Problem set 1.1

1) a) $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

$\vec{v} = 3\vec{u}$ All linear combinations (ALC) of these two vectors is the line spanned by $\vec{u}$.

b) $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ $\vec{u}$ and $\vec{v}$ are linearly independent. ALC is a plane.

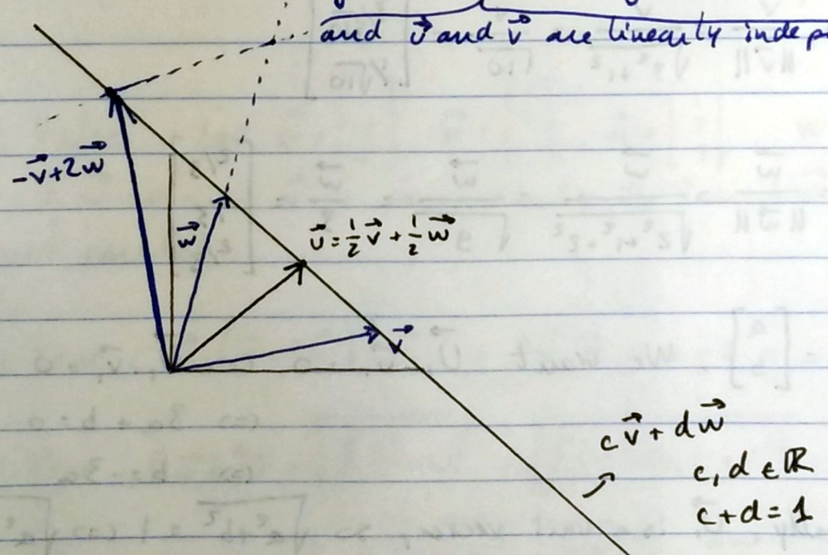
c) $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ $\vec{u}, \vec{v}$, and $\vec{w}$ are all linearly independent. ALC is all of $\mathbb{R}^3$.

5) $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$

$$\vec{u} + \vec{v} + \vec{w} = \begin{bmatrix} 1 - 3 + 2 \\ 2 + 1 - 3 \\ 3 - 2 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0} \qquad 2\vec{u} + 2\vec{v} + \vec{w} = \begin{bmatrix} 2 - 6 + 2 \\ 4 + 2 - 3 \\ 6 - 4 - 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = -\vec{w}$$

From the above equations, we have $\vec{u} + \vec{v} = -\vec{w} \iff \vec{w}$ is a linear combination of $\{\vec{u}, \vec{v}\}$ and $\vec{u}$ and $\vec{v}$ are linearly independant. Therefore, $\vec{u}, \vec{v}, \vec{w}$ lie in a plane.

16)



26) Find c, d such that $c\begin{bmatrix}1\\2\end{bmatrix} + d\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}14\\8\end{bmatrix}$.

$$\begin{cases} 1c + 3d = 14 & (L_1) \\ 2c + d = 8 & (L_2) \end{cases} \iff \begin{cases} 1c + 3d = 14 & (L_1) \\ 5d = 20 & (2L_1 - L_2) \end{cases} \iff \begin{cases} c = 14 - 12 = 2 \\ d = 4 \end{cases} \iff \begin{cases} c = 2 \\ d = 4 \end{cases}$$

Problem set 1.2.

1)

$$\vec{u}.\vec{v} = \begin{bmatrix}-0{,}6\\0{,}8\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix} = -0{,}6.3 + 0{,}8.4 = 1{,}4$$

$$\vec{v}.\vec{w} = \begin{bmatrix}-0{,}6\\0{,}8\end{bmatrix}\cdot\begin{bmatrix}8\\6\end{bmatrix} = -0{,}6.8 + 0{,}8.6 = 0$$

$$\vec{u}.(\vec{v}+\vec{w}) = \begin{bmatrix}-0{,}6\\0{,}8\end{bmatrix}\cdot\begin{bmatrix}11\\10\end{bmatrix} = -0{,}6.11 + 0{,}8.10 = 1{,}4$$

other way

$$\vec{u}.(\vec{v}+\vec{w}) = \vec{U}.\vec{v} + \underbrace{\vec{U}.\vec{w}}_{0} = \vec{U}.\vec{v} = 1{,}4$$

$$\vec{w}.\vec{v} = \begin{bmatrix}8\\6\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix} = 8.3 + 6.4 = 48$$

4) a) $\vec{v}.(-\vec{v}) = -\vec{v}.\vec{v} = \|\vec{v}\|^2 = \sum_{i=1}^{n} v_i^2$

b) $(\vec{v}+\vec{w}).(\vec{v}-\vec{w}) = \vec{v}.\vec{v} - \cancel{\vec{v}.\vec{w}} + \cancel{\vec{w}.\vec{v}} - \vec{w}.\vec{w} = \|\vec{v}\|^2 - \|\vec{w}\|^2$

5) $$\vec{U_1} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{v}}{\sqrt{3^2+1^2}} = \frac{\vec{v}}{\sqrt{10}} = \begin{bmatrix}3/\sqrt{10}\\1/\sqrt{10}\end{bmatrix}$$

$$\vec{U_2} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{\vec{w}}{\sqrt{2^2+1^2+2^2}} = \frac{\vec{w}}{\sqrt{9}} = \frac{\vec{w}}{3} = \begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix}$$

Let $\vec{U}_1 = \begin{bmatrix}a\\b\end{bmatrix}$. We want $\vec{U}_1.\vec{v}_1 = 0 \iff \vec{U}_1.\vec{v}_1 = 0$

$\iff 3a + b = 0$

$\iff b = -3a$

Additionally, $\vec{U}_1$ is a unit vector, so $\sqrt{a^2+b^2} = 1 \iff \sqrt{a^2+(-3a)^2} = 1$

$\iff a^2 + 9a^2 = 1$

$\iff 10a^2 = 1$

$\iff a^2 = \frac{1}{10}$

So we have $\vec{U}_1 = \begin{bmatrix}-1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix}$ or $\vec{U}_1 = \begin{bmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{bmatrix}$

Let $\vec{U}_2 = \begin{bmatrix} c \\ d \\ e \end{bmatrix}$. We want $\vec{U}_2 . \vec{u}_2 = 0 \iff \frac{2}{3}c + \frac{1}{3}d + \frac{2}{3}e = 0 \quad (L_1)$

Additionally, $\vec{U}_2$ is a unit vector, so $\sqrt{e^2+d^2+e^2} = 1 \iff c^2+d^2+e^2 = 1 \quad (L_2)$

From (L_1), one possible solution is $c = \alpha$, $d = 0$, $e = -\alpha$, ~~thus~~ $\alpha \in \mathbb{R}$
Injecting into (L_2), we get $\alpha^2 + 0 + (-\alpha)^2 = 1 \iff 2\alpha^2 = 1$

$$\iff \alpha^2 = \frac{1}{2}$$
$$\iff \alpha = \pm \frac{1}{\sqrt{2}}$$

So we have $\vec{U}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$ or $\vec{U}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

6) a) $\vec{w} \perp \vec{v} \iff \vec{w}.\vec{v} = 0$
$$\iff 2w_1 - w_2 = 0$$
$$\iff w_2 = 2w_1$$
$\vec{w}$ is any vector on the line $y = 2x$.

b) The vectors perpendicular to $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ lie on a <u>plane</u>.

c) The vectors that are perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ lie on a <u>line</u>.

8) a) False. Let $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. We have $\vec{u}.\vec{v} = 0$ and $\vec{u}.\vec{w} = 0$, so $\vec{u}$ is perpendicular to $\vec{v}$ and $\vec{w}$. Yet $\vec{v}$ and $\vec{w}$ are not parallel.

b) True. $\vec{u}.(\vec{v} + 2\vec{w}) = \vec{u}.\vec{v} + 2\vec{u}.\vec{w} = 0 + 2.0 = 0$

c) True. $\|\vec{u} - \vec{v}\| = \sqrt{\sum_{i=1}^{n}(u_i - v_i)^2} = \sqrt{\sum_{i=1}^{n}(u_i^2 - 2u_iv_i + v_i^2)} = \sqrt{\sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} v_i^2 - 2\sum_{i=1}^{n} u_iv_i}$

$$= \sqrt{\underbrace{\|\vec{u}\|^2}_{1} + \underbrace{\|\vec{v}\|^2}_{1} - 2\underbrace{\vec{u}.\vec{v}}_{0}} = \sqrt{2}$$

22) a) $(v_1w_1 + v_2w_2)^2 \leq (v_1^2 + v_2^2)(w_1^2 + w_2^2)$

$\Leftrightarrow \cancel{v_1^2w_1^2} + 2v_1v_2w_1w_2 + \cancel{v_2^2w_2^2} \leq \cancel{v_1^2w_1^2} + v_1^2w_2^2 + v_2^2w_1^2 + \cancel{v_2^2w_2^2}$

b)

$\Leftrightarrow \quad 0 \leq v_1^2w_2^2 - 2v_1v_2w_1w_2 + v_2^2w_1^2$

$\Leftrightarrow \quad 0 \leq (v_1w_2 - v_2w_1)^2$

Problem set 1.3.

6) $\begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & c \end{bmatrix} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ If $c = 3$, we get $\vec{w} = 2\vec{u} + \vec{v}$

$\begin{bmatrix} 1 & 0 & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ If $c = -1$, we get $\vec{w} = \vec{v} - \vec{u}$

$\begin{bmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ If $c = 0$, we get $\vec{w} = 3\vec{u} - \vec{v}$

7) The three dot products $\vec{r}_1 \cdot \vec{x}$, $\vec{r}_2 \cdot \vec{x}$, $\vec{r}_3 \cdot \vec{x}$ equal to 0. Therefore, all three vectors are perpendicular to $\vec{x}$. Since all three vectors lie in a plane, then this plane is perpendicular to $\vec{x}$.

Problem set 2.1.

12) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ y \\ x \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \cancel{2x + y + 3z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$

16) a) $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ b) $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

18) $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

19) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}^{E} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} E E^{-1} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

28) For four linear equations in two unknowns x and y, the row picture shows four lines. The column picture is in ~~four~~-dimensional space. The equations have no solution unless the vector on the right side is a combination of the four columns.

Problem set 2.2.

2) $\begin{cases} 2x + 3y = 1 \\ 10x + 9y = 11 \end{cases} \Leftrightarrow \cancel{\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 10 & 9 & 11 \end{array}\right]} \Leftrightarrow \begin{cases} 2x + 3y = 1 \\ 0x - 6y = 6 \end{cases} \Leftrightarrow \begin{cases} 2x - 3 = 1 \\ y = -1 \end{cases} \Leftrightarrow \begin{cases} x = 2 \\ y = -1 \end{cases}$

$\Leftrightarrow \cancel{\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -6 & 6 \end{array}\right]}$ Verification: $\begin{cases} 2 \cdot 2 + 3 \cdot -1 = 1 \\ 10 \cdot 2 + 9 \cdot -1 = 11 \end{cases}$

If the right side changes to $\begin{bmatrix} 4 \\ 44 \end{bmatrix}$, the new solution is $4 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$

6) Choosing $b = (\frac{1}{2}) 8 = 4$ makes the system singular. Choosing $g = 2 \cdot 16 = 32$ makes the system solvable. Two solutions are $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

7) a) Choosing $a = 2$ breaks elimination permanently.

b) Choosing $a = 0$ breaks elimination temporarily.

11) A system of linear equations can't have exactly two solutions. It can have one solution, no solutions, or infinitely many solutions. If we think of two lines, they either intersect in a unique point (one solution), are parallel (no solutions), or are the same (infinitely many solutions). Two lines cannot intersect at two distinct unique ~~locations~~ points.

a) Let $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$

$A\vec{x} = \vec{b}$ and $A\vec{X} = \vec{b}$.

Any combination $c\vec{x} + (1-c)\vec{X}$, with $c \in \mathbb{R}$ is also a solution.

Proof: $A(c\vec{x} + (1-c)\vec{X}) = cA\vec{x} + (1-c)A\vec{X}$

$= c\vec{b} + (1-c)\vec{b}$

$= \vec{b}$

with $c = \frac{1}{2}$, $\frac{1}{2}\vec{x} + \frac{1}{2}\vec{X}$ is a solution.

12) $\begin{cases} 2x + 3y + z = 8 \\ 4x + 7y + 5z = 20 \\ -2y + 2z = 0 \end{cases} \Leftrightarrow \begin{cases} 2x + 3y + z = 8 \\ 1y + 3z = 4 \\ -2y + 2z = 0 \end{cases} \Leftrightarrow \begin{cases} 2x + 3y + z = 8 \\ 1y + 3z = 4 \\ 8z = 8 \end{cases}$

$\Leftrightarrow z = 1, y = 1, x = 2$

13) $\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & -3 & 5 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 2 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{array}\right] \Leftrightarrow \begin{cases} z = 0 \\ y = 1 \\ x = 3 \end{cases}$

24) $A = \begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$

Elimination will fail for $a = 0$ and $a = 2$.

Problem set 2.3

1) a) $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$

c) $P = P_2 P_1$ where P_1 exchanges rows 1 and 2, and P_2 exchanges rows 2 and 3.

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

2) $E_{32} E_{21} \vec{b} = \begin{bmatrix} 1 \\ -5 \\ 35 \end{bmatrix}$ $E_{21} E_{32} \vec{b} = \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$

when E_{32} comes first, row 3 feels no effect from row 2.

3) $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$$M = E_{32} E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \cancel{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$$

$$U = MA = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

7) a) To invert that step you should <u>add</u> 7 times row <u>1</u> to row <u>3</u>.

b) $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$

c) $EE^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

13) • We know that $EB = [E\vec{b}_1 \ \ E\vec{b}_2 \ \ \dots \ \ E\vec{b}_n]$ with $B = [\vec{b}_1 \ \ \vec{b}_2 \dots \vec{b}_n]$ therefore, if the third column of B is all zero (i.e. $\vec{b}_3 = \vec{0}$), then $E\vec{b}_3 = \vec{0}$, and the third column of EB will be all zero for every E.

• Counter-example: $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$. $EB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

18) $EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$

$FE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ac+b & c & 1 \end{bmatrix}$

$E^2 = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$

$F^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3c & 1 \end{bmatrix}$

$F^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 100c & 1 \end{bmatrix}$

27) a) If $\begin{cases} a = \alpha (\in \mathbb{R}) \\ b = \beta (\in \mathbb{R}) \\ c = \gamma \neq 0 \\ d = 0 \end{cases}$, there is no solution.

b) If $\begin{cases} a = \alpha (\in \mathbb{R}) \\ b = \beta (\in \mathbb{R}) \\ c = 0 \\ d = 0 \end{cases}$, there are infinitely many solutions.

a and b have no effect on the solvability

Problem set 2.4

1) BA is allowed $BA = \begin{bmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{bmatrix} \begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{bmatrix} = \begin{bmatrix} 3&3&3&3&3 \\ 3&3&3&3&3 \\ 3&3&3&3&3 \\ 3&3&3&3&3 \\ 3&3&3&3&3 \end{bmatrix}$

AB is allowed $AB = \begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{bmatrix} \begin{bmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{bmatrix} = \begin{bmatrix} 5&5&5 \\ 5&5&5 \\ 5&5&5 \end{bmatrix}$

ABD is allowed $ABD = \begin{bmatrix} 5&5&5 \\ 5&5&5 \\ 5&5&5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix}$

DBA is not allowed because D is (3×1) and BA is (5×5).

$A(B+C)$ is not allowed because B is (5×3) and C is (5×1).

2) a) To find the third column of $\underset{(n \times n)}{AB}$, you multiply (on the right) by an $(n \times 1)$ vector where all entries are 0 except for the third one, where there's a 1.

ex: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

b) To find the first row of AB $(n \times n)$, you multiply (on the ~~right~~ left) by an $(1 \times n)$ vector where all entries are 0 except for the first one where there is a 1. Ex: $[1\ 0\ 0\ \dots\ 0]$

5) • $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ $A^3 = \begin{bmatrix} 1 & 3b \\ 0 & 1 \end{bmatrix}$ $A^5 = \begin{bmatrix} 1 & 5b \\ 0 & 1 \end{bmatrix}$

$A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$

• $A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ $A^3 = \begin{bmatrix} 8 & 8 \\ 0 & 0 \end{bmatrix}$ $A^5 = \begin{bmatrix} 2^5 & 2^5 \\ 0 & 0 \end{bmatrix}$

$A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$

14) a) True. If A is $(n \times m)$ with $n \neq m$, then $A^2 = AA$ is not computable.

b) ~~True~~ False. Let A be (2×3) and B be (3×2). Both AB and BA are defined, yet ~~none of~~ neither is square.

c) True. If AB and BA are defined, then A is $(m \times n)$ and B is $(n \times m)$. Therefore AB is $(m \times m)$ and BA is $(n \times n)$.

d) ~~True~~ False. If $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, then $AB = B$ and $A \neq I$.

22) • If $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, then $A^2 = -I$

• If $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $BC = 0$

• If $D = \begin{bmatrix} \quad \end{bmatrix}$ and $E = \begin{bmatrix} \quad \end{bmatrix}$, then $DE = -ED$

Problem set 2.5

5)

$U = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ gives $U^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $U^{-1} = U$

8) a) ~~If~~ A can be written as $\begin{bmatrix} a & b & a+b \\ c & d & c+d \\ e & f & e+f \end{bmatrix}$ with $a, b, c, d, e, f \in \mathbb{R}$

Therefore, any vector $\vec{x} = \begin{bmatrix} \alpha \\ \alpha \\ -\alpha \end{bmatrix}$ with $\alpha \in \mathbb{R}$ is a solution to $A\vec{x} = \vec{0}$.

One such vector is $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

b) Since elimination keeps column 1 + column 2 = column 3, after the first round of elimination, the second and third line will be similar (line 3 will be a factor away from line 2). As a consequence, the second round of elimination will lead to a third line that reads $0 = 0$. Therefore, there won't be a third pivot, and A is not invertible.

9) Let's use (3×3) matrices, but this applies to (n×n) matrices. By definition, we have $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ (the first two rows are switched. Let's call this matrix $D = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
Additionally, $D^{-1} = D$, since by switching its first two ~~columns~~ rows we get I.
We know that $B^{-1} = (DA)^{-1} = A^{-1}D^{-1}$, so $B^{-1} = A^{-1}D$. This means that B^{-1} is equal to A^{-1} with the first two ~~rows~~ columns exchanged.

12) $C^{-1} = (AB)^{-1} = B^{-1}A^{-1}$

If we left-multiply both sides by B, we get:

$BC^{-1} = A^{-1}$