Bell Inequality Test

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***Abstract*—The Bell Inequality defines an upper limit of correlation between measurements of separated pairs of particles if the theory of locality holds. Quantum mechanics have been shown to theoretically and experimentally violate this inequality. Here we simulate a Bell Test performed on various entangled states and measuring in various bases. We simulate the use of a single-photon SPAD (single-photon avalanche diode) detector with specifications 10% efficiency, 1000Hz dark count rate, and 4 microseconds dead time. We do this by leveraging Qiskit's close-to-ideal circuits to perform measurements, then performing post-processing and time scaling to simulate the desired noise as well as condense the data to an average rate of 15,000 entanglements per second. We find that pure entanglement states can violate the Bell Inequality, even if they're not maximally entangled, while mixed states do not violate the Bell Inequality even in a variety of bases of measurement.**

***Index Terms*—Bell's Inequality test, CHSH game, Coincidence monitor, Non-maximal entanglement, Quantum communication simulation**

I. INTRODUCTION

Bell's theorem [1] is an important scientific statement that hypothesized that local hidden variables may not be responsible for the connection between entangled particles. Therefore, the changes that affect entangled particles can be communicated instantaneously regardless of distance, faster than light.

The Bell Inequality Test allows an experimental analysis to prove that when measurements are performed independently on the two separated particles of an entangled pair, if the outcomes depend on hidden variables then there would be an upper limit on the correlation between the two measurements.

The thought-experiment that spawned the Bell Theorem is as follows: a pair of particles is prepared and Alice and Bob, separated by a large distance, both receive one half of the pair. Alice randomly chooses one of two measurements to perform on her particle, A_0 or A_1. Both A_0 and A_1 are binary measurements, and can have the result of either +1 or -1. Bob does the same with his particle and chooses one of two measurements, B_0 or B_1 which are also binary. The result of these measurements is recorded in variables a_0, a_1, b_0, b_1. Considering the combination of these terms:

$$a_0b_0 + a_0b_1 + a_1b_0 - a_1b_1 = (a_0 + a_1)b_0 + (a_0 - a_1)b_1$$

we can see that since both a_0 and a_1 take the values of ± 1, then either $a_0 = a_1$ or $a_0 = -a_1$. Therefore, either $(a_0 + a_1)b_0 = 0$ or $(a_0 - a_1)b_1 = 0$. The other value, meanwhile, must equal ± 2.

Over the course of many trials, the average value of this combination of terms must be less than or equal to 2. However, quantum mechanics can violate this inequality and produce a combination of average measured values equal to $2\sqrt{2}$. There have been multiple experimental tests that have proven this violation of the Bell Inequality, essentially proving that quantum mechanics apply over locality. The most important of these were the "loophole-free" experiments [2] [3] [4], in 2015, which experimentally demonstrated the violation of Bell's inequality without making any other assumptions.

In this project, we seek to code a simulation of the Bell Inequality test in which we see the effect of different initial entangled states on the resulting average measurement. The theoretical maximum value that the average measurement is $2\sqrt{2}$, but experimentally this may not be the case. We have simulated the Bell Inequality test with several entangled states and measured in multiple bases.

II. PROBLEM STATEMENT

A. Description

We are posed to simulate a Bell Inequality test in a quantum communication setup, with the aim to calculate the metrics for the quantum state and the entanglement qualities, for various generated states over multiple basis. Based on the simulation, the target is to make a comparison on how different entanglement states affect these metrics.

B. Problem settings

In the given quantum communication setting, we have:
I. An entanglement source (ES): that generates entangled photon pairs over the Poisson distribution with an expected

value of 15000. The simulation is run for the following 4 ES generated states:

1) Case 1: Maximally entangled EPR (Einstein, Podolsky, Rosen) pairs:

$$\left|\Phi^{+}\right\rangle = \frac{1}{\sqrt{2}}(\left|HH\right\rangle s + \left|VV\right\rangle)$$

2) Case 2: Maximally entangled EPR pairs with mixture:

$$\rho = 0.7\left|\Phi^{+}\right\rangle\left\langle\Phi^{+}\right| + 0.15\left|HV\right\rangle\left\langle HV\right| + 0.15\left|VH\right\rangle\left\langle VH\right|$$

3) Case 3: Non-maximally entangled pairs:

$$\left|\phi\right\rangle = \frac{1}{\sqrt{5}}\left|HH\right\rangle + \frac{2}{\sqrt{5}}\left|VV\right\rangle$$

4) Case 4: Non-maximally entangled pairs with mixture:

$$\rho' = 0.7\left|\phi\right\rangle\left\langle\phi\right| + 0.15\left|HV\right\rangle\left\langle HV\right| + 0.15\left|VH\right\rangle\left\langle VH\right|$$

H and V represents horizontal and vertical polarization, respectively.

II. These generated entanglements are sent over to Alice and Bob, who each use a single-photon SPAD detector [5] to measure their part of the entanglement, in various bases. The detector has the following specifications of interest:

a) Efficiency: 10% (i.e., the probability of detecting an incident photon)

b) Dark count rate: 1000 Hz (i.e., the rate at which the detector registers a count due to thermal or other background noise)

c) Dead time: 4 microseconds (i.e., the time during which the detector is unable to register another count after detecting a photon)

III. The system is left to run for 30 seconds and the results of Alice and Bob are aggregated at the co incidence monitor (CM) which is calibrated to 1ns to calculate:

i) Total count rate (number of clicks on a detector per second)

ii) Co-incidence rate (refers to the rate at which two detectors register simultaneous detections)

iii) Fidelity (fidelity is a measure of the ”closeness” of two quantum states) [6]

iv) Bell inequality [7]

C. Experiment setup

Precise control over the ES allows for tailored generation of states with specific characteristics, such as a desired level of entanglement or other properties relevant to the quantum communication protocol being used. We simulate this ES on Qiskit [8] using Ideal Qiskit circuit and relevant density matrix, and run it for the desired iterations. The outcomes and timestamps are measured on Qiskit, and used for post processing.

Since the Qiskit measurement libraries provide close to an ideal detection, error models are simulated through post processing. A 10% efficiency model is applied by sampling over a random sampler. Followed by this, the dark counts (1000Hz), randomized over a uniform distribution, are added, leading to 4 microseconds dead time simulation.

The final outcomes are then used to compute the indicators including total count rate, coincidence rate, fidelity, and Bell Inequality. For count rate, frequency of the final outcomes is determined. For coincidence rate, a binary search based algorithm is used to perform coincidence matching over a 1ns window. Fidelity is computed using the initial state density matrix and formalised final state density matrix based on Quantum Tomography. Finally, Bell Inequality is computed using the code referred from IBM article Local Reality and the CHSH inequality [7].

These measurements and post processing are performed at both Alice and Bobs side. Bob always uses the computational Z and the X basis for his measurements [9], whereas Alice also chooses orthogonal basis but the angle is varied between 0 and 2*π, with respect to Bob's basis.

III. Novel Design

In this project, we have considered two avenues of novelty:

1) Since the Qiskit measurement libraries provide close to an ideal detection, simulating error models into Qiskit was not ideal. As such, the error models simulation was offloaded to a post-processing module, which parsed the measurements, and applied the desired models.

2) Since most current system are not capable to properly simulate 1ns coincidence window coupled with the fact that Qiskit simulations take more than 1 millisecond on average, simulating the exact setup as is challenging. In order to accurately achieve the previously mentioned error models, before post processing, the timestamps of the measurements are scaled down to 30 seconds, and error models are then applied. This allows for accurate application of error models as well as 1ns coincidence window.

IV. Simulation Results

It is important to note that for this section, the reported values are high-level summarization of the data that was collected. The holistically presented data can be referred to in the project's github [10], which is also compiled in the google sheet [11].

For maximally and non-maximally entangled EPR pairs, as well as maximally and non-maximally entangled pairs, the simulation produced results for count rate, coincidence, fidelity, and Bell inequality.

A. Count Rate

Using an SPAD detector of the given specifications, Alice and Bob both have greater rates of dark count compared to

TABLE I: Case 1 Count Rate (Hz)

	Alice	*Bob*
True Count	746.0771063	746.5147492
Dark Count	991.6241218	991.9046778

True and dark count rates by Alice and Bob in hertz for case 1.

TABLE II: Case 2 Count Rate (Hz)

	Alice	*Bob*
True Count	745.9055375	746.337975
Dark Count	991.2028673	991.5976883

Average true and dark count rates by Alice and Bob in hertz for case 2.

the number of detected photons from the entanglement source; however, the rates of false positive coincidence counts is seen to be quite low. This phenomenon is due to the low probability that a dark count would occur within the coincidence window of 1ns. Table 1 shows the rate, in hertz, of true and false positive coincidence rates for maximally entangled EPR pair, Table 2 for non-maximally entangled EPR pairs, Table 3 for maximally entangled pairs, and Table 4 for non-maximally entangled pairs.

The values in each table are similar due to the probabilistic nature of their determination, as explained in the *Experiment Setup* section.

B. Coincidence Rate

While the dark count rate is on average 200Hz greater than true count rate, the respective rates of true positive coincidence counts are much higher than false positives. This highlights that when Alice and Bob share coincidence, there is a significantly low probability they do not share entanglement. As shown in the tables for all 4 cases, the false positive rate accounts for around 1% of coincidence counts. Table 5, Table 6, Table 7, and Table 8 show the average true and false positive coincidence rate for maximally entangled EPR pairs, non-maximally entangled EPR pairs, maximally entangled pairs, and non-maximally entangled pairs respectively.

C. Fidelity

As fidelity is measurement basis dependent, and the measurement bases of Alice is rotated in the Bloch sphere in increments of $\frac{2\pi}{15}$, the fidelity curve follows a sinusoidal pattern. As the bases are rotated, more values of 01 and 10 are measured, thus affecting the fidelity of the resulting state. Table 9 shows the minimum and maximum fidelities with respect to the state. Non-maximally entangled states have a smaller magnitude of fluctuation because non-maximally entangled states have valid states of 01 and 10, where maximally entangled states do not.

D. Bell Inequality

Bell Inequality test has a baseline value of ± 2. An experimental value breaching this threshold is considered to be breaching Bell Inequality. Entanglement is a necessary condition for breaking Bell Inequality, but is not the only condition.

From our experiments, we found that for certain values of measurement bases, for maximally entangled states, the Bell Inequality is breached, as shown in Fig. 1 and Fig. 3. Thus, proving non-locality in those cases.

On the other hand, for non-maximally entangled states, the Bell Inequality is not breached on any of the tested bases, as shown in Fig. 2 and Fig. 4. This can be attributed to the fact that mixed states contain non-entangled parts also with 15% probability each. These non-entangled

TABLE III: Case 3 Count Rate (Hz)

	Alice	*Bob*
True Count	746.0568101	746.2615772
Dark Count	991.5852779	991.709498

Average true and dark count rates by Alice and Bob in hertz for case 3.

TABLE IV: Case 4 Count Rate (Hz)

	Alice	*Bob*
True Count	746.0827327	746.3976341
Dark Count	991.7102598	992.1325985

Average true and dark count rates by Alice and Bob in hertz for case 4.

TABLE V: Case 1 Coincidence Rate (Hz)

	True-Positive	*False-Positive*
00	19.19434524	0.001847326583
01	17.87170772	0.003015872917
10	17.88661877	0.001388888833
11	19.28400383	0.004327485167

True and false positive rates for each measurement base over case 1.

TABLE VI: Case 2 Coincidence Rate (Hz)

	True-Positive	*False-Positive*
00	18.90055556	0.00138888875
01	18.22138889	0.00249999975
10	18.25805556	0.002222222
11	19.08	0.003333333

True and false positive rates for each measurement base over case 2.

TABLE VII: Case 3 Coincidence Rate (Hz)

	True-Positive	*False-Positive*
00	13.26341886	0.0028071975
01	12.82894727	0.002499999833
10	23.21048512	0.004563491833
11	25.14226054	0.001944444333

True and false positive rates for each measurement base over case 3.

TABLE VIII: Case 4 Coincidence Rate (Hz)

	True-Positive	*False-Positive*
00	14.65882617	0.002222222
01	14.62361111	0.00138888875
10	21.80666667	0.002777777583
11	23.08694445	0.003333333083

True and false positive rates for each measurement base over case 4.

TABLE IX: Maximum and Minimum Fidelities for all Cases

	Maximum	***Minimum***
Case 1	99.99960909	0
Case 2	62.33164061	23.37012035
Case 3	99.9999997	0
Case 4	53.32958892	24.782382

Minimum and maximum fidelities, as percentages, for every case.

parts are separable and thus lead to a Bell Inequality value of ≤ 2, leading to the mixed state not breaching the threshold.

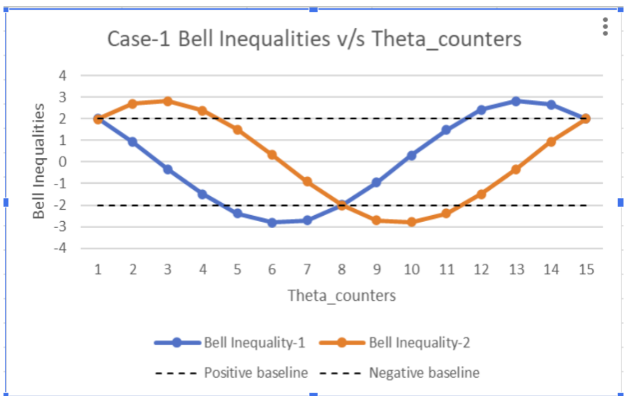

Fig. 1: Bell inequality trend for case 1

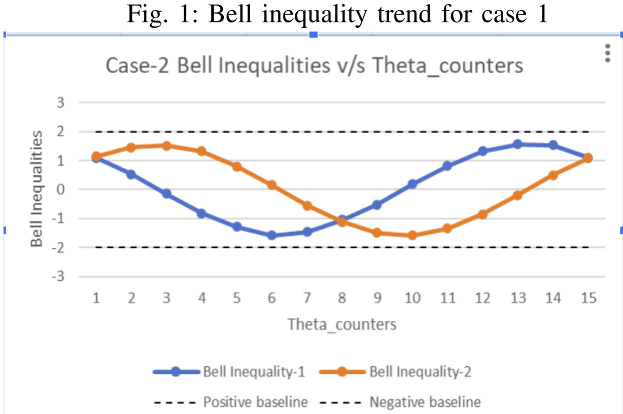

Fig. 2: Bell inequality trend for case 2

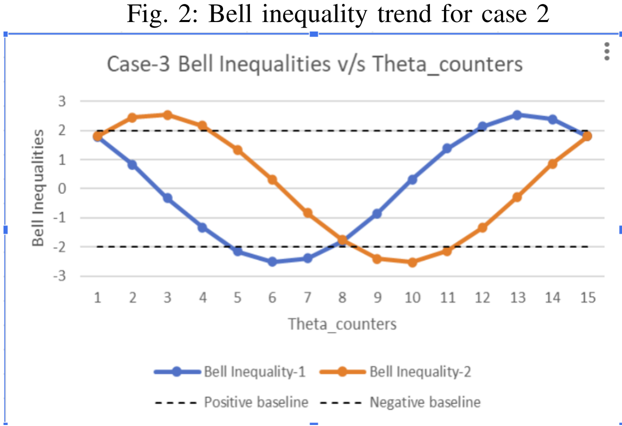

Fig. 3: Bell inequality trend for case 3

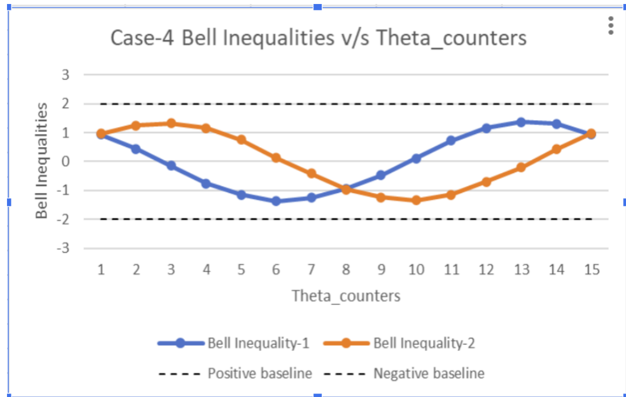

Fig. 4: Bell inequality trend for case 4

V. Conclusion

Here, we have successfully simulated the Bell Inequality test across four initial states including a maximally entangled EPR pair, maximally entangled EPR pair with mixture, non-maximally entangled pair, and non-maximally entangled pair with mixture.

Based on the experiment setup, the results showed that the total count rate is dominated by false positives for all four initial states, however, this does not flow through into the coincidence rates.

Coincidence rates show much higher value for true positives, indicating that even though dark counts were detected at a higher rate, they was not much coincidence found among them.

Fidelity, being measurement basis dependent, followed a sinusoidal curve and maxed out at around 99.99% for entangled states and around 67% and 53% respectively for mixed states. This indicates that for the pure states, the states after measurement are closer to initial state, while they are not for the mixed states.

Bell Inequality results are also dependent on the measurement basis, and also follow a sinusoidal pattern. Pure states were found to breach the Bell Inequality by achieving a value of over 2 for some bases, while mixed states could not. As discussed earlier, this can be accounted for by the fact that mixed states contain separable states.

Based on these results, if Alice and Bob want to breach Bell Inequality, it is suggested that Bob measures in Z and X basis, while Alice measures in orthogonal bases at an angle of $\frac{4\pi}{15}, \frac{6\pi}{15}, \frac{26\pi}{15}, \frac{28\pi}{15}$ with respect to Bob's basis.

VI. Future Work

Based on the simulations, it was noticed that the pure states were successful in breaching the Bell Inequality, and reached a value of greater than 2, however, mixed states didn't breach this barrier. This behavior was attributed to the presence of non-entangled parts in the mixed state.

As such, it would be interesting to do a trend analysis and identifying the impact of changing the probability of the pure state part in the mixed state on the Bell Inequality values.

References

[1] Bell's theorem
[2] Experimental loophole-free violation of a Bell inequality using entangled electron spins separated by 1.3 km
[3] A strong loophole-free test of local realism
[4] Significant-loophole-free test of Bell's theorem with entangled photons
[5] Single photon avalanche detector
[6] Fidelity of quantum states
[7] Local Reality and the CHSH inequality
[8] Qiskit
[9] An entanglement-based wavelength-multiplexed quantum communication network
[10] Github link to the code
[11] Google sheets link to compiled results

VII. Appendix

After the first demo, the following changes were done to the code to bring down the execution time, and take into account various measurement bases for Alice:

A. Measurement bases change

Before the demo, the measurement bases for Alice were fixed as Z and X, however, currently we have moved to a rotation in bloch sphere based approach, where Alice's bases are at an angle θ with respect to Bob's bases. This allows for capturing the performance indicators in various bases allowing a better visualization of Bell Inequality values.

B. Optimization

There were two optimizations carried out. Bulk of time taken was by the measurements generation and coincidence rate calculation:

For measurements generation, we have optimized the code to use multi-processing allowing us to utilize all cores of a given machine, and parallelise the measurement generation. Based on our tests, it has brought down the measurement generation from hours to under 4 minutes on a 16 core machine.

For coincidence rate calculation, instead of using an $O(n^2)$ approach, we moved to a binary search based approach bringing the time for coincidence computation from hours to under 10 seconds on a 16 core machine.

On top of this, we have pre-computed the measurements and provide them with the project code. To use pre-computed measurements, user can specify the mode value as 2 using the command line and utilize the fast results computation.