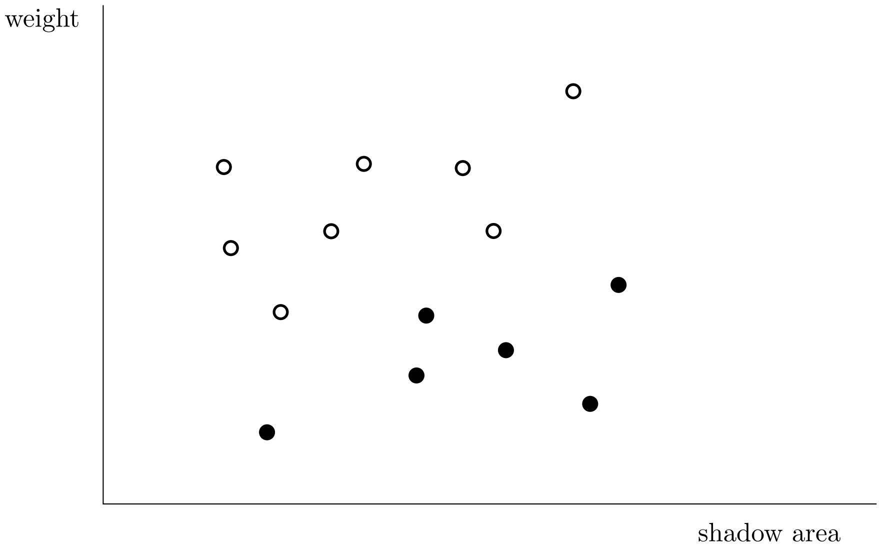
In conclusion, let us recall the useful trick we have learned here and in the previous section:

Objective functions or constraints involving absolute values can often be handled via linear programming by introducing extra variables or extra constraints.

2.5 Separation of Points

A computer-controlled rabbit trap "Gromit RT 2.1" should be programmed so that it catches rabbits, but if a weasel wanders in, it is released. The trap can weigh the animal inside and also can determine the area of its shadow.

These two parameters were collected for a number of specimens of rabbits and weasels, as depicted in the following graph:



(empty circles represent rabbits and full circles weasels).

Apparently, neither weight alone nor shadow area alone can be used to tell a rabbit from a weasel. One of the next-simplest things would be a linear criterion distinguishing them. That is, geometrically, we would like to separate the black points from the white points by a straight line if possible. Mathematically speaking, we are given m white points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$

and n black points $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ in the plane, and we would like to find out whether there exists a line having all white points on one side and all black points on the other side (none of the points should lie on the line).

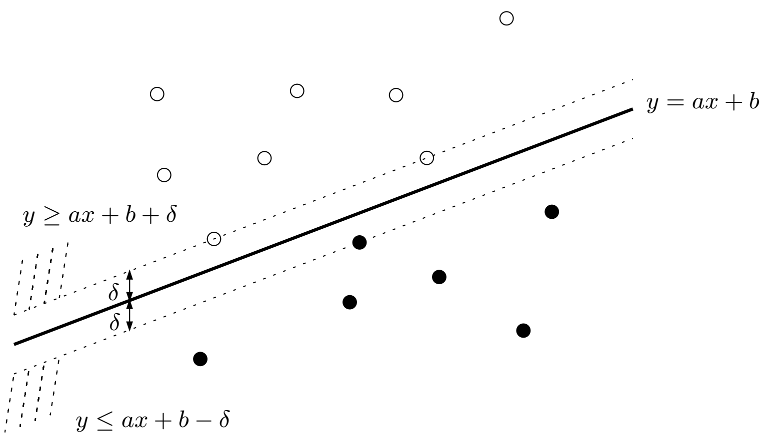
In a solution of this problem by linear programming we distinguish three cases. First we test whether there exists a *vertical* line with the required property. This case needs neither linear programming nor particular cleverness.

The next case is the existence of a line that is not vertical and that has all black points below it and all white points above it. Let us write the equation of such a line as $y = ax + b$, where a and b are some yet unknown real numbers. A point $\mathbf{r}$ with coordinates $x(\mathbf{r})$ and $y(\mathbf{r})$ lies above this line if $y(\mathbf{r}) > ax(\mathbf{r}) + b$, and it lies below it if $y(\mathbf{r}) < ax(\mathbf{r}) + b$. So a suitable line exists if and only if the following system of inequalities with variables a and b has a solution:

$$\begin{aligned} y(\mathbf{p}_i) &> ax(\mathbf{p}_i) + b \qquad && \text{for } i = 1, 2, \ldots, m \\ y(\mathbf{q}_j) &< ax(\mathbf{q}_j) + b \qquad && \text{for } j = 1, 2, \ldots, n. \end{aligned}$$

We haven't yet mentioned strict inequalities in connection with linear programming, and actually, they are not allowed in linear programs. But here we can get around this issue by a small trick: We introduce a new variable δ, which stands for the "gap" between the left and right sides of each strict inequality. Then we try to make the gap as large as possible:

$$\begin{array}{lll} \text{Maximize} & \delta & \\ \text{subject to} & y(\mathbf{p}_i) \geq ax(\mathbf{p}_i) + b + \delta & \text{for } i = 1, 2, \ldots, m \\ & y(\mathbf{q}_j) \leq ax(\mathbf{q}_j) + b - \delta & \text{for } j = 1, 2, \ldots, n. \end{array}$$



This linear program has three variables: a, b, and δ. The optimal δ is positive exactly if the preceding system of strict inequalities has a solution, and the latter happens exactly if a nonvertical line exists with all black points below and all white points above.

Similarly, we can deal with the third case, namely the existence of a nonvertical line having all black points above it and all white points below it. This completes the description of an algorithm for the line separation problem.

A plane separating two point sets in $\mathbb{R}^3$ can be computed by the same approach, and we can also solve the analogous problem in higher dimensions. So we could try to distinguish rabbits from weasels based on more than two measured parameters.

Here is another, perhaps more surprising, extension. Let us imagine that separating rabbits from weasels by a straight line proved impossible. Then we could try, for instance, separating them by a graph of a quadratic function (a parabola), of the form $ax^2 + bx + c$. So given m white points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$ and n black points $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ in the plane, we now ask, are there coefficients $a, b, c \in \mathbb{R}$ such that the graph of $f(x) = ax^2 + bx + c$ has all white points above it and all black points below? This leads to the inequality system

$$\begin{aligned} y(\mathbf{p}_i) &> ax(\mathbf{p}_i)^2 + bx(\mathbf{p}_i) + c \qquad && \text{for } i = 1, 2, \ldots, m \\ y(\mathbf{q}_j) &< ax(\mathbf{q}_j)^2 + bx(\mathbf{q}_j) + c \qquad && \text{for } j = 1, 2, \ldots, n. \end{aligned}$$

By introducing a gap variable δ as before, this can be written as the following linear program in the variables a, b, c, and δ:

$$\begin{array}{lll} \text{Maximize} & \delta & \\ \text{subject to} & y(\mathbf{p}_i) \geq ax(\mathbf{p}_i)^2 + bx(\mathbf{p}_i) + c + \delta & \text{for } i = 1, 2, \ldots, m \\ & y(\mathbf{q}_j) \leq ax(\mathbf{q}_j)^2 + bx(\mathbf{q}_j) + c - \delta & \text{for } j = 1, 2, \ldots, n. \end{array}$$

In this linear program the quadratic terms are coefficients and therefore they cause no harm.

The same approach also allows us to test whether two point sets in the plane, or in higher dimensions, can be separated by a function of the form $f(\mathbf{x}) = a_1\varphi_1(\mathbf{x}) + a_2\varphi_2(\mathbf{x}) + \cdots + a_k\varphi_k(\mathbf{x})$, where $\varphi_1, \ldots, \varphi_k$ are given functions (possibly nonlinear) and $a_1, a_2, \ldots, a_k$ are real coefficients, in the sense that $f(\mathbf{p}_i) > 0$ for every white point $\mathbf{p}_i$ and $f(\mathbf{q}_j) < 0$ for every black point $\mathbf{q}_j$.

2.6 Largest Disk in a Convex Polygon

Here we will encounter another problem that may look nonlinear at first sight but can be transformed to a linear program. It is a simple instance of a geometric *packing problem:* Given a container, in our case a convex polygon, we want to fit as large an object as possible into it, in our case a disk of the largest possible radius.