MSAN 504 — Probability/Statistics — Summer 2016
Homework One

1. Flip a coin three times. Let X be the number of heads observed. Write down a probability mass function $p(x)$ that characterizes the way that X allocates probability to the integers $0, 1, 2$ and 3.

2. The density function of a normal random variable is given by the function `dnorm` in R. You should set the `mean` option to zero, the `sd` option to one, and the `log` option to false. Verify that the function is strictly positive and that when we use R to integrate it over the whole real number line, it integrates to one. Report the code that you used in your homework write-up.

3. F, the distribution function of a random variable X, is given by

$$F(t) = \begin{cases} 0 & t < -1 \\ (1/4)t + 1/4 & -1 \leq t < 0 \\ 1/2 & 0 \leq t < 1 \\ (1/12)t + 7/12 & 1 \leq t < 2 \\ 1 & t \geq 2 \end{cases}$$

(a) Sketch the graph of F.
(b) Calculate the following quantities: $\mathbb{P}(X < 1)$, $\mathbb{P}(X = 1)$, $\mathbb{P}(1 \leq X < 2)$, $\mathbb{P}(X > 1/2)$, $\mathbb{P}(X = 3/2)$, and $\mathbb{P}(1 < X \leq 6)$.

4. Determine if the following is a distribution function: $F(t) = t/(1+t)$ if $t \geq 0$ and 0 if $t < 0$.

5. Let $p(x) = (3/4)(1/4)^x$ for $x = 0, 1, 2, 3, ...$ be the probability mass function of some random variable X. Find F, the distribution function of X, and sketch its graph. (**Hint:** Adding up the probabilities in question will be equivalent to finding the sum of a geometric series.)

6. If X is a random number selected from the first 10 strictly positive integers, what is $\mathbb{E}[X(11 - X)]$?

7. Find the variance and the standard deviation of a random variable X with distribution function

$$F(x) = \begin{cases} 0 & x < -3 \\ 3/8 & -3 \leq x < 0 \\ 3/4 & 0 \leq x < 6 \\ 1 & x \geq 6 \end{cases}$$

8. Suppose that X is a discrete random variable with $\mathbb{E}[X] = 1$ and $\mathbb{E}[X(X-2)] = 3$. Find $\mathrm{Var}(-3X+5)$.

9. A graduate class consists of six students. What is the probability that exactly three of them are born either in April or in October?

10. Suppose that 2.5% of the population of a border town are illegal immigrants. Find the probability that, in a theater of this town with 80 random viewers, there are at least two illegal immigrants.

11. The probability is p that a randomly-chosen light bulb is defective. We screw a bulb into a lamp and switch on the current. If the bulb works, we stop; otherwise, we try another and continue until a good bulb is found. What is the probability that at least n bulbs are required?

12. The time it takes for a student to finish an aptitude test (in hours) has a density function of the form $f(x) = c(x-1)(2-x)$ if $1 < x < 2$ and 0 otherwise.

(a) Calculate the value of c that causes f to be a proper density function.
(b) Calculate the distribution function of the time it takes for a randomly-selected student to finish the aptitude test.
(c) What is the probability that a student will finish the aptitude test in less than 75 minutes? Between 1.5 and 2 hours?

13. Consider again the density function mentioned in problem #12. Determine the mean and standard deviation of the time it takes for a randomly-selected student to finish the aptitude test.

14. Write pseudocode, motivated by the Method of Inverse Transformation, to simulate realizations from a geometric random variable with success parameter p.

15. A discrete random variable is said to be Poisson if its probability mass function is equal to $p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$ for $x = 0, 1, 2, 3, \ldots$. (a) Prove that this probability mass function is indeed a proper probability mass function. (b) Prove the following important recursive relationship: that $p(x+1) = \frac{\lambda}{x+1} p(x)$.

16. Using R and, by doing Internet research, implement an algorithm that generates random variables that are governed by a Poisson distribution. The best algorithm exploits the recursive relationship you proved in problem #15. Let your deliverable be the following: the code that you used to implement the simulation and a histogram of 1000 realizations from a Poisson distribution with $\lambda = 3$.