Final

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Problem 1.0:
Prove

$$L_1 \diamond L_2 = \{xy \mid x \in L_1, y \in L_2, \text{ and } |x|= 2|y|\} \tag{1.1}$$

is not context free.
Let $L_1 = \{0^{2n}1^{2n}\}$ and $L_2 = \{0^n 1^n\}$. Then,

$$L_1 \diamond L_2 = \{0^{2n}1^{2n}0^n1^n \mid x \in L_1, y \in L_2, \text{ and } |x|= 2|y|\} \tag{1.2}$$

Proof.
Towards contradiction assume $L_1 \diamond L_2$ is context-free.

- By the pumping lemma $\exists$ pumping length p.
- Let $w = 0^{2p}1^{2p}0^p1^p \in L_1 \diamond L_2$ and $|w| \geq p$.
- By pumping lemma $0^{2p}1^{2p}0^p1^p = abcde$ s.t:
  1. $|bd| \geq 1$
  2. $|bcd| \leq p$

Case 1: $bcd = 0^{\alpha}1^{\beta}$ (on the left side)

- We pump down then we have either:
  1. $ace = 0^{2p-\alpha}1^{2p}0^p1^p \notin L_1 \diamond L_2$, since $2p - \alpha + 2p = 4p \implies \alpha = 0$ and $1 \leq \alpha \leq p$ $\implies\impliedby$
  2. $ace = 0^{2p}1^{2p-\beta}0^p1^p \notin L_1 \diamond L_2$, since $2p - \beta + 2p = 4p \implies \beta = 0$ and $1 \leq \beta \leq p$ $\implies\impliedby$
  3. $ace = 0^{2p-\alpha}1^{2p-\beta}0^p1^p \notin L_1 \diamond L_2$, since $2p - \alpha + 2p - \beta = 4p \implies \alpha + \beta = 0$ and $1 \leq \alpha + \beta \leq p \implies\impliedby$

Case 2: $bcd = 0^{\alpha}1^{\beta}$ (on the right side)

- We pump up then we have either:
  1. $ace = 0^{2p}1^{2p}0^{p+\alpha}1^p \notin L_1 \diamond L_2$, since $2(p + \alpha + p) = 4p \implies \alpha = 0$ and $1 \leq \alpha \leq p$ $\implies\impliedby$
  2. $ace = 0^{2p}1^{2p}0^p1^{p+\beta} \notin L_1 \diamond L_2$, since $2(p + \beta + p) = 4p \implies \beta = 0$ and $1 \leq \beta \leq p$ $\implies\impliedby$

Problem 1.0 continued on next page. . .

3. $ace = 0^{2p}1^{2p}0^{p+\alpha}1^{p+\beta} \notin L_1 \diamond L_2$, since $2(p+\alpha+p+\beta) = 4p \implies \alpha+\beta = 0$ and $1 \leq \alpha+\beta \leq p \implies\impliedby$

Case 3: $bcd = 1^{\alpha}0^{\beta}$ (middle)

- We pump down then we have either:
  1. $ace = 0^{2p}1^{2p-\alpha}0^{p}1^{p} \notin L_1 \diamond L_2$, since $2p-\alpha+2p = 4p \implies \alpha = 0$ and $1 \leq \alpha \leq p \implies\impliedby$
  2. $ace = 0^{2p}1^{2p}0^{p-\beta}1^{p} \notin L_1 \diamond L_2$, since $2(p-\beta+p) = 4p \implies \beta = 0$ and $1 \leq \beta \leq p \implies\impliedby$
  3. $ace = 0^{2p}1^{2p-\alpha}0^{p-\beta}1^{p} \notin L_1 \diamond L_2$, since $2p-\alpha+2p = 2(p-\beta+p) \implies \beta = \alpha$. This is true if $\alpha = \beta = 0$ but $1 \leq \beta \leq p \implies\impliedby$. We can also have that $\alpha = \beta$ is true if p is even and each is half of p. However this destroys symmetry in L_1 and L_2, $0^{2p}1^{2p-\alpha} \notin L_1$ and $0^{p-\beta}1^{p} \notin L_2 \implies\impliedby$.

□

Problem 2.0:

(a)

Show that

$$HALT = \{(\langle M \rangle, x) \mid M \text{ halts on input } x\} \tag{2.3}$$

is oracle decidable.

Proof.
We construct OBTM $O(\langle M \rangle, x)$:

- O writes $\langle M \rangle$ to machine tape and w to input tape.
- O enters query state:

  1: $x \in L(M)$ then accept.
  2: $x \notin L(M)$ the reject.

The query is immediate therefore if $x \notin L(M)$, we can reject without looping. Therefore O always terminates thus it is a decider for HALT. □

Problem 2.0 continued on next page. . .

(b)

Show that

$$NEQ = \{(\langle M_1 \rangle, \langle M_2 \rangle)\ |L(M_1) \neq L(M_2)\} \tag{2.4}$$

is oracle recognizable.

Proof.
We construct OBTM $O(\langle M_1 \rangle, \langle M_2 \rangle)$:
In class we showed that a multiple tapes can be simulated with a single tape so we split the regular tape into 4 tapes w_1, w_2, w_3, and w_4.

1 Write $\langle M_1 \rangle$ onto w_1

2 Write $\langle M_2 \rangle$ onto w_2

3 Will keep a binary count starting at 0 in w_3.

    - We are assuming that all strings can be converted to binary.

4 Will maintain a tuple starting at $(0, 0)$ in w_4

We will have 4 states S_1, S_{oracle}, S_3, S_4

S_1: Write contents of tape w_1 onto the machine tape and contents of w_3 onto the input tape.

S_{oracle}: Enter query state and write the contents of the first cell in the input tape onto w_4 and move head right.

S_3: Clear the machine tape and write the contents of w_2 onto machine tape. Clear input tape and write w_3 onto input tape.

S_4: Reset tape w_4 to (0,0) and increment tape w_3 by one.

We will begin in state S_1 and transition to S_{oracle}. Then we transition to S_3 and back to S_{oracle}. If the contents of tape $w_4 = (1, 1)$ we transition onto S_4. □

(c)

Problem 3.0: