Introduction to Mathematical Thinking: Week-II

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1 Lecture-III: Analysis of Language Implication

Our next step in becoming more precise about our use of language for use in mathematics is to take a close look at the meaning of the word implies[1].

Most of the benefit from understanding the way our language is used in mathematics comes from trying to figure it out. The benefit in this case is helping to develop your mathematical thinking ability, and extra process of trying to understand the issue that goes out for you.

In mathematics we frequently encouter expresion:

$$\phi \text{ implies } \psi$$

Indeed, implication is the means by which we prove result in mathematics, starting with observations or axioms[2]. So we'd better understand how the word *implies* behaves.

In particular, how does the truth or falsity of a statement (ϕ implies ψ), depend on the truth or falsity of ϕ and ψ?

Well, the obvious answer is:

$$\text{The truth of } \phi \text{, follows from the truth of } \psi$$

Let me give you an example,

$$(\phi : \sqrt{2}) \qquad\qquad (\psi : (0 < 1)$$

[1] implies : strongly suggest the truth or existence of (something not expressly stated).

[2] axiom: a statement or proposition which is regarded as being established, accepted, or self-evidently true.

That ϕ of the statement square root of 2 is a rational; and that ψ of the statement is 0 less than 1. Let's ask ourselves, is the statement "ϕ implies ψ" true?

Well, ϕ is TRUE, and we all know that ψ is true, cause 0 less than 1. So we have both ϕ and ψ are truth.

Does that mean "ϕ implies ψ"? Obviously not. There is no relationship between ϕ and ψ.

The $(\psi : \sqrt{2})$ is take some effort as you see.

And $(\psi : (0 < 1)$ we all know.

So yes, the truth from first statement (ϕ) doesn't follow from truth of (ψ).

Now, we realize there's complexity with implication that we didn't meet before when we were dealing with "AND", with "OR", and with "NOT".

Now we know,

implication involves causality

Causality is an issue of great complexity that philosophers have been discussing for generations.

now we're facing a problem. It didn't arise before, because when we're dealing with conjunction (AND) and disjunction (OR), it didn't matter whether there was any kind of relationship between the two conjuncts or the two disjuncts.

For example, let's look at the sentence:

1. (Julius Caesar is death) $\wedge$ $(1 + 1 = 3)$

2. (Julius Caesar is death) $\vee$ $(1 + 1 = 3)$

Forming a conjunction and disjunction didn't require any kind of relationship between these two sentences. Clearly, they're independent. One's statement about a long dead individual, and the other one is a mathematical statement.

Let examine above statements:

- Understanding the conjunction Expression:
    1. (Julius Caesar is death) is TRUE
    2. $(1 + 1 = 3)$: is FALSE

So, the conjunction is FALSE.

- Understanding the disjunction Expression:
    1. (Julius Caesar is death) is TRUE.
    2. $(1 + 1 = 3)$: is FALSE

So, the disjunction is TRUE.

The fact that there's no meaningful relationship between the two conjuncts in the first case, or the two disjuncts in the second case. Created no lull in determining what the truth value was. It was purely in terms of truth and falsity.

But it's no sitting wit implication, because implication involves *causality*.

So let me just express explicitly,

implication has a truth part and causation part

What we're going to do? Ignore the *causation*–part. We're going to leave that to the philosophers if you like, and we're just going to focus on the *truth*–part.

Throwing away a causation, we can't be left with anything useful, but it turns out, it might seem dangerous thing to do, to throw away this important causation-part implication, but it turns out when we focus on the truth part we left with enough to save our leaves in mathematics.

So much that we're going to give (truth part) a name,

$$(\underline{\text{Julius Caesar is dead}}) \wedge (1 + 1 = 3)$$

We're going to call it **the conditional** or **material conditional**. That's the part we're going to focus on.

So, we're going to split implication into two part:

$$\text{implication} = \quad \text{conditional} \quad + \quad \text{causation}$$

The first part, the conditional, we're going to define entirely in terms of ***truth value***. The second part, we're going to leave to the philosophers.

The symbol that we use normally for **conditional**, at least the symbol I'm going to use, is:

$$\Longrightarrow \tag{1}$$

So I'm going to write conditional expression like this?

$$\phi \implies \psi$$

That's the truth path of ϕ implies ψ.

When we have a conditional expression like above, we call ϕ ***antecedent***, and we call ψ the ***consequent***; And we going to formally define the truth of ϕ conditional ψ in terms of the truth of ϕ, and the truth of ψ, we can write like:

Define the truth of $\phi \implies \psi$ in terms of the truth | falsity of ϕ, ψ

Well, you might worry that by throwing away a *causation*, we're going to be left with a notion that's really of no use whatsoever. That actually is not the case. Even though we're throwing away something of great significance, hanging on the truth-part leaves us something very useful.

And the reason is, whenever we have a genuine implication, which are actually the only circumstances in which we're ultimately going to be interested, whenever we have a genuine implication, the truth behavior of the conditional is the correct one. It really does capture what happens with truth and falsity, when we have genuine implication, we can write it down:

When ϕ <u>does</u> implies ϕ, $\phi \implies \psi$ behaves "correctly".

That probably seems a bit mysterious at this stage, but when we start to look at some examples, I think it should become clear what I mean.

The advantage is that the conditional is always defined.

For real implication, you've got that issue of *causation*. the $(\phi : \sqrt{2})(\psi : (0 < 1)$ example, the truth or falsity wasn't the issue, it was whether there was a relationship between those two statements. Now, that's a complicated issue.

But because we're going to define the conditional, purely in terms of the truth value of the two constituents, the antecedent and the consequent, it turns out that the conditional will always be defined. When we do a genuine implication, the definition of the conditional will agree with the way implication behaves.

When we don't have a genuine implication, the conditional will still be defined, and so we can proceed.

Again, this probably seems very mysterious when I describe it in this way. But as we develop some examples, I hope you'll be able to understand what I'm trying to get at.

Let me take a quiz:

The truth of the conditional $\phi \implies \psi$ is defined in terms of

[ ] The truth of ϕ and ψ

[ ] Whether ϕ causes ψ

[ ] Both

Which is it?

It's number 1.

We define the truth of a conditional in terms of the truth and falsity of the antecedents and the consequent; And because we define the truth of the conditional in terms of truth and falsity in that way, it has a truth table.

ϕ (Conditional)	ψ (Conclusion)	$\phi \implies \psi$ (Implication)
T	T	?
T	F	?
F	F	?
F	F	?

1.1 Fill the First Table

This part We've already looked at.

ϕ (Conditional)	ψ (Conclusion)	$\phi \implies \psi$ (Implication)
T	T	T

We define the conditional (ϕ) as the truth part of implication; And implication has a property that a true implication leads to a true conclusion (ψ) from a true assumption.

So because we take the conditional from real implication, we have truths all the way throughout the top level.

NOTE:

Lets break down the definition of implication and how it works with logical statements, so it makes sense in the contexts.

Understanding implication ($\implies$)

1. **Conditional (ϕ):** This is the "IF" part of the statement.
2. **Conclusion (ψ):** This is the "THEN" part of the statement.

In logic, the implication ($\phi \implies \psi$) read as "IF ϕ, then ψ", this means:

- When ψ (the condition) is TRUE, ϕ (the conclusion) must be TRUE for the entire statement to be true.
- If ϕ is TRUE and ψ is FALSE, the implication is FALSE.
- If *phi* is FALSE, the implication is TRUE regardless of the truth value of ψ.

Truth Table for Implication

Here's how we can visualize this truth table:

ϕ (Conditional)	ψ (Conclusion)	$\phi \implies \psi$ (Implication)
T	T	T
T	F	F
F	T	T
F	F	T

Explanation in Simple Terms

- If both the condition (ϕ) and the conclusion (ψ) are true, the implication is true.
- If the condition (ϕ) is true and the conclusion (ψ) is false, the implication is false. This because the truth of the condition did not lead a true conclusion.
- If the condition (ϕ) is false, the implication is always true, regardless of whether the conclusion (ψ) is true or false. This might seem counterintuitive, but it's because an implication with a false condition doesn't make any promise about the conclusion.

Putting it All Together

Statement: "We define the conditional (ϕ) as the truth part of implication; and implication has a property that a true implication leads to a true conclusion (ψ) from a true assumption."

This means:

- The conditional (ϕ) is the part we assume or check first.
- The implication ($\phi \implies \psi$) says that if our assumption (ϕ) is true, then the conclusion (ψ) must also be true.
- If our assumption (ϕ) is true and leads to a true conclusion(ψ), then the implication ($\phi \implies \psi$) is true.
- If our assumption (ϕ) is false, we don't care about the conclusion (ψ); the implication ($\phi \implies \psi$) is considered true by default.

Simplified Example

Consider the statement "If it rains (ϕ), then the ground will be wet (ψ)."

- If it rains and ground is wet, the implication is true.
- If it rains and ground is not we, the implication is false.
- If it doesn't rain, the implication is true regardless of whether the ground is wet or not.

Let's look at the first row of the truth table above, and I give you some example to observe:

$$(\phi : N > 7) \qquad\qquad (\psi : N^2 > 40)$$

This is consistent with the truth table.

Suppose ϕ is the statement ($N > 87$); And suppose ψ is the statement ($N^2 > 40$). In other words

- If N is bigger than 7, then N^2 is bigger than 40.

In fact, it's bigger than 40 now. So, certainly, in this case ϕ implies *psi* or it is TRUE.

Now let's look at different example.

$$(\text{Julius Caesar is death}) \qquad\qquad (\pi > 3)$$

ϕ is true, ψ is true, According to the truth table, it follows that

$$\phi \implies \psi$$

In other words,

$$\text{(Julius Caesar is death)} \implies (\pi > 3)$$

Now, if you read this as Julius Caesar is dead implies pi bigger than 3, then you're in a nonsensical situation. But remember above statement isn't implication, this is just truth part implication, and in terms of the truth part there's no problem.

In the first example, $((\phi : N > 7)(\psi : N^2 > 40))$ there is meaningful relationship between ϕ and ψ.

When we know that N is bigger than 7, than we can conclude that N^2 is certainly bigger than 40. There's connection between the two statement (condition and conclusion); And in this case, the behavior of the conditional is certainly consistent with what's really going on.

In the second example, there's no connection between the two.

The conditional is true, but it's got nothing to do with one thing following from the other.

The value of doing this (in the second example), is even though has no meaning in terms of implication, its truth value is defined.

In both cases, we have a well-defined truth value. In the first case, it's a meaningful truth value. In the second case, it's purely defined truth value.

But that's not going to cause us any problem, because we're never going to encouter like second case in mathematics. We encounter the first case all the time.

So all we've done is we've extended a notion to be defined under all circumstances; And we've done it in a way that's consistent with the behavior we want when something meaningful is going on.

This is actually quite common in mathematics to extend the domain of definition of something so that it's always defined.

So long as it has the correct behavior, the correct definition for the meaningful cases, and provided we do the definition correctly, it really doesn't cause any problems. In fact, it solves a lot of problems and eliminates a lot of difficulties. If we extend the definition so that it covers all cases.

Is it just something we do in mathematics all the time? May seem strange when you first meet it, but it is a part of modern advanced mathematics. Incidentally, if you think is just playing games, let me mention that the computer system that controls that aircraft that you'll be flying in next time depends upon the fact that expressions like $(\phi \implies \psi)$ are always well defined.

But software control system doesn't depend upon knowing "Julius Caesar is death" or things like that. It doesn't depend on those kind of facts of the world.

Computer systems, by and large, don't depend upon understanding causation, which is just as well, because they don't.

What computer systems depends upon is that things are always accurately and precisely defined.

And this expression, $(\phi \implies \psi)$ occurs all over the places in software systems. So, quite literally, your life depends upon the fact that this is always well-defined. It doesn't depend upon the fact that the computer doesn't know whether "Julius Caesar is death."

1.2 Fill the second table

Okay, time to look at the second table,

ϕ (Conditional)	ψ (Conclusion)	$\phi \implies \psi$ (Implication)
T	F	?

What will the value on $(\phi \implies \psi)$, if ϕ is true, and ψ is false?

When we think about it in terms of genuine implication, because we trying to capture the truth behavior with genuine implication.

So if it was the case that $(\phi \implies \psi$, if that statement was true when we interpret it as real implication, Then the truth of ψ would follow from the truth of ψ. That's how we began remember.

That's real implication means, the truth of ψ will follow from truth from ϕ.

So, if the result of implication is TRUE, then when we have a TRUE of ϕ, we would have TRUE in ψ. But we don't. We've got FALSE in ψ. So, we cannot have a TRUE value in implication ϕ of ψ, because if we put TRUE as a result, the conditional is contrary, it contradicts real implication and we're trying to extend implication to be defined in all cases where there's no causation.

So it has to be FALSE.

ϕ (Conditional)	ψ (Conclusion)	$\phi \implies \psi$ (Implication)
T	F	F

In order that the conditional agrees with real implication, that has to be an FALSE. If it's a truth, then we would have true antecedent and false consequence from a true implication.

Let me write that down just to make sure everyone's following what I'm trying to say.

> If there were a genuine implication "ϕ implies ψ, and if that implication were TRUE then ψ would have to be TRUE if ϕ were TRUE.
>
> So we cannot have ϕ TRUE and ψ FALSE if $\phi \implies \psi$ is TRUE

That means, that in the case where ϕ is true and ψ is false we have a false implication.

1.3 Fill the last two table

ϕ (Conditional)	ψ (Conclusion)	$\phi \implies \psi$ (Implication)
F	T	?
F	F	?

Now if you're like me, you have no intuitions as to what to put above; And the reason you have no intuition is that even though you're used to dealing with implication you've never dealt with an implication where the antecedent (conditional) was false. You're only ever interested in drawing conclusions from true assumptions.

You do have an intuition with:

$$\phi \not\Longrightarrow \psi$$

The reason that's going to help us out, that negation implication ($\not\Longrightarrow$) swap around falsity of ϕ. So corresponding to the "F" for ϕ here when we look at ϕ on ($\phi \not\Longrightarrow \psi$) we'll have truths.

So you are used to having deal with ($\phi \not\Longrightarrow \psi$), So The trick, or at least the idea by which we're going to figure out what goes here, is to stop looking at implication ($\implies$) and look at ($\not\Longrightarrow$). ϕ does not imply ψ if even though ϕ is true, ψ neverthenless false.

That's how you know that ($\phi \not\Longrightarrow \psi$) holds. You know that ϕ doesn't imply ψ if you can check ϕ is true but ψ neverthenless false.

ϕ (Conditional)	ψ (Conclusion)	$\phi \not\Longrightarrow \psi$ (Implication)
T	F	T

That's the how you now that ($\phi \not\Longrightarrow \psi$) holds, you know that ϕ doesn't imply ψ if you can check that ϕ is true but ψ is false.

That's the only circumstance under which you can conclude ($\not\Longrightarrow$) is true. In all other circumstances ($\phi \not\Longrightarrow \psi$) will be false.

Let me write a conclusions:

In all other circumstances $\phi \not\Longrightarrow \psi$ will be false

In all other circumstances $\phi \implies \psi$ will be true

Because ($\neg$) swap false and true, $\phi \not\Longrightarrow \psi$ will be false, and $\phi \implies \psi$ will be true

Let me give you a little quiz?

Which of the following are true?

$\phi \implies \psi$ is true, whenever:

[ ] ϕ and ψ are both true

[ ] ϕ is false and ψ is true

[ ] ϕ and ψ are both false

[ ] ϕ is true and ψ is false

check all that are true!!

Which of these four conditions all the case when $\phi \implies \psi$ is true?

The answer is 1, 2, and 3.

1.4 Summary

We've defined a notion, the conditional, that captures only part of what implies means.

To avoid difficulties, we base our definition solely on the notion of truth and falsity. Our definition agrees with our intuition concerning implication in all meaningful cases.

The definition for a true anticedent is based on analysis of the truth values of genuine implication.

The definition for false antecedent, is based n a truth value analysis of the notion does not imply.

In defining the conditional the way we do, we do not end up with a notion that contradicts a notion of genuine implication.

Rather, we obtain a notion that extends genuine implication to cover those cases where the claim of implication is irrelevant, because the antecedent is false or meaningless when there's no real connection between the antecedent and the consequences.

In the meaningful case where there is a relationship between ϕ and ψ, and in addition, where ϕ is true, namely, the cases covered by the first two rows of the truth table, the truth value of the conditional will be the same as the truth value of the actual implication.

Remember, it's the fact that the conditional always has a well-defined truth value that makes this notion important in mathematics since in mathematics, we can't afford to have statements with undefined truth values floating around.

I've kept assignment three fairly short since I expect you'll need most of your time simply understanding our analysis of implication and the definition of the conditional.

1.5 Last Quiz

Here the last quiz:

Here the last quiz:

If the conditional is true, check the corresponding box.

[ ] $(\pi^2 > 2) \implies (\pi > 1.2)$

[ ] $(\pi^2 < 0) \implies (\pi = 3)$

[ ] $(\pi^2 > 0) \implies (1 + 2 = 3)$

[ ] (The area of a circle of radius is π) $\implies$ (3 is prime)

[ ] (Triangles have four sides) $\implies$ (Squares have five sides)

[ ] (Euclid's brithday was july 4) $\implies$ (Rectangles have four sides)

$(\pi^2 > 0)$	$(\pi > 1.2)$	$(\pi^2 > 0) \implies (\pi > 1.2)$
T	T	T

The answer for the first one is that it's TRUE. The antecedent $(\pi^2 > 2)$ is true and the consequence is true, so the conditional is true.

In fact there's deeper result is going on the first one. Providing you take a positive number, instead of π, any positive number, then if the N^2 of that positive number bigger than 2, that number must be bigger than 1.2. Because the $2^2 = 1.41421356237$.

So for positive numbers, it doesn't have to be (π) it can be anything. Any positive number whose square is bigger than 2, it must be bigger than 1.2.

The first question would be a case of genuine causation and genuine implication. But in terms of the conditional, it's enough that the antecedent its true, then the consequence is true.

$(\pi^2 < 0)$	$(\pi = 3)$	$(\pi^2) \implies (\pi = 3)$
F	F	T

For the second question it's also TRUE. Now the consequence is false $(\pi = 3)$ but the antecedent is false $(\pi^2 < 0)$; And if you have false antecedent, the conditional is always true. π^2 is most certainly not less than 0. So you've got $(\pi^2 < 0)$ is false, $(\pi = 3)$ is false, that makes the conditional true.

$(\pi^2 > 0)$	$(1 + 2 = 3)$	$(\pi^2) \implies (1 + 2 = 3)$
T	F	F

Number three, that's one false. The antecedent is true, and the consequence is false; And you cannot obtain a false conclusion from a true assumption.

(circle radius is π)	(3 is prime)	(circle radius is π) $\implies$ (3 is prime)
T	T	T

The forth is true. The antecedent is true, and the consequence is true.

(Triangles = 4 sides)	(Squares = 5 sides)	(Triangles = 4 sides) $\implies$ (Squares = 5 sides)
F	F	T

Do triangles have four sides? No. Do squares have five sides? No. But anything with a false antecedent is true, so that's true.

You've got conditional false, conclusion false, so that's true.

(Euclid's = july 4)	(Rectangles = 4 sides)	(Euclid's = july 4) $\implies$ (Rectangles = 4 sides)
T	T	T
F	T	T

We don't know Euclid's birthday was. At least, I don't know when Euclid's birthday was. I suspect you either.

Either the consequences is true, or it's false.

Either way, since the consequence is true, the thing is true.

We've going to run down *using two consequences*, so either we have true consequence and true conclusion, in which case, it's true; or we have false consequence and true conclusion, in which case, it's true.

2 Assignment 3 for Lecture-III

2.1 Q: Let *D* be the statement "The dollar is strong", *Y* the statement "The yuan is strong", and *T* statement "New US-China trade agreement signed". Express the main content of each of the following (fictitious) newspaper headlines in logical notation.

Remember, logical notation captures truth, but not the many nuances and inferences of natural language. As before, make sure you could justify and defend your answer.

1. New trade agreement will leas to strong currencies in both countries.
2. Strong Dollar means a weak Yuan

3. Trade agreement fails on news of weak Dollar.
4. If new trade agreement is signed. Dollar and Yuan can't both remain strongly
5. Dollar weak but Yuan strong. Following new trade agreement.
6. If the trade agreement is signed, a rise in the Yuan will result in a fall in the Dollar.
7. New trade agreement means Dollar and Yuan will rise and fall together.
8. New trade agreement will be good for one side. But no one knows which

A:

Let's start by converting each of the statement into logical notation. Then we will create a truth table to illustrate the possible truth values.

1. New trade agreement will leas to strong currencies in both countries.
   - Logical notation: $T \implies (D \wedge Y)$
2. Strong Dollar means a weak Yuan
   - Logical notation: $D \implies \neg Y$
3. Trade agreement fails on news of weak Dollar.
   - Logical notation: $\neg D \implies \neg D$
4. If new trade agreement is signed. Dollar and Yuan can't both remain strongly
   - Logical notation: $T \implies \neg(D \wedge T)$
5. Dollar weak but Yuan strong. Following new trade agreement.
   - Logical notation: $T \implies (\neg D \wedge T)$
6. If the trade agreement is signed, a rise in the Yuan will result in a fall in the Dollar.
   - Logical notation: $T \implies (Y \implies \neg D)$
7. New trade agreement means Dollar and Yuan will rise and fall together.
   - Logical notation: $T \implies (Y \iff \neg D)$

8. New trade agreement will be good for one side. But no one knows which

    - Logical notation: $T \implies (D \oplus Y)$

2.2 Q: Complete the following truth table

ϕ	$\neg\phi$	ψ	$\phi \implies \psi$	$\neg\phi \vee \psi$
T	?	T	?	?
T	?	F	?	?
F	?	T	?	?
F	?	F	?	?

Note: $(\neg)$ has the same binding rules as $(-)$ (minus) in arithmetic and algebra, so $\neg\phi \vee \psi$ is the same as $(\neg\phi) \vee \psi$

A:

ϕ	$\neg\phi$	ψ	$\phi \implies \psi$	$\neg\phi \vee \psi$
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T

Explanation:

1. First row ($\phi = T, \psi = T$)

    - $\neg\phi =$ F
    - $\phi \implies \psi = T \implies T = T$

        (a) When $\phi = T$ and $\phi = T$:

        - Statement: $\phi \implies \psi$
        - Substitution: $T \implies T$
        - Evaluation:
            * According to the truth table, if both ϕ and ψ are true, then the implication $\phi \implies \psi$ is true.
            * Therefore, $T \implies T$ evaluates to T.

        So, $T \implies T = T$ evaluates to T

    - $\neg\phi \vee \psi = F \vee T = T$

2. Second row ($\phi = T, \psi = F$)
    - $\neg\phi = F$
    - $\phi \implies \psi = T \implies F = F$
    - \($\neg\ \phi \lor \psi$ = F $\lor$ F = F)
3. Third row ($\phi = F, \psi = T$)
    - $\neg\phi = T$
    - $\phi \implies \psi = F \implies T = T$
    - \($\neg\ \phi \lor \psi$ = T $\lor$ T = T)
4. Fourth row ($\phi = F, \psi = F$)
    - $\neg\phi = T$
    - $\phi \implies \psi = F \implies F = T$
    - \($\neg\ \phi \lor \psi$ = T $\lor$ F = T)

2.3 Q: What conclusion can you draw from the above table?

From the truth table, we can observer the following:

- The expression $\phi \implies \psi$A and $\neg\phi \lor \psi$ have identical truth value in all cases.
- This demonstrates tat $\phi \implies \psi$ is logical equivalent to $\neg\phi \lor \psi$. This is a fundamental equivalence in propositional logic known as the implication equivalence.

2.4 Q: Complete the following truth table.

Recall that ($\phi \not\Rightarrow \psi$) is another way of writing ($\neg[\phi \implies \psi]$).

ϕ	ψ	$\neg\phi$	$\phi \implies \psi$	$\phi \not\Rightarrow \psi$	$\phi \land \neg\psi$
T	T	?	?	?	?
T	F	?	?	?	?
F	T	?	?	?	?
F	F	?	?	?	?

A:

My first attempt:

ϕ	ψ	$\neg\phi$	$\phi \implies \psi$	$\phi \not\Longrightarrow \psi$	$\phi \wedge \neg\psi$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	T	T	T	T
F	F	T	T	T	F

(This have incorrect answers, take a look at below table.)

My second attempt:

ϕ	ψ	$\phi \not\Longrightarrow \psi$
T	T	F
T	F	T
F	T	F
F	F	F

- I have to fix the negation of the implication (ϕ does not imply ψ)

ϕ	ψ	$\neg\phi$	$\phi \implies \psi$	$\phi \not\Longrightarrow \psi$	$\phi \wedge \neg\psi$
T	T	F	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	F	F

3 Lecture-IV: Analysis of Language Equivalent

This lecture's going to be fairly short as well. The next topic I want to look at is logical equivalence[3].

Equivalence is closely related to implication. Two statements are said to be equivalent, or more fully, logically equivalent, if each implies the other. Equivalence is a central notion in mathematics.

Many mathematical result are proofs that two statement are equivalent. In fact, equivalence is to logic as equations are to arithmetic and algebra; And you already know that equations play a central role in mathematics.

Just as we had introduce a formal version implication that avoids the complex issue of causation, namely the conditional. We have to introduce an analogous version of equivalence. It called ”Biconditional”. Fortunately, we did all the difficult work with implication. Now we can reap the benefits of those efforts.

[3]equivalence: a way of defining when two elements are considered equivalent.

Two statement Φ and Ψ area said to be (logically) equivalent, or just equivalent if each implies the other we call it the biconditional.

The biconditional of ϕ and ψ is denoted by " $\iff$ ".

$$\iff$$

Formally, the biconditional is an abbreviation of:

$$(\phi \implies \psi) \wedge (\psi \implies \phi)$$

Since the conditional is defined it terms of truth values, it follows that biconditional is defined in terms of truth values.

If you work out the truth table for ϕ conditional ψ ($\phi \implies \psi$) and ψ conditional ϕ ($\psi \implies \phi$) then you work out the conjunction, you'll get the truth table for ($\phi \iff \psi$)

If you that, what you will find is that $\phi \iff \psi$ is true, if ϕ and ψ are both true or both false.

One way to show two statements Φ, Ψ are equivalent is to show they have the same truth table.

Actually to avoid confusion, I choose to use capital phi (Φ) and capital psi (Ψ), because I want to use the lower case phi (ϕ) and psi (ψ) for something else.

For example,

$$\text{(i)}\underline{(\phi \wedge \psi) \vee (\neg\psi)} \text{ is equivalent to (ii)}\underline{\phi \implies \psi}$$

$$\text{(i)} = \Phi, \text{(ii)} = \Psi$$

If I was teaching this material at high school level, I'd be very careful to choose different letters to denote everything, but we're looking at college, university level mathematics now; And university mathematicians, professional mathematicians frequently use upper case and lower case symbol in the same context; And part of being able to master university level is actually getting use to disambiguous notations.

What we have to do is to work up truth table for Φ and truth table for Ψ.

ϕ	ψ	$\phi \wedge \phi$	$\neg\phi$	$(\phi \wedge \psi) \vee (\neg\phi)$	$\phi \implies \psi$
T	T	T	F	T	T
T	F	F	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

You can check the last two table above, it's the same.

I should mention that provind equivalence by means of truth tables is very unusual. It's only special case of equivalence.

In general proving equivalence is really quite hard. You have to look at what the two statements mean and develop a proof based on their meaning.

Equivalence itself is not too difficult to notion deal with. What is problematic is mastering the various nomenclatures that are associated with implication.

3.1 Expression $\phi \implies \psi$

There are many different expression we use to describe "$\phi \implies \psi$", some of them intuitively obvious and some of them are actually counter-intuitive when you first meet them.

NOTE:

$$\phi \implies \psi$$

ϕ: antecedent
ψ: consequent

The following all mean "$\phi \implies \psi$":

3.1.1 If ϕ, the ψ

This is the most straightforward way to express an implication, it means that whenever ϕ is true, ψ must also true.

Example:

- ϕ: It is raining.
- ψ: The ground is wet.
- Statement: If it is raining, the ground is wet.

3.1.2 ϕ is sufficient for ψ

This mean that ϕ being true guarantees that ψ is true. If ϕ happens, ψ must also happen.

Example:

- ϕ: You get 90% or more on a test.
- ψ: You pass the test.
- Statement: Getting 90% or more on a test is sufficient for passign the test.

If you score 90% or more, you will definitely pass.

3.1.3 ϕ only if ψ [NOT SAME as "IF ψ THEN ψ"]

Let's clarify the difference between "ϕ only if ψ" and "if ψ, then ϕ". These statement might sound similar, but they different meanings in logic.

1. "ϕ only if ψ":
   - This statement means that ψ true only if ψ is also true.
   - In logical terms, ϕ only if ψ written as: $\phi \implies \psi$.
2. "If ψ, then ϕ":
   - This statement means that if ψ is true, then ϕ must also be true.
   - In logical terms, if ψ, then ϕ is written as: $\psi \implies \phi$

Difference:

Let look at each statement more closely:

- **"ϕ only if ψ"** ($\phi \implies \psi$)
  - This means that whenever ϕ is true, ψ must also true.
  - If ψ false, then ϕ also be false.
  - This does **not** necessarily mean that ψ being true implies that ϕ is true.
- **"if ψ, then ϕ"** ($\psi \implies \phi$)
  - This means that whenever ψ is true, ϕ must also be true.

- If ϕ false, then ψ must also be false.
- This does **not** necessarily means that ϕ being true implies that ψ is true.

Example:

A. Consider the statements:

- ϕ: "It is raining."
- ψ: "The ground is wet."

"ϕ only if ψ" means "It is raining only if ground is wet" ($\phi \implies \psi$)

- This means: If it is raining, then the ground must be wet.
- It does **not** mean: If the ground is wet, then it is raining. (The ground could be wet for other reasons, such as someone watering the garden.)

"if ψ, then ϕ" means "If the ground is wet, then it is raining" ($\psi \implies \phi$)

- This means: If ground is wet, then it is raining.
- It does **not** mean: If it raining, then ground is wet. (The ground being wet is a consequence of rain, but rain can occur without the ground getting wet if, for instance, it is raining in a different area.)

Visualizing with Truth Table

ϕ (Raining)	ψ (Ground is wet)	$\phi \implies \psi$	$\psi \implies \phi$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

B: Consider the statements:

- ϕ: "I go to the Tour de France."
- ψ: "I have a bike."

"ϕ only if ψ" ($\phi \implies \psi$\)

This means: "I go to the Tour de France only if I have a bike."

- In logical terms: $\phi \implies \psi$.

  This implies that if you are going to the Tour de France, you must have a bike. Going to the Tour de France is conditional upon having a bike. If you don't have a bike, you can't go to the tour de France. However, it does **not** imply that having a bike means you are going to the Tour de France. You could have a bike and not go to the Tour de France.

"if ψ, then ϕ" $\psi \implies \phi$

This means: "If I have a bike, then I go to the Tour de France."

- In logical terms: $\psi \implies \phi$.

  This implies that having a bike means you must be going to the tour de France. If you have a bike, then you are definitely participating in the Tour de France. However, it does **not** imply that going to the Tour de France requires having a bike. You could be going to the Tour de France without having a bike, perhaps as spectator or in some other capacity.

Analysis Table of truth

- **"ϕ only if ψ" ($\phi \implies \psi$\)**
  - If ϕ (I go to the Tour de France) is true, and ψ (I have a bike) is true, then the statement is true.
  - If ϕ is true, and ψ is false, then the statement is false because going to the Tour de France should mean you have a bike.
  - If ϕ is false (I don't go to the Tour de France), then the statement is true regardless of ψ, because not going to the Tour de France does no contradict having a bike.
  - If (ϕ\) and ψ are both false, the statement is true because not going to the Tour de France and not having a bike aligns the condition.
- **"if ψ, then ϕ"** $\psi \implies \phi$
  - If ψ (I have a bike) is true, and ϕ (I go to the Tour de France) is true, then the statement is true.
  - If ψ is true and ϕ is false, the statement is false because having a bike should mean you are going to the Tour de France.
  - If ψ (I don't have a bike) is false, the statement is true regardless of ϕ, because not having a bike does not require going to the Tour de France.

- If ψ and ϕ are both false, the statement is true because not having a bike and not going to the Tour de France aligns with the condition.

3.1.4 ψ if ϕ

This is just different way of saying: "If ϕ, then ψ". It mean the same thing as #1.

Example

- ψ: It is raining.
- ϕ: The ground is wet.
- Statement: If it is raining, the ground is wet.

3.1.5 ψ Whenever ϕ

This is just different way of saying that ψ happens every time ϕ happens

Example

- ψ: You press the light switch.
- ϕ: The light turns on.
- Statement: The light turns on whenever you pres the light switch.

3.1.6 ψ is necessary for ϕ

This means that ψ must be true for ϕ to be true. If ϕ is true, then ψ has to be true as well.

Example

- ψ: You can graduate.
- ϕ: You have passed all exam
- Statement: Passing all your exam is necessary for graduating.

3.1.7 Summary

It's important to really master this terminology, because it's used all the time, not just in mathematics, but in science, in analytic reasoning and tough in general.

This is not just mathematical language, this is the language that people use in legal documents, in logical arguments, in analytic arguments, and in discussion and so forth.

So understanding language as it's used is very important in many walks of life; And having introduced terminology commonly associated with implication, we have an associated terminology for equivalence.

3.2 Equivalence Terminology "$\phi \iff \psi$"

"ϕ is equivalent to ψ" is itself equivalent to

By the way this already shows how ubiquitous the equivalence is, because the obvious word to describe this.

The obvious way to describe that ("ϕ is equivalent to ψ") is equivalent to something is to use world "equivalence" as well. It mean, equivalence is just very basic concept in mathematics.

So this sentence ("ϕ is equivalent to ψ") or statement, it's claim is itself equivalent to:

3.2.1 ϕ is necessary and sufficient for ψ

Notice we combining *necessary* and *sufficient*. With necessary we have ψ before ϕ ("ϕ is sufficient fit ϕ"), with sufficient we have ϕ before ψ ("ϕ is sufficient for ψ") that gets us the implication in both directions, and equivalence means *implication in both directions*.

So, the fact that it's in both direction is captured by the fact that we have both sufficient and necessary with below expression:

3.2.2 ϕ if and only if ψ

Similarly with "if and only if" it combines *only if* ("*phi* only if *psi*") where ϕ comes before ψ, with *if*, ("ψ if ϕ") where ψ comes before ϕ. So both in this cases, we have an implication from 'ϕ to ψ', and from ψ to ϕ.

Final remark, this expression is often abbreviated ***IFF***. IFF is standard mathematicians abbreviation for 'if and only if'.

So, 'if and only if' or 'iff' means the two things are equivalent.

Okay, once you've mastered this terminology you should be able to read and make sense of pretty well any mathematics that you come across. That doesn't mean to say you understand the mathematics itself, but at least you should be able to understand what it's talking about.

That's a first step towards understanding the mathematics itself. The rest is really up to you to spend some time mastering the concepts and the associated terminology.

3.3 Quiz Lecture 4

This quiz comes in four parts.

3.3.1 Quiz - Part 1 – 1

Which of the following conditions is necessary for the natural number n to be a multiple of 10?

1. n is a multiple of 5.
2. n is a multiple of 20.
3. n is even and a multiple of 5.
4. $n = 100$.
5. n^2 is multiple of 100.

A:

To answer this tricky question, we have to ask ourselves. "Does n being a multiple of 10 imply the statement?" To be necessary, n being a multiple of 10 has to imply the statement.

1. n is a multiple of 5. [T]
    - A number that is multiple of 10 is also a multiple of 5 because $10 = 2X5$
    - However, not every multiple of 5 is a multiple of 10 (e.g, 15 is multiple of 5 but not 10).
    - Therefore, this condition is not sufficient, **but** it is part of being a multiple of 10.
2. n is a multiple of 20, [F]
    - Any number that is multiple of 20 is also multiple of 10 because $20 = X10$

- This condition is sufficient but not necessary since a multiple of 10 does not have to be a multiple of 20 (e.g, 10 is a multiple of 10 but not 20).

3. n is even and a multiple of 5 [T]

   - If n is even, it divisible by 2.
   - If n is also multiple by 5, then n must be a multiple of $2X5 = 10$.
   - Therefore, this condition is both necessary and sufficient for n to be a multiple of 10.

4. $n = 100$ [F]

   - This is specific case of multiple of 10, but it is not necessary for n to equal 100 to be a multiple of 10 (e.g, 10, 20, and 30 are also multiple of 10).
   - Therefore, this conditions is not necessary.

5. n^2 is multiple of 100. [T]

   - If n^2 is multiple of 100, then n must of 10 because if n were not a multiple of 10, n^2 could not be a multiple of 10.
   - Thus, this condition is necessary for n to be multiple of 10.

After evaluating each condition, the necessary condition for n to be a multiple of 10 is:

1. n is even and a multiple of 5.

   This condition ensure that n is divisible by both 2 and 5, which makes it divisible by 10. Additionally, condition 5 (n^2 is multiple of 100) is also valid, but condition 3 is more straightforward and directly confirms the requirement.

3.3.2 Quiz - Part 1 – 2

Which of the following conditions is sufficient for the natural number n to be a multiple of 10?

1. n is a multiple of 5.
2. n is a multiple of 20.
3. n is even and a multiple of 5.

4. $n = 100$.
5. n^2 is multiple of 100.

A:

1. n is a multiple of 5. [F]
2. n is a multiple of 20. [T]
3. n is even and a multiple of 5. [T]
4. $n = 100$ [T]
5. n^2 is multiple of 100. [T]

3.3.3 Quiz - Part 1 – 3

Which of the following conditions is necessary and sufficient for the natural number n to be a multiple of 10?

1. n is a multiple of 5.
2. n is a multiple of 20.
3. n is even and a multiple of 5.
4. $n = 100$.
5. n^2 is multiple of 100.

A:

	Necessity	Sufficient	N $\wedge$ S
1	T	F	F
2	F	T	F
3	T	T	T
4	F	T	F
5	T	T	T

So the answer is number 3 and number 5.

3.3.4 Quiz - Part 1 – 4

Identify the antecedent in each of the following conditional:

1 If the alarm rings, every one leaves

2 Everyone leaves if the alarm rings

3 Keith cycles only if the sun shines

4 Joe leaves whenever Amy arrives

A:

The following sentence is the antecedent:

1. "The alarm rings"
2. "The alarm rings"
3. "Keith cycles"
4. "Amy arrives"

3.4 Summary

So far, I've distinguish between genuine implication and equivalents, and they're far more counterparts. The conditional and bi-conditional[4], in the daily work however, Mathematicians are very not particular. For instance, we often use the arrow symbol ($\Longrightarrow$) as an abbreviation for implies. On the double headed arrow ($\Longleftrightarrow$) is an abbreviation for is equivalent to.

Although this is very confusing for beginners, it's simply the way a mathematical practice is evolved and there's no getting around it. In fact, once you get used to the notions, it's not all this confusing as it might seem at first and here is why: The conditional and bi-conditional only differ from implication and equivalents in situation that not adrise in the cause of normal mathematical practice. In any real mathematical context, the conditional effectively is implication, and the bi-conditional effectively is equivalent.

So having made note of where the formal notions differ from the everyday ones, mathematicians simply move on and turn their attention to other things. The very act of formulating definitions creates an understanding of implication and equivalence that allows us to use the everyday notion safely.

Of course, computer programmer and people who develop aircraft control systems don't have such freedom. They have to make sure all the notions in their programs are defined and give answers in all circumstances.

Okay, that's the end of the lecture-IV. As I said at the start, it's been a fairly short lecture. My reason for keeping the lecture is that the upcoming

[4]Two statement Φ and Ψ area said to be (logically) equivalent, or just equivalent if each implies the other we call it the biconditional.

is much longer than the others, it has to be.

Implication and equivalence are the heart of mathematics. Mastery of those concepts and of the terminology associated with them is fundamental to mathematical thinking.

You simply have to master implication and equivalence before you can go much further; And there's only one way to achieve mastery, right? Remember the story of the elderly lady who approached a New York City policeman and asked, officer, how do I get to Carnegie Hall? The officer smiled and said, lady, there's only one way, practice, practice, practice. So, I suggest you carve out some time, grab some food and drink, and head off somewhere quiet to complete as much of assignment-IV as you possible can.