Y_1 + (X_1'X_1)^{-1}X_2'Y_2] \tag{7.48}$$
$$= \frac{1}{2}[(X_1'X_1)^{-1}X_1'Y_1 + (X_2'X_2)^{-1}X_2'Y_2] \tag{7.49}$$
$$\hat{\beta} = \frac{1}{2}(\hat{\beta}_1 + \hat{\beta}_2) \tag{7.50}$$