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Problem set 2.5

11) a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

A and B both have 2 pivots, so they are invertible. However, $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has 0 pivots, and is therefore not invertible

b) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

A and B both have 1 pivot, so they are singular. However, $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has 2 pivots, and is therefore invertible.

25) $[A\ I] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = [I\ A^{-1}] \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

29) a) True. A 4x4 matrix is invertible if and only if it has 4 pivots. If it has a row of zeros, then the matrix has at most 3 pivots and is not invertible.

b) False. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. A has ones down the main diagonal. However, A has only one pivot, so is not invertible.

c) True. If A is invertible, then A^{-1} exists. By definition, $(A^{-1})^{-1} = A$, so A^{-1} is invertible. Also $A^2 = AA$, so $(A^2)^{-1} = (AA)^{-1} = A^{-1}A^{-1} = (A^{-1})^2$ and A^{-1} exists.

Problem set 2.6

7) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$ $\quad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow E_{21}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix}$ $\quad E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \rightarrow E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$ $\quad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \rightarrow E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

9) • The matrix $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ has a 0 in the pivot position of the first row, so LU decomposition is impossible.

• $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ This matrix has a 0 in the pivot position of the second row, so LU decomposition is impossible.

For the first matrix, we have $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

For the second matrix, we have $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

10) If $c = 2$, the first step of elimination will produce a 0 in the second pivot position.

$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & c & 0 \\ 0 & 4-2c & 1 \\ 0 & 5-3c & 1 \end{bmatrix}$ In order to get a 0 in the third pivot position, we need $4-2c = 5-3c$

$\Leftrightarrow \boxed{c = 1}$

16) $L\vec{c}=\vec{b} \iff \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}4\\5\\6\end{bmatrix} \iff \begin{cases}c_1=4\\c_2=1\\c_3=1\end{cases} \quad \vec{c}=\begin{bmatrix}4\\1\\1\end{bmatrix}$

$U\vec{x}=\vec{c} \iff \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}4\\1\\1\end{bmatrix} \iff \begin{cases}x_1=3\\x_2=0\\x_3=1\end{cases} \quad \vec{x}=\begin{bmatrix}3\\0\\1\end{bmatrix}$

$A=LU=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix}$

19)

- $A=\begin{bmatrix}1&1&0\\1&2&1\\0&1&2\end{bmatrix} \quad E_{21}=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix} \rightarrow E_{21}A=\begin{bmatrix}1&1&0\\0&1&1\\0&1&2\end{bmatrix}$

$E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix} \rightarrow E_{32}E_{21}A=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}=U$

$L=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix} \quad D=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad L^T=U$

$\Rightarrow A=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$

- $A=\begin{bmatrix}a&a&0\\a&a+b&b\\0&b&b+c\end{bmatrix} \quad E_{21}=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix} \rightarrow E_{21}A=\begin{bmatrix}a&a&0\\0&b&b\\0&b&b+c\end{bmatrix}$

$E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix} \rightarrow E_{32}E_{21}A=\begin{bmatrix}a&a&0\\0&b&b\\0&0&c\end{bmatrix}=U$

$L=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix} \quad D=\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix} \Rightarrow U=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}=L^T$

$\Rightarrow A=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}a&a&0\\0&b&b\\0&0&c\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$

Problem set 3.1

4)
$$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tfrac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad -A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$$

The matrices in the smallest subspace containing A are all 2x2 matrices (M_2).

9) a) $\left\{ \begin{bmatrix} a \\ b \end{bmatrix}, a \geq 2, b \geq 2 \right\}$ $\quad x+y = \begin{bmatrix} a_x \\ b_x \end{bmatrix} + \begin{bmatrix} a_y \\ b_y \end{bmatrix} = \begin{bmatrix} a_x + a_y \\ b_x + b_y \end{bmatrix}, \; a_x + a_y > 2, \; b_x + b_y > 2$

$\frac{1}{2}x = \frac{1}{2}\begin{bmatrix} a_x \\ b_x \end{bmatrix} = \begin{bmatrix} a_x/2 \\ b_x/2 \end{bmatrix}$, $\frac{a_x}{2}$ may be < 2 (if $a_x = 2$ for example)

$\frac{b_x}{2}$ may be < 2 (if $b_x = 2$ for example)

b) $\left\{ \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \beta \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\}$ $\quad cx = \begin{bmatrix} c\alpha \\ 0 \end{bmatrix}$ is in the set

$cy = \begin{bmatrix} 0 \\ c\beta \end{bmatrix}$ is in the set

$x+y = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is not in the set.

10) a) True $\quad c\begin{bmatrix} b_1 \\ b_1 \\ b_3 \end{bmatrix} + d\begin{bmatrix} a_1 \\ a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} cb_1 + da_1 \\ cb_1 + da_1 \\ cb_3 + da_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ \beta \end{bmatrix}$

b) False $\quad \cancel{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + } \quad 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad b_1 \neq 1$

c) False $\quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad 2 \times 1 \times 1 = 2 \neq 0$

~~d) True by definition False All of $\vec{v}$ and $\vec{w}$ does not contain $\vec{0}$~~

d) True by definition

~~e) False~~ $\cancel{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$

e) True $\quad c\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + d\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} ca_1 + db_1 \\ ca_2 + db_2 \\ ca_3 + db_3 \end{bmatrix} \Rightarrow ca_1 + db_1 + ca_2 + db_2 + ca_3 + db_3$

$= c(a_1 + a_2 + a_3) + d(b_1 + b_2 + b_3)$

$= 0$

~~f) True~~ $\cancel{c\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + d\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$ $\cancel{a_1 \leq a_2 \leq a_3 \Leftrightarrow ca_1 \leq ca_2 \leq ca_3}$

$\cancel{b_1 \leq b_2 \leq b_3 \Leftrightarrow db_1 \leq db_2 \leq db_3}$

f) False $\quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} \quad -1 > -2 > -3$ $\quad \cancel{ca_1 + db_1 \leq ca_2 + db_2 \leq ca_3 + db_3}$

17) a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible. $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is invertible. $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is singular. $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is singular. $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not singular.

19) $C(A)$ is the x-axis.
$C(B)$ is the x-y plane.
$C(C)$ is the line spanned by $(1,2,0)$.

20) a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \to \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 + b_1 \end{bmatrix}$

Solvability conditions: $\begin{cases} b_2 - 2b_1 = 0 \\ b_3 + b_1 = 0 \end{cases} \Leftrightarrow \boxed{\begin{cases} b_2 = 2b_1 \\ b_3 = -b_1 \end{cases}}$

$\Rightarrow$ the solutions $\vec{b}$ are $\begin{bmatrix} \alpha \\ 2\alpha \\ -\alpha \end{bmatrix}$, $\alpha \in \mathbb{R}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 4 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \to \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 + b_1 \end{bmatrix}$

Solvability condition: $b_3 + b_1 = 0 \Leftrightarrow \boxed{b_3 = -b_1}$

$\Rightarrow$ the solutions $\vec{b}$ are $\begin{bmatrix} \alpha \\ \beta \\ -\alpha \end{bmatrix}$, $\alpha, \beta \in \mathbb{R}$

23) If we add an extra column $\vec{b}$ to a matrix A, then the column space gets larger unless $\underline{\vec{b} \in C(A)}$.

If we add $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, the column space gets larger.

If we add $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the column space doesn't get bigger.

By definition, $C(A)$ is all linear combinations of the columns of A ($\Leftrightarrow A\vec{x}$). So $\vec{b} \in C(A)$ if $A\vec{x} = \vec{b}$ is solvable. Adding $\vec{b}$ to A does not then make $C(A)$ bigger.

26) If A is any 5 by 5 invertible matrix, then its column space is $\mathbb{R}^5$.

why? If A is invertible, then it has 5 pivots and $A\vec{x}=\vec{b}$ is always solvable. The column space is therefore spanned by the 5 columns, which are all linearly independent.

Problem set 3.2

1)

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow E_{21}A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow E_{32}E_{21}A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Pivot variables: $\{x_1, x_3\}$

Free variables: $\{x_2, x_4, x_5\}$

2) $ss_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad ss_2 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \qquad ss_3 = \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

3) $A\vec{x} = \vec{0} \iff \vec{x} = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \ \alpha, \beta, \gamma \in \mathbb{R}$

The nullspace contains only $\vec{x} = \vec{0}$ when there are no free variables

4) $U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$

True $N(A) = N(U) = N(R)$

9) a) False. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has one free variable.

b) True. An invertible matrix has n pivot variables and 0 free variables

c) True. There are only n total variables.

d) True. There are only m rows, and one pivot at most per row.

21) Done in class $A = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$

26) If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $C(A)$ is the line spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $N(A)$ is the line $y=0$. Both are the same line (x-axis). So $C(A) = N(A)$.

27) The rank-nullity theorem states that, for a (3×3) matrix, the dimensions of the rank and the nullity of the matrix must add up to 3. It is impossible then for the two to be equal. Therefore, $C(A) \neq N(A)$ for any (3×3) matrix

Problem set 3.3

2) a)

$$A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\div 4} R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad r = 1$$

8) $$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad r = 1$$

$$B = \begin{bmatrix} 3 & 9 & -3/2 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 3 & -3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad r = 1$$

17) a) If column j of B is a combination of previous columns of B, then $\vec{b}_j = \alpha\vec{b}_1 + \beta\vec{b}_2 + \dots + \omega\vec{b}_{j-1}$.

$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \dots & \underbrace{A\vec{b}_j} & \dots & A\vec{b}_n \end{bmatrix}$$

$$A\vec{b}_j = A(\alpha\vec{b}_1 + \beta\vec{b}_2 + \dots + \omega\vec{b}_{j-1}) = \alpha A\vec{b}_1 + \beta A\vec{b}_2 + \dots + \omega A\vec{b}_{j-1}$$

$\cancel{= \alpha A\vec{b}_1}$

The column j of AB is therefore the same combination of previous columns of AB. Then AB cannot have more pivot columns than B.
$\Rightarrow \text{rank}(AB) \leq \text{rank}(B)$.

b) If $A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $A_2B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\text{rank}(A_2) = 0$

If $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_1B = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ and $\text{rank}(A_1) = 1$

Problem set 3.4

4)
$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - \frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 1/2 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(left matrix: A | $\vec{b}$; right matrix: R | $\vec{d}$)

$$\vec{x}_p = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} \qquad \vec{ss}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{ss}_2 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{x}_G = \vec{x}_p + \alpha\vec{ss}_1 + \beta\vec{ss}_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + \alpha\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \ \alpha, \beta \in \mathbb{R}$$

6) a)
$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{array}\right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 2R_1 \\ R_4 - 3R_1}} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \end{array}\right]$$

~~Solvability conditions~~:
$\cancel{b_2 - 2b_1 = 0}$
$\cancel{b_4 - 3b_1 = 3(b_3 - 2b_1)}$

$$\left[\begin{array}{cc|c} 1 & 0 & 5b_1 - 2b_3 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_4 + 3b_1 - 3b_3 \end{array}\right]$$

$\cancel{b_2 = 2b_1}$
$\cancel{b_4 = 3(b_3 - b_1)}$

Solvability conditions: $\begin{cases} b_2 - 2b_1 = 0 \\ b_4 + 3b_1 - 3b_3 = 0 \end{cases}$. If both conditions are met, there is only one solution:

$$\vec{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix}$$

13) a) The complete solution is the particular solution plus any linear combination of the null space solutions.

$$\vec{x} = \vec{x_p} + \vec{x_n} \Rightarrow A\vec{x} = \underbrace{A\vec{x_p}}_{\vec{b}} + \underbrace{A\vec{x_n}}_{\vec{0}} = \vec{b}$$

b) A system can have infinitely many particular solutions. If it has nullspace solutions, then the sum of $\vec{x_p}$ and any linear combination of the nullspace solutions $\vec{x_n}$ will also be a particular solution.

c) If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we get $R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

we can find two particular solutions:

- the one with the free variable set to 0 : $\vec{x_{p_0}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- another one : $\vec{x_{p_1}} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

We have $\|\vec{x_{p_0}}\| = 1$ and $\|\vec{x_{p_1}}\| = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} < 1$

d) If A is invertible, then $\vec{x_n} = \vec{0}$ is a solution in the nullspace.

22) If $A\vec{x} = \vec{b}$ has infinitely many solutions, then we can write $A\vec{x_1} = \vec{b}$ and $A\vec{x_2} = \vec{b}$, where $\vec{x_1}$ and $\vec{x_2}$ are two solutions.

$\Rightarrow A\vec{x_1} = A\vec{x_2} \Leftrightarrow A(\vec{x_1} - \vec{x_2}) = \vec{0}$.

If $A\vec{x} = \vec{B}$ has a solution $\vec{x_0}$, then $A\vec{x_0} = \vec{B}$.

$$\Rightarrow \underbrace{A\vec{x_0}}_{\vec{B}} + \underbrace{A(\vec{x_1} - \vec{x_2})}_{\vec{0}} = \vec{B}$$

So $\vec{x_0} + (\vec{x_1} - \vec{x_2})$ is also a solution to $A\vec{x} = \vec{B}$.

However, if $\vec{B} \notin C(A)$, then $A\vec{x} = \vec{B}$ has no solution

Problem

When we multiply an $(m \times n)$ matrix and an $(n \times r)$ matrix, the resulting matrix is $(m \times r)$. For each element in this new matrix, we make $n.m.r$ multiplications and $(n-1).m.r$ additions.

If A is (2×4), B is (4×7), and C is (7×10), we need $4.2.7 = 56$ multiplications to get AB (2×7), and an additional $7.2.10 = 140$ multiplications to get $(AB)C$. In total, we therefore need $56 + 140 = 196$ multiplications to get ABC.