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Problem set 4-4

4) a) Let $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. We have $Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, and $QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I$

↳ has ~~orthogonal~~ orthonormal columns

b) Let $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. $\vec{a} \cdot \vec{b} = 0$, and $\vec{a}$ and $\vec{b}$ are linearly independent.

c) We have $\vec{q_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. We can define $\vec{b}$ and $\vec{c}$ such that they are all linearly independent. For example, $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{c} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. We use Gram-Schmidt to find $\vec{\beta}$ and $\vec{\gamma}$ such that they are all orthogonal.

$\vec{\beta} = \vec{b}$ ($\vec{q_1}$ and $\vec{b}$ are already orthogonal)

$\vec{\gamma} = \vec{c} - \frac{\vec{b}^T \vec{c}}{\vec{b}^T \vec{b}} \vec{b}$ ($\vec{q_1}$ and $\vec{c}$ are already orthogonal)

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{[1 \ -1 \ 0] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{[1 \ -1 \ 0] \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}$$

Now we define $\vec{q_2} = \frac{\vec{\beta}}{\|\vec{\beta}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{q_3} = \frac{\vec{\gamma}}{\|\vec{\gamma}\|} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}$

6) In order for Q to be orthogonal, we need to have $Q^TQ = I$.
In this case, we check $(Q_1Q_2)^T Q_1 Q_2 = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I$. ($Q_2$ orthogonal; Q_1 orthogonal)
So if Q_1 and Q_2 are orthogonal, Q_1Q_2 is also orthogonal.

10) a) When $c_1\vec{q_1} + c_2\vec{q_2} + c_3\vec{q_3} = \vec{0}$, where $\vec{q_1}, \vec{q_2}, \vec{q_3}$ are ~~orthogonal~~ orthonormal, doing the dot product with $\vec{q_1}$ leads to $c_1 = 0$. The dot product with $\vec{q_2}$ leads to $c_2 = 0$. The dot product with $\vec{q_3}$ leads to $c_3 = 0$. Thus $\vec{q_1}, \vec{q_2}, \vec{q_3}$ are linearly independent.

b) $Q = [\vec{q_1} \ \vec{q_2} \ \vec{q_3}]$. Since Q is orthonormal, $Q^TQ = I$.
So if $Q\vec{x} = \vec{0}$, then $Q^TQ\vec{x} = \vec{0} \Leftrightarrow \vec{x} = 0$. $N(Q) = \{\vec{0}\}$ and $\vec{q_1}, \vec{q_2}, \vec{q_3}$ are linearly independent.

19) If $A = QR$, then $A^TA = R^TR$ = lower triangular times upper triangular.

We have $A = [\vec{a}\ \vec{b}]$ with $\vec{a} = (-1, 2, 2)$ and $\vec{b} = (1, 1, 4)$.

We get $\vec{\alpha} = (-1, 2, 2)$ and $\vec{\beta} = \vec{b} - \frac{\vec{\alpha}^T\vec{b}}{\vec{\alpha}^T\vec{\alpha}}\vec{\alpha} = \begin{bmatrix}1\\1\\4\end{bmatrix} - \frac{[-1\ 2\ 2]\begin{bmatrix}1\\1\\4\end{bmatrix}}{[-1\ 2\ 2]\begin{bmatrix}-1\\2\\2\end{bmatrix}}\begin{bmatrix}-1\\2\\2\end{bmatrix} = \begin{bmatrix}2\\-1\\2\end{bmatrix}$

Therefore, $\vec{q_1} = \frac{\vec{\alpha}}{\|\vec{\alpha}\|} = \frac{1}{3}\begin{bmatrix}-1\\2\\2\end{bmatrix}$ and $\vec{q_2} = \frac{\vec{\beta}}{\|\vec{\beta}\|} = \frac{1}{3}\begin{bmatrix}2\\-1\\2\end{bmatrix}$

So $Q = \begin{bmatrix}-1/3 & 2/3\\ 2/3 & -1/3\\ 2/3 & 2/3\end{bmatrix}$ and $R = \begin{bmatrix}q_1^T a & q_1^T b\\ 0 & q_2^T b\end{bmatrix} = \begin{bmatrix}3 & 3\\ 0 & 3\end{bmatrix}$

24) a) $S = N(A)$ where $A = [1\ 1\ 1\ -1]$. We have 3 special solutions, $\vec{ss_1} = (-1, 1, 0, 0)$, $\vec{ss_2} = (-1, 0, 1, 0)$, $\vec{ss_3} = (1, 0, 0, 1)$. So a basis for S is $\{\vec{ss_1}, \vec{ss_2}, \vec{ss_3}\}$.

b) $S^\perp = N(A)^\perp = C(A^T) = \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$. A basis for $S^\perp$ is $\left\{\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}\right\}$.

c) We have $\vec{b_1} = \alpha\vec{ss_1} + \beta\vec{ss_2} + \gamma\vec{ss_3}$ and $\vec{b_2} = \delta\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$. So we need to solve

$$\begin{cases} -\alpha - \beta + \gamma + \delta = 1 \\ \alpha + \delta = 1 \\ \beta + \delta = 1 \\ \gamma - \delta = 1 \end{cases} \Rightarrow$$

~~$\gamma = 2, \beta = 0, \alpha = 0$. We verify that $\vec{b_1} + \vec{b_2} = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\0\\-1\\0\end{bmatrix} + \begin{bmatrix}1\\0\\0\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$~~

We need to solve $B = \begin{bmatrix}-1 & -1 & 1 & 1\\ 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -1\end{bmatrix}\vec{x} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

After elimination, we have $\left[\begin{array}{cccc|c}-1 & -1 & 1 & 1 & 1\\ 0 & -1 & 1 & 2 & 2\\ 0 & 0 & 1 & 3 & 3\\ 0 & 0 & 0 & -4 & -2\end{array}\right]$

So we get $\delta = \frac{1}{2}, \gamma = \frac{3}{2}, \beta = \frac{1}{2}, \alpha = \frac{1}{2}$

We verify that $\vec{b_1} + \vec{b_2} = \frac{1}{2}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + \frac{3}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

So $\vec{b_1} = \begin{bmatrix}1/2\\1/2\\1/2\\3/2\end{bmatrix}$ and $\vec{b_2} = \begin{bmatrix}1/2\\1/2\\1/2\\-1/2\end{bmatrix}$

Problem set 5.1

3) a) False. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $\det(A) = 1$. $I + A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ $\det(I+A) = 5 \neq 1+1 = 2$

b) True. ~~det(AB) = det A det B~~

$\det(PQ) = \det P \cdot \det Q$ Let $P = A$ and $Q = BC$

Then $\det Q = \det(BC) = \det B \cdot \det C$

So $\det(ABC) = \det A \cdot \det B \cdot \det C$

c) False Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det A = ad - bc$, $\det(4A) = 16(ad-bc) = 4^2 \det(A)$

d) False. Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. We have $AB = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

$AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\det(AB - BA) = 1 \neq 0$.

8) a) $|Q^TQ| = |Q^T||Q| = |Q|^2 = 1$, so $|Q| = 1$ or -1.

b) We have $|Q| = 1$ or -1, So $|Q^n| = |Q|^n = 1$ or -1 $\forall n$

28) ~~a) False. Let A = [0 1; 0 1] det A = 0 and A is not invertible.~~
~~Let B = [1 1; 1 1] AB = [1 1; 1 1].~~

a) True. $|AB| = |A| \cdot |B|$ if $|A| = 0$, then $|AB| = 0$.

b) False. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ The pivots are 1 and 1, but $|A| = -1$

c) False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\det A = 0$. Let $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\det B = 0$.

$A - B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\det(A - B) = -1$.

d) True. $|AB| = |A||B| = |B||A| = |BA|$

Problem set 5.2

12) We get $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. $C^T = C$, and $AC^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

We can see that $A^{-1} = \frac{1}{\det A} C^T$

13) a) $C_1 = 0$
$C_2 = -1$
$C_3 = 0$
$C_4 = 1$

b) We can see that $C_4 = -C_2$ (and if we do C_5, we see that $C_5 = -C_3$). We can say that, for $n \in \mathbb{N}_{n>}$ we get $C_n = -C_{n-2}$.
So $C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1$

15) a) We have $E_n = a_{11} C_{11} + a_{12} C_{12} + \dots + a_{1n} C_{1n}$ where a_{ij} are the elements in the matrix. By construction, $a_{1j} = 0$ for $j > 2$.
So $E_n = a_{11} C_{11} + a_{12} C_{12}$. In addition, $a_{11} = a_{12} = 1$.
So $E_n = C_{11} + C_{12}$. Also, it is easy to see that $C_{11} = E_{n-1}$, and $C_{12} = -E_{n-2}$, So we get $E_n = E_{n-1} - E_{n-2}$.

b) $E_1 = \cancel{0}1$, $E_2 = 0$, $E_3 = -1$, $E_4 = -1$, $E_5 = 0$, $E_6 = 1$, $\underset{= E_1}{E_7 = 1}$, $\underset{= E_2}{E_8 = 0}$ ...

c) $E_{100} = E_{100 \% 6} = E_4 = -1$

18) We have $|B_n| = |A_n| - |A_{n-1}|$. We saw in chapter 5-2) that A_n is the $n \times n$ $[-1, 2, -1]$ matrix. It's determinant is $|A_n| = n+1$.

Therefore, $|B_n| = n+1 - (n \cancel{-1} \cancel{+1}) = 1$

Problem set 6.1

4) $|A-\lambda I| = \begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = \lambda^2+\lambda-6 \Rightarrow \lambda_{11}=2,\ \lambda_{21}=-3$

$A-\lambda_{11}I = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} \to \vec{v}_{11} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A-\lambda_{21}I = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \to \vec{v}_{21} = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

$|A^2-\lambda I| = \begin{vmatrix} 7-\lambda & -3 \\ -2 & 6-\lambda \end{vmatrix} = \lambda^2-13\lambda+36 \Rightarrow \lambda_{12}=4=\lambda_{11}^2,\ \lambda_{22}=9=\lambda_{21}^2$

$A^2-\lambda_{12}I = \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \to \vec{v}_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{v}_{11}$

$A^2-\lambda_{22}I = \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \to \vec{v}_{22} = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} = \vec{v}_{21}$

A^2 has the same eigenvectors as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues λ_1^2 and λ_2^2. In this example, $\lambda_1=2$, $\lambda_2=-3$, so $\lambda_1^2=4$, $\lambda_2^2=9$ and $\lambda_1^2+\lambda_2^2=13$.

10) $|A-\lambda I| = \begin{vmatrix} 0.6-\lambda & 0.2 \\ 0.4 & 0.8-\lambda \end{vmatrix} = \lambda^2-1.4\lambda+0.4 \Rightarrow \lambda_1=1$ and $\lambda_2=0.4$

$A-\lambda_1 I = \begin{bmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{bmatrix} \to \vec{v}_1 = a\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ In particular $\vec{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

$A-\lambda_2 I = \begin{bmatrix} 0.2 & 0.2 \\ 0.4 & 0.4 \end{bmatrix} \to \vec{v}_2 = a\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ In particular $\vec{v}_2 = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$.

$|A^\infty-\lambda I| = \begin{vmatrix} 1/3-\lambda & 1/3 \\ 2/3 & 2/3-\lambda \end{vmatrix} = \lambda(\lambda-1) \Rightarrow \lambda_1=1$ and $\lambda_2=0$.

$A^\infty-\lambda I = \begin{bmatrix} -2/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix} \to \vec{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

$A^\infty-\lambda_2 I = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix} \to \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

A^{100} has eigenvalues $\lambda_1=1^{100}=1$ and $\lambda_2=0.4^{100}\approx 0$. So A^{100} is close to A^∞.

13) $\vec{u} = (1/6, 1/6, 3/6, 5/6)$ $\quad P = \begin{bmatrix} 1/36 & 1/36 & 3/36 & 5/36 \\ 1/36 & 1/36 & 3/36 & 5/36 \\ 3/36 & 3/36 & 9/36 & 15/36 \\ 5/36 & 5/36 & 15/36 & 25/36 \end{bmatrix}$

a) $P\vec{u} = \vec{u}$ comes from $(\vec{u}\vec{u}^T)\vec{u} = \vec{u}(\underbrace{\vec{u}^T\vec{u}}_{1})$. Then $\vec{u}$ is an eigenvector with $\lambda = 1$

b) $\vec{v} \perp \vec{u}$, so $\vec{u}^T\vec{v} = 0 \Rightarrow P\vec{v} = (\vec{u}\vec{u}^T)\vec{v} = \vec{u}(\vec{u}^T\vec{v}) = \vec{0}$

c) Three independent eigenvectors of P with $\lambda = 0$ are $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

16) $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$ so $\underline{\det A = \prod_{i=1}^{n} \lambda_i}$ if $\lambda = 0$.

$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ $\det(A - \lambda I) = (\lambda - 1)(\lambda - \frac{1}{2})$ $\det A = .8 \times .7 - .2 \times .3 = 0{,}5 = \lambda_1\lambda_2 = 1 \cdot \frac{1}{2} = 0{,}5$

27) $C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ So $\underline{r = 2}$

We have $\lambda_1 = 0$ and the corresponding eigenvectors are elements of $N(C)$. We get $\vec{v}_1 = (-1, 0, 1, 0)$ and $\vec{v}_2 = (0, -1, 0, 1)$.

$|C - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix} = \lambda^2(\lambda - 2)^2 \Rightarrow \lambda_1 = 0$ and $\lambda_2 = 2$

$C - \lambda_2 I = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{V}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{V}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

For matrix C, we get two eigenvalues ~~eigenvectors~~ with multiplicity 2.

We have $\lambda_1 = 0$ and its associated eigenvectors $\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

and $\lambda_2 = 2$ with its associated eigenvectors $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Problem set 6.2

1) a)

- $A=\begin{bmatrix}1&2\\0&3\end{bmatrix}$

$$|A-\lambda I|=\begin{vmatrix}1-\lambda&2\\0&3-\lambda\end{vmatrix}=(1-\lambda)(3-\lambda) \Rightarrow \lambda_1=1,\ \lambda_2=3$$

$$A-\lambda_1 I=\begin{bmatrix}0&2\\0&2\end{bmatrix}\to\begin{bmatrix}0&2\\0&0\end{bmatrix}\Rightarrow \vec{v}_1=\begin{bmatrix}1\\0\end{bmatrix}$$

$$A-\lambda_2 I=\begin{bmatrix}-2&2\\0&0\end{bmatrix}\Rightarrow \vec{v}_2=\begin{bmatrix}1\\1\end{bmatrix} \qquad \Rightarrow A=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&3\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}$$

- $A=\begin{bmatrix}1&1\\3&3\end{bmatrix}$

$$|A-\lambda I|=\begin{vmatrix}1-\lambda&1\\3&3-\lambda\end{vmatrix}=\lambda(\lambda-4)\Rightarrow \lambda_1=0,\ \lambda_2=4$$

$$A-\lambda_1 I=\begin{bmatrix}1&1\\3&3\end{bmatrix}\to\begin{bmatrix}1&1\\0&0\end{bmatrix}\Rightarrow \vec{v}_1=\begin{bmatrix}1\\-1\end{bmatrix}$$

$$A-\lambda_2 I=\begin{bmatrix}-3&1\\3&-1\end{bmatrix}\to\begin{bmatrix}-3&1\\0&0\end{bmatrix}\Rightarrow \vec{v}_2=\begin{bmatrix}1\\3\end{bmatrix} \qquad \Rightarrow A=\begin{bmatrix}1&1\\-1&3\end{bmatrix}\begin{bmatrix}0&0\\0&4\end{bmatrix}\begin{bmatrix}0&0\\1/4&1/4\end{bmatrix}$$

b) If $A=S\Lambda S^{-1}$, $A^3=S\Lambda^3S^{-1}$; $A^{-1}=S\Lambda^{-1}S^{-1}$

11) a) True. All eigenvalues are non-zero.

b) False. A may not be diagonalizable because $\lambda=2$ has multiplicity 2.

c) False. If $A=\begin{bmatrix}2&0&0\\0&2&0\\0&0&5\end{bmatrix}$, then A is diagonal already.

19) $B=\begin{bmatrix}5&1\\0&4\end{bmatrix}$

$$|B-\lambda I|=\begin{vmatrix}5-\lambda&1\\0&4-\lambda\end{vmatrix}=(5-\lambda)(4-\lambda)\Rightarrow \lambda_1=4,\ \lambda_2=5$$

$$B-\lambda_1 I=\begin{bmatrix}1&1\\0&0\end{bmatrix}\Rightarrow \vec{v}_1=\begin{bmatrix}-1\\1\end{bmatrix}$$

$$B-\lambda_2 I=\begin{bmatrix}0&1\\0&-1\end{bmatrix}\Rightarrow \vec{v}_2=\begin{bmatrix}1\\0\end{bmatrix}$$

$$S=\begin{bmatrix}-1&1\\1&0\end{bmatrix},\ \Lambda=\begin{bmatrix}4&0\\0&5\end{bmatrix},\ S^{-1}=\begin{bmatrix}0&1\\1&1\end{bmatrix}$$

$$\Lambda^k=\begin{bmatrix}4^k&0\\0&5^k\end{bmatrix}.\ S\Lambda^kS^{-1}=\begin{bmatrix}5^k&5^k-4^k\\0&4^k\end{bmatrix}=A^k$$

4

25) $C(A)$ contains eigenvectors with $\lambda=1$.
$N(A^q)$ contains eigenvectors with $\lambda=0$.

Google PageRank problems

Exercise 10:

• We have $(A^2)_{ij} = \sum_k A_{ik}A_{kj} = A_{i1}A_{1j} + A_{i2}A_{2j} + \dots + A_{in}A_{nj}$.

So $(A^2)_{ij} > 0$ if and only if there is a k for which $A_{ik}A_{kj} \neq 0$. This means that we can go from page j to page k in one step, and from page k to page i in one step. Hence page i can be reached from page j in exactly two steps.

• We want to show that $(A^p)_{ij} > 0$ if and only if page i can be reached from page j in exactly p steps. We have already shown that this is true for $p=1$ and $p=2$.
Let's assume this is true for a value q: $(A^q)_{ij} > 0$ if and only if page i can be reached from page j in exactly q steps. We will show then that this is true for $q+1$. Let A^q have entries A^q_{ij}.
Then, we have $(A^{q+1})_{ij} = \sum_k A^q_{ik} A_{ki} = A^q_{i1}A_{1j} + \dots + A^q_{in}A_{nj}$. So $(A^{q+1})_{ij} > 0$ if and only if there is a k for which $A^q_{ik}A_{kj} \neq 0$. This means that ~~page~~ we can go from page j to page k in one step, and from page k to page i in ~~k~~ q steps. Hence page i can be reached from page j in $q+1$ steps exactly.
We have shown that ~~the~~ if the statement is true for ~~some~~ q, then it is true for ~~next~~ $q+1$. So it must be true for any p.
$(A^p)_{ij} > 0$ if and only if page i can be reached from page j in exactly p steps.

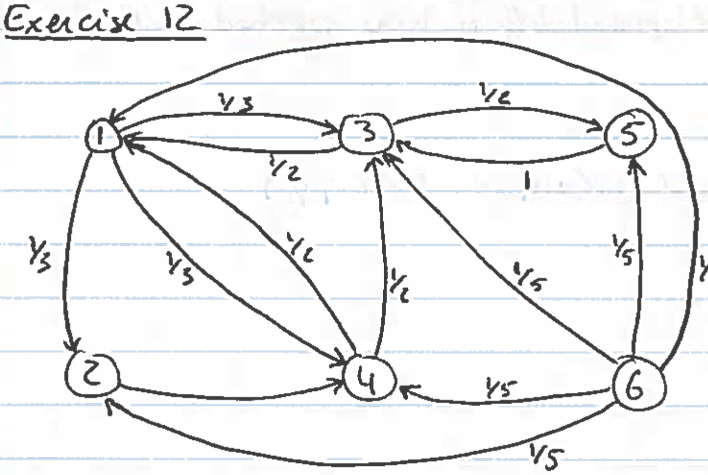
• We have shown that $(A^p)_{ij} > 0$ if and only if page i can be reached from page j in exactly p steps. If $p=0$, $A^p = I$, and $I_{ij} > 0$ if and only if $i=j$ (the page links to itself). $(I + A + A^2 + \dots + A^p)_{ij} > 0$ means that there exists ~~m~~ $1 \leq m \leq p$ such that $(A^m)_{ij} > 0$, ~~therefore~~ page i can be reached from page j in m steps, with $1 \leq m \leq p$.

- If the web is strongly connected, then each page is reachable from any other page in at most $n-1$ steps. We have also shown that if page i is reachable from page j in exactly p steps, then $(A^p)_{ij} > 0$. Therefore, matrix A^p has positive elements for all such ij combinations. So, $\sum_{i=0}^{n-1} A^p$ is the sum of the matrix with positive elements for pages reached in exactly 0 steps, plus those reached in exactly 1 step, ..., plus those reached in exactly $n-1$ steps. Therefore, all elements of $I + A + A^2 + \dots + A^{n-1}$ are positive (strictly) and the matrix is positive.

- We have shown that the matrix $I + A + A^2 + \dots + A^{n-1}$ is positive, so $B = \frac{1}{n}(I + A + A^2 + \dots + A^{n-1})$ is positive too $(n > 0)$. In addition, we know by construction that all matrices A^p $(p \geq 0)$ are column stochastic. Therefore, the sums of the columns in the matrix $I + A + A^2 + \dots + A^{n-1}$ will add to n (there are n total matrices). So $B = \frac{1}{n}(I + A + A^2 + \dots + A^{n-1})$ is column stochastic.

- We have $\vec{x} \in V_1(A)$. Consequently, we also have $\vec{x} \in V_1(A^2)$, $\vec{x} \in V_1(A^3)$, ... In general, we have $\vec{x} \in V_1(A^p)$ $(p \geq 0)$. Therefore, $\vec{x}$ in in ~~only~~ all linear combinations of $\{I, A, A^2, \dots, A^{n-1}\}$. In particular, we have $\vec{x} \in \frac{1}{n}(I + A + A^2 + \dots + A^{n-1}) = B$.
  In addition, we know that $\dim(V_1(B)) = 1$. From the previous statement, $\dim(V_1(A)) \leq \dim(V_1(B)) = 1$. Since $\vec{x} \neq \vec{0}$, $\dim(V_1(A)) \geq 1$. Therefore, we get $\dim(V_1(A)) = 1$.

<u>Exercise 12</u>



We get the link matrix $A = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 & 0 & 1/5 \\ 1/3 & 0 & 0 & 0 & 0 & 1/5 \\ 1/3 & 1/2 & 0 & 1/2 & 1 & 1/5 \\ 1/3 & 1/2 & 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/2 & 0 & 0 & 1/5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\lambda_1 = 1$

We use python to find the eigenvector associated to $\lambda = 1$, and we get $\vec{v}_A = (0.2449, 0.0816, 0.3673, 0.1224, 0.1837, 0)$. Therefore, the pages are ranked in the following way (from most important to least important): 3 - 1 - 5 - 4 - 2 - 6.

We set $\alpha = 0.15$, and $B = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$. We get the matrix

$$H = (1-\alpha)A + \alpha B = \begin{bmatrix} 0.025 & 0.025 & 0.45 & 0.45 & 0.025 & 0.195 \\ 0.30833 & 0.025 & 0.025 & 0.025 & 0.025 & 0.195 \\ 0.30833 & 0.45 & 0.025 & 0.45 & 0.875 & 0.195 \\ 0.30833 & 0.45 & 0.025 & 0.025 & 0.025 & 0.195 \\ 0.025 & 0.025 & 0.45 & 0.025 & 0.025 & 0.195 \\ 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 \end{bmatrix} \quad \lambda_1 = 1$$

The eigenvector associated to $\lambda = 1$ is $\vec{v}_H = (0.2312, 0.0948, 0.3402, 0.1350, 0.1738, 0.025)$. Therefore, the pages are ranked in the following way: 3 - 1 - 5 - 4 - 2 - 6.

We can see that the ranking is the same for matrices H and A, but page 6 had 0 probability of being reached in A, and has 0.025 probability in H.

(python file uploaded to Canvas: Ex12.py)