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Problem set 3.5)

2) Let $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4\ \vec{v}_5\ \vec{v}_6]$

We have $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$

$\xrightarrow{R_4+R_3} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 3 pivots

So the largest possible number of independent vectors among $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6\}$ is 3. For example $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

11) a) The two vectors are linearly dependent: $(1,1,-1) = -1(-1,-1,1)$. So the subspace spanned by them is a line.

b) The subspace spanned by $(0,1,1)$, $(1,1,0)$, and $(0,0,0)$ is the same as the subspace spanned by $(0,1,1)$ and $(1,1,0)$, which is a plane since the vectors are linearly independent.

c) The subspace spanned in this case is all of $\mathbb{R}^3$. For example, three such vectors are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

d) The subspace spanned in this case is all of $\mathbb{R}^3$. ALC of three of these independent vectors span $\mathbb{R}^3$.

~~20)~~

20). The plane $x-2y+3z=0$ is the nullspace of $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{ss}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{ss}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ So $\{\vec{ss}_1, \vec{ss}_2\}$ is a basis for the plane.

. The intersection of the plane $x-2y+3z=0$ with the xy-plane is a line. In addition, we have $\vec{ss}_1$ in both planes. So a basis for this intersection is $\{\vec{ss}_1\}$.

. We need only find a vector ~~such~~ $\vec{n} = (a,b,c)$ such that $\vec{n}\cdot\vec{ss}_1 = \vec{n}\cdot\vec{ss}_2 = 0 \iff 2a+b = -3a+c = 0 \iff 5a+b-c=0$

If $a=1$, $b=-2$, $c=3$, then a basis for this subspace is $\{\vec{n} = (1,-2,3)\}$.

23) $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$ $U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

- A basis for the column space of U is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$

- A basis for $C(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$

- A basis for $R(A)$ is the same as for $R(U)$ and is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

The row spaces stay fixed during elimination.

41) The five possible permutation matrices are

$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, and $P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

We get $P_1 + P_2 + P_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $P_4 + P_5 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Therefore, $I = P_1 + P_2 + P_3 - P_4 - P_5$

Assume $\exists\, c_1, \dots, c_5$ such that $c_1 P_1 + \dots + c_5 P_5 = 0$.

This means $\begin{bmatrix} c_3 & c_1 + c_4 & c_2 + c_5 \\ c_1 + c_5 & c_2 & c_3 + c_4 \\ c_2 + c_4 & c_3 + c_5 & c_1 \end{bmatrix} = 0_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

In particular, we have $c_1 = c_2 = c_3 = 0$ (from the diagonal). Therefore $c_4 = c_5 = 0$ as well.

The only possibility for $c_1 P_1 + \dots + c_5 P_5 = 0$ is that $c_i = 0\ \forall i = 1, \dots, 5$. Therefore $P_1, \dots, P_5$ are linearly independent and form a basis for the subspace of 3×3 matrices with row and column sums all equal.

Problem set 3.6

1) a) the matrix has $\underset{(m)}{7}$ rows, $\underset{(n)}{9}$ columns. $r=5$.

$$\left.\begin{array}{l} \dim(C(A)) = r = 5 \\ \dim(N(A)) = n-r = 4 \\ \dim(C(A^T)) = r = 5 \\ \dim(N(A^T)) = m-r = 2 \end{array}\right\} \begin{array}{l}\text{the sum of the four dimensions is} \\ \cancel{r}+n-\cancel{r}+\cancel{r}+m-\cancel{r} = n+m = 16.\end{array}$$

b) the matrix has $m=3$ rows and $n=4$ columns. $r=3$.

$\dim(C(A)) = r = 3$ and $C(A) \subset \mathbb{R}^3$. So $C(A) = \mathbb{R}^3$

$\dim(N(A^T)) = m-r = 0$ So the left null space is $\{\vec{0}\}$.

4) a) A such that $C(A)$ contains $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$, $C(A^T)$ contains $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\5\end{bmatrix}$.

So since $C(A) \subset \mathbb{R}^3$, $m=3$ A has 3 rows.

since $C(A^T) \subset \mathbb{R}^2$, $n=2$ A has 2 columns.

$$R = \begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix} \longrightarrow \boxed{A = \begin{bmatrix}1&2\\1&2\\2&5\end{bmatrix}}$$

b) A such that $C(A)$ has basis $\{(1,1,3)\}$ and $N(A)$ has basis $\{(3,1,1)\}$.

Since $C(A) \subset \mathbb{R}^3$, $m=3$ A has 3 rows.

Since $N(A) \subset \mathbb{R}^3$, $n=3$ A has 3 columns.

Also, $r=1$ and $\dim(N(A)) = 1$, which is <u>impossible</u>, since $\dim(N(A)) = n-r = 2$

c) A such that $\dim(N(A)) = 1 + \dim(N(A^T))$.

We know that $\dim(N(A^T)) = m-r$, and $\dim(N(A)) = n-r$

So we need to find A such that $n-\cancel{r} = 1+m-\cancel{r} \iff n = 1+m$

A is any 2×3 matrix with two pivots. For example

$$\boxed{A = \begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}}$$

d) A such that $N(A^T)$ contains $(1,3)$ and $C(A^T)$ contains $(3,1)$.

$\dim(N(A^T)) = m-r = 1$

$\dim(C(A^T)) = r = 1$. So $m=2$

Since $C(A^T) \subset \mathbb{R}^2$, $n=2$

$\cancel{A = \begin{bmatrix}3&1\\a&b\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow}$

We have two cases:

- $A = \begin{bmatrix} 3 & 1 \\ a & b \end{bmatrix}$ then we must have $\begin{bmatrix} 3 & a \\ 1 & b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} 3 + 3a = 0 \\ 1 + 3b = 0 \end{cases} \iff \begin{cases} a = -1 \\ b = -\frac{1}{3} \end{cases}$

  then A exists and $\boxed{A = \begin{bmatrix} 3 & 1 \\ -1 & -\frac{1}{3} \end{bmatrix}}$

- $A = \begin{bmatrix} a & b \\ 3 & 1 \end{bmatrix}$ then we must have $\begin{bmatrix} a & 3 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} a + 9 = 0 \\ b + 3 = 0 \end{cases} \iff \begin{cases} a = -9 \\ b = -3 \end{cases}$

  Then A exists and $\boxed{A = \begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}}$

e) A such that $C(A^T) = C(A)$ and $N(A) \neq N(A^T)$.

14) $A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}}_{U}$ We have $m = 3$ and $n = 4$, and $r = 3$

- A basis of $C(A)$ is $\left\{ \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

- A basis of $C(A^T)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

- We have $U \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $\vec{SS_1} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

  A basis of $N(A)$ is $\left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

- $\dim(N(A^T)) = m - r = 0$. So a basis of $N(A^T)$ is $\{\vec{0}\}$.

24) $A^T \vec{y} = \vec{d}$ is solvable if $\boxed{\vec{d} \in C(A^T)}$. The solution $\vec{y}$ is unique when the <u>left nullspace</u> contains only $\vec{0}$.

25) a) True. $\dim(C(A^T)) = \dim(C(A)) = r$, and r is the number of pivots in the matrix.

b) False. If A is $(m \times n)$ and $m \neq n$, then $\dim(N(A^T)) = m - r$, and $\dim(N((A^T)^T)) = \dim(N(A)) = n - r$.

c) False. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ then $C(A) = C(A^T) = \mathbb{R}^2$

d) True. If $A^T = -A$, then $C(A^T) = C(-A) = C(A)$

26) If $AB = C$, the rows of C are combinations of the rows of $\underline{B}$. So the rank of C is not greater than the rank of $\underline{B}$. Since $B^T A^T = C^T$, the rank of C is also not greater than the rank of $\underline{A}$.

Problem set 4.1)

3) a) We try $A = \begin{bmatrix} 1 & 2 & a \\ 2 & -3 & b \\ -3 & 5 & c \end{bmatrix}$ such that $\begin{cases} 1+2+a=0 \\ 2-3+b=0 \\ -3+5+c=0 \end{cases} \Leftrightarrow \begin{cases} a=-3 \\ b=1 \\ c=-2 \end{cases}$

We get $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$. We can verify that we get $R = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$

So A answers the problem

~~b) We try~~ $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 5 \\ a & b & c \end{bmatrix}$ ~~with where~~ $\begin{cases} 1+2-3=0 \\ 2-3+5=0 \\ a+b+c=0 \end{cases}$

~~In addition, we get~~ $R = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 0 & 0 & c+3a-\frac{b-2a}{2}\cdot 11 \end{bmatrix}$

~~We need~~ $c+3a-\frac{b-2a}{2}\cdot 11 = 0 \Leftrightarrow 14a - \frac{11}{2}b + c = 0$

~~We need to solve~~ $\begin{cases} 14a - \frac{11}{2}b + c = 0 \\ a + b + c = 0 \end{cases} \Leftrightarrow \begin{cases} 13a - \frac{13}{2}b = 0 \\ a + b + c = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{b}{2} \\ c = -\frac{3}{2}b \end{cases}$

~~So~~ $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 5 \\ 1/2 & 1 & -3/2 \end{bmatrix}$ ~~answers the problem.~~

b) We know that $N(A) \perp C(A^T)$. However, we have $1 \cdot 2 + 2 \cdot (-3) + (-3) \cdot 5 \neq 0$. So no such matrix exists.

c) If $A^T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, then $\begin{bmatrix}1\\0\\0\end{bmatrix}\in N(A^T)$. Therefore, the first column of A^T is $(0,0,0)$. This means that the first row of A is $[0\ 0\ 0]$, which is impossible if $Ax=\begin{bmatrix}1\\1\\1\end{bmatrix}$ has a solution.

d) If every row is orthogonal to every column, then $A\cdot A=0 \iff A^2=0$
For example, $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ answers the problem.

e) No such matrix exists. If all columns add to 0, then the sum of all elements of A adds to 0. However, since all rows add to 1, then the sum of all elements of A adds to 1. We have a contradiction.

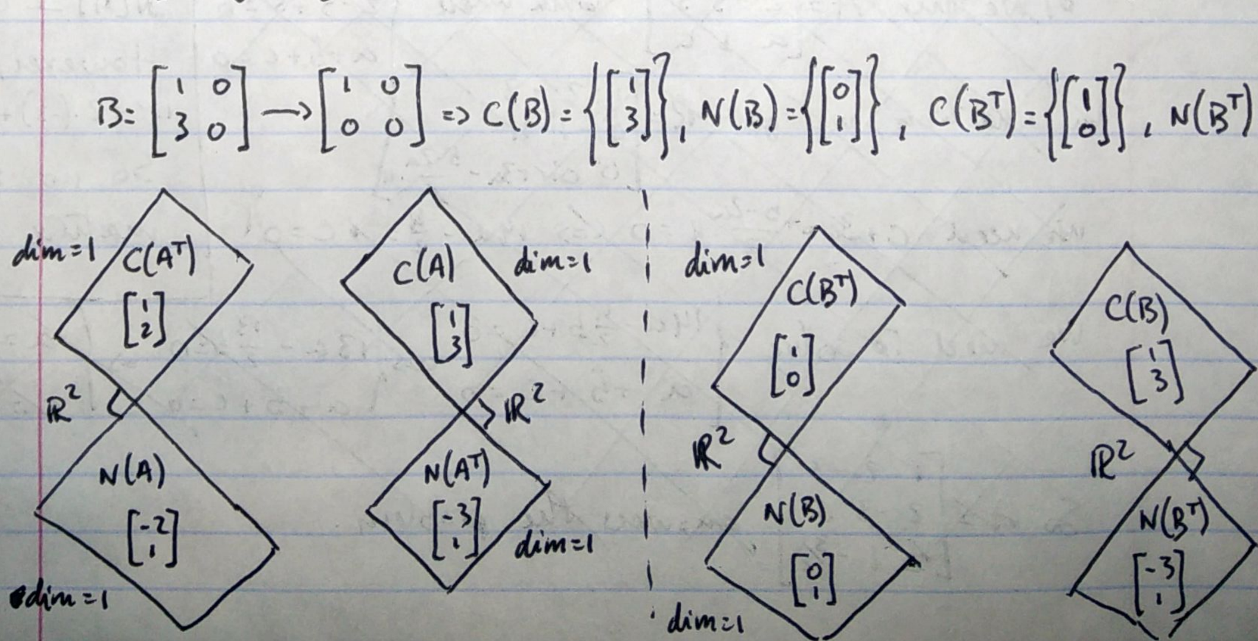
4) If $AB=0$, then the columns of B are in the <u>nullspace</u> of A. The rows of A are in the <u>left nullspace</u> of B.

If A and B were 3×3 matrices of rank 2, then $\dim(N(A)) = \dim(N(A^T)) = \dim(N(B)) = \dim(N(B^T)) = 2$, which is impossible because $m=n=3$ and $r=2$ (then $m-r=n-r=1$).

7) If <u>$y_1=1$</u>, <u>$y_2=1$</u>, and <u>$y_3=-1$</u>, we get $1(\cancel{x_1}-\cancel{x_2})+1(\cancel{x_2}-\cancel{x_3})-1(\cancel{x_1}-\cancel{x_3})=1$
$\iff 0=1$

11) $A=\begin{bmatrix}1&2\\3&6\end{bmatrix}\longrightarrow\begin{bmatrix}1&2\\0&0\end{bmatrix} \Rightarrow C(A)=\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\},\ N(A)=\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\},\ C(A^T)=\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\},\ N(A^T)=\left\{\begin{bmatrix}-3\\1\end{bmatrix}\right\}$

$B=\begin{bmatrix}1&0\\3&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&0\\0&0\end{bmatrix} \Rightarrow C(B)=\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\},\ N(B)=\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\},\ C(B^T)=\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\},\ N(B^T)=\left\{\begin{bmatrix}-3\\1\end{bmatrix}\right\}$



21) S is spanned by $\vec{v}=(1,2,2,3)$ and $\vec{w}=(1,3,3,2)$. $\vec{v}$ and $\vec{w}$ are linearly independent, so S is a plane in $\mathbb{R}^2$. Therefore, $S^\perp$ is also a plane in $\mathbb{R}^2$.

If we have A such that $C(A^T) = S = \{\vec{v}, \vec{w}\}$, then we only need to find $N(A)$, since ~~AA~~ $C(A^T)^\perp = N(A) = S^\perp$.

We get $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

So $\vec{ss}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{ss}_2 = \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow N(A) = S^\perp = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Problem set 4.2)

1) a) We get $\vec{p} = \dfrac{[1\ 1\ 1] \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}{[1\ 1\ 1] \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \dfrac{5}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$

So $\vec{e} = \vec{b} - \vec{p} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ and $\vec{e} \cdot \vec{a} = -\frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 = 0$

So $\vec{e} \perp \vec{a}$

11) b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$. We also have $A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

So $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$ and $A^T\vec{b} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$

So we need to solve $\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \Rightarrow \boxed{\hat{x} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}}$

Therefore $\vec{p} = A\hat{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \vec{b}$

We get $\vec{e} = \vec{b} - \vec{p} = \vec{0}$

13) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$C(A) = \mathbb{R}^3$, so $\vec{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$ and $\vec{e} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$. Therefore $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (we remove the last coordinate of $\vec{b}$)

21) $P = A(A^TA)^{-1}A^T$

$\Rightarrow P^2 = A(A^TA)^{-1}\cancel{A^T \cdot A(A^TA)^{-1}}A^T$

$= A(A^TA^{-1})A^T$

$= P$

Geometrically, when we do $P\vec{b}$, we project $\vec{b}$ onto $\vec{a}$, so $P\vec{b}$ is on $\vec{a}$. Therefore, by ~~exa~~ any further projection won't do anything i.e, $P(P\vec{b}) = P\vec{b}$, so $P^2 = P$. The vector $P\vec{b}$ is in the column space so its projection is <u>itself</u>.

<u>Problem set 4.3)</u>

1) we have $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. So $A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$

$A^TA = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^T\vec{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$

So we need to solve $\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 8 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 36 \\ 40 \end{bmatrix} = \hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

So the best straight line, we get ~~[illegible]~~ $\boxed{b = 1 + 4t}$

therefore, from figure 4.9, we get $\begin{cases} p_1 = 1 + 4.0 = 1 \\ p_2 = 1 + 4.1 = 5 \\ p_3 = 1 + 4.3 = 13 \\ p_4 = 1 + 4.4 = 17 \end{cases} \Rightarrow \begin{cases} e_1 = |1 - 0| = 1 \\ e_2 = |5 - 8| = 3 \\ e_3 = |13 - 8| = 3 \\ e_4 = |17 - 20| = 3 \end{cases}$

12) a) $\vec{a}^T\vec{a} = m$ and $\vec{a}^T\vec{b} = \sum_{i=1}^{m} b_i$

So $\vec{a}^T\vec{a}\,\hat{x} = \vec{a}^T\vec{b} \iff m\hat{x} = \sum_{i=1}^{m} b_i \iff \hat{x} = \dfrac{\sum_{i=1}^{m} b_i}{m}$. $\hat{x}$ is the mean of the $\vec{b}$'s.

b) $\vec{e} = \vec{b} - \vec{a}\hat{x} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} - \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \hat{x} = \begin{bmatrix} b_1 - \hat{x} \\ \vdots \\ b_m - \hat{x} \end{bmatrix}$

So $\|\vec{e}\|^2 = (b_1 - \hat{x})^2 + \dots + (b_m - \hat{x})^2 = \sum_{i=1}^{m} (b_i - \hat{x})^2$

and $\|\vec{e}\| = \sqrt{(b_1 - \hat{x})^2 + \dots + (b_m - \hat{x})^2} = \sqrt{\sum_{i=1}^{m} (b_i - \hat{x})^2}$

c) $\vec{e} = \vec{b} - \vec{p} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{e} \cdot \vec{p} = -2\cdot 3 - 1\cdot 3 + 3\cdot 3 = 0$ so $\vec{e} \perp \vec{p}$.

We have $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $P = A(A^TA)^{-1}A^T$. $A^TA = 3$, $(A^TA)^{-1} = \frac{1}{3}$

So $P = \frac{1}{3} AA^T = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$.

16) If we know the average $\hat{x}_9$ of $b_1, \dots, b_9$, we can quickly find the average $\hat{x}_{10}$ with one more number b_{10} by doing the following calculation

$\boxed{\hat{x}_{10} = \frac{9}{10}\hat{x}_9 + \frac{1}{10} b_{10}}$ $\frac{9}{10}$ multiplies $\hat{x}_9$ in computing $\hat{x}_{10}$.

~~26) We need to solve~~ $\cancel{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}}$

$\cancel{\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 3 \\ 1 & 0 & -1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & -1 & -1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 3 \end{array}\right]}$

26) We have $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$. So $A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $A^T\vec{b} = \begin{bmatrix} 8 \\ -3 \\ -3 \end{bmatrix}$

So we need to solve $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ -3 \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} 2 \\ -3/2 \\ -3/2 \end{bmatrix}$

At the center of the square, we get $2 + (-\frac{3}{2}) \cdot 0 + (-\frac{3}{2}) \cdot 0 = 2 = \frac{0+1+3+4}{4}$