Name: Solutions

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Fall 2016 MSAN 604: Quiz 2

Thursday November 3rd, 2016

Question 1

(a) Consider the stationary time series $\{X_t\}$, and assume that it can be suitably modeled by an ARMA process with the following structural relationship:

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \phi_3 X_{t-3} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$$

Write this structural relationship in the form $\phi^p(B)X_t = \theta^q(B)\varepsilon_t$, where B is the back-shift operator. Be sure to identify p and q, and appropriately define the generating functions $\phi^p(B)$ and $\theta^q(B)$.

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$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \phi_3 X_{t-3} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$$

$$X_t - \phi_1 B X_t - \phi_2 B^2 X_t - \phi_2 B^3 X_t = \varepsilon_t + \theta_1 B \varepsilon_t + \theta_2 B^2 \varepsilon_t$$

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3) X_t = (1 + \theta_1 B + \theta_2 B^2)\varepsilon_t \quad ✓ \quad ✓$$

$$\phi^3(B) X_t = \theta^2(B)\varepsilon_t \quad ✓$$

$$\therefore p = 3 \text{ and } q = 2 \Rightarrow \{X_t\} \sim ARMA(3,2)$$

(b) Consider the ARMA process given by $\phi^1(B)X_t = \theta^3(B)\varepsilon_t$. By expanding the generating functions and applying the back-shift operator, represent this process using the "expanded notation".

$$\phi^1(B) X_t = \theta^3(B)\varepsilon_t$$

$$(1 - \phi B) X_t = (1 + \theta_1 B + \theta_2 B^2 + \theta_3 B^3)\varepsilon_t \quad ✓ \quad ✓$$

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$$X_t - \phi B X_t = \varepsilon_t + \theta_1 B \varepsilon_t + \theta_2 B^2 \varepsilon_t + \theta_3 B^3 \varepsilon_t$$

$$X_t - \phi X_{t-1} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} \quad ✓$$

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Question 2

For the process shown below, identify whether it is stationary or invertible or both. Show your work. Where appropriate, you may use the fact that the zeros of $ax^2 + bx + c$ are given by $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$.

$$X_t + 1.9X_{t-1} + 0.88X_{t-2} = \varepsilon_t + 0.2\varepsilon_{t-1} + 0.7\varepsilon_{t-2}, \text{ where } \{\varepsilon_t\} \sim WN(0, \sigma^2)$$

$\phi(z) = 1 + 1.19z + 0.88z^2$

Roots: $\phi(z) = 0$ when

$$z = \frac{-1.9 \pm \sqrt{1.9^2 - 4(0.88)}}{2(0.88)}$$

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$$= \frac{-1.9 \pm \sqrt{0.09}}{1.76}$$

$z_1 = -0.91$, $z_2 = -1.25$

$|z_1| = 0.91$, $|z_2| = 1.25$

Since $|z_1| < 1$, $\{X_t\}$ is not stationary ✓ (worth 1.5)

$\theta(z) = 1 + 0.2z + 0.7z^2$

Roots: $\theta(z) = 0$ when

$$z = \frac{-0.2 \pm \sqrt{0.2^2 - 4(0.7)}}{2(0.7)}$$

$$= \frac{-0.2 \pm \sqrt{2.76}\, i}{1.4}$$

$z_1 = -0.14 - 1.19i$, $z_2 = -0.14 + 1.19i$

$|z_1| = \sqrt{(-0.14)^2 + (-1.19)^2} = 1.198$

$|z_2| = \sqrt{(-0.14)^2 + (1.19)^2} = 1.198$

Since $|z_1| = |z_2| > 1$, $\{X_t\}$ is invertible. ✓ (worth 1.5)

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Question 3

(a) Without providing technical detail, explain (in your own words) the difference between an *autocorrelation function* and a *partial autocorrelation function*.

The autocorrelation function calculates the correlation between observations at different time points (lags).

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The partial autocorrelation function also calculates the correlation between observations at different lags, but it does so after having accounted for any possible relationships with observations at intermediate lags. What results is a quantification of the linear association at two time points, corrected for associations at intermediate time points.

(b) For each of the following processes, describe what you would expect to see on an ACF plot, and on a PACF plot.

i. AR(p)

$\frac{1}{2}$ each

- ACF: (mixed) exponential decay
- PACF: p initial spikes, then negligibly small spikes thereafter

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ii. MA(q)

$\frac{1}{2}$ each

- ACF: q initial spikes, then negligibly small spikes thereafter
- PACF: (mixed) exponential decay

iii. ARMA(p,q)

$\frac{1}{2}$ each

- ACF: q spikes and exponential decay
- PACF: p spikes and exponential decay