MSAN 504 — Probability/Statistics — Summer 2016
Homework Five

1. Let $X_1, ...X_n$ be independent exponential random variables with **different** means $1/\lambda_1, 1/\lambda_2, ..., 1/\lambda_n$. Find the cumulative distribution function of $Z_1 = \min\{X_1, ..., X_n\}$ and $Z_2 = \max\{X_1, ..., X_n\}$. (**Hint:** Use the definition of the cumulative distribution function!)

2. Let $X_1, ...X_n$ be independent geometric random variables with common success parameter p. Find the cumulative distribution function of $Z_1 = \min\{X_1, ..., X_n\}$ and $Z_2 = \max\{X_1, ..., X_n\}$.

3. Let X and Y be continuous random variables with joint probability density function given by $f(x, y) = x + y$ if $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and 0 otherwise. Calculate $f_{X|Y}(x|y)$, compute $\mathbb{P}(0.25 < X < 0.5|Y = 0.2)$, and then determine $\mathbb{E}[X|Y = y]$.

4. Let the joint probability density of X and Y be bivariate normal. For what values of α is the variance of $\alpha X + Y$ minimized?

5. Let X be the height of a man and Y the height of his daughter (both in inches). Suppose that the joint probability density function of X and Y is bivariate normal with the following parameters: $\mu_X = 71$, $\mu_Y = 60$, $\sigma_X = 3$, $\sigma_Y = 2.7$, and $\rho = 0.45$. Find the probability that the height of the daughter, of a man who is 70 inches tall, is at least 59 inches.

6. Let $\mathbf{X}$ be a multivariate Gaussian random vector with mean vector $\boldsymbol{\mu} = [2, -3, 1]$ and variance-covariance matrix $\boldsymbol{\Sigma}$, which is given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

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Suppose that $Y = 3X_1 - 2X_2 + X_3$, where X_i is the i^{th} component of $\mathbf{X}$. What is the distribution of Y?

7. Let $X_1, ..., X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Solve the MLE problem simultaneously for both μ and σ.

8. Show that the two components of your answer to problem #7, i.e., $\widehat{\mu}$ and $\widehat{\sigma}^2$, are independent. (Arguably, your answer to problem #9 will immediately solve #8, but in case you are unable to do problem #9, there is another way to do problem #8.)

9. Use the theorem we stated at the end of class on Tuesday to determine the asymptotic distribution of the vector $(\widehat{\mu}, \widehat{\sigma}^2)$. You may need to look up a clean version of the multidimensional verson of this theorem. **Hint:** In this case, the inverse of the Fisher information

will be the inverse of a particular matrix. The form of that matrix is given by the picture below! You'll need to compute a matrix of second-order partial derivatives of the log-likelihood function, take the expected values of the entries in that matrix after plugging in the random vector $\mathbf{X}$, multiply by -1, and then do a matrix inversion.

THEOREM 14.4 Cramér–Rao Lower Bound

Assuming that the density of y_i satisfies the regularity conditions R1–R3, the asymptotic variance of a consistent and asymptotically normally distributed estimator of the parameter vector $\boldsymbol{\theta}_0$ will always be at least as large as

$$[\mathbf{I}(\boldsymbol{\theta}_0)]^{-1} = \left(-E_0\left[\frac{\partial^2 \ln L(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}_0\, \partial \boldsymbol{\theta}_0'}\right]\right)^{-1} = \left(E_0\left[\left(\frac{\partial \ln L(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}_0}\right)\left(\frac{\partial \ln L(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}_0}\right)'\right]\right)^{-1}.$$