2) $P(Y=1 \mid X=x) \rightarrow \text{Logit}\left(\eta(x:\beta)\right) = \beta^T x$

$$\Leftrightarrow \log \frac{\eta(x:\beta)}{1-\eta(x:\beta)} = \beta^T x$$

e^x is monotonically increasing

a)

$$\Leftrightarrow \frac{\eta(x:\beta)}{1-\eta(x:\beta)} = e^{\beta^T x}$$

$$\Leftrightarrow \eta(x:\beta) = e^{\beta^T x} - e^{\beta^T x}\eta(x:\beta)$$

$$\Leftrightarrow \left(1+e^{\beta^T x}\right)\eta(x:\beta) = e^{\beta^T x}$$

$$\Leftrightarrow \boxed{\eta(x:\beta) = \frac{e^{\beta^T x}}{1+e^{\beta^T x}} \times \frac{e^{-\beta^T x}}{e^{-\beta^T x}} = \frac{1}{1+e^{-\beta^T x}}}$$

b) $$\frac{\partial}{\partial\beta}\eta(x:\beta) = \frac{\partial}{\partial\beta}\left[\frac{e^{\beta x}}{1+e^{\beta x}}\right]$$

$$= \frac{xe^{\beta x}\left(1+e^{\beta x}\right) - e^{\beta x}\cdot xe^{\beta x}}{\left(1+e^{\beta x}\right)^2}$$

$$= \frac{xe^{\beta x} \cancel{+ xe^{2\beta x}} \cancel{- xe^{2\beta x}}}{\left(1+e^{\beta x}\right)^2}$$

$$\frac{\partial}{\partial\beta}\log\left(\eta(x:\beta)\right) = \frac{\partial}{\partial\beta}\left[\log\left(\frac{e^{\beta x}}{1+e^{\beta x}}\right)\right]$$

$$= \frac{\frac{\partial}{\partial\beta}\left[\frac{e^{\beta x}}{1+e^{\beta x}}\right]}{\frac{e^{\beta x}}{1+e^{\beta x}}}$$

$$= \frac{x\cancel{e^{\beta x}}}{\left(1+e^{\beta x}\right)^{\cancel{2}}} \cdot \frac{\cancel{1+e^{\beta x}}}{\cancel{e^{\beta x}}}$$

$$= \frac{x}{1+e^{\beta x}}$$

$$\frac{\partial^2}{\partial\beta^2}\log\left(\eta(x:\beta)\right) = \frac{\partial}{\partial\beta}\left[\frac{x}{1+e^{\beta x}}\right]$$

$$= \frac{-x^2 e^{\beta x}}{\left(1+e^{\beta x}\right)^2} < 0 \quad \forall x, \beta$$

This means that the solution (estimate $\hat{\beta}$) is the one for which the observed data would have the highest probability of occurrence.