A Theory of Efficient Price Cycles Under Imperfect Monitoring

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Abstract

This paper studies collusion between firms whose actions are constrained by behavioural or organizational concerns. I introduce a novel solution concept – pathwise ex post equilibrium (PXE) – for repeated games with imperfect public monitoring. In equilibrium, prescribed actions are required to be optimal for any possible realization of the signals. The resulting equilibria are ex post incentive compatible, have no punishment on the path of play, and are robust to different choice rules such as regret minimization and worst-case payoff maximization. When applied to a model with price setting firms, PXE yields sharp predictions about equilibrium outcomes: either firms play the same price in every period or prices follow a cyclical pattern consistent with the *rockets and feathers* price cycles documented in retail gasoline markets worldwide. I show that the observed cycles can serve as a robust and efficient mechanism for redistributing profits between firms and can therefore help sustain collusive agreements in the absence of explicit cash transfers. Price cycles may thus reflect deliberate choices rather than random shocks or coordination failures.

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1 Introduction

A rich theoretical literature studies collusion between firms in many different settings. However, most existing models assume that the firms are expected utility maximizers. In practice, firms typically face many additional constraints. For example, agency problems may result in employees making decisions that don't maximize their firm's expected profit; firms with external creditors may adopt a worst-case criterion to minimize the risk of bankruptcy; or ex post incentive compatibility may be necessary to foster trust and ensure the stability of a cartel.

This paper is motivated by the fact that retail gasoline prices in many cities exhibit a distinctive and persistent "rockets and feathers" pattern – a large and synchronous increase in pump prices is followed by many small and asynchronous decreases until prices approach marginal cost and the pattern repeats. Often called Edgeworth cycles or price cycles, this phenomenon has been studied extensively in the empirical literature but existing theories with expected utility maximizing firms struggle to offer a satisfactory explanation. Textbook Bertrand competition predicts marginal cost pricing. Competitive behaviour in the tradition of Maskin and Tirole (1988) can generate the observed price cycles, but only under strong assumptions that limit how often a firm can update its posted price. Folk theorem constructions suggest that colluding firms can sustain equilibria that yield more profit for both firms. Thus posing a significant challenge for any collusion based theory of price cycles. Random price wars caused by imperfect monitoring are also not consistent with the observed dynamics.[1]

This paper suggests that price cycles are the result of collusion between firms that face behavioural and organizational constraints. To study such firms, I model a repeated game with imperfect public monitoring. The firms post prices but do not observe their rival's posted price. They monitor each other using using a randomly updated price comparison app. The app generates public signals, which I call the observed prices. If the app updates, the observed price is equal to the rival's actual posted price. But if the app does not update, an outdated price is observed. The monitoring technology thus creates observational delays – changes in the rival's posted price are not observed until the price comparison app updates.

[1] A more detailed description of the observed price cycles and existing theories is contained in the next section.

I assume that the firms play actions which minimize their worst-case loss from not deviating. The loss from not deviating is defined as the difference between a firm's realized profit and the profit it could have obtained by playing a different action. Equilibria are identified using a novel solution concept called *pathwise ex post equilibrium* (PXE). PXE requires that equilibrium actions are optimal for any possible realization of the public signals (observed prices). As a result, there is no loss from not deviating. In other words, even after the worst-case realization of the public signals, equilibrium payoffs are greater than the payoffs after any deviation. Equilibrium actions are therefore ex post incentive compatible – a commonly used robustness criterion in the mechanism design literature.[2]

When studying collusion under imperfect monitoring, there are several reasons for focusing on equilibria that are ex post incentive compatible. First, firms are typically not certain if their rival has deviated or if they are observing an incorrect signal. So incentive compatibility requires the firms to sometimes initiate punishment even when no deviation has occurred. In practice, decision makers who are employees may be very reluctant to start a costly price war when there is no conclusive evidence of cheating. As a result, punishments may be delayed, undermining the credibility of the collusive agreement.

Second, the possibility of a price war can itself undermine the viability of collusion. That is because even a brief price war can be extremely costly and firms with external creditors, risk-averse managers, or short-term financial constraints may evaluate continuation payoffs using a worst-case criterion. In such cases, the risk of a price war may outweigh the expected gains from cooperation and make it impossible to sustain collusion, even when it is profitable on average.

Third, even in the absence of price wars, future payoffs are typically uncertain because they depend on the realization of noisy public signals. Many studies suggest that human decision makers facing uncertainty do not necessarily maximize expected utility.[3] They find evidence that decision makers often prioritize other objectives, such as minimizing future regret. Strategies that are not ex post incentive compatible may therefore invite second-guessing by the decision maker. This

[2]See for example Bergemann and Välimäki (2010) and Athey and Segal (2013).

[3]See for example Filiz-Ozbay and Ozbay (2007), Strack and Viefers (2021), Jhunjhunwala (2021), and Fioretti et al. (2022).

is particularly salient in my setting because collusion is illegal in most industries and thus not enforceable via formal contracts.

Consequently, PXE are credible predictors of equilibrium outcomes under a variety of decision rules and heuristics that are used by humans in practice. In particular, firms that minimize regret and firms that maximize worst-case payoffs both find it optimal to follow their equilibrium strategy in a PXE. Moreover, equilibrium actions are ex post incentive compatible so even employees who are concerned about justifying their actions in the future are not hesitant to follow strategies that are a PXE.

Under the assumed monitoring technology, PXE requires that delaying a price change is never profitable. More specifically, if firm i is expected to change its price between periods t and $t+1$, the new price cannot be less profitable than the old price. As a result, PXE do not require on-path punishments to ensure incentive compatibility. Firm i does not benefit by deviating and delaying the price change, so firm j does not need to provide incentives using continuation play. Even when the price comparison app does not update and the old price is observed, there is no need for firm j to initiate punishment.

In retail gasoline markets, the own price elasticity of demand facing an individual firm is typically very high. Consequently, price decreases are generally compatible with PXE but a price increase is feasible only when the old price is sufficiently close to marginal cost. Otherwise, the decrease in demand outweighs the increase in margin and makes a price increase unprofitable.

The behavioural considerations motivating PXE thus rationalize one key feature of price cycles – prices increase only when margins are sufficiently close to zero because earlier price increases are not ex post incentive compatible. That is because raising its price from an intermediate level is not myopically profitable for a firm. A collusive agreement must therefore use continuation play to ensure incentive compatibility. But continuation play is a function of the realized public signals and hence uncertain. So the loss from not deviating is positive after a bad signal realization and the price increase will not appear optimal in hindsight.

I use this result to characterize the entire set of price paths that may be observed in a PXE. One possibility is that the vector of prices becomes constant in finite time. The only other possibility is that the prices exhibit a cyclical pattern consistent with the price cycles observed in retail gasoline markets. Formally, this cyclical pattern consists of countably many blocks of finite length. Within each block, prices are non-increasing and every block ends with a period of low prices. Crucially, all price increases occur at the transition between blocks, when prices are sufficiently low. So any sequence of prices which fits this pattern will closely resemble the observed gasoline price cycles.

PXE thus rules out a large set of price paths. However, equilibria in which both firms play the same vector of prices in every period are not ruled out. Such constant price equilibria are a natural and intuitive benchmark when studying collusion between price setting firms. Any theory of price cycles must therefore explain why the firms do not coordinate on a constant cartel price. In my setting, that is because the set of payoff vectors which can be achieved via constant prices is too limited. For example, under the Bertrand model of competition, the firms have equal market share in any equilibrium with constant prices. However, in practice, one of the firms may demand a larger share of the gains from cooperation.[4] Existing papers typically sidestep this issue by allowing for cash transfers or assuming that the firms can split demand in any ratio they want. However, neither assumption is reasonable in the retail gasoline market. Beyond their choice of posted price, gas stations have practically no control over where a consumer purchases gasoline. And since collusion is illegal in most markets, the firms have a strong incentive to avoid direct transfers if they carry an increased risk of detection and prosecution.

To formalize this intuition, I show that the firms can use price cycles to achieve payoff vectors that cannot be achieved by any constant price equilibrium. By switching between periods with different relative prices, the firms shift demand and achieve an implicit transfer of profit, without any explicit cash payments. The importance of such transfers for sustaining long term cooperation is already well documented in the literature. In the retail gasoline context, there is additional evidence supporting my claim that firms use price cycles precisely because they want to redistribute profits. Clark and Houde (2013) documented evidence from the prosecution of a gas station cartel in Quebec, Canada. Former cartel members testified that they used asynchronous price changes to

[4] A formal model of bargaining between the firms is beyond the scope of this paper. But it is intuitive that some firms may have more bargaining power by virtue of their size, geographic location, access to credit etc.

transfer surplus and ensure the stability of the cartel. Clark and Houde (2013) also used data on prices and station characteristics to show that this mechanism can be used to transfer substantial amounts and can significantly reduce the frequency of deviations.

I also provide an explicit construction of strategies that are a PXE and generate the price cycles observed in retail gasoline markets. I show that there is no other equilibrium which can guarantee a higher payoff to both firms. The constructed price cycle is thus efficient and expands the set of equilibrium payoff vectors. Intuitively, the price cycle acts as a compromise between two distinct equilibria with constant prices. The price leader prefers the cycle over any equilibrium with equal prices because it is able to undercut the follower in some of the periods. The follower prefers the cycle over any equilibrium with unequal prices because it is able to match the leader in some of the periods.

To summarize, my results show that the firms use a price cycle when they wish to redistribute profits without explicit cash payments. The behavioural considerations formalized by PXE allow price increases only when the margins are sufficiently close to zero. This results in the distinctive rockets and feathers pattern observed in the data. Moreover, I show that price cycles are an efficient and robust transfer mechanism. They do not rely on a knife-edge choice of parameters and they do not require assumptions about sticky prices or unobserved cost shocks.

This paper thus contributes to three distinct but related fields. First, I build on the rich empirical literature that has meticulously documented and analyzed price cycles across numerous markets all over the world.[5] Foundational work in this area has established the key stylized facts that any successful theory must explain: the rockets and feathers pattern; adjustment delays during the undercutting phase; and the fact that prices increase only when they are close to marginal cost. While this body of work provides compelling evidence of the phenomenon, the underlying mechanism remains a subject of active debate. My primary contribution here is to offer a novel explanation for the observed price cycles. I demonstrate why they might emerge as an equilibrium outcome in a setting with rational firms competing under a realistic form of imperfect monitoring.

Second, my work engages with the classic literature on collusion in a repeated game with

[5] Some of this literature is surveyed in the next section.

imperfect monitoring. I study how the set of equilibrium outcomes changes when firms are guided by considerations beyond expected utility maximization. My focus is on behavioural and organizational concerns that shape decision making under uncertainty. But the equilibria I identify are robust and credible predictions of firm behaviour across a range of environments and selection criteria. The retail gasoline market provides a well-documented illustration of these mechanisms, but the framework applies more broadly to settings in which monitoring is imperfect, punishment is costly, and sustained collusion requires more than just positive expected payoffs.

Finally, I contribute to the literature on equilibrium refinements and behavioural game theory. The idea that human decision makers are motivated by a desire to avoid regret has a long history in economics. My solution concept – PXE – formalizes this concept within a repeated game with imperfect monitoring. In some ways, PXE is an extension of ex post perfect equilibria (XPE) as defined by Carroll (2024) and Krasikov and Lamba (2023). Whereas XPE is defined in a stochastic game with exogenous state transitions, PXE is defined in a setting where endogenous actions affect the uncertainty. PXE provides a powerful selection criterion in any dynamic game where cooperation must be sustained under uncertainty and the players desire robust and stable agreements. Because equilibrium actions are always ex post optimal, PXE is particularly valuable for analyzing settings in which trust and perceived fairness are important. The solution concept also extends to games with imperfect private monitoring, making it a flexible tool for understanding cooperation in a wide range of environments.

2 Description of Price Cycles

Empirical data on gasoline price cycles dates back to at least 1993. Castanias and Johnson (1993) found evidence of price cycles in Los Angeles between 1968 and 1972. Since then, a plethora of papers have documented gasoline price cycles in many different cities at many different points in time. Some examples include Eckert (2003), Noel (2007a), and Noel (2007b) in Canada; Lewis (2012) in the United States; Foros and Steen (2013) in Norway; and Wang (2009a), Byrne (2012), de Roos and Katayama (2013), and Byrne and de Roos (2019) in Australia. A more complete survey can be found in Eckert (2013).

While there are some differences, the overall shape and structure of price cycles is remarkably consistent across time and space. Fig. 1 reproduces a graph from Byrne et al. (2025) to illustrate a typical gasoline price cycle.[6]

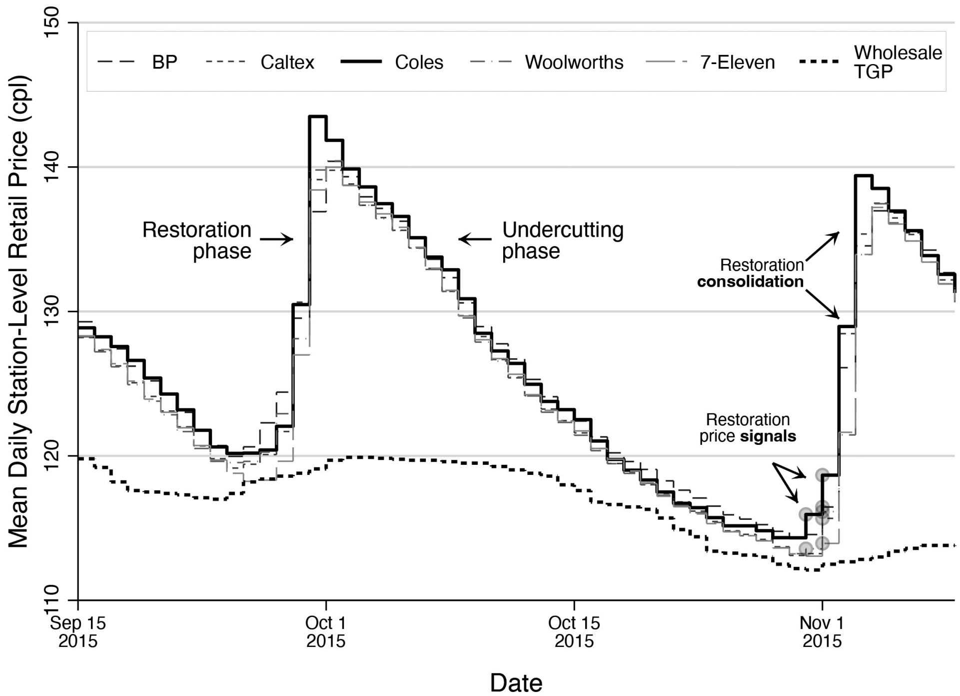


Figure 1: Example of a Price Cycle

Price cycles are characterized by two distinct phases. In the undercutting phase, stations decrease their prices asynchronously and in small increments. Notably, some stations act as price leaders by lowering their prices earlier and consequently enjoy elevated market shares for an extended period of time. This is puzzling because, in most jurisdictions, competing stations face minimal restrictions on how frequently they can update their own price. Stations can, and frequently do, adjust posted prices multiple times in a single day. So one would expect rapid price matching. Yet, in practice, competing stations often delay their response, allowing the price leaders to temporarily benefit from lower prices. This behaviour implies the existence of strategic considerations beyond simple short-run profit maximization.

In stark contrast, the restoration phase occurs abruptly and simultaneously. After a sustained

[6]Used with the permission of University of Chicago Press - Journals; permission obtained through Copyright Clearance Center, Inc.

period of incremental price cuts, retail prices collectively jump back to a higher common level. Crucially, this coordinated price restoration occurs only after margins approach marginal cost. Intermediate margin levels typically fail to trigger a price increase. This is also surprising because one expects an earlier price increase to raise profits for all firms.

In addition, it is important to keep in mind that gasoline is a largely homogeneous good. Individual gas stations face an extremely high own-price elasticity of demand – even small differences in prices often cause very large differences in market share. For example, Wang (2009b) estimates the elasticity to be as high as -18.77 for some gas stations. Given the high elasticity of demand, it is natural to expect that prices in a competitive market will be close to marginal cost. However, that is not the case in the retail gasoline market. As Fig. 1 illustrates, retail prices are frequently much higher than the wholesale price and changes in the two prices are often uncorrelated.

Maskin and Tirole (1988) showed that competitive behaviour can generate price cycles under Bertrand competition. However, their result relies on an alternating moves assumption under which each firm is only allowed to update its price in every other period. This assumption introduces a form of price stickiness that drives the slow and asynchronous nature of the undercutting phase. In periods when firm i is allowed to change its price, it undercuts firm j by the smallest increment possible. Firm i is able to do so because firm j is constrained to repeating its previous price and cannot respond to firm i's anticipated price. In retail gasoline markets firms are typically allowed to update their prices in every period and this mechanism breaks down. Firms respond to their rival's anticipated price and competition leads to marginal cost pricing.

Collusion based theories of price cycles also face significant challenges. In particular, if the firms are colluding, why are the prices frequently close to marginal cost? One would expect that the cartel can raise profits for both firms by effecting a price increase earlier in the cycle. On the surface, it appears suboptimal that prices increase only when margins are close to zero.

One possibility is that the firms are trying to sustain a high cartel price but imperfect monitoring leads to on-path price wars that manifest as price cycles. However, genuine price wars typically feature rapid and aggressive price cuts as firms actively attempt to undercut their rival. This is inconsistent with the observed price cycles which exhibit asynchronous price cuts and significant adjustment delays. On-path price wars in the spirit of Green and Porter (1984) thus do not appear to be a plausible explanation.

It is also worth noting that changes in marginal cost tend to be highly correlated in the retail gasoline industry. That is because the largest component of marginal cost is the wholesale price of gasoline, which is largely determined by global oil prices. Therefore, any theory which relies on unobserved cost shocks will struggle to model gasoline price cycles. In particular, a decrease in marginal cost may explain why a gas station lowers its price but it cannot explain why the rival delays matching the new price.

3 Model

I study a game with two firms indexed by $i \in \{1, 2\}$. Time is discrete and indexed by $t \in \{1, 2, \dots\}$. The firms have a common discount rate $0 < \delta < 1$. Both firms sell a single homogeneous good at a constant marginal cost given by c.

3.1 Actions and Signals

Define $p_i^t \in \mathcal{P}$ to be the actual price posted by firm i in period t. I assume that $\mathcal{P} = \{\epsilon, 2\epsilon, \dots, \overline{p}\}$ where $\epsilon > 0$ represents the smallest admissible change in price. $\overline{p}$ can be interpreted as the consumers' maximum willingness to pay for the good. Assuming that prices are discrete is both realistic and analytically convenient. Retail prices are always quoted in finite increments and there are many practical concerns preventing infinitely small changes.[7]

Monitoring is imperfect so firms do not observe the actual price posted by their rival. Instead, at the end of each period, the firms observe two independent public signals. The signal generating process is given by:

[7] In the United States, ϵ is typically one-tenth of a cent for gas stations.

$$
\begin{aligned}
(y_1^1, y_2^1) &= (p_1^1, p_2^1) \\
y_1^t &= \begin{cases} p_1^t & \text{with probability } 1-\gamma \\ y_1^{t-1} & \text{with probability } \gamma \end{cases} \\
y_2^t &= \begin{cases} p_2^t & \text{with probability } 1-\gamma \\ y_2^{t-1} & \text{with probability } \gamma \end{cases}
\end{aligned}
$$

where $0 < \gamma < 1$ is fixed and common knowledge. The signals are independent so it is possible that $y_1^t = p_1^t$ while $y_2^t = y_2^{t-1}$ and vice-versa. I will use $y^t = (y_1^t, y_2^t) \in \mathcal{Y} \times \mathcal{Y}$ to denote the public signal profile observed at the end of period t. Note that $\mathcal{Y} = \mathcal{P}$.

This monitoring technology is consistent with a price comparison app that updates at random times. When a firm observes its rival's price on the app, it does not know if the app updated in that period. So the observed price may be accurate or it may be an older price. In particular, when a firm changes its posted price, the rival does not always learn the new price immediately. This informational friction captures a common challenge faced by firms in many industries – changes in actions are often observed with a delay.

3.2 Demand

In order to focus on collusion between the firms, I abstract away from a formal model of consumer choice. However, I assume that the demand for firm i in period t is a function of p_i^t, y_i^t, and y_j^t.

One interpretation is that consumers don't observe the posted prices so they use the public signals to decide where to shop. Demand is thus a function of both public signals. If they make a purchase, consumers have to pay the actual price posted by their chosen firm. Consumers learn the posted price before completing their purchase and can alter their consumption decision if the actual price is different from the observed signal. Demand for firm i is therefore affected by p_i^t as well.

Formally, the demand for firm i in period t is given by $D_i(p_i^t, y_1^t, y_2^t)$ and I assume:

Assumption 1. $D_i(p_i^t, y_1^t, y_2^t)$ is uniformly bounded by M.

Assumption 2. $D_i(p_i^t, y_1^t, y_2^t)$ is weakly increasing in y_j^t.

Assumption 3. There exists $\eta > 0$ such that:

$D_i(p_i^t, y_1^t, y_2^t) \geq (1+\eta)\ D_i(y_i^t, y_1^t, y_2^t)$ if $p_i^t < y_i^t$.

$D_i(p_i^t, y_1^t, y_2^t) \leq (1-\eta)\ D_i(y_i^t, y_1^t, y_2^t)$ if $p_i^t > y_i^t$.

Assumption 2 states that the demand for firm i is increasing in its rival's signal. The implicit assumption is that some consumers switch to firm i when the observed price signal for firm j increases.

Assumption 3 states that the demand for firm i is affected by p_i^t even when the public signals don't update. That is because consumers pay p_i^t and can alter their consumption decision if the posted price turns out to be different from the observed signal.

The stage payoff for firm i in period t is given by:

$$u_i(p_i^t, y_i^t, y_j^t) = D(p_i^t, y_i^t, y_j^t)(p_i^t - c)$$

Note that the demand and stage payoff functions are the same for both firms. However, I use subscripts to clarify which firm's demand / payoff is being referred to.

3.3 Histories and Strategies

Define $h^t = (y^1, y^2, \ldots, y^{t-1})$ to be the public history at the beginning of period t. Let $\mathcal{H}^t$ be the set of all public histories of length $t-1$. Then $\mathcal{H}^\infty$ is the set of all infinite public histories with generic element $h = (y^1, y^2, \ldots)$.

Let s_i be a public strategy for firm i. Then s_i is a measurable mapping from the set of public histories to the set of actions. Throughout this paper, I restrict attention to public strategies.

Let $\mathcal{Y}_i(s_i \mid h^t)$ be the set of individual signals that are consistent with the public strategy s_i and the public history h^t. That means every element of $\mathcal{Y}_i(s_i \mid h^t)$ is in the support of y_i^t when firm i plays according to s_i after h^t is realized.

I say that $h \in \mathcal{H}^\infty$ is *consistent* with the public strategy profile $s = (s_1, s_2)$ if, for every t, there is a positive probability that the public history at the beginning of period t is given by the first $t-1$ elements of h when the firms play according to s.

I use $\mathcal{H}^\infty(s)$ to denote the set of all infinitely long histories that are consistent with s. I say that $h \in \mathcal{H}^\infty(s \mid h^t)$ if for every $\tau > t$, there is a positive probability that the public history at the beginning of period τ is given by the first $\tau - 1$ elements of h when the firms play according to the continuation strategy profile $s \mid h^t$ after h^t is realized.

3.4 Payoffs and Preferences

Fix any public history h^t. Assume that the firms play according to the public strategy profile s. Then the *guaranteed* payoff for firm i at the beginning of period t is given by:

$$g_i^t(s \mid h^t) = \inf_{h \in \mathcal{H}^\infty(s|h^t)} (1-\delta) \sum_{\tau \geq t} \delta^{\tau - t}\, u_i\big(s_i(h^\tau), y_i^\tau, y_j^\tau\big)$$

where (y_i^τ, y_j^τ) is the τ element of $h \in \mathcal{H}^\infty(s \mid h^t)$ and h^τ is equal to the first $\tau - 1$ elements of h.

Lemma 1. The following exists for every i, every s, and every h^t:

$$\min_{h \in \mathcal{H}^\infty(s|h^t)} (1-\delta) \sum_{\tau \geq t} \delta^{\tau - t}\, u\big(s_i(h^\tau), y_i^\tau, y_j^\tau\big)$$

Since the minimum always exists, I can redefine guaranteed payoffs using the minimum instead of the infimum. The proof is contained in the appendix but the intuition is that the set of infinitely long histories is compact. The subset that is consistent with s is also compact and the objective function is continuous over that subset. And the minimum of a continuous function on a compact set always exists.

Notice that the guaranteed payoffs at time t depend only on the public history and the choice of continuation strategies. Moreover, there is no ambiguity so each firm's prior is a singleton and the rectangularity condition from Epstein and Schneider (2003) is trivially satisfied. As a result, guaranteed payoffs can be written as:

$$g_i^t(s \mid h^t) = \min_{(y_i^t, y_j^t) \in \mathcal{Y}(s \mid h^t)} \Big\{ (1-\delta)\ u\big(s_i(h^t), y_i^t, y_j^t\big) + \delta\ g_i^{t+1}(s \mid h^t, y_i^t, y_j^t) \Big\} \tag{1}$$

That is because:

$$g_i^{t+1}(s \mid h^t, y_i^t, y_j^t) = \min_{h \in \mathcal{H}^\infty(s \mid h^t, y_i^t, y_j^t)} (1-\delta) \sum_{\tau \geq t+1} \delta^{\tau-t-1}\ u\big(s_i(h^\tau), y_i^\tau, y_j^\tau\big)$$

Notice that $h \in \mathcal{H}^\infty(s \mid h^t)$ if $h \in \mathcal{H}^\infty(s \mid h^t, y_i^t, y_j^t)$ and $(y_i^t, y_j^t) \in \mathcal{Y}(s \mid h^t)$. So the RHS of (1) can be written as:

$$\begin{aligned}
&= \min_{(y_i^t, y_j^t) \in \mathcal{Y}(s \mid h^t)} \Big\{ (1-\delta)\ u\big(s_i(h^t), y_i^t, y_j^t\big) + \min_{h \in \mathcal{H}^\infty(s \mid h^t, y_i^t, y_j^t)} (1-\delta) \sum_{\tau \geq t+1} \delta^{\tau-t-1}\ u\big(s_i(h^\tau), y_i^\tau, y_j^\tau\big) \Big\} \\
&= \min_{h \in \mathcal{H}^\infty(s \mid h^t)} (1-\delta) \sum_{\tau \geq t} \delta^{\tau-t}\ u\big(s_i(h^\tau), y_i^\tau, y_j^\tau\big) \\
&= g_i^t(s \mid h^t)
\end{aligned}$$

Dynamic consistency is satisfied because, conditional on s and h^t, the value of $g_i^{t+1}(s \mid h^t)$ is a deterministic function of (y_i^t, y_j^t). So, at the beginning of period t, each firm can correctly anticipate what its guaranteed payoff can be starting in period $t+1$. In particular, information revealed in period t does not change the value of any $g_i^{t+1}(s \mid h^t, y_i^t, y_j^t)$.

With a slight abuse of notation I define:

$$g_i^t(s \mid h^t, y_j^t) = \min_{y_i^t \in \mathcal{Y}_i(s_i \mid h^t)} \Big\{ (1-\delta)\ u_i(s_i(h^t), y_i^t, y_j^t) + \delta\ g_i^{t+1}(s \mid h^t, y_i^t, y_j^t) \Big\}$$

Now define:

$$r_i(s, h^t) = \max_{y_j^t \in \mathcal{Y}_j(s_j \mid h^t)} \max_{\hat{s}_i \neq s_i} \Big\{ g_i^t(\hat{s}_i, s_j \mid h^t, y_j^t) - g_i^t(s_i, s_j \mid h^t, y_j^t) \Big\}$$

Then $r_i(s, h^t)$ can be interpreted as the maximum *regret* experienced by firm i at the end of period t when both firms play according to the public strategy profile s. It is the maximum payoff that firm i can gain by deviating in period t. I assume that the firms want to minimize $r_i(s, h^t)$. So after h^t is realized, the optimization problem for firm i is to pick a strategy s_i such that:

$$s_i \in \arg\min_{s_i'} r_i(s_i', s_j, h^t)$$

Remark 1. When the firms are expected utility maximizers, they minimize expected regret instead of maximum regret.

Remark 2. The realization of y_j^t depends only on the public history h^t and firm j's strategy s_j. Neither is affected by firm i's action in period t. That is why firm i treats the realization of y_j^t as fixed when evaluating regret at the end of period t.

3.5 Equilibria

Definition 1. A profile of public strategies s is a *pathwise ex post equilibrium* (PXE) if for both i and every public history h^t:

$$r_i(s, h^t) \leq 0$$

In practice, this definition can be difficult to work with. So I will prove that the following definition is equivalent and use it for most of my subsequent results:

Definition 2. A profile of public strategies s is a PXE if for both i, every public history h^t, every $y_j^t \in \mathcal{Y}_j(s_j \mid h^t)$, and every one shot deviation $\hat{a}_i^t \neq s_i(h^t)$:

$$\min_{y_i^t \in \mathcal{Y}_i(s_i | h^t)} \left\{ \begin{aligned} &(1-\delta)\ u_i\big(a_i^t, y_i^t, y_j^t\big) \\ &+\ \delta\ g_i^{t+1}\big(s \mid h^t, y_i^t, y_j^t\big) \end{aligned} \right\} \geq \min_{\hat{y}_i^t \in \mathcal{Y}_i(\hat{s}_i | h^t)} \left\{ \begin{aligned} &(1-\delta)\ u_i\big(\hat{a}_i^t, \hat{y}_i^t, y_j^t\big) \\ &+\ \delta\ g_i^{t+1}\big(s \mid h^t, \hat{y}_i^t, y_j^t\big) \end{aligned} \right\} \tag{2}$$

where $a_i^t = s_i(h^t)$ is the prescribed action for firm i.

Lemma 2. A public strategy profile s is a PXE if and only if it satisfies the condition in Definition 2.

The left-hand side of Condition (2) is firm i's guaranteed payoff in equilibrium. The right-hand side is its guaranteed payoff after a deviation in period t. Both terms condition on the same realization of y_j^t. Thus, PXE requires that, for any realization of y_j^t, the guaranteed payoff in equilibrium is no less than the guaranteed payoff after a deviation.

PXE is closely related to ex post perfect equilibrium (XPE) as defined by Carroll (2024) and Krasikov and Lamba (2023). Like XPE, PXE requires that equilibrium actions are ex post incentive compatible. In other words, the prescribed actions in period t must be optimal for any possible realization of the public signals in period t. Intuitively, ex post incentive compatibility means that actions are dominant with respect to the set of public signals that may be realized in equilibrium. This requirement is much stronger than standard sequential rationality, which only requires that actions are incentive compatible in expectation.

However, PXE and XPE differ in some subtle but important ways. XPE is defined in a stochastic game framework, where the uncertainty is over the exogenous realization of future stage games. In contrast, the uncertainty in my setting is over the realization of future public signals, which depend on the firms' endogenous actions.

Because the realization of public signals depends on endogenous choices, formalizing ex post incentive compatibility is challenging in this setting. In equilibrium, firms never deviate, so the sequence of public signals following a deviation is never observed. Incentive compatibility therefore relies on beliefs about what would have happened if a deviation had occurred. PXE assumes that firms evaluate deviation payoffs under the worst possible realization of public signals. This is ultimately a modelling choice and an argument can be made for alternative criteria. However, most subsequent results are unchanged if firms instead use the *expected* sequence of public signals.

PXE's restrictions make it a credible predictor of behaviour in many settings. Because equilibrium actions are ex post incentive compatible, PXE are easy to sustain even when the decision maker is an employee who might have to justify her choices in the future. She can be certain that the prescribed action will appear optimal, regardless of which public signals are realized. So the employee knows that she won't be penalized for following the equilibrium strategy.

A related benefit is that firms playing a PXE experience no regret on the path of play. Because equilibrium actions are optimal for any possible realization of the public signals, there is no realization such that a deviation would have yielded a higher guaranteed payoff. Consequently, the firms never lose payoff by not deviating and never wish to revise past actions.

Finally, PXE ensures that the worst-case payoffs in equilibrium exceed those after any deviation. If the equilibrium requires a firm to sacrifice some payoff today, it knows that, even in the worst case, future compensation will offset today's loss. Consequently, even firms that maximize worst-case payoffs, instead of expected payoffs, will not deviate if the strategy profile is a PXE.

Remark 3. Any dominant strategy equilibrium of the stage game is always a PXE. For example, when the stage game is prisoner's dilemma, it is a PXE for both players to defect after every history.

In general, a mixed strategy equilibrium of the stage game cannot be supported in a PXE. For example, when the stage game is rock paper scissors, there are no off-punishments that make it ex post incentive compatible for both players to randomize over their three actions. Intuitively, if player 1's randomization results in rock and player 2's randomization results in paper, player 1 regrets not deviating and playing scissors.

Definition 3. An equilibrium with strategy profile s is *dominated* by an equilibrium with strategy profile s' if $g_i(s') \geq g_i(s)$ for all i with at least one strict inequality.

An equilibrium is *efficient* if it is not dominated by any other equilibrium.

Definition 4. An equilibrium is *stationary* if $s(h^t) = p = (p_i, p_j)$ for all on-path histories.

Note that a stationary equilibrium need not have $p_i = p_j$.

4 General Results

All proofs are included in the appendix but I provide some intuition where appropriate.

Lemma 3. Fix a mixed strategy equilibrium σ. There exists a pure strategy equilibrium s such that $g(s) = g(\sigma)$.

I will restrict attention to pure strategies for the remainder of this paper. Lemma 3 establishes that doing so is without loss of generality. The intuition for Lemma 3 comes from the fact that the guaranteed payoffs in any equilibrium are determined by the worst-case sequence of public signals. If σ is a mixed strategy equilibrium, the worst-case sequence of signals under σ must be consistent with some pure strategy profile that is in the support of σ. The proof of Lemma 3 shows that this pure strategy profile can also be sustained in equilibrium and can therefore replicate the guaranteed payoffs under σ.

Proposition 1. Fix a strategy profile s and a public history h^t. Then s is a PXE only if:

$$s_i(h^t) = p_i^t \neq y_i^{t-1} \implies u_i(p_i^t, y_i^{t-1}, y_j^t) \geq u_i(y_i^{t-1}, y_i^{t-1}, y_j^t)$$

where y_j^t is any public signal which is realized with positive probability.

Proposition 1 formalizes the main restriction imposed by PXE on equilibrium strategies. A firm will not change its price if the new price is less profitable than the old price. Conversely, if the equilibrium requires a firm to update its price, the new price must yield a higher stage payoff than the old price.

Suppose this condition is not satisfied – the equilibrium requires firm i to update its price but p_i^t yields a strictly lower stage payoff. Firm i has an incentive to deviate and delay the price change i.e. play y_i^{t-1} again. To play p_i^t and sacrifice payoff in period t, firm i must be incentivized using continuation play. However, any sequence of public signals which may be realized after y_i^{t-1} may also be realized after firm i plays p_i^t. That is because the price comparison app can fail to update and display y_i^{t-1} even when firm i plays p_i^t. So the worst-case continuation payoff after playing p_i^t cannot be higher than the worst-case continuation payoff after deviating to y_i^{t-1}. This violates ex post incentive compatibility since the total guaranteed payoff for firm i is strictly higher if it deviates.

Remark 4. In equilibrium, p_i^t and y_i^{t-1} are the only signals that are observed with positive probability. Proposition 1 establishes that p_i^t always yields weakly higher stage payoff. So firm i already has an incentive to play p_i^t over y_i^{t-1}. Firm j does not need to provide incentives using continuation play. In particular, firm i will not deviate even if its continuation payoffs are the same under both signal realizations. That is why I say that there is no punishment on the path of play in a PXE.

Lemma 4. If $y_i^{t-1} > \underline{p} = c + (1-\eta)(\overline{p} - c)$ then $p_i^t \leq y_i^{t-1}$ in every PXE.

When demand is elastic, it is profitable for a firm to raise its price only when the old price is sufficiently close to marginal cost. Otherwise, the decrease in demand outweighs the increase in margin and profit decreases. Lemma 4 formalizes this intuition and derives an upper bound on the old price. When its old price is greater than $\underline{p}$, it is never profitable for a firm to raise its price. So the new price must be weakly lower. I stress that the derived bound is a function of the elasticity parameter η. When η is close to zero, the restriction imposed by Lemma 4 may be trivial. But when demand is very elastic, as is typically the case in the retail gasoline market, $\underline{p}$ will be close to marginal cost.[8]

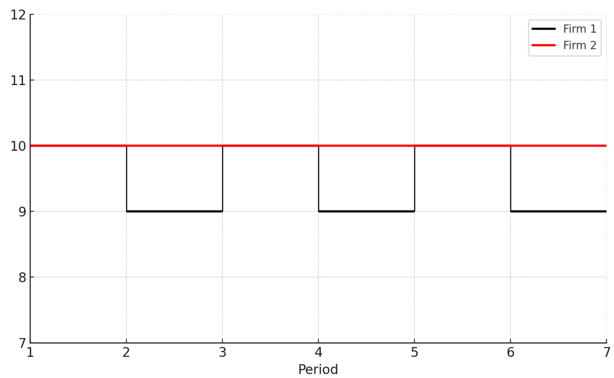
Lemma 4 thus rationalizes one of the key features of the observed price cycles – prices increase only when they are sufficiently close to marginal cost. The reason being that the behavioural considerations formalized by PXE rule out price increases from intermediate levels. So while they may be inefficient in an unconstrained setting, the observed strategies can be optimal given the PXE constraints.

[8] I remind the reader that some studies estimate the own price elasticity of demand to be as high as -18.77 for gas stations.

Example 1. Consider the following profile of on-path strategies:

$$p_1^t = \begin{cases} 10 & \text{if } t \text{ is odd} \\ \\ 9 & \text{if } t \text{ is even} \end{cases} \qquad \text{and} \qquad p_2^t = 10$$

When the firms play according to these strategies, one possible sequence of observed prices is:



When $c = 0$ and η is sufficiently large, the on-path strategies in Example 1 *cannot* be supported in a PXE. That is because $p_i^t = 9$ is more profitable than $p_i^t = 10$ for firm 1. So there are no off-path strategies that can make it ex post incentive compatible for firm 1 to raise its price from 9 to 10.

The significance of Example 1 is that it shows how PXE can rule out an intuitive way for the firms to redistribute profits. When the prices are already high, PXE prevents further increases. So any change in market share requires at least one firm to lower its price. Consequently, when the prices are high, the firms have only two choices. Either they can maintain the current price and get the same vector of payoffs in every subsequent period; or at least one firm must lower its price to a level which might appear suboptimal.

Remark 5. In a more general setting, PXE restricts transitions between action profiles. In particular, if it is feasible to switch from an action profile a to a different action profile a', it is typically not feasible to switch from a' to a. That is because of the restriction imposed by Proposition 1. If a player is willing to switch from a to a' then she must prefer a' over a. But that means she does not prefer a over a' and is therefore unwilling to make that switch.[9]

[9]The only exception is when the player changing her action is indifferent between the two action profiles.

Proposition 2. Fix any $h \in \mathcal{H}^\infty(s)$ where s is a PXE.

Then either there exists some T such that $y^t = y^T$ for every $t \geq T$.

Or there exists a strictly increasing sequence of natural numbers (τ_n) such that:

1. $\tau_1 = 1$.
2. y^t is non-increasing when $\tau_n < t < \tau_{n+1}$.
3. There exists i such that $y_i^t \leq \underline{p}$ when $t = \tau_n - 1$ for any $n \geq 2$.

Note that y^t is a vector. So y^t non-increasing means that $y_i^t \leq y_i^{t-1}$ for all i.

Proposition 2 uses Lemma 4 to characterize the set of price paths which may be observed in a PXE. It establishes that any sequence of observed prices will be (eventually) stationary or it will follow the cyclical pattern observed in gasoline markets. More specifically, any non-stationary sequence of prices can be split into a countable number of blocks such that, within each block, prices are non-increasing and all price increases happen at the transition between blocks, when prices are sufficiently low.

5 Efficiency of Price Cycles

The results in Section 4 provide some intuition for why price cycles are observed in retail gasoline markets. By changing their posted price, the firms are able to shift demand and alter the distribution of profits. But the constraints imposed by PXE do not allow for arbitrary changes in price. In particular, prices can increase only when they are sufficiently close to marginal cost. This forces most changes to be price cuts and leads to the distinctive rockets and feathers pattern.

However, it is not obvious that price cycles are efficient. Because margins are often close to zero in a price cycle, one may be concerned that both firms can increase their profit by playing an equilibrium with a stationary cartel price. This section shows that that is not the case. Price cycles can be used to achieve a vector of payoffs that is not dominated by any stationary equilibrium.

To fix ideas, I will use the following example throughout this section:

Example 2. Fix a $p^{min} < \overline{p}$. On path actions are given by:

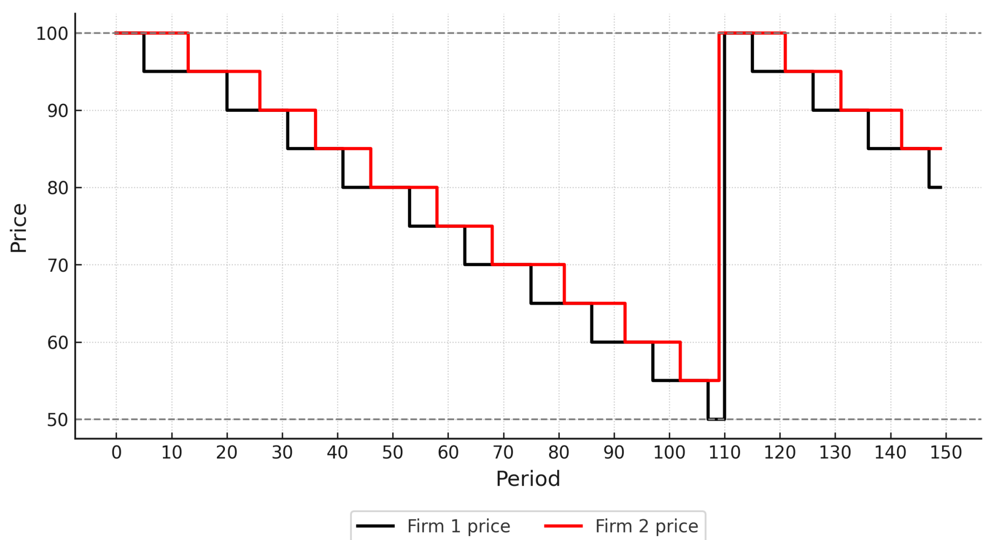
$$(p_1^1, p_2^1) = (\overline{p}, \overline{p})$$

$$p_1^t = \begin{cases} \overline{p} & \text{if } (y_1^{t-1}, y_2^{t-1}) = (p^{min}, p^{min}) \\ y_1^{t-1} - \epsilon & \text{if } y_1^\tau = y_2^\tau \text{ for all } t - k_1 \leq \tau \leq t-1 \\ y_1^{t-1} & \text{otherwise} \end{cases}$$

$$p_2^t = \begin{cases} \overline{p} & \text{if } (y_1^{t-1}, y_2^{t-1}) = (p^{min}, \overline{p}) \\ y_1^{t-1} & \text{if } p^{min} < y_1^\tau < y_2^\tau \text{ for all } t - k_2 \leq \tau \leq t-1 \\ y_2^{t-1} & \text{otherwise} \end{cases}$$

Fig. 2 shows a sequence of prices that may be observed when the firms play according to the strategies in Example 2.

Figure 2: Sequence of Observed Prices



In this example, both firms start by posting $\overline{p}$ as their price in the first period.[10] The firms post the same price in the first k_1 periods of the game. In period k_1+1, firm 1 undercuts its rival and lowers its price to $\overline{p} - \epsilon$. Firm 2 does not react immediately and continues to post $\overline{p}$. After k_2 periods

[10]Recall that $\overline{p}$ is the upper bound of the set of prices.

of unequal prices, firm 2 also lowers its price by ϵ and both firms post $\overline{p} - \epsilon$. This process repeats until both firms start posting p^{min}. At that point, firm 1 raises its price to $\overline{p}$, firm 2 matches, and the cycle restarts.

As is clear from Fig. 2, the resulting sequence of prices matches the price cycles observed in retail gasoline markets. So I will start by proving that the strategies in Example 2 can be supported in a PXE under mild assumptions. First define:

$$
\begin{aligned}
D(y_1^t, y_2^t) &= D_1(y_1^t, y_1^t, y_2^t) + D_2(y_2^t, y_1^t, y_2^t) \\
\alpha_i(y_1^t, y_2^t) &= \frac{D_i(y_i^t, y_1^t, y_2^t)}{D(y_1^t, y_2^t)}
\end{aligned}
$$

$D(y_1^t, y_2^t)$ is the total market demand when both public signals are accurate. $\alpha_i(y_1^t, y_2^t)$ is the market share for firm i when both signals are accurate.

Assumption 4. $D_i(p_i^t, y_i^t, y_j^t) > 0$ for every p_i^t, y_i^t, and y_j^t.

Then there exists $\psi \in (\eta, 1)$ such that for every $p_i^t > y_i^t$:

$$
D_i(p_i^t, y_i^t, y_j^t) \geq (1 - \psi) D_i(y_i^t, y_i^t, y_j^t)
$$

Assumption 5. For every $h \geq \epsilon$:

$$
\frac{D(y_i^t - h, y_j^t) - D(y_i^t, y_j^t)}{D(y_i^t, y_j^t)} < \alpha_i(y_i^t, y_j^t) \frac{h}{\max\{y_i^t, y_j^t\}}
$$

Assumption 6.

$$
\epsilon < \frac{\eta}{1 + \eta}(1 - \psi)(\overline{p} - c)
$$

Assumption 4 requires only that demand is always positive. The existence of an appropriate ψ is a consequence of the fact that the price grid is finite. Assumption 4 thus implies that there is an upper bound on the change in demand when firm i raises its price but the public signal does not update. Contrast this with Assumption 3 which imposes a lower bound on the change in demand. Collectively, the two assumptions require that, even when the public signal does not update, an increase in price results in strictly lower demand but demand never falls to zero.

Assumption 5 is an inelasticity assumption. Intuitively, it requires that the change in total market demand is always less than the market share weighted change in price. Note that Assumption 5 does not contradict my earlier claim that the own price elasticity of demand is very high for individual firms in the retail gasoline market. The reason is that, when firm i lowers its price, most of the additional demand comes from consumers who were previously purchasing from firm j. So the increase in total demand is small even though the increase firm i's demand is large.

I also stress that Assumption 5 restricts how demand changes when the public signals are accurate. In comparison, Assumption 3 and Assumption 4 restrict how demand reacts when the public signals do not update. The assumptions are therefore not directly comparable.

Assumption 5 has two useful implications:

Corollary 1. Total profits are increasing in prices when both public signals are accurate. Additionally, for any $h \geq \epsilon$:

$$u_i(p_i, p_j) - u_i(p_i - h, p_j - h) > 0$$

If both public signals are accurate, Corollary 1 says that total profits are maximized when the prices are $(\overline{p}, \overline{p})$. Moreover, both firms receive strictly lower profits if they implement the same price cut.

Assumption 6 requires that the elasticity of demand (captured by η) is large relative to the smallest feasible change in prices (captured by ϵ). When demand is smooth, which I do not assume, η approaches zero as ϵ approaches zero. In that setting, Assumption 6 places a restriction on the relative rate at which the two variables approach zero. When demand is not smooth – a common feature in many models – Assumption 6 becomes trivial.

Given these assumptions, I can pick $p^{min} > c$ such that:

$$p^{min} \leq c + (1 - \psi)(\overline{p} - c)$$
$$p^{min} \geq c + \epsilon + \frac{\epsilon}{\eta}$$

Lemma 5. There exists $\delta^* < 1$ such that the on-path actions in Example 2 can be supported in a PXE for any k_1 and k_2 when $\delta > \delta^*$.

The formal proof is contained in the appendix but the off-path punishments are very simple. If firm i deviates and posts a price that is lower than the prescribed price, firm j responds by lowering its own price. Firm i's demand is decreasing in y_j^t so firm i's payoffs in the deviation phase are lower than its payoffs on path. For a sufficiently high δ, lower demand in the punishment phase thus deters deviations.

The equilibrium in Example 2 is also efficient under some additional assumptions:

Assumption 7. $D(y_1^t, y_2^t)$ is concave on $\mathcal{Y} \times \mathcal{Y}$.

Assumption 8. The slope of $\alpha_i(y_1^t, y_2^t)D(y_1^t, y_2^t) = D_i$ with respect to y_i^t is weakly decreasing in $\mid y_1^t - y_2^t \mid$.

Assumption 9. For every $y_i^t < p_i^t$:

$$D_i(p_i^t, y_i^t, y_j^t) \leq D_i(p_i^t, p_i^t, y_j^t)$$

Assumption 7 is a standard concavity assumption. Assumption 8 requires that, if both public signals are accurate, each firm's demand is most sensitive to price changes when the two prices are equal. Conversely, when the difference in prices is large, a unilateral price change by firm i leads to a relatively small change in its demand. Assumption 8 is consistent with a world in which some consumers are more price sensitive than others. When firm i undercuts its rival by ϵ, a mass of consumers shift their demand from firm j to firm i. The consumers who continue to purchase from firm j are less price sensitive so further price cuts by firm i result in smaller changes in market shares. Intuitively, Assumption 8 is satisfied when the demand for firm i, conditional on the price of firm j, has a sigmoid shape centred on firm j's price.

Notice that Assumption 8 is a weaker version of the assumption imposed by the Bertrand model of price competition. Under Bertrand competition, the firm with the lower price serves the entire market. So D_i is vertical when $y_1^t = y_2^t$ and horizontal everywhere else. Such an extreme form of demand is consistent with Assumption 8 but is unnecessary for my results. Also note that Assumption 8 imposes no restriction on the elasticity of individual demand. Intuitively, it is a restriction on the second derivative of demand while elasticity is a function of the first derivative.[11]

Assumption 9 is a very strong assumption that is imposed to rule out one counterintuitive scenario – total profits being maximized when both price signals are at their lowest level. As stated, Assumption 9 requires that a firm does not benefit if its observed price signal decreases while the actual posted price remains constant. The intuition for this assumption is that a lower price signal may change where a consumer shops but the quantity actually purchased is determined by the posted price.

I stress that Assumption 7 and Assumption 8 are needed only to ensure that the specific equilibrium in Example 2 is efficient. All other results are valid when these assumptions do not hold. Moreover, even when Assumption 7 or Assumption 8 is violated, there typically exists a price cycle equilibrium which is efficient. In that scenario, the key difference will be that the firms may lower their price by more than ϵ at a time.

Proposition 3. Under Assumptions 1 - 9, there exists $k^* < \infty$ such that the equilibrium in Example 2 is efficient when $k = \min\{k_1, k_2\} > k^*$.

Intuitively, a price cycle can be efficient because it allows the firms to switch between periods with different relative prices. If, instead, the firms play a stationary equilibrium with equal prices, firm 1's payoff will be strictly lower because it will not enjoy any periods with extra demand. Similarly, if the firms play a stationary equilibrium with different prices, firm 2's payoff will be strictly lower because there will be no periods with equal demand.

The proof of Proposition 3 relies on the fact that PXE rules out arbitrary changes in price. In particular, it is not ex post incentive compatible for firm 1 to raise its price when the public signals are $(\overline{p} - \epsilon, \overline{p})$. That is why, in the constructed example, the sequence of prices is:

[11] I do not assume that demand is differentiable. This discussion of derivatives is only for the purposes of providing intuition.

$$(\overline{p},\overline{p});(\overline{p}-\epsilon,\overline{p});(\overline{p}-\epsilon,\overline{p}-\epsilon);\ldots$$

and not:

$$(\overline{p},\overline{p});(\overline{p}-\epsilon,\overline{p});(\overline{p},\overline{p});\ldots$$

The alternate sequence is more profitable for both firms but it is not within the set of PXE outcomes. That is why it does not dominate the constructed example.

The next step is proving that it is efficient for firm 1 to lower its price in the first place. When firm 2 eventually matches the price cut, both firms end up with the same market shares but at a lower price. Nevertheless, it is profitable for firm 1 to lower its price if the phase with unequal prices is sustained for sufficiently long. The reason is that future payoffs are discounted. So the extra demand today is more valuable to firm 1 than the lost margin in the future. The same argument can be used to show why firm 2 matches firm 1's price, even though that causes firm 1 to eventually lower its price further.

Perhaps the most striking feature of this result is the fact that the constructed equilibrium is efficient even though the firms play $(\overline{p}-\epsilon,\overline{p}-\epsilon)$, which is strictly dominated (in the static sense) by $(\overline{p},\overline{p})$. The reason is that any additional periods with the prices at $(\overline{p},\overline{p})$ must come at the beginning of the game – because PXE prevents the firms from returning to $(\overline{p},\overline{p})$ when the signals are $(\overline{p}-\epsilon,\overline{p})$. Any additional period with the prices at $(\overline{p},\overline{p})$ therefore delays the first period in which firm 1 can undercut its rival. Because of discounting, this delay causes a discrete drop in firm 1's payoff. By instead switching from $(\overline{p}-\epsilon,\overline{p})$ to $(\overline{p}-\epsilon,\overline{p}-\epsilon)$ later in the game, the firms can achieve a smaller transfer of payoff from firm 1 to firm 2.

In effect, the set of payoffs that can be guaranteed with only two distinct price vectors is discrete and sparse. Playing a third price vector later in the game allows the firms to make smaller adjustments to their relative payoffs. The equilibrium therefore trades a small amount of total surplus for finer control over the distribution of surplus across firms. Under PXE, this trade-off preserves overall efficiency. No alternative sequence of prices can make both firms better off.

6 Conclusion

This paper introduces a novel solution concept – pathwise ex post equilibrium (PXE) – to study how behavioural and organizational concerns shape equilibrium outcomes in a repeated game with imperfect monitoring. My results suggest that the rockets and feathers price cycles commonly observed in retail gasoline markets may not be caused by competitive undercutting or random shocks. Instead, they can be a deliberate choice, serving as a robust and efficient mechanism for the firms to redistribute profits and sustain collusive agreements.

The model assumes a realistic form of imperfect monitoring that reflects the realities of the retail gasoline market – firms observe their rival's price through randomly updated and potentially delayed public signals, such as online price-tracking applications. Under these conditions, PXE imposes a critical restriction on firm behaviour: a firm will only change its price if the new price is guaranteed to be at least as profitable as the old one, regardless of the signal observed by its rival.

This restriction generates the key dynamics of a price cycle. When demand is elastic, price decreases are generally compatible with PXE but price increases are feasible only when the current price is sufficiently close to marginal cost. In particular, small price increases from intermediate levels are ruled out. This finding explains both the sharp, coordinated price restorations from a low base (the rockets) and the subsequent phase of slow, asynchronous price decreases (the feathers). Any pathwise ex post equilibrium must therefore feature constant prices or exhibit this cyclical pattern.

Crucially, my results establish that price cycles are not just feasible but can also be Pareto efficient. If the firms differ in their bargaining power, a stationary cartel price may not be agreeable to all parties. Price cycles provide a dynamic mechanism to redistribute profits over time without resorting to illegal cash payments. By allowing for periods of price leadership where one firm enjoys a temporary market share advantage, the cycle facilitates a division of collusive profits that can be more desirable for all firms. This theoretical conclusion is supported by empirical evidence from the prosecution of a gas station cartel – former cartel members testified that adjustment delays were used specifically to redistribute surplus within the cartel.

In conclusion, this paper makes a significant contribution by demonstrating that the puzzling and persistent phenomenon of gasoline price cycles can be understood as a rational, robust, and efficient collusive strategy. By integrating a behaviourally motivated solution concept with a realistic form of imperfect monitoring, the analysis furthers our understanding of how non-expected utility considerations affect collusive outcomes in an uncertain environment. The PXE solution concept is itself a valuable theoretical tool with broader applicability. It can be used for analyzing cooperation in any setting where the players may be concerned about ex post incentive compatibility, regret minimization, or worst-case payoffs.

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Appendix

Demand Function. The following is an example of a demand function that satisfies all nine assumptions in this paper. I stress that this is only an example and the assumptions allow for more general functions.

To simplify notation, I will omit the time superscripts when no ambiguity exists.

The total number of consumers in the market is given by $Q > 0$. Each consumer's baseline demand is one unit as long as the actual price is weakly less than $\overline{p}$.[12]

[12] Recall that $\overline{p}$ is the upper bound of the set of prices.

The firms are spatially differentiated and the consumers have to pay a travel cost that is increasing in their distance to the firm. Consumers want to minimize their total cost i.e. travel cost plus the actual price.

When both public signals are accurate:

$$\alpha_i(y_1, y_2) = \frac{1}{1 + \exp\left(\beta[y_i - y_j]\right)}$$
$$D_i(y_i, y_1, y_2) = \frac{Q}{1 + \exp\left(\beta[y_i - y_j]\right)}$$

This is a standard logit function with $\beta > 0$ representing consumer sensitivity to the difference in prices. I note that $\beta \to \infty$ approximates Bertrand demand.

Assumption 5 is trivial because total demand is fixed when the signals are accurate. Assumption 7 is also satisfied for the same reason. Assumption 8 is satisfied because logit demand has a sigmoid shape centred on $y_1 = y_2$.

When the signals are not accurate, the observed prices serve as an anchor or reference for the consumers. If the actual price of their chosen firm is greater than the observed price, consumers experience a negative expectation shock and purchase less than one unit. Conversely, if the actual price is lower, consumers purchase more.[13]

Formally, the quantity purchased by each consumer who picks firm i is given by:

$$1 - \psi\left(1 - (1 - \frac{\eta}{\psi})^{\frac{p_i - y_i}{\epsilon}}\right) \quad \text{if} \quad p_i > y_i$$
$$1 + \psi\left(1 - (1 - \frac{\eta}{\psi})^{\frac{y_i - p_i}{\epsilon}}\right) \quad \text{if} \quad p_i < y_i$$

Where $r(p_i, y_i) = \psi\left(1 - (1 - \frac{\eta}{\psi})^{\frac{|p_i - y_i|}{\epsilon}}\right)$ is the *expectation shock*. It decays geometrically in the difference between the actual price and the observed price. In particular, each consumer purchases exactly $1 - \eta$ units when $p_i - y_i = \epsilon$. Combining everything gives:

[13] The assumption is that demand is reference dependent. See Kőszegi and Rabin (2006) for a treatment of the underlying consumer utility. See Popescu and Wu (2007) and Heidhues and Kőszegi (2008) for examples of reference dependent demand in the literature.

$$D_i(p_i, y_1, y_2) = \frac{Q}{1 + \exp\left(\beta[y_i - y_j]\right)}\Big[1 - r(p_i, y_i)\Big] \quad \text{if} \quad p_i > y_i$$
$$D_i(p_i, y_1, y_2) = \frac{Q}{1 + \exp\left(\beta[y_i - y_j]\right)}\Big[1 + r(p_i, y_i)\Big] \quad \text{if} \quad p_i < y_i$$

Notice that $r(p_i, y_i) \in [0, \psi)$ where $\psi < 1$. So Assumption 1 and Assumption 4 are satisfied because demand is always positive and bounded above. Assumption 2 is satisfied because demand is increasing in the rival's signal. Assumption 3 is satisfied by construction.

It is easy to see that Assumption 9 also holds when:

$$\eta \geq 1 - \exp(-\beta\, \epsilon)$$
$$\psi \geq 1 - \exp(-\beta\, \overline{p})$$

$\square$

Proof of Lemma 1. First note that every $h \in \mathcal{H}^\infty$ is an element of $\left(\mathcal{Y} \times \mathcal{Y}\right)^{\mathbb{N}}$.

$\mathcal{Y}$ is finite so it is compact under the discrete topology. $\left(\mathcal{Y} \times \mathcal{Y}\right)^{\mathbb{N}}$ is a countable product of compact sets so it is compact by Tychonoff's theorem.

Now fix any strategy profile s. $\mathcal{H}^\infty(s)$ is a subset of $\left(\mathcal{Y} \times \mathcal{Y}\right)^{\mathbb{N}}$.

Let $C_t(h^t) \subseteq \mathcal{H}^\infty$ be the open cylinder such that the first $t-1$ elements of every $h \in C_t(h^t)$ are equal to h^t. Then:

$$\left[\mathcal{H}^\infty(s)\right]^C = \bigcup_t \bigcup_{h^t \notin \mathcal{H}^t(s)} C_t(h^t)$$

The complement of $\mathcal{H}^\infty(s)$ is therefore a union of open sets. So $\mathcal{H}^\infty(s)$ is closed under the product topology. It is thus compact.

For a fixed $h \in \mathcal{H}^\infty(s)$ define:

$$f_i^T(s,h) = \sum_{t\geq 1}^{T} \delta^{t-1}\, u_i\big(s_i(h^t), y^t\big)$$
$$f_i(s,h) = \sum_{t\geq 1}^{\infty} \delta^{t-1}\, u_i\big(s_i(h^t), y^t\big)$$

Let d be any metric that induces the product topology on $\mathcal{H}^\infty$. For instance:

$$d(h, \hat{h}) = \sum_{t=1}^{\infty} \frac{1}{2^t}\, \mathbb{1}(y^t \neq \hat{y}^t)$$

where $\mathbb{1}$ is the indicator function; y^t is the t-th element of h; and $\hat{y}^t$ is the t-th element of $\hat{h}$.

Then $f_i^T(s,h)$ is a continuous function of h for every finite T. That is because f_i^T depends only on the first T elements of h and the distance between h and $\hat{h}$ is bounded away from zero when the first T elements are not the same. Furthermore:

$$\Big|\; f_i^T(s,h) - f_i(s,h) \;\Big| \leq M\delta^T$$

So $f_i^T(s,h) \to f_i(s,h)$ as $T \to \infty$.

Therefore $f_i(s,h)$ is continuous because it is the uniform limit of partial sums and each partial sum is continuous.[14]

The minimum of a continuous function on a compact set exists. So $\min_h f_i(s,h)$ exists.

Notice that $\min_h f_i(s,h) = \min_{h\in\mathcal{H}^\infty(s)} \sum_{t\geq 1} \; \delta^{t-1}\; u_i\big(s_i(h^t), y^t\big)$.

Exactly the same argument can be used for any strategy profile and any initial history h^t.

□

Proof of Lemma 2. It is obvious that Definition 1 implies Definition 2. So I will only prove the converse.

Suppose the condition in Definition 2 is satisfied. Then for both i; every h^t; and any one shot

[14] The uniform limit of continuous functions is continuous.

deviation $\tilde{s}_i$:

$$\min_{y_j^t \in \mathcal{Y}_j(s_j|h^t)} \left\{ g_i^t(s_i, s_j \mid h^t, y_j^t) - g_i^t(\tilde{s}_i, s_j \mid h^t, y_j^t) \right\} \geq 0$$

$$\min_{y_j^t \in \mathcal{Y}_j(s_j|h^t)} g_i^t(s_i, s_j \mid h^t, y_j^t) - \min_{y_j^t \in \mathcal{Y}_j(s_j|h^t)} g_i^t(\tilde{s}_i, s_j \mid h^t, y_j^t) \geq 0$$

$$g_i^t(s_i, s_j \mid h^t) - g_i^t(\tilde{s}_i, s_j \mid h^t) \geq 0$$

Recall that guaranteed payoffs can be written recursively and are "continuous at infinity" because of discounting . So Blackwell (1965) says that the usual one shot deviation principle holds. For completeness, I provide a proof in the next section.

That means $g_i^t(s_i, s_j \mid h^t) - g_i^t(\hat{s}_i, s_j \mid h^t) \geq 0$ for an arbitrary $\hat{s}_i \neq s_i$. So for every $y_j^t \in \mathcal{Y}_j(s_j \mid h^t)$:

$$\begin{aligned}
g_i^t(\hat{s}_i, s_j \mid h^t, y_j^t) &= \min_{\hat{y}_i^t \in \mathcal{Y}_i(\hat{s}_i|h^t)} \left\{ (1-\delta)\, u_i\big(\hat{a}_i^t, \hat{y}_i^t, y_j^t\big) \;+\; \delta\, g_i^{t+1}\big(\hat{s}_i, s_j \bigm| h^t, \hat{y}_i^t, y_j^t\big) \right\} \\
&\leq \min_{\hat{y}_i^t \in \mathcal{Y}_i(\hat{s}_i|h^t)} \left\{ (1-\delta)\, u_i\big(\hat{a}_i^t, \hat{y}_i^t, y_j^t\big) \;+\; \delta\, g_i^{t+1}\big(s_i, s_j \bigm| h^t, \hat{y}_i^t, y_j^t\big) \right\} \\
&\leq \min_{y_i^t \in \mathcal{Y}_i(s_i|h^t)} \left\{ (1-\delta)\, u_i\big(a_i^t, y_i^t, y_j^t\big) \;+\; \delta\, g_i^{t+1}\big(s_i, s_j \bigm| h^t, y_i^t, y_j^t\big) \right\} \\
&= g_i^t(s_i, s_j \mid h^t, y_j^t)
\end{aligned}$$

But that shows that $r_i(s \mid h^t) \leq 0$ for every h^t.

□

One Shot Deviation Principle. Fix a public strategy profile $s = (s_i, s_j)$ and a public history h^t.

Assume that s_i is not optimal for firm i after h^t. So there exists $\hat{s}_i$ such that:

$$g_i^t(\hat{s}_i, s_j \mid h^t) - g_i^t(s_i, s_j \mid h^t) > 2\mu$$

for some $\mu > 0$. Suppose (for a contradiction) there is no profitable one shot deviation for firm i after h^t.

Fix N such that $M\delta^N < \mu$. Then the payoff difference between s_i and $\hat{s}_i$ is less than μ after period $t+N$.[15]

Define $\tilde{s}_i$ to be a strategy that copies $\hat{s}_i$ until period $t+N$ and copies s_i starting in period $t+N+1$. Define $\tilde{s}_i^{(\tau)}$ to be a one shot deviation that copies $\hat{s}_i$ in period τ and copies s_i everywhere else. First note that:

$$g_i^t(\tilde{s}_i, s_j \mid h^t) - g_i^t(s_i, s_j \mid h^t) > \mu$$

Let h^{t+N} be any public history at the beginning of period $t+N$. After h^{t+N} is realized, $\tilde{s}_i$ is a one shot deviation relative to s_i. That is because $\tilde{s}_i$ coincides with s_i starting in period $t+N+1$. So by assumption:

$$g_i^t(\tilde{s}_i, s_j \mid h^{t+N}) \leq g_i^t(s_i, s_j \mid h^{t+N})$$

Now consider any h^{t+N-1}. The previous inequality gives:

$$\begin{aligned} g_i^t(\tilde{s}_i, s_j \mid h^{t+N-1}) &\leq g_i^t(\tilde{s}_i^{(t+N-1)}, s_j \mid h^{t+N-1}) \\ &\leq g_i^t(s_i, s_j \mid h^{t+N-1}) \end{aligned}$$

The second inequality again comes from the assumption that there is no profitable one shot deviation. Repeating this inductively gives:

$$g_i^t(\tilde{s}_i, s_j \mid h^t) \leq g_i^t(s_i, s_j \mid h^t)$$

This is a contradiction. So there must be a profitable one shot deviation for firm i when s_i is not optimal.

□

Proof of Lemma 3. Fix any mixed strategy PXE. Let σ be the strategy profile. Fix any public

[15] I am discounting all payoffs to the beginning of period t.

history h^t.

There exists some pure strategy s_i such that s_i is in the support of σ_i and:

$$g_j^t(\sigma_i, \sigma_j \mid h^t) = g_j^t(s_i, \sigma_j \mid h^t)$$

Meaning s_i is the worst strategy for firm j after h^t is realized. Next note that:

$$g_i^t(s_i, \sigma_j \mid h^t) = g_i^t(\sigma_i, \sigma_j \mid h^t)$$

That is because mixing requires firm i to be indifferent between all strategies in the support of σ_i.

Now define a strategy σ_i^* such that $\sigma_i^*(h^t) = s_i(h^t)$ and $\sigma_i^* = \sigma_i$ after all other histories.

So firm i plays the worst action for firm j after h^t is realized but continues to play according to σ_i after all other histories.

Recall that PXE requires ex post incentive compatibility. Therefore:

$$\min_{y_i^t \in \mathcal{Y}_i(\sigma_i|h^t)} g_j^t(\sigma_i, \sigma_j \mid h^t, y_i^t) - g_j^t(\sigma_i, \hat{\sigma_j} \mid h^t, y_i^t) \geq 0$$

$$\min_{y_i^t \in \mathcal{Y}_i(\sigma_i^*|h^t)} g_j^t(\sigma_i^*, \sigma_j \mid h^t, y_i^t) - g_j^t(\sigma_i, \hat{\sigma_j} \mid h^t, y_i^t) \geq 0$$

$$\min_{y_i^t \in \mathcal{Y}_i(\sigma_i^*|h^t)} g_j^t(\sigma_i^*, s_j \mid h^t, y_i^t) - g_j^t(\sigma_i, \hat{\sigma_j} \mid h^t, y_i^t) \geq 0$$

where s_j is any strategy in the support of σ_j and $\hat{\sigma_j} \neq \sigma_j$ is any deviation. That means any strategy in the support of σ_j is optimal even when firm i plays according to σ_i^*.

I stress the importance of ex post incentive compatibility here. In general, s_j will not remain optimal after firm i switches from σ_i to σ_i^*. However, ex post incentive compatibility requires that s_j is optimal for every possible realization of y_i^t. Firm i's switch to σ_i^* only restricts the support of y_i^t so s_j remains optimal.

Now define σ_j^* analogously to σ_i^* i.e. σ_j^* plays the worst action for firm i after h^t and coincides with σ_j everywhere else.

The argument above shows that (σ_i^*, σ_j^*) is a PXE and the guaranteed payoffs after every public history are the same as they are under (σ_i, σ_j)

Repeating this procedure for every public history yields a pure strategy profile $s^* = (s_i^*, s_j^*)$ such that s^* is a PXE and $g(s^*) = g(\sigma)$.

$\square$

Proof of Proposition 1. Fix any history h^t such that $s_i(h^t) = p_i^t \neq y_i^{t-1}$.

Fix any y_j^t that is realized with positive probability.

Consider a one-shot deviation under which firm i plays $p_i^t = y_i^{t-1}$.

Firm i's observed price at the end of period t is certain to be $y_i^t = y_i^{t-1}$.

So the guaranteed payoff after this deviation is:

$$(1-\delta)\ u_i(y_i^{t-1}, y_i^{t-1}, y_j^t) + \delta\ g_i^{t+1}(s \mid h^t, y_i^{t-1}, y_j^t)$$

Compare this to the guaranteed payoff in equilibrium:

$$\min_{y_i^t} (1-\delta)\ u_i(p_i^t, y_i^t, y_j^t) + \delta\ g_i^{t+1}(s \mid h^t, y_i^t, y_j^t) \leq (1-\delta)\ u_i(p_i^t, y_i^{t-1}, y_j^t) + \delta\ g_i^{t+1}(s \mid h^t, y_i^{t-1}, y_j^t)$$

Condition (2) requires that the guaranteed payoff in equilibrium is greater than the guaranteed payoff after the one shot deviation:

$$(1-\delta)\ u_i(p_i^t, y_i^{t-1}, y_j^t) + \delta\ g_i^{t+1}(s \mid h^t, y_i^{t-1}, y_j^t) \geq (1-\delta)\ u_i(y_i^{t-1}, y_i^{t-1}, y_j^t) + \delta\ g_i^{t+1}(s \mid h^t, y_i^{t-1}, y_j^t)$$
$$u_i(p_i^t, y_i^{t-1}, y_j^t) \geq u_i(y_i^{t-1}, y_i^{t-1}, y_j^t)$$

This establishes the desired inequality and completes the proof.

$\square$

Proof of Lemma 4. To simplify notation, I will omit the time superscripts when there is no ambiguity.

Let (y_i, y_j) be the vector of public signals realized at the end of period $t-1$.

Assume $y_i > c + (1-\eta)(\overline{p} - c)$. Then for any $p_i > y_i$:

$$\begin{aligned}
& & y_i &> c + (1-\eta)(\overline{p} - c) \\
&\implies & y_i - c &> (1-\eta)(\overline{p} - c) \\
&\implies & D_i(y_i, y_1, y_2)(y_i - c) &> (1-\eta) D_i(y_i, y_1, y_2)(\overline{p} - c) \\
& & &\geq (1-\eta) D_i(y_i, y_1, y_2)(p_i - c) \\
& & &\geq D_i(p_i, y_1, y_2)(p_i - c) \\
&\implies & D_i(y_i, y_1, y_2)(y_i - c) &> D_i(p_i, y_1, y_2)(p_i - c) \\
&\implies & u_i(y_i, y_1, y_2) &> u_i(p_i, y_1, y_2)
\end{aligned}$$

Therefore, Proposition 1 says that it is not PXE compatible for firm i to play p_i in period t.

So if $y_i^{t-1} > c + (1-\eta)(\overline{p} - c) = \underline{p}$ then $p_i^t \leq y_i^{t-1}$ in any PXE.

□

Proof of Proposition 2. Define $\tau_1 = 1$. Define $\tau_{n+1} = \min\{t > \tau_n \colon y_i^t > y_i^{t-1}\}$ for some i.

First consider the possibility that (τ_n) is a finite sequence.

After $\max\{\tau_n\}$ each component of y^t forms a non-increasing sequence on a finite set $\mathcal{Y}$.

So only finitely many decreases are possible. Hence, there exists some T such that $y^t = y^T$ for all $t \geq T$.

Next consider the possibility that (τ_n) is an infinite sequence.

By construction, $y^t \leq y^{t-1}$ when $\tau_n < t < \tau_{n+1}$ for any n.

Now fix $t = \tau_n$ for any $n \geq 2$. We know that $y_i^t > y_i^{t-1}$ for some i.

Given the signal generating process, it must be the case that $p_i^t = y_i^t > y_i^{t-1}$.

Then Lemma 4 says that $y_i^{t-1} \leq \underline{p}$.

$\square$

Proof of Corollary 1. To simplify notation, I will omit the time superscripts when there is no ambiguity.

Fix any p_i and p_j. When both signals are accurate, total profits are given by:

$$u_1(p_i, p_j) + u_2(p_i, p_j) = (p_i - c)\ D_i(p_i, p_i, p_j) + (p_j - c)\ D_j(p_j, p_i, p_j)$$

Then for any $h \geq \epsilon$:

$$\begin{aligned}
& u_1(p_i, p_j) + u_2(p_i, p_j) - u_1(p_i - h, p_j) - u_2(p_i - h, p_j) \\
&= (p_i - c)\ D_i(p_i, p_i, p_j) - (p_i - h - c)\ D_i(p_i - h, p_i - h, p_j) \\
&\qquad + (p_j - c)\ D_j(p_j, p_i, p_j) - (p_j - c)\ D_j(p_j, p_i - h, p_j) \\
&= (p_i - c)\Big[D_i(p_i, p_i, p_j) - D_i(p_i - h, p_i - h, p_j)\Big] + h\ D_i(p_i - h, p_i - h, p_j) \\
&\qquad + (p_j - c)\Big[D_j(p_j, p_i, p_j) - D_j(p_j, p_i - h, p_j)\Big] \\
&\geq (\max\{p_i, p_j\} - c)\Big[D(p_i, p_j) - D(p_i - h, p_j)\Big] + h\ D_i(p_i - h, p_i - h, p_j)
\end{aligned}$$

The last inequality comes from the facts that $D(p_i, p_j) < D(p_i - h, p_j)$ and $D = D_i + D_j$. Recall that Assumption 5 gives:

$$\begin{aligned}
D(p_i - h, p_j) - D(p_i, p_j) &< \frac{h}{\max\{p_i, p_j\}}\ \alpha_i(p_i, p_j)\ D(p_i, p_j) \\
D(p_i, p_j) - D(p_i - h, p_j) &> \frac{-h}{\max\{p_i, p_j\}}\ \alpha_i(p_i, p_j)\ D(p_i, p_j) \\
&= \frac{-h}{\max\{p_i, p_j\}}\ D_i(p_i, p_i, p_j)
\end{aligned}$$

Substituting:

$$
\begin{aligned}
& u_1(p_i,p_j)+u_2(p_i,p_j)-u_1(p_i-h,p_j)-u_2(p_i-h,p_j) \\
& \geq h\ D_i(p_i-h,p_i-h,p_j)-(\max\{p_i,p_j\}-c)\ \frac{h}{\max\{p_i,p_j\}}\ D_i(p_i,p_i,p_j) \\
& \geq h\ D_i(p_i-h,p_i-h,p_j)-(\max\{p_i,p_j\}-c)\ \frac{h}{\max\{p_i,p_j\}}\ D_i(p_i-h,p_i-h,p_j) \\
& = \frac{c}{\max\{p_i,p_j\}}\ h\ D_i(p_i-h,p_i-h,p_j) > 0
\end{aligned}
$$

Thus proving that total profits decrease when either firm lowers its price and the signals update.

Now compare $u_i(p_i,p_j)$ and $u_i(p_i-h,p_j-h)$. Note that $\alpha_i(p_i,p_j)=\alpha_i(p_i-h,p_j-h)=\alpha_i$. So:

$$
\begin{aligned}
& u_i(p_i,p_j)-u_i(p_i-h,p_j-h) > 0 \\
\iff \quad & \alpha_i\ D(p_i,p_j)(p_i-c) > \alpha_i\ D(p_i-h,p_j-h)(p_i-h-c) \\
\iff \quad & D(p_i,p_j)-D(p_i-h,p_j-h) > \frac{-h}{p_i-c}\ D(p_i-h,p_j-h) \\
\iff \quad & D(p_i-h,p_j-h)-D(p_i,p_j) < \frac{h}{p_i-c}\ D(p_i-h,p_j-h)
\end{aligned}
$$

Applying Assumption 5 gives:

$$
\begin{aligned}
D(p_i-h,p_j-h)-D(p_i,p_j) &= D(p_i-h,p_j-h)-D(p_i,p_j-h)+D(p_i,p_j-h)-D(p_i,p_j) \\
&< \alpha_i(p_i,p_j-h)\ \frac{h}{\max\{p_i,p_j-h\}}\ D(p_i,p_j-h) \\
&\quad + \alpha_j(p_i,p_j)\ \frac{h}{\max\{p_i,p_j\}}\ D(p_i,p_j) \\
&\leq \frac{h}{\max\{p_i,p_j-h\}}\ D(p_i-h,p_j)\Big[\alpha_i(p_i,p_j-h)+\alpha_j(p_i,p_j)\Big] \\
&\leq \frac{h}{\max\{p_i,p_j-h\}}\ D(p_i-h,p_j)\Big[\alpha_i(p_i,p_j)+\alpha_j(p_i,p_j)\Big] \\
&= \frac{h}{\max\{p_i,p_j-h\}}\ D(p_i-h,p_j) \\
&\leq \frac{h}{\max\{p_i,p_j-h\}}\ D(p_i-h,p_j-h) \\
&\leq \frac{h}{p_i-c}\ D(p_i-h,p_j-h) > 0
\end{aligned}
$$

This shows that a symmetric price cut lowers the profit for both firms.

$\square$

Proof of Lemma 5. I will start by proving that all of the price changes in Example 2 are compatible with PXE.

Note that every price cut is exactly ϵ. So each price decrease satisfies the restriction imposed by Proposition 1 if:

$$\begin{aligned}
& D_i(y_i^t, y_1^t, y_2^t)\ (y_i^t - c) \leq (1+\eta)\ D_i(y_i^t, y_1^t, y_2^t)\ (y_i^t - \epsilon - c) \\
& \qquad \leq D_i(y_i^t - \epsilon, y_1^t, y_2^t)\ (y_i^t - \epsilon - c) \\
\implies \quad & y_i^t - c \leq (1+\eta)\ (y_i^t - \epsilon - c) \\
\implies \quad & c + \epsilon + \frac{\epsilon}{\eta} \leq y_i^t
\end{aligned}$$

Recall that $c + \epsilon + \frac{\epsilon}{\eta} \leq p^{min}$ so an ϵ price cut is feasible whenever $y_i^t \geq p^{min}$.

Price increases to $\overline{p}$ are compatible with PXE if:

$$\begin{aligned}
& D_i(y_i^t, y_1^t, y_2^t)\ (y_i^t - c) \leq (1-\psi)\ D_i(y_i^t, y_1^t, y_2^t)\ (\overline{p} - c) \\
& \qquad \leq D_i(\overline{p}, y_1^t, y_2^t)\ (\overline{p} - c) \\
\implies \quad & y_i^t - c \leq (1-\psi)\ (\overline{p} - c) \\
\implies \quad & y_i^t \leq c + (1-\psi)\ (\overline{p} - c)
\end{aligned}$$

Recall that $p^{min} \leq c + (1-\psi)\ (\overline{p} - c)$ so a price increase to $\overline{p}$ is feasible when $y_i^t \leq p^{min}$.

Satisfying both inequalities simultaneously requires:

$$\begin{aligned}
c + \epsilon + \frac{\epsilon}{\eta} &\leq c + (1-\psi)\ (\overline{p} - c) \\
\epsilon + \frac{\epsilon}{\eta} &\leq (1-\psi)\ (\overline{p} - c) \\
\epsilon &\leq \frac{\eta}{1+\eta}\ (1-\psi)\ (\overline{p} - c)
\end{aligned}$$

This is ensured by Assumption 6. Thus, all of the price changes in Example 2 are PXE compatible.

For the off-path punishments, I will first provide an intuitive description and then prove that the punishments support a PXE.

If $y_i^t = y_i^{t-1}$ then there is no punishment.

If $y_i^t = p_i^t \geq p^{min}$ then the firms lower their prices to $(p^{min}, p^{min} - \epsilon)$.

The firms maintain $(p^{min}, p^{min} - \epsilon)$ for n periods; after n periods, they increase their prices to $(\overline{p}, \overline{p})$; the firms return to playing the price cycle thereafter.

Playing $(p^{min}, p^{min} - \epsilon)$ gives firm i lower profit than what it receives on path after any history

So playing $(p^{min}, p^{min} - \epsilon)$ for sufficiently long is enough to outweigh the one period gain from any deviation.

Two remarks are in order here. First, given the choice of p^{min}, these punishments are PXE compatible by the same arguments detailed earlier in this proof.

Second, the threat of additional punishment enforces incentives during the punishment phase. In particular, if the punishing firm (firm j) deviates then firm i lowers its price to $p^{min} - \epsilon$.

To see that these punishments deter deviations, first note that, in the punishment phase, the stage payoff for firm i is given by:

$$u_i(p^{min}, p^{min}, p^{min} - \epsilon)$$

Corollary 1 says that for any $p > p^{min}$:

$$u_i(p, p, p - \epsilon) > u_i(p^{min}, p^{min}, p^{min} - \epsilon)$$

As a result, on the path of play, the stage payoff for firm i is never less than $u_i(p^{min}+\epsilon, p^{min}+\epsilon, p^{min})$. So for both i and any public history h^t:

$$g_i^t(s \mid h^t) \geq u_i(p^{min} + \epsilon, p^{min} + \epsilon, p^{min}) > u_i(p^{min}, p^{min}, p^{min} - \epsilon)$$

I stress that this relation does not depend on the value of k_1 and k_2. In particular, even when the firms play $(\overline{p}-\epsilon,\overline{p})$ in every period (which happens when $k_1=0$ and $k_2=\infty$) the guaranteed payoffs for firm 2 satisfy:

$$g_2^t(s \mid h^t) = u_2(\overline{p},\overline{p}-\epsilon,\overline{p}) > u_2(p^{min},p^{min}-\epsilon,p^{min})$$

Now recall that demand is bounded by M. So the one period deviation gain for firm i is bounded above by $(\overline{p}-c)\ M$.

Incentive compatibility is therefore satisfied when:

$$\begin{aligned}
(1-\delta^n)\ u_i(p^{min}+\epsilon,p^{min}+\epsilon,p^{min}) &\geq (1-\delta)\ M\ (\overline{p}-c) \\
&\quad + (\delta-\delta^n)\ u_i(p^{min},p^{min},p^{min}-\epsilon) \\
\implies \qquad (1-\delta^n)\ g_i^t(s \mid h^t) &\geq (1-\delta)\ M\ (\overline{p}-c) \\
&\quad + (\delta-\delta^n)\ u_i(p^{min},p^{min},p^{min}-\epsilon)
\end{aligned}$$

This requires:

$$\frac{1-\delta}{\delta-\delta^n} \leq \frac{u_i(p^{min}+\epsilon,p^{min}+\epsilon,p^{min})-u_i(p^{min},p^{min},p^{min}-\epsilon)}{M\ (\overline{p}-c)-u_i(p^{min}+\epsilon,p^{min}+\epsilon,p^{min})}$$

Define for the remainder of this proof:

$$\Upsilon = \frac{u_i(p^{min}+\epsilon,p^{min}+\epsilon,p^{min})-u_i(p^{min},p^{min},p^{min}-\epsilon)}{M\ (\overline{p}-c)-u_i(p^{min}+\epsilon,p^{min}+\epsilon,p^{min})}$$

Note that $\Upsilon > 0$. So there exists a finite n that deters deviations to $p_i^t \geq p^{min}$ when $\delta > \frac{1}{1+\Upsilon}$.

Now suppose $y_i^t = p_i^t < p^{min}$.

Then the off-path punishment requires firm j to lower its price to $p^{min}-\epsilon$ while firm i continues to play p_i^t. The firms revert to playing the price cycle after n periods at $(p_i^t,p^{min}-\epsilon)$.

Since $p_i^t < p^{min}$:

$$
\begin{aligned}
u_i(p_i^t, p_i^t, p^{min} - \epsilon) &\leq u_i(\overline{p}, \overline{p}, p^{min} - \epsilon) \\
&\leq u_i(\overline{p}, \overline{p}, p^{min}) \\
&< u_i(p^{min} + \epsilon, p^{min} + \epsilon, p^{min})
\end{aligned}
$$

The second inequality holds because firm i's demand is strictly increasing in firm j's signal. The third inequality holds because it is not profitable for firm i to raise its price from $p^{min} + \epsilon$ to $\overline{p}$.

The same argument as before can therefore be used to show that a finite n deters deviations to $p_i^t < p^{min}$ when $\delta > \frac{1}{1+\Upsilon'}$ where:

$$
\Upsilon' = \frac{u_i(p^{min} + \epsilon, p^{min} + \epsilon, p^{min}) - u_i(\overline{p}, \overline{p}, p^{min} - \epsilon)}{M\ (\overline{p} - c) - u_i(p^{min} + \epsilon, p^{min} + \epsilon, p^{min})}
$$

So we require:

$$
\delta^* = \max\left\{\frac{1}{1+\Upsilon}; \frac{1}{1+\Upsilon'}\right\}
$$

□

Proof of Proposition 3. Let s be the strategy profile in Example 2.

Fix any $h^* \in \mathcal{H}^\infty(s)$. Recall that $h^* = (y^1, y^2, \dots)$ is a sequence of vectors of public signals.

Define $v(s, h^*)$ to be the vector of actual payoffs, discounted to the beginning of the first period, when the firms play according to s and h^* is realized. Similarly, define $v^t(s, h^*)$ to be the vector of continuation payoffs, discounted to the beginning of period t.

Lemma 5 implies:

$$
u_i(p_i^t, y_1^t, y_2^t) \geq u_i(y_i^t, y_1^t, y_2^t)
$$

when $p_i^t \neq y_i^t$ and s is being played. So for the remainder of this proof, I will compute $v(s, h^*)$ and $v^t(s, h^*)$ assuming that the public signals reflect the actual prices.

Now fix any $s' \neq s$ such that s' is a PXE. I will show that $g(s')$ does not dominate $v(s, h^*)$. This is sufficient because I am allowing $h^* \in \mathcal{H}^\infty$ to be any.

Recall that $g(s')$ is the worst-case vector of payoffs when s' is played. So I can limit attention to the case when the public signals update immediately after any price decrease but never update after a price increase. In particular, I will show that the s' does not dominate s when the public signals are always weakly lower than the posted prices.

Now define:

$$u(y_1, y_2) = \Big(\max_{p \geq y} u_1(p_1, y_1, y_2), \max_{p \geq y} u_2(p_2, y_1, y_2) \Big)$$
$$V = \Big\{ u(y_1, y_2) \colon (y_1, y_2) \in \mathcal{Y} \times \mathcal{Y} \Big\}$$
$$V(y_1, y_2) = \Big\{ u(y_1', y_2') \colon y_i' \leq y_i \Big\}$$

So $u(y_1, y_2)$ is the vector of maximum stage payoffs when the observed signals are (y_1, y_2) and the public signals are weakly lower than the actual prices.

V is the set of all possible $u(y_1, y_2)$. $V(y_1, y_2)$ is the subset of V when the signals are constrained to be weakly less than (y_1, y_2). Note that I use $co(A)$ to denote the convex hull of a set A.

Corollary 1 says that total profit is increasing in both prices when the signals are accurate. Assumption 9 ensures that total profit is not higher when the signals are lower than the actual prices. As a result, $u(\overline{p}, \overline{p})$ is an efficient extreme point of $co(V)$ because it uniquely maximizes the sum of profits.

So there exists a positive weight vector $\lambda^1 = (\lambda_1^1, \lambda_2^1)$ such that $\lambda_i^1 > 0$ for both i and:

$$\lambda^1 \, u(\overline{p}, \overline{p}) = \max_{\hat{u} \in V} \lambda^1 \, \hat{u}$$

And since V is finite there exists $\mu^1 > 0$ such that:

$$\lambda^1 \, u(\overline{p}, \overline{p}) - \lambda^1 \, \hat{u} \geq \mu^1$$

where $\hat{u} \neq u(\overline{p}, \overline{p})$ is any element of V. Recall that $k = \min\{k_1, k_2\}$ and $(\overline{p}, \overline{p})$ is played in the first k periods under s. So:

$$\lambda^1 \; u(s, h^*) \geq (1 - \delta^k) \; \lambda^1 \; u(\overline{p}, \overline{p})$$

Recall that any k can be sustained in equilibrium as long as $\delta > \delta^*$. So for the remainder of this proof I assume $\delta \in (\delta^*, 1)$. Now pick k large enough to ensure that:

$$(1 - \delta) \; \frac{\mu^1}{\lambda^1 \; u(\overline{p}, \overline{p})} > \delta^k$$

Then:

$$\begin{aligned} \lambda^1 \; u(\overline{p}, \overline{p}) - (1 - \delta) \; \mu^1 &< (1 - \delta^k) \; \lambda^1 \; u(\overline{p}, \overline{p}) \\ &\leq \lambda^1 \; v(s, h^*) \end{aligned}$$

Notice that $\lambda^1 \; g(s') \leq \lambda^1 \; u(\overline{p}, \overline{p}) - (1 - \delta) \; \mu^1$ if s' does not play $(\overline{p}, \overline{p})$ in the first period.

That is why $\lambda^1 \; g(s') < \lambda^1 \; v(s, h^*)$ if s' does not play $(\overline{p}, \overline{p})$ in the first period.

So s' cannot dominate s if it does not play $(\overline{p}, \overline{p})$ in the first period.

Intuition. Suppose s' plays something other than $(\overline{p}, \overline{p})$ in the first period. Then at least one firm is strictly worse off in period 1 because $u(\overline{p}, \overline{p})$ is efficient. For s' to dominate, it must compensate that firm in the future. But the compensation must come after period k. By picking a sufficiently large k, I can ensure that extra profit after period k cannot compensate for the profit lost in the first period. The remainder of the proof will extend this argument to every subsequent period. I thus show that s' cannot dominate s if it plays a different action in any period.

Now let $t > 1$ be the first period in which s' and s diverge. Since s and s' coincide in every previous period, I need only show that s' does not dominate in the subgame starting in period t.

Note that there are two possibilities:

1. $y_1^{t-1} = y_2^{t-1}$.

2. $y_1^{t-1} = y_2^{t-1} - \epsilon$.

I will focus only on the case when $y_1^{t-1} = y_2^{t-1}$ because the second case is completely symmetric.

First consider the sub-case when $s(h^t) = (y_1^{t-1}, y_2^{t-1}) = (y, y)$.

Then s plays no more than two distinct action profiles between periods t and $t + k$.[16] The action profiles are (y, y) and $(y - \epsilon, y)$.

I claim that $u(y, y)$ and $u(y - \epsilon, y)$ form an efficient edge of $V(y, y)$ i.e. there exists $\lambda^t = (\lambda_1^t, \lambda_2^t)$ such that $\lambda_i^t > 0$ for both i and:

$$\lambda^t\ u(y, y) = \lambda^t\ u(y - \epsilon, y) = \max_{\hat{u} \in V(y,y)} \lambda^t\ \hat{u}$$

I will take this as given for now. The next proof in this section proves it.

Now define $\Lambda^t = \lambda^t + \rho^t d^t$ where $d^t = u(y, y) - u(y - \epsilon, y)$ and $\rho^t > 0$. Intuitively, λ^t is a vector of positive weights that supports both $u(y, y)$ and $u(y - \epsilon, y)$. Λ^t is a small perturbation of λ^t such that $u(y, y)$ is preferred over $u(y - \epsilon, y)$ but $u(y - \epsilon, y)$ is still preferred over every other element of $V(y, y)$.

Formally, if ρ^t is sufficiently small, there exists $\mu^t > 0$ such that:

$$\Lambda^t\ u(y - \epsilon, y) = \Lambda^t\ u(y, y) - \mu^t$$

And for all $\hat{u} \in V(y, y)$ but $\hat{u} \notin \{u(y, y), u(y - \epsilon, y)\}$:

$$\Lambda^t\ \hat{u} \leq \Lambda^t\ u(y, y) - 2\mu^t$$

Since s plays $u(y, y)$ in period t and either $u(y, y)$ or $u(y - \epsilon, y)$ in every period until $t + k$:

$$(1 - \delta)\ \mu^t + (1 - \delta^k)\ (\Lambda^t u(y, y) - \mu^t) \leq \Lambda^t\ v^t(s, h^*)$$

[16] I am focusing on the case when the signals are always accurate under s. So there is no distinction between what is played and what is observed.

Recall that I am focusing on the case when the public signals never increase under s'. So if s' plays anything lower than (y, y) in period t, the signals will permanently fall below (y, y) and payoffs can never be $u(y, y)$ in any subsequent period. Furthermore, it is not PXE compatible for s' to play anything higher than (y, y) in period t so $s'(h^t) \neq (y, y)$ means that $s'(h^t) < (y, y)$.

Consequently, the best s' can do is play $(y - \epsilon, y)$ in every period. That gives:

$$\Lambda^t g(s' \mid h^t) \leq \Lambda^t u(p, p) - \mu^t$$

So if we pick k large enough to ensure that:

$$(1 - \delta) \, \frac{\mu^t}{\Lambda^t \; u(y, y) - \mu^t} > \delta^k \tag{3}$$

Then:

$$\Lambda^t g(s' \mid h^t) < \Lambda^t v^t(s, h^*)$$

Thus s' cannot dominate s if it plays anything other than (y, y) in period t.

Next consider the case when $s(h^t) = (y_1^{t-1} - \epsilon, y_2^{t-1}) = (y - \epsilon, y)$.

Now the only action profile played between periods t and $t + k$ is $(y - \epsilon, y)$.

$u(y - \epsilon, y)$ is also an efficient extreme point of $co\big(V(y, y)\big)$. So the same argument as before can be used to show that s' cannot dominate s if it plays anything other than $(y - \epsilon, y)$ in period t.

Repeating this exercise for every t gives that s' cannot dominate s if it plays a different action profile in any period.

Therefore, there is no $s' \neq s$ such that s' dominates s.

Note that the required k^* is the smallest k such that Condition (3) is satisfied for every t.

$\square$

Proof of Efficient Edge. This proof focuses on the case when the public signals are accurate. So I will simplify notation by using $u_i(p_i, p_j) = u_i(p_i, p_1, p_2)$ and $\boldsymbol{u}(p_i, p_j) = (u_1, u_2)$.

First note that $\boldsymbol{u}(y, y)$ and $\boldsymbol{u}(y-\epsilon, y)$ are two points in $\mathbb{R}^2$. So there exists λ^t such that:

$$\begin{aligned}
\lambda^t \, \boldsymbol{u}(y, y) &= \lambda^t \, \boldsymbol{u}(y-\epsilon, y) \\
\lambda_1^t \, u_1(y, y) + \lambda_2^t \, u_2(y, y) &= \lambda_1^t \, u_1(y-\epsilon, y) + \lambda_2^t \, u_2(y-\epsilon, y) \\
\frac{\lambda_2^t}{\lambda_1^t} &= \frac{u_1(y-\epsilon, y) - u_1(y, y)}{u_2(y, y) - u_2(y-\epsilon, y)}
\end{aligned}$$

Both $u_1(y-\epsilon, y) - u_1(y, y)$ and $u_2(y, y) - u_2(y-\epsilon, y)$ are positive so I can pick $\lambda_1 > 0$ and $\lambda_2 > 0$.

To simplify notation, I will drop the superscripts and use $\lambda^t = \lambda = (\lambda_1, \lambda_2)$ for the remainder of this proof.

I will show that $\lambda \, \hat{\boldsymbol{u}} < \lambda \, \boldsymbol{u}(y, y)$ for all $\hat{\boldsymbol{u}} \in V(y, y)$ and $\hat{\boldsymbol{u}} \notin \{\boldsymbol{u}(y, y), \boldsymbol{u}(y-\epsilon, y)\}$.

Corollary 1 makes it clear that $\lambda \, \boldsymbol{u}(y-m\epsilon, y-m\epsilon) < \lambda \, \boldsymbol{u}(y, y)$ for any $m \geq 1$. Because both firms' profit is lower after a symmetric price cut.

Next I consider larger price cuts by firm 1. Fix any $m \geq 0$ and note:

$$\begin{aligned}
& \lambda \, \boldsymbol{u}(y-(m+1)\epsilon, y) - \lambda \, \boldsymbol{u}(y-m\epsilon, y) \\
= \quad & \lambda_1 \, u_1(y-(m+1)\epsilon, y) + \lambda_2 \, u_2(y-(m+1)\epsilon, y) \\
& - \lambda_1 \, u_1(y-m\epsilon, y) - \lambda_2 \, u_2(y-m\epsilon, y) \\
= \quad & \lambda_1 \, (y-(m+1)\epsilon-c) \, \alpha_1(y-(m+1)\epsilon, y) \, D(y-(m+1)\epsilon, y) \\
& + \lambda_2 \, (y-c) \, \alpha_2(y-(m+1)\epsilon, y) \, D(y-(m+1)\epsilon, y) \\
& - \lambda_1 \, (y-m\epsilon-c) \, \alpha_1(y-m\epsilon, y) \, D(y-m\epsilon, y) \\
& - \lambda_2 \, (y-c) \, \alpha_2(y-m\epsilon, y) \, D(y-m\epsilon, y)
\end{aligned}$$

Now define:

$$A_m = \alpha_1(y-m\epsilon, y) \qquad D_m = D(y-m\epsilon, y) \qquad \Pi_m = y-m\epsilon-c$$

Then:

$$\lambda\, \boldsymbol{u}(y-(m+1)\epsilon, y) - \lambda\, \boldsymbol{u}(y-m\epsilon, y) = -\lambda_1\, \epsilon\, A_{m+1}\, D_{m+1} + \lambda_2\, \Pi_0 \left(D_{m+1} - D_m\right) + \left(A_{m+1}D_{m+1} - A_m D_m\right)\left(\lambda_1 \Pi_m - \lambda_2 \Pi_0\right)$$

Notice that:

1. $A_{m+1}\, D_{m+1}$ is increasing in m.
2. $D_{m+1} - D_m$ is decreasing in m. By Assumption 7.
3. $A_{m+1}D_{m+1} - A_m D_m$ is decreasing in m. By Assumption 7.
4. Π_m is decreasing in m.

That means all three terms are decreasing in m so $\lambda\, \boldsymbol{u}(y-(m+1)\epsilon, y) - \lambda\, \boldsymbol{u}(y-m\epsilon, y)$ is decreasing as well.

Now recall that $\lambda\, \boldsymbol{u}(y-(m+1)\epsilon, y) - \lambda\, \boldsymbol{u}(y-m\epsilon, y) = 0$ when $m = 0$. So for all $m > 0$:

$$\begin{aligned} \lambda\, \boldsymbol{u}(y-(m+1)\epsilon, y) - \lambda\, \boldsymbol{u}(y-m\epsilon, y) &< 0 \\ \lambda\, \boldsymbol{u}(y-(m+1)\epsilon, y) &< \lambda\, \boldsymbol{u}(y-m\epsilon, y) \\ &\leq \lambda\, \boldsymbol{u}(y-\epsilon, y) = \lambda\, \boldsymbol{u}(y, y) \end{aligned}$$

Thus $\lambda\, \boldsymbol{u}(y-m\epsilon, y) < \lambda\, \boldsymbol{u}(y, y)$ if $m > 1$.

Finally consider a price cut by firm 2. I will show that:

$$\lambda\, \boldsymbol{u}(y, y) > \lambda\, \boldsymbol{u}(y, y-\epsilon)$$

Recall that total profits are increasing in both prices. So:

$$\frac{u_1(y, y) - u_1(y, y-\epsilon)}{u_2(y, y-\epsilon) - u_2(y, y)} > 1$$

Note that I am assuming $u_2(y, y - \epsilon) - u_2(y, y) > 0$. The proof is trivial when this is not true.

For the same reason as above:

$$\frac{u_1(y-\epsilon, y) - u_1(y, y)}{u_2(y, y) - u_2(y-\epsilon, y)} = \frac{\lambda_2}{\lambda_1} < 1$$

This gives:

$$\frac{u_1(y, y) - u_1(y, y-\epsilon)}{u_2(y, y-\epsilon) - u_2(y, y)} > \frac{\lambda_2}{\lambda_1}$$

$$\lambda_1 \; u_1(y, y) - \lambda_1 \; u_1(y, y-\epsilon) > \lambda_2 \; u_2(y, y-\epsilon) - \lambda_2 \; u_2(y, y)$$

Rearranging and combining terms gives the desired inequality.

$\square$