Utility Maximization Under Endogenous Uncertainty

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1 Introduction

The existence of an expected utility maximizing action is a minimal requirement for any model of decision making under uncertainty. Kennan (1981) showed that such an optimal action exists under remarkably weak regularity conditions: compactness of the action set, upper semi-continuity of the utility function, and finite expected value of perfect information.

However, Kennan's result assumes that the probabilities of uncertain events are fixed ex-ante. Many economic settings violate that assumption: lifestyle choices affect an individual's health outcomes, R&D spending affects a firm's profits, and the level of effort affects a worker's wage. In each of these examples, the agent's action (lifestyle / investment / effort) alters the distribution of a utility-relevant random variable. The importance of such endogenous uncertainty is highlighted, for example, by Hansen and Sargent (2001). They explicitly consider environments where agents' decisions affect the underlying uncertainty but simply assume the existence of an optimal policy. Similar implicit assumptions are widespread across economics. The goal of this paper is to fill that gap by extending existence theorems to explicitly accommodate decision-dependent uncertainty.

The analysis presented here is most closely related to some existence results in the principal–agent literature. In standard models, the contract induces a mapping from the agent's effort level to a distribution over wages. The principal selects a contract to maximize her ex-ante expected utility, and the agent responds by choosing effort to maximize his own expected utility. Early contributions by Holmström (1979) and Shavell (1979) established existence under compact action sets and continuous outcome distributions. However, subsequent work clarified that stronger regularity assumptions are typically necessary. In particular, Rogerson (1985) demonstrated that the widely-used first-order approach is valid only if the output distribution satisfies both the monotonelikelihood-ratio property and the convexity-of-distribution-function condition. These assumptions ensure that the agent's expected utility is concave in effort, thereby guaranteeing the existence of an optimal (typically interior) choice.

This paper extends foundational existence results in decision theory to settings where the agent's choice influences the probability of different outcomes. By formulating appropriate continuity conditions – most notably, a version of upper semi-continuity for choice-dependent probability measures – we provide general conditions under which an optimal action exists. We employ a topological proof that does not rely on the concavity or monotonicity assumptions commonly imposed in earlier literature. As a result, our approach yields a more general existence result and allows expected utility theory to be applied in a broader class of problems with decision-dependent uncertainty.

2 Notation and Assumptions

Let A be the set of alternatives available to the agent. Let Ω be the (possibly infinite) set of possible realizations of a random variable. The associated probabilities are given by the measure m_a. The subscript $a \in A$ signifies that the measure is a function of the agent's choice. Note that we are not requiring the support of the random variable to be the same for every choice of a.

Let $u(a, \omega)$ be the agent's utility function. We want to prove that the following maximization problem has at least one solution:

$$\max_a v(a) = \max_a \mathbb{E}\big[u(a, \omega)\big]$$

This problem can be written as:

$$\max_a v(a) = \max_a \int_\Omega u(a, \omega) dm_a(\omega)$$

Our result requires the following assumptions:

1. A is a compact and first countable topological space.

2. $u(\cdot, \omega)$ is a set of equicontinuous and real valued functions on A.

3. $u(a, \cdot)$ is a real valued random variable for all $a \in A$ and is integrable.

4. For any integrable function g and any sequence $a_n \to a$ the measure satisfies:

$$\int_{\Omega} g dm_a(\omega) \geq \limsup_n \int_{\Omega} g dm_{a_n}(\omega)$$

Assumptions 1 and 3 are ubiquitous in the literature. Assumption 2 is a stronger version of the standard continuity assumption. While it may appear restrictive at first, we point out that Assumption 2 will be satisfied whenever Ω is compact and $u(a, \omega)$ is continuous. These two assumptions are very common in the literature and are satisfied by most economic problems of interest.

Assumption 4 is our main assumption. It applies the notion of upper semi-continuity to the space of probability measures. Strictly speaking, Assumption 4 is stronger than necessary – the given condition does not need to hold for all integrable functions. Our proof remains unchanged as long as the given condition holds for any subset of integrable functions which includes $u(a, \omega)$.

This paper does not prove necessity but it is easy to see that the existence of a maximizing action is not guaranteed without some version of Assumption 4. We provide some conditions under which Assumption 4 holds and show that it aligns closely with an intuitive notion of upper semi-continuity.

We note that our assumptions do not impose monotonicity or concavity on $v(a)$. This makes our result broadly applicable and ensures existence in many settings which are not covered by existing results. Moreover, unlike Kennan (1981), our proof does not use Fatou's Lemma so we do not need to assume the existence of a dominating function.

3 Results

Our main result can be stated as:

Theorem 1. *The expected utility maximization problem has at least one solution whenever Assumptions 1 - 4 are satisfied.*

Proof. We have assumed that A is compact (Assumption 1). So a solution to the maximization problem is guaranteed if we can show that $v(a)$ is upper semi-continuous. That requires the following for any $a_n \to a$:

$$\int_{\Omega} u(a, \omega) dm_a(\omega) \geq \limsup_n \int_{\Omega} u(a_n, \omega) dm_{a_n}(\omega)$$

The right hand side of the inequality can be written as:

$$\begin{aligned} \limsup_n \int_\Omega u(a_n,\omega) dm_{a_n}(\omega) \leq & \limsup_n \int_\Omega u(a,\omega) dm_{a_n}(\omega) \\ & + \limsup_n \int_\Omega \Big[u(a_n,\omega) - u(a,\omega) \Big] dm_{a_n}(\omega) \end{aligned} \tag{1}$$

Applying Assumption 4 to the first term in (1) gives:

$$\limsup_n \int_\Omega u(a,\omega) dm_{a_n}(\omega) \leq \int_\Omega u(a,\omega) dm_a(\omega)$$

Recall that $u(\cdot,\omega)$ is an equicontinuous set of functions on a compact space (Assumptions 1 and 2). Therefore $u(\cdot,\omega)$ are uniformly equicontinuous. Then:

$$\epsilon_n := \sup_\omega \Big| u(a_n,\omega) - u(a,\omega) \Big| \to 0$$

Applying this to the second term in (1) gives:

$$\begin{aligned} \limsup_n \int_\Omega \Big[u(a_n,\omega) - u(a,\omega) \Big] dm_{a_n}(\omega) & \leq \limsup_n \int_\Omega \sup_\omega \Big| u(a_n,\omega) - u(a,\omega) \Big| dm_{a_n}(\omega) \\ & = \limsup_n \int_\Omega \epsilon_n dm_{a_n}(\omega) \\ & = \limsup_n \epsilon_n m_{a_n}(\Omega) \\ & = 0 \end{aligned}$$

Combining everything gives:

$$\begin{aligned} \limsup_n \int_\Omega u(a_n,\omega) dm_{a_n}(\omega) \leq & \limsup_n \int_\Omega u(a,\omega) dm_{a_n}(\omega) \\ & + \limsup_n \int_\Omega \Big[u(a_n,\omega) - u(a,\omega) \Big] dm_{a_n}(\omega) \\ & \leq \int_\Omega u(a,\omega) dm_a(\omega) + 0 \end{aligned}$$

This proves that $v(a)$ is upper semi-continuous and completes the proof of existence.

$\square$

We mentioned earlier that Assumption 4 is our main assumption. The remainder of this section provides some intuition for it and establishes sufficient conditions for Assumption 4.

Assume hereafter that Ω has a σ-finite measure μ such that m_a is absolutely continuous with respect to μ for every a. Then the Radon–Nikodym theorem tells us that each m_a induces a density function f_a. Lemma 1 allows us to establish Assumption 4 using these densities.

Lemma 1. *Assumption 4 is satisfied whenever f_a is a continuous function of a.*

Proof.

$$\begin{aligned}
\limsup_n \int_\Omega u(a,\omega)dm_{a_n}(\omega) &= \limsup_n \int_\Omega u(a,\omega)f_{a_n}(\omega)d\mu(\omega) \\
&\leq \int_\Omega \limsup_n u(a,\omega)f_{a_n}(\omega)d\mu(\omega) \\
&= \int_\Omega u(a,\omega)f_a(\omega)d\mu(\omega) \\
&= \int_\Omega u(a,\omega)dm_a(\omega)
\end{aligned}$$

□

Lemma 1 proves that Assumption 4 holds whenever the random variable ω admits a density which is a continuous function of the choice variable a.[1]

Importantly, Lemma 1 makes it obvious that our result applies when Ω is the real line and ω is normally distributed with the mean and / or the variance given by a continuous function of a. Similar arguments can be used to show that our assumptions are satisfied by most continuous random variables which are commonly used by economists.

The restriction imposed by Assumption 4 is more substantive when we focus on discrete random variables. Consider the following example:

Example 1.

$$\omega \mid a \leq 1 = \begin{cases} 0, & \textit{with probability } \frac{1}{2} \\ 1, & \textit{with probability } \frac{1}{2} \end{cases}$$

$$\omega \mid a > 1 = 1, \ \textit{with probability 1}$$

[1] The density need only be upper semi-continuous if $u(a,\omega)$ is always non-negative.

Notice that the density of this random variable is not continuous at $a = 1$. It is easy to show that Assumption 4 is not satisfied and the existence of a maximizer is therefore not guaranteed by our result. Indeed, if $u(a, \omega)$ is decreasing in a and increasing in ω then it is clear that the expected utility maximization problem need not have a solution.

In Example 1, the existence of a maximizer is not guaranteed because the random variable "jumps" at $a = 1$. This is precisely what is ruled out by Assumption 4. In general, utility maximization requires that the random variable is a well-behaved function of the choice variable. That is why we argue that some version of Assumption 4 is necessary.

However, intuition suggests that jumps are not necessarily an impediment to expected utility maximization. We illustrate this with the following example:

Example 2.

$$\omega \mid a < 1 = \begin{cases} 0, & \textit{with probability } \frac{1}{2} \\ 1, & \textit{with probability } \frac{1}{2} \end{cases}$$

$$\omega \mid a \geq 1 = 1, \ \textit{with probability 1}$$

Example 2 is consistent with the decision problem faced by workers in many principal-agent models. If a is the worker's level of effort and ω is her wage, we expect at least one solution to exist. However, the density of this random variable is also discontinuous at $a = 1$ and our result does not apply.

Nevertheless, we believe that this example highlights the importance of our result. That is because the existence of a maximizer is contingent on the choice of objective function. If $u(a, \omega)$ is increasing in a and decreasing in ω then it is easy to see that a maximizer need not exist. Unless additional assumptions are imposed on the objective function, the existence of a solution is not guaranteed and should not be assumed.

Motivated by this example, we establish the sufficiency of a much weaker condition when the objective function is monotone:

Lemma 2. *Suppose $u(a, \omega)$ is increasing in ω. Then Assumption 4 is satisfied if for every $a_n \to a$ either of the following holds:*

1. $\limsup_n f_{a_n} = f_a$
2. $\limsup_n f_{a_n}$ *is first order stochastically dominated by* f_a *for all* $a_n \to a$.

Proof. Fix any sequence $a_n \to a$. We want to show that:

$$\limsup_n \int_\Omega u(a,\omega) f_{a_n}(\omega) d\mu(\omega) \leq \int_\Omega u(a,\omega) f_a(\omega) d\mu(\omega)$$

If $\limsup_n f_{a_n} = f_a$ then this is immediate from the proof of Lemma 1. Otherwise:

$$\begin{aligned}
\limsup_n \int_\Omega u(a,\omega) f_{a_n}(\omega) d\mu(\omega) &= \limsup_n \mathbb{E}\big[u(a,\omega_{a_n})\big] \\
&\leq \mathbb{E}\big[u(a,\omega_a)\big]
\end{aligned}$$

The inequality comes from the fact that $u(a,\omega)$ is increasing in ω and ω_a first order stochastically dominates ω_{a_n}.

A symmetric argument can be used to show $\limsup_n f_{a_n}$ first order stochastically dominating f_a is sufficient when $u(a,\omega)$ is decreasing in ω.

$\square$

Applying this lemma to Example 2 gives the expected result – a maximizer exists if $u(a,\omega)$ is increasing in ω.

4 Conclusion

This paper establishes a general existence result for optimal decision-making when choices influence the probabilities of uncertain outcomes. We introduce a continuity condition – a version of upper semi-continuity for choice-dependent probability measures – which ensures upper semi-continuity of expected utility. Our topological proof does not require standard assumptions such as concavity of preferences or monotonicity of outcome distributions. Additionally, we identify sufficient conditions, including continuity of densities and stochastic dominance criteria, that allow the main assumption to be verified in relevant economic contexts. These findings expand the applicability of expected utility theory in settings with endogenous uncertainty.