Homework 2

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Problem 1.0:

$$Q = \begin{pmatrix} \hat{\beta} \\ \hat{Y} \end{pmatrix} = \begin{pmatrix} (X^{'}X)^{-1}X^{'}Y \\ (I-H)Y \end{pmatrix} \tag{1.1}$$

$$Q = LY \tag{1.2}$$

$$Var(Q) = Var(LY) = LVar(Y)L^{'} \tag{1.3}$$

$$= \sigma^2 LL^{'} \tag{1.4}$$

$$= \sigma^2 \begin{pmatrix} (X^{'}X)^{-1}X^{'} \\ (I-H) \end{pmatrix} \begin{pmatrix} (X^{'}X)^{-1}X^{'} & (I-H) \end{pmatrix} \tag{1.5}$$

$$= \sigma^2 \begin{pmatrix} (X^{'}X)^{-1} & (X^{'}X)^{-1}X^{'}(I-H) \\ (I-H)X(X^{'}X)^{-1} & (I-H) \end{pmatrix} \tag{1.6}$$

$$Var(Q) = \sigma^2 \begin{pmatrix} (X^{'}X)^{-1} & 0 \\ 0 & (I-H) \end{pmatrix} \tag{1.7}$$

$\boxed{\text{Therefore } \hat{Y} \text{ and } e \text{ are independent.}}$

Problem 2.0:

$$\hat{\beta}_c = \hat{\beta} - (X^{'}X)^{-1}c^{'}[c(X^{'}X)^{-1}c^{'}]^{-1}(c\hat{\beta} - \gamma) \tag{2.8}$$

$$\tag{2.9}$$

let $A = (X^{'}X)^{-1}c^{'}[c(X^{'}X)^{-1}c^{'}]^{-1}$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - Var(A(c\hat{\beta} - \gamma)) \tag{2.10}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - AVar(c\hat{\beta} - \gamma)A^{'} \tag{2.11}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - A[cVar(\hat{\beta})c^{'} - Var(\gamma)]A^{'} \tag{2.12}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - AcVar(\hat{\beta})c^{'}A^{'} + AVar(\gamma)A^{'} \tag{2.13}$$

$$Var(\hat{\beta}_c) = Var(\hat{\beta}) - AcVar(\hat{\beta})c^{'}A^{'} + AVar(c\beta)A^{'} \tag{2.14}$$

$$Var(\hat{\beta}_c) = \sigma^2(X^{'}X)^{-1} - Ac[\sigma^2(X^{'}X)^{-1}]c^{'}A^{'} \tag{2.15}$$

$$\boxed{Var(\hat{\beta}_c) = \sigma^2(X^{'}X)^{-1} - \sigma^2(X^{'}X)^{-1}c^{'}[c^{'}(X^{'}X)^{-1}c]^{-1}c(X^{'}X)^{-1}} \tag{2.16}$$

Problem 2.0 continued on next page...

Problem 3.0:

We have that $S^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1}$ and $S_e^2 = \frac{e^{'}e}{n-k-1} = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2 - \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{n-k-1}$

$$R_a^2 = \frac{S^2 - s_e^2}{S^2} \tag{3.17}$$

$$= \frac{n-1}{\sum_{i=1}^{n}(y_i - \bar{y})^2}\left[\frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1} - \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-k-1} + \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{n-k-1}\right] \tag{3.18}$$

$$= \frac{n-1}{\sum_{i=1}^{n}(y_i - \bar{y})^2}\left[\frac{(n-k-1)\sum_{i=1}^{n}(y_i - \bar{y})^2 - (n-1)\sum_{i=1}^{n}(y_i - \bar{y})^2 + (n-1)\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{(n-1)(n-k-1)}\right] \tag{3.19}$$

$$= \left[\frac{(n-k-1) - (n-1) + (n-1)\frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}}{n-k-1}\right] \tag{3.20}$$

$$= 1 - \frac{(n-1) - (n-1)R^2}{n-k-1} \tag{3.21}$$

$$\boxed{R_a^2 = 1 - \frac{n-1}{n-k-1}(1 - R^2)} \tag{3.22}$$

Problem 4.0:

a

$$var\left(\frac{(n-k-1)S_e^2}{\sigma^2}\right) = 2(n-k-1) \tag{4.23}$$

$$\frac{(n-k-1)^2}{\sigma^4}Var(S_e^2) = 2(n-k-1) \tag{4.24}$$

$$\frac{n-k-1}{\sigma^4}Var(S_e^2) = 2 \tag{4.25}$$

$$Var(S_e^2) = \frac{2\sigma^4}{n-k-1} \tag{4.26}$$

Problem 4.0 continued on next page...

b

We have that $S^2 = \frac{e^{'}e}{n-k+1}$ so then $\frac{(n-k+1)S^2}{\sigma^2} = \frac{e^{'}e}{\sigma^2}$ we have shown in class that $\frac{e^{'}e}{\sigma^2} = \frac{\varepsilon^{'}(I-H)\varepsilon}{\sigma^2} \sim \chi^2_{n-k-1}$.

$$E[S^2] = \frac{E[e^{'}e]}{n-k+1} \tag{4.27}$$
$$= \frac{\sigma^2(n-k-1)}{(n-k+1)} \tag{4.28}$$

$$Var\left(\frac{(n-k+1)S^2}{\sigma^2}\right) = 2(n-k-1) \tag{4.29}$$
$$\frac{(n-k+1)^2}{\sigma^4}Var(S^2) = 2(n-k-1) \tag{4.30}$$
$$Var(S^2) = \frac{2\sigma^4(n-k-1)}{(n-k+1)^2} \tag{4.31}$$

c

Bias for S_e^2

$$B_1 = \sigma^2 - \sigma^2 \tag{4.32}$$
$$B_1 = 0 \tag{4.33}$$

MSE for S_e^2

$$MSE_1 = \frac{2\sigma^4}{n-k-1} \tag{4.34}$$

Bias for S^2

$$B_2 = \frac{\sigma^2(n-k-1)}{n-k+1} - \sigma^2 \tag{4.35}$$
$$= \sigma^2\frac{n-k-1-(n-k+1)}{n-k+1} \tag{4.36}$$
$$B_2 = \frac{-2\sigma^2}{n-k+1} \tag{4.37}$$

MSE for S^2

$$MSE_2 = \frac{2\sigma^4(n-k-1)}{(n-k+1)^2} + \frac{4\sigma^4}{(n-k+1)^2} \tag{4.38}$$
$$= \frac{2\sigma^4}{(n-k+1)^2}(n-k-1+2) \tag{4.39}$$
$$MSE_2 = \frac{2\sigma^4}{n-k+1} \tag{4.40}$$

Problem 4.0 continued on next page...

$\boxed{\text{The } MSE_2 \text{ is smaller and so we should use } S^2}$.

Problem 5.0:

$$(Y - Xc)^{'}(Y - Xc) - (Y - X\hat{\beta})^{'}(Y - X\hat{\beta}) \tag{5.41}$$

$$Y^{'}Y - Y^{'}Xc - c^{'}X^{'}Y + c^{'}X^{'}Xc - Y^{'}Y - Y^{'}Y + Y^{'}X\hat{\beta} + \hat{\beta}^{'}X^{'}Y - \hat{\beta}^{'}X^{'}X\hat{\beta} \tag{5.42}$$

$$\tag{5.43}$$

We use the normal equations to substitute for $Y^{'}X$ and $X^{'}Y$.

$$-\hat{\beta}^{'}X^{'}Xc - c^{'}X^{'}X\hat{\beta} + c^{'}X^{'}Xc + \hat{\beta}X^{'}X\hat{\beta} + \hat{\beta}^{'}X^{'}X\hat{\beta} - \hat{\beta}X^{'}X\hat{\beta} \tag{5.44}$$

$$c^{'}X^{'}Xc - \hat{\beta}^{'}X^{'}Xc - c^{'}X^{'}X\hat{\beta} + \hat{\beta}^{'}X^{'}X\hat{\beta} \tag{5.45}$$

$$(c^{'}X^{'}X - \hat{\beta}^{'}X^{'}X)(c - \hat{\beta}) \tag{5.46}$$

$$(c^{'} - \hat{\beta}^{'})X^{'}X(c - \hat{\beta}) \tag{5.47}$$

$$\boxed{(c - \hat{\beta})^{'}X^{'}X(c - \hat{\beta})} \tag{5.48}$$

Problem 6.0:

a

1

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = (X^{'}X)^{-1}X^{'}Y \tag{6.49}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \tag{6.50}$$

$$\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} 8 \\ -11 \\ 7 \end{pmatrix} \tag{6.51}$$

2

$$\sigma^2(X^{'}X){-1} = \begin{pmatrix} 2\sigma^2 & 0 & 0 \\ 0 & 3\sigma^2 & -\sigma^2 \\ 0 & -\sigma^2 & \sigma^2 \end{pmatrix} \tag{6.52}$$

$\boxed{\text{Therefore } Cov(\hat{\beta}_1, \hat{\beta}_2) = 0}$

Problem 6.0 continued on next page...

b

$$X = \begin{pmatrix} 1 & 3 & 1 & -1 & 1 \\ 1 & 4 & 1 & 1 & -1 \\ 1 & 5 & -1 & 1 & 1 \\ 1 & 6 & 0.5 & 0.2 & 0.3 \\ 1 & 8 & 0.8 & 0.1 & 0.1 \\ 1 & 9 & 0.3 & 0.5 & 0.2 \\ 1 & 10 & 0.2 & 0.3 & 0.5 \\ 1 & 13 & 0.1 & 0.6 & 0.3 \end{pmatrix} \tag{6.53}$$

$\boxed{x_2 + x_3 + x_4 = 1 \implies \text{ matrix is not full rank and we cannot invert } X^{'}X}$.

Problem 7.0:

a

$E[\hat{y}_{n+1}]$**:**

$$E[\hat{y}_{n+1}] = E[x^{'}_{n+1}\hat{\beta}] \tag{7.54}$$

$$= x^{'}_{n+1}E[\hat{\beta}] \tag{7.55}$$

$$\boxed{E[\hat{y}_{n+1}] = x^{'}_{n+1}\beta} \tag{7.56}$$

$E[y_{n+1}]$**:**

$$\boxed{E[y_{n+1}] = E[x^{'}_{n+1}\beta + \varepsilon_{n+1}] = x^{'}_{n+1}\beta} \tag{7.57}$$

b

$$E(\tilde{y}_{n+1} - y_{n+1}) = 0 \tag{7.58}$$

$$E[\tilde{y}_{n+1}] - E[y_{n+1}] = 0 \tag{7.59}$$

$$a^{'}E[Y] - x^{'}_{n+1}\beta = 0 \tag{7.60}$$

$$a^{'}X\beta - x^{'}_{n+1}\beta = 0 \tag{7.61}$$

$$\boxed{(a^{'}X - x^{'}_{n+1})\beta = 0 \implies a^{'}X = x^{'}_{n+1}} \tag{7.62}$$

Problem 7.0 continued on next page. . .

c

$$Var(\hat{y}_{n+1}) = Var(x^{'}_{n+1}\hat{\beta}) \tag{7.63}$$
$$= x^{'}_{n+1} Var(\hat{\beta}) x_{n+1} \tag{7.64}$$
$$= x^{'}_{n+1}[\sigma^2 (X^{'}X)^{-1}] x_{n+1} \tag{7.65}$$
$$\boxed{Var(\hat{y}_{n+1}) = \sigma^2 x^{'}_{n+1}(X^{'}X)^{-1} x_{n+1}} \tag{7.66}$$

$$\boxed{Var(\tilde{y}_{n+1}) = Var(a^{'}Y) = \sigma^2 a^{'}a} \tag{7.67}$$

d

$$Var(\tilde{y}_{n+1}) - Var(\hat{y}_{n+1}) = 0 \tag{7.68}$$
$$\sigma^2 a^{'}a - \sigma^2 x^{'}_{n+1}(X^{'}X)^{-1}x_{n+1} = \sigma^2 a^{'}a - \sigma^2 a^{'}x(X^{'}X)^{-1}x^{'}a \tag{7.69}$$
$$\sigma^2 a^{'}(I - X(X^{'}X)^{-1}X^{'})a = \sigma^2 a^{'}(I-H)a \tag{7.70}$$
$$\sigma^2 a^{'}(I-H)(I-H)^{'}a = \sigma^2((I-H)^{'}a)^{'}(I-H)^{'}a \tag{7.71}$$

We notice that $((I-H)^{'}a)^{'}(I-H)^{'}a$ is of the form $(M^{'}l)^{'}(M^{'}l)$ which implies that $(I-H)$ is at least postive semidefinite so then

$$\sigma^2((I-H)^{'}a)^{'}(I-H)^{'}a \geq 0 \tag{7.72}$$

which implies that

$$Var(\tilde{y}_{n+1}) - Var(\hat{y}_{n+1}) \geq 0 \tag{7.73}$$
$$Var(\tilde{y}_{n+1}) \geq Var(\hat{y}_{n+1}) \tag{7.74}$$

Problem 8.0:

a

$$e^{'}e = ((I-H)Y)^{'}((I-H)Y) \tag{8.75}$$
$$= Y^{'}(I-H)^{'}(I-H)Y \tag{8.76}$$
$$\boxed{e^{'}e = Y^{'}(I-H)Y} \tag{8.77}$$

b

$$
\begin{aligned}
tr[(I-H)E[Y^{'}Y]] &= tr[(I-H)(\sigma^2 I + X\beta\beta^{'}X^{'})] && (8.78)\\
&= tr[\sigma^2 I + X\beta\beta^{'}X^{'} - \sigma^2 H - HX\beta\beta^{'}X^{'}] && (8.79)\\
&= \sigma^2 tr(I) - \sigma^2 tr(H) && (8.80)\\
\boxed{tr[(I-H)E[Y^{'}Y]] = \sigma^2(n-k-1)} & && (8.81)
\end{aligned}
$$

c

$$
\begin{aligned}
Var(e) &= Var((I-H)Y) && (8.82)\\
&= \sigma^2(I-H)(I-H)^{'} && (8.83)\\
&= \sigma^2(I-H) && (8.84)\\
Var(e) &= \sigma^2(I - X(X^{'}X)^{-1}X^{'}) && (8.85)
\end{aligned}
$$

Therefore $Var(e_i) = \sigma^2(1 - x_i^{'}(X^{'}X)^{-1}x_i)$

d

$$
\begin{aligned}
Var(\hat{Y}) &= Var(HY) && (8.86)\\
&= HVar(Y)H^{'} && (8.87)\\
&= H(\sigma^2 I)H^{'} && (8.88)\\
&= \sigma^2 H && (8.89)\\
Var(\hat{Y}) &= \sigma^2 X(X^{'}X)^{-1}X^{'} && (8.90)
\end{aligned}
$$

Therefore $Var(e_i) = \sigma^2 - Var(\hat{Y})$