

## Motivation:

- Analyze, model and **predict** data that is observed in a sequential order
- Data is no longer independent, and so standard inferential procedures don't work anymore/are invalid
- Decompose dependent data into independent components
- We care less about finding relationships between a response variable and covariates. We typically want to forecast a response using just its past values.

*Regression\_Example.R* and *ConsIndex.txt*

## Definitions

An **observed time series**  $\{x_t : t \in T\}$  is a collection of observations of a variable of interest over time.

A **time series** is a stochastic process indexed by time. Specifically, we have a sequence of random variables  $\{X_t : t \in T\}$ , where  $T$  is an index of time points.

- if  $T$  is a discrete set, i.e.  $T = \{1, 2, 3, \dots\}$ , then  $\{X_t\}$  is a **discrete time series**.
- if  $T$  is a continuous interval, i.e.  $T = \{t > 0\}$ , then  $\{X_t\}$  is a **continuous time series**.

A **time series model** is the specification of the joint distribution of the random variables  $\{X_t : t \in N\}$ :  $P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n)$  for  $-\infty < x_1, x_2, \dots, x_n < \infty$  and  $n \in N$ . But, in general, we can't hope to estimate all of the parameters in such a model with the data we've observed.

**But**, most of the information about a distribution is contained in the first two moments:

- First Moments:  $E[X_t]$ ,  $t = 1, 2, \dots$  -> *means*
- Second Moments:  $E[X_t X_{t+h}]$ ,  $t = 1, 2, \dots$  and  $h = 0, 1, 2, \dots$  -> *variances/covariances*

Main take-away: we don't need the whole joint distribution. Our modeling will be based on **second-order properties**.

$\{x_t\}$  —observed from—>  $\{X_t\}$

## Zero Mean Models

### IID Noise

If  $\{X_1, X_2, \dots, X_k\}$  are iid random variables with  $E[X_t] = 0$ ,  $t = 1, 2, \dots, k$ , then  $P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) = \text{independence} = \prod_{t=1}^k P(X_t \leq x_t) = \text{identically distributed} = \prod_{t=1}^k F(x_t)$ . In this special case, the joint distribution is defined by one marginal distribution with zero mean.

### White noise

A white noise process is a sequence of **uncorrelated** (not necessarily independent!) random variables  $\{X_t\}$  each with mean 0, and finite variance  $\sigma^2$ .

We denote this by  $\{X_t\} \sim WN(0, \sigma^2)$ .

- $E[X_t] = 0$
- $Var(X_t) = \sigma^2$  finite

- $Cov(X_i, X_j) = 0$  for  $i \neq j$

(note: IID noise is a subset of White noise)

## Classical Time Series Decomposition

$$X_t = m_t + s_t + \epsilon_t$$

- $m_t$ : trend term (average change in  $X_t$  over time)
- $s_t$ : seasonal term (regular periodic fluctuations)
- $\epsilon_t$ : error (unexplained variation in  $X_t$ 's)

Lecture 1.pptx

### Example

Consider average seasonal temperature over many years where we wish to fit a model of the form  $X_t = m_t + s_t + \epsilon_t$ .

Here, we assume  $m_t$  is a polynomial in  $t$ , and  $s_t$  can be represented with indicator/dummy variables:

- $W_1 = \begin{cases} 1 & \text{if spring} \\ 0 & \text{otherwise} \end{cases}$
- $W_2 = \begin{cases} 1 & \text{if fall} \\ 0 & \text{otherwise} \end{cases}$
- $W_3 = \begin{cases} 1 & \text{if winter} \\ 0 & \text{otherwise} \end{cases}$

$$X_t = \sum_{i=0}^p \beta_i t^i + \sum_{j=1}^3 \alpha_j W_j + \epsilon_t, \epsilon_t \sim N(0, \sigma^2) \text{ (iid)}$$

- We typically estimate  $\alpha$ 's and  $\beta$ 's using OLD, which implies that we are making OLS assumptions (which still may not be valid).
- If the assumptions are invalid, then we use the **Box-Jenkins** class of models (i.e. AR, MA, ARMA, SARIMA)

AirPassengers Analysis.R

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## Recap

- Time series  $\rightarrow \{X_t : t \in N\}$   $\leftarrow$  a time series model puts constraints on the first and second moments of these random variables.
- Observed time series  $\rightarrow \{x_t : t \in N\}$ .

# Stationarity

## Strict stationarity

A time series  $\{X_t\}$  is said to be **strictly stationary** if the joint distribution of  $X_{t_1}, X_{t_2}, \dots, X_{t_n}$  is the same as that of  $X_{t_1+h}, X_{t_2+h}, \dots, X_{t_n+h}$  for all  $n, h, t_1, t_2, \dots, t_n \in N$ .

i.e., a strictly stationary time series preserves **all** statistical properties over time shift.

### Problems:

- We often can't specify the joint distribution of these random variables and so this assumption is usually impossible to check.
- Also, this assumption tends to be too strict and is not often met.

This motivates the need for a weaker version of stationarity.

But first...

Let  $\{X_t\}$  be a time series.

- The **mean function** of  $\{X_t\}$  is  $\mu_X(t) = E(X_t)$ ,
- The **covariance function** of  $\{X_t\}$  is  $\gamma_X(r, s) = Cov(X_r, X_s) = E(X_r X_s) - \mu_X(r)\mu_X(s)$ .

## Weak stationarity

A time series  $\{X_t\}$  is **weakly stationary** if  $E(X_t^2) < \infty$  and:

- $\{X_t\}$  is  $\mu_X(t) = E(X_t)$  is independent of  $t$ ,
- $\gamma_X(t, t+h) = Cov(X_t, X_{t+h})$  is independent of  $t$  for all  $h$ .
  - covariance depends on  $h$  but not  $t$

### Remarks:

- Strict stationarity  $\implies$  weak stationarity
- From now on, "stationarity" means *weak* stationarity
- For a stationary time series  $\{X_t\}$ :
  - $E(X_t) = \mu_X$
  - $Cov(X_t, X_{t+h}) = \gamma_X(t, t+h) = \gamma_X(0, h) = \gamma_X(h)$

## Definitions

Let  $\{X_t\}$  be a stationary time series.

- The **autocovariance function** (ACVF) of  $\{X_t\}$  at **lag**  $h$  is  $\gamma_X(h)$ .
- The **autocorrelation function** (ACF) of  $\{X_t\}$  at **lag**  $h$  is  $\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = Corr(X_t, X_{t+h})$ .
- $\gamma_X(h) = \gamma_X(-h)$ .

$$\begin{aligned} \text{(Reminder: } Corr(X, Y) &= \frac{Cov(X, Y)}{SD(X)SD(Y)} = \frac{Cov(X_t, X_{t+h})}{\sqrt{Var(X_t)Var(X_{t+h})}} = \frac{Cov(X_t, X_{t+h})}{\sqrt{Cov(X_t, X_t)Cov(X_{t+h}, X_{t+h})}} = \\ &= \frac{Cov(X_t, X_{t+h})}{\sqrt{\gamma_X(0)\gamma_X(0)}} = \frac{\gamma_X(h)}{\gamma_X(0)} \end{aligned}$$

## Examples

### First Order Autoregression: AR(1)

Assume  $\{X_t\}$  is a stationary time series satisfying the equations

$$X_t = \phi X_{t-1} + Z_t$$

for  $t \in \mathbb{Z}$ ,  $|\phi| < 1$  and  $Z_t \sim WN(0, \sigma^2)$ . Also assume  $Z_t$  and  $X_s$  are uncorrelated for all  $s < t$ . Calculate the ACVF and ACF of  $\{X_t\}$ .

- $E(X_t) = \phi E(X_{t-1}) + E(Z_t) \rightarrow E(X_t) = \phi E(X_{t-1}) \rightarrow E(X_t) = 0$  since  $\{X_t\}$  is stationary.
- $\gamma_X(h) = Cov(X_t, X_{t-h}) = E(X_t X_{t-h}) = E(\phi X_{t-1} X_{t-h} + Z_t X_{t-h}) = \phi E(X_{t-1} X_{t-h}) + E(Z_t X_{t-h}) = \phi E(X_{t-1} X_{t-h}) = \phi \gamma_X(h-1) = \phi^h \gamma_X(0)$  (assume  $h > 0$ ).

By stationarity,  $\gamma_X(h) = \gamma_X(-h)$  so  $\gamma_X(h) = \phi^{|h|} \gamma_X(0)$ .

- $\gamma_X(0) = Cov(X_t, X_t) = E(X_t^2) = E(\phi^2 X_{t-1}^2 + 2\phi X_{t-1} Z_t + Z_t^2) = \phi^2 E(X_{t-1}^2) + 2\phi E(X_{t-1} Z_t) + E(Z_t^2) = \phi^2 \gamma_X(0) + \sigma^2 \Rightarrow \gamma_X(0) = \frac{\sigma^2}{1-\phi^2}$

$$\therefore \gamma_X(h) = \frac{\phi^{|h|} \sigma^2}{1-\phi^2} \text{ for } h \in \mathbb{Z}$$

$$\therefore \rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \phi^{|h|} \text{ for } h \in \mathbb{Z}$$

**ACF signature for AR(1) is exponential decay.**

### First Order Moving Average: MA(1)

Consider process  $X_t = Z_t + \theta Z_{t-1}$  where  $t \in \mathbb{N}$  and  $\{Z_t\} \sim WN(0, \sigma^2)$  and  $\theta \in \mathbb{R}$ . Show  $\{X_t\}$  is stationary and derive its ACF.

- $\mu_X = E(X_t) = E(Z_t) + \theta E(Z_{t-1}) = 0$  for all  $t$ .
- $\gamma_X(h) = Cov(X_t, X_{t+h}) = Cov(Z_t + \theta Z_{t-1}, Z_{t+h} + \theta Z_{t+h-1}) = Cov(Z_t, Z_{t+h}) + \theta Cov(Z_t, Z_{t+h-1}) + \theta Cov(Z_{t-1}, Z_{t+h}) + \theta^2 Cov(Z_{t-1}, Z_{t+h-1})$

$$\gamma_X(h) = \begin{cases} \sigma^2(1 + \theta^2) & \text{if } h = 0 \\ \theta \sigma^2 & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases} \quad \leftarrow \text{independent of } t.$$

(Reminder:  $Cov(X + Y, W + Z) = Cov(X, W) + Cov(X, Z) + Cov(Y, W) + Cov(Y, Z)$ )

$\therefore \{X_t\}$  is stationary.

- $\gamma_X(0) = \sigma^2(1 + \theta^2)$

and  $\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1 & \text{if } h = 0, \\ \frac{\theta}{1+\theta^2} & \text{if } h = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$

**ACF signature of MA(1) is a spike for  $h = 0, 1$  and then nothing for  $h > 1$ .**

We've seen that the ACF can provide information regarding which model may be appropriate for an observed time series. To do this in practice, we need a sample estimate of the ACF.

## Definitions

Let  $x_1, x_2, \dots, x_n$  be our observed time series.

- the **sample mean** is  $\hat{\mu}_x = \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ ,

- the **sample autocovariance** is  $\hat{\gamma}_x(h) = \frac{1}{n} \sum_{t=1}^{n-h} (x_{t+h} - \bar{x})(x_t - \bar{x})$ ,
- the **sample autocorrelation** is  $\hat{\rho}_x(h) = \frac{\hat{\gamma}_x(h)}{\hat{\gamma}_x(0)}$

Note:

- $\theta$ , a Greek letter, denotes a parameters (unknown number),
- $\hat{\theta}$ , is a sample estimate of  $\theta$  (known number),
- $\tilde{\theta}$ , is an estimator, a random variable.

The sample ACF can be used to investigate the “uncorrelatedness” in a time series. For example, we might use this to evaluate the uncorrelated assumption in residuals.

(Reminder: independence  $\Rightarrow$  uncorrelated; uncorrelated  $\nRightarrow$  independence)

For stationary time series,  $\tilde{\rho}(h) \sim N(0, \frac{1}{n})$  ( $n$  = number of data points).

Consequently, an approximate 95% confidence interval for  $\rho_x(h)$  is  $\pm \frac{1.96}{\sqrt{n}}$ .

If  $\tilde{\rho}(h)$  falls outside these limits, for any  $h$ , we judge this to be significant.

*SACF Examples.R*

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## Recap

- Autocovariance function (ACVF):  $\gamma_X(h) = Cov(X_t, X_{t-h})$  for all  $h \in Z$
- Autocorrelation function (ACF):  $\rho_X(h) = Corr(X_t, X_{t-h}) = \frac{\gamma_X(h)}{\gamma_X(0)}$ 
  - Properties of ACVF:
    - \*  $\gamma_X(0) = Var(X_t)$
    - \*  $\gamma_X(-h) = \gamma_X(h)$
    - \*  $|\rho_X(h)| \leq 1$

Why is stationarity important?

In order to build a model that forecasts with any accuracy, we require an assumptions that something doesn't vary with time. After accounting for deterministic trend and/or seasonality, we hope that the remaining randomness can be described as stationary.

In the Box-Jenkins class of models, we can use AR (autoregressive), MA (moving average), and ARMA models to model stationary time series.

First, notation:

**Backshift operator:**  $B$ , where  $BX_t = X_{t-1}$  i.e.  $B^2X_t = X_{t-2}$ .

Generally,  $B^n X_t = X_{t-n}$  and  $B^0 = I$

## MA(q) Process

A process/time series  $\{X_t\}$  is called a **moving average process of order  $q$**  if

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots + \theta_q \epsilon_{t-q}$$

where  $\{\epsilon_t\} \sim WN(0, \sigma^2)$  and  $\theta_1, \theta_2, \dots, \theta_q$  are constants.

Remarks:

- MA(q) processes are stationary (*exercise*: prove this!)
- An MA(q) process is **q-correlated** (i.e.,  $\rho_X(h) = \text{Corr}(X_t, X_{t-h}) = 0$  for  $h > q$  and not necessarily 0 for  $h \leq q$ )

Thus, the **ACF signature of an MA(q) process is non-zero spikes for  $h = 0, 1, 2, \dots, q$  and then no spikes for ever after.**

- An MA(q) process can be denoted as:  $X_t = \epsilon_t + \theta_1 B^1 \epsilon_t + \dots + \theta_q B^q \epsilon_t = (1 + \sum_{s=1}^q \theta_s B^s) \epsilon_t = \theta^q(B) \epsilon_t$

where  $\theta^q(z) = 1 + \sum_{s=1}^q \theta_s z^s$  is the **generating function**.

- An MA(q) is **invertible** if the complex roots of  $\theta^q(z)$  lie outside the unit circle. i.e. For all  $z$  such that  $\theta^q(z) = 0$ , then  $|z| > 1$ .

### Example

$$X_t = \epsilon_t + 0.2\epsilon_{t-1} + 0.7\epsilon_{t-2}$$

$$\theta(z) = 1 + 0.2z + 0.7z^2$$

$$\text{The roots of } \theta(z) \text{ are } z = \frac{-0.2 \pm \sqrt{0.2^2 - 4(0.7)(1)}}{2(0.7)} = \frac{-0.2 \pm \sqrt{2.76}i}{1.4} \Rightarrow z = -0.14 \pm 1.19i$$

$$|z| = \sqrt{(-0.14)^2 + (1.19)^2} = 1.198 > 1$$

So  $\{X_t\}$  is invertible.

Reminders:

- The zeros of a quadratic of the form  $ax^2 + bx + c$  are  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- $c = a + ib \Rightarrow |c| = \sqrt{a^2 + b^2}$

## AR(p) Process

The process  $\{X_t\}$  is called an **autoregressive process of order p** if

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots + \phi_p X_{t-p} + \epsilon_t$$

where  $\epsilon_t \sim WN(0, \sigma^2)$  and  $\phi_1, \phi_2, \dots, \phi_p$  are constants.

- An AR(p) process can be denoted as:

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \dots - \phi_p X_{t-p} = \epsilon_t$$

$$\Leftrightarrow X_t - \phi_1 B^1 X_t - \phi_2 B^2 X_t - \dots - \phi_p B^p X_t = \epsilon_t$$

$$\Leftrightarrow (1 - \sum_{r=1}^p \phi_r B^r) X_t = \epsilon_t$$

$$\Leftrightarrow \phi^p(B) X_t = \epsilon_t$$

where  $\phi^p(z) = 1 - \sum_{r=1}^p \phi_r z^r$  is the **generating function**.

- An AR(p) process is **stationary** if the complex roots of  $\phi^p(z)$  lie outside the unit circle. i.e. For all  $z$  such that  $\phi^p(z) = 0$ , we require  $|z| > 1$ .

### Example

$$X_t = \phi X_{t-1} + \epsilon_t \Rightarrow (1 - \phi B)X_t = \epsilon_t$$

$$\phi(z) = 1 - \phi z \Rightarrow \phi(z) = 0 \text{ if } z = \frac{1}{\phi}$$

For stationarity, we need  $|z| > 1 \Rightarrow |\frac{1}{\phi}| > 1 \Rightarrow |\phi| > 1$ .

## Partial Autocorrelation Function(PACF)

For a stationary process, the ACF of lag  $h$  measures the correlation between  $X_t$  and  $X_{t+h}$ . This correlation could be due to a direct connection between  $X_t$  and  $X_{t+h}$ , but it may also be influenced by observations at intermediate lags:  $X_{t+1}, X_{t+2}, \dots, X_{t+h-1}$ .

The PACF of lag  $h$  measures the correlation between  $X_t$  and  $X_{t+h}$  once the influence of the intermediate lags has been removed/accounted/controlled for.

We remove this effect using **linear predictors**:

$$\hat{X}_t = \text{Pred}(X_t | X_{t+1}, X_{t+2}, \dots, X_{t+h-1})$$

$$\hat{X}_{t+h} = \text{Pred}(X_{t+h} | X_{t+1}, X_{t+2}, \dots, X_{t+h-1})$$

where this prediction is commonly based on a linear regression.

Thus, for a stationary time series  $\{X_t\}$ , the **partial autocorrelation function of lag  $h$**  is:  $\alpha_X(h) =$

$$\begin{cases} \text{Corr}(X_t, X_t) = 1, & \text{if } h = 0 \\ \text{Corr}(X_t, X_{t+1}) = \rho_X(1), & \text{if } h = 1 \\ \text{Corr}(X_t, X_{t+h}) = \text{Corr}(X_t - \hat{X}_t, X_{t+h} - \hat{X}_{t+h}) & \text{if } h > 1 \end{cases}$$

(assume without loss of generality that  $h \geq 0$ )

### Example

Derive the PACF of an AR(1) process  $X_t = \phi X_{t-1} + \epsilon_t$ .

$$\alpha_X(h) = \begin{cases} 1 & \text{if } h = 0 \\ \rho(1) = \phi & \text{if } h = 1 \end{cases}$$

If  $h = 2$ :

$$\begin{aligned} \bullet \quad \alpha(2) &= \text{Corr}[X_t - \hat{X}_t, X_{t+2} - \hat{X}_{t+2}] = \text{Corr}[X_t - f(X_{t+1}), X_{t+2} - \phi X_{t+1}] = \text{Corr}[X_t - f(X_{t+1}), \epsilon_{t+2}] = \\ &= \text{Corr}[X_t, \epsilon_{t+2}] - \text{Corr}[f(X_{t+1}), \epsilon_{t+2}] = 0 - 0 = 0 \end{aligned}$$

We can see that  $\alpha(h) = 0$  for any  $h \geq 2$ .

So **PACF for an AR(1) has non-zero spikes for  $h = 0, 1$  and is zero for all  $h \geq 2$ .**

Remarks:

- If  $\{X_t\} \sim AR(p)$ , then the PACF satisfies  $\alpha(h) = 0$  for all  $h > p$  and  $\alpha(h) \neq 0$  necessarily for  $h \leq p$ .
- Whereas an ACF can be used to determine the order of an MA process, a PACF can be used to determine the order of an AR process.

## ARMA(p,q) Process

$\{X_t\}$  is an **autoregressive moving average process of orders p and q** if

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \dots - \phi_p X_{t-p} = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots + \theta_q \epsilon_{t-q}$$

$$\phi^p(B)X_t = \theta^q(B)\epsilon_t$$

where  $\{\epsilon_t\} \sim WN(0, \sigma^2)$  and  $\phi^p(z)$  and  $\theta^q(z)$  are the AR and MA generating functions, and **we require them to have distinct roots**.

Remark:

- ARMA(p, 0) = AR(p)
- ARMA(0, q) = MA(q)

**Example: ARMA(1,2)**

$$\phi^1(B)X_t = \theta^2(B)\epsilon_t \Rightarrow (1 - \phi B)X_t = (1 + \theta_1 B + \theta_2 B^2)\epsilon_t \Rightarrow X_t - \phi X_{t-1} = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$$

We require  $\begin{cases} \phi^1(z) = 1 - \phi z \\ \theta^2(z) = 1 + \theta_1 z + \theta_2 z^2 \end{cases}$

	ACF	PACF
MA(q)	Spike for $h \leq q$ and negligibly small spikes for $h > q$	Exponential decay
AR(p)	Exponential decay	Spikes for $h \leq p$ and "nothing" for $h > p$
ARMA(p,q)	q spikes then decay	p spikes then decay

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## ARMA Stationarity and Invertibility Conditions

**ARMA(p,q):**  $\phi(B)X_t = \theta(B)\epsilon_t$ ,  $\{\epsilon_t\} \sim WN(0, \sigma^2)$

where  $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$

and  $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$ .

$\{X_t\} \sim ARMA(p, q)$  is stationary if

- $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p \neq 0$  for all  $z$  such that  $|z| \leq 1$  i.e., the modulus of all roots/zeros are  $> 1$  (the complex roots of this generating function lie outside the unit circle in the complex plane).

A **causal** ARMA process is stationary.

$\{X_t\} \sim ARMA(p, q)$  is invertible if

- $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q \neq 0$  for all  $z$  such that  $|z| \leq 1$  i.e., the modulus of all roots/zeros are  $> 1$  (the complex roots of this generating function lie outside the unit circle in the complex plane).

This criterion is equivalent to requiring that  $\epsilon_t$  can be written as an infinite weighted sum of the  $X_t$ 's.

**Note:** we require an MA(q) process, or the MA component of an ARMA(p,q) process to be invertible so that

- (i) we can estimate the  $\theta$ 's in the model



- (ii) we can forecast with the model

For the sake of usefulness, we'll restrict attention to ARMA(p,q) models that are stationary/causal and invertible.

### Example (quiz question)

#### ARMA(2,1)

$$\phi^2(B)X_t = \theta^1(B)\epsilon_t$$

- Represent this in “expanded notation”

$$(1 - \phi_1 B - \phi_2 B^2)X_t = (1 + \theta B)\epsilon_t$$

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = \epsilon_t + \theta \epsilon_{t-1}$$

$$\text{Let } \phi_1 = 0.75, \phi_2 = -0.5625, \theta = 1.25$$

- Is  $\{X_t\}$  stationary?

$$\phi^2(z) = 1 - 0.75z + 0.5625z^2$$

$$\Rightarrow \phi^2(z) = 0 \Leftrightarrow z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0.75 \pm \sqrt{0.75^2 - 4(0.5625)(1)}}{2(0.5625)} = 2\left(\frac{1 \pm \sqrt{-3}}{3}\right) = \frac{2 \pm 2i\sqrt{3}}{3}$$

$$\Rightarrow z_1 = \frac{2}{3} - \frac{2\sqrt{3}}{3}i, z_2 = \frac{2}{3} + \frac{2\sqrt{3}}{3}i$$

$$\Rightarrow |z_1| = |z_2| = \sqrt{(2/3)^2 + (2\sqrt{3}/3)^2} = \frac{4}{3}$$

$$\text{So } |z_1| = |z_2| > 1.$$

$\therefore$  this ARMA(2,1) process is stationary.

- Is  $\{X_t\}$  invertible?

$$\theta(z) = 1 + 1.25z$$

$$\Rightarrow \theta(z) = 0 \Leftrightarrow 1 + 1.25z = 0 \Leftrightarrow z = -0.8$$

$$\Rightarrow |z| = 0.8 < 1.$$

$\therefore$  this ARMA(2,1) process is **not** invertible.

**Note:** also, the roots of the polynomials are **distinct**, so the process is in fact an ARMA(2,1) process, and not a simpler one.

In practice, with an observed time series, we decide whether it is stationary and/or invertible with “**unit root tests**”.

### “Box-Jenkins Approach”

- **Identification:** identify the orders of the model  $\leftarrow$  use ACF/PACF plot
- **Estimation:** estimate the parameters of the model identified in step 1
- **Verification:** ensure that the model is appropriate  $\leftarrow$  residual diagnostics

## Estimating ARMA(p,q) Models

**Goal:** estimate  $\phi_1, \phi_2, \dots, \phi_p, \theta_1, \theta_2, \dots, \theta_q, \sigma^2$  in a stationary and invertible ARMA(p,q) process:  $\phi^p(B)X_t = \theta^q(B)\epsilon_t$ .

- we assume that  $\{X_t\}$  has zero mean, or has been “mean-corrected”

These parameters are estimated using the observed time series  $\{x_1, x_2, \dots, x_n\}$ .

Many methods (cf. Chap. 5) exist for doing this, but we’ll just focus, at a high level, on Maximum Likelihood and Least Squares.

### Maximum Likelihood method for ARMA(p,q)

- We have to make distributional assumptions, which may not be valid
- We typically assume  $\{X_t\}$  is a “Gaussian” time series i.e.,  $\vec{X} = (X_1, X_2, \dots, X_n)^T \sim MVN$ 
  - This seems limiting, but in practice it’s not bad

$L(\phi_1, \phi_2, \dots, \phi_p, \theta_1, \theta_2, \dots, \theta_q, \sigma^2 | \vec{x}) = \frac{1}{(2\pi)^n |\Gamma_n|^{1/2}} \exp \left\{ -\frac{1}{2} \vec{x}^T \Gamma_n^{-1} \vec{x} \right\}$  where  $\vec{x} = (x_1, x_2, \dots, x_n)^T$  and  $\Gamma_n = Cov(\vec{X}, \vec{X}^T)$  (this is the matrix with  $\gamma_X(0)$  in the diagonal,  $\gamma_X(1)$  in the 1-subdiagonals,  $\dots$ ,  $\gamma_X(n-1)$  in the two corners)

We want to find the values of the parameters that maximize this function in light of the observed data. We typically numerically maximize  $l(\cdot)$ , the log-likelihood function to find  $(\hat{\phi}_1, \hat{\phi}_2, \dots, \hat{\phi}_p, \hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_q, \hat{\sigma}^2)$ .

The usual asymptotic properties of MLE exist here as well. This is the basis for CI calculations.

### Least Squares method for ARMA(p,q)

The goal is to minimize  $S(\phi, \theta) = S(\phi_1, \phi_2, \dots, \phi_p, \theta_1, \theta_2, \dots, \theta_q)$  rather than maximize  $L(\cdot)$ , where

$$S(\phi, \theta) = \sum_{j=1}^n \frac{(X_j - \hat{X}_j)^2}{r_{j-1}}$$

where  $E[(X_j - \hat{X}_j)^2] = \sigma^2 r_{j-1} \rightarrow$  LSE of  $\sigma^2$  is  $\hat{\sigma}^2 = \frac{S(\hat{\phi}, \hat{\theta})}{n-p-q}$  ( $r_{j-1} = Var(X_j - \hat{X}_j)$ )

LSE is good because no distributional assumptions need to be made.

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## Order Selection

- Choose  $p$  and  $q$  “optimally”.
- Use “goodness of fit” methods to compare different models.
  - $l(\hat{\theta}, \hat{\phi}) =$  maximize log-likelihood.  $\leftarrow$  we want this to be big
  - $\hat{\sigma}^2 =$  estimate of the error variance  $\sigma^2$ .  $\leftarrow$  we want this to be small
  - $AIC = -2l(\hat{\theta}, \hat{\phi}) + 2(p+q+1)$ .  $\leftarrow$  we want this to be small
- It’s sensible to consider all 3 of these, but an “optimal” model for one may not be the “optimal” model according to another.

- A disadvantage to LSE is that we don't have a likelihood function and so  $l(\hat{\theta}, \hat{\phi})$  and  $AIC$  are not available goodness of fit metrics in this case.

We can compare the fit of two models using a **likelihood ratio test (LRT)**.

$$\begin{cases} H_0 : & \text{null and alternative models fit equally well} \\ H_a : & \text{alternative model fits better than the null} \end{cases}$$

Note: the null model is simpler (has fewer parameters) relative to the alternative.

$$D = -2 \log \left( \frac{L(\text{null model})}{L(\text{alt. model})} \right) \sim \chi^2_{(m_A - m_0)}$$

null model has  $m_o$  parameters, alt.model has  $m_A$  parameters,  $m_A > m_o$ .

$$D = -2(l(\text{null model}) - l(\text{alt. model}))$$

Larger values of  $D$  lead to rejection of  $H_0$ . p-value =  $P(W \geq D) = 1 - P(W < D)$  where  $W \sim \chi^2_{(m_A - m_o)}$ .

*ARMA Fitting Example.R*

## Verification (Residual Diagnostics)

Suppose  $\{X_t\}$  is a time series and we believe an ARMA(p,q) model can model it. We'll call the **fitted value** at time  $t$   $\hat{X}_t$ . We define the residuals as

$$\hat{\epsilon}_t = X_t - \hat{X}_t \quad \text{for } t = 1, 2, \dots, n$$

If assumptions are met, the residual time series  $\{\hat{\epsilon}_t\}$  should behave like the white noise sequence that generated the ARMA(p,q) process. In particular, we should find that  $\{\hat{\epsilon}_t\}$

- have approximately zero mean
- have a constant variance
- are uncorrelated (/ independent iff  $\{\epsilon_t\} \sim IID(0, \sigma^2)$ )
- are normally distributed (if  $\{\epsilon_t\} \sim N(0, \sigma^2)$ )  $\leftarrow$  only if you are using MLE

We can either work with the residuals,  $\hat{\epsilon}_t$ , or the **standardized residuals**,  $\hat{r}_t = \frac{\hat{\epsilon}_t}{\hat{\sigma}}$  (expect the variance to be 1).

## Informal Diagnostics (plots)

- Plot  $\hat{\epsilon}_t$  vs.  $t$  (or  $\hat{r}_t$  vs.  $t$ )
  - change of variability with time? i.e. heteroskedasticity
    - \* this checks ii.
  - check whether points are scattered symmetrically around 0
    - \* this checks i.
  - systematic trends in the residuals can suggest correlation
    - \* this checks iii.
  - check for outliers (using  $\hat{r}_t$  is sensible)

- ACF of  $\hat{e}_t$  (or  $\hat{r}_t$ )
  - use this to check whether residuals seem to be correlated
    - \* this checks iii.
  - should see no significant spikes for  $h > 0$
- QQ-plot or histogram
  - use this to check whether the residuals seem normally distributed
    - \* this checks iv.

## Formal Diagnostics (hypothesis tests)

- To check  $E[\epsilon_t]$ , do a one-sample t-test of the residuals
- To check heteroskedasticity, we can use Bartlett's Test or Levene's Test
  - these tests require us to partition the data set (the residuals) into  $k$  groups. The goal is to look for homogeneity of variance among these groups
  - $$\begin{cases} H_0 : \sigma_1^2 = \sigma_2^2 = \dots = \sigma_k^2 \\ H_a : \sigma_i^2 \neq \sigma_j^2 \end{cases} \quad \text{for some } i \neq j$$
  - Bartlett's test is sensitive to non-normality, but Levene's test isn't. But if the data are normally distributed, Bartlett's test is more powerful
- To check uncorrelatedness, we're interested in testing
  - $$\begin{cases} H_0 : \rho(1) = \rho(2) = \dots = \rho(H) = 0 \\ H_a : \rho(h) \neq 0 \end{cases} \quad \text{for some } h = 1, 2, \dots, H$$
  - we prefer this test as opposed to using an ACF because it avoids the **multiple hypothesis testing problem**
  - we use “Portmanteau” tests in this case. We consider the **Ljung-Box** test
- To check for normality, use the **Shapiro-Wilk** test where
  - $$\begin{cases} H_0 : \{\hat{e}_t\} \sim N(0, \sigma^2) \\ H_a : \{\hat{e}_t\} \not\sim N(0, \sigma^2) \end{cases}$$

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*Residual Diagnostics Example.R*

## Non-stationary models

**Idea:** transform a non-stationary time series so that the new series is stationary, and hence can be modeled by an ARMA process.

## ARIMA (AutoRegressive Integrated Moving Average)

ARIMA processes model non-stationary time series that, after finitely many “differences”, become an ARMA.

If  $d$  is a non-negative integer, then  $\{X_t\}$  is an ARIMA(p,d,q) process if  $Y_t = (1 - B)^d X_t$  is a stationary ARMA(p,q) process.

This definition implies that  $\{X_t\}$  can be written as

$$\phi^*(B)X_t = \theta^q(B)\epsilon_t, \quad \{\epsilon_t\} \sim WN(0, \sigma^2)$$

where  $\phi^*(B) = (1 - B)^d \phi^p(B)$

$$\begin{aligned} \{Y_t\} \sim ARMA(p, q) &\Rightarrow \phi^p(B)Y_t = \theta^q(B)\epsilon_t \\ &\Rightarrow \phi^p(B)(1 - B)^d X_t = \theta^q(B)\epsilon_t \end{aligned}$$

$\phi^*(z) = (1 - z)^d \phi^p(z) \Rightarrow z = 1$  is a root of  $\phi^*(z)$

$\Rightarrow \{X_t\}$  is not stationary unless  $d = 0$ , in which case  $\{X_t\} \sim ARMA(p, q)$ .

### What is differencing? (it allows us to model trend)

Notation:  $\nabla = (1 - B)$  and  $\nabla^d = (1 - B)^d$

$$\nabla X_t = (1 - B)X_t = X_t - BX_t = X_t - X_{t-1}$$

$$\nabla^2 X_t = (1 - B)^2 X_t = (1 - 2B + B^2)X_t = X_t - 2BX_t + B^2 X_t = X_t - 2X_{t-1} + X_{t-2}$$

Remark: if  $\{X_t\}$  exhibits polynomial trend of the form  $m_t = \sum_{i=0}^d \alpha_i t^i$ , the time series  $Y_t = \nabla^d X_t = (1 - B)^d X_t$  will no longer have a trend component.

### Examples

a)  $X_t = c + bt + \epsilon_t$

$$\nabla X_t = (1 - B)X_t = (1 - B)(c + bt + \epsilon_t) = c + bt + \epsilon_t - B(c + bt + \epsilon_t) = c + bt + \epsilon_t - c - b(t - 1) - \epsilon_{t-1} = b + \nabla \epsilon_t$$

b)  $X_t = c + bt + at^2 + \epsilon_t$

$$\begin{aligned} \nabla^2 X_t &= (1 - B)^2 X_t = (1 - 2B + B^2)X_t = (1 - 2B + B^2)(c + bt + at^2 + \epsilon_t) \\ &= (c + bt + at^2) - 2(c + b(t - 1) + a(t - 1)^2) + (c + b(t - 2) + a(t - 2)^2) + \nabla^2 \epsilon_t \\ &= \dots \\ &= 2a + \nabla^2 \epsilon_t \end{aligned}$$

In these cases, and in all cases generally, after an appropriate number of differences  $d$ ,  $\nabla^d X_t$  no longer has any trend.

Once we've identified a suitable value of  $d$ , we model the resulting time series with an ARMA(p,q) process.

### But how do we choose $d$ ?

- Informally: plot of the time series or ACF plot
  - difference until the transformed time series “looks” stationary
  - difference until the transformed time series exhibits rapid decay (as opposed to slow decay)
- Formally: **unit root tests**

- tests that identify whether an observed time series has a root on or “near” the unit circle
- if the AR component has a unit root, then the time series needs to be differenced

### Dickey-Fuller Test

Motivate with AR(1):  $X_t = \phi X_{t-1} + \epsilon_t$

$$\nabla X_t = X_t - X_{t-1} = (\phi X_{t-1} + \epsilon_t) - X_{t-1} = (\phi - 1)X_{t-1} + \epsilon_t = \phi^* X_{t-1} + \epsilon_t$$

Using OLS, we test  $H_0 : \phi^* = 0$  ( $\Leftrightarrow \phi = 1$ ) vs  $H_a : \phi^* \neq 0$  ( $\Leftrightarrow \phi \neq 1$ )

So the null hypothesis in the DF-test is that the time series is non-stationary. So if we reject  $H_0$  (p-value  $< \alpha$ ), we conclude the time series is stationary.

- if we reject  $H_0 \rightarrow$  stop differencing
- if we fail to reject  $H_0 \rightarrow$  difference again

The **Augmented Dickey-Fuller (ADF)** generalizes this for any order  $p$  in AR(p).

This type of differencing **doesn't remove seasonality**. To remove seasonality, we need **seasonal differencing**.

*ARIMA-SARIMA Examples.R*

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## Recap

$$\nabla = (1 - B)$$

$\nabla X_t = X_t - X_{t-1} \leftarrow$  ordinary differencing removes trend but not seasonality.

### Seasonal Differencing

**Notation:**  $(1 - B^k) = \nabla_k \leftarrow$  **this is not the same as**  $\nabla^k = (1 - B)^k$ .

$$\nabla_k X_t = (1 - B^k)X_t = X_t - B^k X_t = X_t - X_{t-k}$$

We call this **lag- $k$  differencing**.

$k = 1 \Leftrightarrow$  ordinary differencing.

In R,

- $\nabla^d X_t = (1 - B)^d X_t = \text{diff}(x, \text{difference}=d)$
- $\nabla_k X_t = (1 - B^k)X_t = \text{diff}(x, \text{lag}=k)$
- $\nabla_k^D X_t = (1 - B^k)^D X_t = \text{diff}(x, \text{lag}=k, \text{difference}=D)$

**Idea:** seasonal effects  $s_t$  manifest themselves as  $s_t = s_{t+m}$ , where  $m$  is the **period** of the seasonal effect.

### Example

$$X_t = s_t + \epsilon_t, \{\epsilon_t\} \sim WN(0, \sigma^2)$$

If we take a lag- $m$  difference (**seasonal difference**), we will remove the seasonal effect:

$$\nabla_m X_t = \nabla_m s_t + \nabla_m \epsilon_t = (1 - B^m)s_t + (1 - B^m)\epsilon_t = (s_t - s_{t-m}) + (\epsilon_t - \epsilon_{t-m}) = \epsilon_t - \epsilon_{t-m} \text{ since } s_t = s_{t-m}$$

We hope that after 1 or 2 applications of seasonal differencing, we're left with a stationary time series that can be modeled by an ARMA(p,q) process.

So we use seasonal differencing to eliminate seasonal effects and ordinary differencing to eliminate trend. Order of differencing does not matter.

### But how do we choose $m$ ?

We can use an ACF plot, which will exhibit seasonality, by counting the number of lags between “peaks”. This number corresponds to the period.

### SARIMA (Seasonal AutoRegressive Integrated Moving Average)

If  $Y_t = (1 - B)^d(1 - B^s)^D X_t$  is stationary, and hence can be modeled by an ARMA(p,q) process, we say  $\{X_t\}$  is SARIMA. In particular,

$$\{X_t\} \sim SARIMA(p, d, q) \times (P, D, Q)_s$$

where:

- $s$  is the period of the seasonal effect
- $d$  is the number of ordinary differences necessary to remove trend
- $D$  is the number of seasonal differences necessary to remove seasonality
- $p, q$  are the AR and MA orders of the **within-period** process
- $P, Q$  are the AR and MA orders of the **between-period** process

**Typically,  $d, D = 1$  or  $2$  and  $p, P, q, Q < 3$**

We can write this in generating function notation as follows:

$$\phi^p(B)\Phi^P(B^s)Y_t = \theta^q(B)\Theta^Q(B^s)\epsilon_t$$

where  $\{\epsilon_t\} \sim WN(0, \sigma^2)$

**Idea:**

- the data within a period is a time series,
- the data between periods is a time series,
- these two time series have different ARMA representations.

### Order Selection

**Step 1:** Choose  $d, D, s$  such that  $Y_t = (1 - B)^d(1 - B^s)^D X_t$  is stationary.

**Step 2:** Examine the ACF and PACF plots of  $\{Y_t\}$  for  $p, P, q, Q$ .

- $p$  and  $q$  are chosen such that  $\rho(1), \rho(2), \dots, \rho(s-1)$  and  $\alpha(1), \alpha(2), \dots, \alpha(s-1)$  are compatible with ARMA(p,q).
- $P$  and  $Q$  are chosen such that  $\rho(k s)$  and  $\alpha(k s)$  where  $k = 1, 2, \dots$  are compatible with ARMA(P,Q).

**Step 3:** Fit models in the neighborhood of these choices and use goodness of fit metrics to choose an optimal model.

## Full Box-Jenkins Approach

1. Check for non-constance variance and apply transformation if necessary.
2. Check for seasonal and non-seasonal trends.
3. Use differencing to make the time series stationary.
4. Identify  $p, P, q, Q$  from the ACF and PACF plots of the differenced data.
5. Fit the proposed model and iterate to an optimal one.
6. Check residuals to assess assumptions.
7. Forecast into the future.

*ARIMA-SARIMA Examples.R*

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*chemical.txt*

$$\text{RMSE} = \sqrt{\frac{\sum_{t=1}^n (x_t - \hat{x}_t)^2}{n}}$$

*wine.txt*

**Homework: look at 2 datasets**

- USAccDeaths
- ldeaths

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11/17/16