# Assignment 4 Solutions

$\boxed{1}$ $\{X_t\} \sim ARCH(\ell) \rightarrow \quad X_t = \sigma_t \varepsilon_t$

$$\sigma_t^2 = \omega + \alpha_1 X_{t-1}^2 + \cdots + \alpha_\ell X_{t-\ell}^2$$

where $\omega > 0$, $\alpha_j \geq 0$, $\{\varepsilon_t\} \sim IID(0,1)$, $\varepsilon_t \perp X_s \ \forall s<t$

$\boxed{a}$
$$\begin{aligned}
E(X_t) &= E[E(X_t \mid X_{t-1}, \ldots, X_{t-\ell})] \\
&= E[E(\sigma_t \varepsilon_t \mid X_{t-1}, \ldots X_{t-\ell})] \\
&= E[\sigma_t E(\varepsilon_t \mid X_{t-1}, \ldots X_{t-\ell})] \leftarrow \text{since given the history } X_{t-1}, \ldots X_{t-\ell},\ \sigma_t \text{ is constant} \\
&= E[\sigma_t E(\varepsilon_t)] \quad \text{since } \varepsilon_t \perp X_s \ \forall s<t \\
&= E[\sigma_t \times 0] \\
&= E[0] \\
&= 0
\end{aligned}$$

Alternative Solution:

Since $\varepsilon_t \perp X_s \ \forall s<t$, it implies $\varepsilon_t$ is independent of $\sigma_t$.

As such
$$\begin{aligned}
E(X_t) &= E(\sigma_t \varepsilon_t) \\
&= E(\sigma_t)E(\varepsilon_t) \\
&= E(\sigma_t) \times 0 \\
&= 0
\end{aligned}$$

$\boxed{b}$
$$\begin{aligned}
Var(X_t \mid X_{t-1}, \ldots X_{t-\ell}) &= Var(\sigma_t \varepsilon_t \mid X_{t-1}, \ldots, X_{t-\ell}) \\
&= \sigma_t^2 Var(\varepsilon_t \mid X_{t-1}, \ldots, X_{t-\ell}) \quad \text{since } \sigma_t \text{ is constant given history} \\
&= \sigma_t^2 Var(\varepsilon_t) \quad \text{since } \varepsilon_t \perp X_s \ \forall s<t \\
&= \sigma_t^2 \quad \text{since } Var(\varepsilon_t) = 1
\end{aligned}$$

$\boxed{\text{(c)}}$ $Cov(X_t, X_{t+h}) = E(X_t X_{t+h}) - \cancelto{0}{E(X_t)}\, \cancelto{0}{E(X_{t+h})}$ by part (a)

$$= E(\sigma_t \varepsilon_t \sigma_{t+h} \varepsilon_{t+h})$$
$$= E[E(\sigma_t \sigma_{t+h} \varepsilon_t \varepsilon_{t+h} \mid X_{t-1}, \ldots, X_{t-\ell})]$$
$$= E[\sigma_t \sigma_{t+h} E(\varepsilon_t \varepsilon_{t+h} \mid X_{t-1}, \ldots, X_{t-\ell})]$$
$$= E[\sigma_t \sigma_{t+h} E(\varepsilon_t \varepsilon_{t+h})]$$
$$= E[\sigma_t \sigma_{t+h} E(\varepsilon_t) E(\varepsilon_{t+h})]$$
$$= E[\sigma_t \sigma_{t+h} \times 0 \times 0]$$
$$= E[0]$$
$$= 0.$$