M-body entanglement in fermion systems

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We show that one-body entanglement, which measures the departure of a fermionic state from a Slater Determinant and is determined by the mixedness of the single particle density matrix (SPDM), can be considered as a quantum resource. We define the free states and operations for this resource through majorization, and show that pure state conversion via these operations will in general imply a decrease in one-body entanglement. It is then demonstrated that fermion linear optics operations, which include one-body unitary transformations and measurements of the occupancy of single particle modes, are free operations implying a majorization relation between the initial and postmeasurement SPDMs. Finally, it is shown that the ensuing resource is consistent with a model of fermionic quantum computation which requires correlations beyond antisymmetrization. A general bipartite-like formulation of one-body entanglement is also discussed.

I. INTRODUCTION

A remarkable feature of quantum mechanics is the existence of correlations between quantum systems that cannot be emulated by their classical counterpart. Entanglement is the most celebrated manifestation of such correlations and it has been object of intense research in quantum physics, and in particular within the field of quantum information theory. Particle indistinguishability is another fundamental feature of quantum mechanics, lying at the heart of condensed matter physics and quantum field theories. An interesting problem combining these two fundamental concepts is that of the study and quantification of correlations between indistinguishable particles, a topic that has received increasing attention in the last years. Indistinguishability poses a nontrivial difficulty in the study of quantum correlations, because the notion of entanglement is intimately connected with that of local operations, and the latter are possible only if the constituents of the system can be distinguished. Different approaches to this problem have been considered, like mode entanglement

II. FORMALISM

We consider a SP space $\mathcal{H}$ of finite dimension d, spanned by fermion operators c_i, $c_i^\dagger$, $i=1,\ldots,d$ satisfying the anticommutation relations

$$\{c_i, c_j^\dagger\} = \delta_{ij},\ \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0\,. \tag{1}$$

We also define the M-fermion creation operators

$$C_\alpha^{M\dagger} = c_{i_1}^\dagger \ldots c_{i_M}^\dagger\,, \tag{2}$$

where $i_1 < i_2 < \ldots < i_M$ and $\alpha = (i_1,\ldots,i_M)$ labels all distinct sets of M SP states sorted in increasing order ($1 \le \alpha \le \binom{d}{M}$). These operators satisfy

$$\langle 0|C_\alpha^{(M)} C_{\alpha'}^{(M')\dagger}|0\rangle = \delta^{MM'}\delta_{\alpha\alpha'}\,, \tag{3}$$

$$\sum_\alpha C_\alpha^{(M)\dagger} C_\alpha^{(M)} = \binom{\hat{N}}{M}\,, \tag{4}$$

where $\hat{N} = \sum_i c_i^\dagger c_i$ is the fermion number operator and $\binom{N}{M} = \frac{N!}{M!(N-M)!}$. The ensuing states $C_\alpha^{M\dagger}|0\rangle$ are Slater Determinants (SDs) forming, for $0 \le M \le d$ and $1 \le \alpha \le \binom{d}{M}$, an orthonormal basis of the full 2^d-dimensional Fock space associated to $\mathcal{H}$.

A normalized pure state $|\Psi\rangle$ of N fermions ($\hat{N}|\Psi\rangle = N|\Psi\rangle$) can then be written as

$$|\Psi\rangle = \frac{1}{N!}\sum_{i_1,\ldots,i_N} \Gamma_{i_1\ldots i_N}\, c_{i_1}^\dagger \ldots c_{i_N}^\dagger |0\rangle \tag{5}$$

$$= \sum_\alpha \Gamma_\alpha C_\alpha^{(N)\dagger}|0\rangle\,, \tag{6}$$

where $\Gamma_{i_1\ldots i_N}$ is a rank-N antisymmetric tensor and

$$\Gamma_\alpha = \langle 0|C_\alpha^{(N)}|\Psi\rangle = \Gamma_{i_1\ldots i_N} \tag{7}$$

for $\alpha = (i_1,\ldots,i_N)$ (and $i_1 < i_2 < \ldots < i_N$), with

$$\sum_\alpha |\Gamma_\alpha|^2 = \frac{1}{N!}\sum_{i_1,\ldots i_N} |\Gamma_{i_1\ldots i_N}|^2 = 1\,. \tag{8}$$

Thus $|\Gamma_\alpha|^2$ is the probability of finding the N SP states α occupied in $|\Psi\rangle$.

A. The $(M, N-M)$ bipartite representation of an N-fermion state

For $0 \le M \le N$ we can also rewrite Eq. (5) as

$$|\Psi\rangle = \binom{N}{M}^{-1} \sum_{\alpha,\beta} \Gamma_{\alpha\beta}^{(M)} C_\alpha^{(M)\dagger} C_\beta^{(N-M)\dagger}|0\rangle\,, \tag{9}$$

where $\alpha = (i_1,\ldots,i_M)$, $\beta = (i_{M+1},\ldots,i_N)$ and

$$\Gamma_{\alpha\beta}^{(M)} = \langle 0|C_\beta^{(N-M)} C_\alpha^{(M)}|\Psi\rangle = \Gamma_{i_1\ldots i_M i_{M+1}\ldots i_N}\,, \tag{10}$$

with $1 \le \alpha \le \binom{d}{M}$, $1 \le \beta \le \binom{d}{N-M}$ and

$$\sum_{\alpha,\beta} |\Gamma_{\alpha\beta}^{(M)}|^2 = \binom{N}{M}\,. \tag{11}$$

We will denote Eq. (9) as the $(M, N-M)$-representation of the N-fermion state (5). It is a bipartite-like expansion of $|\Psi\rangle$ in orthogonal M- and $(N-M)$-fermion states, leading to a $\binom{d}{M}\times\binom{d}{N-M}$ matrix representation $\Gamma^{(M)}$ of the original tensor Γ in (5). Of course, decompositions $(M, N-M)$ and $(N-M, M)$ are equivalent, with

$$\Gamma^{(N-M)} = (-1)^{M(N-M)}(\Gamma^{(M)})^T\,, \tag{12}$$

due the antisymmetry of Γ, where T denotes transpose. Eq. (6) is the trivial $(N,0)$ representation $\Gamma^{(N)}_\alpha = \Gamma_\alpha$.

From the antisymmetry of Γ it also follows that

$$C^{(M)}_\alpha|\Psi\rangle = \sum_\beta \Gamma^{(M)}_{\alpha\beta} C^{(N-M)\dagger}_\beta|0\rangle\,, \tag{13}$$

which represents the (unormalized) state of remaining $N-M$ fermions when the M SP states α are occupied in $|\Psi\rangle$. Eqs. (13) and (3) imply that the *M-body density matrix*, whose elements are defined as

$$\rho^{(M)}_{\alpha\alpha'} := \langle\Psi|C^{(M)\dagger}_{\alpha'}C^{(M)}_\alpha|\Psi\rangle\,, \tag{14}$$

can be expressed in terms of $\Gamma^{(m)}$ as

$$\rho^{(M)} = \Gamma^{(M)}(\Gamma^{(M)})^\dagger\,, \tag{15}$$

i.e., $\rho^{(M)}_{\alpha\alpha'} = \sum_\beta \Gamma^{(M)}_{\alpha\beta}\Gamma^{(M)\,*}_{\alpha'\beta}$. In particular, $\rho^{(M)}_{\alpha\alpha}$ is the probability of finding the M SP states α occupied in $|\Psi\rangle$. Eq. (15) is a positive semidefinite $\binom{d}{M}\times\binom{d}{M}$ matrix which determines the average of any M-body operator:

$$\langle\Psi|\sum_{\alpha,\alpha'} O^{(M)}_{\alpha\alpha'}C^{(M)\dagger}_\alpha C^{(M)}_{\alpha'}|\Psi\rangle = \mathrm{Tr}\,\rho^{(M)}O^{(M)}\,. \tag{16}$$

Its trace is given by

$$\mathrm{Tr}\,\rho^{(M)} = \binom{N}{M}\,, \tag{17}$$

as implied by (4) or (11). We also notice that

$$C^{(N-M)}_\beta|\Psi\rangle = \sum_\alpha \Gamma^{(N-M)}_{\beta\alpha}C^{(M)\dagger}_\alpha|0\rangle\,. \tag{18}$$

Hence, using (12) the partner $(N-M)$-body DM, of elements $\rho^{(N-M)}_{\beta\beta'} = \langle\Psi|C^{(N-M)\dagger}_{\beta'}C^{(N-M)}_\beta|\Psi\rangle$, is

$$\rho^{(N-M)} = (\Gamma^{(M)})^T\Gamma^{(M)*}\,, \tag{19}$$

which shows it has *the same non-zero eigenvalues* as the M-body DM (15), as discussed below in more detail.

In particular, for $M=1$, $C^{(1)\dagger}_\alpha = c^\dagger_i$ and we recover from (9) the $(1, N-1)$ representation [?]

$$|\Psi\rangle = \frac{1}{N}\sum_{i,\alpha}\Gamma^{(1)}_{i\alpha}c^\dagger_i C^{(N-1)\dagger}_\alpha|0\rangle\,. \tag{20}$$

where the $n\times\binom{n}{N-1}$ matrix $\Gamma^{(1)}$ determines the *one-body* DM (also denoted as SPDM) $\rho^{(1)}_{ii'} = \langle\Psi|c^\dagger_{i'}c_i|\Psi\rangle$ through

$$\rho^{(1)} = \Gamma^{(1)}{\Gamma^{(1)}}^\dagger\,. \tag{21}$$

It will then have the same non-zero eigenvalues as the $(N-1)$-body DM $\rho^{(N-1)}$.

We finally remark that Eq. (9) is a particular case of the more general *k-partite representation* of the state (5),

$$|\Psi\rangle = \tfrac{M_1!\ldots M_k!}{N!}\sum_{\alpha_1,\ldots,\alpha_k}\Gamma^{M_1\ldots M_k}_{\alpha_1\ldots\alpha_k}C^{(M_1)\dagger}_{\alpha_1}\ldots C^{(M_k)\dagger}_{\alpha_k}|0\rangle, \tag{22}$$

where $\Gamma^{M_1\ldots M_k}_{\alpha_1\ldots\alpha_k} = \langle 0|C^{(M_k)}_{\alpha_k}\ldots C^{(M_1)}_{\alpha_1}|\Psi\rangle = \Gamma_{i_1\ldots i_N}$ for $\alpha_1 = (i_1,\ldots,i_{M_1})$, …, $\alpha_k = (i_{N-M_k+1},\ldots,i_N)$ and $\sum_{j=1}^k M_j = N$, with $k\le N$ and $1\le\alpha_k\le\binom{d}{M_k}$. The basic expansion (5) corresponds to $k=N$ and $M_j = 1\,\forall\,j$.

B. The $(M, N-M)$ Schmidt representation

We can now employ the singular value decomposition of the matrix $\Gamma^{(M)}$,

$$\Gamma^{(M)} = U^{(M)}D^{(M)}V^{(M)\dagger}\,, \tag{23}$$

$$D^{(M)}_{\nu\nu'} = \sqrt{\lambda^{(M)}_\nu}\,\delta_{\nu\nu'}\,, \tag{24}$$

where $U^{(M)}, V^{(M)}$ are $\binom{d}{N}\times\binom{d}{M}$ and $\binom{d}{N-M}\times\binom{d}{N-M}$ unitary matrices and $\sqrt{\lambda^{(M)}_\nu}$ are the singular values of $\Gamma^{(M)}$, i.e. $\lambda^{(M)}_\nu$ are the eigenvalues of $\Gamma^{(M)}\Gamma^{(M)\dagger} = \rho^{(M)}$ or equivalently $\Gamma^{(M)T}\Gamma^{(M)*} = \rho^{(N-M)}$, which have the same spectrum (except for the number of zero eigenvalues). It then becomes possible to rewrite Eq. (9) in the Schmidt-like diagonal form

$$|\Psi\rangle = \binom{N}{M}^{-1}\sum_{\nu=1}^{n_M}\sqrt{\lambda^{(M)}_\nu}A^{(M)\dagger}_\nu A^{(N-M)\dagger}_\nu|0\rangle\,, \tag{25}$$

where n_M is the rank of $\Gamma^{(M)}$ (the number of non-zero eigenvalues $\lambda^{(M)}_\nu$) and

$$A^{(M)\dagger}_\nu = \sum_\alpha U^{(M)}_{\alpha\nu}C^{(M)\dagger}_\alpha\,, \tag{26}$$

$$A^{(N-M)\,\dagger}_\nu = \sum_\beta V^{(M)*}_{\beta\nu}C^{(N-M)\,\dagger}_\beta\,, \tag{27}$$

are operators creating, respectively, M and $N-M$ fermions in generally "collective" states (i.e., non-SD's for $M\ge 2$). Nevertheless, since they are unitarily related to the original operators $C^{(M)\dagger}_\alpha$ and $C^{(N-M)\dagger}_\beta$, they still satisfy

$$\langle 0|A^{(M)}_\nu A^{(M')\dagger}_{\nu'}|0\rangle = \delta^{MM'}_{\nu\nu'}\,, \tag{28}$$

$$\sum_\nu A^{(M)\dagger}_\nu A^{(M)}_\nu = \binom{\hat N}{M}\,, \tag{29}$$

$$\langle 0|A^{(N-M)}_{\nu'}A^{(M)}_\nu|\Psi\rangle = \sqrt{\lambda^{(M)}_\nu}\,\delta_{\nu\nu'}\,, \tag{30}$$

$\forall\,M$ satisfying $1\le M\le N-1$. Moreover, Eq. (13) leads to

$$A^{(M)}_\nu|\Psi\rangle = \sqrt{\lambda^{(M)}_\nu}A^{(N-M)\,\dagger}_\nu|\Psi\rangle\,, \tag{31}$$

such that $A_\nu^{(N-M)\dagger}|0\rangle$ is the state of remaining $N-M$ fermions when M fermions are measured to be in the "normal" or "natural" state $A_\nu^{(M)\dagger}|0\rangle$. Eqs. (28)–(31) also imply that the normal operators $A_\nu^{(M)}$, $A_\nu^{(N-M)}$ are those diagonalizing the M- and $N-M$-body DM's:

$$\langle\Psi|A_{\nu'}^{(M)\dagger}A_\nu^{(M)}|\Psi\rangle=\lambda_\nu^{(M)}\delta_{\nu\nu'}=\langle\Psi|A_{\nu'}^{(N-M)\,\dagger}A_\nu^{(N-M)}|\Psi\rangle\,. \tag{32}$$

For $M=1$ we recover from (26)–(27) the diagonal representation of the one-body DM $\rho^{(1)}$ [**?**], with $A_\nu^{(1)\dagger}=\sum_i U_{i\nu}c_i^\dagger=c_\nu^\dagger$ the operators creating a fermion in the ensuing natural orbitals.

In the trivial case $M=N$, $\rho^{(N)}$ has just a single eigenvalue $\lambda_1^{(N)}=1$ associated with the operator $A_1^{(N)\,\dagger}$ creating the state: $|\Psi\rangle=A_1^{(N)\,\dagger}|0\rangle$. On the other hand, in a SD, which can be always written as $C_1^{(N)\dagger}|0\rangle$ by a suitable choice of the operators c_i, $\rho^{(M)}$ has of course just $\binom{N}{M}$ non-zero eigenvalues $\lambda_\nu^{(M)}=1$ associated with the occupied SP states.

C. The eigenvalues of the M-body DM

While the eigenvalues $\lambda_\nu^{(1)}=\langle\Psi|c_\nu^\dagger c_\nu|\Psi\rangle$ of $\rho^{(1)}$ always lie in the interval $[0,1]$ (as $c_\nu^\dagger$ are standard fermion operators), for $m\geq 2$ those of $\rho^{(M)}$ can be *greater than 1*, as the collective normal operators $A_\nu^{(M)\dagger}$ may exhibit boson-like features for even $M\geq 2$, or in general features of a boson+fermion creation operator for odd $M\geq 3$.

For instance, let us define, assuming even SP dimension d, the collective pair creation operator

$$A^\dagger=\frac{1}{\sqrt{d/2}}\sum_{i=1}^{d/2}c_{2i-1}^\dagger c_{2i}^\dagger \tag{33}$$

which satisfies

$$[A,A^\dagger]=1-2\hat{N}/d\,. \tag{34}$$

We then consider, for $0\leq k\leq d/2$, the normalized states

$$|\Psi_k\rangle=\frac{1}{k!\sqrt{(\frac{2}{d})^k\binom{d/2}{k}}}(A^\dagger)^k|0\rangle \tag{35}$$

which contain k of such pairs and hence $N=2k$ fermions. They satisfy $A^\dagger|\Psi_{k-1}\rangle=\sqrt{k(1-2\frac{k-1}{d})}|\Psi_k\rangle$ and

$$A|\Psi_k\rangle=\sqrt{k(1-2\tfrac{k-1}{d})}|\Psi_{k-1}\rangle\,. \tag{36}$$

Since all SP states have the same probability of being occupied, it is apparent that the SPDM $\rho^{(1)}$ has in these states just a single degenerate eigenvalue $\lambda_\nu^{(1)}=N/d$, such that it is uniformly mixed, i.e. proportional to the identity and hence diagonal in *any* SP basis:

$$\rho^{(1)}=\frac{2k}{d}\mathbb{1}\,. \tag{37}$$

In contrast, from Eqs. (35)–(36) it follows that $A^\dagger A|\Psi_k\rangle=k(1-2\frac{k-1}{d})|\Psi_k\rangle$, implying that the two-body DM $\rho^{(2)}$ has one large nondegenerate eigenvalue

$$\lambda_1^{(2)}=\langle\Psi_k|A^\dagger A|\Psi_k\rangle=k(1-2\tfrac{k-1}{d})\geq 1\,, \tag{38}$$

associated with the normal operator $A^\dagger$, which satisfies $\lambda_1^{(2)}>1$ for $2\leq k\leq d/2-1$ [$\lambda_1^{(2)}=1+(k-1)(1-\frac{2k}{d})$], while all remaining $\binom{d/2}{2}-1$ eigenvalues are small and identical (see Appendix):

$$\lambda_2^{(2)}=\frac{4k(k-1)}{d(d-2)}\leq 1\,, \tag{39}$$

such that $\lambda_1^{(2)}+(\binom{d}{2}-1)\lambda_2^{(2)}=\binom{N}{2}$ (Eq. (17)). The first eigenvalue (38) is maximum in the half-filled case $k=[\frac{d+2}{4}]$, where $\lambda_1^{(2)}\approx d(1+2/d)^2/8$ increases linearly with d for large d, and can then become arbitrarily large. For $d\gg k$, $\lambda_1^{(2)}\approx k=N/2$ is just the number of pairs, whereas $\lambda_2^{(2)}\approx(2k/d)^2$ becomes very small. In fact, for $d\to\infty$ at fixed $\hat{N}$, $A^\dagger$ becomes a true boson: $[A,A^\dagger]\to 1$ (Eq. (34)) while $|\Psi_k\rangle\to\frac{(A^\dagger)^k}{\sqrt{k!}}|0\rangle$ (Eq. (35)), which is a state of k bosons.

Eigenvalues of higher order DM's can also be >1. For instance, in an odd state $|\Psi_k^o\rangle=c_{d+1}^\dagger|\Psi_k\rangle$, where we have enlarged the SP space with one additional state, the largest eigenvalue of the three-body DM $\rho^{(3)}$ is again given by Eq. (38), i.e., $\lambda_1^{(3)}=\langle\Psi_k^o|c_{d+1}^\dagger A^\dagger A c_{d+1}|\Psi_k^o\rangle=\lambda_1^{(2)}\geq 1$. While three fermions could in principle be associated to a composite fermion due the half-integer spin, this eigenvalue corresponds physically to the number of bosons times the number of fermions (1) in a boson-fermion system, and can then also exceed 1. The eigenvalues of $\rho^{(3)}$ in the even state (35) can be analytically determined too (see Appendix). Its largest eigenvalue,

$$\lambda_1^{(3)}=\frac{2k(k-1)(1-\frac{2}{d}(k-1))}{d-2}\,, \tag{40}$$

while smaller than (38), still exceeds 1 for $1+\sqrt{d/2}<k<d/2$, with $\lambda_1^{(3)}\approx 2d/27$ at its maximum $k\approx d/3+1$ for large d.

From Eqs. (35)–(36) it also follows that $(A^\dagger)^m A^m|\Psi_k\rangle\propto|\Psi_k\rangle$ for $m\leq k$. Thus, the largest eigenvalue of the $2m$-body DM $\rho^{(2m)}$ in the state (35) is determined by the normalized operator $A_1^{(2m)\dagger}=(A^\dagger)^m/[m!\sqrt{(\frac{2}{d})^m\binom{d/2}{m}}]=\frac{1}{\sqrt{\binom{d/2}{m}}}\sum_\mu C_\mu^{(2m)}$, where $C_\mu^{(2m)}=\prod_i(c_{2i-1}^\dagger c_{2i}^\dagger)^{n_{i_\mu}}$, $n_{i_\mu}=0,1$, creates $2m$ fermions in a series of contiguous pairs $(2i-1,2i)$:

$$\lambda_1^{(2m)}=\langle\Psi_k|\frac{(A^\dagger)^m A^m}{k!^2(\frac{2}{d})^k\binom{d/2}{k}}|\Psi_k\rangle=\frac{\binom{k}{m}\binom{d/2-k+m}{m}}{\binom{d/2}{m}}\,, \tag{41}$$

which generalizes Eq. (38). It is larger than 1 for $m<k<d/2$, with $\lambda_1^{(2m)}\approx\binom{k}{m}$ for $d\gg k$. Similarly, in the partner odd state $|\Psi_k^o\rangle=c_{d+1}^\dagger|\Psi_k\rangle$, $\lambda_1^{(2m+1)}=\lambda_1^{(2m)}$.

We finally notice that it may also occur, of course, that all eigenvalues of $\rho^{(m)}$ are smaller than 1. For instance, an $N = d/2$ GHZ-like fermion state

$$|\Psi_{d/2}\rangle = \frac{1}{\sqrt{2}}(c_1^\dagger \dots c_{d/2}^\dagger + c_{d/2+1}^\dagger \dots c_d^\dagger)|0\rangle \tag{42}$$

leads to eigenvalues $\lambda_\nu^{(m)} = 1/2 \; \forall \; m$ in the range $1 \leq m \leq N-1$, $2\binom{N}{m}$ degenerate (all remaining ones are 0).

III. M-BODY ENTANGLEMENT

The Schmidt-like decomposition (25), or equivalently the original bipartite expansion (9), can be considered as the starting point for defining M-body entanglement: Given two pure states $|\Psi\rangle$, $|\Phi\rangle$ of N fermions, we will say that $|\Psi\rangle$ is *not less* $(m, N-m)$ entangled (or simply M-body entangled) than $|\Phi\rangle$ if the following majorization relation is satisfied by the corresponding M-body DMs:

$$\rho_\Psi^{(M)} \prec \rho_\Phi^{(M)}\,, \tag{43}$$

which is equivalent to

$$\boldsymbol{\lambda}^{(M)}(|\Psi\rangle) \prec \boldsymbol{\lambda}^{(M)}(|\Phi\rangle) \tag{44}$$

where $\boldsymbol{\lambda}^{(M)}(|\Psi\rangle)$ denotes the sorted (in decreasing order) spectrum of $\rho_\Psi^{(M)}$. i.e., the square of the sorted singular values of the $M-(N-M)$ decomposition ScDc. Explicitly, this means that all inequalities

$$\sum_{\nu=1}^{j}(\lambda_\nu^{(M)}(|\Psi\rangle) \leq \sum_{\nu=1}^{j}\lambda_\nu^{(M)}(|\Phi\rangle)\,,\;\; j = 1, \dots, \frac{d}{M}\,, \tag{45}$$

are to be satisfied, with equality for $j = \binom{d}{M}$. Of course, one may likewise employ the partner $N-m$-body DM matrix $\rho^{(N-M)}$ in (43) since $\rho^{(M)}$ and $\rho^{(N-M)}$ have the same nonzero eigenvalues. Then, a given N fermion state $|\Psi\rangle$ is more M-body entangled than another N fermion state $|\Phi\rangle$ if the eigenvalues of $\rho^{(M)}$ are more spread than those of ${\rho'}^{(M)}$. Of course, majorization provides just a partial order, entailing that two states may be uncomparable according to this definition. For $M = 1$ we recover the concept of one-body entanglement, discussed in [**?**].

Associated with this definition, we may introduce an $M - body$ entanglement measure in two ways: The first one is just

$$\begin{aligned} E_f^{(M)}(|\Psi\rangle) &= S_f(\rho^{(M)}) = S_f(\rho^{(N-M)}) \\ &= \sum_\nu f(\lambda_\nu^{(M)}) \end{aligned} \tag{46}$$

where $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a strictly concave function satisfying $f(0) = 0$. Such choice is particularly suitable for defining a one-body entanglement entropy, since in this case $\lambda_\nu^{(1)} \in [0,1]$ and standard entropic measures can be employed. In any case, any measure (46) will satisfy

$$E_f^{(M)}(|\Psi\rangle) \geq E_f^{(M)}(|\Phi\rangle) \tag{47}$$

whenever (43) is satisfied.

We will here consider instead entropies defined on the normalized

Whenever $f(1) = 0$, like in (43), these entropies will always vanish in a SD, where $\rho^{(m)}$ has $\binom{N}{m}$ eigenvalues $\lambda_\nu^{(m)} = 1$ (and remaining ones 0). Nonetheless, we notice that for arbitrary distributions, Eq. (43) may become negative since now $\lambda_\nu^{(m)}$ can be > 1. This would correspond to an hypothetical $\rho^{(m)}$ less mixed than a SD, like a pure boson-like condensate leading to a single non-zero eigenvalue equal to $\binom{N}{m}$.

The second way is to consider a normalized density $\rho^{(m)}/\binom{N}{m}$ such that $\lambda_\nu^{(m)}/\binom{N}{m} \in [0,1] \; \forall \; \nu$ and $\sum_\nu \lambda_\nu^{(m)}/\binom{N}{m} = 1$. In this case we can employ any entropic measure intended for standard probabilities:

$$E_{f_n}^m(|\Psi\rangle) = S_f\left(\frac{\rho^{(m)}}{\binom{N}{m}}\right) = \sum_\nu f\left(\frac{\lambda_\nu^{(m)}}{\binom{N}{m}}\right) \tag{48}$$

where $f : [0,1] \to \mathbb{R}$ is any strictly concave function satisfying $f(0) = f(1) = 0$ (and hence $f(p) > 0$ for $p \in (0,1)$). These entropies are always non-negative and are just a particular case of the previous entropies (46) $(f(p) \to f_n(p) = f(p/\binom{N}{m}))$. We will denote this entropes as normalized m-body entanglement entropy

In any case, these entanglement entropies satisfy the inequality

$$\rho^{(m)} \prec {\rho'}^{(m)} \Rightarrow E_f^m(|\Psi\rangle) \geq E_f^m(|\Psi'\rangle) \tag{49}$$

i.e., they will be larger for a more entangled state.

A. Measurements decreasing the m-body entanglement

In a previous work we have shown that 1-body entanglement, i.e., the one quantified by the mixedness of the SPDM, is non increasing under Fermion Linear Optics (FLO) operations. These operations include one-body unitary transformations $c_i \to \hat{U}^\dagger c_i \hat{U} = \sum_j U_{ij} c_j$, with $U^\dagger U = \mathbb{1}$, and also measurements of the occupancy of a SP state $|k\rangle \in \mathcal{H}$, described by projection operators

$$\mathcal{P}_k = c_k^\dagger c_k\,, \quad \mathcal{P}_{\bar{k}} = c_k c_k^\dagger = \mathbb{1} - \mathcal{P}_k\,. \tag{50}$$

satisfying $\mathcal{P}_k^\dagger \mathcal{P}_k + \mathcal{P}_{\bar{k}}^\dagger \mathcal{P}_{\bar{k}} = \mathcal{P}_k + \mathcal{P}_{\bar{k}} = \mathbb{1}$. One-body unitary transformations just lead to a unitary transformation of $\rho^{(1)}$ and also all $\rho^{(m)}$, so that they will not affect their eigenvalues. Consequently, they are also free operations for *all m-body entanglements* $(1 \leq m \leq n-1)$, leaving them invariant. We mention that general unitary transformations $C_\alpha^m \to \sum_{\alpha'} W_{\alpha\alpha'} C_{\alpha'}^m$, with $W^\dagger W = \mathbb{1}$, just lead to $\rho^{(m)} \to W\rho^{(m)}W^\dagger$ and hence will leave the spectrum of $\rho^{(m)}$ unchanged. One-body unitary transformations correspond to $W_{\alpha.\alpha'} = \epsilon_{i_1\dots i_m} U_{\alpha_1\alpha'_{i_1}} \dots U_{\alpha_m\alpha'_{i_m}}$ with ϵ the fully antisymmetric tensor.

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This is not the case, however, for the measurements (50) when considering $m \geq 2$-body entanglement. Under these measurements, the SPDM satisfies the majorization relation [**?**]

$$\rho^{(1)} \prec p_k \rho_k^{(1)} + (1-p_k)\rho_{\bar{k}}^{(1)}\,, \tag{51}$$

where $p_k = \langle\Psi|c_k^\dagger c_k|\Psi\rangle$ is the probability of the state k being occupied in $|\Psi\rangle$ $p_{\bar{k}} = 1-p_k$ that of being empty, and $\rho_k^{(1)}$, $\rho_{\bar{k}}^{(1)}$ are the SPDM's determined by the post-selected states $|\Psi_k\rangle = c_k^\dagger c_k|\Psi\rangle/\sqrt{p_k}$ and $|\Psi_{\bar{k}}\rangle = c_k c_k^\dagger|\Psi\rangle/\sqrt{1-p_k}$, These measurement reduces our ignorance about $\rho^{(1)}$ and hence cannot increase, on average, the mixedness of $\rho^{(1)}$ [**?**]. Eq. (51) implies the inequality

$$E_f^1(|\Psi\rangle) \geq p_k E_f^1(|\Psi_k\rangle) + p_{\bar{k}} E_f^1(|\Psi_{\bar{k}}\rangle) \tag{52}$$

indicating that one-body entanglement entropies will on average decrease after this measurement.

However, Eq. (51) *does not hold*, in general, for $\rho^{(m)}$ with $m \geq 2$, implying that m-body entanglement will not necessarily decrease after such measurement. A simple analytic example is provided in the Appendix. Essentially, measurement of the occupancy of a SP state can reduce, for instance, the number of "collective bosons" in a state like (35), implying a lower maximum eigenvalue $\lambda_1^{(m)}$ in the postmeasurement states and hence violation of the inequality (51) for $\rho^{(m)}$ with $m \geq 2$. This result is expected as these measurements do not necessarily increase our knowledge of the m-body DM.

Instead, we will now show, for states $|\Psi\rangle$ with fixed fermion number N, that the quantum operation described by Krauss operators of the form

$$\mathcal{M}_k^{(1)} = \frac{c_k}{\sqrt{N}}\,, \tag{53}$$

which satisfy $\sum_k \mathcal{M}_k^{(1)\dagger}\mathcal{M}_k^{(1)} = \hat{N}/N = \mathbb{1}$ within the subspace of states $|\Psi\rangle$ satisfying $\hat{N}|\Psi\rangle = N|\Psi\rangle$, do not increase m-body entanglement. This this operation, with d distinct outcomes, corresponds for instance to the detection of a single fermion through its momentum or SP energy (labelled by k), annihilating it at the measurement.

Proof: Let ρ be the state of a N fermion system upon which the operation is performed. After outcome k is obtained the state of the system is $\rho_k = p_k^{-1}\,\mathcal{M}_k^{(1)}\rho\mathcal{M}_k^{(1)\dagger}$, with $p_k = \operatorname{Tr}\rho\mathcal{M}_k^{(1)\dagger}\rho\mathcal{M}_k^{(1)}$, and its associated m-body DM has elements

$$\rho_k^{(m)}{}_{\alpha\alpha'} = p_k^{-1}\operatorname{Tr}\mathcal{M}_k^{(1)}\rho\mathcal{M}_k^{(1)\dagger}C_{\alpha'}^{(m)}C_\alpha^{(m)} \tag{54}$$

$$= p_k^{-1}\operatorname{Tr}\rho\, C_{\alpha'}^{(m)}\mathcal{M}_k^{(1)\dagger}\mathcal{M}_k^{(1)}C_\alpha^{(m)}, \tag{55}$$

where the last line holds because operators $C_\alpha^{(m)}$ and $\mathcal{M}_k^{(1)}$ either commute or anticommute. It follows then that

$$\sum_k p_k\rho_k^{(m)} = \frac{N-m}{N}\rho^{(m)}\,, \tag{56}$$

implying that the eigenvalue vectors of $\rho^{(m)}$ and $\rho_k^{(m)}$, $\boldsymbol{\lambda}(\rho^{(m)})$ and $\boldsymbol{\lambda}(\rho_k^{(m)})$ respectively, satisfy the majorization relation

$$\boldsymbol{\lambda}(\rho^{(m)}) \prec \sum_k p_k\boldsymbol{\lambda}(\rho_k^{(m)}), \tag{57}$$

and therefore that the operation does not increase m-body entanglement.

For $1 \leq m < N$ (56) can be rewritten as

$$\sum_k p_k\left(\frac{\rho_k^{(m)}}{\binom{N-1}{m}}\right) = \frac{\rho^{(m)}}{\binom{N}{m}}\,, \tag{58}$$

where $\rho_k^{(m)}/\binom{N-1}{m}$, $\rho^{(m)}/\binom{N}{m}$ are density matrices normalized to 1. Majorization relation (57) is clearly preserved by this normalization, i.e., it holds that

$$\boldsymbol{\lambda}\left(\frac{\rho^{(m)}}{\binom{n}{m}}\right) \prec \sum_k \frac{p_k}{N}\boldsymbol{\lambda}\left(\frac{\rho_k^{(m)}}{\binom{N-1}{m}}\right) \tag{59}$$

This leads to the general entropic inequality

$$S_f\left(\frac{\rho^{(m)}}{\binom{N}{m}}\right) \geq S_f\left[\sum_k \frac{p_k}{N}\boldsymbol{\lambda}\left(\frac{\rho_k^{(m)}}{\binom{N-1}{m}}\right)\right) \tag{60}$$

$$\geq \sum_k \frac{p_k}{N} S_f\left(\frac{\rho_k^{(m)}}{\binom{N-1}{m}}\right), \tag{61}$$

which shows that the m-body entanglement entropy (48) exhibits a monotonic behavior under these operations.

B. m-body density matrices as post-measurement states and generalized majorization relations

If the result of previous measurement is unknown, the post-measurement state (with no pos-selection)

$$\rho' = \sum_k \mathcal{M}_k^{(1)}|\Psi\rangle\langle\Psi|\mathcal{M}_k^{(1)\dagger} = \frac{1}{N}\sum_k c_k|\Psi\rangle\langle\Psi|c_k^\dagger\,, \tag{62}$$

is, remarkably, proportional to the $m = (N-1)$-body density operator:

$$\hat{\rho}^{(N-1)} := \sum_{\alpha,\alpha'} \rho_{\alpha\alpha'}^{(N-1)} C_\alpha^{(N-1)\,\dagger}|0\rangle\langle 0|C_{\alpha'}^{N-1} \tag{63}$$

$$= \sum_k c_k|\Psi\rangle\langle\Psi|c_k^\dagger\,. \tag{64}$$

The operator (63) is the unique mixed state of $N-1$ fermions satisfying

$$\operatorname{Tr}\left(\hat{\rho}^{(N-1)}C_{\alpha'}^{(N-1)\,\dagger}C_\alpha^{N-1}\right) = \rho_{\alpha\alpha'}^{(N-1)}$$

$$= \langle\Psi|C_{\alpha'}^{(N-1)\,\dagger}C_\alpha^{N-1}|\Psi\rangle \tag{65}$$

$\forall\ \alpha,\alpha'$, and by Eq. (56) for $m = N-1$, it must then coincide exactly with (64), i.e., with $N\rho'$, since it is also an $(N-1)$-fermion mixed state leading to the same mean values.

These results, together with Eqs. (58)–(61), can be extended to l-body measurements, in which l fermions are detected and annihilated. We then consider the operators annihilating l fermions as measurement operators,

$$\mathcal{M}^{(l)}_\beta = \frac{C^l_\beta}{\sqrt{\binom{N}{l}}} = \frac{c_{\beta_1}\ldots c_{\beta_l}}{\sqrt{\binom{N}{l}}}\,. \tag{66}$$

which satisfy $\sum_\beta \mathcal{M}^{(l)\dagger}_\beta \mathcal{M}^{(l)}_\beta = \mathbb{1}$ within the subspace of N-fermion states since

$$\sum_\beta C^{l\,\dagger}_\beta C^l_\beta = \frac{1}{l!}\sum_{\beta_1,\ldots,\beta_l} c^\dagger_{\beta_l}\ldots c^\dagger_{\beta_1} c_{\beta_1}\ldots c_{\beta_l} = \binom{\hat{N}}{l} \tag{67}$$

The proof follows that given in the previous section. After outcome β is obtained the state of the system is $\rho_\beta = p_\beta^{-1}\mathcal{M}^{(l)}_\beta \rho \mathcal{M}^{(l)\dagger}_\beta$, with $p_\beta = \mathrm{Tr}\,\rho\mathcal{M}^{(l)\dagger}_\beta\mathcal{M}^{(l)}_\beta$, and its associated m-body DM will be

$$\rho^{(m)}_\beta = \sum_{\alpha,\alpha'} \mathrm{Tr}\left(\mathcal{M}^{(l)}_\beta \rho \mathcal{M}^{(l)\dagger}_\beta C^{(m)\dagger}_{\alpha'} C^{(m)}_\alpha\right) C^{(m)\dagger}_\alpha |0\rangle\langle 0| C^{(m)}_{\alpha'}\,. \tag{68}$$

As before, because the operators $\mathcal{M}^{(l)}_\beta$ and $C^{(m)}_\alpha$ either commute or anticommute it holds that

$$\sum_\beta p_\beta \rho^{(m)}_\beta = \frac{\binom{N-m}{l}}{\binom{N}{l}}\rho^{(m)} \tag{69}$$

and after normalization by $\binom{N-l}{m}$

$$\sum_\beta p_\beta \frac{\rho^{(m)}_\beta}{\binom{N-l}{m}} = \frac{\rho^{(m)}}{\binom{N}{m}}. \tag{70}$$

Eq. (70) then implies the majorization relation

$$\boldsymbol{\lambda}\left(\frac{\rho^{(m)}}{\binom{n}{m}}\right) \prec \sum_\beta p_\beta \boldsymbol{\lambda}\left(\frac{\rho^{(m)}_\beta}{\binom{N-l}{m}}\right) \tag{71}$$

which leads to the general entropic inequality

$$S_f\left(\frac{\rho^{(m)}}{\binom{N}{m}}\right) \geq S_f\left[\sum_\beta p_\beta \boldsymbol{\lambda}\left(\frac{\rho^{(m)}_\beta}{\binom{N-l}{m}}\right)\right] \tag{72}$$

$$\geq \sum_k \frac{p_\beta}{\binom{N}{l}} S_f\left(\frac{\rho^{(m)}_\beta}{\binom{N-l}{m}}\right) \tag{73}$$

The average post-measurement m-body entanglement is never larger than the initial normalized m-body entanglement, for the present l-fermion measurements ($m \leq N-l$).

Besides, Eqs. (63)–(64) are generalized to

$$\begin{aligned}\hat{\rho}^{(N-l)} :&= \sum_{\alpha,\alpha'} \rho^{(N-l)}_{\alpha\alpha'} C^{(N-l)\,\dagger}_\alpha |0\rangle\langle 0| C^{N-l}_{\alpha'} &(74)\\ &= \sum_\beta C^l_\beta |\Psi\rangle\langle\Psi| C^{l\,\dagger}_\beta \\ &= \frac{1}{l!}\sum_{\beta_1,\ldots,\beta_l} c_{\beta_1}\ldots c_{\beta_l}|\Psi\rangle\langle\Psi| c^\dagger_{\beta_l}\ldots c^\dagger_{\beta_1}\,, &(75)\end{aligned}$$

where the expression (75) follows from Eq. (67) for $m = N-l$ and commutation relations between operators $\mathcal{M}^{(l)}_\beta$ and $C^{(m)}_\alpha$. This result indicates that after this l-body measurement is performed, the post-measurement state when the outcome is unkown (no postselection) is proportional to the $N-l$-body density matrix.

APPENDIX

1. Eigenvalues of two- and three-body density matrices in the states $|\Psi_{2k}\rangle$

We derive here the eigenvalues of the first three DM's in the states (35). The elements of the one-body DM in these states are obviously

$$\langle\Psi_k| c^\dagger_{i'} c_j |\Psi_k\rangle = \delta_{ij} N/d \tag{A.1}$$

implying $\rho^{(1)} = \frac{N}{d}\mathbb{1}$. It is then the maximally mixed SPDM compatible with the total fermion number N, being hence diagonal in *any* SP basis with a single d-fold degenerate eigenvalue N/d. Therefore, these states lead to maximum one-body entanglement compatible with a given value of N.

On the other hand, the elements of the two-body DM are blocked in two submatrices. The first one, comprising the contiguous pair creation operators $c^\dagger_{2i-1}c^\dagger_{2i}$ that form the operator $A^\dagger$ of Eq. (33), has elements

$$\langle\Psi_{2k}| c^\dagger_{2i-1} c^\dagger_{2i} c_{2j} c_{2j-1} |\Psi_{2k}\rangle = \alpha\delta_{ij} + \beta(1-\delta_{ij}) \tag{A.2}$$

where, using $k = N/2$ and assuming d even,

$$\alpha = \frac{\binom{d/2-1}{k-1}}{\binom{d/2}{k}} = \frac{2k}{d}\,,\quad \beta = \frac{\binom{d/2-2}{k-1}}{\binom{d/2}{k}} = \frac{2k(d-2k)}{d(d-2)}\,.$$

Hence, since this $\frac{d}{2}\times\frac{d}{2}$ block is just $(\alpha-\beta)\mathbb{1} + M$, with M a rank one matrix with all elements equal to β, it has just two distinct eigenvalues: a nondegenerate eigenvalue

$$\lambda^{(2)}_1 = \alpha + (\frac{d}{2}-1)\beta = k(1-2(k-1)/d)\,, \tag{A.3}$$

which is precisely that associated with the collective pair creation operator $A^\dagger$:

$$\langle\Psi_{2k}| A^\dagger A |\Psi_{2k}\rangle = \lambda^{(2)}_1\,, \tag{A.4}$$

and a $n/2-1$-fold degenerate smaller eigenvalue

$$\lambda_2^{(2)} = \alpha - \beta = \frac{4k(k-1)}{d(d-2)}\,. \tag{A.5}$$

The other block comprises the remaining $\binom{d}{2}-\frac{d}{2}$ pair creation operators $c_i^\dagger c_j^\dagger$ and is directly diagonal, with elements $\lambda_3^{(2)} = \frac{\binom{d/2-2}{k-2}}{\binom{d/2}{k}} = \lambda_2^{(2)}$. Thus, the final result is one large nondegenerate eigenvalue $\lambda_1^{(2)} \geq 1$, plus $\binom{d}{2}-1$ smaller identical eigenvalues $\lambda_2^{(2)}$, satisfying

$$\lambda_1^{(2)} + \left(\binom{d}{2}-1\right)\lambda_2^{(2)} = \binom{N}{2}\,. \tag{A.6}$$

The same procedure can be applied to determine the eigenvalues of the three-body DM $\rho^{(3)}$. For creation of three fermions with two of them in one of the contiguous pairs $2i-1, 2i$, we obtain d identical $(\frac{d}{2}-1)\times(\frac{d}{2}-1)$ blocks of elements

$$\langle\Psi_{2k}|c_{2i-1}^\dagger c_{2i}^\dagger c_j^\dagger c_k c_{2l} c_{2l-1}|\Psi_{2k}\rangle = \delta_{jk}[\gamma\delta_{il}+\eta(1-\delta_{il})] \tag{A.7}$$

where $k \neq 2l, 2l-1$ and $j \neq 2i,\ 2i-1$, with

$$\gamma = \frac{\binom{d/2-2}{k-2}}{\binom{d/2}{k}}\,,\quad \eta = \frac{\binom{d/2-3}{k-2}}{\binom{d/2}{k}}\,.$$

Hence, each of these d blocks has a large non-degenerate eigenvalue

$$\lambda_1^{(3)} = \gamma + (\frac{d}{2}-2)\eta = \frac{2k(k-1)(1-2(k-1)/d)}{d-2} \tag{A.8}$$

$$= \langle\Psi_{2k}|A_j^{(3)\dagger}A_j^{(3)}|\Psi_{2k}\rangle, \tag{A.9}$$

with $A_j^{(3)\dagger} = \frac{1}{\sqrt{d/2-1}}\sum_{i\neq j} c_{2i-1}^\dagger c_{2i}^\dagger c_j^\dagger$ a collective operator, and $d/2-2$ identical eigenvalues

$$\lambda_2^{(3)} = \gamma - \eta = \frac{8k(k-1)(k-2)}{d(d-2)(d-4)}\,. \tag{A.10}$$

associated to orthogonal operators $A_\nu^{(3)\dagger}$. On the other hand, remaining $\binom{d}{3}-d(\frac{d}{2}-1)$ triplets $c_i^\dagger c_j^\dagger c_k^\dagger$ belonging to distinct pairs lead to a diagonal block in $\rho^{(3)}$ with identical diagonal elements $\frac{\binom{d/2-3}{k-3}}{\binom{d/2}{k}} = \lambda_2^{(3)}$. Therefore, there are d eigenvalues equal to $\lambda_1^{(3)}$ plus $\binom{d}{3}-d$ eigenvalues equal to $\lambda_2^{(3)}$, satisfying

$$d\lambda_1^{(3)} + \left(\binom{d}{3}-d\right)\lambda_2^{(2)} = \binom{N}{3}\,. \tag{A.11}$$

It should be noticed that while $\lambda_2^{(3)} \leq 1$, $\lambda_1^{(3)} \geq 1$ for $1+\sqrt{d/2} \leq k \leq d/2$, reaching its maximum for $k \approx d/3$ for $d \geq 6$, where $\lambda_1^{(3)} \approx 2d/27$.

2. Lack of majorization of $\rho^{(2)}$ under single mode occupancy measurement

We will now prove that in the states (35), measurement of the occupancy of one SP mode through the operators $\mathcal{P}_k = c_k^\dagger c_k$ and $\mathcal{P}_{\bar{k}} = c_k c_k^\dagger$, will break Eq. (51) for $\rho^{(2)}$. We will prove in fact that the largest eigenvalue (A.3) of $\rho^{(2)}$ is greater than that of $\rho_k^{(2)}$ and $\rho_{\bar{k}}^{(2)}$, implying

$$\lambda_1^{(2)} > p_k\lambda_{1k}^{(2)} + (1-p_k)\lambda_{1\bar{k}}^{(2)}\,, \tag{A.12}$$

which breaks the first majorization inequality in (51).
Proof: If state k is measured to be occupied, which will occur with probability $p_k = N/d$, the associated contiguous pair $(k, k+1)$ (k odd) or $(k-1, k)$ (k even) becomes "freezed" and the maximum eigenvalue $\lambda_k^{(2)}$ of the ensuing DM $\rho_k^{(2)}$ will arise from the remaining $N-2$ fermions occupying the other $d-2$ SP states. Consequently, using Eq. (A.3) for $d \to d-2$ and $N \to N-2$,

$$\lambda_{1k}^{(2)} = \frac{N-2}{2(d-2)}(d+2-N) < \frac{N}{2d}(d+2-N) = \lambda_1^{(2)}$$

where the inequality holds for $N < d$. Similarly, if state k is found to be empty, a similar reasoning leads to

$$\lambda_{1\bar{k}}^{(2)} = \frac{N}{2(d-2)}(d-N) < \frac{N}{2d}(d+2-N) = \lambda_1^{(2)}$$

where the inequality holds for $N > 2$. These two results imply Eq. (A.12) and hence violation of the majorization relation (51) for $m = 2$. Analogous results can be obtained for $m = 3$ in the same states (35).