Homework # 4 Answer key

1) a) $$\phi^*(x) = \begin{cases} -1, & \text{if } \eta(x_1, x_2) < 0.5 \\ +1, & \text{if } \eta(x_1, x_2) > 0.5 \end{cases}$$

$$= \begin{cases} -1, & \text{if } x_1 + x_2 < 1 \\ +1, & \text{if } x_1 + x_2 > 1 \end{cases}$$

b) $\cancel{f(x_1, x_2) = \int_0^1 \int_0^1}$

Since X_1 and X_2 are independent, it follows that

$$f(x_1, x_2) = 1\{x_1 \in (0,1)\} \cdot 3x_2^2 \, 1\{x_2 \in (0,1)\}$$

$$= 3x_2^2 \, 1\{x_1, x_2 \in (0,1)\}$$

c) We can now use the law of total probability

$$P(Y=+1) = \int_0^1\int_0^1 P(Y=+1 \mid x_1, x_2) f(x_1, x_2) \, dx_1 dx_2$$

$$= \int_0^1\int_0^1 \frac{3x_2^2(x_1 + x_2)}{2} \, dx_1 dx_2$$

$$= \int_0^1 \frac{3x_2^3}{2} \, dx_2 + \int_0^1\int_0^1 \frac{3x_2^2 x_1}{2} \, dx_1 dx_2$$

$$= \frac{3}{8} x_2^4 \Big|_0^1 + \int_0^1 \left(\frac{3}{4} x_2^2 x_1^2 \Big|_{x_1=0}^{x_1=1} \right) dx_2$$

$$= \frac{3}{8} + \int_0^1 \frac{3}{4} x_2^2 \, dx_2 = \frac{3}{8} + \frac{3}{12} = \frac{15}{24}$$

1) continued...

d) $$f(x_1, x_2 \mid Y=+1) = \frac{P(Y=+1 \mid x_1, x_2)\, f(x_1, x_2)}{P(Y=+1)}$$

$$= \frac{3(x_1 + x_2)\, x_2^2}{2\left(\frac{15}{24}\right)}$$

$$= \frac{12}{5}(x_1 + x_2)\, x_2^2\ \mathbb{1}\{x_1, x_2 \in (0,1)\}$$

2) a) $$\operatorname{logit}(\eta(x:\beta)) = \beta^T x$$

$$\Rightarrow \log\left(\frac{\eta(x:\beta)}{1-\eta(x:\beta)}\right) = \beta^T x$$

$$\Rightarrow \frac{\eta(x:\beta)}{1-\eta(x:\beta)} = \exp\{\beta^T x\}$$

$$\Rightarrow \eta(x:\beta) + \eta(x:\beta)\exp\{\beta^T x\} = \exp\{\beta^T x\}$$

$$\Rightarrow \eta(x:\beta) = \frac{\exp\{\beta^T x\}}{1+\exp\{\beta^T x\}} \checkmark$$

$$= \frac{1}{1+\exp\{-\beta^T x\}}$$

(by dividing the numerator & denominator by $\exp\{\beta^T x\}$)

b) Note that $\log(\eta(x:\beta)) = -\log\left(1 + \exp\{-\beta^T x\}\right)$

So, $$\frac{\partial \log(\eta(x:\beta))}{\partial \beta} = \frac{-1}{1+\exp\{-\beta^T x\}} \cdot -x\exp\{-\beta^T x\}$$

$$= \frac{x\exp\{-\beta^T x\}}{1+\exp\{-\beta^T x\}} = \frac{x}{1+\exp\{\beta^T x\}}$$

2) b) continued...

Thus, $\frac{\partial^2}{\partial \beta^2}(\log(\tau(x;\beta))) = -x(1+\exp\{\beta^T x\})^{-2} \cdot x e^{\beta^T x}$

$$= \frac{-x^2 \exp\{\beta^T x\}}{(1+\exp\{\beta^T x\})^2}$$

Since $x^2 > 0$ for all $x \neq 0$ and $\exp\{\beta^T x\} > 0$ for all $x \neq 0$, it follows that

$$\frac{\partial^2}{\partial \beta^2}(\log(\tau(x;\beta))) \leq 0 \quad \text{for all } x.$$

Moreover, $\frac{\partial^2}{\partial \beta^2}[\log(\tau(x;\beta))] < 0$ for all β. (1)

(1) implies that $\log(\tau(x;\beta))$ is a concave function of β and therefore must have a unique maximum. Hence, maximum likelihood estimators for β are unique.

c) i) $P(\text{receives an A} \mid x_1 = 40, X_2 = 3.5)$

$$= \frac{1}{1+\exp\{6-0.05(40)-3.5\}} = \frac{1}{1+\exp\{0.5\}}$$

$= 0.378$

ii) We solve for x_1 w/ $0.5 = \frac{1}{1+\exp\{-.05x_1\}\exp\{2.5\}}$

$\Rightarrow 0.5 + 0.5\exp\{-.05x_1\}\exp\{2.5\} = 1$

2) c) ii) continued..

$\Rightarrow \quad \frac{1}{\exp\{2.5\}} = \exp\{-.05\, x_1\}$

$\Rightarrow \quad \exp\{-2.5\} = \exp\{-.05\, x_1\}$

$\Rightarrow \quad x_1 = 50$ hours.