UChicago Honors Analysis Notes: 20700

Notes by Agustín Esteva, Lectures by Luis Silvestre, Books by Charles Pugh and Hubbard & Hubbard

Academic Year 2024-2025

Contents

1 Lectures

1.1 Monday, Sept 30: The sup Property of $\mathbb{R}$.

Definition 1. A **Dedekind cut** $A|B$ are two sets in $\mathbb{R}$ such that $A, B \subset \mathbb{Q}$ and:

- $A \cap B = \emptyset$;
- $A \cup B = \mathbb{Q}$;
- $A, B \neq \emptyset$;
- If $a \in A$ and $b \in B$, then $a < b$.
- A contains no largest element.

$\mathbb{R}$ is made up of all the Dedekind cuts. Dedekind cuts are terrible and unwieldy (see PSET 1), but at least the following theorem, a very important one, is easy to prove with them.

Theorem 1. (sup property of $\mathbb{R}$) Suppose that $X \subset \mathbb{R}$ is nonempty and bounded above, then $c = \sup X$ exists.

Proof. Since $X \subset \mathbb{R}$, then we let

$$X = \{A_\alpha | B_\alpha\},$$

where $A_\alpha | B_\alpha$ is the collection of cuts in X. Let $\mathscr{A} = \bigcup_\alpha A_\alpha$ and $\mathscr{B} = \bigcap_\alpha B_\alpha$. From here, it is easy to show that $\mathscr{A} | \mathscr{B} = \sup X$. ☐

This is how most of the proofs go in this class, we state the main idea, and leave the verification of the idea to the reader. The ease of the above proof is, from what I can tell, why we care about Dedekind cuts. We go on to construct $\mathbb{R}$ with them, which is a pain. Alternatively, we could have constructed $\mathbb{R}$ with the completion of the Cauchy sequences in $\mathbb{Q}$, but then Theorem 1 becomes a pain to prove. In my experience, this second method is much more natural, and it should be provided as an exercise to anyone:

We now formalize this with the completeness of $\mathbb{R}$.

Definition 2. A sequence (a_n) is **Cauchy** if for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $n, m \geq N$, we have that

$$|a_n - a_m| < \epsilon.$$

Theorem 2. $\mathbb{R}$ is **complete** in the sense that every Cauchy sequence converges to some limit in $\mathbb{R}$.

This was not proved in class, but it follows pretty inmediately from Theorem 1:

Proof. Let $(a_n) \in \mathbb{R}$ be Cauchy and let $\epsilon > 0$. Let X be its image. Since (a_n) is Cauchy, its image must be bounded (exercise). Thus, X is bounded and nonempty. Create a new set:

$$S = \{s \in X \ : \ a_n < s \quad \text{finitely often}\}.$$

Let $b = \sup S$, indeed, this is by definition its limsup (see Def 3) We claim that $a_n \to b$. We have that by the sup property, there exists some $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then

$$|a_n - b| < \frac{\epsilon}{2}.$$

Let $N_2 \in \mathbb{N}$ be the natural from $(a_n)'s$ Cauchyness. Then take $N = \max\{N_1, N_2\}$ and we get that if $n \geq N$:

$$|a_n - b| \leq |a_n - a_N| + |a_N - b| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

$\square$

In the proof, we showed that if (a_n) is Cauchy, then it must converge to its $\limsup$. Later we will show that if (a_n) has a convergent subsequence and it is Cauchy, then it must converge. To see a metric space that is not complete, consider $\mathbb{Q}$ and consider the Cauchy Sequence:

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142, \ldots$$

Which would converge to $\sqrt{2}$, but $\sqrt{2} \notin \mathbb{Q}$!!! Or maybe consider $\mathbb{R} \setminus \{0\}$ and the sequence $\frac{1}{n}$. Cauchy, but does not converge to a limit in $\mathbb{R} \setminus \{0\}$ since $0 \notin \mathbb{R} \setminus \{0\}$!!!

Example 1.1. (From Rudin). *This exercise is not that easy.*

Let X be a metric space.

(a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X *equivalent* if

$$\lim_{n \to \infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.

(c) Prove that the resulting metric space X^* is complete.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^*.

(e) Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X.

Definition 3. Let (x_n) be a sequence, then we say that

$$\limsup_{n\to\infty} x_n = \inf_{m\to\infty}\{\sup\{x_m\ ;\ m \geq n\}\} = \lim_{n\to\infty} \sup_{n\geq m}\{x_m\}$$

It immediately follows that

$$\lim_{n\to\infty} a_n = a \qquad \Longleftrightarrow \qquad \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a.$$

To prove this, consider that subsequences converge to their mother sequence and for the back way, take $\frac{\epsilon}{2}$.

1.2 Wed, Oct 2: Continuity of Real-Valued Functions

Definition 4. Let X be a metric space. The following are equivalent:

- We say $f : X \to \mathbb{R}$ with $X \subset \mathbb{R}$ is **continuous** at some $x \in X$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in X$ with $|y - x| < \delta$, we have that $|f(x) - f(y)| < \epsilon$.
- We say $f : X \to \mathbb{R}$ with $X \subset \mathbb{R}$ is continuous at some $x \in X$ if when $x_n \to x$, we have that $f(x_n) \to f(x)$.

Proof. To show these are equivalent, consider first (1) $\to$ (2). Suppose $x_n \to x$. By the $\epsilon - \delta$ definition, we have that for any $\epsilon > 0$, there exists some $\delta > 0$ such that if $y \in X$ with $|y - x| < \delta$, then $|f(x) - f(y)| < \epsilon$. Since $x_n \to x$, we have that for large n, $|x_n - x| < \delta$, and thus $|f(x_n) - f(x)| < \epsilon$. Because this is true for any ϵ, then $f(x_n) \to f(x)$.
To show (2) $\to$ (1), we suppose it does not fulfill the $\epsilon - \delta$ criterion. That is, for some $\epsilon > 0$, we have that for all $\delta > 0$, if $y \in X$ with $|y - x| < \delta$, we have that $|f(x) - f(y)| \geq \epsilon$. Let $\delta_n = \frac{1}{n}$, then we can build a sequence x_n such that $|x_n - x| < \frac{1}{n}$. Thus, we have that $x_n \to x$ but $f(x_n) \not\to f(x)$, a contradiction! $\square$

After this, we discussed the EVT and IVT. I hate these proofs, so I will not write them up. They really just amount to using Theorem 1 and the definition of continuity, but we will end up proving them later in much cooler ways. It is still nice to see that you can prove them using continuity alone without the fancy machinery.

Definition 5. Let $x, y \in \mathbb{R}^n$. The **dot product** of x and y is a scalar such that

$$(x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Remark 1. The dot product is bilinear, symmetric, and positive definite. The dot product is a special case of an inner product, There exists inner products in infinite-dimensional vector spaces, such as continuous functions:

$$(f, g) = \int_a^b f(x) g(x) dx$$

Remark 2. You might see some people use $\langle \cdot, \cdot \rangle$ instead of $(\cdot, \cdot)$ for dot products. This might seem like a good notation until you get to next quarter when it becomes terrible notation.

Definition 6. A **norm** on a vector space V is any function $|| \; || : V \to \mathbb{R}$ such that fo any $v, w \in V$ and $\lambda \in \mathbb{R}$, we have that

- $||v|| \geq 0$ and $||v|| = 0$ if and only if $v = 0$;
- $||\lambda v|| = |\lambda| ||v||$;
- $||v + w|| \leq ||v|| + ||w||$.

Remark 3. Every inner product defines a norm with $||v|| = \sqrt{(v, v)}$, but the converse fails. For a partial converse, it might interest the reader (and it might be a good exam question (!)) to read up on how to induce the norm with the parallelogram law satisfied by any dot product: https://math.stackexchange.com/questions/21792/norms-induced-by-inner-products-and-the-parallelogram-law

We now come to the most important inequality of all time ever (until you get to Hölder's Inequality):

Theorem 3. (Cauchy-Shwarz Inequality) For all $x, y \in \mathbb{R}^m$, we have that

$$(x, y) \leq |x||y|.$$

Proof. Let $z = x + ty$, where $t \in \mathbb{R}$. Then we have that by bilinearity:

$$(z, z) = (x, x) + 2t(x, y) + t^2(y, y) = c + bt + at^2.$$

Since the expression must be nonnegative, then we must have a nonpositive discriminant, i.e, $b^2 - 4ac \leq 0$. Thus, we have that

$$4(x, y)^2 - 4(x, x)(y, y) \leq 0,$$

and so

$$(x, y) \leq \sqrt{(x, x)}\sqrt{(y, y)} = |x||y|.$$

□

This is a special case of the Hölder inequality, which becomes very important next quarter!

1.3 Fri, Oct 4: Metric Spaces and Topology

We stop talking about $\mathbb{R}$ and start talking about more interesting, more general metric spaces.

Definition 7. A metric space (M, d) is a set M equipped with a metric d that satisfies the same requirements as Definition 6.

Example 1.2.

- The usual Euclidean Metric on $\mathbb{R}^n$:

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

- The discrete metric on any set M :

$$d(x,y) = 1 \quad \forall x, y \in M.$$

  Except when $x = y$ in which case $d(x,y) = 0$. This metric is extremely useful for coming up with counterexamples. As you will see later, every set is both open and closed here, so any function from this space is continuous.

- Let $M = \{(x_n)\ ;\ (x_i) \text{ is a bounded sequence in } \mathbb{R}\}$, then

$$d((x_n),(y_n)) = \sup |x_i - y_i|.$$

- Let $M = C^0([a,b],\mathbb{R})$ be the space of real-valued continuous functions defined on $[a,b]$. Then we define the sup $-$*norm* to be

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

Now we start talking about topological properties of abstract metrics. For the following, let (M, d) be a metric space and suppose $X \subset M$

Definition 8. A **limit point** of X is some x such that there exists a sequence $x_n \in X$ such that $x_n \to x$.

Definition 9. We say that x is an **interior point** of X if there exists some $r > 0$ such that

$$B_r(x) \subset M,$$

where $B_r(x) := \{q \in M\ :\ d(q,x) < r\}$ is the **ball of radius** r around x.

Definition 10. We say that X is **open** if for all $x \in X$, x is an interior point of X.

Definition 11. We say that X is **closed** if X^c (the complement of X) is open. This is equivalent (prove) to saying that if x is a limit point of X, then $x \in X$.

The topological criterion for continuity is very easy to state:

Proposition 1. We say that $f : X \to Y$ is continuous if and only if for all closed $F \subset Y$, $f^{-1}(F)$ is closed in X.

I leave the simple proof as an enjoyable exercise. Hint: Use the sequential definition of continuity instead of $\epsilon - \delta$.

Definition 12. We say that $f : X \to Y$ is a **homeomorphism** if f is continuous and if $f^{-1} : Y \to X$ is continuous.

Definition 13. An **exterior point** y of X is such that there exists some $r > 0$, such that $B_r(y) \subset X^c$.

Definition 14. A **boundary point** x of X is a a point such that for all $r > 0$, we have that

$$B_r(x) \not\subset X \qquad B_r(x) \not\subset X^c.$$

What is the boundary of $\mathbb{Q}$?

Proposition 2.

- The arbitrary union of open sets is open, the arbitrary intersection of closed sets is closed.
- The finite intersection of closed sets is open, the finite union of closed sets is closed.

Proof.

- Let G_α be open for all $\alpha \in \mathscr{A}$, we claim that

$$\mathscr{G} = \bigcup_{\alpha \in \mathscr{A}} G_\alpha$$

is open. Let $x \in \mathscr{G}$, then $x \in G_\alpha$ for some α, and thus there exists some $r_\alpha > 0$ such that

$$B_{r_\alpha}(x) \subset G \implies B_{r\alpha}(x) \subset \mathscr{G}.$$

Thus, $\mathscr{G}$ is open. Using DeMorgan's law, let F_α be closed for all $\alpha \in \mathscr{A}$, then if

$$\mathscr{F} = \bigcap_{\alpha \in \mathscr{A}} F_\alpha \implies \mathscr{F}^c = \left(\bigcap_{\alpha \in \mathscr{A}} F_\alpha \right)^c = \bigcup_{\alpha \in \mathscr{A}} F_\alpha^c,$$

which is open by the above. Thus, $\mathscr{F}$ is open.

- Let G_i be open for all $i \in [n]$. We claim that

$$\mathscr{G} = \bigcap_{i=1}^{n} G_i$$

is open. Let $x \in \mathscr{G}$, then we have that $x \in G_i$ for all $i \in [n]$, and thus for each i, there exists some $r_i > 0$ such that $B_{r_i}(x) \subset G_i$. Take the minimum of such r_i, and we obtain the result.

$\square$

Remark 4. Consider

$$\bigcap_{i=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}).$$

We have that each set is open, but the intersection is just $\{0\}$, which is a singleton set and thus closed.

For the other counterexample, consider

$$\bigcup_{i=1}^{n} [\frac{1}{n}, 1 - \frac{1}{n}] = (0, 1).$$

1.4 Mon, Oct 7: Sequential Compactness

Recall that $X \subset M$ is **bounded** if there exists some $R \geq 0$ such that $S \subset B_R(x)$ for any $x \in S$. We now talk about what Charles Pugh calls "the single most important concept in Real Analysis." He obviously has not heard about Fat Cantor sets.

Definition 15. We say that $X \subset M$ is **compact** if for any $(x_n) \in X$, there exists a subsequence (x_{n_k}) that converges to some $x \in X$.

We can go from the infinite to the finite with compactness. Its effects will echo across every page of these notes. Worship it.

Proposition 3. Compact sets are closed and bounded.

Proof. Suppose $X \subset M$ is compact. Let x be a limit point of X, then there exists some $(x_n) \in X$ with $x_n \to x$. By compactness, there exists some $(x_{n_k}) \to x$ and $x \in X$, but since subsequences converge to the same limit as their mothers, we have that $x_n \to x$. To prove bounded, we suppose it's not. Thus, let $x_0 \in X$, then create a sequence such that $d(x_n, x_0) \geq n$. But then by compactness we have that there exists some subsequence such that $d(x_{n_k}, x_0) \to d(x_n, x_0) \not< \infty$. □

We try (and succeed) to prove the Heine-Borel theorem in $\mathbb{R}$. We make heavy use of the sup property of the reals.

Lemma 1. $[a, b] \subset \mathbb{R}$ is compact.

Proof. We use Theorem 1. Let $(x_n) \in X$, and define

$$S := \{s \in [a, b] \ : \ x_n < s \quad \text{finitely often}\}.$$

By Theorem 1, we have that there exists some $s = \sup S$. Take $\epsilon = \frac{1}{k}$, then by the sup property, there must exist infinitely many x_n such that

$$x_n \in (s - \epsilon, s + \epsilon),$$

and thus just choose (x_{n_k}) to be so. Then we have that $d(x_{n_k}, s) < \epsilon$. □

Lemma 2. Suppose A, B are compact, then $A \times B$ is compact.

Proof. Let $(x_n) \in A \times B$. Thus, we have that $x_n = (a_n, b_n)$ for any $n \in \mathbb{N}$. Since A is compact, we have that there exists some $(a_{n_k}) \to a$. Since B is compact, we have that there exists some $(b_{n_j}) \to b$. Sample the a_{n_k} from the n_j and sample the b_{n_j} from the n_k, then we have subsubsequences of both which converge at the same rate to (a, b). Thus, we have created a subsequence $(x_{n_\ell}) \to (a, b) \in A \times B$. □

The moral of the proof is that we can make convergent sequences converge at the same rate by sampling from each other.

Proposition 4. Suppose $X \subset M$ is closed and M is compact, then X is compact.

Proof. Let (x_n) be a sequence in X. Thus, $(x_n) \in M$ and since M compact, we have that $x_{n_k} \to x$ with $x \in M$. Thus, x is a limit point of X, but since X is closed, it must be that $x \in X$. □

We now arrive to a very useful theorem, Heine(!)-Borel.

Theorem 4. (Heine-Borel). A set $X \subset M$ is compact if and only if it is closed and bounded.

Proof. We proved the forward direction in Proposition 3. For the backwards direction, we note that any bounded set fits inside of an $[a, b]$ box. Thus, since X is closed, we use proposition 4 and claim X is compact. $\square$

Remark 5. (Generalize Heine-Borel) A compact set is complete and **totally bounded**, i.e, for any $\epsilon > 0$, there exists a finite covering of the set by ϵ balls.

Now we get to the cool proof of the extreme value theorem, which states the continuous functions achieve their extrema.

Theorem 5. (EVT) Suppose X is compact and $f : X \to Y$ is continuous, then we have that $f(X)$ is compact.

Proof. Let $(y_n) \in f(X)$. Thus, there exists $(x_n) \in X$ such that $f(x_n) = y_n$. By compactness of X, we have that $(x_{n_k}) \to x$ with $x \in X$. Continuity carries limits, so we have that $y_n = f(x_n) \to f(x)$. Thus, $f(X)$ is compact. $\square$

Since $f(X)$ is compact, then it is closed and bounded. By bounded it must have a sup and and inf, and by closed those sups and infs must be in $f(X)$.

1.5 Wed Oct 9: Uniformly continuity and Connectedness

We arrive at a stronger notion of continuity.

Definition 16. We say that $f : M \to N$ is **uniformly continuous** if for any $\epsilon > 0$, there exist a $\delta > 0$ such that if $x, y \in M$ with $d(x, y) < \delta$, we have that $d(f(x), f(y)) < \epsilon$.

In words, the δ depends not on our choice of x. Uniform continuity obviously implies continuity.

Theorem 6. Suppose X is compact and $f : X \to Y$ is continuous, then f is uniformly continuous.

Proof. Suppose not, then we have that there exist some $\epsilon > 0$ such that if $\delta > 0$ and $x, y \in X$ with $d(x, y) < \delta$, then $d(f(x), f(y)) \geq \epsilon$. Let $\epsilon = \frac{1}{n}$ and let $(x_n), (y_n)$ such that $d(x_n, y_n) < \frac{1}{n}$. Then $d(f(x_n), f(y_n)) > \epsilon$. There exists (x_{n_k}) and (y_{n_k}) such that $d(x_{n_k}) \to x \in X$. Thus, we have that $d(y_{n_j}) \to x$. Thus, by continuity, we have that $f(x_{n_k}), f(y_{n_j}) \to f(x)$ and thus $d(f(x_n), f(y_n)) < \epsilon$. ☐

Here we implicitly used another fact without realizing it was a fact. f is uniformly continuous if it sends Cauchy sequences to Cauchy sequences.

Corollary 1. Suppose $f : M \to N$ is bijective and continuous, then if M is compact, we have that f is homeomorphism.

We leave this simple proof as an exercise of applying previous results. Hint: Use the topological description of continuity. This corollary hints that compactness is a topological property. Thus, (a, b) is not homeomorphic to $[a, b]$ since one is not compact and the other is.

Definition 17. A metric space M is **disconnected** if $M = A \sqcup B$ with A, B clopen and nonempty.

Remark 6. You will see in some textbooks alternative definitions of diconnected sets. The most common one I have seen is by splitting M with A, B open. However, since A is open, then $A^c = B$ is closed, and thus B is both open and closed. Similar for A. Another common one is by just saying that there exists some nonempty $A \subset M$ clopen.

A metric space M is connected if it isn't disconnected.

Proposition 5. The interval $[a, b] \subset \mathbb{R}$ is connected.

Proof. As usual, we use the sup property of the reals. First, suppose $[a, b]$ is disconnected, then let $[a, b] = A \sqcup B$ and define

$$S := \{x \ : \ (a, x) \subset A\}.$$

Let $c = \sup S$. It takes little work to show that c is a limit point of both A and B, and thus by closedeness of both, $c \in A, B$, a contradiction to their disjointness. ☐

Time for the IVT!

Proposition 6. Suppose $f : M \to N$ is continuous with M connected and $N = A \sqcup B$, then either $f(N) \subset A$ or $f(N) \subset B$.

Proof. Suppose not, then take $f^{-1}(A)$ and $f^{-1}(B)$. Continuity leads to clopen, and we have nonempty by assumption. Thus, $M = f^{-1}(A) \sqcup f^{-1}(B)$. A contradiction. ☐

Corollary 2. (IVT) Suppose $f : M \to N$ is continuous with M connected, then $f(M)$ is connected.

Remark 7. Any path connected set is connected, and in $\mathbb{R}^n$, any connected set is path connected. The only connected sets in $\mathbb{R}$ are the intervals and $\mathbb{R}$.

Example 1.3. From Pugh, the topologists sine curve is a compact connected set that is not path connected. Let $M = G \sqcup Y$, with

$$G = \{(x, y) \ : \ y = \sin\left(\frac{1}{x}\right), \quad x \in (0, \frac{1}{\pi})\}$$

$$Y = \{(0, y) \ : \ y \in [-1, 1]\}$$

1.6 Fri, Oct 11: Covering Compactness and the Cantor Set

We see know a more unwieldy yet weirdly useful definition of compactness. It is more clear now how to go to the finite, but less clear to prove the previous results.

Definition 18. A set X is **covering compact** if for all open covers $\mathscr{G}$ of X, there exists a finite open subcover.

Theorem 7. Let M be a metric space. M is sequentially compact if and only if it is covering compact.

Proof. One implication is much easier than the other.

- ($\Longleftarrow$) Let $(x_n) \in M$ be some sequence, and assume it does not have a convergent subsequence. Thus, for all $x \in M$, there exist some $\epsilon > 0$ such that $x_n \in B_{\epsilon_x}(x)$ only finitely often. Cover M with these $B_{\epsilon_x}(x)$ balls, then by covering compactness, there exists some finite covering of M by $B_{\{}\epsilon_i\}(x_i)$, but then we still have that $x_n \in B_{\{}\epsilon_i\}(x_i)$ only finitely often. How is it that infinitely many things fit into finitely many things? The pigeonhole principle states that x_n must be in some ball infinitely often, a contradiction!

- ($\Longrightarrow$) Let $M \subset \bigcup_{\alpha \in \mathscr{A}} G_\alpha$ be an open cover of M. Let $\rho : M \to \mathbb{R}$ such that
$$\rho(x) := \sup\{r \ : \ B_r(x) \subset G_\alpha\}.$$
We claim that ρ is continuous. To do this, convince yourself that
$$\rho(y) \geq \rho(x) - d(x,y), \qquad \rho(x) \geq \rho(y) - d(x,y) \implies |d(\rho(y), \rho(x))| \leq d(x,y).$$
Theorem 5, we have that ρ achieves its minimum, and thus there exist some $x_0 \in M$ such that
$$\min \rho = \rho(x_0) = \rho_0 > 0.$$
Thus, for all $x \in M$ we have that $B_{\rho_0}(x) \subset G_\alpha$. This is what some books call a Lebesgue number. Let $p_1 \in M$, then we have that there exist some α_1 such that $B_{\rho_0}(p_1) \subset G_{\alpha_1}$. Let $p_2 \in M \setminus B_{\rho_0}(p_1)$, then there exists some α_2 such that $B_{\rho_0}(p_2) \subset G_{\alpha_2}$. If this process continuous infinitely often, then for any $n, m \in \mathbb{N}$, we have that $d(p_n, p_m) \geq \rho_0$, and thus (p_n) does not have a convergent subsequence, a contradiction. Thus, we can cover M by finitely many G_{α_i}.

$\square$

Luis gave a half hearted discussion of Cantor sets, so I will do the same. For an in depth discussion, Pugh glazes Cantor for the last part of Chapter 2 in his book, so go there.

Definition 19. We construct the middle-thirds **cantor set** by taking out intervals. Start with $C^0 = [0,1]$, then $C^1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, i.e, $C^0 \setminus (\frac{1}{3}, \frac{2}{3})$. Continue this way by taking out middle-thirds. Then the Cantor set is
$$C = \bigcap C^n$$

We give a few extremely useful properties with zero proof for them! Do them as exercises or something.

Proposition 7. The Cantor set C has the following properties:

- Compact
- No interior (C is the boundary)

- Nowhere Dense in the sense that it contains no interval.
- Uncountable
- The Cantor set is **measure zero** (for all $\epsilon > 0$, there is a countable covering of C by open intervals (a_i, b_i)) such that
$$\sum_{i=1}^{\infty}(b_i - a_i) < \epsilon$$
- C is totally disconnected in the sense that any point $c \in C$ has arbitrary small clopen neighborhoods.

1.7 Mon, Oct 14: Differentiability

We now talk about the infinitesimal!

Definition 20. Suppose $f : (a, b) \to \mathbb{R}$. We say f is **differentiable** at some $x \in (a, b)$ if

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. If it exists, then we say that $f'(x)$ is the **derivative** of f at x. We say f is differentiable on (a, b) if it is differentiable for all $x \in (a, b)$.

Lemma 3. Suppose f has a max/min at some $x \in (a, b)$, then $f'(x) = 0$.

Proof. WLOG, suppose f has a local max at x. Then we have that $f(y) - f(x) \leq 0$, and thus

$$\frac{f(y) - f(x)}{y - x} \leq 0 \leq \frac{f(y) - f(x)}{y - x},$$

where $y \to x$ (hand-wavy, but that's Luis!). $\square$

Now we get to Rolle's theorem. Important!

Theorem 8. (Rolle) Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a) = f(b)$, then there exists some $\lambda \in (a, b)$ such that $f'(\lambda) = 0$.

Proof. The proof amounts to understanding that either f is constant on $[a, b]$, or else by continuity there must be a peak and/or crest somewhere on it. Since $[a, b]$ is compact, $f([a, b])$ is compact. Let $\lambda = \max f([a, b])$. If $\lambda \in (a, b)$, then by Lemma 3, we have that $f'(\lambda) = 0$. If it is not, then consider $\rho = \min f([a, b])$ and run it back. $\square$

A very important theorem. The function achieves its derivative in some form.

Theorem 9. Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $\lambda \in (a, b)$ such that

$$f'(\lambda) = \frac{f(b) - f(a)}{b - a}.$$

Proof. We 'normalize' the function in some way by taking the ends and putting them on the same line. I.e, let $g : [a, b] \to \mathbb{R}$ such that

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right].$$

Now we have that $g(a) = g(b) = 0$, and so we apply Rolle's theorem and find some $\lambda \in (a, b)$ such that

$$g'(\lambda) = 0 \implies f'(\lambda) = \frac{f(b) - f(a)}{b - a}.$$

$\square$

We now arrive at one of my favorite theorems for differentiability, the idea of Darboux continuity. That is, the derivative of a function never has a jump discontinuity. That is sick.

Theorem 10. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) and continuous on $[a, b]$. If $f'(a) \leq D \leq f'(b)$, then there exists some $\gamma \in (a, b)$ such that $f'(\gamma) = D$.

Proof. Unfortunately, the proof shown in class is not too informative. The proof in Pugh chapter 3 is much better. Let $g(x) = f(x) - Dx$. Then $g'(a) = f'(a) - D \leq 0$ and $g'(b) = f'(b) - D \geq 0$, and thus since g is continuous, we must have that it achieves its minimum and thus for some $\gamma \in (a, b)$,

$$g'(\gamma) = 0 \implies f'(\gamma) = D.$$

$\square$

We arrive at the one-dimension Inverse-Function Theorem. First, we say that $f \in C([a, b], \mathbb{R})$ if $f : [a, b] \to \mathbb{R}$ has a continuous derivative on $[a, b]$.

Theorem 11. (Inverse-Function) Suppose $f \in C([a, b], \mathbb{R})$ Suppose that for some $x \in (a, b)$, $f'(x) \neq 0$, then for some small neigborhood of x, we have that f restricted to this neighborhood is bijective and $f^{-1}(x)$ is differentiable and

$$(f^{-1})'(f(p)) = \frac{1}{f'(p)}.$$

Proof. Suppose $f'(x) \neq 0$. WLOG, suppose $f'(x) > 0$. By continuity of f', there exists some $\epsilon > 0$ such that for any $p \in B_\epsilon(x)$, we have that $f'(p) \geq 0$. Thus, f is strictly increasing on $B_\epsilon(x)$ (prove using MVT) and is thus injective. Thus, we have found a bijection, and thus we have that $f^{-1}|_{f(B_\epsilon(x))}$ exists. Now we use the chain rule:

$$f^{-1}(f(x)) = x \implies (f^{-1})'(f(x))f'(x) = 1.$$

$\square$

1.8 Wed, Oct 16: Taylor Series and Integration

Definition 21. We say f is **analytic** on (a,b) if for any $x \in (a,b)$, we can express $f(x+h)$ as a convergent series, where $h > 0$:

$$f(x+h) = \sum_{r=1}^{\infty} a_r h^r.$$

Definition 22. Suppose $f \in C^\infty$ (all derivatives of f are continuous), then the **Taylor Series** of f at x is

$$f(x+h) = \sum_{r=0}^{\infty} \frac{f^{(r)}(x)}{r!} h^r$$

Definition 23. Suppose $f \in C^r$. The **Taylor Polynomial** of f at x is

$$P(h) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} h^i.$$

Theorem 12. (Taylor Remainder Theorem) Suppose $f \in C^r$, then

$$\lim_{h \to 0} \frac{f(x+h) - P_r(h)}{h^r} \to 0.$$

Suppose $f \in C^{r+1}$, then $f(x+h) - P_r(h) = \frac{f^{(r+1)(\theta)}}{(r+1)!} h^{r+1}$ for some $\theta \in (0,h)$.

Proof. We present the proof for the second statement first, making heavy use of the mean value theorem. Define a function $g : [a,b] \to \mathbb{R}$ such that

$$g(t) = f(x+t) - P(t) - R(h)\frac{t^{r+1}}{(r+1)!}.$$

Then $g(0) = 0$ (show this) and $g(h) = 0$ (convince yourself of this!). Thus, we can use MVT: there exists some $\theta_1 \in (0,h)$ such that

$$g(0) - g(h) = g'(\theta_1)(h) \implies g'(\theta) = 0.$$

Consider that $g'(h) = 0$, so use MVT to find some $\theta_2 \in (0, \theta_1)$ such that

$$g(\theta_2) = 0.$$

We can repeat this process $r+1$ times. Then $g(\theta_{r+1}) = f^{(r+1)}(x + \theta_{r+1}) - R(h) = 0$. This proves the second statement. To prove the first statement, consider that if $f \in C^r$, then by what we showed above, we have that $f(x+h) = P_{r-1}(h) + \frac{f^{(r)}(\theta)h^r}{r!}$. Morevoer, we have that $f^{(r)}$ is continuous, and thus as $\theta \to x$, we have that $f^{(r)}(\theta) - f^{(r)}(x) \to 0$.

$$\lim_{h \to 0} \frac{f(x+h) - P_r(h)}{h^r} = \lim_{h \to 0} \frac{P_{r-1}(h) + f^{(r)}(\theta)\frac{h^r}{r!} - (P_{r-1}(h) + f^{(r)}(x)\frac{h^r}{r!})}{h^r} \to 0.$$

$\square$

I don't find these proofs entertaining. I don't find the result amusing. I don't like Taylor series if I am being honest with y'all. We introduce integrals!

Definition 24. (Riemann-Integrability) Let P, T be partition of $[a, b]$ such that if $P = \{x_i\}$ and $T = \{t_i\}$ then

$$a = x_0 \leq t_1 \leq x_1 \leq \ldots x_{n-1} \leq t_n \leq x_n = b.$$

The **Riemann Sum**, is

$$R(f, P, T) = \sum_{i=1}^{n} f(t_i)[x_i - x_{i-1}].$$

We say a function is **Riemann-Integrable** if there exists some $I \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$||P|| < \delta \implies |R(f, P, T) - I| < \epsilon.$$

Here $||P||$ denotes the **mesh size** of P, i.e, the largest length between any x_i and x_{i-1}.

Definition 25. (Darboux-Integrability) Let f be bounded. Let P be a partition of $[a, b]$. The **lower sum** of f is

$$L(f, P) = \sum_{i=1}^{n} \inf_{[x_{i-1}, x_i]} f(t)(x_i - x_{i-1})$$

and the **upper sum** of f is

$$U(f, P) = \sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]} f(t)(x_i - x_{i-1}).$$

We say that f is **Darboux-integrable** if for any $\epsilon > 0$, there exists a partition P such that

$$|U(f, P) - L(f, P)| < \epsilon.$$

Remark 8. Darboux-integrability and Riemann-integrability are equivalent.

This is pretty much all Luis had to say about integrals. There is one extremely useful result, which he stated without proof.

Theorem 13. (Riemann-Lebesgue) We can say that f is integrable if and only if it is bounded and if the set of discontinuities has measure zero (or equivalently, is a zero set).

1.9 Fri, Oct 18: Midtem I

We had an awesome midterm! Here it is!

Write explicit examples of the following, whenever possible:

(a) A subset of $\mathbb{R}$ that has no interior and no exterior points.

(b) A subset of $\mathbb{R}^2$ that is homeomorphic to the open square but is not homeomorphic to the open circle.

(c) A continuous bijective function $f : M \to N$ so that N is connected but M is not.

(d) A continuous function $f : (0, 1) \to \mathbb{R}$ whose image is compact.

(e) A metric space M with infinitely many points that contains exactly four subsets with empty boundary.

(f) Two disjoint nonempty closed sets A and B in $\mathbb{R}$ such that

$$\inf\{|x - y| \ : \ x \in A, \ y \in B\} = 0.$$

Proof. As a merciful overlord, we provide answers:

(a) $\mathbb{Q}$.

(b) Impossible.

(c) $f : [0, 1) \cup \{2\} \to [0, 1]$ such that

$$f(x) = \begin{cases} x, & x \in [0, 1) \\ 1, & x = 2 \end{cases} .$$

(d) $f(x) = \{420\}$.

(e) $\mathbb{R} \setminus \{0\}$.

(f) $A = \mathbb{N}$, $B = \{n - \frac{1}{n} \ : \ n \in \mathbb{N}\}$.

□

Theorem 14. Given two nonempty sets A and B and a metric space (M, d), we define the distances between A and B as

$$d(A, B) = \inf\{d(x, y) \ : \ x \in A, \ y \in B\}.$$

Prove that if A and B are compact and disjoint, then $d(A, B) > 0$.

Proof. Suppose not. Thus, there exist sequences $(a_n) \in A$ and $(b_n) \in B$ such that $d(a_n, b_n) \to 0$. From compactness, we have that there exists some convergent subsequence $(a_{n_k}) \to a$ with $a \in A$. Thus, we have that $d(a_{n_k}, a) \to 0$, which implies that if we sample (b_n) with n_k, we get that $d(b_{n_k}, a) \to 0$, implying that a is a limit point of B, and thus since B is closed (Heine-Borel), we get that $a \in B$, a contradiction to the fact that the sets are disjoint. □

1.10 Mon, Oct 21: Series

We say that a sequence converges if its partials sums converge.

Proposition 8. (Comparison test) Suppose $|a_k| < b_k$ for any k. If $\sum b_k$ converges, then $\sum |a_k|$ converges. We say that $|a_k|$ **converges absolutely** here.

Proof. Using Cauchy's convergence criterion, we have that for any $\epsilon > 0$:

$$\left|\sum_{k=m}^{n} |a_k|\right| = \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k \leq \epsilon.$$

$\square$

We come now to the integral test.

Theorem 15. (Integral test) Give $0 \leq a_k$, if $f(x) \geq a_k$ for $x \in [k-1, k]$ and $\int_0^\infty f(x)dx < \infty$, then the series converges. Suppose $g(x) \leq a_k$ for $x \in [k-1, k]$ and $\int_0^\infty g(x)dx$ diverges, then the series diverges.

Proof. This proof is not bad and amounts to showing that the partial sums are increasing and bounded. A picture is much more useful here. Look at Wikipedia or at next quarter's PSETs! $\square$

Example 1.4. We prove that $\sum \frac{1}{n^p}$ connverges if $p > 1$ and diverges if $p \leq 1$.

Proof. We compare it to the integral

$$\int \frac{1}{x^p} = \begin{cases} \frac{x^{1-p}}{1-p}, & p \neq 1 \\ \ln(x), & p = 1 \end{cases}.$$

Thus, if $p < 1$, we have that the integral diverges. Conversely, if $p > 1$, the integral converges. $\square$

Remark 9. As an important example, we have that for power series,

$$\sum_{k=0}^{\infty} \lambda^k = \begin{cases} \frac{1}{1-\lambda}, & |\lambda| < 1 \\ \infty, & |\lambda| > 1 \end{cases}$$

Theorem 16. (Root test) Let $\sum a_k$ be a series. The series converges absolutely if $\lambda = \limsup |a_k^{\frac{1}{k}}| < 1$. It diverges if $\lambda > 1$.

Proof. Since $\rho < 1$, we have that for k large, we have that there exists some $\beta \in \mathbb{R}$

$$\sqrt[k]{|a_k|} \leq \beta < 1 \implies |a_k| \leq \beta^k < 1.$$

$\sum \beta^k$ converges by a simple geometric series. Thus, we have that the series converges by the comparison test. It diverges if $\lambda > 1$ since the terms would not go to 0. $\square$

We never use the ratio test since the root test is strictly stronger that it, but here it is regardless.

Theorem 17. Let $\lambda = \limsup \left|\frac{a_k}{a_{k+1}}\right|$ and $\rho = \liminf \left|\frac{a_{k+1}}{a_k}\right|$. The series converges if $\lambda < 1$ and diverges if $\rho > 1$.

Proof. We will prove for when it converges. We have that there exists some K large such that

$$\left|\frac{a_{K+1}}{a_K}\right| < 1 \implies |a_{K+1}| \leq |a_K| < 1 \implies |a_k| \leq |a_K|^{K-k},$$

the latter of which series converges by geometric series, and thus the series converges by comparison. □

The alternating series test sucks but is lowkey important.

Theorem 18. (AST) Suppose $|a_n| \to 0$ monotonically and a_n alternates, then $\sum a_n$ converges.

Example 1.5. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k}$ converges.

Definition 26. A **power series** is defined as the series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

Theorem 19. A power series converges for all $|x| < R$, where R is the **radius of convergence** such that

$$R = \frac{1}{\limsup \sqrt[k]{|a_k|}}$$

Proof. This is just a straight up application of the ratio test. If $|x| < R$, then $|x| < \frac{1}{\limsup \sqrt[k]{|a_k|}}$ Thus, we have that applying the root test to $f(x)$, we have that

$$\limsup \sqrt[k]{|a_k x|} = |x| \limsup \sqrt[k]{|a_k|} < 1.$$

□

1.11 Wed, Oct 23: Convergence of Functions

This is my favorite chapter in 20700, and it all has to do with spaces of functions.

Definition 27. Suppose that $f_n : X \to Y$ for any $n \in \mathbb{N}$. We say that $f_n \to f$ **uniformly** if for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $n \geq N$ and $x \in X$, we have that

$$d(f_n(x), f(x)) < \epsilon.$$

Remark 10. This is equivalent to convergence in the sup metric, i.e, $d(f_n, f) \to 0$ as $n \to \infty$.

We can imagine an ϵ tube around f. Does there exist some N such that for large n, we have that every f_n fits in this tube:

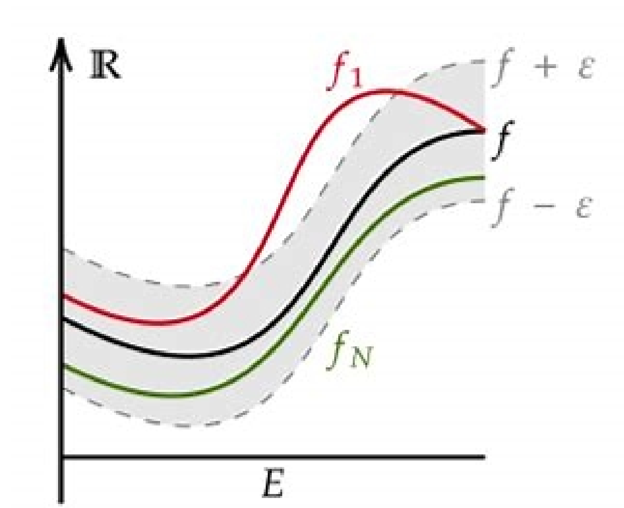


Figure 1: Epsilon-Tube

Definition 28. We say that $f_n \to f$ **pointwise** if for all $\epsilon > 0$, for each $x \in X$, there exists some $N_x \in \mathbb{N}$ such that if $n \geq N_x$, then we have that

$$d(f_n(x), f(x)) < \epsilon.$$

You can see now that uniform convergence is strictly stronger than pointwise convergence. The classic theorem to begin with is when do we have continuity of f?

Theorem 20. Suppose f_n is continuous for all n. If $f_n \to f$ unif. then f is continuous.

Proof. Let $\epsilon > 0$ and $x \in X$. Since $f_n \to f$ uniformly, we have that there exists some $N \in \mathbb{N}$ such that if $n \geq N$, we have that

$$d(f_N(x), f(x)) < \frac{\epsilon}{3} \qquad \forall x \in X.$$

Since f_N is continuous, then there exists some $\delta > 0$ such that if $y \in X$ with $d(x, y) < \delta$, then

$$d(f_N(x), f_N(y)) < \frac{\epsilon}{3}.$$

Thus,

$$d(f(x), f(y)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}.$$

$\square$

Pointwise convergence is not enough to satisfy continuity as per the following example.

Example 1.6. Let $f_n : [0,1] \to \mathbb{R}$ with $f_n = x^n$. To show that

$$\lim_{n\to\infty} f_n = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases} \quad ,$$

consider that if $x \in [0,1)$, then $\lim_{n\to\infty} x^n = 0$ and if $x = 1$, then the limit is simply 1. To show that the convergence is not uniform, we must show that the N is dependent on both ϵ and x. Thus, let $\epsilon = \frac{1}{2}$, then assume there exists some N for uniform convergence. This N will then work for any x, say $x = (\frac{2}{3})^{\frac{1}{N}}$, but then

$$d(f_N(x), f(x)) = |x^N - 0| = x^N = (\frac{2}{3})^{\frac{1}{N}^N} = \frac{2}{3} > \epsilon.$$

Definition 29. We say the **space of bounded functions** is C_b and the **space of continuous functions** is $C^0 = C^0([a,b], \mathbb{R})$.

Remark 11. Note that by the EVT, $C^0 \subset C_b$. Also not that by Theorem 20, if f is a limit point of C^0, then we have a sequence of continuous functions converging (with respect to the sup metric, i.e, uniform convergence) to f implying that f is in fact continuous. Thus, C^0 is closed.

Theorem 21. C_b is complete.

Proof. Let (f_n) be Cauchy in C_b. For any x in the domain, we have that $(f_n(x))$ is a bounded Cauchy sequence in $\mathbb{R}$, and thus must converge. For each x, we let $\lim_{n\to\infty} f_n(x) = f(x)$. We claim that the convergence is also uniform! Let $\epsilon > 0$. By Cauchyness, we have that there exists some N such that if $n, m \geq N$, we have

$$d(f_n, f_m) < \frac{\epsilon}{2}.$$

By pointwise convergence, exists some $m \geq N$ dependent on x such that

$$d(f_{m(x)}, f(x)) < \frac{\epsilon}{2}.$$

Thus, we use the N from Cauchy and we know that if $n \geq N$, we have that:

$$d(f_n(x), f(x)) \leq d(f_n(x), f_{m(x)}(x)) + d(f_{m(x)}, f(x)) < \epsilon.$$

$\square$

Note that we are using a different m for each x in this proof, but it doesn't matter at the end of the day because we know that given some N, then for any x, we can find this m by the pointwise convergence.

Corollary 3. C^0 is complete.

For this one-liner, use Remark 10 and Theorem 21.
What else does uniform convergence imply?

Theorem 22. Suppose $f_n \to f$ uniformly with each f_n integrable. Then f is integrable. Moreover,

$$\int f_n \to \int f, \quad \text{uniformly}$$

The spirit of the proof is that we can bound the upper and lower sums of f by upper and lower sums of f_n. Pugh presents a cleaner proof using the Riemann-Lebesgue Integrable Criterion:

Proof. Since each f_n is integrable, then each f_n has countably many discontinuities. Call this set D_n. Then each f_n is continuous for any $x \in [a,b] \setminus D_n$, and thus Theorem 20 states that f must be continuous for any $x \in [a,b] \setminus \bigcup D_n$. We have that f is bounded by Theorem 21. Thus by the Riemann-Lebesgue Theorem, f is integrable.
Let $\epsilon > 0$. From uniform convergence, there exists some N such that if $n \geq N$, we have that $d(f_n, f) < \frac{\epsilon}{b-a}$

Get the N from the uniform convergence of f_n, then

$$\left| \int_a^x f_n(t)dt - \int_a^x f(t)dt \right| = \left| \int_a^x (f_n(t) - f(t))dt \right| \leq \int_a^x |f_n(t) - f(x)|dt \leq d(f_n, f)(b-a) < \epsilon.$$

□

Luis forgot (didn't care) to mention the Weierstrass-M test, but here it is. It is a useful criteria for deciding when a power series converges.

Theorem 23. (Weierstrass-M test) Let $(f_n) \in C_b$. Suppose that for each n, $||f_n|| \leq M_n$, where M_n forms a convergent series. Then $\sum |f_n|$ converges uniformly.

Proof. We use our most powerful theorem, C_b is complete. We must show that the series is Cauchy, or better, that the partial sums, $F_n = \sum_{k=0}^n f_k$ are Cauchy. Suppose $m > n$, then we have that

$$d(F_n, F_m) \leq d(F_m, F_{m-1}) + \cdots + d(F_{n+1}, F_n) = \sum_{k=n+1}^{m} ||f_k|| \leq \sum_{k=n+1}^{m} M_k.$$

□

Looking back on this with the foresight of functional analysis, the M-test reminds me a ton of the Banach-Steinhaus Uniform Boundedness Principle (foreshadowing).

1.12 Fri, Oct 25: Arzela-Ascoli

The second half of the lecture fucks. Persevere!

Theorem 24. Suppose f_n is differentiable for all n. Suppose $f_n \to f$ uniformly and f_n' converges uniformly. Then f is differentiable and moreover $f_n' \to f'$ uniformly.

Proof. We weaken the statement and assume $f_n \in C^1$ just to make the proof enjoyable. We apply FTC and Theorem 22. Note that all the following convergenes are uniform.

$$f_n(x) = f_n(a) + \int_a^x f_n'(t)dt \to f(a) + \int_a^x g(t)dt \implies f'(x) = g(x).$$

We leave the proof of the actual statement to the gods. $\square$

We can now use the Weirstrass-M test!

Proposition 9. A power series converges uniformly on $[-r, r]$, where $|r| < R$ is the radius of convergence.

Proof. Let $\sum c_k x^k$ be a power series. We have that for large k, $\frac{1}{\sqrt[k]{c_k}} < \beta < R$, and thus we have that for large k :

$$x^k c_k \le \left(\frac{x}{\beta}\right)^k,$$

where the right side forms a geometric power series. Thus, we have our result by Weirstrass-M. $\square$

Now we get to the good stuff. First, a definition.

Definition 30. Let $\mathscr{E}$ be a family of functions. We say $\mathscr{E}$ is equicontinuous if for any $\epsilon > 0$, there exists some $\delta > 0$ such that if $d(x, y) < \delta$ and $f \in \mathscr{E}$, then

$$d(f(x), f(y)) < \epsilon.$$

To make sense of this definition, think of it as uniform continuity of a bunch of functions. There is some δ that determines the continuity of every function in the family.

Example 1.7. Suppose $f_\alpha : [a, b] \to \mathbb{R}$ and $||f_\alpha|| < A$ for any α. Then we have by the MVT that if $\alpha \in \mathscr{A}$ and $x, y \in [a, b]$ then

$$|f_\alpha(x) - f_\alpha(y)| \le |f'(\lambda)||x - y| \le A|x - y|.$$

Thus, the family is equicontinuous since they are governed by a **modulus of continuity**.

Here is a very very important theorem with a very very cool name.

Theorem 25. (Arzela-Ascoli) Any bounded equicontinuous sequence of functions in $C^0([a, b], \mathbb{R})$ has a uniformly convergent subsequence.

Proof. Since $[a, b]$ is compact, there exists some countably dense subset $D \subset [a, b]$ (exercise). Let $D = \{d_1, d_2, \dots\}$. Consider that $(f_n(d_1))$ forms a bounded sequence of reals, and thus by Bolzano-Weierstrass must have a convergent subsequence $(f_{n,1}(x))$ which converges at d_1. Consider that $(f_{n,1})(d_2)$ forms a bounded sequences of reals, and must have a convergent subsequence $(f_{n,2}(x))$ which converges at d_2. However, since this subsequence is a subsequence of a sequence which converges at d_1, then it also converges at d_1. Keep going in this way, then consider the sequence $(f_{i,i})(x)$. That is, we are consider the following diagonal sequence:

$$\begin{array}{cccccc} f_{1,1}(x) & f_{2,1}(x) & f_{3,1}(x) & f_{4,1}(x) & f_{5,1}(x) & \cdots \\ & f_{2,2}(x) & f_{3,2}(x) & f_{4,2}(x) & f_{5,2}(x) & \cdots \\ & & f_{3,3}(x) & f_{4,3}(x) & f_{5,3}(x) & \cdots \\ & & & f_{4,4}(x) & f_{5,4}(x) & \cdots \\ & & & & f_{5,5}(x) & \cdots \\ & & & & & \ddots \end{array}$$

For any $d_k \in D$, $f_{i,i}(x)$ converges at d_k. Thus, we have that for any d_k, there exists some large N such that if $i, j \geq N$, we have that

$$d(f_{i,i}(d_k), f_{j,j}(d_k)) < \frac{\epsilon}{3}.$$

We have that each $f_{i,i}$ is continuous on a compact set (and thus uniformly continuous), and so there exists some $\delta > 0$ such that if

$$d(x,y) < \delta \implies d(f_{i,i}(x), f_{i,i}(y)) < \frac{\epsilon}{3}.$$

For any $x \in [a,b]$, there exists some $d_k \in D$ such that $d(x, d_k) < \delta$. Thus, for any $\epsilon > 0$, there exists this δ such that if $x, y \in [a,b]$, then

$$d(f_{i,i}(x), f_{j,j}(x)) \leq d(f_{i,i}(x), f_{i,i}(d_k)) + d(f_{i,i}(d_k), f_{j,j}(d_k)) + d(f_{j,j}(d_k), f_{j,j}(y)) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}.$$

Again, the first and last distances in the inequality above are bounded due to uniform continuity of the individual f_n, and the middle one is bounded from equicontinuity.

We have that $(f_{i,i})$ is Cauchy, and thus converges since C^0 is complete. □

This theorem has countless consequences, but the most important is the following.

Theorem 26. (Heine-Borel) $\mathscr{E} \subset C^0$ is compact if and only if it is equicontinuous, closed, and bounded.

Proof. We prove both directions.

- ($\implies$) Suppose $\mathscr{E}$ is compact. Closed and bounded are immediate. We can find a finite $\epsilon - 3$ dense subset of $\mathscr{E}$ by compactness (exercise). Let $\{f_{k_{\frac{\epsilon}{3}}} : k \in [n]\}$ be such a subset. Thus, if $f \in \mathscr{E}$, then there exists some f_k such that $d(f, f_k) < \frac{\epsilon}{3}$. Suppose $x, y \in [a,b]$ and $d(x,y) < \delta$, where δ is the one from uniform continuity, then

$$d(f(x), f(y)) \leq d(f(x), f_k(x)) + d(f_k(x), f_k(y)) + d(f_k(y), f(y)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}.$$

  Again, the middle distance is bounded by uniform continuity, and outer ones by the dense subset.

- ($\impliedby$) Let $(f_n) \in \mathscr{E}$. By equicontinuity and boundedness, Arzela-Ascoli tells us that we have a convergent subsequence (f_{n_k}) and closedeness tells us the limit lies within $\mathscr{E}$.

□

Remark 12. f_n converges uniformly if and only if f_n are equicontinuous and converge pointwise.

The proof of the remark is an $\epsilon/3$ mess, but try it out.

1.13 Mon, Oct 28: Weierstrass Approximation Theorem

We discuss convolutions and the Weierstrass Approximation Theorem. It is a cool result with an unfortunately not cool (in my opinion) proof.

Definition 31. A **convolution** is defined by

$$f * g = \int_{\mathbb{R}} f(x-y)g(y)dr = \int_{\mathbb{R}} g(x-y)f(y)dy.$$

Remark 13. Suppose $f \in C^1$ and g have support (set where g outputs nonzero values) on $[a,b]$. Then

$$(f*g)'(x) = \lim_{h\to 0} \frac{1}{h} \int f(x+h-y)g(y)dy - \int f(x-y)g(y)dy = \int f'(x-y)g(y)dy = (f'*g)(x)$$

Proposition 10. The convolution of a not smooth function can be made smooth.

I apologize for this, Luis was pretty bad with it and I don't care enough about convolutions to fix his lecture.

Proof. Let $f : [a,b] \to \mathbb{R}$ be smooth with $f(a) = f(b) = 0$ with support on (a,b). Let (g_n) such that g_n has support on $[-\frac{1}{n}, \frac{1}{n}]$ and $\int_{\mathbb{R}} g_n(y)dy = 1$. Then

$$\begin{aligned} |(f*g_n)(x) - f(x)| &= \int_{\mathbb{R}} |f(x-y) - f(x)| - g_n(y)dy \\ &= \int_{-\delta}^{\delta} f(x-y) - f(x)g_n(y)dy + \int_{\mathbb{R}\setminus[-\delta,\delta]} |f(x-y) - f(x)|g_n(y)dy \\ &\leq \epsilon + C\int_{\mathbb{R}\setminus[-\delta,\delta]} g_n dy \to 0. \end{aligned}$$

□

Theorem 27. (Weirstrass Approximation Theorem) Polynomials are dense in $C^0([a,b],\mathbb{R})$.

The proof follows from the above and also the fact that if P is a polynomial, then $f * P$ is a polynomial.

1.14 Wed, Oct 30: Banach Contraction Principle and Picard's Theorem for ODEs

This lecture is awesome sauce, so strap in.

Definition 32. We say $f : M \to M$ is a **contraction** if there exists some $k < 1$ such that for any $x, y \in M$:

$$d(f(x), f(y)) \leq kd(x, y).$$

Definition 33. We say f has a **fixed point** at some x if $f(x) = x$.

Theorem 28. (Brouwer) Suppose $f : B_1(0) \to B_1(0)$ is continuous, then f has a fixed point.

The proof for this will hopefully be on the last page of this document. We turn to another fixed-point theorem, which has an awesome proof.

Theorem 29. (Banach) Suppose that M is complete and $f : M \to M$ is a contraction. Then f has a fixed point.

The idea for this proof is that eventually enough contraction results in a single point.

Proof. Let $x_0 \in M$. Let $x_1 = f(x_0)$, $x_2 = f(f(x_0))$, and $x_n = f^{(n)}(x_0)$. We wish to show that (x_n) is Cauchy. Consider that for any n :

$$d(x_n, x_{n+1}) = d(f(x_{n-1}, f(x_n)) \leq kd(x_{n-1}, x_n) \leq k^n d(x_0, x_1).$$

Thus, if $n > m$, we have that

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, d_{n+2}) + d(x_{m-1}, x_m) \leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \ldots k^{m-1} d(x_0, x_1) = d(x_0, x_1)(\frac{k^n}{1-k}).$$

Since $k < 1$, we have that as $n \to \infty$, $d(x_n, x_m) \to 0$, and thus (x_n) is Cauchy, and by the completeness of M, we have $(x_n) \to x_\infty \in M$. We claim that $f(x_\infty) = x_\infty$. That is, x_∞ is our fixed point. To show this, consider that

$$d(f(x_\infty), x_\infty) \leq d(f(x_\infty), f(x_n)) + d(f(x_n), x_n) + d(x_n, x_\infty) \leq kd(x_\infty, x_n) + kd(x_\infty, x_n) + k^{n-1}(x_0, x_1) + d(x_\infty, x_n) \to 0.$$

□

How can we guarantee the existence and uniqueness for a solution of an ODE?

Theorem 30. (Picard) Suppose $F : \Omega \to \mathbb{R}^d$ with Ω open is Lipshitz. Suppose $x_0 \in \Omega$, then there exists some unique trajectory $\gamma : (a, b) \to \Omega$ that if $\gamma(0) = x_0$, then $\gamma'(t) = F(\gamma(t))$.

Proof. Solving the ODE in the problem statement is equivalent to finding the solution to

$$\gamma(t) = x_0 + \int_0^t F(\gamma(s))ds.$$

We can only find this solution locally, so restrict F to some compact $\overline{B_r(x_0)}$. By continuity of F, we have that F is locally bounded here by some M. Choose τ small such that

$$\tau M \leq r, \qquad \tau L < 1.$$

Define : $\mathscr{M} \to \mathbb{R}$, where

$$\mathscr{M} = \{f : [-\tau, \tau] \to \Omega \ : \ f \text{ is continuous}\}.$$

With respect to d_{sup}, $\mathscr{M}$ is complete. Define m by

$$m(\gamma(t)) = p + \int_0^t F(\gamma(s))ds.$$

By Banach's Contraction principle, it suffices to show that m is a contraction and it maps functions to $\mathscr{M}$. Let $\sigma, \gamma \in \mathscr{M}$, then we have that

$$d(m(\sigma), m(\gamma)) = \sup_t \left| \int_0^t F(\gamma(s)) - F(\sigma(s))ds \right| \leq \tau \sup_t |F(\gamma(t)) - F(\sigma(t))| \leq \tau L d(\gamma(t), d(\sigma(t))).$$

Since $\tau L < 1$, we have that m is a contraction. Suppose $\gamma \in \mathscr{M}$. Then we have that

$$m(\gamma(t)) = x_0 + \int_0^t F(\gamma(s))ds \leq x_0 + \tau M \implies d(x_0, m(\gamma(t))) \leq r \implies m(\gamma) \in \overline{B_r(x_0)}.$$

Thus, we have the existence of some γ that satisfies the conditions. $\square$

Suppose we have a river (Ω) and some rock thrown at some point in the river ($x_0 \in \Omega$). Picard guarantees us that if the velocity/flow of the river is Lipshitz, then we can put some boat where the rock landed who's path ($\gamma(t)$) will move in the flow of the river in some time interval $t \in [-\tau, \tau]$.

Example 1.8. Suppose $f(x) = x^2$. Then f is not global Lipshitz, and so we have the consequence that the solution blows up in local time. I.e, we can find a solution in local but not global time:

$$\gamma'(t) = (\gamma(t))^2 \implies \gamma(t) = \frac{1}{2-t},$$

which blows up near $x = 2$.

Example 1.9. Suppose $f(x) = \sqrt{x}$, thn $\gamma(t) = 0$ or $\gamma(t) = \frac{1}{4}t^2$. However, since we do not have Lipshitz, we also have that

$$\gamma(t) = \begin{cases} \frac{(x-a)^2}{4}, & x > a \\ 0, & x < 0 \end{cases}$$

and so our solution is not unique!

1.15 Fri, Nov 1: Baire Category Theorem and Consequences

The best theorem of all time. If I were a lemma I would go out with Baire Category Theorem, even though I hardly know him!

Theorem 31. Let (G_n) be a countable collection of open dense sets in M, a complete metric space. Then

$$\bigcap G_n$$

is dense in M.

Proof. Let $p \in M$ and $\epsilon > 0$. Since $B_\epsilon(p_0)$ is open, then there exists some $r_0 > 0$ and $p_1 \in M$ such that

$$\overline{B_{r_0}(p_1)} \subset B_\epsilon(p_0).$$

Since G_1 is open, there exists some $r_1 > 0$ and $p_2 \in M$ such that

$$\overline{B_{r_1}(p_2)} \subset B_{\frac{r_0}{2}} \cap G_1,$$

where the intersection is nonempty by the density of G_1. Make the sequence (p_n) such that

$$\overline{B_{r_{n-1}}(p_n)} \subset B_{\frac{r_{n-2}}{2}} \cap G_1 \cap G_2 \cdots \cap G_{n-1}$$

We have that $r_n \to 0$ since $r_{n+1} \leq \frac{r}{2}$ and so

$$\lim_{n\to\infty} \text{diam}\overline{B_{\frac{r}{2}}(p_n)} \to 0,$$

implying that (p_n) is Cauchy. Moreover, we have that each $(p_n) \in B_\epsilon(p_0)$. Since M is complete, we have that $(p_n) \to p_\infty \in M$, and we must also have that $p_\infty \in B_\epsilon(p_0)$ and also that for each n, $p_\infty \in G_n$. Thus, for any $p_0 \in M$, and $\epsilon > 0$, there exists some $p_\infty \in M$ such that

$$p_\infty \in B_\epsilon(p_0) \cap \bigcap G_n,$$

implying that the intersection is dense in M. ☐

Corollary 4. Suppose M is complete and

$$M = \bigcap F_n$$

where each F_n is closed. Then at least one F_i must have some nonempty interior.

Proof. Suppose not. That is, suppose we can write

$$M = \bigcup F_n$$

where F_n closed has an empty interior for any n. Thus, we have that

$$\emptyset = M^c = (\bigcup F_n)^c = \bigcap F_n^c$$

is an intersection of open dense sets (the compliment of empty interior is a dense set). Thus, we have by Baire's theorem that $\emptyset$ must be dense in M. Ridiculous! ☐

Baire Category Theorem allows us to prove some pretty insane stuff. Luis of course didn't prove them, but they are still cool!

Theorem 32. The set of continuous and nowhere-differentiable functions is general in C^0. That is, this set is dense in C^0.

Sketch: define

$$G_n := \{f \in C^0([a,b], \mathbb{R}) \ : \ \forall x \in [a, b - \frac{1}{n}], \exists y > x \ s.t. |f(y) - f(x)| \geq n|x - y|\}.$$

We claim that G_n is open and that G_n is dense in C^0.

1.16 Mon, Nov 4: The Derivative in Higher Dimensions

So get ready for the worst thing in analysis. In $1-D$, we defined the derivative of a function $f:\mathbb{R}\to\mathbb{R}$ to be the limit

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x) \iff f(x+h)=f'(x)h+f(x),$$

where we take the right hand in the limit as $h\to 0$.

Definition 34. The **total derivative** of $f:\mathbb{R}^n\to\mathbb{R}^m$ at $x\in\mathbb{R}^n$ if it exists, is the limit linear transform Df_x such that

$$f(x+h)=f(x)+Df_x+R(h), \quad \lim_{|h|\to 0}\frac{|R(h)|}{|h|}\to 0,$$

where $||$ is the vector norm and $R(h)$ is an error term.

Since Df_x is a linear operator, it will be useful to talk about some linear algebra. In particular, lets define the stretch a linear transform T applies to vectors.

Definition 35. Suppose $T:V\to W$ is a linear transformation, we define the **operator norm** of T to be

$$||T||=\sup_{v\in V}\frac{|T(v)|_W}{|v|_V},$$

where $|\cdot|_W$ and $|\cdot|_V$ are the norms of vectors in V and W.

You might have seen it with different definitions, we remark that this is equivalent to saying that

$$||T||=\sup_{v\in V}|T(v)|.$$

In particular, we have the following theorem:

Theorem 33. The following are equivalent:

(a) $||T||$ is finite.

(b) T is uniformly continuous.

(c) T is continuous at the origin (or any point in V for that matter).

Proof. Let $\epsilon>0$. There exists a $\delta=\frac{\epsilon}{||T||}$ such that if $v,v'\in V$ with $|v-v'|<\delta$, then

$$|T(v)-T(v')|=|T(v-v')|\leq ||T|||v-v'.|$$

Two obviously proves three. Suppose T is continuous at 0. Let $v\in V$. Then let $\epsilon=1$, and so we have that that for any $\delta>0$, if $|u|<\delta$, then

$$|T(u)|\leq 1.$$

Thus, if we let $u=\frac{\delta v}{2|v|}$, then

$$\frac{|T(v)|}{|v|}=\frac{T(u)}{|u|}\leq\frac{2}{\delta}<\infty.$$

□

Moreover, we have the following theorem:

Theorem 34. Suppose W is a finite dimensional vector space, then any linear transform $T : \mathbb{R}^n \to W$ is continuous. In fact, if it is an isomorphism then it is a homeomorphism.

Definition 36. We define the **partial derivative** of $f_j : \mathbb{R}^n \to \mathbb{R}$ at x to be

$$(Df_j)_x(e_i) = D_i f_j(x) = \partial_i f_j(x) = \frac{\partial f_j}{\partial x_i}(x) = \lim_{|h|\to 0} \frac{f_j(x_1, x_2, \ldots, x_i + h, \ldots, x_n) - f_j(x)}{|h|} = \lim_{|h|\to 0} \frac{f_j(p + te_i) - f_j(p)}{|h|}.$$

Remark 14. In high school calculus textbooks, the total derivative of f at x would be denoted by the $m \times n$ matrix of partials:

$$Df_x = \begin{bmatrix} D_1 f_1 & D_2 f_1 & \cdots & D_n f_1 \\ D_1 f_2 & D_2 f_2 & \cdots & D_n f_2 \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f_m & D_2 f_m & \cdots & D_n f_m \end{bmatrix}.$$

Theorem 35. If every $\partial_i f_j(x)$ exists and is continuous for all $i \in [n]$ and $j \in [m]$, then f is differentiable.

This proof is severely unenjoyable and I refuse to type it up for now.

Proposition 11. The total derivative satisfies the following properties:

(a) $D(f + g) = Df + Dg$

(b) $D(f \circ g) = Df \circ Dg$

(c) $D(f \cdot g) = Df \cdot g + f \cdot Dg$ (where $\cdot$ is the dot product).

We give the best possible version of the Mean Value Theorem we can.

Theorem 36. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable on U and $[p, q] \subset U$, then

$$|f(q) - f(p)| \leq \sup_{x\in U} Df_x |q - p|.$$

Proof. Let $u \in \mathbb{R}^n$ be the unit vector, then if $t \in [0, 1]$, we get that

$$g(t) = \langle u, f(p + t(q - p))\rangle$$

is differentiable and real valued, and thus we use the 1-D MVT to get some $\theta \in [0, 1]$ such that

$$g(1) - g(0) = \langle u, f(q) - f(p)\rangle = g'(\theta)$$

and using Proposition 11, we get that

$$g'(\theta) = \langle u, Df_{p+\theta(q-p)}(q - p)\rangle \leq \sup_{x\in U} Df_x |q - p|.$$

Thus, we have that it has norm less than the desired quantity. $\square$

We have that Df is a linear transform $T : \mathbb{R}^n \to \mathbb{R}^m$. The second derivative $D^2 f$ is a function $D^2 f : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m)$, where L is the space of linear transforms like T. Thus,v, w can take in two vectors $D^2(v)(w)$ and output a vector in $\mathbb{R}^m$.

Theorem 37. If $f : U \to \mathbb{R}^m$ is C^1 and if the segment $[p, q] \subset U$, then

$$f(q) - f(p) = T(q - p),$$

where T is the average derivative

$$T = \int_0^1 (Df)_{p+t(q-p)} dt.$$

Definition 37. Assume $f : \mathbb{R}^n \to \mathbb{R}$ is second differentiable, then we say that $D^2 f_x$ is the **Hessian** of f. That is, the matrix of second partial derivatives.

Theorem 38. The Hessian is symmetric. That is, for any $v, w \in \mathbb{R}^n$ we have that

$$(D^2 f)_x(v)(w) = (D^2 f)_p(w)(v).$$

Remark 15. This does imply that partial double derivative are symmetric since

$$(D^2 f_x)(e_i)(e_j) = \frac{\partial f_k}{\partial x_i \partial x_j} = \frac{\partial f_k}{\partial x_j x_i} = (D^2)_x(e_j)(e_i)$$

Have fun proving this one.

1.17 Wed, Nov 6: C^r Space

Definition 38. We say $f \in C^r(U, \mathbb{R}^m)$ if the rth derivative of f exists and is continuous.

Definition 39. The C^r norm of a function $f \in C^r$ is defined by

$$||f||_{C^r} = \max\{\sup_{x \in U} |f(x)|, \ldots, \sup_{x \in U} |D^{(r)}(x)|\}.$$

Definition 40. We say that a series f_k is C^r **uniformly convergent** to some f ($f_k \to f$ C^r unif) if

$$\begin{gathered} f_k \to f \quad \text{unif.} \\ (Df_k) \to Df \quad \text{unif} \\ \vdots \\ (D^{(r)} f_k) \to D^{(r)} f \quad \text{unif} \end{gathered}$$

Definition 41. We say that a series f_k is C^r **Cauchy** if for all $\epsilon > 0$, there exits some $N \in \mathbb{N}$ such that if $k, \ell > N$, then

$$\begin{gathered} ||f_k - f_\ell|| < \epsilon \\ ||(Df_k) - Df_\ell|| < \epsilon \\ \vdots \\ ||(D^{(r)} f_k) - (D^{(r)} f_\ell)|| < \epsilon \end{gathered}$$

Theorem 39. C^r is a **Banach Space,** i.e, a complete normed vector space.

Proof. Let f_k be C^r Cauchy. Since C^0 is complete, we have that $f_k \to f$ uniformly. We also know that $(Df_k) \to G$ uniformly. We claim that $(Df_k) \to (Df)$ uniformly (i.e, $(Df_k) = G$). Let $p \in U$ and consider $q \in U$ such that $[p, q] \subset U$. By Theorem 37, we have that

$$f_k(q) - f_k(p) = \int_0^1 (Df_k)_{p+t(q-p)} dt(q-p) \to f(q) - f(p) = \int_0^1 G(p + t(q-p)) dt(q-p).$$

Proceed by induction. $\square$

1.18 Fri, Nov 8: The Inverse and Implicit Function Theorem

Both Pugh and Luis have godawful explanations for this. I defer to the Youtube Channel "The Bright Side of Mathematics."

Theorem 40. Let $U, V \subset \mathbb{R}^n$ and $f \in C^1(U, V)$ and $x_0 \in U$. If $(Df)_{x_0}$ is invertible, then f is a local C^1 diffeomorphism at x_0.

Proof. (STEP 1) If $y_0 = f(x_0)$, then define $\hat{f} = f - y_0$ and define $g(x) = \hat{f}(x + x_0)$. Thus, we have that $\hat{f}(0) = 0$and $(Df)_0$ is invertible. Define $h(x) = (Df)_0^{-1} g(x)$, then $(Dh)_0 = \mathbf{I}$.

(STEP 2) Now that we have normalized the problem, we can begin. Let $z(x) = h(x) - x$, then $(Dz)_0 = \mathbf{0}$. Since $z \in C^1$, then by continuouity we choose $\epsilon > 0$ such that

$$\left| \frac{\partial f_i(x)}{\partial x_j} \right| \leq \frac{1}{2} \frac{1}{n^2}, \qquad \forall i, j \in [n], \quad \forall x \in \overline{B_\epsilon(0)}$$

and $(D_h)(x)$ is invertible for all $x \in B_\epsilon(0)$.

(STEP 3) Now that we have restricted our function to a convex ball, we can use the Mean Value Theorem when we define $a : [0, 1] \to \mathbb{R}$ by

$$a(t) = z_i(p + t(q - p)),$$

where $p, q \in B_\epsilon(0)$. Thus, we have that if

$$z(q) - z(p) = a(1) - a(0) = a'(\theta) = \langle \nabla z_i(p + \theta(q - p)), p - q \rangle.$$

Thus, we have by Cauchy-Schwartz that

$$|z(q) - z(p)| \leq ||\nabla z_i(p + \theta(q - p))||^2 ||p - q||^2 = \sum_j \left| \frac{\partial z_i}{\partial x_j} \right|^2 ||q - p||^2 \leq \frac{1}{4} \frac{n}{n^4} ||q - p||$$

Thus, we have that

$$||z(q) - z(p)||^2 = \sum_i |z_i(q) - z_i(p)|^2 \leq \frac{1}{16} \frac{1}{n^2} ||p - q||^2,$$

thus, z is a contraction since

$$||z(q) - z(p)||| \leq \frac{1}{2} ||q - p||.$$

(STEP 4) Fix an element $y \in \overline{B_\epsilon(0)}$ and define $z_y : \overline{B_\epsilon(0)} \to \overline{B_\epsilon(0)}$ by

$$z_y(x) = y - z(x).$$

Thus, we have that

$$||z_y(q) - z_y(p)|| \leq \frac{1}{2} ||q - p||,$$

and by Banach's fixed point theorem, there exists some unique fixed point x such that

$$z_y(x) = x \iff y - z(x) = x \iff y - (h(x) - x) = x \iff y = h(x).$$

Thus, we have found a bijection in this closed ϵ ball.

The rest of the proof is left up to the reader. $\square$

Remark 16. We have a nice formula for the inverse, that being

$$(Dh^{-1})_y = \left((Dh)_{h^{-1}(y)}\right)^{-1}$$

Theorem 41. Suppose $U \subset \mathbb{R}^n \times \mathbb{R}^m$ is open, and $F : U \to \mathbb{R}^m$ is C^1. Let $z_0 = (x_0, y_0) \in U$ with $F(z_0) = 0$ and

$$DF_{z_0} = \begin{bmatrix} \frac{\partial F_1(z_0)}{\partial x_1} & \frac{\partial F_1(z_0)}{\partial x_2} & \cdots & \frac{\partial F_1(z_0)}{\partial x_n} & \frac{\partial F_1(z_0)}{\partial x_{n+1}} & \cdots & \frac{\partial F_1(z_0)}{\partial x_m} \\ \frac{\partial F_1(z_0)}{\partial x_1} & \frac{\partial F_2(z_0)}{\partial x_2} & \cdots & \frac{\partial F_2(z_0)}{\partial x_n} & \frac{\partial F_2(z_0)}{\partial x_{n+1}} & \cdots & \frac{\partial F_2(z_0)}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m(z_0)}{\partial x_1} & \frac{\partial F_m(z_0)}{\partial x_2} & \cdots & \frac{\partial F_m(z_0)}{\partial x_n} & \frac{\partial F_m(z_0)}{\partial x_{n+1}} & \cdots & \frac{\partial F_m(z_0)}{\partial x_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{bmatrix}.$$

If $\frac{\partial F}{\partial y} \neq \mathbf{0}$, then there exists open $V_1 \subset \mathbb{R}^n$ and $V_2 \subset \mathbb{R}^m$ such that $z_0 \in V_1 \times V_2$ and a map $g \in C^1(V_1, V_2)$ such that

$$g(x, g(x)) = 0.$$

Proof. Define $f : U \to \mathbb{R}^n \times \mathbb{R}^m$ by $f(x, y) = \begin{pmatrix} x \\ f(x, y) \end{pmatrix}$. Thus $f \in C^1(U, \mathbb{R}^n \times \mathbb{R}^m)$ and

$$Df_{z_0} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \frac{\partial F(z_0)}{\partial x} & \frac{\partial F(z_0)}{dy} \end{bmatrix}$$

Thus, we have that $\det Df_{z_0} = 1 \cdot \det DF_{z_0} \neq 0$. We can apply the inverse function theorem to f. We get open set $U;, V' \subset \mathbb{R}^n$ and $f^{-1} : V' \to U'$ is C^1 and $f(z_0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \in R = V_1' \times V_2' \subset V'$. Let $S = f^{-1}(R)$, we know it is open and $z_0 \in S = V_1 \times_2$. Thus, $f^{-1} : V_1 \times V_2 \to V_1 \times V_2$ with

$$\begin{pmatrix} x \\ y' \end{pmatrix} \to \begin{pmatrix} x \\ h(x, y') \end{pmatrix}$$

, where

$$h : V_1 \times V_2' \to V_2$$

is C^1. Thus, define $g : V_1 \to V_2$ such that $g(x) = h(x, 0) \in C^1$. $\square$

1.19 Mon, Nov 11: Lagrange Multipliers

Theorem 42. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \to \mathbb{R}$ is C^1 for $j \in [m]$ (i.e, $g : \mathbb{R}^n \to \mathbb{R}^m$). Suppose f has a local extremum at x_0 with the constraint $g(x) = 0$ and $\text{rank}(Dg_{x_0}) = m$. Then there are real numbers, called **Lagrange Multipliers** $\lambda_j \in \mathbb{R}$ such that

$$\nabla f(x_0) = \sum_{j=1}^{m} \lambda_j \nabla g_j(x_0)$$

We will prove this for the $m = 1$, $n = 2$ case, which states that if $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ is C^1, then if f has a local extremum at x_0 with the constraint of $g(x) = 0$, and $\nabla g(x_0) \neq 0$, then there exists some $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda \nabla g(x_0).$$

Proof. By the implicit function theorem, we have that $g(x_1, x_2) = 0$ can locally be written as either

$$g(x_1, \gamma(x_2)) = 0, \qquad \text{or} \qquad g(\beta(x_2), x_2) = 0.$$

Let's concentrate on the first. Then for all $x_1 \in U$, we write $g(x_1, \gamma(x_2)) = 0$. By assumption, we have that that by the chain rule

$$0 = \frac{d}{dx_1} g(x_1, \gamma(x_1)) = \langle \nabla g(x_1, \gamma(x_1)), \begin{pmatrix} 1 \\ \gamma'(x) \end{pmatrix} \rangle. \tag{1}$$

We have that the function $\hat{f} : U \to \mathbb{R}$ with $\hat{f}(x_1) = f(x_1, \gamma(x_1))$ has a local extremum at x_{0_1}. Thus, we have that

$$0 = \hat{f}'(x_{0_1}) = \langle \nabla f(x_{0_1}, \gamma(x_{0_1})), \begin{pmatrix} 1 \\ \gamma'(x_{0_1}) \end{pmatrix} \rangle$$

Because (1) holds for any $x \in U$, then we have the ∇f and ∇g are orthagonal to each other, and so there must exist some real λ such that

$$\nabla f = \lambda \nabla g.$$

$\square$

You can see that the method of Lagrange Multipliers is an application of the implicit function theorem.

1.20 Wed, Nov 13: Multivariate Integration

We mostly talked about integration from $\mathbb{R}^2 \to \mathbb{R}$.

Definition 42. Suppose $R \subset \mathbb{R} \times \mathbb{R}$ is a rectangular region. Then we partition R into $R_{i,j} = [x_{i-1}, x_i) \times [y_{j-1}, y_j)$ such that $R = \bigcup_j \bigcup_i R_{i,j}$. Then we take a sampling of points S inside each $R_{i,j}$ and we call the Riemann Sum to be

$$R(f, P, S) = \sum f(s_{i,j})|R_{i,j}| = \sum f(s_{i,j})(x_i - x_{i-1})(y_i - y_{i-1}).$$

We say f is **Riemann Integrable** if there exists some $I \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists some partition P such that

$$|P - I| < \epsilon.$$

Definition 43. We define the **upper sum** and the **lower sum** to be

$$U(f, P) = \sum \max f|R_{i,j}|, \qquad L(f, P) = \sum \min f|R_{i,j}|.$$

We say that f is**Darboux integrable** if for any $\epsilon > 0$, there exists partitions P such that

$$|U(f, P) - L(f, P)| < \epsilon \iff \inf U = \sup L.$$

Remark 17. Notationally, we say that

$$\underline{\int} f dxdy = \sup L, \qquad \overline{\int} f dxdy = \inf U$$

Sliced Bread.

Theorem 43. (Fubini's Theorem) If f is integrable over $R = [a, b] \times [c, d]$, then

$$\int_R f dxdy = \int_c^d \left[\int_a^b f dx\right] dy.$$

Proof.

$$U = \inf\{\int_R gdxdy;\ g \geq f, \quad g \text{ constant on } R_{i,j}\}$$

$$L = \sup\{\int_R gdxdy;\ g \leq f, \quad g \text{ constant on } R_{i,j}\}$$

Thus,

$$\int_c^d \left[\int_a^b gdx\right] dy = \sum g(s_{i,j})|R_{i,j}|.$$

For any g with $g \leq f$ we have

$$\underline{\int_c^d} \left[\underline{\int_a^b} f dx\right] dy \geq \underline{\int_c^d \int_a^b} gdxdy = \int_R gdxdy \leq \int_R f dxdy$$

and thus

$\square$

We now pivot to a very important theorem in multidimensional calculus:

Theorem 44. (Inverse Function Theorem) Suppose $\varphi : U \to W$ is a C^1 differeomrphism with U, W open. Suppose further that f is integrable and that $R \subset U$. Then we have that

$$\int_{\varphi(R)} f = \int_R f(\varphi)|\det(D\varphi)|$$

Proof. No. $\square$

1.21 Fri, Nov 15: Line Integrals and Green's Theorem

We leave the rigour for a few lectures to define some important topics in vector calculus. Suppose $\gamma : [a,b] \to \mathbb{R}^2$ and let f be an integrable function. If τ is the unit tangent, then we define the **line integral** of F, vector field, along the curve C parameterized by γ to be

$$\int_C F d\mathbf{s} = \int_C F \cdot \tau ds = \int_{\gamma([a,b])} F \cdot \frac{\gamma'(t)}{|\gamma'(t)|} dt = \int_a^b F(\gamma(t)) \cdot |\gamma'(t)| \tag{2}$$

Theorem 45. The line integral is independent of parametrization.

Proof. Suppose $\alpha : [a,b] \to C$ and $\beta : [c,d] \to C$ are two parametrization of a curve C. Define $\gamma : \beta^{-1} \circ \alpha$. Thus, we have that $\gamma : [a,b] \to [c,d]$. Thus, we compute using the change of variables formula:

$$\begin{aligned}\int_C F ds &= \int_c^d f(\beta(t))|\beta'(t)|dt \\ &= \int_a^b f(\beta(\gamma))|\beta'(\gamma(t))|\gamma'(s)dt \\ &= \int_a^b f(\alpha(s)|\alpha'(s)|ds\end{aligned}$$

□

Using this theorem, we can provide an alternate parametrization to (2). Let $\gamma(t) = (t, g(t))$ parameterize C. Then we have

$$\int_C f dx = \int f(x, g(x))dx = \int_C (f, 0) \cdot \tau ds$$

$$\int_C f dy = \int_C (0, f) \cdot \tau ds$$

Now we get that

$$\int_C F \cdot \tau ds = \int_C F_1 dx + F_2 dy.$$

Theorem 46. (Gradient Integral Theorem or the Fundamental Theorem for Line Integrals) Suppose $f : \mathbb{R}^d \to \mathbb{R}$ and we have a curve C from $p \in \mathbb{R}^d$ to $q \in \mathbb{R}^d$, then

$$\int_C \nabla f \cdot \tau ds = \int_C \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n = f(q) - f(p)$$

Proof. Clear by FTC: if γ is a paremeterization of C, then

$$\int_C \nabla f \cdot \tau ds = \int_a^b \nabla f(\gamma(t)) \cdot \gamma'(t)dt = \int_a^b \frac{d}{dt}(\gamma(t)) = f(\gamma(b)) - f(\gamma(a))$$

□

Theorem 47. Suppose $F : \Omega \to \mathbb{R}^d$ for $\Omega \subset \mathbb{R}^d$ open, then the following are equivalent:

(a) $\int_C F \cdot \tau ds$ depends only on the endpoints.

(b)

$$\oint_C F \cdot \tau ds = 0$$

if C is a loop.

(c) There exists some $\rho : \Omega \to \mathbb{R}$ such that $F = \nabla\rho$.

(d) If Ω is convex, then we have that $\nabla \times F = 0$.

Theorem 48. (Green's Theorem) Suppose $D \subset \mathbb{R}^2$ is simple bounded by the curve C and F is a vector field. Then

$$\int_C F_1 dx + F_2 dy = \int_D (D_2 F_1 - D_1 F_2) dx dy.$$

Remark 18. Luis didn't prove Theorem 47, but he showed a nice little geometric proof for Theorem 48 that I cannot stand because I know Stokes Theorem exists. Note that in $2-D$, we have that

$$D_2 F_1 - D_1 F_2 = \nabla \times F,$$

meaning we have $1 \to 4$ in Theorem 47. $3 \to 4$ is equally clear since we show in a pset that $ddf = 0$ and thus $\boldsymbol{\nabla} \times \nabla\rho = 0$.

1.22 Fri, Nov 17: Midterm II

We had another midterm! Thank god I'm god!

Example 1.10. Write explicit examples of the following. No justification is required.

(a) A differentiable function on $\mathbb{R}$ that is not C^1.

(b) A function $f : \mathbb{R}^2 \to \mathbb{R}$ which has directional derivatives and in every direction, but is not differentiable.

Proof. No justification is great.

(a)
$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

(b)
$$f(x) = \begin{cases} \frac{x^3 y}{x^4 + y^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

□

Proposition 12. Consider the $C([0,1], \mathbb{R})$ with the sup-metric, and let A be the subspace

$$A := \{f \in C([0,1], \mathbb{R}) \ : \ f \text{is differentiable everywhere}, |f| \leq 1, \ |f'| \leq 1\}.$$

- Is A a closed subset of $C([0,1])$.
- Is A and open subset of $C([0,1])$.
- Is the closure of A compact on $C([0,1])$.

Proof. We give the best proofs of all time.

- **No.** Consider a sequence of $(f_n) \in A$ which uniformly converges to the absolute value function translated to have the crest at $\frac{1}{2}$. Call this f. Then $f_n \to f$ unif but $f \notin A$. Do a little doodle for this o get full credit.

- **No.** By Baire Category Theorem, we know that the set of continuous but non-differentiable functions is dense in $C([0,1])$. Thus, for any $f \in A$, any $r > 0$, we have that there exists some g continuous but nondifferentiable such that
$$g \in B_r(f).$$
Thus, A is not open.

- By the generalized Heine Borel (Theorem 26), we have that $\overline{A} \subset C$ is compact if and only if it is equicontinuous, bounded, and closed. Since A is bounded, then $\overline{A}$ is bounded. Obviously $\overline{A}$ is closed. Since $|f'| \leq 1$, then we have that for any $\epsilon > 0$, there exists a $\delta = \epsilon$ such that if $|x - y| < \delta$ for any $x, y \in A$ and $f \in A$, then by MVT,
$$|f(x) - f(y)| = f'(\theta)|x - y| \leq |x - y| < \epsilon.$$
Thus, A is equicontinuous. To show $\overline{A}$ is equicontinuous, let $\epsilon > 0$. It suffices to show that if f is a LP of A, then equicontinuity follows. Thus, suppose $(f_n) \to f$ with $f_n \in A$. Then we have that
$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3},$$
where the first and last terms are bounded by the fact that $f_n \to f$ uniformly and the middle term is bounded by the equicontinuity of the f_n.

$\square$

Proposition 13. Let α and β be two given continuous functions on $[0,1]$. Assume that for all $x \in [0,1]$, $|\alpha(x)| < 1$. prove that there exists a unique continuous function $f : [0,1] \to \mathbb{R}$ such that for all $x \in [0,1]$,
$$f(x) - \int_0^x \alpha(y) f(y) dy = \beta(x).$$

To prove this, used the fixed point theorem.

1.23 Mon, Nov 20: Surface Integrals and the Divergence Theorem

Now that we have talked about line integrals, we can beging talking about surface integrals. Suppose that $S \subset \mathbb{R}^3$ be a surface parameterized by some $\alpha : Q \to \mathbb{R}^3$, then the **surface integral** of some function f is defined to be

$$\int_S f dA = \int_Q f(\alpha(x,y)) \left| \frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{y} \right| dxdy.$$

If $F : S \to \mathbb{R}^3$ is a vector field and $S \subset \mathbb{R}^3$ is parameterized by α, then the **flux** of F is

$$\int_S F \cdot ndA = \int_Q F(\alpha(t)) \cdot \left(\frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y} \right) dxdy = \int_Q \det\left(F(\alpha), \frac{\partial \alpha}{\partial x}, \frac{\partial \alpha}{\partial y} \right) dxdy. \tag{3}$$

Theorem 49. The Flux of (3) is independent of parametrization.

The proof for this carries a similar flavor to Theorem 45.

Theorem 50. The flux in (3) can be written by

$$\int_S F \cdot ndA = \int_Q F_1 dydz - F_2 dzdx + F_3 dxdy$$

Remark 19. To sketch a proof for this, we parameterize the surface by $\alpha(x,y) = (x, y, g(x,y))$ and proceed as in Theorem 45!

Theorem 51. (Divergence Theorem) Suppose $D \subset \mathbb{R}^3$ is a simple region. Then if F is a vector field, then

$$\int_{\partial D} F \cdot ndA = \int_D \nabla \cdot F dxdydz$$

1.24 Fri, Nov 22: Differential Forms

Luis had a terrible treatment of differential forms, so we differ to Pugh for his..., wait his stuff is also terrible?? We take heavy inspiration from Hubbard and Hubbards *Vector Calc, Linear Algebra, and Differential Forms*' Chapter 6.

Definition 44. A **k-form** on $\mathbb{R}^n$ is a function φ that takes k vectors in $\mathbb{R}^n$ and returns a number $\varphi(v_1, \ldots, v_k)$ such that φ is multilinear and alternating and antisymmetric.

Remark 20. We usually denote $1-$forms as dx_1, etc. A two form as $dx_1 \wedge dx_2$, etc. We provide meaning to his wedge operator later.

Example 1.11.

$$dx_1 \wedge dx_2 \left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \\ 2 \end{bmatrix} \right) = \det \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} = -8$$

Note that the two form gives the signed area of the parallelogram they span. Similarly, $dx_1 \wedge dx_2 \wedge dx_4$ gives the (x_1, x_2, x_4) component of signed volume of the parralelipied spanned by three vectors.

Remark 21. A zero form is just a number in $\mathbb{R}^n$.

Remark 22. $dx_i \wedge dx_i = 0$. Just look at the determinant!

Remark 23.

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

More generally, on k form, it is the sign of the permutations needed to get to the elementary form.

Definition 45. An **elementary k-form** on $\mathbb{R}^n$ is an expression of the form

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where $1 \leq i_1 < \cdots i_k \leq n$ which when evaluated on the vectors $v_1, \ldots, v_k$, gives the determinant of the $k \times k$ matrix obtained by selecting rows $i_1, \ldots, i_k$ of the matrix who's columns are $v_1, \ldots v_k$. The only elementary $0-$form is just 1.

The following theorem is quite important. Also very cool.

Theorem 52. The space of $k-$forms in $\mathbb{R}^n$ forms the vector space $A^k(\mathbb{R}^n)$. I.e, we can add $k-$forms and multiply them by scalars. Moreover, the elementary $k-$forms form a basis for $A^k(\mathbb{R}^n)$. I.e, any $k-$form in $A^k(\mathbb{R}^n)$ can be uniquely written by

$$\varphi = \sum_{1 \leq i_1 < \cdots i_k \leq n} a_{i_1, \ldots, i_k} dx_{i_1} \wedge \cdots dx_{i_k}$$

where

$$a_{i_1, \ldots, i_k} dx_{i_1} = \varphi(e_{i_1}, \ldots, e_{i_k}).$$

The dimension of $A^k(\mathbb{R}^n)$ is

$$\binom{n}{k}$$

Definition 46. Let φ be a $k-$form and ω be an $\ell-$form, both on $\mathbb{R}^n$. The **wedge product** $\varphi \wedge \omega$ is a $(k + \ell)-$form. It is defined as

$$\varpi \wedge \omega(v_1, \ldots, v_{k+\ell}) = \sum_{\sigma \in \mathrm{Perm}(k,\ell)} \varphi(v_{\sigma(1)}, \ldots v_{\sigma(k)}) \omega(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

It is fairly painful to prove the following properties:

Proposition 14. The wedge product satisfies the following:

(a) Distributivity: $\varphi \wedge (\omega_1 + \omega_2) = \varphi \wedge \omega_1 + \varphi \wedge \omega_2$.

(b) Associativity: $\varphi \wedge (\omega_1 \wedge \omega_2) = (\varphi \wedge \omega_1) \wedge \omega_2$.

(c) Skew commutativity: $\varphi \wedge \omega = (-1)^{k\ell}(\omega \wedge \varphi)$.

Definition 47. A **k-differential form** on an open subset $U \subset \mathbb{R}^n$ is a function that takes $k-$vectors $v_1, \dots, v_k$ anchonred at a point $x \in \mathbb{R}^n$ (i.e, a parallelogram $P_x(v_1, \dots, v_k)$ and returns a number. It is a multilinear and antisymmetric a function of $\mathbf{v}$.

Example 1.12.

$$\cos(xz)dx \wedge dy \left(P_{\left(\begin{bmatrix}1\\2\\\pi\end{bmatrix}\right)} \left(\begin{bmatrix}1\\0\\1\end{bmatrix} \begin{bmatrix}2\\2\\3\end{bmatrix} \right) \right) = \cos(1 \cdot \pi) \det \begin{bmatrix}1 & 2\\0 & 2\end{bmatrix} = -2$$

The entire reason of differential forms is to integrate them, and so we define how to do it with the change of variables formula.

Definition 48. Let $[\gamma(U)]$ be a parametrized domain in $\mathbb{R}^n$. The **integral of the k-differential form** φ over $[\gamma(U)]$ is

$$\int_{[\gamma(U)]} \varphi = \int_U \varphi \left(P_{\gamma(U)} \left(D\gamma(\mathbf{u}) \right) \right) |d\mathbf{u}|$$

Example 1.13. Let $k = 1$ and $n = 2$ and $\gamma(u) = \begin{pmatrix} R\cos u \\ R\sin u \end{pmatrix}$, with U being the interval $(0, \alpha)$. Integrating $xdy - ydx$ over $[\gamma(U)]$, then we have

$$\int_{[\gamma(U)]} xdy - ydx = \int_0^\alpha xdy - ydx \left(P_{\left(\begin{bmatrix}R\cos u\\R\sin u\end{bmatrix}\right)} \begin{bmatrix}-R\sin u\\R\cos u\end{bmatrix} \right) du = \int_0^\alpha (R^2\cos^2 u + R^2 \sin^2 u)du = R^2\alpha.$$

Example 1.14. Integrate $dx \wedge dy + ydx \wedge dz$ over $[\gamma(S)]$ where

$$\gamma\begin{pmatrix}s\\t\end{pmatrix} = \begin{pmatrix}s+t\\s^2\\t^2\end{pmatrix}, \qquad S = \{\begin{pmatrix}s\\t\end{pmatrix} | s \in [0,1], t \in [0,1]\} :$$

$$\int_{[\gamma(S)]} dx \wedge dy + ydx \wedge dz = \int_S dx \wedge dy + ydx \wedge dz \left(P_{\begin{pmatrix} s+t \\ s^2 \\ t^2 \end{pmatrix}} \left(\begin{bmatrix} 1 \\ 2s \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2t \end{bmatrix} \right) \right) dsdt$$

$$= \int_0^1 \int_0^1 \det \begin{bmatrix} 1 & 1 \\ 2s & 0 \end{bmatrix} + s^2 \det \begin{bmatrix} 1 & 1 \\ 0 & 2t \end{bmatrix} dsdt$$

$$= \int_0^1 \int_0^1 (-2s + 2s^2 t) dsdt$$

$$= \int_0^1 (-1 + \frac{2}{3}t) dt$$

$$= \frac{-2}{3}$$

Going over to vector calculus, we see how amazing differential forms really are.

Definition 49. The **work form** W_F of a vector field $F = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$ is the 1−form defined by

$$W_F(P_x(v)) = F(x) \cdot v = F_1 dx + \dots F_n dx_n.$$

Example 1.15. Any differential 1form is the work form of a vector field. The one form $ydx - xdy$ is the work form of $F = \begin{bmatrix} -y \\ x \end{bmatrix}$

Remark 24. Work forms should be integrated over oriented curves.

Definition 50. The **work** of a vector field F along an oriented curve C is

$$\int_C W_F$$

Example 1.16. What is the work of $F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}$ over the heliz orientedd by the tangent vector field

$$\mathbf{t} = \begin{bmatrix} \sin t \\ \cos t \\ 1 \end{bmatrix}$$

and parameterized by

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$

for $t \in (0, 4\pi)$

$$\int_0^{4\pi} ydx_1 + -xdx_2 \left(P_{\begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}} \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix} \right) dt = \int_0^{4\pi} -\sin^2(t) - \cos^2(t) dt = -4\pi$$

Definition 51. The **flux-form** is the differential $2-form$

$$\Phi_F(P_x(v,w)) = \det[F(x), v, w] = F_1 dy \wedge dz - F_2 dz \wedge dz + F_3 dx \wedge dy$$

Remark 25. The flux form of a vector field associates to a parallelogram the flow of the vector field through it.

Definition 52. The **flux** of a vector field F over an oriented surface S is $\int_S \Phi_F$.

Example 1.17. The flux of the vector field $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} x \\ y^2 \\ z \end{bmatrix}$ through the parameterized domain $\binom{u}{v} \to \begin{pmatrix} u^2 \\ uv \\ v^2 \end{pmatrix}$ for $0 \leq u, v \leq 1$ is:

$$\int_0^1 \int_0^1 xdy \wedge dz - y^2 dz \wedge dx + zdx \wedge dy \left(P_{\begin{pmatrix} u^2 \\ uv \\ v^2 \end{pmatrix}} \left(\begin{bmatrix} 2u \\ v \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ u \\ 2v \end{bmatrix} \right) \right) dudv =$$

$$= \int_0^1 \int_0^1 2u^2v^2 - 4u^3v^3 + 2u^2v^2$$
$$= \frac{7}{36}$$

Definition 53. The **mass form** M_f is the $3-$form defined by

$$M_f(P_x(v_1, v_2, v_3)) = f(x) \det[v_1, v_2, v_3] = f dx \wedge dy \wedge dz$$

We are ready to talk about the exterior derivative!

Definition 54. The **exterior derivative** d of a $k-$form φ, denoted $d\varphi$, takes a $k+1$ parallelogram and returns a number, i.e,

$$d\varphi(P_x(v_1, \ldots, v_{k+1})) = \lim_{h \to 0} \frac{1}{h^{k+1}} \int_{\partial P_x(hv_1, \ldots, hv_{k+1})} \varphi$$

Proposition 15. Let φ be a $k-$form:

(a) The exterior derivative defines a $(k+1)-$form.

(b) The exterior derivative is linear over $\mathbb{R}$. That is,

$$d(a\varphi + b\omega) = ad\varphi + bd\omega.$$

(c) The exterior derivative of a constant form is 0.

(d) Let f be a zero form, then

$$df = \sum_{i=1}^{n} (Df(e_i)) dx_i.$$

(e) If f is a function, then

$$d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = df \wedge dx_{i_1} \cdots \wedge dx_{i_k}.$$

Example 1.18. Compute the exterior derivative of the following 2−form on $\mathbb{R}^4$:

$$\psi = x_1x_2dx_2 \wedge dx_4 - x_2^2dx_3 \wedge dx_4.$$

We have

$$d\psi = d(x_1x_2dx_2 \wedge dx_4) - d(x_2^2dx_3 \wedge dx_4.) = x_2dx_1 \wedge dx_2 \wedge dx_4 - 2x_2dx_2 \wedge dx_3 \wedge 2x_4$$

Theorem 53. For any k−form φ :

$$d(d\varphi) = 0$$

Theorem 54. The exterior derivative of wedge product is

$$d(\varphi \wedge \omega) = d\varphi \wedge \omega + (-1)^k\varphi \wedge \omega.$$

Recall from vector calculus class:

Definition 55. Let $U \subset \mathbb{R}^3$ be an open set, $f : U \to \mathbb{R}$ be C^1 and F be a C^1 vector field on U.

- The **gradient** of f is given by
$$\nabla f = \begin{bmatrix} Df(e_1) \\ Df(e_2) \\ Df(e_3) \end{bmatrix}$$

- The **curl** of F is given by
$$\boldsymbol{\nabla} \times F = \nabla \times F = \begin{bmatrix} DF_3(e_2) - DF_2(e_3) \\ DF_1(e_3) - DF_3(e_1) \\ DF_2(e_1) - DF_2(e_1) \end{bmatrix}$$

- The **divergence** of F is given by
$$\nabla \cdot F = DF_1(e_1) + DF_2(e_2) + DF_3(e_3).$$

We can now relate these concepts to differential forms:

Theorem 55. Let f be a function and F be a vector field. Then

(a)
$$df = W_{\nabla f} = D_1f dx_1 + D_2f dx_2 + D_3f dx_3$$

(b)
$$dW_F = \Phi_{\nabla \times F}$$

(c)
$$d\Phi_F = M_{\nabla \cdot F}$$

We use the previous theorem and Theorem 21 to show two consequences.

Corollary 5.

$$\nabla \times \nabla f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\nabla \cdot (\nabla \times F) = 0$$

We now get to the most beautiful theorem in Real Analysis:

Theorem 56. (Generalized Stokes Theorem) Let X be a compact piece-with-boundary of a $k-$dimensional oriented manifold $M \subset \mathbb{R}^n$. Give the boundary ∂X of X the boundary orientation, and let φ be a $(k-1)-$ form defined on an open set containing X. Then

$$\int_{\partial X} \varphi = \int_X d\varphi$$

We can intuit two corollaries from this

Corollary 6. (Green's Theorem) Let S be a bounded region in $\mathbb{R}^2$ bounded by some curve C. Let F be a vector field defined on a neighborhood of S. Then

$$\int_S dW_F = \int_C W_F.$$

Remark 26. To get the usual form of Green's theorem, just let $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$, then

$$W_F = F_1 dx + F_2 dy, \qquad dW_F = dF \wedge dx + dF_2 \wedge dy = (D_1F_2 - D_2F_1) dx \wedge dy.$$

Thus, we have that

$$\int_C F_1 dx + F_2 dy = \int_S (D_1F_2 - D_2F_1) dx \wedge dy$$

Corollary 7. (Stokes Theorem) Let S be an oriented surface in $\mathbb{R}^3$, bounded by a curve C that is given the boundary orientation. Let φ be a $1-$form defined on a neighborhood of S, then

$$\int_S d\varphi = \int_C \varphi.$$

Remark 27. To get the usual form of the divergence theorem, remind yourself that if we write $\phi = W_F$, then we get that

$$\int_S dW_F = \int_S \Phi_{\nabla \times F}.$$

Note that the classical version is

$$\int \int_S (\nabla \times F) \cdot N dx dy = \int_C F \cdot T dx.$$

Corollary 8. (Divergence Theorem) Let X be a bounded domain in $\mathbb{R}^3$. Let its boundary ∂X be a union of surfaces S_i. Let φ be a $2-$form fild defined on a neighborhood of X, then

$$\int_X d\varphi = \sum \int_{S_i} \varphi.$$

Remark 28. To extract the classical version, let $\varphi = \Phi_F$, then we have that $d\varphi = M_{\nabla . F}$. We then rewrite

$$\int_X M_{\nabla \cdot F} = \int \int \int_X \nabla \cdot F dx dy dz = \sum \int_S F \cdot N dx dy$$

Theorem 57. (Fundamental Theorem of Line Integrals) Suppose $F : U \to \mathbb{R}^n$ for $U \subset \mathbb{R}^n$ open, then the following are equivalent:

(a)

$$\int_{\gamma([a,b])} W_{\nabla F} = f(\gamma(b)) - f(\gamma(a))$$

(b)

$$F = \nabla f$$

for some f.

(c)

$$\nabla \times F = 0$$

and U is convex.

Mon, Dec 2: $k-$cells and pullbacks

We talked about a lot of which I wrote about in the previous section. Here, we more formally define the $k-$cell, that which we integrate forms over.

Definition 56. A **k-cell** is a parametrization $\varphi : I^k \to \mathbb{R}^n$ and we can integrate over a $k-$cell by

$$\int_{\varphi} \omega \int_{I^k} \omega(\varphi(u))(\frac{\partial \varphi}{\partial u_1}, \ldots, \frac{\partial \varphi}{\partial u_k}) du_1 du_2 \ldots du_k.$$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

Definition 57. The **pushforward** of φ, is where $\varphi : Q \to \mathbb{R}^n$, is defined as

$$T_* \varphi = T \circ \varphi : Q \to \mathbb{R}^m.$$

The more important one to us is the pullback, which is defined as

Definition 58. The **pullback** of a $k-$form in $\mathbb{R}^m$ is a $k-$form denoted by $T^*\omega$ and defined by

$$\int_{\varphi} T^* \omega = \int_{T_* \varphi} \omega.$$

Remark 29. Thus, we have that $T^*(x)(v_1, \ldots, v_k) = \omega(x)(DT_x v_1, \ldots, DT_x v_k)$.

1.25 Wed, Dec 4: Pullbacks and Stokes' Theorem

Proposition 16. (a)
$$T^*(\omega_1 \wedge \omega_2) = T^*\omega_w \wedge T^*\omega_2.$$

(b)
$$T^*(d\omega) = d(T^*\omega)$$

Proof. We don't prove the first because fuck you. We verify for when ω is a zero form ($\omega(y) = f(y)$.) Then we have that by the chain rule:

$$\begin{aligned}
T^*\omega = f(T(x)) \implies d(T^*(\omega)) \\
&= \frac{\partial f \circ T}{\partial x_1} dx_1 + \cdots + \frac{\partial f \circ T}{\partial x_n} dx_n \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \frac{\partial f_i}{\partial y_i} \frac{\partial T_i}{x_j} \right) dx_j
\end{aligned}$$

The other equality we get straight from the definition of the exterior derivative:

$$d\omega = \frac{\partial f}{\partial y_1} dy_1 + \cdots + \frac{\partial f}{\partial y_n},$$

and so plugging in the definition of $T^*\omega$ yields the above. For a general $k-$form, induct. $\square$

Now we get to Stoke's Theorem, but we must handwave the fuck out of some shit beforehand. Let Q be a cube in $\mathbb{R}^n$ and let

$$d\omega = \sum f_i \hat{dx_i}, \qquad \hat{dx_i} = dx_1 \wedge \cdots \wedge d_{i-1} \wedge d_{i+1} \wedge \cdots dx_n.$$

Observe that if $\omega = f_1(x) dx_2 \wedge dx_3 \cdots dx_n$, then by FTC:

$$f_1(1, x_2, \dots) - f_1(0, x_2, \dots, x_n) = \int_0^1 \frac{\partial f_1}{\partial x_1}(t, x_2, \dots, x_n) dt,$$

and thus

$$\int_{x_1=1} f_1(x) dx_2 \wedge dx_3 \cdots dx_n - \int_{x_1=0} f_1(x) dx_2 \wedge dx_3 \cdots dx_n = \int_Q \frac{\partial f_1}{\partial x_1} dx_2 \wedge \cdots dx_n.$$

Theorem 58. Let M be a $k-$orientable (the $k-$cells don't cancel out) in $\mathbb{R}^n$ and ω be a $k-1$ form, then

$$\int_{\partial M} \omega = \int_M d\omega.$$

Proof. This is not right, Luis is smoking something. For each $k - cell$ that makes up the manifold, we have that

$$\int_{\partial \varphi} \omega = \int_{\partial \varphi} \varphi^*(\omega)$$

and

$$\int_M d\omega = \int_Q \varphi^* d\omega = \int_Q d(\varphi^*\omega)$$

$\square$

1.26 Fri, Dec 6: Translating Stokes, Hairy Ball Theorem, and Brower's Theorem

r/mildlyinteresting

1.27 Thu, Dec 13: Final

This final was ok. If only Luis had graded mine correctly!

Write explicit examples of the following. One is not possible. No justification is required.

(a) A metric space that is not complete.

$\mathbb{R} \setminus \{0\}$

(b) A finite connected set with more than one point.

Impossible

(c) A compact subset $C([0,1])$ that has infinitely many elements.

$\mathscr{F} = \{f_n : [0,1] \to \mathbb{R} \ : \ f_n(x) = \frac{1}{n}\sin(x)\}$, $\boxed{\mathscr{F}}$

(d) An ODE of the form $x'(t) = f(x(t))$ with $x(0) = x_0$and $f : \mathbb{R} \to \mathbb{R}$ a C^1 function that has a solution in some interval $t \in (-a, b)$ but not on the full real line $t \in \mathbb{R}$.

$x'(t) = (x(t))^2$

(e) A 1−form ω in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $d\omega = 0$. However, there is no 0−form ω_1 such that $d\omega_1 = \omega$.

$r = \sqrt{x^2 + y^2}$, $\omega = \frac{-y}{r^2}dx + \frac{x}{r^2}dy$

Proposition 17. Let K be a compact set and $f : K \to K$ continuous. Assume that $d(f(x), f(y)) < d(x,y)$ for all $x, y \in K$. Prove that f has a fixed point in K. Show an example in which the conclusion fails if the set K is closed but not necessarily compact.

Proposition 18. Suppose that a sequence of continuous functions $f_n : [a,b] \to \mathbb{R}$ converges monotonically down to zero. Prove that the convergence is uniform.

Proposition 19. We say that two curves γ_0 and γ_1 with the same endpoints are homotopoic in some open set $\Omega \subset \mathbb{R}^n$ if there exists a 2-cell $\varphi : [0,1]^2 \to \mathbb{R}^n$ such that

(a) For all $s \in [0,1], \gamma_0(0) = \gamma_1(0) = \varphi(0, s)$.

(b) For all $s \in [0,1], \gamma_0(0) = \gamma_1(0) = \varphi(1, s)$.

(c) For all $t \in [0,1], \gamma_0(t) = \varphi(t, 0) = \gamma_1(t) = \varphi(t, 1)$.

Assume that ω is a 1-form in Ω such that $d = 0$ and γ_1 is homotpoic to γ_2. Prove that

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$$

Disclaimer: Some solutions might be wrong

1.28 PSET 1

PSET 1: Problem 1

Example 1.19. Let $\mathbf{x} = A|B$, $\mathbf{x}' = A'|B'$ be cuts in $\mathbb{Q}$. We defined

$$\mathbf{x} + \mathbf{x}' = (A + A')|(\mathbb{Q}(A + A')).$$

(a)

Example 1.20. Show that although $B + B'$ is disjoint from $A + A'$, it may happen in degenerate cases that $\mathbb{Q} \neq (A + A') \cup (B + B')$

Proof. Suppose $A|B = \mathbf{x}$, where $\mathbf{x}$ is irrational and let $(-\mathbf{x}) = A'|B'$. Specifically, suppose $A|B = \{r \in \mathbb{Q}|r \leq 0\} \cup \{r \in \mathbb{Q}|r^2 < 2\}|\{r \in \mathbb{Q}|r^2 \geq 2\}$. Then

$$A'|B' = \{r \in \mathbb{Q}| - r \notin A \text{ but } -r \text{ is not a first point of } \mathbb{Q} \setminus A\}|(\mathbb{Q}A').$$

We claim that $B + B' \not\ni 0$. Note that since $B' = \mathbb{Q}A'$, then $B' = \{r \in \mathbb{Q}|r \geq 0\} \cup \{r \in \mathbb{Q}|r^2 \leq 2\}$. Let $b' \in \{r \in \mathbb{Q}|r^2 > 0, \}$ then it is evident that if $b \in B$, then $b + b' > 0$. Suppose $b' \in \{r \in \mathbb{Q}|r^2 \leq 2\}$, then since $\sqrt{2} \notin \mathbb{Q}$, we have that $(b')^2 < 2$. Similarly, $b^2 > 2$. Therefore,

$$b^2 - (b')^2 > 0, \implies |b| > |b'|.$$

Thus, $b + b' > 0$ for any $b, b' \in (B + B')$. Since $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$ (look at problem 2), then $(A + A') = \{r \in \mathbb{Q}|r < 0\}$, and so $0 \notin (A + A')$. Thus, $(A + A') \cup (B + B') = \mathbb{Q}0$. ☐

(b)

Example 1.21. Infer that the definition of $x + x'$ as $(A + A')|(B + B')$ would not be correct.

Proof. By part (a), there exists counterexamples that show that $(A+A') \cup (B+B') \neq \mathbb{Q}$, contradicting the first part of the definition of a Dedekind cut. ☐

(c)

Example 1.22. Why did we not define $x \cdot x' = (A \cdot A')|\text{rest of } \mathbb{Q}$?

Proof. Take $(A \cdot A') = (a \cdot a'|a \in A, a' \in A')$. Let $A, A' \subset \{r \in \mathbb{Q}|r < 0\}$, then for all $\alpha \in (A \cdot A')$, $\alpha = a \cdot a' > 0$ because both a, a' are less than 0. Thus, if $F =$rest of $\mathbb{Q}$, then $0 \in F$, and so $0 < \alpha$, a contradiction to the definition of a cut! ☐

PSET 1: Problem 2

Example 1.23. Let $\mathbf{x}$ be a cut. Prove that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

Proof. Let $\mathbf{x} = A|B$. By definition,

$$(-\mathbf{x}) = A'|B' = \{-r \notin A \text{ but } -r \text{ is not a first point of } \mathbb{Q} \setminus A\}|(\mathbb{Q}B').$$

Suppose $\mathbf{x} + (-\mathbf{x}) = E|F$.

- Let $e \in E$, then $e = a + a'$ for some $a \in A$, $a' \in A'$. Thus, since $-a' \notin A$, then $-a' > a$, and so $0 > a + a' = e$. Thus, since $e \in \{r \in \mathbb{Q}|r < 0\}$, then $E \subseteq \{r \in \mathbb{Q}|r < 0\}$ and thus $E|F \leq \mathbf{0}$.

- Let $z \in \mathbf{0}$. We then note that $(\frac{-z}{10}) > 0$ and that there exists some $n \in \mathbb{Z}$ such that $(n)(\frac{-z}{10}) \in A$ and $(n+1)(\frac{-z}{10}) \notin A'$.[1] Since $(n+1)(\frac{-z}{10}) < (n+10)\frac{-z}{10}$, then the latter is not in A and thus $(n+10)(\frac{z}{10}) \in A'$. Consider that
$$A + A' \ni (n)(\frac{-z}{10}) + (n+10)(\frac{z}{10}) = \frac{-nz}{10} + \frac{nz}{10} + z = z.$$
Thus, $z \in A + A'$ and so $\mathbf{0} \leq E|F$.

- Alternative for the second part: Let $x = \sup(A|B)$ and $-x = \sup(A'|B')$. Note that since $-(-x) = \sup -(A'|B') = \sup -(-(A|B)) = \sup(A|B) = x$. Thus $x + (-x) = 0$.

  Let $z \in \{r \in \mathbb{Q}|r < 0\}$. Evidently, $z < 0$. We want to show there exists some $e = a + a'$ such that $z < a + a' < 0$. To do this, we note that for any $\epsilon > 0$, there exist $a \in A$ and $a' \in A'$ such that $-x - \frac{\epsilon}{2} \leq a' < -x$ and $x - \frac{\epsilon}{2} \leq a < x$, and thus
$$-\epsilon \leq e < 0.$$
  Since ϵ can be arbitrarily smaller than $|z|$, we are done. Thus, $z \in E$ for all z and so $\mathbf{0} \leq E|F$.

$\square$

REFLECTIONS:

[1] This is a corollary of the Archimidean property and quite annoying to prove. Email me if you need this proof or look at my github for corollary 6.14: https://github.com/agustinestevah/Calculus-IBL-Scripts/blob/main/Script%207.tex

PSET 1: Problem 3

Example 1.24. A multiplicative inverse of a nonzero cut $\mathbf{x} = A|B$ is a cut $C|D$ such that $\mathbf{x} \cdot \mathbf{y} = \mathbf{1}$.

(a)

Example 1.25. If $\mathbf{x} > \mathbf{0}$, what are $C|D$?

Proof.

$$C = \{r \in \mathbb{Q}\ ;\ r \le 0\} \cup \{r \in \mathbb{Q}\ ;\ \frac{1}{r} \notin A, \frac{1}{r} \neq \inf B\}$$

and

$$D = \mathbb{Q} \setminus C.$$

First we must prove that $C|D$ is a cut:

(i) $0 \in C$ and thus $C \neq \emptyset$. Let $a \in A$ and $\frac{1}{n} < a$. Let $a \in A$ and $\frac{1}{n} < a$, then $n \notin C$ since if it were, then $\frac{1}{n} \notin A$, which is a contradiction to the fact that $\frac{1}{n} < a$. Thus, $C \neq \mathbb{Q}$ and so $D \neq \emptyset$. By definition, $\mathbb{Q} = C \sqcup D$.

(ii) Suppose $c \in C$ and $d \in D$. Suppose that $d \le c$. Then $a < \frac{1}{c} \le \frac{1}{d}$, where $a \in A$. Thus, $\frac{1}{d} \notin A$ and so $d \in C$, which is a contradiction to the fact that C and D are disjoint.

(iii) Suppose $c \in C$. Then $a < \frac{1}{c}$ for all $a \in A$. Let $b = \inf B$, then $b < \frac{1}{c} - \frac{1}{n} < \frac{1}{c}$ for large n, implying that $\frac{1}{c} - \frac{1}{n} \notin A$. Thus, $C \ni \frac{1}{\frac{1}{c} - \frac{1}{n}} = c \cdot \frac{n}{n-c}$, which for large n is greater than c. Thus, C has no maximal point.

Suppose $\mathbf{x} \cdot \mathbf{y} = E|F$.

- Let $e \in E$, then $e \in \{r \in \mathbb{Q}|r \le 0\} \cup \{ac|a \in A, c \in C, a, c > 0\}$. If e is in the first set, then evidently it is in $\{r \in \mathbb{Q}|r < 1\}$. If e is in the latter set, then $e = ac$ where $a \in A$ and $c \in C$. Since $\frac{1}{c} \notin A$, then $\frac{1}{c} > a$, and thus $e = ac < 1$. Therefore, $E \subseteq \{r \in \mathbb{Q}|r < 1\}$ and so $E|F \le \mathbf{1}$.
- Let $z \in \mathbf{1}$. Then $z < 1$.
  - If $z \le 0$, then just take $a \in A$ to be 0, then $a \cdot a' = 0$ and so
  $$z \le a \cdot a' \implies z \in E \implies \mathbf{1} \le E|F.$$
  - If $0 < z < 1$, then choose $a \in A$ such that $0 < a$. Let $q \in \mathbb{Q}$ such that $0 < q < \frac{a}{n}$, where $n \in \mathbb{N}$. By the same Archimidean logic as the above problem, there exists some $m \in \mathbb{Z}$ such $m > n$ and that $mq \in A$ but $(m+1)q \notin A$. We wish to show that $\frac{z}{mq} \in C$. Since $z < 1$, then if we choose n large, m must also be large and thus $z < 1 - \frac{1}{m+1}$. Thus,
  $$\frac{z}{mq} < \frac{1}{mq}(1 - \frac{1}{m+1}) = \frac{1}{(m+1)q} \in C.$$
  Therefore,
  $$A \cdot C \ni mq \cdot \frac{z}{mq} = z.$$
  Thus $z \in A \cdot C$ and thus $\mathbf{1} \le E|F$.

□

(b)

Example 1.26. If $x < \mathbf{0}$, what are $C|D$?

Proof.
$$C = \{r \in \mathbb{Q} | -\frac{1}{r} \in -A, \frac{1}{r} \neq \inf B\}$$
Note that
$$-C = \{r \in \mathbb{Q} | r \leq 0\} \cup \{r \in \mathbb{Q} | r = -b, b \in B, b \neq \inf D\}.$$
It takes little convincing and a few set manipulations to show this is equivalent to
$$-C = \{r \in \mathbb{Q} | r \leq 0\} \cup \{r \in \mathbb{Q} | \frac{1}{r} \notin -A, \frac{1}{r} \neq \inf -B\},$$
where $-B = \mathbb{Q} - A$. This means that by definition, $-\mathbf{y} = -C|(\mathbb{Q} - C)$ is the multiplicative inverse to $-\mathbf{x} = -A|(\mathbb{Q} - A)$. Thus, by part 1, $\mathbf{x} \otimes \mathbf{y} = (-\mathbf{x}) \otimes (-\mathbf{y}) = \mathbf{1}$. □

(c)

Example 1.27. Prove that x uniquely determines y.

Proof. Suppose $x \cdot y = \mathbf{1}$ and $x \cdot y' = \mathbf{1}$. Thus, $x \cdot y = x \cdot y'$ and $y \cdot x \cdot y = y \cdot x \cdot y'$. By associativity, $\mathbf{1} \cdot y = \mathbf{1} \cdot y'$. It suffices to show that $\mathbf{1} \cdot (A|B) = A|B$ for any cut.

Example 1.28. Let $\mathbf{x} = A|B$.

(i) If $\mathbf{x} > \mathbf{0}$, then $\mathbf{x} \otimes \mathbf{1} = \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, x \in \mathbf{1}, a > 0, x > 0\}$.:
  (a) Proving that $\mathbf{x} \subset \mathbf{x} \otimes \mathbf{1}$:
    (a) For all $a \in A$ where $a \leq 0$, we evidently have that $a \in \mathbf{x} \otimes \mathbf{1}$.
    (b) For all $a \in A$ where $0 < a$, there exists some $a < a'$ such that $a' \in A$ and $a' \neq 0$. Therefore, $a \cdot a'^{-1} < a' \cdot a'^{-1}$. Thus $\frac{a}{a'} < 1$, and so $\frac{a}{a'} \in \mathbf{1}$. Thus, because $\frac{a}{a'} \cdot a' = a$, then for all $a \in A$ where $a > 0$, $a \in \mathbf{x} \otimes \mathbf{1}$.

    Thus, $\mathbf{x} \subset \mathbf{x} \otimes \mathbf{1}$
  (b) Proving that $\mathbf{x} \otimes \mathbf{1} \subset A$. Let $a \in \mathbf{A}$ and $x \in \{r \in \mathbb{Q} | r < 1\}$.
    (a) If $a \cdot x \geq 0$, then since $x < 1$, we have $ax \leq a$. Thus, for all $ax \in \mathbf{x} \otimes \mathbf{1}$, where $ax \geq 0$, $ax \leq a$. Thus, $ax \in A$.
    (b) If $a \cdot x < 0$, then since $a \cdot x \in \mathbf{0}$, and then $\mathbf{0} < A$, then $\mathbf{0} \subset A$, so then $ax \in A$ for all $ax < 0$.

    Thus, $\mathbf{x} \otimes \mathbf{1} \subset A$

  Therefore, if $\mathbf{x} > \mathbf{0}$, then $\mathbf{x} \otimes \mathbf{1} = A$

(ii) If $\mathbf{x} < \mathbf{0}$, then since $-\mathbf{x} > \mathbf{0}$, then $-\mathbf{x} \otimes \mathbf{1} = -\mathbf{x}$. Therefore, $\mathbf{x} \otimes \mathbf{1} = -(-\mathbf{x}) = \mathbf{x}$.

□

PSET 1: Problem 4

Example 1.29. Let $b = \sup S$, where $S \subset \mathbb{R}$ is nonempty and bounded.

(a)

Example 1.30. Given $\epsilon > 0$, show there exists an $s \in S$ with

$$b - \epsilon \leq s \leq b.$$

Proof. Suppose not. Then since S is nonempty and b is an upper bound, all $s \in S$ are such that $s \leq b - \epsilon \leq b$, implying that $b - \epsilon$ is an upper bound. A contradiction to the fact that b is the least upper bound! $\square$

(b)

Example 1.31. Can $s \in S$ always be found so that $b - \epsilon < s < b$?

Proof. No. Consider the case when $S = \{b\}$. Evidently, $b = \sup S$. However, there does not exist some $s \in S$ such that $s < b$. $\square$

(c)

Example 1.32. If $\mathbf{x} = A|B$ is a cut in $\mathbb{Q}$, show that $x = \sup A$.

Proof. We need to show that $\mathbf{x}$ is greater than or equal to all $\mathbf{a} = A_i|B_i \in A$, where $\mathbf{a} = \{r \in \mathbb{Q}|r < a\}|\{r \in \mathbb{Q}|r \geq a\}$ denotes the rational cuts in A and $a \in \mathbb{Q}$ denotes the rational being cut. Let $\mathbf{a} \in A$, then for all $\alpha \in A_i$, $\alpha < a$, and thus $\alpha' \in A$ (I cannot bold α in latex, so α' is the rational cut for α.) This shows that $A_i \subseteq A$ for all i and thus $\mathbf{a} \leq \mathbf{x}$ for all $\mathbf{a} \in A$.

Now we need to show that there does not exist some $\mathbf{y} = C|D$ such that $\mathbf{y} < \mathbf{x}$ and $A_i \subseteq C$ for all A_i. Suppose there does exist such $\mathbf{y}$, then since $\mathbf{y} < \mathbf{x}$, $C \subset A$, and thus there exists some $a \in \mathbb{Q}$ such that $a \notin C$ but $a \in A$. Thus, let $\mathbf{a} = \{r \in \mathbb{Q}|r < a\}$. Since $a \notin C$, then for all $q \in C$, $q < a$, and so $C \subset \{r \in \mathbb{Q}|r < a\}$, and thus $\mathbf{y} < \mathbf{a}$, contradicting the fact that $\mathbf{y}$ is an upper bound of A. $\square$

PSET 1: Problem 5

Example 1.33. Prove $\sqrt{2} \in \mathbb{R}$ by showing that $x \cdot x = 2$, where $x = A|B$ is a cut in $\mathbb{Q}$ where $A = \{q \in \mathbb{Q} | r \leq 0 \text{ or } r^2 < 2\}$.

(a) Lemma 1: Given $y \in \mathbb{R}$, $n \in \mathbb{N}$, and $\epsilon > 0$, show that for some $\delta > 0$, if $u \in \mathbb{R}$ and $|u - y| < \delta$, then $|u^n - y^n| < \epsilon$.

Proof. Consider that

$$u^n - y^n = (u - y) \sum_{i=0}^{n-1} u^{n-1-i} y^i.$$

Let $\epsilon > 0$ and proceed by induction:

(i) For the $k = 1$ case, take $\delta_1 = \epsilon$.

(ii) For the $k = 2$ case, take $\delta_2 = \min\{1, \frac{\epsilon}{2|y|+1}\}$ Thus, if $u \in \mathbb{R}$ with $|u - y| < \delta$, then

$$|u^2 - y^2| = |u - y||u + y| < \delta(|u + y|) < \delta(2|y| + 1) < \epsilon.$$

(iii) For $k = n$, assume that there exists some $u \in \mathbb{R}$ with $|u - y| < \delta_n$ such that $|u^n - y^n| < \epsilon$.

(iv) For $k = n + 1$, take $\delta = \min\{\delta_1, \ldots, \delta_n, \frac{\epsilon}{2|y^n|+1}\}$ Thus, if $u \in \mathbb{R}$ with $|u - y| < \delta$, then if we take the ϵ above to be 1 and $\frac{\epsilon}{2}$, then

$$\begin{aligned} |u^{n+1} - y^{n+1}| &= |u - y| \left| \sum_{i=0}^{n} u^{n-1-i} y^i \right| \\ &\leq |u - y|((|u^n + y^n|) + \frac{|u^n - y^n|}{|u - y|}) \\ &< |u - y|(2|y^n| + 1) + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

□

(b)

Example 1.34. Lemma 2: Given $x > 0$ and $n \in \mathbb{N}$, prove that there is a unique $y > 0$ such that $y^n = x$. That is, $y = \sqrt[n]{x}$ exists and is unique.

Proof. Let $S := \{s \in \mathbb{R} | s^n \leq x\}$. Note that since $x > 0$ and $0^n = 0$, then $0 \in S$. Moreover, $\lceil \sqrt[n]{x} \rceil$ is an upper bound for S. Let $y = \sup S$.

$$y = \sup S.$$

(i) Suppose $y^n < x$. By Lemma 1, there exists some $u \in \mathbb{R}$ and $\delta > 0$ such that if $|y - u| < \delta$, then $|y^n - u^n| < x - y^n$. Note that this implies that $u^n < x$. However, consider that for k large, $\frac{1}{k} < \delta$, and thus if $u = y + \frac{1}{k}$, then $u \in \mathbb{R}$ and

$$|y - u| = |\frac{1}{k}| < \delta \implies u^n < x.$$

Thus, $u \in S$ but $y < u$, a contradiction!

(ii) Suppose $y^n > x$. By Lemma 1, there exists some $u \in \mathbb{R}$ and $\delta > 0$ such that if $|y-u| < \delta$, then $|y^n - u^n| < y^n - x$. Note that this implies that $x < u^n$ However, consider that for k large, $\frac{1}{k} < \delta$, and thus if $u = y - \frac{1}{k}$, then $u \in \mathbb{R}$. Moreover, since $u < y$, then $u^n < y^n$ and so $u \in S$. Note that

$$|y-u| = |\frac{1}{k}| < \delta \implies x < u^n.$$

Thus, $u \notin S$. A contradiction!

Thus, $y^n = x$. Note that suprema are unique as a corollary to problem 4 and thus $y = \sup S$ is unique. □

(c) Solution to the problem:

Proof. By Lemma 2 and problem 4, it will suffice to show that $\sqrt{2} = \sup\{s \in \mathbb{R} | s^2 \leq 2\} = \sup A$, where $A = \{r \in \mathbb{Q} | \{q \in \mathbb{Q} | r \leq 0 \text{ or } r^2 < 2\}\}$. Evidently, it suffices to show that $\sup\{s \in \mathbb{R}|s^2 \leq 2\} = \sup\{r \in \mathbb{Q}|r^2 < 2\}$. Suppose not, then since $\sqrt{2}$ is an upper bound for the latter, it must be the case that $\sup\{r \in \mathbb{Q}|r^2 < 2\} < \sup\{s \in \mathbb{R}|s^2 \leq 2\}$. Thus, there exists some $z \in \mathbb{R}$ with $z < y$ and $z^2 \leq 2$ such for all $r \in \mathbb{Q}$ with $r^2 < 2$, we have that $r < z$. However, by Lemma 1, if there exists some $\delta > 0$ and $q \in \mathbb{Q}$ such that if $|z - q| < \delta$, then $|z^2 - q^2| < 2 - z^2$. Since there exists some $q' \in \mathbb{Q}$ such that $q' > z$ and $|z - q'| < \delta$, then

$$|z^2 - (q')^2| < 2 - z^2 \implies (q')^2 < 2.$$

Thus, $q' \in \{r \in \mathbb{Q}|r^2 < 2\}$ but $q' > z$, a contradiction! □

PSET 1: Problem 6

Example 1.35. Formulate the definition of the greatest lower bound of a set of real numbers. State a greatest lower bound property of $\mathbb{R}$ and show it is equivalent to the least upper bound property of $\mathbb{R}$.

Proof. We say that *the greatest lower bound, or infemum*, of a nonempty set $S \subset \mathbb{R}$ is $s = \inf S$ if it satisfies the following conditons:

- s is a lower bound: for all $x \in S$, $s \leq x$.
- For all lower bounds b, $s \leq b$.

. We can state a greatest lower bound property of $\mathbb{R}$: If S is a non-empty subset of $\mathbb{R}$ that is bounded below, then in $\mathbb{R}$ there exists a greatest lower bound for S.

Proof. :

- Suppose the l.u.b. property is met. We want to show that if $S \subset \mathbb{R}$ is bounded below and nonempty, then there exists some $s' = \inf S$. Suppose not. That is, if b is a lower bound of S, there exists some other lower bound of S greater than b. Let $\mathscr{B}$ be the set of lower bounds of S. Note that it is obviously nonempty. Let $s \in S$, then for any $b \in \mathscr{B}$, $b \leq s$ since b is a lower bound of S. Thus, $\mathscr{B}$ is bounded above. By the l.u.b property, there exists some $\beta = \sup \mathscr{B}$. Thus, for all $b \in \mathscr{B}$, $b \leq \beta$. Suppose that β is not a lower bound of S, then there exists some $s \in S$ such that $s < \beta$. However, since $\beta = \sup \mathscr{B}$, by problem 4 one can make ϵ small enough such that $s < b \leq \beta$, implying that b is not a lower bound. Contradiction! Thus, $\beta = \sup \mathscr{B}$. Contradiction! Thus, there exists some g.l.b of S.
- The equivalence for the other way is similar to the above.

□

□

PSET 1: Problem 7

Example 1.36. Prove that limits are unique, i.e., if (a_n) is a sequence of real numbers that converges to a real number b and also converges to a real number b', then $b = b'$.

Proof. Suppose not. That is, $\lim_{n\to\infty} a_n = b$ and $\lim_{n\to\infty} a_n = b'$ with $b \neq b'$. Let $\epsilon > 0$. For the latter, there exists some N_1 such that all a_n with $n \geq N_1$ are $\frac{\epsilon}{2}$ close to b. For the latter, there exists some N_2 such that all a_n with $n \geq N_2$ are $\frac{\epsilon}{2}$ close to b'. Take $N = \max\{N_1, N_2\}$. if $n > N$, then $|a_n - b| < \frac{\epsilon}{2}$ and $|a_n - b'| < \frac{\epsilon}{2}$ imply by triangle inequality that $|b - b'| \leq |b - a_n| + |a_n - b'| < \epsilon$. Since ϵ is arbitrarily small, we have that $|b - b'| = 0$. ☐

PSET 1: Problem 8

Example 1.37. Prove that real numbers correspond bijectively to decimal expansions not terminating in an infinite strings of 9's as follows.

$$x \to N.x_1x_2\ldots,$$

where $N = x_0 = \lfloor x \rfloor$ and

$$x_n = \lfloor 10^n \left(x - \sum_{i=0}^{n-1} \frac{x_i}{10^i} \right) \rfloor$$

for $n > 0$ For example, if $x = 6.45$, then

$$x_0 = N = \lfloor x \rfloor = 6$$
$$x_1 = \lfloor 10^1 \left(x - \sum_{i=0}^{n-1} \frac{x_i}{10^i} \right) \rfloor = \lfloor 10(x - (x_0)) \rfloor = 4$$
$$x_2 = \lfloor 10^2 \left(x - \sum_{i=0}^{n-1} \frac{x_i}{10^i} \right) \rfloor = \lfloor 100(x - (x_0 + \frac{x_1}{10})) \rfloor = 5$$

(a)

Example 1.38. Show that x_k is a digit between 0 and 9.

Proof. We proceed by inducting:

(i) For $n = 1$, note that since $x_0 \leq x < x_0 + 1$, then $0 \leq x - x_0 < 1$, and so $0 \leq 10(x - x_0) < 10$, producing the result when taking the floor of $10(x - x_0)$.

(ii) Assume that for $n = k - 1$, we have that

$$x_{k-1} = \lfloor 10^{k-1} \left(x - \sum_{i=0}^{k-2} \frac{x_i}{10^i} \right) \rfloor$$

is a digit between 0 and 9. Let $0 \leq \epsilon < 1$ denote the remainder from taking the floor:

$$x_{k-1} = 10^{k-1} \left(x - \sum_{i=0}^{k-2} \frac{x_i}{10^i} \right) + \epsilon_{k-1}$$

(iii) Consider that

$$\begin{aligned}
x_k &= \lfloor 10^k \left(x - \sum_{i=0}^{k-1} \frac{x_i}{10^i} \right) \rfloor \\
&= \lfloor 10(10^{k-1} \left(x - (\sum_{i=0}^{k-2} \frac{x_i}{10^i} + \frac{x_{k-1}}{10^{k-1}}) \right))\rfloor \\
&= \lfloor 10(10^{k-1} \left(x - \sum_{i=0}^{k-2} \frac{x_i}{10^i} - \frac{x_{k-1}}{10^{k-1}}) \right))\rfloor \\
&= \lfloor 10(10^{k-1} \left(x - \sum_{i=0}^{k-2} \frac{x_i}{10^i} \right) - x_{k-1}))\rfloor \\
&= \lfloor 10(x_{k-1} + \epsilon_{k-1} - x_{k-1}) \\
&= \lfloor 10(\epsilon_{k-1}) \rfloor
\end{aligned}$$

Which is clearly a digit between 0 and 9.

□

(b)

Example 1.39. Show that for each k there exists an $\ell \geq k$ such that $x_\ell \neq 9$.

Proof. Assume that there exists some k such that for all $\ell > k$, we have that $x_\ell = 9$. Thus, we can say that

$$x = \sum_{i=0}^{k} \frac{x_i}{10^i} + \sum_{i=k+1}^{\infty} \frac{x_i}{10^i}.$$

Thus, for $k = 0$, we have that every

$$x = x_0 + \sum_{i=k+1}^{\infty} \frac{9}{10^i} = x_0 + \frac{1}{10^0} = x_0 + 1,$$

which is a contradiction. For a more general $k > 0$, we have that

$$x = \sum_{i=0}^{k} \frac{x_i}{10^i} + \sum_{i=k+1}^{\infty} \frac{9}{10^i} = \sum_{i=0}^{k} \frac{x_i}{10^i} + \frac{1}{10^k} = \sum_{i=0}^{k-1} \frac{x_i}{10^i} + \frac{x_k + 1}{10^k}$$

Rearranging:

$$x_k = 10^k \left(x - \sum_{i=0}^{k-1} \frac{x_i}{10^i} \right) - 1$$

Which is a contradiction to the definition given in (iii) above! □

(c)

Example 1.40. Conversely, show that for each such expansion $x_0.x_1.x_2\ldots$ not terminating in an infinite string of 9's, the set

$$X = \{x_0, x_0 + \frac{x_1}{10^1}, x_0 + \frac{x_1}{10^1} + \frac{x_2}{10^2}, \ldots\}$$

is bounded an its supremum is a real number x with decimal expansion $N.x_1.x_2\ldots$

Proof. Consider that either $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}$. That is, either there exists some k such that for every $\ell > k$, $x_\ell = 0$, or else $x_\ell \in \{n \in \mathbb{N}_0 | n \leq 9\}$ (with of the terminating $9'$s.)

(i) For the first case, consider that since each x_i for $i \geq 0$ is positive, we have

$$\sum_{i=0}^{n} \frac{x_i}{10^i} \leq \sum_{i=0}^{n+1} \frac{x_i}{10^i}. \tag{4}$$

However, since after some k, every $\ell > k$ has the property that $x_\ell = 0$, then the set is finite and thus contains its supremum. In fact we claim that

$$N.x_1.x_2\ldots x_k000\cdots = \sum_{i=1}^{k} \frac{x_i}{10^i} = \sup X.$$

Note that by (1), $\sum_{i=1}^{k} \frac{x_i}{10^i}$ is an upper bound. Since $\sum_{i=1}^{k} \frac{x_i}{10^i} \in X$, then $\sum_{i=1}^{k} \frac{x_i}{10^i} = N.x_1x_2\ldots x_k \cdots =$ $\sup X$.

(ii) For the second case, where $, x_1, x_2, \ldots$ are infinite sequence of integers between 0 and 9 not terminating in infinite 9's, consider that (1) still holds. Moreover, note that

$$N.x_1x_2\cdots = \sum_{i=0}^{\infty} \frac{x_i}{10^i}.$$

By (1), if $s \in X$, then $s \leq \sum_{i=0}^{\infty} \frac{x_i}{10^i}$ and thus $N.x_1x_2\ldots$ is an upper bound for X. We wish to find some $s = \sum_{i=0}^{k} \frac{x_i}{10^i} \in X$ such that $x_0.x_1x_2\cdots - \frac{1}{10^n} \leq s \leq N.x_1x_2\ldots$ for any $n \in \mathbb{N}$. [2]

(a) Suppose $x_n = 0$. Then it is easy to see that

$$\sum_{i=0}^{\infty} \frac{x_i}{10^i} - \frac{1}{10^n} = N.x_1x_2\ldots(x_{n-1}-1)x_n\cdots < N.x_1x_2\ldots$$

We claim that

$$N.x_0.x_1x_2\ldots(x_{n-1}-1) + \sum_{i=n}^{\infty} \frac{x_i}{10^i} \leq \sum_{i=0}^{n-1} \frac{x_i}{10^i}.$$

It suffices to show then that

$$\frac{(x_{n-1}-1)}{10^{n-1}} + \sum_{i=n}^{\infty} \frac{x_i}{10^i} \leq \frac{x_{n-1}}{10^{n-1}}.$$

Thus, it suffices to show that

$$\sum_{i=n}^{\infty} \frac{x_i}{10^i} \leq \frac{1}{10^{n-1}}.$$

Consider that because there exists some $x_\ell \neq 9$ where $\ell > n$, then

$$\sum_{i=n}^{\infty} \frac{x_i}{10^i} < \sum_{i=n}^{\infty} \frac{9}{10^i} \leq \frac{1}{10^{n-1}}.$$

Note that the last inequality is actually an equality[3]. Thus,

$$\sum_{i=0}^{\infty} \frac{x_i}{10^i} - \frac{1}{10^n} \leq \sum_{i=0}^{n-1} \frac{x_i}{10^i} \leq \sum_{i=0}^{\infty} \frac{x_i}{10^i}.$$

(b) If $x_n \neq 0$, then one can similarly show that

$$\sum_{i=0}^{\infty} \frac{x_i}{10^i} - \frac{1}{10^n} \leq \sum_{i=0}^{n} \frac{x_i}{10^i} \leq \sum_{i=0}^{\infty} \frac{x_i}{10^i}.$$

[2] This is because for any $\epsilon > 0$, one can use the Archimedean property and the fact that $n < 10^n$ to show that Problem 4 a holds and thus there does not exist an upper bound smaller than $N.x_1x_2\ldots$

[3] Here we are using the fact that

$$\sum_{i=1}^{\infty} \frac{9}{10^i} = 1.$$

Note that one can see this by:

$$\sum_{i=1}^{\infty} \frac{9}{10^i} = 9\sum_{i=1}^{\infty} (\frac{1}{10})^i = 9 \cdot \frac{1}{9}$$

$\square$

(d)

Example 1.41. For a general base K, $x \in \mathbb{R}$ has a decimal expansion $N.x_1x_2\dots$, where $N = x_0 = \lfloor x \rfloor$ and

$$x_n = \lfloor K^n \left(x - \sum_{i=0}^{n-1} \frac{x_i}{K^i} \right) \rfloor$$

for $n > 0$. Repeat (a-c) with a general base K.

Proof. We proceed by inducting:

(i) For $n = 1$, note that since $x_0 \le x < x_0 + 1$, then $0 \le x - x_0 < 1$, and so $0 \le K(x - x_0) < K$, producing the result when taking the floor of $K(x - x_0)$.

(ii) Assume that for $n = k - 1$, we have that

$$x_{k-1} = \lfloor K^{k-1} \left(x - \sum_{i=0}^{k-2} \frac{x_i}{K^i} \right) \rfloor$$

is a digit between 0 and 9. Let $0 \le \epsilon < 1$ denote the remainder from taking the floor:

$$x_{k-1} = K^{k-1} \left(x - \sum_{i=0}^{k-2} \frac{x_i}{K^i} \right) + \epsilon_{k-1}$$

(iii) Consider that

$$\begin{aligned}
x_k &= \lfloor K^k \left(x - \sum_{i=0}^{k-1} \frac{x_i}{K^i} \right) \rfloor \\
&= \lfloor K(K^{k-1} \left(x - (\sum_{i=0}^{k-2} \frac{x_i}{K^i} + \frac{x_{k-1}}{K^{k-1}}) \right)) \rfloor \\
&= \lfloor K(K^{k-1} \left(x - \sum_{i=0}^{k-2} \frac{x_i}{K^i} - \frac{x_{k-1}}{K^{k-1}}) \right)) \rfloor \\
&= \lfloor K(K^{k-1} \left(x - \sum_{i=0}^{k-2} \frac{x_i}{10^i} \right) - x_{k-1})) \rfloor \\
&= \lfloor K(x_{k-1} + \epsilon_{k-1} - x_{k-1}) \\
&= \lfloor K(\epsilon_{k-1}) \rfloor
\end{aligned}$$

Which is clearly a digit between 0 and $K - 1$.

$\square$

(e)

Example 1.42. Show that for each k there exists an $\ell \ge k$ such that $x_\ell \ne K - 1$.

Proof. Suppose not. Thus, there exists some k such that for all $\ell \ge k$, $\ell = K - 1$. Thus, $x \to N.x_1 \dots x_k(K-1)(K-1)\dots$, which is a contradiction to the fact that the decimal expansions do not terminate in infinite strings of $(K - 1)$'s. $\square$

Example 1.43. Conversely, show that for each such expansion $x_0.x_1.x_2\dots$ not terminating in an infinite string of $K-1$'s, the set

$$X = \{x_0, x_0 + \frac{x_1}{K^1}, x_0 + \frac{x_1}{K^1} + \frac{x_2}{K^2}, \dots\}$$

is bounded an its supremum is a real number x with decimal expansion $N.x_1.x_2\dots$

Proof. This proof is exactly the same as (c), just replace every 10 with a K (it will take up too much space to write it out). $\square$

PSET 1: Problem 9

Let $b(R)$ and $s(R)$ be the number of cubes in $\mathbb{R}^m$ which intersect with a ball and sphere of radius R, centered at the origin.

(i)

Example 1.44. Let $m = 2$ and calculate the following:

$$\lim_{R\to\infty} \frac{s(R)}{b(R)}, \qquad \lim_{R\to\infty} \frac{(s(R))^2}{b(R)}.$$

Proof. We will need to bound both $s(R)$ and $b(R)$. Let's begin with $s(R)$. Split the circle into 8 parts (subdivide the quadrants by the $y = x$ and $y = -x$ lines), it is easy to see that if we simply count the number of cubes in one octant, add 4 from the cubes on the diagonal, then this will suffice.

Consider the number of lines on $\mathbb{R}^2$ the circle passes through. We claim that when the circle passes through no lattice points, this number is $R - 1$. Thus, since each time the circle passes by a line, it means it just passed through a unique unit cube, then there are $R - 1$ unit cubes in the quadrant. To see this fact, consider that in the first octant, the circle passes

$$R - \lceil \frac{\sqrt{2}}{2} R \rceil$$

vertical lines, while it passes

$$\lfloor \frac{\sqrt{2}}{2} R \lfloor$$

horizontal lines. Thus, it passes

$$R - \lceil \frac{\sqrt{2}}{2} R \rceil + \lfloor \frac{\sqrt{2}}{2} R \lfloor = R - 1$$

lines in the first quadrant.[4]

Thus, there is a total of

$$s(R) = 8(R - 1) + 4 = 8R - 4$$

squares which intersect the edge of the circle.[5]

We shall just bound $b(R)$, which is easy:

$$\pi R^2 \leq b(R) \leq \pi R^2 + s(R).$$

Using the squeeze theorem:

$$\frac{8R - 4}{\pi R^2} \leq \frac{s(R)}{b(R)} \leq \frac{8R - 4}{\pi R^2 + 8R - 4} \implies 0 \leq \lim_{R\to\infty} \frac{s(R)}{b(R)} \leq 0.$$

Thus, $\lim_{R\to\infty} \frac{s(R)}{b(R)} = 0$. Moreover,

$$\frac{64R^2 - 64R + 16}{\pi R^2} \leq \frac{(s(R))^2}{b(R)} \leq \frac{64R^2 - 64R + 16}{\pi R^2 + 8R - 4} \implies \frac{64}{\pi} \leq \lim_{R\to\infty} \frac{(s(R))^2}{b(R)} \leq \frac{64}{\pi},$$

and thus $\lim_{R\to\infty} \frac{(s(R))^2}{b(R)} = \frac{64}{\pi}$. ∎

[4] The $\frac{\sqrt{2}}{2} R$ factor comes from the fact that the circle is limited to the first quadrant, and thus cannot cross the $\frac{\sqrt{2}}{2} R X \frac{\sqrt{2}}{2} R$ square formed around the origin.

[5] The lattice points would form a problem, but the squares are defined to be half open in the half that is most convinient to us (always pointing towards origin), and so it is fine to count each lattice point intersection as 2 squares (the one from which it came from, and the other intersecting), and so we still end with $R - 1$ squares.

(ii)

Example 1.45. Take $m \geq 3$, what exponent k makes the limit

$$\lim_{r\to\infty} \frac{(s(R))^k)}{b(R)}$$

interesting?

Proof. Consider that $s(R) \propto \{\text{Surface Area}\}$ for all $m \geq 2$. This is because the cubes are constrained to be along the surface of the sphere. Similarly, $b(R) \propto \{\text{Volunme}\}$ The volume of an m dimensional ball is $V(R) = V_m R^m$, and since surface area is the derivative of the volume, $A(R) = mV_m R^{m-1}$. Thus, the exponent that will make it interesting is when $s(R)$ and $b(R)$ have the same exponent, which is $k = \frac{m}{m-1}$. □

(iii)

Example 1.46. Let $c(R)$ be the number of integer unit cubes contained in the unit ball of radius R, centered at the origin. Calculate

$$\lim_{R\to\infty} \frac{c(R)}{b(R)}.$$

Proof. Evidently, $c(R) = b(R) - s(R)$. Thus,

$$\frac{b(R) - s(R)}{b(R)} \leq \frac{c(R)}{b(R)} \leq \frac{b(R)}{b(R)},$$

since $s(R)$ is of power $m-1$ and $b(R)$ is of power m, then

$$1 \leq \lim_{R\to\infty} \frac{c(R)}{b(R)} \leq 1,$$

and thus $\displaystyle\lim_{R\to\infty} \frac{c(R)}{b(R)} = 1.$ □

(iv)

Example 1.47. Shift the ball to a new, arbitrary center (not on the integer lattice) and recalculate the limits.

Proof. All the limits remain the same. This is clear in part (ii) and (iii), where the fact that it was centered at the origin was not used. The lower and upper bounds for $b(R)$ in part (i) will also remain the same. Thus, the only thing to show is that $s(R)$ remains linear with the linear coefficient remaining as 8 as $R \to \infty$.
$s(R)$ remains linear because it still needs to be proportional to the surface area of the sphere. Next, we need to show that

$$\lim_{R\to\infty} \frac{s(R)}{8R} = 1.$$

To do this, we can recreate our argument from part a, but this time by accounting for shifts in the x and y directions. However, note that the number of vertical lines the circle will intersect in the first quadrant will be at at most a change of one line. The same is true for the horizontal lines. Thus, in the limit, $s(R)$ changes only by a small factor, and thus $\lim_{R\to\infty} \frac{s(R)}{8R} = 1$. □

1.29 PSET 2

PSET 2: Problem 1

Example 1.48. For $p, q \in S^1$, the unit circle in the plane, let

$$d_a(p,q) = \min\{|\measuredangle(p) - \measuredangle(q)|, 2\pi - |\measuredangle(p) - \measuredangle(q)|\},$$

where $\measuredangle(z) \in (0, 2\pi]$ is the angle z makes with the positive x axis.

Proof. :

(a) Let $p = q$, then

$$d_a(p,q) = \min\{|\measuredangle(p) - \measuredangle(q)|, 2\pi - |\measuredangle(p) - \measuredangle(q)|\} = \min\{0, 2\pi\} = 0.$$

Let $p \neq q$, then since we are dealing with absolute value, it suffices to show $|\measuredangle(p) - \measuredangle(q)| \leq 2\pi$. However, by definition, the furthest apart the angles can be is if (without loss of generality), $\measuredangle(q) = 2\pi$ and $\measuredangle(p) = \epsilon > 0$ for small ϵ, and thus $|\measuredangle(p) - \measuredangle(q)| < 2\pi$.

(b) We have that by definition of absolute value,

$$\begin{aligned} d_a(p,q) &= \min\{|\measuredangle(p) - \measuredangle(q)|, 2\pi - |\measuredangle(p) - \measuredangle(q)|\} \\ &= \min\{|\measuredangle(q) - \measuredangle(p)|, 2\pi - |\measuredangle(q) - \measuredangle(p)|\} \\ &= d_a(q,p) \end{aligned}$$

(c) Let $p, q, r \in S$. We wish to bound $d(p,r)$. This problem reduces to a few cases, but we will work, without loss of generality by shifting p to be at $(1,0)$. and r to be above or on the x axis. Thus, $d(p,r) = \measuredangle(r)$ Thus, we only have two cases (note that for this problem it makes the arithmetic simpler if we let $\measuredangle(z) \in [0, 2\pi)$).

(i) If q is above or on the x axis, then evidently: $d(p,q) = \measuredangle(q)$ Note also that since p is above the axis as well, $d(q,r) = |\measuredangle(q) - \measuredangle(r).|$. Thus, we have that

$$\begin{aligned} d(p,r) &= \measuredangle(r) \\ &= \measuredangle(r) - \measuredangle(q) + \measuredangle(q) \\ &\leq |\measuredangle(r) - \measuredangle(q)| + \measuredangle(q) \\ &= d(q,r) + \measuredangle(q,p). \end{aligned}$$

(ii) If q is below the x axis, then we have that $d(p,q) = 2\pi - \measuredangle(q)$. Moreover, we have that either $d(q,r) = \measuredangle(q) - \measuredangle(r)$, or $d(q,r) = 2\pi - (\measuredangle(q) - \measuredangle(r)) = 2\pi - \measuredangle(q) + \measuredangle(r)$

(a) If the former, then since r is above the $x-$axis:

$$\begin{aligned} d(p,r) &= \measuredangle(r) \\ &\leq 2\pi - \measuredangle(r) \\ &= 2\pi - \measuredangle(r) + \measuredangle(q) - \measuredangle(q) \\ &= 2\pi - \measuredangle(q) + \measuredangle(q) - \measuredangle(r) \\ &= d(p,q) + \measuredangle(q,r). \end{aligned}$$

(b) If the latter, then a similar argument can be applied. However, one could also bootleg the first case by noting that we can compare $d(q,r)$ by moving q to $(1,0)$ and repeating the arguments above.

$\square$

PSET 2: Problem 2

Example 1.49. For $p, q \in [0, \frac{\pi}{2})$, let
$$d_s(p,q) = \sin|p-q|.$$
Use your calculus talent to decide whether d_s is a metric.

Proof. :

(a) If $p = q$, then by calculus knowledge, $d_s(p,q) = \sin(0) = 0$. If $p \neq q$, then since $\sin(x)$ is increasing in the interval, then $d_s(p,q) \geq d_s(p,p) = 0$.

(b) $d_s(p,q) = d_s(q,p)$ because subtraction is commutative inside the absolute value.

(c) Let $p, q, r \in (0, \frac{\pi}{2}$, then since sin is increasing in the interval,

$$\begin{aligned} d_s(p,r) &= \sin|p-r| \\ &= \sin|p-q+q-r| \\ &= \sin|p-q|\cos|q-r| + \cos|p-q|\sin|q-r| \\ &\leq \sin|p-q| + \sin|q-r| \\ &= d_s(p,q) + d_s(q,r) \end{aligned}$$

Where the inequality holds because for $x \in [0, \frac{\pi}{2})$, we have that $\cos(x) \leq 1$.

$\square$

PSET 2: Problem 3

Example 1.50. Prove that every convergent sequence (p_n) in a metric space M is bounded.

Proof. Let $p_n \to p$. Let $\epsilon = 1$, then there exists some $N \in \mathbb{N}$ such that if $n \geq N$, we have that $d(p_n, p) < 1$. Thus, we have that for all $n \geq N$, p_n are bounded by the ball $B_1(p)$. It will suffice to show that for $n = 1, 2, \ldots, N-1$, p_n is also bounded. Consider that if we let $r = 1 + \max\{d(p_1, p), d(p_2, p), \ldots, d(p_{n-1}, p)\}$, then we have that for all $n \in \mathbb{N}$, $p_n \in M_r p$. $\square$

PSET 2: Problem 4

Example 1.51. A sequence (x_n) in $\mathbb{R}$ *increases* if $n < m$ implies $x_n \leq x_m$. It *strictly increases* if $x_n < x_m$. A sequence is *monotone* if it increases or it decreases. Prove that every sequence in $\mathbb{R}$ which is monotone and bounded converges in $\mathbb{R}$.

Proof. Suppose x_n is increasing (the proof for when it is decreasing is similar). Let $\{x_n\}$ be the set of the values of (x_n). Evidently, since (x_n) is bounded then $\{x_n\}$ is also bounded. Moreover, $x_1 \in \{x_n\} \neq \emptyset$. Since we are in $\mathbb{R}$, then by the least upper bound property, $s = \sup\{x_n\}$ exists. We claim that $(x_n) \to s$. Assume not, that is, there exists some $\epsilon > 0$ such that if $n \in \mathbb{N}$, $d(x_n, s) \geq \epsilon$. Thus, we have that for all $n \in \mathbb{N}$, $s - \epsilon \geq x_n$, implying that s is not the least upper bound, a contradiction! $\square$

PSET 2: Problem 5

Example 1.52. Let (x_n) be a sequence in $\mathbb{R}$.

(a)

Example 1.53. Prove that (x_n) has a monotone subsequence.

Proof. Let

$$S := \{x_{n_k} |\, \forall \ell > n_k,\, x_\ell \leq x_{n_k}\}$$

Either S is finite or it is infinite.

(i) Suppose S is finite, then we claim that (x_n) has a strictly increasing subsequence. Take the greatest k such that $x_{n_k} \in S$. Note that it must be the case that $x_{n_k+1} \leq x_{n_k}$, as otherwise, we would have that $x_{n_k} \notin S$. Define

$$S_1 := \{x_{n_i} | \forall i > n_k + 1,\, x_i > x_{n_k+1}\}.$$

Observe that $S_1 \neq \emptyset$, for if it were, then $x_{n_k+1} \in S$. Take the first i such that $x_{n_i} \in S_1$, then $n_k + 1 < n_i$ and $x_{n_k+1} < x_{n_i}$. Let

$$S_2 := \{x_{n_j} | \forall j > n_i,\, x_j > x_i.\}.$$

Similarly, $S_2 \neq \emptyset$ for if it were, then $x_i \in S$. Take the first j such that $x_{n_j} \in S_2$, then $n_i < n_j$ and $x_{n_k+1} < x_{n_i} < x_{n_j}$. Because we can continue this process infinitely, then we have built a strictly increasing subsequence.

(ii) If S is infinite, then we can take our decreasing sequence to be (x_{n_k}). To see this, assume it is strictly increasing, then for some $j < \ell$, we have that $x_{n_j} < x_{n_\ell}$, but then we have that by definition of the set, $x_{n_j} \notin S$.

□

(b)

Example 1.54. How can you deduce that every bounded sequence in $\mathbb{R}$ has a convergent subsequence?

Proof. It is evident that if the parent sequence is bounded, then any subsequence must also be bounded. By a, every bounded sequence in $\mathbb{R}$ has a monotone subsequence. By problem 4, since this monotone sequence is bounded, then it must converge in $\mathbb{R}$. □

(c)

Example 1.55. Infer that you have a second proof of the Bolzano-Weierstrass Theorem in $\mathbb{R}$.

Proof. By part b, every bounded sequence in $\mathbb{R}$ has a convergent subsequence. This is literally the Bolzanno-Weirstrass Theorem in $\mathbb{R}$. □

(d)

Example 1.56. What about the Heine-Borel Theorem?

Proof. We want to show that if $X \subset \mathbb{R}$ is closed and bounded, then it is compact. Let (x_n) be a sequence in X. Since X is bounded and $(x_n) \in X$, then the sequence is bounded. Thus, by part c, there exists a convergent subsequence $(x_{n_k}) \to x$. Since X is closed, then $x \in X$, and thus X is compact. □

PSET 2: Problem 6

Let (p_n) be a sequence and $f : \mathbb{N} \to \mathbb{N}$ be a bijection. The sequence (q_k) with $q_k = p_{f(k)}$ is a *rearrangement* of (p_n).

(a)

Example 1.57. Are limits of a sequence unaffected by rearrangement?

Proof. Suppose $p_n \to p$. We claim that if f is a bijection, that $p_{f(k)} \to p$. Since $p_n \to p$, then for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, we have that $|p_n - p| < \epsilon$. Thus, there exists infinitely many $i \in \mathbb{N}$ such that $|p_i - p| \leq \epsilon$. Suppose $p_{f(k)} \not\to p$, then if N is large, $|p_N - p| \geq \epsilon$. Thus, there exists finite j such that $|p_j - p| \leq \epsilon$. Thus, since f is bijective, we must have an injective correspondence between the infinite such i and the finitely many j, which is absurd. Thus, $p_{f(k)} \to p$. $\square$

(b)

Example 1.58. What if f is an injection?

Proof. Suppose $p_n \to p$ and $f : \mathbb{N} \to \mathbb{N}$ is an injection. We claim that as $k \to \infty$, $f(k) \to \infty$. Suppose not, then as $k \to \infty$, $f(k) \to n$ for some n, which is ridiculous since this means that after some large N, $n \geq N$ are sent to 0, and is thus not injective. Thus, since

$$\lim_{n\to\infty} p_n = \lim_{f(k)\to\infty} p_n,$$

then the limit is not affected.

However, consider the sequence $a_n = (-1)^n$, and the injection $f : \mathbb{N} \to \mathbb{N}$ that sends

$$f(k) = 2k$$

Evidently, f is an injection. However, $a_{f(k)} = (-1)^{2k} = 1$ for any k, and thus $\lim_{k\to\infty} a_k = \lim_{k\to\infty} a_{f(k)} = 1$. $\square$

> Reflections: We can think of f as sampling a subsequence from the original sequence. Thus, if the original sequence converges, then the rearrangement will converge, but if it doesn't, then we can sometimes sample a converging sequence from it.

(c)

Example 1.59. What if f is a surjection?

Proof. Suppose $p_n = \frac{1}{n}$ and $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$f(k) = \begin{cases} 1, & k \text{ is odd} \\ \frac{k}{2} + 1, & k \text{ is even} \end{cases}.$$

Then f is a surjection since for every $n \in \mathbb{N}$, there exists some $2(n-1) \in \mathbb{N}$ such that $f(2n-2) = n$. Moreover, we have that $\frac{1}{n} \to 0$, but $\lim_{k\to\infty} p_{f(k)}$ does not exist. $\square$

PSET 2: Problem 7

Example 1.60. Assume that $f : M \to N$ is a function from one metric space to another which satisfies the following condition: if a sequence $(p_n) \in M$ converges, then the sequence $(f(p_n)) \in N$ converges. Prove that f is continuous.

Proof. We claim that $f(p_n) \to f(p)$. Let x_n be a sequence such that $x_n = p$ for even n and $x_n = p_{\frac{n-1}{2}}$ for odd n.[6] Thus, x_n alternates between foreshawdowing the limit at p during the even n and a slower p_n in the odd n. We take x_{n_k} to be the subsequence of x_n that samples from even ns, and thus $x_{n_k} = p$ for all k and $f(x_{n_k}) = f(p)$, and so the subsequence converges to $f(p)$. Since a subsequence converges to the same limit as its mother sequence, we have that $f(x_n) \to f(p)$. We know that odd subsequence converges to the same limit, and that the odd subsequence is just (p_n), and thus we have shown that $f(p_n) \to f(p)$. $\square$

[6] For $n = 1$ just let $x_n = p_1$, it doesn't matter.

PSET 2: Problem 8

Example 1.61. Which capital letters of the Roman alphabet are homeomorphic? Are any isometric? Explain.

Proof. Homeomorphic groups with explanations:

(a) {C, G, J, L, M, N, S, U, V, W, Z} One can stretch all these into a single line.

(b) {A,R} You can compress the bubble in the R and more the right line up and it will become A.

(c) {P,Q} For this one, I am assuming that the line in the Q is not inside the O. Then it is fairly obvious.

(d) {E, F, T, Y} Can all become F with minimal bending.

(e) {D,O} Not gonna explain.

(f) $\{X\}$

(g) {K,H, I} The I and K are obvious, the K one can just move the right line to the end and then extend.

(h) {B} Homeomorphic to 8.

$\square$

Proof. By problem 14.b, if a function is not a homeomorphism, then it is not an isometry. Thus, it suffices to show there exists isometries within the homeomorphic groups. Two objects are isometric up to reflections and translation and rotations. Thus, if we place all the letters in $\mathbb{R}^2$ with the euclidean metric, we only have two isometry groups:

(a) $\{N, Z\}$, because we can flip N 90 degrees to get Z.

(b) $\{M, W\}$, because we can rotate M 180 degrees to get W (this lowkey depends on the font, on this font no).

You cannot rotate, reflect, or translate any other letter to become exactly another. $\square$

PSET 2: Problem 9

Example 1.62. If every closed and bounded subset of a metric space M is compact, does it follow that M is complete? (Proof or counterexample.)

Proof. Let (a_n) be Cauchy in M. Then we claim that the sequence forms a bounded subset of M. Take $\epsilon = 1$, then there is some $N \in \mathbb{N}$ such that if $n, m \geq N$, we have that $d(p_n, p_m) < 1$. Let $r > 1 + \max\{d(p_1, p_2), \ldots, d(p_1, p_N)\}$, then for any $i \in \mathbb{N}$

$$d(p_1, p_i) \leq d(p_1, p_N) + d(p_N, p_i) \leq r,$$

and thus the sequence is contained in $B_r(p_1)$. Take the closure of $B_r(p_1)$ to be $\overline{B_r(p_1)}$, then since $B_r(p_1) \subset \overline{B_r(p_1)}$, the sequence is contained in a closed and bounded subset of M. Thus, since $\overline{B_r(p_1)}$ is compact, then (a_n) has a convergent subsequence $a_{n_k} \to a$, where $a \in M$. Thus, it suffices to show that if (a_n) is Cauchy and it has a convergent subsequence, then (a_n) must converge. Observe that since (a_n) is Cauchy, then there exists some $N_1 \in \mathbb{N}$ such that $n > N_1$, then $d(a_n, a_{n_k}) < \frac{\epsilon}{2}$ (since n_k is implicitly greater than n). Since (a_{n_k}) converges to a, then there exists some N_2, such that if $n > N_2$, we have that $d(a_{n_k}, a) < \frac{\epsilon}{2}$. Thus if $N = \max\{N_1, N_2\}$ and $n \geq N$,

$$d(a_n, a) \leq d(a_n, a_{n_k}) + d(a_{n_k}, a) \leq \epsilon.$$

□

PSET 2: Problem 10

Example 1.63. A map $f : M \to N$ is said to be *open* if for all open $U \subset M$, we have that $f(U)$ is open in N.

(a)

Example 1.64. If f is open, is it continuous?

Proof. **Not necessarily.** Let $Id : (M, d) \to (N, d')$, where d is the euclidean metric and d' is the discrete metric. Let U be open in M, then $id(U) = U$ is open in N since we can write $U = \bigcup_{\alpha \in \mathscr{A}} u_\alpha$, where $\{u_\alpha\}$ is every point in U. Since $B_{\frac{1}{2}}(u_\alpha) = \{u_\alpha\}$, and the ball of radius $\frac{1}{2}$ is open in N, then $\{u_\alpha\}$ is open. Thus, since unions of open sets are open, U is open in N and thus id is open.
Let $\{x\} \subset N$. Then $\{x\}$ is open in N. Since $id^{-1}(\{x\}) = \{x\}$ is closed in M, then id is not continuous. ☐

(b)

Example 1.65. If f is a homeomorphism, is it open?

Proof. **Yes.** Let $U \subset M$ be open. Since f is bijective, there exists $L \subset N$ such that $f^{-1}(L) = U$. Suppose L is closed, then by continuity, U is closed. Thus, L is open, and so $f(U) = f(f^{-1}(L)) = L$ is open. ☐

(c)

Example 1.66. If f is an open continuous bijection, is a homeomorphism?

Proof. **Yes.** Suppose $U \subset M$ is open. Since f is bijective, there exists some $L \subset N$ such that $f^{-1}(L) = U$. Since f is open, then $f(U) = f(f^{-1}(L)) = L$ is open. Thus, $f^{-1}(U)$ is continuous. ☐

(d)

Example 1.67. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and surjective, is it open?

Proof. **Not necessarily.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous surjection. We claim that we can map $(0, 1) \mapsto [0, 1]$. Let $a < b$ with $a, b \in (0, 1)$ such that $f(a) = 0$ and $f(b) = 1$, then by the IVT, $f([a, b]) = [0, 1]$, and so if we let $0 < x < a$ be mapped by $f(a) = 0$ and $b < x < 1$ be mapped by $f(x) = 1$, we are sending an open set to a closed set. ☐

(e)

Example 1.68. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous, open, and a surjection, must it be a homeomorphism?

Proof. **Yep.** By (c), it suffices to show that f is an injection. Suppose $f(x) = f(y)$ with $x < y$, then by the extreme value theorem, there exists m and M maximum and minimum achieved in the interval $f((x, y))$. Then by the IVT, $f((x, y)) = [m, M]$, and thus f is not open. Therefore, f must be an injection. ☐

(f)

Example 1.69. What happens in (e) if $\mathbb{R}$ is replaced by the unit circle S^1.

Proof. The same is not true. Consider $f : S^1 \to S^1$, where if $z \in S^1$, then $f(z) = z^2$. Obviously, f is continuous and open and a surjection. However, f is not injective since if $f(z_1) = f(z_2) = (-1, 0)$, then $z_1 = (0, 1)$ and $z_2 = (0, -1)$ give $z_1^2 = (0, 1)(0, 1) = (-1, 0)$ and $z_2^2 = (0, -1)(0, -1) = (-1, 0)$. Thus f is not injective. ☐

PSET 2: Problem 11

Example 1.70. Let Σ be the set of all infinite ones and zeroes. Let $a = (a_n)$ and $b = (b_n)$ be in Σ, then define the metric

$$d(a, b) = \sum \frac{|a_n - b_n|}{2^n}$$

(a)

Example 1.71. Prove that Σ is compact.

Proof. Let $(B_k) = ((b_{n_1}), (b_{n_2}), \dots)$, where $(b_{n_k}) \in \Sigma$ for all k. We wish to find some converging subsequence (B_{k_j}) of (B_k).

Let (x_{n_1}) be first sequence of (B_{k_j}) such that it is sampled from (B_k) by choosing a sequence (b_{n_k}) with $b_{1_k} = 1$. If no such sequence in (B_k) exists, then choose the sequence (b_{n_k}) with $b_{2_k} = 1$. If no such sequence exists then keep repeating the process (this process terminates since otherwise, we would have every sequence in (B_k) be $(000\dots)$)

Suppose every sequence in (B_k) is of the form $(00\dots1\dots)$, where $1 = b_{j_k}$ from the process above (every sequence is all zeroes then a 1 at the same position). Then we repeat the process above to change (x_{n_1}) to be sampled as the sequence (b_{n_k}) with $b_{j+1_k} = 1$. If no such sequence exists, then keep going as in above to find one that exists. Thus, $(x_{n_1}) = (00\dots100\dots1\dots)$. Suppose every sequence in (B_k) is of the same form, then we repeat the process to change (x_{n_1}).

At some point, there will exist some (b_{n_k}) who will not be of the same form (i.e, will have have the same sequence of numbers as (x_{n_k}) but will have a 0 instead of a 1 in the pivotal position (the important position we have talked about above)). Sample (x_{n_2}) with such a sequence. Then if $(x_{n_1}) = (00\dots100\dots1\dots)$, we shall have that $(x_{n_2}) = (00\dots100\dots0\dots)$.

Repeat the entire process to find the subsequence (B_{k_j}). We claim that this subsequence converges. To see this, consider

$$d((x_{n_k}), (x_{n_{k+1}})) = \sum \frac{|x_{i_k} - x_{i_{k+1}}|}{2^i} \leq \frac{1}{2^{k-1}}.$$

This is because $|x_{i_k} - x_{i_{k+1}}| \leq 1$ and (as shown in the last PSET), $\sum_{i=n}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n-1}}$.

Let $\epsilon > 0$. By the Archimidean property, there exists some K large such that if $k, k' \geq K$, we have that $d((x_{n_k}), (x_{n_{k'}}) < \epsilon$. Therefore, (B_{k_j}) is Cauchy.

Let $p_1 = \lim\limits_{k \to \infty} (x_{1_k})$. By construction of the sequence, this limit is well defined and is either 0 or 1. For k' large, we have that $|x_{1_{k'}} - p_1| = 0$. Similarly, $p_n = \lim\limits_{k \to \infty} (x_{n_k})$ is well defined. We claim that $B_{k_j} \to p_n$. Suppose not, then for some $\epsilon > 0$ and k' large, we have that $d((x_{n_{k'}}), (p_n)) \geq \epsilon$. However, we have that

$$\sum \frac{|x_{i_{k'}} - p_i|}{2^i} \leq 0 + \frac{1}{2^{k'-1}},$$

where the 0 comes from all the terms that agree (as in the example above), and the other term comes from the terms that don't. Evidently, for k' large enough, this difference is $< \epsilon$.

Therefore, since Σ is complete (there was nothing special about our sequence so we can generalize our convergence argument) and our subsequence is Cauchy, then it converges to some limit in Σ, and thus Σ is compact. $\square$

(b)

Example 1.72. Prove that Σ is homeomorphic to the Cantor set.

Proof. Let $C_0 = [0, \frac{1}{3}]$, $C_1 = [\frac{2}{3}, 1]$, and $C^1 = C_0 \cup C_1$. Let $C_{00} = [0, \frac{1}{9}]$, and let $C^2 = C_{00} \cup C_{01} \cup C_{10} \cup C_{11}$. Keep going with this construction.
Let $\omega \in \Sigma$, then by the previous construction, we have that

$$C_{\omega_1} \supset C_{\omega_1 \omega_2} \supset \cdots \supset C_{\omega_1 \ldots \omega_n} \supset \ldots$$

Define a function $p : \Sigma \to C$ such that

$$p(\omega) = \bigcap_{n \in \mathbb{N}} C_{\omega|n},$$

where $\omega|n = \omega_1 \omega 2, \ldots, \omega_n$. Note that p is well defined because the intersection of a nested sequence of compact non-empty sets is compact and nonempty and since the diameter of $C_{\omega_1 \ldots \omega_n} \to 0$ as $n \to \infty$, we have that the intersection is a single point.

Let $c \in C$, then because Σ is the infinite sequences of ones and zeros, there exists some $\omega = \omega(p)$ such that if $c \in C_\alpha$ of C^n, $\alpha = \omega|n$. Thus, $p(\omega) = c$ and so p is surjective.

Suppose $c_1 \neq c_2 \in C$, then for some n large, they lie in different C_α for some C^n, and their corresponding sequences will truncate to different values at n. Thus, p is injective

We wish to show that p is continuous at ω'. For any $\epsilon > 0$, there exists some large n such that if $\omega \in \Sigma$ and $d(\omega, \omega') < \frac{1}{2^n}$, then we have by work in part a that $\omega|[n-1] = \omega'|[n-1]$. Thus, $p(\omega), p(\omega') \in C_{\omega_1 \ldots \omega_{n-1}}$, and so $d(p(w), p(w')) \leq \frac{1}{3^{n-1}} < \epsilon$ for large n. Because this is true for any $\omega \in \Sigma$, then p is continuous.

Thus, since $p : \Sigma \to C$ is a continuous surjection and Σ is compact, then by Theorem 42 in Pugh (proved in class), p is homeomorphic. $\square$

1.30 PSET 3

PSET 3: Problem 1

Example 1.73. A map $f : M \to N$ is said to be *open* if for all open $U \subset M$, we have that $f(U)$ is open in N.

(a)

Example 1.74. If f is open, is it continuous?

Proof. **Not necessarily.** Let $Id : (M, d) \to (N, d')$, where d is the euclidean metric and d' is the discrete metric. Let U be open in M, then $id(U) = U$ is open in N since we can write $U = \bigcup_{\alpha \in \mathscr{A}} u_\alpha$, where $\{u_\alpha\}$ is every point in U. Since $B_{\frac{1}{2}}(u_\alpha) = \{u_\alpha\}$, and the ball of radius $\frac{1}{2}$ is open in N, then $\{u_\alpha\}$ is open. Thus, since unions of open sets are open, U is open in N and thus id is open.
Let $\{x\} \subset N$. Then $\{x\}$ is open in N. Since $id^{-1}(\{x\}) = \{x\}$ is closed in M, then id is not continuous. □

(b)

Example 1.75. If f is a homeomorphism, is it open?

Proof. **Yes.** Let $U \subset M$ be open. Since f is bijective, there exists $L \subset N$ such that $f^{-1}(L) = U$. Suppose L is closed, then by continuity, U is closed. Thus, L is open, and so $f(U) = f(f^{-1}(L)) = L$ is open. □

(c)

Example 1.76. If f is an open continuous bijection, is a homeomorphism?

Proof. **Yes.** Suppose $U \subset M$ is open. Since f is bijective, there exists some $L \subset N$ such that $f^{-1}(L) = U$. Since f is open, then $f(U) = f(f^{-1}(L)) = L$ is open. Thus, $f^{-1}(U)$ is continuous. □

(d)

Example 1.77. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and surjective, is it open?

Proof. **Not necessarily.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous surjection. We claim that we can map $(0, 1) \mapsto [0, 1]$. Let $a < b$ with $a, b \in (0, 1)$ such that $f(a) = 0$ and $f(b) = 1$, then by the IVT, $f([a, b]) = [0, 1]$, and so if we let $0 < x < a$ be mapped by $f(a) = 0$ and $b < x < 1$ be mapped by $f(x) = 1$, we are sending an open set to a closed set. □

(e)

Example 1.78. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous, open, and a surjection, must it be a homeomorphism?

Proof. **Yep.** By (c), it suffices to show that f is an injection. Suppose $f(x) = f(y)$ with $x < y$, then by the extreme value theorem, there exists m and M maximum and minimum achieved in the interval $f((x, y))$. Then by the IVT, $f((x, y)) = [m, M]$, and thus f is not open. Therefore, f must be an injection. □

(f)

Example 1.79. What happens in (e) if $\mathbb{R}$ is replaced by the unit circle S^1.

Proof. The same is not true. Consider $f : S^1 \to S^1$, where if $z \in S^1$, then $f(z) = z^2$. Obviously, f is continuous and open and a surjection. However, f is not injective since if $f(z_1) = f(z_2) = (-1, 0)$, then $z_1 = (0, 1)$ and $z_2 = (0, -1)$ give $z_1^2 = (0, 1)(0, 1) = (-1, 0)$ and $z_2^2 = (0, -1)(0, -1) = (-1, 0)$. Thus f is not injective. $\square$

PSET 3: Problem 2

Example 1.80. Consider a function $f : M \to \mathbb{R}$. Its graph is the set

$$G := \{(p, y) \in M \times \mathbb{R} | y = f(p)\}$$

(a)

Example 1.81. Prove that if f is continuous then its graph is closed (as a subset of $M \times \mathbb{R}$).

Proof. Let $s = (p, y)$ be a limit point of G. Thus, there exists a sequence $(s_n) \in G$ such that $s_n \to s$. Since $s_n \in G$ for all n, we have that $s_n = (p_n, f(p_n)) \to (p, y)$. We have $p_n \to p$, then because f is continuous, $f(p_n) \to f(p)$, and thus we use the fact that limits are unique to show that $y = f(p)$. Thus, $s = (p, f(p))$. Therefore, $s \in G$. $\square$

(b)

Example 1.82. Prove that if f is continuous and M is compact then its graph is compact.

Proof. Let $(s_n) \in G$. We want to find a convergent subsequence (s_{n_k}) such that it converges to a limit in G. Suppose $s_n = (p_n, f(p_n))$ for each n, then since M is compact and (p_n) is a sequence in M, we have (p_{n_k}) convergent sequence to some $p \in M$. Thus, $s_{n_k} = (p_{n_k}, f(p_{n_k})) \to (p, y) = s$. Since G is closed (part a), we have that $s \in G$. $\square$

(c)

Example 1.83. Prove that if the graph of f is compact then f is continuous.

Proof. Suppose $(p_n) \in M$ is convergent with $p_n \to p$. We claim $f(p_n)$ converges. Let $(s_n) = (p_n, f(p_n))$, then since G is compact, there exists a convergent subsequence $(s_{n_k}) = (p_{n_k}, f(p_{n_k})) \to (p', y')$. Note that $s = (p', y') \in G$ because G is closed). However, since (p_{n_k}) is a convergent subsequence of a convergent sequence, then we necessarily must have that $p' = p$, and thus $f(p) = y'$. Thus, $f(p_{n_k}) \to f(p)$. It suffices to show that $f(p_n) \to f(p)$. Suppose not, then there exists some $\epsilon > 0$ such that for N large enough, if $n \geq N$, we have that $d(f(p_n), f(p)) > \epsilon$. Consider $(s_{n_j}) = (p_{n_j}, f(p_{n_j}))$ as the subsequence of (s_n) such that $d(f(p_{n_j}), f(p)) > \epsilon$. By compactness, there exists some sub-subsequence that converges to some (p, γ), where evidently, $\gamma \neq f(p)$. Thus $(p, \gamma) \notin G$, which is a contradiction to the fact G is closed. $\square$

p

(d)

Example 1.84. What if the graph is merely closed? Give an example of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed.

Proof. Consider the function $f(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 0, & = 0. \end{cases}$ Evidently, the function is discontinuous at $x = 0$. We need to show that there does not exist some $y \in \mathbb{R}$ such that $(0, y)$ is a limit of $G = \{(p, y) \in \mathbb{R} \times \mathbb{R} | y = \frac{1}{p^2}\}$. Suppose not, that is, there exists some $y \in \mathbb{R}$ such that there is some sequence $s_n = (x_n, y_n)$ in G that converges to $(0, y)$. Thus, $s_n = (x_n, \frac{1}{x_n^2})$. Thus, there exists some $N \in \mathbb{N}$ such that for $n \geq N$, we have that

$$d(s_n, (0, y)) = \sqrt{(x_n - 0)^2 + (y_n - y)^2} < \epsilon.$$

However, if we take n large, then $x_n \to 0$, and thus $\frac{1}{x_n} \to \infty$. then $d(s_n, (0, y)) > \epsilon$ since

$$\begin{aligned} d(s_n, (0, y)) &= \sqrt{(x_n - 0)^2 + (y_n - y)^2} \\ &= \sqrt{(x_n)^2 + (\frac{1}{x_n^2} - y)^2} \end{aligned}$$

Thus, the graph G only has limit points where it can achieve them, and is thus closed. $\square$

PSET 3: Problem 3

Example 1.85. Suppose that (K_n) is a nested sequence of compact nonempty sets, $K_1 \supset K_2 \supset \dots$, and $K = \bigcap K_n$. If for some $\mu > 0$, $\text{diam} K_n \geq \mu$ for all n, is it true that $\text{diam} K \geq \mu$

Proof. **Yes.** Let $x_n, y_n \in K_n$ with $d(x_n, y_n) \geq \mu$. We know these exist by assumption. Evidently, $(x_n) \in K_1$, and since K_1 is compact, there exists some convergent subsequence $x_{n_k} \to x$. Since K_1 is closed, then $x \in K_1$. We must have that $x \in K_2$ since except for maybe a few terms of x_{n_k}, $(x_{n_k}) \in K_2$, for if it weren't, then $x_n \notin K_n$. Continuing with this logic, $x \in K_n$ for all n, and thus $x \in K$. Similarly, $y \in K$. Thus, for k large.

$$\begin{aligned}
\mu &\leq d(x_{n_k}, y_{n_k}) \\
&\leq d(x, x_{n_k}) + d(x, y) + d(y_{n_k}, y) \\
&< \frac{\epsilon}{2} + d(x, y) + \frac{\epsilon}{2} \\
&= d(x, y) + \epsilon.
\end{aligned}$$

Because this is true for all $\epsilon > 0$, we have that $\text{diam} K \geq \mu$. Note that here we use the fact that we can sample from the same subsequence by taking sub subsequences as shown in class. $\square$

PSET 3: Problem 4

Example 1.86. The distance from a point p in a metric space M to a nonempty subset $S \subset M$ is defined to be $\text{dist}(p, S) = \inf\{d(p, s) : s \in S\}$.

(a)

Example 1.87. Show that p is a limit of S if and only if $\text{dist}(p, S) = 0$.

Proof.

- ($\Longrightarrow$) Suppose p is a limit of S, and suppose for the sake of contradiction that $\text{dis}(p, S) = \mu$ for some $\mu > 0$. Thus, there does not exist any $s \in S$ such that $d(p, s) < \mu$. Since p is a limit of S, then there exists some sequence $s_n \in S$ such that $s_n \to p$, and thus there exists some n large enough such that
$$d(p, s_n) < \frac{\mu}{2}.$$
Oops!
- ($\Longleftarrow$) Suppose $\text{dist}(p, S) = 0$. For all $n \in \mathbb{N}$, there exists some $s_n \in S$ such that
$$d(p, s_n) < \frac{1}{n},$$
as otherwise, $s \geq \frac{1}{n}$ for all $s \in S$, and so $\text{dist}(p, S) \geq \frac{1}{n}$. Thus, we have built a sequence $s_n \to p$ with all $s_n \in S$, and so p is a limit of S.

$\square$

(b)

Example 1.88. Show that $p \to \text{dist}(p, S)$ is a uniformly continuous function of $p \in M$.

Proof. Suppose $p, q \in M$ and let $\epsilon > 0$. There exists $\delta = \frac{\epsilon}{2}$ such that if $p, q \in M$ with $d(p, q) < \delta$, we have the following. For all $s \in S$, we have that

$$d(p, s) \leq d(p, q) + d(q, s), \qquad d(q, s) \leq d(q, p) + d(p, s)$$

and thus

$$\text{dist}(p, S) \leq d(p, q) + \text{dist}(q, S), \qquad \text{dist}(q, S) \leq d(p, q) + \text{dist}(p, S)$$

Thus, combining the inequalities,

$$d(\text{dist}(q, S), \text{dist}(q, S)) \leq 2d(p, q) < 2\frac{\epsilon}{2}.$$

$\square$

PSET 3: Problem 5

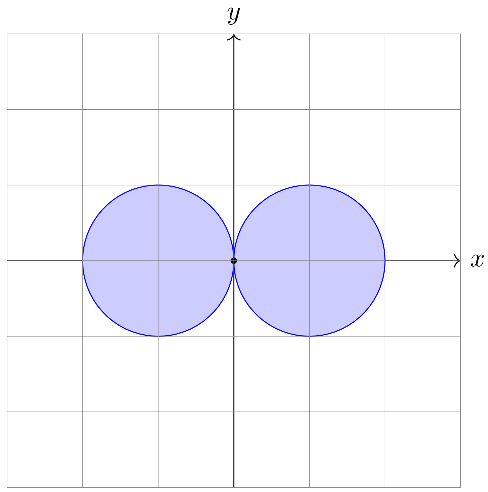
Example 1.89. Prove that the 2-sphere is not homeomorphic to the plane.

Proof. S^2 is compact since it fits inside a box ($[0,1]^3$) in $\mathbb{R}^2$. The plane is not compact. Thus, they are not homeomorphic. $\square$

PSET 3: Problem 6

Example 1.90. If S is connected, is the interior of S connected? Prove this or give a counterexample.

Proof.



Consider the union of two balls who intersect at the origin. Evidently, each ball is connected since they are path connected. The union of connected sets sharing a common point is connected (Theorem in book). Thus, the union is connected. Consider that the interior of the union does not contain the origin since any ball around the origin contains points in the plane not in either ball. Thus, we can express the interior of the union as two open balls which are disjoint and nonempty, and thus the interior of the union is disconnected. □

PSET 3: Problem 7

Example 1.91. A function $f : (a, b) \to \mathbb{R}$ satisfies a $\alpha-$Hölder condition of order α if $\alpha > 0$, H is a constant, and for all $u, x \in (a, b)$, we have

$$|f(u) - f(x)| \leq H|u - x|^\alpha$$

(a)

Example 1.92. Prove that an $\alpha-$Hölder function defined on (a, b) is uniformly continuous and infer that it extends uniquely to a continuous function defined on $[a, b]$. Is the extended function $\alpha-$Hölder?

Proof. Let $\epsilon > 0$. For all $u, x \in (a, b)$, there exists a $\delta = \min\{1, \frac{\epsilon}{|H|+1}\}$ such that if $|u - x| < \delta$, we have that by the $\alpha-$Hölder condition,

$$|f(u) - f(x)| \leq H|u - x|^\alpha < H|u - x| < |H|\frac{\epsilon}{|H| + 1} < \epsilon.$$

To extend this function to $[a, b]$, we need to define $f(a)$ and $f(b)$. To do this, let

$$f(a) = \lim_{x \to a^+} f(x), \qquad f(b) = \lim_{x \to b^-} f(x).$$

Since limits are unique, then this is a unique extension on $[a, b]$. Continuity comes directly from the construction (See PSET 2). We claim that the extended function is $\alpha-$Hölder
Let $x \in (a, b]$, we want to show that

$$|f(a) - f(x)| \leq H|a - x|^\alpha.$$

By continuity, for all $\epsilon > 0$, there exists some $\delta > 0$ and $u \in (a, x)$ such that if $x - a < \delta$, we have that $|f(a) - f(x)| < \epsilon$. Thus,

$$\begin{aligned}
|f(a) - f(x)| &\leq |f(a) - f(u)| + |f(u) - f(x)| \\
&< \epsilon + H|u - x|^\alpha \\
&\leq \epsilon + H|a - x|^\alpha
\end{aligned}$$

Where the last inequality is justified since $|u - x| \leq |a - x|$ because $u \in (a, x)$. Thus, as $\epsilon \to 0$, we have that f is $\alpha-$Hölder continiuos at a. A similar argument can be shown for b. □

(b)

Example 1.93. What does $\alpha-$Hölder continuity mean when $\alpha = 1$?

Proof. If $\alpha = 1$, we have that for all $u, x \in (a, b)$,

$$|f(u) - f(x)| \leq H|u - x|,$$

and thus f satisfies a global Lipschitz condition. Specifically, if f is differentiable, then $|f'(x)| \leq H$ for all $x \in (a, b)$. Thus, the derivative is bounded for all $x \in (a, b)$. □

(c)

Example 1.94. Prove that $\alpha-$Hölder continuity when $\alpha > 1$ implies that f is constant.

Proof. Let $x \in (a, b)$. Because f is $\alpha-$Hölder, then

$$|f(u) - f(x)| \leq H|u - x|^{\alpha}.$$

Thus,

$$0 \leq \left| \frac{f(u) - f(x)}{u - x} \right| \leq H|u - x|^{\alpha - 1}$$

Because $\alpha > 1$, then $|u - x|^{\alpha-1}$ is a well defined denominator. Consider that by continuity:

$$\lim_{u \to x} (0) = 0, \qquad H \lim_{u \to x} |u - x|^{\alpha - 1} = H(0) = 0,$$

where the second limit comes from the fact that for u close to x, because $\alpha - 1 > 0$, we have that $|u - x| > |u - x|^{\alpha-1}$. Thus, by the squeeze theorem, we have that

$$\lim_{u \to x} \left| \frac{f(u) - f(x)}{u - x} \right| = |f'(x)| = 0$$

It suffices to show that if $f'(x) = 0$ for any $x \in (a, b)$, we have that f is constant. Let $y < x \in (a, b)$ By mean value theorem, we have that $f(y) - f(x) = f'(\lambda)(x - a) = 0$, and thus $f(y) = f(x)$. $\square$

PSET 3: Problem 8

Example 1.95. For each $r \geq 1$, find a function that is C^r but not C^{r+1}.

Proof. Let $f_r : \mathbb{R} \to \mathbb{R}$ be defined by $f_r(x) = x^r|x|$. We claim that f_r is C^r but not C^{r+1}. We proceed by induction on $\mathbb{N}$ over the variable r.

(a) For $r = 1$, we have that $f_1(x) = x|x|$. We claim that $f'(x) = 2|x|$. To see this, let $x > 0$, then $f_1'(x) = 2x$. If $x < 0$, then $f_1'(x) = -2x$. If $x = 0$, then $f_1'(0) = 0$. Thus, $f_1'(x) = 2|x|$, which is continuous everywhere. Evidently, $f_1''(x)$ is not differentiable at $x = 0$.

(b) Assume $f_r(x) = x^r|x|$. is C^r. Note that by similar work done above, $f_r^{(r)}(x) = r!|x|$. Also assume that $f_r(x)$ is not C^{r+1}

(c) We claim that $f_{r+1}(x) = x^{(r+1)}|x|$ is C^{r+1}. We have that, $f_{r+1}^{(r+1)}(x) = (r+1)!|x| = r!|x| + (r+1)|x|$, which is continuous by hypothesis. Evidently, since $(r+1)|x|$ is not differentiable, we have that $f_{r+1}^{(r+2)}(x)$ is not continuous at $x = 0$, and thus $f_{r+1} \notin C^{r+1}$.

$\square$

1.31 PSET 4

PSET 4: Problem 1

Example 1.96. Let $f : (a, b) \to \mathbb{R}$ be given.

(a)

Example 1.97. If $f''(x)$ exists, prove that

$$\lim_{h\to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x).$$

Proof. We can use Taylor Series:

$$f(x-h) = f(x) - f'(x)h + \frac{1}{2}f''(x)h^2 + R(h).$$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + R(h)$$

Thus,

$$f(x-h) - 2f(x) + f(x+h) = f''(x)h^2 + 2R(h),$$

and so

$$\lim_{h\to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x) + \lim_{h\to 0} \frac{R(h)}{h^2} = f''(x),$$

where the last equality holds by the Taylor Remainder Theorem.

Alternatively, we can sketch a proof using $L'Hopital$:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} &= \lim_{h\to 0} \frac{-f'(x-h) + f'(x+h)}{2h} \\
&= \frac{1}{2} \lim_{h\to 0} \frac{f'(x) - f'(x-h)}{h} + \frac{1}{2} \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \frac{1}{2}(f''(x) + f''(x))
\end{aligned}$$

To finish the proof, we need to be careful such that h is small enough such that f' exists around a small neighborhood of x. Moreover, this derivation should really be going backwards, with the existence of the limits coming from the existence of f''. $\square$

(b)

Example 1.98. Find an example that this limit can exist even when f”(x) fails to exist.

Proof. Consider $f(x) = x|x|$. By work done in PSET 3, $f'(x)$ is not differentiable at $x = 0$, and thus $f''(0)$ fails to exist. However, consider that:

$$\begin{aligned}
\lim_{h\to 0} \frac{(x-h)|x-h| - 2x|x| + (x+h)|x+h|}{h^2} &= \lim_{h\to 0} \frac{(-h)|(-h)| + h|h|}{h^2} \\
&= \lim_{h\to 0} \frac{-h^2 + h^2}{h^2} \\
&= \lim_{h\to 0} \frac{0}{h^2} \\
&= 0
\end{aligned}$$

$\square$

PSET 4: Problem 2

Example 1.99. Define $e : \mathbb{R} \to \mathbb{R}$ by

$$e(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0, \\ 0, & x \leq 0 \end{cases}.$$

(a) Prove that e is smooth.

Proof. Evidently, e is smooth for $x < 0$. Suppose $x > 0$, then by my calculus talent and the chain rule, we have that $e'(x) = \frac{e^{-\frac{1}{x}}}{x^2}$. Assume that $e^{(r)}(x) = (-1)^r \frac{e^{-\frac{1}{x}}}{x^{r+1}}$. Then we have that

$$e^{(r+1)}(x) = \frac{d}{dx} e^{(r)}(x) = (-1)^{r+1} \frac{e^{-\frac{1}{x}}}{x^{r+2}}. \tag{5}$$

The rth derivative is clearly continuous and exists for any r if $x > 0$. Similarly, if $x < 0$, we have that $e^{(x)}(x) = 0$. Thus, it suffices to show that the derivative is continuous at $x = 0$ for any rth derivative. Consider the case when $r = 1$, then if we let $y = \frac{1}{x}$, we have that as $x \to 0$, $y \to \infty$, and moreover,

$$\lim_{x \to 0^+} e'(x) = \lim_{x \to 0^+} \frac{e^{-\frac{1}{x}} - e(0)}{x} = \lim_{y \to \infty} \frac{y}{e^y} = \lim_{y \to \infty} \frac{1}{e^y} = 0 = \lim_{x \to 0^-} e(x),$$

where $L'hopital's$ rule was used in the third to last equality. Thus, $e'(x)$ is continuous at 0. Assume this holds for the rth derivative, that is

$$\lim_{x \to 0^+} e^{(r)}(x) = \lim_{x \to 0^+} \frac{(-1)^r e^{\frac{-1}{x}}}{x^{r+1}} = (-1)^r \lim_{x \to 0^+} \frac{y^{r+1}}{e^y} = 0.$$

Then using the derivative rule we derived in (1) :

$$\lim_{x \to 0^+} e^{(r+1)}(x) = \lim_{x \to 0^+} (-1)^{r+1} \frac{e^{-\frac{1}{x}}}{x^{r+2}} = (-1)^{r+1} \lim_{y \to \infty} \frac{y^{r+2}}{e^y} = (-1)^{r+1}(r+2) \lim_{y \to \infty} \frac{y^{r+1}}{e^y} = 0$$

where the last equality holds by the inductive step and the second to last by $L'hopital$. Thus, the rth derivative exists and is continuous for any r, and thus e is smooth. □

(b)

Example 1.100. Prove that e is not analytic.

Proof. By the above solution, $e^{(r)}(0) = 0$ for all r. Consider some $0 < h < \epsilon(h)$ small and assume $e(x)$ is analytic, then for $x = 0$,

$$e(x+h) = e(h) = \sum_{r=0}^{\infty} \frac{e^r(0)}{r!} h^r = 0,$$

which is a contradiction, since $e(h) \neq 0$ for h since it is a strictly increasing function for $x > 0$. □

(c)

Example 1.101. Show that the bump function

$$\beta(x) = e^2 e(1-x) e(x+1)$$

is smooth, identically zero outside the interval $(-1, 1)$, positive inside the interval $(-1, 1)$, and takes value 1 at $x = 0$.

Proof. Consider that $\beta^{(r)}(x) = e^2[e^{(r)}(1-x)e(x+1)+e(1-x)e^{(r)}(x+1)]$. Since $e(x)$ is smooth, then it is continuous everywhere. Since $1-x$ is continuous everywhere, then $e(1-x)$ continuous everywhere. Similarly, $e(x+1)$ is continuous everywhere. A similar argument can be used to show that $e^{(r)}(1-x)$ and $e^{(r)}(x+1)$ are continuous everywhere. Note that these derivatives must necessarily exist because if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 1-x$ and $g : \mathbb{R} \to \mathbb{R}$ is defined by $e(x)$, then we know that $e^{(r)}(x)$ is differentiable everywhere, and is thus differentiable at $1-a$. We also know that $f'(a)$ exists. Thus, we have that $(g(f(a)))'$ exists for all $a \in \mathbb{R}$ by the chain rule. Thus, we have that $e^{(r)}(1-x)$ exists for all $r \in \mathbb{N}$, $x \in \mathbb{R}$. Similar for $e^{(r)}(x+1)$.

Suppose $x < -1$, then

$$\beta(x) = e^2 e(1-x)e(x+1) = e^2 e(a)e(b),$$

where $a > 0$ and $b < 0$. Thus, we have that $e(a)$ is positive by part b, and that $e(b) = 0$ by definition, and thus $\beta(x) = 0$. Similar argument for $x > 1$. Suppose $x \in (-1, 1)$, then

$$\beta(x) = e^2 e^{-\frac{1}{1-x}} e^{-\frac{1}{x+1}} = e^{2-\frac{1}{1-x}-\frac{1}{x+1}} = e^{\frac{2x^2}{x^2-1}}.$$

Since $x \in (-1, 1)$, we have that $x^2 - 1 > 0$, and thus $\gamma = \frac{2x^2}{x^2-1} > 0$, and thus $e^{\gamma} > 0$. Suppose $x = 0$, then

$$\beta(0) = e^2 e(1)e(1) = e^2(e(1))^2 = e^2(e^{-\frac{1}{1}})^2 = e^2 e^{-2} = 1$$

□

(d)

Example 1.102. For $|x| < 1$, show that

$$\beta(x) = e^{\frac{2x^2}{x^2-1}}$$

Proof. Solved in part (c). □

PSET 4: Problem 3

Example 1.103. D_k refers to the set of points with oscillation $\geq \frac{1}{k}$.

(a)

Example 1.104. Prove that D_k is closed.

Proof. Let x_k be a limit point of D_k. Thus, there exists some sequence $(x_n) \in D_k$ such that $x_n \to x_k$. Since $x_n \in D_k$ for all n, then we have that for any $n \in \mathbb{N}$:

$$\limsup_{t \to x_n} f(t) - \liminf_{t \to x_n} f(t) \geq \frac{1}{k}. \tag{6}$$

Thus, we have that $\operatorname{osc}_{x_n}(f) \geq \frac{1}{k}$ for any n. We claim that $\operatorname{osc}_{x_k}(f) \geq \frac{1}{k}$. Thus, we claim that

$$\limsup_{t \to x_k} f(t) - \liminf_{t \to x_k} f(t) \geq \lim_{n \to \infty} \limsup_{t \to x_n} f(t) - \lim_{n \to \infty} \limsup_{t \to x_n} f(t).$$

To see this, it suffices to notice that

$$\lim_{n \to \infty} \limsup_{t \to x_n} f(t) \leq \limsup_{t \to x_k} f(t), \qquad \lim_{n \to \infty} \liminf_{t \to x_n} \geq \liminf_{x_n \to x_k}$$

Let $\epsilon > 0$. Because $x_n \to x_k$ and $t \to x_k$, we have that for n large:

$$d(t, x_n) \leq d(x_n, x_k) + d(x_k, t) < \epsilon.$$

Thus, we have that convergence of $t \to x_n$ as $n \to \infty$ implies that $t \to x_k$. In other words, $t \to x_n$ for all n and $x_n \to x_k$ implies that $t \to x_k$. Consider the diameter definition: For any $r \geq 0$, we have that there exists some x_n such that $d(x_n, x_k) \leq \frac{r}{2}$. Thus, we must have that for large n, $B_r(x_n) \subset B_r(x)$, and thus

$$\lim_{n \to \infty} \lim_{r \to 0} \sup_{s,t \in B_r(x_n)} f(t) \leq \lim_{r \to 0} \sup_{s,t \in B_r(x_k)} f(t).$$

We have a similar result for the infemum. Thus, because for any $x \in \mathbb{N}$ we have (2), then as $n \to \infty$, we will still have that

$$\limsup_{t \to x_k} f(t) - \liminf_{t \to x_k} f(t)) \geq \lim_{n \to \infty} \limsup_{t \to x_n} f(t) - \lim_{n \to \infty} \liminf_{t \to x_n} f(t) \geq \frac{1}{k},$$

and thus $x_k \in D_{\frac{1}{k}}$, and thus $D_{\frac{1}{k}}$ is closed. □

(b)

Example 1.105. Infer that the discontinuity set of f is a countable union of closed sets (This is called the F_σ set)

Proof. Consider that the set D of discontinuity points filters itself as the countable union:

$$D = \bigcup_{k=1}^{\infty} D_{\frac{1}{k}},$$

where each $D_{\frac{1}{k}}$ is closed. To note this, consider it suffices to show there does not exist $x \in D_{\frac{1}{k}}$ such that f is continuous at x. This is clear since we have that since $x \in D_{\frac{1}{k}}$, then $\operatorname{osc}_x(f) \geq \frac{1}{k}$, and thus f is discontinuous at x. Note that this filtration is also given by the book in the proof of the Riemann Lebesgue. □

(c)

Example 1.106. Infer from (b) that the set of continuity points is a countable intersection of open sets.

Proof. Consider that if D is the set of discontinuities, then $[a, b]D = D^c$ is the set of continuity points. Thus, using DeMorgan's Law:

$$D^c = \left(\bigcup_{k=1}^{\infty} D_{\frac{1}{k}} \right)^c = \bigcap_{k=1}^{\infty} (D_{\frac{1}{k}})^c.$$

Since each $D_{\frac{1}{k}}$ is closed, then $(D_{\frac{1}{k}})^c$ is open. $\square$

PSET 4: Problem 4

Example 1.107. We say that $f : (a, b) \to \mathbb{R}$ has a *jump discontinuity* at $c \in (a, b)$ if

$$\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$$

or

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) \neq f(c)$$

An *oscillating discontinuity* is any non-jump discontinuity.

(a)

Example 1.108. Show that $f : \mathbb{R} \to \mathbb{R}$ has at most countably many jump discontinuities.

Proof. Suppose f has a jump discontinuity at some $b \in \mathbb{R}$. By definition, we have that both limits exist but

$$\lim_{x\to b^-} f(x) \neq \lim_{x\to b^+} f(x).$$

We claim that the set of points D_1 such that

$$|\lim_{x\to b^-} f(x) - \lim_{x\to b^+} f(x)| \geq 1$$

is countable. Let $b \in D_1$. Suppose WLOG that $\lim_{x\to b^-} f(x) < \lim_{x\to b^+} f(x)$. Let $S_b = \{f(x)|x < b\}$ and $L_b = \{f(x)|b < x\}$. Evidently, neither is nonempty and by our assumption, we have that they are respectively bounded above and below by each other. Let $s_b = \sup S_b$ and $l_b = \inf L_b$. Notice that $l_b - s_b \geq 1$ because of the discontinuity, and thus we have that either $s_b \leq f(b) < l_b$ or $s_b < f(b) \leq l_b$. If we flip the inequality in our WLOG assumption, then we have the same, except now we must also switch what we define as our sup and inf. Thus, by continuity, there exists some n large such that $x \in (b - \frac{1}{n}, b)$ and $y \in (b, b + \frac{1}{n})$ with $|f(x) - f(y)| \geq 1$. Define

$$D_{1,n} := \{b \in \mathbb{R} : \ x \in (b - \frac{1}{n}, b), y \in (b, b + \frac{1}{n}) \text{ and } |f(z) - f(y)| \geq 1\}.$$

We claim that for large enough n, there does not exist some $b' \neq b$ such that $b' \in D_1$ and $b' \in D_{1,n}$. Thus, we claim that b is isolated. By pure existence of the left limit, we have that there exists some δ_L and $x \in \mathbb{R}$ such that if $0 < b - x < \delta$, we have that $|f(b) - f(x)| < \frac{1}{2}$. Similarly we have that there exits some δ_R and $x \in \mathbb{R}$ such that if $0 < x - b < \delta_R$, then $|f(b) - f(x)| < \frac{1}{2}$. Thus, if we consider $\text{osc}_x f$ for any $x \in (b - \delta, b + \delta)\{a\}$, where $\delta = \min\{\delta_L, \delta_R\}$, then we have that as $\delta \to 0$, if $y \in \mathbb{R}$ with $0 < |x - y| < \delta$, then we have that $|f(x) - f(y)| < 1$, Evidently, we have that $\text{osc}_b f > 1$. Thus, every jump discontinuity $b \in D_1$ is contained in some open interval with no other jump discontinuities who's oscillation is greater than 1.

There exists some distinct $q \in \mathbb{Q}$ for each such isolated interval, and thus since $\mathbb{Q}$ is countable, we can count the number of intervals containing a jump discontinuity with oscillation greater than 1. Because the number of intervals is at most countable and each interval contains at most one jump discontinuity of such oscillation, we have that there are at most countably many jump discontinuities of oscillation greater than 1 of f.

Define

$$D_{\frac{1}{k},n} = \{b \in \mathbb{R} : \ y \in (b - \frac{1}{n}, x), z \in (b, b + \frac{1}{n}) \text{ and } |f(z) - f(y)| \geq \frac{1}{k}\}.$$

We can do the exact proof as above, but this time fixing $\epsilon = \frac{1}{2k}$, and thus for n large, if $b \in D_{\frac{1}{k},n}$ then b is isolated. That is, there does not exist any other $b' \in D_{\frac{1}{k}}$ in a small region around b. Thus, we have that the set of jump discontinuities, D, is as follows:

$$D = \bigcup_{k\in\mathbb{N}} \bigcup_{n\geq N} D_{\frac{1}{k},n},$$

where N is the smallest natural that makes every set $D_{\frac{1}{k},n}$ isolated. Because the right hand side is obviously at most countable, and each set in the right hand side contains at most one jump discontinuity then we have that D is at most countable.

□

(b)

Example 1.109. What about the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x > 0) \\ 0, & x \leq 0 \end{cases} .$$

Proof. Evidently, the function f has no discontinuities for any $x \in \mathbb{R}_-$. Consider when $x = 0$. We claim that $\limsup_{t\to 0^+} f(t) = 1$ and $\liminf_{t\to 0^+} f(t) = -1$. Consider that

$$\limsup_{t\to 0^+} f(t) = \inf_{\delta\to 0^+} \sup[f(t) : \ |t| < \delta].$$

Thus, it suffices to show that for any $\delta > 0$, there exists some $t < \delta$ such that $f(t) = 1$. For any $\delta > 0$, there exists some $N \in \mathbb{N}$ such that $\frac{2}{N} < \delta$, and thus if $n > N$, we have that $\frac{2}{\pi(1+4n)} < \frac{2}{n} < \frac{2}{N} < \delta$. We also have that $f(\frac{2}{\pi(1+4n)}) = 1$. Thus, we have that

$$\lim \sup_{t\to 0^+} f(t) = 1.$$

Similarly we can always find some $N \in \mathbb{N}$ such that $\frac{2}{\pi(1+2n)} < \delta$ and thus if for any $f(\frac{2}{\pi(1+2n)}) = -1$, and thus

$$\liminf_{t\to 0^+} f(t) = -1.$$

Thus, $\lim_{t\to 0^+} f(t)$ does not exist since the limsup and liminf do not equal each other, and so we have an oscillating discontinuity at $x = 0$. Thus, we have that $\text{osc}_0(f) = 2$.

We do not have any other jump discontinuities, since $\sin\left(\frac{1}{x}\right)$ is continuous for all $x > 0$. One can see this because $\sin(y)$ is continuous for any $y \in \mathbb{R}$ and $\frac{1}{x}$ is continuous for any $x > 0$. Thus, f still has countably many discontinuities. □

(c)

Example 1.110. What about the characteristic function of the rationals?

Proof. Define $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ such that

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} .$$

We claim that $\chi_{\mathbb{Q}}$ has no jump discontinuities, but instead has all oscillating discontinuities. Suppose it did have some jump discontinuity at some $x \in \mathbb{R}$. Thus,

$$L := \lim_{t \to x^-} f(t), \qquad R := \lim_{t \to x^+} f(t)$$

both exist. However, if $\epsilon = \frac{1}{2}$, then for any $\delta > 0$, since there exists some $q \in \mathbb{Q}$ and some $r \in \mathbb{R}\mathbb{Q}$ such that $q, r \in (x - \delta, x)$, then we have that

$$\limsup_{\delta \to 0} f(x - \delta) - \liminf_{\delta \to 0} f(x - \delta) = 1 - 0 = 1.$$

Thus, since $\limsup \neq \liminf$, then the left hand limit cannot exist. Similarly, the right hand limit does not exist. Thus $\chi_{\mathbb{Q}}$ has no jump discontinuities but instead has oscillating discontinuities everywhere. $\square$

PSET 4: Problem 5

Example 1.111. Suppose that $f : \mathbb{R} \to [-M, M]$ has no jump discontinuities. Does f have the intermediate value property?

Proof. Consider $f : \mathbb{R} \to [-1, 3]$ defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x > 0 \\ 3, & x \leq 0 \end{cases}$$

We proved in the last problem that f has no jump discontinuities and that $f((0, \infty) = [-1, 1]$. However, f does not take the value 2, and so f does not take the IVP. $\square$

PSET 4: Problem 6

(a)

Example 1.112. Define the oscillation for a function from one metric space to another, $f : M \to N$.

Proof. The oscillation for a function at a point $x \in M$ can be defined by

$$\text{osc}_x f = \lim_{r\to 0}\left(\sup_{t,s\in B_r(x)} d(f(t), f(s))\right) = d(\limsup_{t\to x} f(t), \liminf_{t\to x} f(t)).$$

Note that the rightmost equality exists only if there exists some notion of distance between functions that defines the lim sup and lim inf . ☐

(b)

Example 1.113. Is it true that f is continuous at a point if and only if its oscillation is zero there. Prove or disprove.

Proof. **Yes.**

(i) ($\implies$) : Suppose f is continuous at x. Let $\epsilon > 0$. There exists a $\delta > 0$ such that if $y \in \mathbb{R}$ with $d(x, y) < \delta$, we have that $d(f(x), f(y)) < \frac{\epsilon}{2}$. Thus, as $r \to 0$, we have that $r < \delta$, and thus for all $t \in B_r(x)$, $d(x, t) < r < \delta$, and so $d(f(t), f(x)) < \frac{\epsilon}{2}$. Thus, if $s, t \in B_r(x)$, we have that

$$d(f(s), f(t)) \le d(f(s), f(x)) + d(f(x), f(s)) < \epsilon.$$

Thus, $\sup(d(f(s), f(t)) < \epsilon$. Because this is true for all ϵ, then we have that a $\text{osc}_x(f) = 0$.

(ii) ($\impliedby$) : Suppose $\text{osc}_x(f) = 0$, and assume that f is not continuous at x. Thus, there exists some $\epsilon > 0$ such that if $\delta > 0$ and $y \in \mathbb{R}$ with $d(x, y) < \delta$, we have that $d(f(x), f(y)) < \epsilon$. Thus, we can take $\delta \to 0$ and still we have that for any $s, t \in B_{\frac{\delta}{2}}(x)$, we have that

$$d(s, t) \le d(s, x) + d(x, t) < \delta,$$

and thus $d(f(s), f(t)) \ge \epsilon$, implying that $\sup(f(s), f(t)) \ge \epsilon$. Thus, $\text{osc}_x(f) \ne 0$, a contradiction.

☐

(c)

Example 1.114. Is the set of points at which the oscillation of f is $\ge \frac{1}{k}$ closed in M?

Proof. **Yes.** This proof follows exactly as in Problem 3, and so we omit some details that we clarify there:

Let

$$D_k = \{x \in \mathbb{R} | \text{osc}_x(f) \ge \frac{1}{k}\}$$

Let x_k be a limit point of D_k. Thus, there exists some sequence $(x_n) \in D_k$ such that $x_n \to x_k$. Since $x_n \in D_k$ for all n, then we have that for any $n \in \mathbb{N}$, $\text{osc}_{x_n} f \ge \frac{1}{k}$. For any $r \ge 0$, we have that there exists some x_n such that $d(x_n, x_k) \le \frac{r}{2}$. Thus, we must have that for large n, $B_r(x_n) \subset B_r(x_k)$, and thus

$$\frac{1}{k} \le \lim_{n\to\infty} \lim_{r\to 0} \sup_{s,t\in B_r(x_n)} f(t) \le \lim_{r\to 0} \sup_{s,t\in B_r(x_k)} f(t).$$

Thus $D_{\frac{1}{k}}$ is closed. ☐

PSET 4: Problem 7

(a)

Example 1.115. Prove that the integral of the Zeno's staircase function is $\frac{2}{3}$

Proof. Geometric argument:

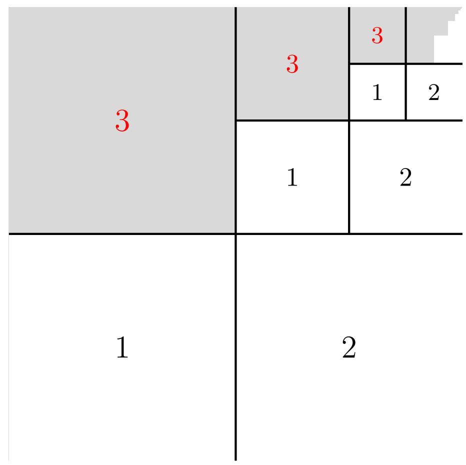


Figure 76: Zeno's staircase

Thus, we have that the area is $\frac{2}{3}$.

Zeno's staircase is constructed as follows:

$$Z(x) = \sum_{k=1}^{n} \frac{1}{2^k}, x \in [\sum_{k=1}^{n-1} \frac{1}{2^k}, \sum_{k=1}^{n} \frac{1}{2^k}).$$

For example, for $n = 1$, we have that $Z(x) = \frac{1}{2}$ for $x \in [0, \frac{1}{2})$. and $Z(x) = \frac{3}{4}$ for $x \in [\frac{1}{2}, \frac{3}{4})$. We have that Z is integrable on $[0, 1]$ by the Riemann-Lebesgue, since the set of discontinuities is countable, since each one happens every $\frac{2^n-1}{2^n}$. Thus, we have that

$$\begin{aligned}
\int_0^1 Z(x) &= \int_0^{\frac{1}{2}} Z(x) + \int_{\frac{1}{2}}^{\frac{3}{4}} Z(x) + \int_{\frac{3}{4}}^{\frac{7}{8}} Z(x) + \dots \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \int_{\sum_{k=1}^{n-1} \frac{1}{2^k}}^{\sum_{k=1}^{n} \frac{1}{2^k}} \sum_{k=1}^{n} \frac{1}{2^k} dx \\
&= \lim_{n\to\infty} \left(\frac{1}{2}\frac{1}{2} + \frac{1}{4}\frac{3}{4} + \frac{1}{8}\frac{7}{8} + \dots + \frac{1}{2^n} \sum_{k=1}^{n} \frac{1}{2^k} \right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{2^k} \frac{2^k - 1}{2^k} \right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{2^k - 1}{2^{2k}} \right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} (\frac{1}{2})^k - \sum_{k=1}^{n} (\frac{1}{4})^k \right)
\end{aligned}$$

$$= 2 - \frac{4}{3}$$
$$= \frac{2}{3}.$$

Where we used the geometric series formula. $\square$

(b)

Example 1.116. What about the Devil's staircase?

Proof. We claim that the devil's staircase satisfies the following for all $c \in [0, 1]$

$$H(x) + H(1 - x) = 1.$$

Let $x \in C$, then x can be uniquely expressed by

$$x = 0.\omega_1\omega_2, \dots,$$

where ω_i is 0, 1 or ,2 The cantor function will send $x \in C$ to

$$H(x) = \sum_{i=1}^{\infty} \frac{\omega_i/2}{2^i}.$$

If $x \notin C$, H has equal values at the endpoints of the discarded gap intervals and so we extend H to them by letting it be constant on each. Thus, suppose $x \in [0, 1]$, then we have that $1 - x = 0.\omega_1'\omega_2', \dots,$ where

$$\omega_i' = 2 - \omega_i$$

Thus, we have that if $x \in C$, then $1 - x \in C$ (still just expressed in 0 and 2) and using the geometric series convergence formula, we have that

$$H(x) + H(1 - x) = \sum_{i=1}^{\infty} \frac{\omega_i/2}{2^i} + \sum_{i=1}^{\infty} \frac{\omega_i'/2}{2^i} = \sum_{i=1}^{\infty} \frac{\omega_i/2}{2^i} + \sum_{i=1}^{\infty} \frac{(2 - \omega_i)/2}{2^i} + \sum_{i=1}^{\infty} \frac{2}{2^{i+1}} = 1.$$

We ignore $x \in [0, 1]C$ since we just extend the values at the discarded gap intervals, and so we can extend this result to those points. Thus,

$$\int_0^1 H(x) + \int_0^1 H(1 - x)dx = \int_0^1 1dx = 1,$$

and thus using $u = 1 - x$ and $\frac{du}{dx} = -1$, we have

$$\int_0^1 (H(x) + H(1 - x))dx = \int_0^1 H(x)dx - \int_1^0 H(u)dx = \int_0^1 H(x)dx + \int_0^1 H(u)dx = 2\int_0^1 H(x)dx.$$

Thus, we have that $\int_0^1 H(x)dx = \frac{1}{2}$. $\square$

PSET 4: Problem 8

(a)

Example 1.117. Construct a function $f : [-1, 1] \to \mathbb{R}$ such that

$$\lim_{r\to 0}\left(\int_{-1}^{-r} f(x)dx + \int_{r}^{1} f(x)dx\right)$$

exists but the integral $\int_{-1}^{1}$ does not exist.

Proof. Consider $f : [-1, 1] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

We have that

$$\lim_{r\to\infty}\left(\int_{-1}^{-r} \frac{1}{x}dx + \int_{r}^{1} \frac{1}{x}dx\right) = 0$$

because of symmetry. However, when considering the full interval, 0 must be included in some subinterval, and thus we would have that "height" of the Riemann rectangle would be infinite, implying that $\int_{-1}^{1} f$ does not exist. □

(b)

Example 1.118. Do the same for a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{R\to\infty}\int_{-R}^{R} g(x)dx$$

exists but $\int_{-\infty}^{\infty} g(x)dx$ fails to exist.

Proof. Consider $g : \mathbb{R}$ to $\mathbb{R}$ such that

$$g(x) = x^3.$$

Because of symmetry and for the same reason as above, we have that the first integral exists and equals 0:

$$\lim_{R\to\infty}\int_{-R}^{R} g(x)dx = \lim_{R\to\infty}\left(\int_{-R}^{0} g(x)dx + \int_{0}^{R} g(x)dx\right) = 0.$$

However, we have that

$$\left(\int_{-\infty}^{0} g(x)dx + \int_{0}^{\infty} g(x)dx\right) = \infty - \infty,$$

which is indeterminate. □

PSET 4: Problem 9

(a)

Example 1.119. If $\sum a_n$ converges and (b_n) is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Proof. We claim that if $(c_n) \to 0$ is monotone decreasing, then $\sum a_k c_n$ converges. By Luis' hint during office hours, we have that:

$$\begin{aligned}\sum_{k=1}^{n-1} A_k(c_k - c_{c+1}) &= A_1(c_1 - c_2) + A_2(c_2 - c_3) + \dots \\ &= \sum_{k=1}^{n-1} a_k c_k - A_{n-1}c_n,\end{aligned}$$

which can be proved by induction. We have that since A_n converges, then it is bounded by some $|A_k| \leq A$. Thus, we note that since $c_k - c_{k+1} \geq 0$ for all k, then

$$\begin{aligned}\left|\sum_{k=m+1}^{n} a_k c_k\right| &= \left|\sum_{k=1}^{n-1} A_k(c_k - c_{k+1}) + A_{n-1}c_n - (\sum_{k=1}^{m} A_k(c_k - c_{k+1}) + A_m c_{m+1})\right| \\ &= \left|\sum_{k=m+1}^{n-1} A_k(c_k - c_{k+1}) + A_{n-1}c_n + A_m c_{m+1}\right| \\ &\leq \left|\sum_{k=m+1}^{n-1} A(c_k - c_{k+1}) + Ac_n + Ac_{m+1}\right| \\ &\leq A\left(\sum_{k=m+1}^{n-1} (c_k - c_{k+1}) + c_n + c_{m+1}\right) \\ &= A\left(2c_{m+1}\right) \\ &< 2A\frac{\epsilon}{2A}.\end{aligned}$$

The last inequality comes from the fact that $c_n \to 0$. Suppose that b_n is monotone increasing and bounded. Thus, it converges to some b. Let $c_n = b - b_n$. Then we have that $c_n \to 0$ and c_n is monotonically decreasing. Thus, by the work above, we have that

$$\sum_{k=1}^{n} a_k(b - b_k) = b\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_k b_k$$

converges. Since $b\sum_{k=1}^{n} a_k$ converges, then $\sum_{k=1}^{n} a_k b_k$ converges. Suppose b_n is monotone decreasing and bounded, then $b_n - b \to 0$ and $c_n = b_n - b$ is monotone decreasing. Thus, we can use the same strategy as above to show that $\sum a_k b_k$ converges. □

(b)

Example 1.120. If the monotonicity condition is dropped, or replaced by the assumption that $\lim_{n\to\infty} b_n = 0$, find a counter-example to convergence of $\sum a_n b_n$.

Proof. Suppose the monotonicity example is dropped. Then let $b_n = \frac{(-1)^n}{n^{\frac{1}{2}}}$ and let $a_n = b_n$. We know that $a_n \to 0$ and that $\sum a_n$ converges by alternating series test. We have that b_n is not monotonic and $b_n \to 0$. Thus, we have that

$$\sum a_n b_n = \sum \frac{(-1)^n}{n^{\frac{1}{2}}} \frac{(-1)^n}{n^{\frac{1}{2}}} = \sum \frac{1}{n},$$

which does not converge. $\square$

PSET 4: Problem 10

Example 1.121. An infinite product is an expression $\prod c_k$ where $c_k > 0$. The nth partial product is $C_n = c_1 \cdots c_n$. If C_n converges to a limit $C \neq 0$ then the product converges to C. Write $c_k = 1 + a_k$. If each $a_k \geq 0$ or each $a_k \leq 0$ prove that $\sum a_k$ converges if and only if $\prod c_k$ converges.

Proof. We claim that if $|a_k| \leq 1$, we have that

$$1 + \sum_{k=1}^{n}(a_k) \leq \prod_{k=1}^{n}(1 + a_k) \leq e^{\sum_{k=1}^{n}(a_k)} \tag{7}$$

To see the first inequality, consider that we can expand the right side:

$$\begin{aligned}\prod_{k=1}^{n}(1 + a_k) &= (1 + a_1)(1 + a_2)(1 + a_3) \cdot \cdots \cdot (1 + a_n) \\ &= (1 + a_1 + a_2 + a_1a_2)(1 + a_3) \cdot \cdots \cdot (1 + a_n) \\ &= 1 + a_1 + a_2 + a_1a_2 + a_3 + a_1a_4 + a_2a_3 + a_1a_2a_3 \cdot \cdots \cdot (1 + a_n) \\ &= 1 + \sum a_n + K,\end{aligned}$$

where K is some constant. That is, the product of the sum contains the sum in it. To see the second inequality, we need to show that

$$\ln(1 + a_k) \leq a_k. \tag{8}$$

To see this, consider that this is equivalent to showing that

$$1 + a_k \leq e^{a_k} \iff 0 \leq e^x - (1 + x) = f(x)$$

for $|x| < 1$ (since $c_k > 0$.) Consider that by taking derivatives, we have that

$$f'(x) = e^x - 1, \qquad f''(x) = e^x.$$

Thus, $f(x)$ has a critical point at $x = 0$, and $f(x)$ is concave up, so $x = 0$ is the local minimum for $-1 < x < 1$. Thus, $0 \leq e^x - (1 + x)$ for all $|x| < 1$. Thus, we have (4), implying that

$$\sum_{k=1}^{n}\ln(1 + a_k) \leq \sum_{k=1}^{n} a_k.$$

Thus, using logarithm rules, we have that

$$\ln\left(\prod_{k=1}^{n}(1 + a_k)\right) \leq \sum_{k=1}^{n} a_k \iff \prod_{k=1}^{n}(1 + a_k) \leq e^{\sum_{k=1}^{n} a_k}.$$

- ($\implies$) : Suppose $\sum a_k$ converges, then since $c_k = 1 + a_k > 0$, we must have that $|a_k| < 1$ for large k. By (3), we have that for large m,

  $$0 \leq \prod_{k=m+1}^{n} c_k \leq e^{\sum_{k=m+1}^{n} a_k},$$

  and thus since ln is continuous and monotonic, we have that

  $$\ln\left(\prod_{k=1}^{n} c_k\right) \leq \sum_{k=1}^{n} a_k,$$

and thus

$$\sum (\ln(1+a_k)) \leq e^{\sum_{k=m+1}^n a_k}.$$

Thus, if $a_k \geq 0$ for all k, then $|\ln(1+a_k)| = \ln(1+a_k) \leq a_k$, and thus the series converges by the comparison test. If $a_k \leq 0$ for all k, then $\ln(1+a_k)$ since $a_k \to 0$, we have that $\ln(1+a_k)$ is increasing. Thus, if $\sum_{k=m+1}^n \ln(1+a_k)$ is an increasing sequence of partial sums that is bounded above, and thus converges. Thus, since ln is continuous, we have that

$$\lim_{n\to\infty} \ln\left(\prod_{k=m+1}^{n} c_k\right) = \ln\left(\lim_{n\to\infty}\prod_{k=m+1}^{n} c_k\right),$$

and so we must have that $\prod c_k$ converges.

- ($\Longleftarrow$): Suppose $\prod c_k$ converges. Since $c_k = 1 + a_k > 0$, then we must have for large k, $|a_k| < 1$. Thus, by (3), we have that for large m,

$$1 + \sum_{k=m+1}^{n} a_k \leq \prod_{k=m+1}^{n} c_k \leq e^{\sum a_k}.$$

Thus, we have that if $a_k \geq 0$ for all k, then the tail of $\sum a_k$ is increasing and bounded above and thus converges. If $-1 < a_k \leq 0$ for all k, then $\sum_{k=1}^n a_k = A_n$ is decreasing, and thus e^{A_n} is decreasing and positive, and thus the tail of e^{A_n} is decreasing and bounded below by our product, and thus converges.

□

Reflections: An alternative method would be to use *L'hopital*.

$$\lim_{k\to\infty} \frac{\ln(1+a_k)}{a_k} = \lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{1}{1+x} = 1,$$

and thus we have that for large enough k, $\ln(1+a_k) = a_k$. Thus, for large n, m :

$$\sum_{k=m+1}^{n} \ln(1+a_k) = \sum_{k=m+1}^{n} a_k + \epsilon,$$

1.32 PSET 5

PSET 5: Problem 1

Example 1.122. Suppose that $f_n : [a, b] \to \mathbb{R}$ and $f_n \to f$ uniformly. Which of the following discontinuity properties of the function f_n carries over to f?

(a)

Example 1.123. No discontinuities

Proof. **Yes.** If f_n has no discontinuities, then it is continuous. Since $f_n \to f$ unif., we have that f is continuous, and thus has no discontinuities. □

(b)

Example 1.124. At most ten discontinuities.

Proof. **Yes.** Suppose f has a discontinuity point $d \in [a, b]$. Suppose that there are infinitely many $n \in \mathbb{N}$ such that f_n is continuous at d, then create a subsequence of such f_{n_k}. Evidently, we have that $f_{n_k} \to f$ uniformly, and thus by the same theorem used in the part above, f is continuous at d, which is a contradiction. Thus, there exists some $N \in \mathbb{N}$ such that if $n \geq N$, then f_n is discontinuous at d. Suppose f has more than 10 discontinuities. Let d_1 be the first such discontinuity, then there exists some $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then f_n has a discontinuity at d_1. Continue this process for all k discontinuities of f, then for $N = \max\{N_1, \ldots, N_k\}$, if $n \geq N$, we have that f_n has more than ten discontinuities, which is a contradiction. ☐

(c)

Example 1.125. At least ten discontinuities.

Proof. **No.** Let $f_n : [a, b] \to \mathbb{R}$ and $Z_n = \{a + \frac{b-a}{i}\}_{i=1}^{N}$, where $N > 10$. such that

$$f_n(x) = \begin{cases} 0, & x \notin Z_n \\ \frac{1}{n}, & x \in Z_n \end{cases} .$$

Evidently, $f_n(x)$ has $N > 10$ discontinuities for each n. However, we have that $f_n(x) \to 0$ uniformly, which has no discontinuities. ☐

(d)

Example 1.126. Finitely many discontinuities

Proof. **No.** Consider a sequence of functions which has an increasing number of discontinuities and uniformly converges to a function with infinitely many discontinuities. Consider $f_n : [0, 1] \to \mathbb{R}$ such that

$$f_n(x) = \begin{cases} 0, & x < \frac{1}{n} \\ \frac{1}{n}, & x \geq \frac{1}{n} \end{cases} .$$

Thus,

$$f_1(x) = \begin{cases} 0, & x < 1 \\ 1, & x = 1 \end{cases} , \qquad f_2(x) = \begin{cases} 0, & x < \frac{1}{2} \\ \frac{1}{2}, & x \in [\frac{1}{2}, 1) \\ 1, & x = 1 \end{cases} , f_3(x) = \begin{cases} 0, & x < \frac{1}{3} \\ \frac{1}{3}, & x \in [\frac{1}{3}, \frac{1}{2}) \\ \frac{1}{2}, & x \in [\frac{1}{2}, 1) \\ 1, & x = 1 \end{cases}$$

Evidently, each f_n has n discontinuities. We claim that $f_n \to f$ uniformly, where

$$f = \begin{cases} 0, & x = 0 \\ x', & x > 0 \end{cases} ,$$

where $x' = \frac{1}{k}$, where k is the the smallest natural such that $\frac{1}{k} < x$. To see this, let $\epsilon > 0$, then there exists some $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ and thus, if $n \geq N$ and $x > \frac{1}{n}$, then we have that $f_n(x) = x' = f(x)$. Else, if $0 < x \leq \frac{1}{n}$, then $f(x) - f_n(x) = x' = \frac{1}{k}$, where we have by assumption that $k > n \geq N$. Thus, for some N, if $n \geq N$, we have

$$|f(x) - f_n(x)| < \epsilon, \quad \forall x \in [0, 1].$$

Evidently, $f(x)$ has infinitely many discontinuities. ☐

(e)

Example 1.127. Countably many discontinuities, all of the jump type.

Proof. **Yes.** Suppose not, then f either had uncountably many jump discontinuities, or it has some oscillating discontinuity. The first case is a contradiction by last PSET, since a function can only have countably many jump discontinuities. The second is a contradiction by the last part. $\square$

(f)

Example 1.128. No jump discontinuities.

Proof. **No.** Define $f_n : [-1,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{n}\sin\left(\frac{1}{x}\right), & x > 0 \\ 420, & x \leq 0 \end{cases}.$$

We proved in the last PSET that f_n has no jump discontinuities. We claim that $f_n \to f$ uniformly, where

$$f = \begin{cases} 0, & x > 0 \\ 420, & x \leq 0 \end{cases}.$$

It is obviously true for any $x \leq 0$, so consider the case when $x > 0$. Let $\epsilon > 0$, and let $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Thus, we have that if $n \geq N$:

$$|f_n(x) - f(x)| = |\frac{1}{n}\sin\left(\frac{1}{x}\right)| \leq \frac{1}{n} < \epsilon,$$

for any $x > 0$. Thus, since $f(x)$ has a jump discontinuity at $x = 0$, we are done. $\square$

(g)

Example 1.129. No oscillating discontinuities.

Proof. **Yes.** Suppose (f_n) is a sequence with no jump discontinuities, and suppose that f has at least one jump discontinuity at some point $d \in [a,b]$. Thus, we have that there exists some $\epsilon > 0$ such that

$$\lim_{r\to 0}\left(\sup_{s,t\in B_r(d)} d(f(s), f(t) \geq \epsilon\right).$$

Thus, since we are in $[a,b] \subset \mathbb{R}$ we have that

$$|\limsup_{x\to d} f(x) - \liminf_{x\to d} f(x)| \geq \epsilon.$$

We claim that there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, we have that

$$|\limsup_{x\to d} f_n(x) - \liminf_{x\to d} f_n(x)| \geq \epsilon.$$

Suppose not, that is, there exists infinitely many $n \in \mathbb{N}$ such that

$$|\limsup_{x\to d} f_n(x) - \liminf_{x\to d} f_n(x)| < \epsilon.$$

Take a subequence (f_{n_k}) of such (f_n), then we have that $(f_{n_k}) \to f$ uniformly, and thus

$$\begin{aligned}
|\limsup_{x\to d} f(x) - \liminf_{x\to d} f(x)| \leq& \\
&\leq |\limsup_{x\to d} f(x) - \limsup_{x\to d} f_{n_k}(x)| + \\
&+ |\limsup_{x\to d} f_{n_k}(x) - \liminf_{x\to d} f_{n_k}(x)| + \\
&+ |\liminf_{x\to d} f_{n_k}(x) - \liminf_{x\to d} f_{n_k}(x)|.
\end{aligned}$$

Thus, it suffices to show that the first term is less than $\frac{\epsilon}{3}$. However, this comes straight from uniform convergence, since the difference is bounded above by $\frac{\epsilon}{3}$. Thus, we have a contradiction, and so there must exist some $N \in \mathbb{N}$ such that if $n \geq N$, we have that

$$\mathrm{osc}_d(f_n(x)) \geq \epsilon,$$

which contradicts the fact that (f_n) has no jump discontinuities. $\square$

PSET 5: Problem 2

Example 1.130. Is the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \cos(n+x) + \log\left(1 + \frac{1}{\sqrt{n+2}} \sin^2(n^n x)\right)$$

equicontinuous?

Proof. We claim that it suffices to show that each term is equicontinuous. Why? Suppose $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n = g_n + h_n$, where g_n and h_n are equicontinuous and defined on the same domain. Let $\epsilon > 0$ and $x, y \in \mathbb{R}$, then there exists a $\delta = \min\{\delta_g, \delta_h\}$ such that if $|x-y| < \delta$ and $n \in \mathbb{N}$, we have that

$$|f_n(x) - f_n(y)| = |g_n(x) + h_n(x) - g_n(y) - h_n(y)| \leq |g_n(x) - g_n(y)| + |h_n(x) - h_n(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

Where δ_f is the distance needed such that if $|x-y| < \delta_f$, we have by equicontinuity of g that for any $n \in \mathbb{N}:$ $|g_n(x) - g_n(y)| < \frac{\epsilon}{2}$. Ditto for h. Thus, the sequence of f_n is equicontinuous.

Consider first $g_n(x) = \cos(n+x)$. We claim that it is equicontinuous. It suffices by the book to show that g_n' is uniformly bounded. Here we use the fact that cosine is everywhere differentiable. Thus, for any $n \in \mathbb{N}:$

$$g_n'(x) = -\sin(n+x) \leq 1.$$

Thus, by the MVT, we have that if $s, t \in \mathbb{R}$ and $n \in \mathbb{N}:$

$$|g_n(s) - g_n(t)| \leq 1|x-y| < \epsilon.$$

Thus, $\cos(n+x)$ is equicontinuous.

We want to show that $h_n = \log\left(1 + \frac{1}{\sqrt{n+2}} x\right)$ is uniformly convergent and we claim that this shows that h_n is uniformly equicontinuous since we are working in a compact space of $[1, 1 + \frac{1}{\sqrt{3}}]$ and since each f_n is continuous.

To prove this claim, let $h_n \to h$ uniformly on a compact space X. Compactness gives totally bounded, that is for any $\epsilon > 0$, we have a finite covering of X by ϵ balls.

Since $h_n \to h$ uniformly, then for some $N \in \mathbb{N}$, if $n \geq N$, we have that $|h_n(x) - h(x)| < \frac{\epsilon}{3}$ for all $x \in X$. Thus, h_n is Cauchy, and thus we have that

$$|h_n(x) - h_N(x)| < \frac{\epsilon}{3}.$$

Since X is compact and h_n is continuous, then h_n is uniformly continuous. Thus, there exists some $\delta > 0$ such that if $|x-y| < \delta$, we have that $|h_N(x) - h_N(y)| < \frac{\epsilon}{3}$. Thus for any $n \geq N$, we have:

$$|h_n(x) - h_n(y)| \leq |h_n(x) - h_N(x)| + |h_N(x) - h_N(y)| + |h_N(y) - h_n(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}.$$

Thus, we have that (h_n) is equicontinuous for all $n \geq N$, leaving only finitely many to be dealt with, which is no problem, since we can just take the maximum N that makes each one possible.

We claim that $h_n \to 0$ unif. on $[1, 1+\frac{1}{\sqrt{3}}]$. To see this, let $\epsilon > 0$ and $x, y \in X$. If $\frac{1}{\sqrt{N}} < \epsilon$ and $n > N$, then using the inequality from last pset that $\ln(1+a_k) \leq a_k$:

$$\begin{aligned}
|h_n - 0| &= \log\left(1 + \frac{1}{\sqrt{n+2}} \sin^2(n^n x)\right) \\
&\leq \frac{1}{\sqrt{n+2}} \sin^2(n^n x) \\
&\leq \frac{1}{\sqrt{n+2}}(1) \\
&< \epsilon.
\end{aligned}$$

Thus, $h_n \to 0$ unif. Prove that this implies that it is equicontinuous. $\square$

PSET 5: Problem 3

Example 1.131. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and the sequence $f_n(x) = f(nx)$ is equicontinuous, what can be said about f?

Proof. We have that f is constant. Suppose not, and for some $x, y \in \mathbb{R}$, we have that $|f(x) \neq f(y)|$. Let $\epsilon = \frac{|f(x)-f(y)|}{2}$, and take any $\delta > 0$. For n large enough, we have that $|\frac{x}{n} - \frac{y}{n}| < \delta$ and thus

$$|f_n(\frac{x}{n}) - f_n(\frac{y}{n})| = |f(x) - f(y)| \geq \epsilon.$$

Thus, (f_n) is not equicontinuous, which is a contradiction. $\square$

PSET 5: Problem 4

Example 1.132. A continuous, strictly increasing function $\mu : (0, \infty) \to (0, \infty)$ is a *modulus of continuity* if $\mu(s) \to 0$ as $s \to 0$. A function $f : [a, b] \to \mathbb{R}$ has a modulus of continuity μ if $|f(s) - f(t)| \leq \mu(|s - t|)$ for all $s, t \in [a, b]$.

(a)

Example 1.133. Prove that a function is uniformly continuous if and only if it has a modulus of continuity.

Proof. :

- ($\implies$) Define $\mu : (0, \infty) \to (0, \infty)$ by:

$$\mu(x) = \sup_{|s-t|<x} d(f(s), f(t)) + x.$$

(i) To prove that μ is well defined, take some $x \in (0, \infty)$. Obviously, the domain is well defined. Consider that if $x \in (0, \infty)$ since $[a, b]$ is connected, there exist $s, t \in [a, b]$ such that $|s-t| < x$. Thus, we have that since a metric is always non-negative and $x \in (0, \infty)$, then $\mu(x)$ exists and is positive. Note that we have that $\sup_{|s-t|\leq x} d(f(s), f(t)) < \infty$ because f is uniformly continuous, and thus continuous, and thus achieves its maximum and minimum over $[a, b]$, which are both finite. Thus, $\mu \in ((0, \infty) \times (0, \infty))$. Suppose $x = x'$, then since sup is unique, we have that $\mu(x) = \mu(x')$. Thus, μ is well defined.

(ii) To prove that μ is a strictly increasing function, let $x, y \in (0, \infty)$ with $x < y$. Consider that by work done in a previous PSET, we have that:

$$\sup_{|s-t|<x} d(f(s), f(t)) \leq \sup_{|s-t|<y} d(f(s). \tag{9}$$

Thus, we have that $\mu(x) < \mu(y)$.

(iii) To prove that as $x \to 0$, we have $\mu(x) \to 0$, consider some sequence $(x_n) \to 0$. Since we have that f is uniformly continuous, then there exists some $\delta_k > 0$ such that if $|s - t| < \delta_k$, we have $|f(s) - f(t)| < \frac{1}{k}$. Consider that since $x_n \to 0$, then for any $k \in \mathbb{N}$, we have that for n large, $x_n \leq \delta_k$. Thus, we have that for any k,

$$\mu(x_n) = \sup_{|s-t|<x_n} d(f(s), f(t)) + x_n \leq \sup_{|s-t|<\delta_k} d(f(s), f(t)) + x_n < \frac{1}{k} + x_n \to 0$$

as $n \to \infty$.

(iv) To prove that μ is continuous,it will suffice to show that

$$\gamma(x) = \sup_{|s-t|<x} d(f(s), f(t))$$

is continuous. We will first show that

$$\lim_{s\to t^-} \gamma(s) = \gamma(t).$$

Take a sequence $\delta_n \to \delta^+$, then we have that there exist $x, y \in [a, b]$ such that $|x - y| < \delta$, and so there exists large enough n such that $|x - y| < \delta_n$. Thus, we have that since γ is increasing:

$$\sup_{|x-y|<\delta_n} d(f(x), f(y)) = \gamma(\delta_n) \leq \gamma(\delta) \implies \gamma(\delta_n) \to \gamma(\delta).$$

Assume it is not right continuous at some $s \in [a,b]$. In particular, there exists some ϵ such that for all $\delta > 0$, we have that if $0 < y - s < \delta$, then $|\gamma(y) - \gamma(s)| \geq \epsilon$. In other words, we have that

$$|\gamma(s+\delta) - \gamma(x)| \geq \epsilon.$$

In particular, this means that there exist $x, y \in [a,b]$ such that $|x-y| < s+\delta$ and

$$|d(f(x), f(y)) - \gamma(s)| \geq \epsilon. \tag{10}$$

Since f is uniformly continuous, there exists some $\delta_f > 0$ such that if

$$|p,q| < \delta_f \implies d(f(p), f(q)) < \frac{\epsilon}{2}.$$

Fix the δ above to be δ_f. Find some $\alpha, \beta \in [a,b]$ such that $[\alpha, \beta] \subset [x,y]$ and $d(\alpha, x) < \frac{\delta}{2}$, and $d(\beta, y) < \frac{\delta}{2}$ and $d(\alpha, \beta) \leq s$. Then we have that

$$d(f(x), f(\alpha)) < \frac{\epsilon}{2}, \quad d(f(\beta), f(y)) < \frac{\epsilon}{2}.$$

By (2), since $|\alpha - \beta| \leq s$, we have that

$$||f(x) - f(y)| - |f(\alpha) - f(\beta)|| \geq \epsilon,$$

but by reverse triangle inequality we have that

$$\begin{aligned}
||f(x) - f(y)| - |f(\alpha) - f(\beta)|| &\leq |f(x) - f(\alpha) - f(y) - f(\beta)| \\
&\leq |f(x) - f(\alpha)| + |f(y) - f(\beta)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

A contradiction! Thus, $\gamma(x)$ is continuous, and adding a continuous function is still continuous, and so μ is continuous.

(v) To prove that μ satisfies the main shabang of being modulus of continuity, consider that since f is continuous, then if $x, y \in [a,b]$, it is clear that since the maximum of f over $[x,y]$ is attained somewhere in $[x,y]$ then

$$d(f(x), f(y)) \leq \sup_{|s-t| \leq |x-y|} d(f(s), f(t)).$$

Thus, we have that

$$d(f(x), f(y)) \leq \sup_{|s-t| \leq |x-y|} d(f(s), f(t)) + |x-y|.$$

- ($\Longleftarrow$) Suppose $f : [a,b] \to \mathbb{R}$ has a modulus of continuity μ. Thus, let $\epsilon > 0$ and let $s, t \in [a,b]$. We claim that exists some $\delta > 0$ such that $\mu(\delta) < \frac{\epsilon}{2}$. Suppose not, then for all $x > 0$, we have that $\mu(\delta) \geq \frac{\epsilon}{2}$, which is a contradiction to the fact that $\mu(x) \to 0$ as $x \to 0$. Thus, if $|s-t| < \delta$, then $\mu(|s-t|) < \mu(\delta) = \frac{\epsilon}{2}$, and so

$$|f(s) - f(t)| < \epsilon.$$

$\square$

(b)

Example 1.134. Prove that a family of functions is equicontinuous if and only if its members have a common modulus of continuity.

Proof. Let $\mathscr{E}$ be a family of functions index by α such that $f_\alpha \in \mathscr{E}$.

- ($\Longrightarrow$) Define $\mu : (0, \infty) \to (0, \infty)$ by:

$$\mu(x) = \sup_{|s-t|<x, f_\alpha \in \mathscr{E}} d(f_\alpha(s), f_\alpha(t)) + x.$$

Note that first we take the sup over every pair $s, t \in [a, b]$ such that $|a - b| < x$ and then we take the sup over all $f_\alpha \in \mathscr{E}$

(i) To prove that μ is well defined, take some $x \in (0, \infty)$. Obviously, the domain is well defined. Consider that if $x \in (0, \infty)$ since $[a, b]$ is connected, there exist $s, t \in [a, b]$ such that $|s-t| < x$. Thus, we have that since a metric is always non-negative and $x \in (0, \infty)$, then $\mu(x)$ exists and is positive. Note that we have that $\sup_{|s-t|\leq x} d(f_\alpha(s), f_\alpha(t)) < \infty$ because f is uniformly continuous, and thus continuous, and thus achieves its maximum and minimum over $[a, b]$, which are both finite. Note that we have that $\sup_{|s-t|<x, f_\alpha \in \mathscr{E}} < \infty$ by repeated use of triangle inequality: Since each f_α is uniformly continuous (equicontinuity in a compact set), then each of them achieve their maximum, and so the supremum of all of them reaches its maximum and is bounded by sum of all the individual maximums. Thus, $\mu \in ((0, \infty) \times (0, \infty))$. Suppose $x = x'$, then since sup is unique, we have that $\mu(x) = \mu(x')$. Thus, μ is well defined.

(ii) To prove that μ is a strictly increasing function, let $x, y \in (0, \infty)$ with $x < y$. Consider that by work done in a previous PSET, we have that:

$$\sup_{|s-t|<x} d(f_\alpha(s), f_\alpha(t)) \leq \sup_{|s-t|<y} d(f(s). \tag{11}$$

Because this is true for any $f_\alpha \in \mathscr{E}$, then we have that $\mu(x) < \mu(y)$.

(iii) To prove that as $x \to 0$, we have $\mu(x) \to 0$, consider some sequence $(x_n) \to 0$. Since we have that $\mathscr{E}$ is equicontinuous, then there exists some $\delta_k > 0$ such that if $|s - t| < \delta_k$, we have for any $f_\alpha \in \mathscr{E}$, $|f_\alpha(s) - f_\alpha(t)| < \frac{1}{k}$. Consider that since $x_n \to 0$, then for any $k \in \mathbb{N}$, we have that for n large, $x_n \leq \delta_k$. Thus, we have that for any k,

$$\mu(x_n) = \sup_{|s-t|<x_n, f_\alpha \in \mathscr{E}} d(f_\alpha(s), f_\alpha(t)) + x_n \leq \sup_{|s-t|<\delta_k, f_\alpha \in \mathscr{E}} d(f_\alpha(s), f_\alpha(t)) + x_n < \frac{1}{k} + x_n \to 0$$

as $n \to \infty$.

(iv) To prove that μ is continuous,it will suffice to show that

$$\gamma(x) = \sup_{|s-t|<x, f_\alpha \in \mathscr{E}} d(f_\alpha(s), f_\alpha(t))$$

is continuous. We notice however, that our proof in (a) works for any $f_\alpha \in \mathscr{E}$, and thus taking the sup over $\mathscr{E}$ yields the desired results.

(v) To prove that μ satisfies the main shabang of being modulus of continuity, consider that since f is continuous, then if $x, y \in [a, b]$, it is clear that since the maximum of f over $[x, y]$ is attained somewhere in $[x, y]$ then

$$d(f_\alpha(x), f_\alpha(y)) \leq \sup_{|s-t|\leq|x-y|} d(f_\alpha(s), f_\alpha(t)).$$

Thus, because this is true for any $f_\alpha \in \mathscr{E}$, we have that

$$d(f(x), f(y)) \leq \sup_{|s-t|\leq|x-y|, f_\alpha} d(f(s), f(t)) + |x - y|.$$

- ($\Longleftarrow$) Suppose $\mathscr{E}$ has a common modulus of continuity. That is, for any $f \in \mathscr{E}$, if $x, y \in [a, b]$ we have that $|f(x) - f(y)| \leq \mu(|x - y|)$. Let $\epsilon > 0$. By the same logic as above, there exists a $\delta > 0$ such that $\mu(\delta) < \epsilon$. Thus, if $|x - y| < \delta$, then we have that for any $f \in \mathscr{E}$,

$$|f(x) - f(y)| \leq \mu(|x - y|) < \mu(\delta) < \epsilon$$

$\square$

PSET 5: Problem 5

Example 1.135. Consider a modulus of continuity $\mu(s) = Hs^\alpha$, where $0 < \alpha \leq 1$ and $0 < H < \infty$. A function with this modulus of continuity is said to be α-Hölder,with $\alpha-$Hölder constant H.

(a)

Example 1.136. Prove that the set $C^\alpha(H)$ of all continuous functions defined on $[a, b]$ which are α-Hölder and have α-Hölder constant $\leq H$ is equicontinuous.

Proof. Let $\epsilon > 0$ and let $x, y \in [a, b]$. Let $\delta = (\frac{\epsilon}{H})^{\frac{1}{\alpha}}$. If $f \in C^\alpha(H)$, and $|x - y| < \delta$, then we have by Hölder condition that

$$|f(x) - f(y)| \leq H|x - y|^\alpha < H\left((\frac{\epsilon}{H})^{\frac{1}{\alpha}}\right)^\alpha < \epsilon.$$

Thus, $C^\alpha(H)$ is equicontinuous. □

(b)

Example 1.137. Replace $[a, b]$ with (a, b). Is the same thing true?

Proof. **Yes**. Nothing in the proof above changes. □

(c)

Example 1.138. Replace $[a, b]$ with $\mathbb{R}$. Is the same thing true?

Proof. **Yes.** Again, nothing changes from above. □

(d)

Example 1.139. What about $\mathbb{Q}$.

Proof. **Yes.** Nothing in the proof above changes. □

(e)

Example 1.140. What about $\mathbb{N}$

Proof. **Yes.** Nothing in the proof above changes. □

PSET 5: Problem 6

Example 1.141. Suppose that (f_n) is an equicontinuous sequence in $C^0[[a,b],\mathbb{R}]$ and $p \in [a,b]$ is given.

(a)

Example 1.142. If $(f_n(p))$ is a bounded sequence of real numbers, prove that (f_n) is uniformly bounded.

Proof. Since $(f_n(p))$ is bounded, let $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $|f_n(p)| < M$. Since (f_n) is equicontinuous and $p \in [a,b]$, then there exists a $\delta > 0$ such that if $|x-p| < \delta$, we have that for all $n \in \mathbb{N}$:

$$|f_n(x) - f_n(p)| < 1 \iff |f_n(x)| < f_n(p) + 1 < M + 1.$$

Thus, $f_n(x)$ is bounded for all $x \in (p-\delta, p+\delta)$, $n \in \mathbb{N}$. Let $y \in [a,b]$ and suppose $|y-p| \geq \delta$. Without loss of generality, suppose that $y > p$. Thus, we have that $p + \frac{k\delta}{2} > y$ for some $k \in \mathbb{N}$, and thus we can partition $[p,y]$ into $P = \{p, p+\frac{\delta}{2}, p+\delta, \ldots, p+\frac{k\delta}{2}\}$. We have that $y \in P$, and if $p_1 \in (p+\frac{\delta}{2}, p+\delta], p_2 \in (p+\delta, p+\frac{3\delta}{2}], \ldots, y \in (p+\frac{(k-1)\delta}{2}, p+\frac{k\delta}{2}]$, then we have that $|p_{i-1} - p_i| < \delta$, and

$$|f_n(p) - f_n(y)| \leq |f_n(p) - f_n(p_1)| + |f_n(p_1) + f_n(p_2)| + \cdots + |f_n(p_{k-2}) - f_n(y)|.$$

We showed that $f_n(p_1)$ is uniformly bounded by $M+1$. Thus, since $|p_1 - p_2| < \delta$, we have that

$$|f_n(p_1) - f_n(p_2)| < 1 \implies |f_n(p_2)| < f_n(p_1) + 1 < M + 2.$$

We can use this same strategy for every p_i and thus

$$|f_n(p_{i-1})| < M + i \implies |f_n(y)| < M + k - 2.$$

Thus, we have that for any $y \in [a,b]$ and $n \in \mathbb{N}$, $|f_n(y)| < N$, where $N \in \mathbb{N}$. $\square$

(b)

Example 1.143. Reformulate the Arzelá-Ascoli theorem with the weaker boundedness hypothesis in (a).

Proof. Suppose $(f_n) \in C^0([a,b],\mathbb{R})$ are a sequence of equicontinuous functions with X compact. If there exists some $p \in [a,b]$ such that $f_n(p)$ forms a bounded sequence of real numbers, then there exist some uniformly convergent subsequence of (f_n). $\square$

(c)

Example 1.144. Can $[a,b]$ be replaced with (a,b)? $\mathbb{Q}$?, $\mathbb{R}$?, $\mathbb{N}$?

Proof. **Well, yes, but not for 3 of them.** It cannot be replaced by $\mathbb{N}$ or $\mathbb{Q}$ because these are not connected metric spaces. $\mathbb{R}$ because of example on problem 10.

(i) It cannot be replaced with $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{N}$ because of the example in Problem 9, part c. In this example, we have that $f_n : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_n(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{n}x, & x > 0. \end{cases}$$

We show there that (f_n) is not equicontinuous. However, we obviously have that for any $x \in \mathbb{R}$, $f_n(x)$ is a bounded sequence of real numbers. We could replace the proof with $f_n : \mathbb{Q} \to \mathbb{Q}$ or $f_n : \mathbb{N} \to \mathbb{N}$ and suffer no consequences, it still will not be equicontinuous because the domains are unbounded.

(ii) It is fine to replace with (a, b) since the proof above does not change.

$\square$

Example 1.145. What is the correct generalization.

Proof. Evidently, we know by the previous problem that X, our domain, has to be bounded. Moreover, we know that it has to be connected in order to be able to 'jump' between partitions as we did in the proof above (hence why $\mathbb{N}$ does not work in general). However, we need a stronger notion, that of totally boundedness in order for any open set around a point $p \in X$ to contain other points $x \in X$. $\square$

PSET 5: Problem 7

Example 1.146. If M is compact and A is dense in M, prove that for each $\delta > 0$ there is a finite subset $\{a_1, \dots, a_k\} \subset A$ which is δ-*dense* in M in the sense that each $x \in M$ lies within distance δ of at least one of the points $a_1, \dots, a_k$.

Proof. Consider $\{B_{\frac{\delta}{2}}(x) | x \in M\}$. By compactness,

$$M \subset \bigcup_{x \in M} B_{\frac{\delta}{2}}(x) \implies M \subset \bigcup_{i=1}^{n} B_{\frac{\delta}{2}}(x_i).$$

Since A is dense in M, then for each i, there exists some $a_i \in A$ such that $a_i \in B_{\frac{\delta}{2}}(x_i)$. Suppose $x \in M$. Then $x \in B_{\frac{\delta}{2}}(x_i)$ for some i, and thus

$$d(x, a_i) \leq d(x, x_i) + d(x_i, a_i) \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

$\square$

PSET 5: Problem 8

Example 1.147. Suppose that $\mathscr{E} \subset C^0$ is equicontinuous and bounded.

(a)

Example 1.148. Prove that $\sup(f(x), f \in \mathscr{E})$ is a continuous function of x.

Proof. Since $\mathscr{E}$ is equicontinuous, then there exists some $\delta > 0$ such that if $|x-y| < \delta$ and $f \in \mathscr{E}$, then

$$|f(x) - f(y)| < \frac{\epsilon}{2} \iff f(x) < f(y) + \frac{\epsilon}{2} \implies \sup(f(x)) < \sup(f(y)) + \epsilon.$$

Identically,

$$|f(y) - f(x)| < \frac{\epsilon}{2} \iff f(y) < f(x) + \frac{\epsilon}{2} \implies \sup(f(y)) < \sup(f(x)) + \epsilon.$$

Thus, we have that

$$\sup(f(x)) - \sup(f(y)) < \epsilon, \qquad \sup(f(y)) - \sup(f(x)) < \epsilon,$$

and so

$$|\sup(f(x)) - \sup(f(y))| < \epsilon.$$

$\square$

(b)

Example 1.149. Show that (a) fails without equicontinuity.

Proof. Let $\mathscr{E} = \{\sin(nx) | n \in \mathbb{N}\}$. We claim that $\mathscr{E}$ is not equicontinuous. Let $\epsilon = \frac{1}{2}$ and suppose $\delta > 0$. If $x = 0$ and $|y| = \frac{\pi}{2n} < \delta$ for some large n, then we have that for any $n \in \mathbb{N}$:

$$|\sin(nx) - \sin(ny)| = |\sin(ny)| = |\sin\left(n\frac{\pi}{2n}\right)| = 1.$$

We claim that $\sup(\sin(nx), n \in \mathbb{N})$ is not continuous at $x = 0$. Let $\epsilon = \frac{1}{2}$, then if $\delta > 0$, we have that $y = \frac{2\pi}{n}$, and thus $|x - y| = |y| < \delta$ for large n. Thus, $\sup(\sin(ny)) = 1$ and thus

$$|\sup((\sin(nx)) - \sup(\sin(ny))| = 1 \geq \frac{1}{2},$$

and thus $\sup(\sin(nx))$ is not a continuous function at $x = 0$. $\square$

(c)

Example 1.150. Show that this continuous-sup property does not imply equicontinuity.

Proof. Consider $\mathscr{E} = \{x^n | n \in \mathbb{N}\}$, with $[a, b] = [0, 1]$. Evidently, we have that for any $x \in [0, 1]$. $\sup(f(x), f \in \mathscr{E}) = x$, which is continuous, and thus the continuous sup property is satisfied. $\mathscr{E}$ is not continuous though, since consider $\epsilon = \frac{1}{2}$, and take $\delta > 0$. There exists $x = 1$ and $y = \frac{1}{2}^{\frac{1}{n}}$ such that for any large enough $n \in \mathbb{N}$, $|x - y| = |1 - y| < \delta$. However, consider that

$$|f_n(x) - f_n(y)| = |x^n - y^n| = |1 - (\frac{1}{2})^{\frac{1}{n}^n}| = \frac{1}{2}$$

$\square$

(d)

Example 1.151. Assume that the continuous-sup property is true for each subset $\mathscr{F} \subset \mathscr{E}$. Is $\mathscr{E}$ equicontinuous? Give a proof or counterexample.

Proof. Use the solution from part c. $\square$

PSET 5: Problem 9

Example 1.152. Suppose that a sequence of continuous functions $f_n : [a,b] \to \mathbb{R}$ converge monotonically down to a continuous function f. That is, for each $x \in [a,b]$, we have that $f_1(x) \geq f_2(x) \geq \dots$ and $f_n(x) \to f(x)$ as $n \to \infty$.

(a)

Example 1.153. Prove that convergence is uniform.

Proof. **This proof is wrong, and it's pretty subtle why. Can you find the mistake?** Let $D = \{d_1, d_2, \dots\}$ be a dense subset of $[a,b]$. Let $\epsilon > 0$. Since each f_n is continuous, then there exists some $\delta_n > 0$ such that if $x \in [a,b]$ with $|x - d_k| < \delta_n$ for some $d_k \in D$ we have that $|f_n(x) - f_n(d_k)| < \frac{\epsilon}{3}$. Since f is continuous, then there exists some $\delta > 0$ such that if $x \in [a,b]$ with $|x - d_k| < \delta$ for some $d_k \in D$, we have that $|f(x) - f(d_k)| < \frac{\epsilon}{3}$. Let $\delta_m = \min(\delta, \delta_n)$. By problem 7, there exists a finite subset $D_m = \{d_1, \dots, d_{k_m}\} \subset D$ such that for any $x \in [a,b]$, there exists some $i \in [k_m]$ such that $|x - d_i| < \delta_m$.

Since $\lim_{n\to\infty} f_n(d_i) = f(d_i)$, then there exists some $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, we have that if we call d_1 to be d_{i_1}, then

$$|f_n(d_{i_1}) - f(d_{i_1})| < \frac{\epsilon}{3}.$$

Either we have that this N_1 works for any $d_i \in D_m$, or we have that for some $d_{i'} \in D_m$, there exists some $N > N_1$ such that

$$|f_N(d_{i'}) - f(d_{i'})| \geq \frac{\epsilon}{3}.$$

Let the first such i' be called i_2, then since $\lim_{n\to\infty} f_n(d_{i_2}) = f(d_{i_2})$, we have that there exists some $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$|f_n(d_{i_2}) - f(d_{i_2})| < \frac{\epsilon}{3}.$$

Note that $N_2 > N_1$. Since f_n is monotonically decreasing, we have that any d_i that 'worked' with the N_1 will still 'work' with N_2. Either we have that this N_2 works for any $d_i \in D_m$, or we continue this process until we run out of $i \in [k_m]$. Because we can do this at most a finite amount of times, then for each D_m, we can take we can take $N_m = \max\{N_{i_1}, N_{i_2}, \dots\}$. Thus, if $n \geq N_m$ then

$$f_n(d_i) - f(d_i) < \epsilon, \qquad \forall d_i \in D_m.$$

Let $x \in [a,b]$. There exists some $d_i \in D_m$ for some large m such that such that $|x - d_i| < \delta$. Thus, if $n \geq N_m$:

$$|f_n(x) - f(x)| \leq |f_n(x) - f_n(d_i)| + |f_n(d_i) - f(d_i)| + |f(d_i) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

□

Proof. **Here is the correct solution.** Let $g_n(x) = f_n(x) - f(x)$. Evidently, $g_n \to 0$ pointwise. Let $\epsilon > 0$, and consider now

$$H_n = \{x \in [a,b] \mid g_n(x) < \epsilon\} = g_n^{-1}(-\infty, \epsilon).$$

H_n is open by continuity of g, and note that since $g_n(x) \to 0$, we must have that $H_n \subset H_{n+1}$ for any n. We thus have:

$$[a,b] \subset \bigcup_n H_n,$$

and so by compactness there exists some finite subcover with H_{n_i}. But since the sets are increasing upwards, we have that $\bigcup_{i=1}^k H_{n_i} = H_{n_k}$. Thus, $[a,b] \subset H_{n_k}$. In other words, we have that if $x \in [a,b]$, we have that if $\ell \geq n_k$, then $g_\ell(x) < \epsilon$. □

(b)

Example 1.154. What if the sequence is increasing instead of decreasing?

Proof. This is fine. It should be pretty clear from the proof that as long as $f_n : [a, b] \to \mathbb{R}$ converge monotonically up to a continuous function f, the proof still works. To bootstrap the argument, just take $-(f_n)$, then this is a monotonically decreasing function that converges pointwise to f, and by part (a) we have that it converges uniformly to f. Thus, (f_n) converges uniformly to f. $\square$

(c)

Example 1.155. What if you replace $[a, b]$ with $\mathbb{R}$?

Proof. **No.** Consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by:

$$f_n(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{n}x, & x > 0 \end{cases}.$$

We obviously have that for any $x \in \mathbb{R}$, $f_1(x) \geq f_2(x) \geq \dots$. We claim that for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} f_n(x) = 0.$$

This is obvious for $x \leq 0$. Let $\epsilon > 0$ and $x > 0$. There exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{x}$ and thus if $n \geq N$:

$$|f_n(x)| = |\frac{1}{n}x| < \epsilon.$$

Assume for the sake of contradiction that $f_n \to 0$ uniformly. Thus, for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $|\frac{1}{n}x| < \epsilon$. However, take $\epsilon = 1$, then for any N, we can take $x = 2N$ and thus we have that

$$|f_n(x)| = |\frac{1}{N}2N| = 2 \geq \epsilon.$$

$\square$

(d)

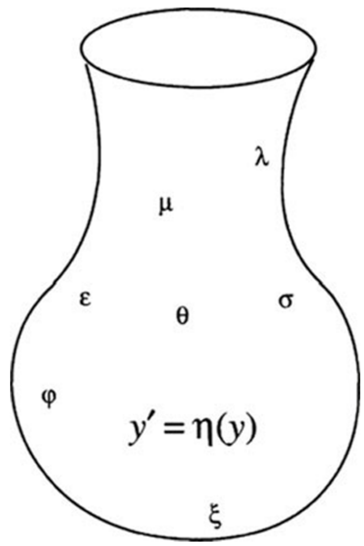
Example 1.156. What if you replace $[a, b]$ with a compact metric space or $\mathbb{R}^m$?

Proof. Replacing it with a compact metric space if fine, since you can always find some finite subset of a dense subset that is $\delta-$dense by Problem 7. It did not require it to be an interval in $\mathbb{R}$.

Replacing it with $\mathbb{R}^m$ is not fine, since you run into the same problem as you did in part (c). $\square$

PSET 5: Problem 10

Example 1.157. What is the joke?



Proof. There are three jokes that we could think of. None of them funny.

(a) There is no joke. Absurdist humor. Kind of like Dada art, but with humor. Pugh sounds like poo is the joke. This is like the button in season 2 of Lost that does nothing (I haven't finished the show it might do something) even though they are told the world will blow up if they don't keep pushing it.

(b) $y' = \eta(y)$ sounds like "y prime ate a y." Pugh was born in 1940, and he thinks fatshaming y is real funny. This is still Biden's America Godamit.

(c) This is a reference to the poem "Ode to a Grecian Urn." This is not funny for multiple reasons:

   (i) This is a Dad Joke. And not a good one.

   (ii) Poetry.

   (iii) I only laugh at brainrot with Peter Griffin underneath.

□

1.33 PSET 6

PSET 6: Problem 1

Example 1.158. Suppose that $f : M \to M$ and for all $x, y \in M$, if $x \neq y$ then $d(f(x), f(y)) < d(x, y)$. Such an f is a *weak contraction.*

(a)

Example 1.159. Is a weak contraction a contraction? (Proof or counterexample.)

Proof. **No.** Consider $f : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ such that $f(x) = x^2$.[7] f is a contraction because for any $x, y \in [0, \frac{1}{2}]$, we have that $|x + y| \leq 1$, and thus if $x \neq y$, then

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| < |x - y|.$$

Thus, f is a weak contraction. Suppose f is a contraction as well. Then there exists some $k < 1$ such that $d(f(x), f(y)) \leq kd(x, y)$. However, take $x = k$ and $y = \frac{1-k}{2}$, then we have that

$$|x + y| > k \implies |x + y||x - y| > k|x - y| \implies |f(x) - f(y)| > k|x - y|,$$

and thus f is not a contraction. □

(b)

Example 1.160. If M is compact is a weak contraction a contraction?

Proof. **No.** The above example works. □

(c)

Example 1.161. If M is compact, prove that a weak contraction has a unique fixed point.

Proof. Since f is a contraction and $f : M \to M$, then we claim that $f(M) \subset M$. Since f is a contraction, we have that there exists some $\delta > 0$ such that if $d(x, y) < \frac{\delta}{2}$, then $d(f(x), f(y)) < \frac{\delta}{2}$. Cover M by δ balls. Then if $y \in B_\delta(x) \subset M$, we have that $d(f(x), f(y)) < \frac{\delta}{2}$, and thus $f(x)$ and $f(y)$ are in (possible another) δ ball of M, and so $f(M) \subset M$. Since f is continuous and M is compact, then $f(M)$ is compact. We can induct on this process and notice that

$$M \supset f(M) \supset f^2(M) \supset \dots$$

with each set compact. We now claim that if

$$X = M \cap \bigcap_{n \in \mathbb{N}} f^n(M),$$

then X is our set of fixed point. To see this, notice that each set is compact and nonempty, and thus X is compact and nonempty. We now wish to show that $f(X) = X$. One inclusion is easy. If $x \in f(X)$, then since $f(X) \subset X$ by the above logic, $x \in X$. Suppose now that $x \in X$. Thus, $x \in M \cap \bigcap_{n \in \mathbb{N}} f^n(M)$. Since $x \in f(M)$, then there exists some $m_1 \in M$ such that $f(m_1) = x$. Similarly, there exists some $m_2 \in M$ such that $f^2(m_2) = x$. Take the sequence

$$y_1 = m_1, \ y_2 = f(m_2), \ \dots \ y_n = f^{n-1}(m_n).$$

We have by compactness of M that it has some convergent subsequence $(y_{n_k}) \to y_\infty$. We claim that $f(y_\infty) = x$. To see this, consider that since f is continuous, we have that $f(y_{n_k}) \to f(y_\infty)$. However, we have by construction that

$$f(y_{n_k}) = f(f^{n_{k-1}}(m_{n_k})) = x \implies f(y_\infty) = x.$$

[7]Note that f does not need to be a surjection in order to be a contraction, which is good because $f([0, \frac{1}{2}]) = [0, \frac{1}{4}]$

Moreover, we have that $y_\infty \in f^n(M)$ for every $n \in \mathbb{N}$ by closedeness, and thus

$$y_\infty \in X \implies f(y_\infty) \in f(X) \implies x \in f(X).$$

It suffices to show that $(X) = 0$. Suppose not, then $(X) > 0$. Thus, since X is compact, we have that there must exist $x_1, x_2 \in X$ such that $d(x_1, x_2) > 0$. However, we have proved that $f(x_1) = x_1$ and $f(x_2) = x_2$, and thus

$$d(f(x_1), f(x_2)) = d(x_1, x_2) > 0,$$

which is a contradiction to the fact that f is a contraction. Thus, $(X) = 0$ and thus X is a single point and thus we have that there exists a unique $x \in X$ such that $f(x) = x$. $\square$

Reflections: The following is a proof I am currently in the process of fixing, but have not figured out how:

Let $x_0 \in M$. Let $x_n = f^n(x_0)$, where $f^n(x_0) = (f \circ f \circ f \circ \cdots \circ f)(x_0)$, with f composite itself n times. Thus, since $f^n(x_0) \in M$ for any n, then by compactness, $(x_n) \in M$ has a convergent subsequence $x_{n_k} \to x_\infty$. We claim that x_∞ is a fixed point. To see this, notice that since (x_{n_k}) is convergent, then it is Cauchy, and thus for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $n_k, m_k \geq N_1$, we have $d(x_{n_k}, x_{m_k}) < \frac{\epsilon}{3}$. Since $x_{n_k} \to x_\infty$, then there exists some N_2 such that if $n_k \geq N_2$, then $d(x_{n_k}, x_\infty) < \frac{\epsilon}{3}$. Take $N = \min\{N_1, N_2\}$, then we have if $x_{n_k} > N$,

$$\begin{aligned} d(x_\infty, f(x_\infty)) &\leq d(x_\infty, x_{n_k}) + d(x_{n_k}, f(x_{n_k})) + d(f(x_{n_k}), f(x_\infty)) \\ &< \frac{\epsilon}{3} + d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_k}, x_\infty) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}. \end{aligned}$$

The second term of the second inequality follows by definition of (x_{n_k}), and the last term of the second inequality follows from the fact that f is a contraction. Suppose f has another unique point at some $p \in M$, then $|f(p) - f(x_\infty)| = |p - x_\infty| \not< |p - x_\infty|$, and thus f is not a contraction.

PSET 6: Problem 2

Example 1.162. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and its derivative satisfies $|f'(x)| < 1$ for all $x \in \mathbb{R}$.

(a)

Example 1.163. Is f a contraction?

Proof. Consider $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} x - \arctan(x) \end{cases}$$

We claim without proof that $f'(x) = 1 - \frac{1}{x^2+1}$. Thus, we have that $f'(x) < 1$ for all $x \in \mathbb{R}$, but as $x \to \infty$, $f'(x) \to 1$. Suppose f is a contraction, then there exists some $k < 1$ such that if $x, y \in \mathbb{R}$, then

$$|f(x) - f(y)| \leq k|x - y| \implies \frac{|f(x) - f(y)|}{|x - y|} = |f'(\theta)| \leq k$$

for some $\theta \in (x, y)$. However, taking $x = 0$ and $y = k + 1$, then we have that

$$|f(x) - f(y)| = |k + 1 - \arctan(k + 1)|$$

$\square$

(b)

Example 1.164. Is f a weak contraction?

Proof. **Yes.** Let $x, y \in \mathbb{R}$, then since f f is differentiable on (x, y) and continuous on $[x, y]$, there exists some $\theta \in (y, x)$ such that

$$|f(y) - f(x)| = f'(\theta)|x - y| < |x - y|$$

since $f'(\theta) < 1$. $\square$

(c)

Example 1.165. Does it have a fixed point?

Proof. **No.** $\square$

PSET 6: Problem 2

Example 1.166. Give an example to show that the fixed-point in Brouwer's Theorem need not be unique.

Proof. Let B^1 be the closed unit ball in $\mathbb{R}^1$, and let $f : B^1 \to B^1$ such that $f(x) = x$. Obviously, f is continuous. Every point in B^1 is a fixed point, and thus there is no uniqueness. $\square$

PSET 6: Problem 3

(a)

Example 1.167. Give an example of a function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that for each fixed x, then function $y \to f(x,y)$ is a continuous function of y, and for each fixed y, the function $x \to f(x,y)$ is a continuous function of x, but f is not continuous.

Proof. Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Clearly, $x \to f(x,y)$ is continuous for all fixed $y \neq 0$. Take some sequence $(x_n, 0) \to (0,0)$. By examining the function, it is clear that $f(x_n, 0) = f(0,0) = 0$. Same for $y \to f(x,y)$. To prove that f is not continuous at $(0,0)$, take the sequence $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$. We want to show that $f(\frac{1}{n}, \frac{1}{n})$ does not converge to $f(0,0) = 0$. To see this, consider that

$$f(\frac{1}{n}, \frac{1}{n}) = \frac{\frac{1}{n}\frac{1}{n}}{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\frac{1}{n^2}}{\frac{2}{n^2}} = \frac{1}{2}.$$

□

(b)

Example 1.168. Suppose in addition that the set of functions

$$\mathscr{E} = \{x \to f(x,y) \ \ y \in [0,1]\}$$

is equicontinuous. Prove that f is continuous.

Proof. Let $(x_n, y_n) \to (x,y)$, where $(x_n) \in [0,1]$ and $(y_n) \in [0,1]$. We want to show that $f(x_n, y_n) \to f(x,y)$. Thus, it suffices to show that for any $\epsilon > 0$, we have n large such that

$$d(f(x_n, y_n), f(x,y)) < \epsilon.$$

Since $\mathscr{E}$ is equicontinuous, then for any $\epsilon > 0$, we have that there exists a $\delta > 0$ such that if $|x - t| < \delta$, then for any f such that f is a function that sends $x \to f(x,y)$ with y fixed, $|f(x,y) - f(t,y)| < \frac{\epsilon}{2}$, Since $x_n \to x$, then we have that for large n, $|x - x_n| < \delta$. Since for each fixed x, function $y \to f(x,y)$ is a continuous function of y, then we have that if $(y_n) \to y$, then $f(x, y_n) \to f(x,y)$. Thus, for large enough n, we have that $d(f(x, y_n), f(x,y)) < \frac{\epsilon}{2}$

$$d(f(x_n, y_n), f(x,y)) \leq d(f(x_n, y_n), f(x, y_n)) + d(f(x, y_n), f(x,y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

□

PSET 6: Problem 4

Example 1.169. Let $T : V \to W$ be a linear transformation and let $p \in V$ be given. Prove that the following are equivalent.

(a) T is continuous at the origin.

(b) T is continuous at p.

(c) T is continuous at at least one point of V.

Proof. Suppose T is continuous at the origin, then we claim that T is continuous. To see this, we will first show that $||T|| < \infty$. Let $\epsilon = 1$, then there exists a $\delta > 0$ such that if $u \in V$ and $|u| < \delta$, then

$$|T(u)| < 1.$$

Let $v \in V$ nonzero, then let $\lambda = \frac{\delta}{2|v|}$, and thus $u = \lambda v$. $|u| = \frac{\delta}{2}$ and due to the properties of linear transforms and norms, we have that:

$$\frac{|T(v)|}{|v|} = \frac{|T(\frac{u}{\lambda})|}{|\frac{u}{\lambda}|} = \frac{|T(u)|}{u} < \frac{1}{|u|} = \frac{2}{\delta}.$$

Thus, $||T|| < \infty$. Let $v, v' \in V$ with $|v - v'| < \frac{\epsilon}{||T||}$, then

$$|T(v) - T(v')| = |T(v - v')| \leq ||T|||v - v'| < \epsilon,$$

and thus T is uniformly continuous. Thus, we have b and c.

Suppose c, then T is continuous at some $u \in V$. Let $\epsilon > 0$, then get the $\delta > 0$ from the continuity of u. Thus, if $|v| < \delta$, then we let $v = u - (u + \frac{\delta}{2})$. Notice that we have that $|u - (u + \frac{\delta}{2})| = \frac{\delta}{2} < \delta$, and thus

$$|T(u) - T(u + \frac{\delta}{2})| < \epsilon.$$

Because T is a linear transform, we also have that

$$|T(u) - T(u + \frac{\delta}{2})| = |T(u - (u + \frac{\delta}{2}))| = |T(v)| < \epsilon.$$

Thus, we have that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $v < \delta$, then $|T(v)| < \epsilon$, and thus T is continuous at the origin. $\square$

PSET 6: Problem 5

Example 1.170. Let $\mathscr{L}$ be the vector space of continuous linear transformations from a normed space V to a normed space W. Show that the operator norm makes $\mathscr{L}$ a normed space.

Proof. Suppose $T, T' \in \mathscr{L}$ and let $\lambda \in \mathbb{F}$. Note that $||T||$ is well defined since it is finite since f is continuous.

(a)
$$||T|| = \sup\{\frac{|T(v)|_W}{|v|_V}, v \neq 0.\}$$

Since $T : V \to W$, then $T(v) \in W$, and thus since W is a normed space, we have that $|T(v)|_W \geq 0$ for all $T(v) \in W$. Similarly, we have that $|v|_V \geq 0$ for all $v \in V$. Thus, $||T|| \geq 0$. Suppose T is the zero transformation, then $T(v) = 0$ for any $v \in V$. Thus, we have that

$$||T|| = \sup\{\frac{|T(v)|_W}{|v|_V}, v \neq 0.\} = \sup\{\frac{0}{|v|_V}, v \neq 0.\} = 0.$$

(b) Since $|T(v)|_W$ is a norm in W, then if λ is a scalar, we have that $|\lambda T(v)|_W = |\lambda||T(v)|_W$. Similarly for V.

$$||\lambda T|| = \sup_{v \in V}\{\frac{|\lambda T(v)|_W}{|v|_V} \ ; \ v \neq 0\} = \sup_{v \in V}\{\frac{|\lambda||T(v)|_W}{|v|_V} \ ; \ v \neq 0\} = |\lambda| \sup\{\frac{|T(v)|_W}{|v|_V}, v \neq 0.\}.$$

Thus, $||\lambda T|| = |\lambda|||T||$.

(c) Since W is a normed space, we have that $|T(v) + T'(v)|_W \leq |T(v)|_W + |T(v)|_W$.

$$\begin{aligned}
||T + T'|| &= \sup\{\frac{|T(v) + T'(v)|_W}{|v|_V} \ ; \ v \neq 0\} \\
&\leq \sup\{\frac{|T(v)| + |T'(v)|_W}{|v|_V} \ ; \ v \neq 0\} \\
&\leq \sup\{\frac{|T(v)|}{|v|_V} \ ; \ v \neq 0\} + \sup\{\frac{|T'(v)|_W}{|v|_V} \ ; \ v \neq 0.\} \\
&= ||T|| + ||T'||
\end{aligned}$$

The last inequality comes from the fact that $\sup(f(x) + g(x)) \leq \sup(f(x)) + \sup(g(x))$.[8]

□

[8] Proved on PSET 5, but $f(x) \leq \sup f(x)$ and $g(x) \leq \sup g(x)$ imply that $f(x) + g(x) \leq \sup f(x) + \sup g(x)$ for all x.

PSET 6: Problem 6

Example 1.171. Two norms $| \ |_1$ and $| \ |_2$ on a vector space are *comparable* if there are positive constants c and C such that for all nonzero vectors in V we have

$$c \leq \frac{|v|_1}{|v|_2} \leq C.$$

(a)

Example 1.172. Prove that comparability is an equivalence relation on norms.

Proof. It will suffice to show the three properties of an equivalence relation. Let $| \ |_1, | \ |_2$ be norms on a vector field V, and let $v \in V$.

(i) (Reflexive) We want to show that $| \ |_1$ is comparable to itself. This is clear, since we have that

$$\frac{|v|_1}{|v|_1} = 1 \implies \frac{1}{2} \leq \frac{|v|_1}{|v|_1} \leq 2,$$

and thus $| \ |_1$ is comparable to itself.

(ii) (Symmetry) We want to show that if $| \ |_1$ is comparable to $| \ |_2$, then $|v|_2$ is comparable to $|v|_1$. By assumption, c and C are constants such that

$$c \leq \frac{|v|_1}{|v|_2} \leq C \implies \frac{1}{C} \leq \frac{|v|_2}{|v|_1} \leq \frac{1}{c}.$$

Since $\frac{1}{C}$ and $\frac{1}{c}$ are positive constants, then $| \ |_2$ is comparable to $| \ |_1$.

(iii) (Transitive) Suppose $| \ |_1$ is comparable to $| \ |_2$ and $| \ |_2$ is comparable to $| \ |_3$, then there exists positive constants c, C and c', C' such that

$$c \leq \frac{|v|_1}{|v|_2} \leq C, \qquad c' \leq \frac{|v|_2}{|v|_3} \leq C'.$$

Thus, we have that since everything is positive,

$$cc' \leq \frac{|v|_1}{|v|_2}\frac{|v|_2}{|v|_3} \leq CC' \implies cc' \leq \frac{|v|_1}{|v|_3} \leq CC',$$

and thus $| \ |_1$ is comparable to $| \ |_3$.

$\square$

(b)

Example 1.173. Prove that any two norms on a finite-dimensional vector space are comparable.

Proof. Let V be a finite dimensional vector space and $| \ |_1, | \ |_2$ be norms on V. Let $T : (V, | \ |_1) \to (V, | \ |_2)$ be the identity map.[9] By Corollary 4 on the book, we have that T is continuous (and indeed, a homoemorphism), and thus by Theorem 2, $||T|| < \infty$. Thus, we have that

$$\sup_{v \in V} \frac{|T(v)|_2}{|v|_1} < \infty. \tag{12}$$

[9] T is a linear transform because $T(\alpha v + w) = \alpha v + w = \alpha T(v) + T(w)$.

In particular, since we are dealing with the identity map, we have that there exists some positive C constant such that for all nonzero vectors $v \in V$,

$$\frac{|v|_2}{|v|_1} \leq C.$$

Now consider T^{-1}. This is also continuous because T is a homoeomorphism, and so $||T^{-}1|| < \infty$. Thus, we have that

$$\sup_{v \in V} \frac{|T(v)|_1}{|v|_2} < \infty.$$

In particular, since we are dealing with the identity map, there exists some positive c constant such that for all nonzero vectors $v \in V$,

$$\frac{|v|_1}{|v|_2} \leq c. \tag{13}$$

Combining (1) and (2) we find that

$$\frac{1}{C} \leq \frac{|v|_1}{|v|_2} \leq c,$$

and thus $|\ |_1$ and $|\ |_2$ are comparable. □

(c)

Example 1.174. Consider the norms

$$|f|_{L^1} = \int_0^1 |f(t)|dt, \qquad |f|_{C^0} = \max\{f(t)\ :\ t \in [0,1]\},$$

defined on the infinite-dimensional vector space $C^0([0,1],\mathbb{R})$. Show that the norms are not comparable by finding functions $f \in C^0([0,1],\mathbb{R})$, whose integral norm is small but whose C^0 is 1.

Proof. Suppose $|\ |_{L^1}$ and $|\ |_{C^0}$ are comparable, then there exists some positive c, C such that for any $f \in C^0([0,1],\mathbb{R})$,

$$c \leq \frac{\int_0^1 f(t)dt}{\max\{f(t)\ ;\ t \in [0,1]\}} \leq C.$$

Consider a sequence of functions $f_n : [0,1] \to \mathbb{R}$.

$$f_n(x) = x^n.$$

Each f_n is continuous, and each achieves their maximum at $x = 0$ at $f(x) = 1$. However, as $n \to \infty$, we claim that $\int_0^1 |f_n(t)|dt \to 0$. To see this, use the FTC:

$$\left|\int_0^1 |t^n|dt\right| = \int_0^1 |t^n|dt = \int_0^1 t^n dt = \frac{1}{n+1} \to 0.$$

Thus, for any $c > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, we have that

$$|f_n|_{L^1} < c,$$

and thus we have a contradiction since for any $c > 0$, we have that for large n,

$$\frac{\int_0^1 f_n(t)dt}{\max\{f_n(t)\ :\ t \in [0,1]\}} = \int_0^1 t^n dt = \frac{1}{n+1} < c.$$

□

PSET 6: Problem 7

Example 1.175. Let $|\ | = |\ |_{C^0}$ be the supremum norm on C^0 as defined ini Problem 6. Define an integral transformation $T : C^0 \to C^0$ by

$$T : f \to \int_0^x f(t)dt.$$

(a) Show that T is linear, continuous, and find its norm.

Proof. (i) (Linear) We want to show that if $f, g \in C^0$ and $\alpha \in \mathbb{R}$, then $T(\alpha f + g) = \alpha T(f) + T(g)$. Thus, we use the linearity of the integral:

$$\begin{aligned} T(\alpha f + g) &= \int_0^x \alpha f(t) + g(t)dt \\ &= \int_0^x \alpha f(t)dt + \int g(t)dt \\ &= \alpha \int_0^x f(t)dt + \int g(t)dt \\ &= \alpha T(f) + T(g). \end{aligned}$$

(ii) (Continuous) To show that T is continuous, then by Theorem 2, it will suffice to show that $||T|| < \infty$. Since f is continuous on $[0,1]$, it achieves its maximum on it. Thus, for any $f \in C^0$, we have that

$$\left| \int_0^x f(t)dt \right| \leq \max\{f(t) \ : \ t \in [0,1]\}.$$

Thus, for any $f \in C^0$, we have that

$$\begin{aligned} |T(f)| &= |\int_0^1 f(t)dt|_{C^0} \\ &\leq |\max\{f(t) \ : \ t \in [0,1]\}|_{C^0} \\ &= \max\{f(t) \ : \ t \in [0,1]\} \\ &= |f|_{C^0} \end{aligned}$$

Thus, for any $f \in C^0$:

$$\begin{aligned} \frac{|T(f)|_{C^0}}{|f|_{C^0}} &\leq \frac{|f|_{C^0}}{|f|_{C^0}} \\ &= 1 \end{aligned}$$

Thus, because this is true for any $f \in C^0$, we have that $||T|| < 1 < \infty$.

(iii) (Norm) We defined the usual operator norm on T :

$$||T|| = \sup_{f \in C^0} \frac{|T(f)|_{C^0}}{|f|_{C^0}}.$$

$\square$

(b) Let $f_n(t) = \cos(nt)$, $n = 1, 2, \ldots$ What is $T(f_n)$?

Proof. We use the fundamental theorem of calculus!

$$T(f_n) = \int_0^x \cos(nt)dt = \frac{1}{n}\sin(nx)$$

□

(c) Is the set of functions $K = \{f_n : n \in \mathbb{N}\}$ closed? Bounded? Compact?

Proof. We shall check each condition.

(i) (Closed) Not closed since K has no limit points. Suppose it is closed though! Then $f_n(t) \to f$ with $f \in K$. Thus, we have that for large n,

$$|f_n - f|_{C^0} < \epsilon,$$

and thus using the reverse triangle, we have that

$$||f_n|_{C^0} - |f|_{C^0}| \leq |f_n - f|_{C^0}, \epsilon.$$

Since $|f_n|_{C^0} = 1$, then we have that $|f|_{C^0} = 1$. Since $f_n(t) \to f$ and since T is continuous, we now have that $T(f_n) \to T(f)$. By work in the following section, we have that $T(f(n)) \to 0$, and thus by the same logic as above, $|T(f)|_{C^0} = 0$. However, this is a contradiction, since

$$T(f) = \int_0^1 f(t)dt$$

and $|f(t)|_{C^0} = 1$, which, since T is a linear transform, implies that T only sends the zero vector to the zero vector!

(ii) For any $n \in \mathbb{N}$, we have that

$$|\cos(nt)|_{C^0} = 1,$$

and thus f_n is uniformly bounded since for any $t \in [0,1]$, $n \in \mathbb{N}$, $f_n(t) \leq 1$.

(iii) We claim that K is not compact. By Arzela-Ascoli, it suffices to show that K is not equicontinuous. Let $\epsilon = \frac{1}{2}$ and take $x = 0$ and $y = \frac{\pi}{2n}$ then for all $\delta > 0$, if n large, we have that $|x - y| < \delta$, but

$$|f_n(x) - f_n(y)|_{C^0} = |\cos(n0) - \cos\left(n\frac{\pi}{2n}\right)|_{C^0} = |1 - \cos\left(\frac{\pi}{2}\right)|_{C^0} = 1.$$

Thus, K is not equicontinuous, and thus not compact.

□

(d)

Example 1.176. Is $T(K)$ compact? How about its closure?

Proof. We make heavy use of Arzela-Ascoli.

(i) ($T(K)$) We claim that $T(K)$ is not compact. To do this, it suffices by Arzela-Ascoli to show that it is not closed. Consider that $T(K) = \{\frac{1}{n}\sin(nx) \ : \ n \in \mathbb{N}\}$ by part (b). We claim that $z(x) = 0$ is a limit point of $T(k)$, but $z(x) \notin T(k)$. We claim that $T(f_n) \to z(x)$ uniformly. To see this, let $\epsilon > 0$, then for n large, we have that

$$|T(f_n(x)) - 0| = \left|\frac{1}{n}\sin(nx)\right| \leq \frac{1}{n} < \epsilon.$$

Evidently, we have that $z(x) \notin T(K)$ since $z(x) \neq \frac{1}{n}\sin(nx)$ for any $n \in \mathbb{N}$. Thus, $T(K)$ is not closed.

(ii) ($\overline{T(K)}$) By definition, $\overline{T(K)}$ is closed. Let $\overline{(T(f_n)) \in T(K)}$. Thus, $T(f_n) = \frac{1}{n}\sin(nx)$, which converges, and thus any subsequence of it converges to a function which is in the closure. Thus, the closure is compact. Evidently, since $T(K)$ is uniformly bounded, then $\overline{T(K)}$ is uniformly bounded. Thus, by Arzela-Ascoli, we have compactness.

$\square$

PSET 6: Problem 8

Example 1.177. Let $f : U \to \mathbb{R}^m$ be differentiable, $[p, q] \subset U \subset \mathbb{R}^n$, and ask whether the direct generalization of the one-dimensional Mean Value Theorem is true: Does there exist a point $\theta \in [p, q]$ such that

$$f(q) - f(p) = Df_\theta(q - p)? \tag{14}$$

(a)

Example 1.178. Take $n = 1$, $m = 2$, and examine the function $f(t) = (\cos(t), \sin(t))$ for $t \in [\pi, 2\pi]$. Take $p = \pi$ and $q = 2\pi$. Show that there is no $\theta \in [p, q]$ that satisfies (3).

Proof. Suppose there does exist some $\theta \in [\pi, 2\pi]$ such that

$$f(2\pi) - f(\pi) = \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix} = Df_\theta(\pi).$$

Since θ exists, we have that

$$Df_\theta = \begin{bmatrix} \frac{\partial f_1}{\partial \theta} & \frac{\partial f_2}{\partial \theta} \end{bmatrix} = \begin{bmatrix} -\sin(\theta) & \cos(\theta) \end{bmatrix},$$

thus, we have that

$$\begin{bmatrix} 2 & 0 \end{bmatrix} = \begin{bmatrix} -\pi\sin(\theta) & \pi\cos(\theta) \end{bmatrix} \implies \theta = \frac{3\pi}{2}.$$

However, since $\theta = \frac{3\pi}{2}$, then $-\pi\sin(\theta) = \pi \neq 2$, which is a contradiction. $\square$

(b)

Example 1.179. Assume the set of derivatives

$$(Df)_x \in \{\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m) \ : \ x \in [p, q]\}$$

is convex. Prove there exists $\theta \in [p, q]$ which satisfies (28).

Proof. We use two facts from googling support plane:

(i) If X is compact convex and Y is closed convex and $X \cap Y = \emptyset$, there exists a hyperplane $H_{u,\alpha} = \{x \mid \alpha = \langle u, x\rangle\}$ such that for all $x \in X$ and $y \in Y$, we have that

$$\langle u, x\rangle < \alpha < \langle u, y\rangle.$$

(ii) If X is convex and non-singular and $x_0 \in$ rel. bd(X), then there exists a hyperplane $H_{u,\alpha}$ such that $x_0 \in H_{u,\alpha}$ and for all $x \in X$, $\langle u, x\rangle \leq \alpha$ and $X \not\subset H_{u,\alpha}$.

Define

$$\mathscr{A} := \{Df_x(q - p) \ : \ x \in [p, q]\}.$$

Note that $\mathscr{A}$ is convex since if $t \in [0, 1]$ we have that by the linearity of the derivative:

$$tDf_x(q - p) + (1 - t)Df_y(q - p) = [tDf_x + (1 - t)Df_y](q - p),$$

where the inside of the bracket is a convex combination of the derivatives, which are convex, and thus is a derivative itself. We claim that $f(p) - f(q) \in \mathscr{A}$. To see this, let $X = \{f(p) - f(q)\}$ and $Y = \overline{\mathscr{A}}$. Suppose $f(q) - f(p) \notin Y$, then we have that $X \cap Y = \emptyset$, and from fact (1) we have a hyperplane $H_{u,\alpha}$ such that

$$\langle u, x\rangle < \alpha < \langle u, Df_x(q - p)\rangle.$$

We now claim that if $U \subset \mathbb{R}^m$, then there exists some $z_u \in [p, q]$ such that

$$\langle u, f(q) - f(p) \rangle = \langle u, Df_{z_u}(q - p) \rangle.$$

Let $u \in \mathbb{R}^m$ and let

$$F_u(t) = \langle u, f\left(t(q) - (1-t)p\right)\rangle$$

and apply one-dimensional MVT and Leibniz product rule:

$$F_u(1) - F_u(0) = \langle u, f(q) - f(p)\rangle = F_u'(\theta) = \langle u, Df_{\theta q + (1-\theta)p}(q-p)\rangle = \langle u, D_{z_u}(q-p)\rangle.$$

Note here that $\theta \in [0, 1]$ and thus $z_u = \theta q + (1 - \theta)p \in [p, q]$.

Thus, if $f(q) - f(p) \in \mathscr{A}$, then we are done. If it is not in $\mathscr{A}$, then either $f(q) - f(p) \in$ rel. bd.$(\mathscr{A})$ or $f(q) - f(p) \in \{rel.int\}(\mathscr{A})$. If the latter, then we are done since it is still in the closure. If the former, then by fact (ii), we have that there exists a hyperplane $H_{u,\alpha}$ such that $f(q) - f(p) \in H_{u,\alpha}$ and $\langle u, f(q) - f(p)\rangle = \alpha$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(t) = \langle u, f(tq + (1-t)p)\rangle - \langle u, f(q) - f(p)\rangle t.$$

F is differentiable, and thus using the product rule and the chain rule and the Leibniz product rule and fact (ii) and squeez theorem (jk!):

$$F'(t) = \langle u, D_{tq+(1-t)p}(q-p)\rangle - \langle u, f(q) - f(p)\rangle = \langle u, D_{z_u}(q-p)\rangle - \alpha \leq 0.$$

Now, we basically win, since by fact (ii), we again have that $\mathscr{A} \not\subset H_{u,\alpha}$, then there exists some $t' \in [p, q]$ such that that since $F'(t) \leq 0$ for all t and

$$F'(t') < 0 \implies F(1) < F(0).$$

However, by the very definition F, we have that

$$F(1) - F(0) = \langle u, f(p)\rangle - \langle u, f(q) - f(p)\rangle + \langle u, f(q)\rangle = 0.$$

A contradiction! Thus, $f(q) - f(p) \in \mathscr{A}$ and we are done. $\square$

PSET 6: Problem 9

Example 1.180. Assume that U is a connected open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is differentiable everywhere on U. If $(Df)_p = 0$ for all $p \in U$, show that f is constant.

Proof. Let $p \in U$. Define:

$$A := \{x \ : \ f(x) = f(p)\}.$$

We want to show that A is equal to U. To do this, we prove that A is clopen and that $A \neq \emptyset$. The latter is obvious since $p \in A$. To prove that A is closed, consider that $A = f^{-1}\{f(p)\}$. Since f is differentiable on U, then it is continuous on U, and thus we have that since $\{f(p)\}$ is closed in $\mathbb{R}^m$ (since it is a single point), then $f^{-1}\{f(p)\}$ is closed in U. To prove that A is open, we must show that for any $a \in A$, there exists some $r > 0$ such that

$$B_r(a) \subset A.$$

Since U is open and $a \in U$, then there exists some $r' > 0$ such that

$$B_{r'}(a) \subset U.$$

Thus, let $b \in B_{\frac{r'}{2}}(a)$, then $b \in U$ and $[a, b] \subset U$. Thus, we have by the multivariate MVT that

$$|f(b) - f(a)| \leq M|b - a|, \quad M = \sup\{(Df)_x \ x \in [a, b]\} = 0.$$

Thus,

$$f(b) = f(a) = f(p) \implies b \in A.$$

Thus, A is open. Thus we have that A is clopen, and thus $A = U$. Thus, for all $x \in U$, $f(x) = f(p)$, and so f is constant on U. ☐

1.34 PSET 7

PSET 7: Problem 1

Example 1.181. Assume that U is a connected open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is second differentiable everywhere on U. If $(D^2 f)_p = 0$ for all $p \in U$, what can you say about f.

(a) *Proof.* We can say that $(Df)_p$ is constant for all $x \in U$. To show this, we can run it back. Let $p \in U$ and define

$$A_2 = \{x \ : \ Df_x(y) = Df_p(y) \ \forall y \in U\}.$$

We wish to show that A_2 is both open and closed. Since $(D^2 f)_p$ exists for all $p \in U$, then we have that Df_p is continuous for any p. Thus, we have that since $A_2 = Df_p^{-1}\{(Df_p(y)\}$, and Df is continuous, then A_2 is closed. U is open, and so if $a \in A_2 \subset U$, we have that there exists some $r > 0$ such that

$$B_r(a) \subset U,$$

and thus if $b \in B_{\frac{r}{2}}(a)$, then $b \in U$ and $[a, b] \subset U$, and thus we have by the multivariate MVT that for any $u, w \in U$:

$$|Df_b(u) - Df_a(u)| \leq M|b - a|, \qquad M = \sup\{(D^2 f)_\theta(u)(w) \ : \ \theta \in [a, b]\} = 0.$$

Thus, we have that for any $y \in [a, b]$,

$$Df_b(y) = Df_a(y) = Df_p(y).$$

Thus, we have that $b \in A_2$. Thus, for all $x \in U$, $(Df)_x$ is constant and thus we can now talk about the behavior of f. Let $B = (Df)_x$ for all $x \in U$. We claim that f is linear. To show this, consider the set

$$A = \{x \ : \ f(x) = Bx\}.$$

We will again show that A is clopen. To show that it is closed, we note that $A = f^{-1}\{B(x)\}$, where f is continuous because Df exists and so A is closed. To show it is open, take some $p \in A$. we do the same process above and take some $q \in B_{\frac{r}{2}(p)}$. Use the C^1 MVT:

$$f(q) - f(p) = \int_0^1 (Df)_{p+t(q-p)} dt(q-p) = \int_0^1 Bdt(q-p) = B(q-p) = Bq - Bp.$$

Thus, we have that since $p \in A$, $f(p) = B(p)$, and thus by the above, $f(q) = B(q)$, and so $q \in A$. Thus, A is open. Since A is both open and closed, we have that $A = U$, and so f is linear for all $x \in U$.

□

(b)

Example 1.182. Generalize for higher derivatives.

Proof. For higher derivatives one can induct. We showed the base case, when $(D^1 f)_p = 0$ for any $p \in U$, implies that f is constant. I.e, f is an $n-1$ degree polynomial and thus f_i is an $n-1$ degree polynomial. Now we assume that if $(D^{(n-1)}(f))_p = 0$ for any $p \in U$, we have that f_i is an $n-2$ degree polynomial since $(D^{n-2}f)$ is constant. Now consider the case when $(D^{(n)}f)_p = 0$ for any $p \in U$. We wish to show that $(D^{n-1}f)$ is constant, and thus by the inductive hypothesis we will have that f_i is an $n-1$ degree polynomial. To do this, we claim that

$$(D^n f)_p = 0, \implies \frac{\partial^n f_i}{\partial x_{j_1} \partial x_{j_2} \dots \partial x_{j_n}}(p) = 0 \implies \frac{\partial^{n-1} f_i}{\partial x_{j_1} \partial x_{j_2} \dots \partial x_{j_{n-1}}}(x) = c_{i,j_1,j_2,\dots,j_{n-1}}, \quad \forall x \in U$$

for all $i \in [m]$. Note that this is just saying that since the total derivative is zero, then the matrix the total derivative represents must be also zero, and thus $(D^{(n-1)}f)$ is the same constant matrix for any $x \in U$. The first implication is clear, to show the second one, we proceed as with the $D^2(f)$ case. Fix $p \in U$.

$$A_{n-1,i} := \{x \ : \ \frac{\partial^{n-1} f_i}{\partial x_{j_1} \partial x_{j_2} \dots \partial x_{j_{n-1}}}(x) = \frac{\partial^{n-1} f_i}{\partial x_{j_1} \partial x_{j_2} \dots \partial x_{j_{n-1}}}(p), x \in U\}.$$

We wish to show $A_{n-1,i}$ is clopen. To show it is closed is easy, since we have that the partial must be continuous since the total exists, and thus we apply the method from the previous part. To show that it is open, we re-do the proof from above, and use the Multivariate MVT on some $b \in B_{\frac{r}{2}}(a)$, where $a \in A_{n-1,i}$:

$$|\frac{\partial^{n-1} f_i}{\partial x_{j_1} \partial x_{j_2} \dots \partial x_{j_{n-1}}}(b) - \frac{\partial^{n-1} f_i}{\partial x_{j_1} \partial x_{j_2} \dots \partial x_{j_{n-1}}}(a)| \leq M|b-a| = 0$$

since $(D^n f)_x = 0$ for all $x \in U$, we have that $b \in A_{n-1,i}$ and so

$$\frac{\partial^{n-1} f_i}{\partial x_{j_1} \partial x_{j_2} \dots \partial x_{j_{n-1}}}(x)$$

is constant for all $x \in U$, and because this holds for any partial in the Hessian, we have that $(D^{n-1}f)_x$ is constant for all $x \in U$, and so by our inductive hypothesis, f_i is $n-1$ degree polynomial. □

PSET 7: Problem 2

Example 1.183. Assume that $f : [a,b] \times Y \to \mathbb{R}^m$, is continuous, Y is an open subset of $\mathbb{R}^n$, the partial derivatives

$$\frac{\partial f_i(x,y)}{\partial y_j}$$

exist, and they are continuous. Let $D_y f$ be the linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ which is represented by the $m \times n$ matrix of partials.

(a)

Example 1.184. Show that

$$F(y) = \int_a^b f(x,y)dx$$

is of class C^1 and

$$(DF)_y = \int_a^b (D_y f)dx.$$

Proof. Since the partial derivatives exist and are continuous, we have that f is C^1. Consider that

$$(DF)_y = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \cdots & \frac{\partial F_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \frac{\partial F_m}{\partial y_2} & \cdots & \frac{\partial F_m}{\partial y_n} \end{bmatrix}$$

We wish to show that for any i, j we have that

$$\frac{\partial F_i}{\partial y_j} = \int_a^b \frac{\partial f_i(x,y)}{\partial y_j} dx.$$

Consider that

$$f_i : [a,b] \times Y \to \mathbb{R},$$

and $\frac{\partial f_i(x,y_j)}{\partial y_j}$ is continuous. By Theorem 14, we have that

$$F_i(y) = \int_a^b f_i(x,y)dx$$

is of class C^1 and

$$\frac{\partial F_i}{\partial y_j} = \int_a^b \frac{\partial f_i(x,y_j)}{\partial y_j} dx.$$

Thus, we have that $\frac{\partial F_i}{\partial y_j}$ exists and is continuous for any i, j, and so $(DF)_y$ is is continuous and can be expressed as desired. Thus, we have that F is C^1. $\square$

(b)

Example 1.185. Generalize (a) to higher-order differentiability.

Proof. We wish to show the following. Suppose $f : [a, b] \times Y \to \mathbb{R}^m$ is continuous, Y is an open subset of $\mathbb{R}^n$, the *nth* partial derivatives exist with respect to everything except for x are continuous and $D_y f$ is the linear transform represented by the $m \times n$ matrix of partials. Then

$$F(y) = \int_a^b f(x, y)dx$$

is of class C^n and

$$(D^{(n)}F)_y = \int_a^b (D_y^{(n)} f)dx.$$

We induct on the nth derivative. We proved the base case above. Suppose the $n-1$ partial derivatives with respect to everything (except maybe x) are continuous and $D_y^{(n-1)} f$ is the linear transform represented by the matrix of partials. Then

$$F(y) = \int_a^b f(x, y)dx$$

is of class C^{n-1} and

$$(D^{(n-1)}F)_y = \int_a^b (D_y^{(n-1)} f)dx.$$

Now suppose Suppose the nth partial derivatives with respect to everything (except maybe x) are continuous and $D_y^{(n)} f$ is the linear transform represented by the matrix of partials. We have that the entries of $(D^{(n)}F)_y$ are

$$\frac{\partial^n F_i}{\partial y_{j_1} \partial y_{j_2} \dots \partial y_{j_n}}.$$

We use the fact that $f_i : [a, b] \times Y \to \mathbb{R}$ is C^n since all its nth order partials exist and are continuous. We wish to show that for any $i, j, \dots,$ we have that

$$\frac{\partial^n F_i}{\partial y_{j_1} \partial y_{j_2} \dots \partial y_{j_n}} = \int_a^b \frac{\partial^n f_i(x, y)}{\partial y_{j_1} \partial y_{j_2} \dots \partial y_{j_n}} dx. \tag{15}$$

Note that by the inductive hypothesis, we have that for any i or y :

$$(D^{(n-1)}F)_y = \int_a^b (D_y^{(n-1)} f)dx \implies \frac{\partial^{n-1} F_i}{\partial y_{j_1} \partial y_{j_2} \dots \partial y_{j_n}} = \int_a^b \frac{\partial^{n-1} f_i(x, y)}{\partial y_{j_1} \partial y_{j_2} \dots \partial y_{j_n}} dx$$

But since the *nth* partials exist and are continuous, we can just apply Theorem 14 to this monstrosity and get the result of (1). Thus, we have that Thus, we have that $\frac{\partial F_i}{\partial y_j}$ exists and is continuous for any i, j, and so $(D^{(n)}F)_y$ is is continuous and can be expressed as desired. Thus, we have that F is C^n. $\square$

PSET 7: Problem 3

Example 1.186. Let $S \subset M$ be given.

(a)

Example 1.187. Define the characteristic function $\chi_S : M \to \mathbb{R}$.

Proof.

$$\chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}.$$

$\square$

(b)

Example 1.188. If M is a metric space, show that $\chi_S(x)$ is discontinuous at x if and only if x is a boundary point of S.

Proof.

- ($\implies$) Suppose $\chi_S(x)$ is discontinuous at x. Thus, we have that $\text{osc}_x(f) \geq \kappa$. Thus, we have that

$$\lim_{r \to 0} \sup_{s,t \in B_r(x)} d(\chi_S(s), \chi_S(t)) =: \lim_{r \to 0}(\chi_S(B_r(x))) \geq \kappa.$$

Since $\chi_S(M) = \{0, 1\}$, we have that if $\chi_S(x)$ is discontinuous at x, then

$$\lim_{r \to 0}(\chi_S(B_r(x))) \geq \frac{1}{2}.$$

That is, for any $r > 0$, there exists some $(x_n) \to x$ such that for large enough n, there exists some $x_n \in S$ and some $x_n \in S^c$, as otherwise, we would have that for radius small enough, $(\chi_S(B_r(x))) = 0$. Thus, we have that for any $r > 0$, there exist $s \in S$ and $s^c \in S^c$ such that $s \in B_r(x)$ and $s^c \in B_r(x)$. Thus, $x \in \partial S$.

- ($\impliedby$) Suppose $x \in \partial S$. For any $r > 0$, we have that there exist $s_r \in S$ and $s_r^c \in S^c$ such that $s_r, s_r^c \in B_r(x)$. Take $r = \frac{1}{n}$, and choose a sequence $(x_n) \to x$ such that $x_{2n} = s_{2n}$ with $s_{2n} \in B_{\frac{1}{2n}}(x)$ and $x_{2n+1} = s_{2n+1}^c$ with $s_{2n+1} \in B_{\frac{1}{2n+1}}(x)$. Thus, we have that since $\chi_S(x_{2n}) = 1$ and $\chi_S(x_{2n+1}) = 0$, then

$$\lim_{r \to 0}(\chi_S(B_r(x))) = 1.$$

Thus, we have that χ_S is discontinuous at x.

$\square$

PSET 7: Problem 4

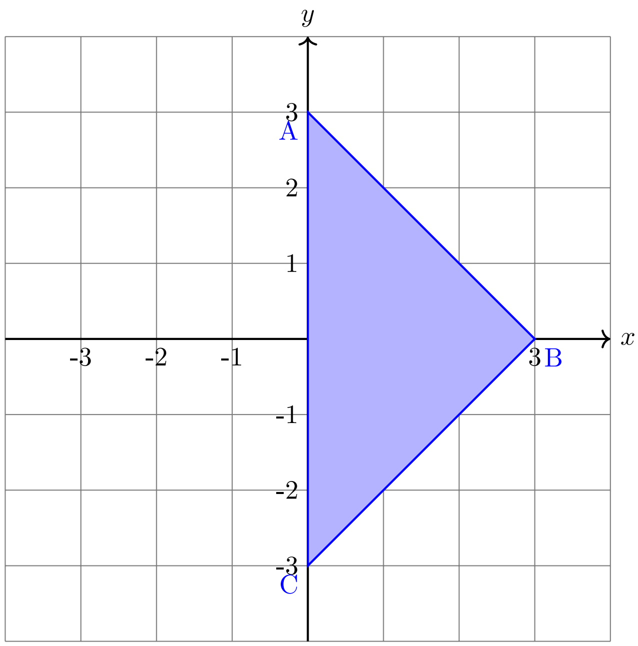
Example 1.189. A region R in the plane is of *type 1* if there are smooth functions $g_1 : [a,b] \to \mathbb{R}$ $g_2 : [a,b] \to \mathbb{R}$ such that $g_1(x) \le g_2(x)$ and

$$R = \{(x,y) \ : \ a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\}.$$

R is *type 2* if the roles of x and y can be reversed, and it is *simple* if it is both type 1 and type 2.

(a)

Example 1.190. Give an example of a region that is type 1 but not type 2



Proof.

We have that it is type 1 because for any $(x,y) \in R$, we have that if $0 \le x \le 3$, then there exists smooth functions

$$g_1(x) = -3 + x, \qquad g_2(x) = 3 - x$$

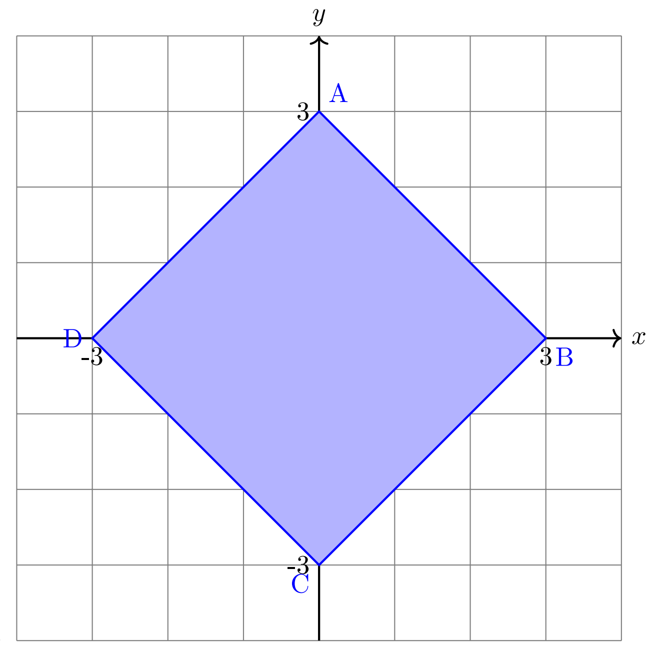
such that $g_1(x) \le y \le g_2(x)$.

However, it is not type 2 because if you take the point $(3,0) \in R$, it cannot be bounded to the right by a smooth function because of the sharp corner (in fact, any point along $y = 0$ works great). $\square$

(b)

Example 1.191. Give an example of a region that is neither type 1 nor type 2.

Proof.



It is clear by the same reasoning as above that this is neither type 1 nor type 2. ☐

(c)

Example 1.192. Is every simple region starlike? convex?

Proof. No consider a function that looks like nearly-headless Nick's moustache as in the Figure below. I apologize for the bad drawing, I am no artist (only consider the moustache curve, not the circle around it or the weird shape intersecting it, that's my bad). We have that this shape is pretty obviously not starlike and thus not convex. To show it is simple, simply split it up into vertical and horizontal lines. Every such line only intersect the curve at a single point (if it doesn't, then that's because I'm bad at drawing). Moreover, every line intersects with a smooth curve (no sharp corners, etc). ☐

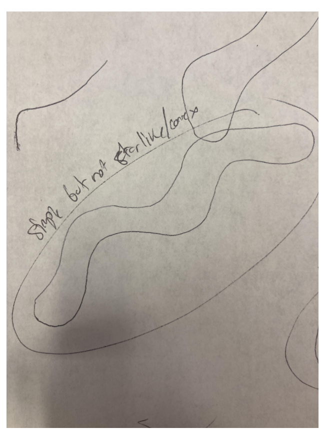


Figure 2: Neither starlike nor convex simple region.

(d)

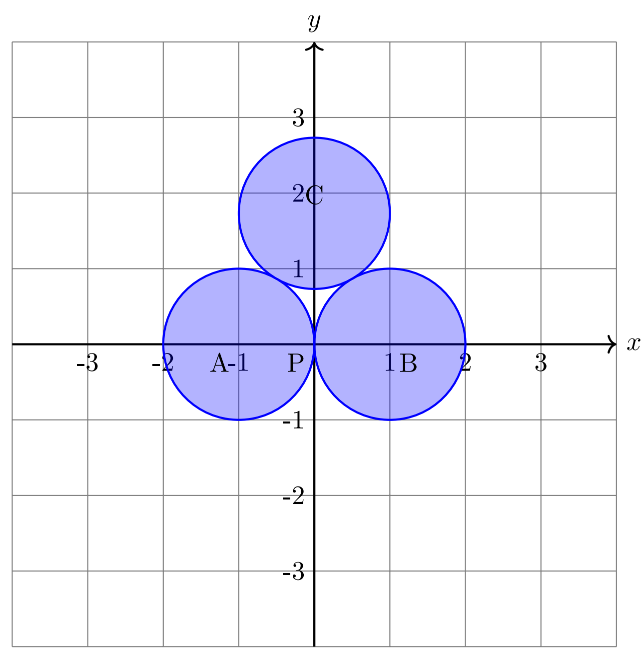
Example 1.193. If a convex region is bounded by a smooth simple closed curve, is it simple?

Proof. **Yes.** Let R be such a region. Intuitively, we have that it is bounded by no sharp corners and it does not have any holes or significant dips or dents. We wish to show that if $(x, y) \in R$, then there exists smooth curves such that $g_1(x) \leq y \leq g_2(x)$. We claim that C, the boundary of R, is such a curve. We have smoothness by assumption. Since C is closed and bounded (since it is closed), then it must be compact, and so it achieves its min/max in the horizontal and its min/max in the vertical. Call the x components of the horizontal min/max a and b and the y components of the vertical directions min/max c, d. Thus, we have that $R \subset C \subset [a, b] \times [c, d]$. Thus, we have that $a \leq x \leq b$. Consider the vertical line across x. Since C is simple, we have that it intersects with C at most twice. Since the curve is smooth, we implicit function theorem to implicitly define the function this vertical line intersects. Since R is convex, we must have that for any y in this vertical line inside the boundary, $y \in R$. Thus, y is bounded by the smooth curve above and below. This shows that it is Type I. To show it is Type 2, take some y with $c \leq y \leq d$, then we have that any horizontal line across y intersects with C at most twice, and this if x in the line inside the boundary, x is bounded by the smooth curve to the left and right. Thus, R is Type II. $\square$

(e)

Example 1.194. Give an example of a region that divides into three simple subregions but not into two.

Proof.



Here, we have the region of $D = D_1 \cup D_2 \cup D_3$, where D_i is a disk of radius 1 intersecting the other 2 at exactly one point. Evidently, we can split it into D_1, D_2, D_3, each one simple. Evidently, splitting it into two subregions creates at least one region which is not simple, since there do not exist smooth curves to bound anything else than each D_i. $\square$

(f)

Example 1.195. If a region is bounded by a smooth simple closed curve C then it need not divide into a finite number of simple subregions. Find an example.

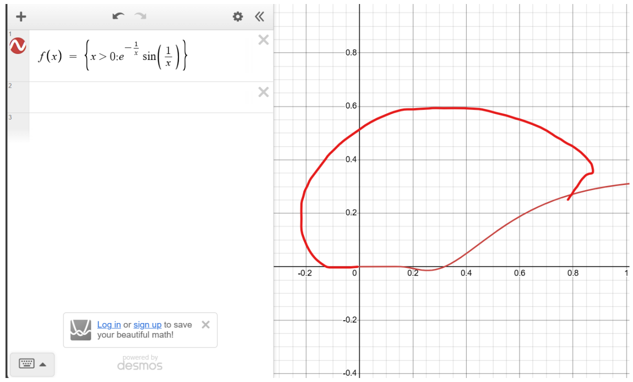


Figure 3: $e^{\frac{-1}{x}}\sin\left(\frac{1}{x}\right)$

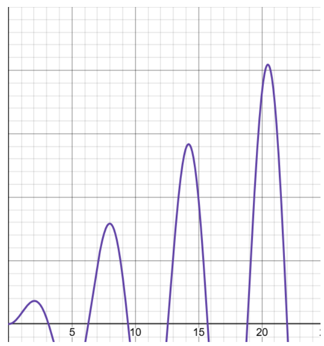


Figure 4: Some subregion contains bumps like these

Proof. Consider the region bounded by the curve $f(x) = e^{\frac{-1}{x}}\sin\left(\frac{1}{x}\right)$ connected at the origin by a smooth curve that goes all the way around to the other end. An image to keep in mind is Figure 2 above. Suppose that it can be divided into a finite number of simple subregions, then near the origin, there must exist some subregion containing more than one bump (in fact, pigeonhole says that it will contain an infinite number of bumps). I.e, we have a subregion bounded below by something like Figure 3 above. This subregion is evidently not Type II though. $\square$

(g)

Example 1.196. Infer that the standard proof of Green's Formulas for simple regions (as, for example, in J. Stewart's Calculus) does not immediately carry over to the general planar region R with smooth boundary; i.e., cutting R into simple regions can fail.

Proof. As shown above, it is sometimes impossible to cut a general planar region R into simple regions, and thus his proof does not carry over. $\square$

(h)

Example 1.197. Is there a planar region bounded by a smooth simple closed curve such that for every linear coordinate system (i.e., a new pair of axes), the region does not divide into finitely many simple subregions? In other words, is Stewart's proof of Green's Theorem doomed?

Proof. **No.** His proof is not doomed. Marrs' proof is aight, but see Desmond's proof for a better one (he stole it from me). $\square$

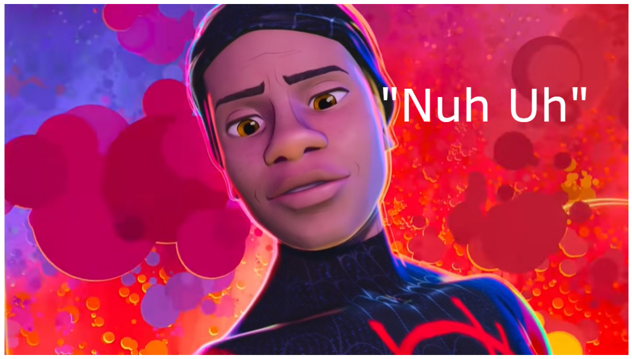


Figure 5: Is Stewart's proof doomed?

(i)

Example 1.198. Show that if the curve C in (f) is analytic, then no such example exists.

Proof. It suffices to show that if C is analytic, then it cannot have infinitely many bumps around a point. That is, we cannot have a situation as in (f) where there were an infinite number of zeros of the curve that lead up to the origin. We do this by contradiction. That is, suppose that for some $(x, y) \in C$, we have that for any neighborhood around (x, y), C has infinitely many zeros. It is no loss of generality to call the point the origin and look at the zeros of the curve crossing the $x-$axis. Thus, for any $h > 0$, we have that C intercepts the axis infinitely many times withing $(0, 0)$ and $(h, 0)$. Since f is analytic, we can express it as

$$f(h) = \sum_{k=0}^{\infty} a_k h^k$$

for $h > 0$ (suppose further that $|h| < r$, where $r > 0$ is the radius of convergence). By the hint, we have that since f is nonconstant around the origin (since it is smooth then it must be nonconstant since it intersects with the axis infinitely many times), and thus for each $h > 0$, there is some derivative of f which is nonzero. Our contradiction is that f is constantly zero around the origin. Note that we assume that $f(0) = 0$, implying that $a_0 = 0$. Consider that as $h \to 0$, we have that

$$\lim_{h\to 0} f(h) = \lim_{h\to 0}\left[\sum_{k=0}^{\infty} a_k h^r\right] = \lim_{h\to 0}\left[a_0 + a_1 h + \sum_{k=2}^{\infty} a_k h^r\right] = 0,$$

and so for h small enough, we have that since the series is absolutely convergence for $|h| < r$, then the tail of the series will still absolutely converge, and thus since $h \to 0$, we have that

$$a_1 = -h \sum_{k=2}^{\infty} a_k h^{r-2} \to 0 \implies a_1 = 0.$$

We induct on this process to show that for all $k \in \mathbb{N}$, $a_k = 0$. Thus, we have that $f(h) \equiv 0$ for $|h| < r$, but then for any $r \in \mathbb{N}$ (sorry for abuse of notation), we have that since all $a_k = 0$, then

$$f^{(r)}(h) = \sum_{k=r}^{\infty} \frac{k!}{(k-r)!} a_k h^{k-r} = 0,$$

which is a contradiction to the hint. $\square$

Example 1.199.

Figure 6: Jeremy Allen White as a Region that does Not Divide into Two

PSET 7: Problem 5

Example 1.200. Does there exist a continuous mapping from the circle to itself that has no fixed-point? What about the 2-torus? The 2-sphere?

Proof. The following maps are all isometries, and are thus continuous.

- (circle) Rotate each point in the circle $180°$. We have showed in PSET 2 that this map is continuous. It evidently has no fixed point.
- (2-torus) For any point on the torus, the point lies along a circle (which would be the cross section of the torus at the point- think of slicing the torus with a knife at the point). Rotate the whole circle $180°$ around the torus. We can also think of this map as just rotating the entire torus around its center $180°$.
- (2-sphere) map each point in the sphere to be at its anti-pole. This map is continuous for similar reasons to the first. Evidently, there is no fixed point.

$\square$

1.35 PSET 8

PSET 8: Problem 1

Example 1.201. Let $R_t = (t, t+2\pi) \times (-\frac{1}{2}, \frac{1}{2}) \subset \mathbb{R}^2$. For each t, parameterize the Möbius band by $\gamma_t : R_t \to \mathbb{R}^3$ as

$$\gamma_t(\theta, r) = \begin{pmatrix} (1 + r\sin\left(\frac{\theta}{2}\right))\cos\theta \\ (1 + r\sin\left(\frac{\theta}{2}\right))\sin\theta \\ r\cos\left(\frac{\theta}{2}\right) \end{pmatrix}.$$

Let F be a vector field such that

$$F = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and show that the surface integral

$$\int\int_{R_t} F \cdot (D_\theta(\gamma(\theta, r) \times D_r(\gamma(\theta, r)))) \, d\theta dr$$

depends on t. Evaluate the surface integral for $t \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$. Why are the values for $t = 0$ and $t = 2\pi$ related?

Proof. This is equivalent to integrating over $\det(F, D_\theta(\gamma), D_r(\gamma))$, that is, we are integrating the flux form Φ_F over the curve R_t applied to the vectors as being the partials. Thus, we have that using the change of variable formula, if we let M denote the band and we let

$$D_1\gamma(\theta, r) = \begin{bmatrix} \frac{r}{2}\cos\left(\frac{\theta}{2}\right)\cos(\theta) - r\sin\left(\frac{\theta}{2}\right)\sin(\theta) - \sin(\theta) \\ \frac{r}{2}\cos\left(\frac{\theta}{2}\right)\sin(\theta) + r\sin\left(\frac{\theta}{2}\right)\cos(\theta) + \cos(\theta) \\ \frac{-r}{2}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

and

$$D_r(\gamma(\theta, r)) = \begin{bmatrix} \sin\left(\frac{\theta}{2}\right)\cos(\theta) \\ \sin\left(\frac{\theta}{2}\right)\sin(\theta) \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$\begin{aligned}
\int_M \Phi_F &= \int_{R_t} \Phi_F \left(P_{\gamma(\theta,r)}(D_1\gamma(\theta, r), D_r(\gamma(\theta, r)))\right) d\theta dr \\
&= \int_t^{t+2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \wedge dz - dx \wedge dz + dx \wedge dy \left(P_{\begin{pmatrix} (1 + r\sin\left(\frac{\theta}{2}\right))\cos\theta \\ (1 + r\sin\left(\frac{\theta}{2}\right))\sin\theta \\ r\cos\left(\frac{\theta}{2}\right) \end{pmatrix}} (D_1\gamma(\theta, r), D_2\gamma(\theta, r)) \right) \\
&= \int_t^{t+2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos\left(\frac{\theta}{2}\right)(\cos(\theta) + \sin(\theta)) - \sin\left(\frac{\theta}{2}\right) + \frac{r}{2}(\sin(\theta) - \cos(\theta)) d\theta dr
\end{aligned}$$

We use Fubini's theorem to first evaluate the dr integral:

$$\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos\left(\frac{\theta}{2}\right)(\cos(\theta) + \sin(\theta)) - \sin\left(\frac{\theta}{2}\right) dr + [\sin(\theta) - \cos(\theta)] \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{r}{2} dr \\
&= \cos\left(\frac{\theta}{2}\right)(\cos(\theta) + \sin(\theta)) - \sin\left(\frac{\theta}{2}\right).
\end{aligned}$$

Thus we need only evaluate the following:

$$\begin{aligned}\int_M \Phi_F &= \int_t^{t+2\pi} \cos\left(\frac{\theta}{2}\right)(\cos(\theta)+\sin(\theta)) - \sin\left(\frac{\theta}{2}\right) d\theta \\ &= -\frac{4}{3}\left(\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)\right)^3\end{aligned}$$

Which is obviously dependent on t. Here is the godawful calculation that simplifies the integral!

$$D_\theta(\gamma(\theta,r)) \times D_r(\gamma(\theta,r)) = \begin{bmatrix} \frac{r}{2}\cos\left(\frac{\theta}{2}\right)\cos(\theta) - r\sin\left(\frac{\theta}{2}\right)\sin(\theta) - \sin(\theta) \\ \frac{r}{2}\cos\left(\frac{\theta}{2}\right)\sin(\theta) + r\sin\left(\frac{\theta}{2}\right)\cos(\theta) + \cos(\theta) \\ \frac{-r}{2}\sin\left(\frac{\theta}{2}\right) \end{bmatrix} \times \begin{bmatrix} \sin\left(\frac{\theta}{2}\right)\cos(\theta) \\ \sin\left(\frac{\theta}{2}\right)\sin(\theta) \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= \det\left|\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{r}{2}\cos\left(\frac{\theta}{2}\right)\cos(\theta) - r\sin\left(\frac{\theta}{2}\right)\sin(\theta) - \sin(\theta) & \frac{r}{2}\cos\left(\frac{\theta}{2}\right)\sin(\theta) + r\sin\left(\frac{\theta}{2}\right)\cos(\theta) + \cos(\theta) & \frac{-r}{2}\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)\cos(\theta) & \sin\left(\frac{\theta}{2}\right)\sin(\theta) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}\right|$$

$$\begin{aligned}&= \cos\left(\frac{\theta}{2}\right)\cos(\theta) - \frac{r\cos^2\left(\frac{\theta}{2}\right)\cos(\theta)}{2} + r\cos\left(\frac{\theta}{2}\right)\cos(\theta)\sin\left(\frac{\theta}{2}\right) - \\ &- \cos^2(\theta)\sin\left(\frac{\theta}{2}\right) - \frac{r\cos(\theta)\sin^2\left(\frac{\theta}{2}\right)}{2} - r\cos^2(\theta)\sin^2\left(\frac{\theta}{2}\right) \\ &+ \cos\left(\frac{\theta}{2}\right)\sin(\theta) + \frac{r\cos^2\left(\frac{\theta}{2}\right)\sin(\theta)}{2} + r\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\sin(\theta) + \\ &+ \frac{r\sin^2\left(\frac{\theta}{2}\right)\sin(\theta)}{2} - \sin\left(\frac{\theta}{2}\right)\sin^2(\theta) - r\sin^2\left(\frac{\theta}{2}\right)\sin^2(\theta). \\ &= \cos\left(\frac{\theta}{2}\right)\cos(\theta) - \frac{r}{2}\cos(\theta) + \frac{r}{2}\sin^2(\theta) - \sin\left(\frac{\theta}{2}\right) - r\sin^2(\frac{\theta}{2}) + \cos\left(\frac{\theta}{2}\right)\sin(\theta) \\ &\quad + \frac{r}{2}\sin(\theta) + \frac{r}{2}\sin^2(\theta) \\ &= \cos\left(\frac{\theta}{2}\right)\cos(\theta) - \frac{r}{2}\cos(\theta) - \sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin(\theta) + \frac{r}{2}\sin(\theta) \\ &= \cos\left(\frac{\theta}{2}\right)(\cos(\theta)+\sin(\theta)) - \sin\left(\frac{\theta}{2}\right) + \frac{r}{2}(\sin(\theta)-\cos(\theta))\end{aligned}$$

□

t	Surface Integral Value
0	$\frac{-4}{3}$
$\frac{\pi}{2}$	$-\frac{8\sqrt{2}}{3}$
π	$\frac{-4}{3}$
$\frac{3\pi}{2}$	0
2π	$\frac{4}{3}$

Table 1: Values of the surface integral for various t.

Proof. $t=0$ and $t=2\pi$ are related because we have gone one full rotation across the Mobius strip and they change in sign is due to the non-orientability of the strip. □

PSET 8: Problem 2

Example 1.202. Let $\gamma : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\gamma(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right)$$

be a parametrization for a surface $\Sigma \subset \mathbb{R}^3$

Example 1.203.

(a) Show that $\Sigma \subset S^2$.

Proof. Let $p \in \Sigma$. Since Σ is parameterized by γ, we have that there exist $u, v \in \mathbb{R}^2$ such that

$$\gamma(u,v) = p = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right).$$

It suffices to show that $||p|| = 1$ or equivalently, $||p||^2 = 1$ To do this, consider that

$$\begin{aligned} ||p||^2 &= (\frac{2u}{1+u^2+v^2})^2 + (\frac{2v}{1+u^2+v^2})^2 + (\frac{-1+u^2+v^2}{1+u^2+v^2})^2 \\ &= \frac{4u^2+4v^2+(-1+u^2+v^2)^2}{(1+u^2+v^2)^2} \\ &= \frac{4u^2+4v^2+1-2(u^2+v^2)+(u^2+v^2)^2}{(1+u^2+v^2)^2} \\ &= \frac{2u^2+2v^2+1u^4+2u^2v^2+v^4}{(1+u^2+v^2)^2} \\ &= 1 \end{aligned}$$

Thus, we have that $p \in S^2$. $\square$

Example 1.204.

(b) Show that α is a bijection from $\mathbb{R}^2$ to $S^2 \setminus (0,0,1)$. The parametrization γ is known as stereographic projection, and can be viewed geometrically as follows: take a line L in $\mathbb{R}^3$ that connects the north pole $(0,0,1)$ and a point $(u,v,0)$. Then $\gamma(u,v)$ is the point of intersection of L and $S^2 \setminus (0,0,1)$.

Proof. Suppose $\gamma(u,v) = \gamma(u',v')$, we wish to show that $(u,v) = (u',v')$.

$$\gamma(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right) = \left(\frac{2u'}{1+u'^2+v'^2}, \frac{2v'}{1+u'^2+v'^2}, \frac{-1+u'^2+v'^2}{1+u'^2+v'^2} \right)$$

and so

$$\begin{aligned} \frac{2u}{1+u^2+v^2} &= \frac{2u'}{1+u'^2+v'^2} \\ \frac{2v}{1+u^2+v^2} &= \frac{2v'}{1+u'^2+v'^2} \\ \frac{-1+u^2+v^2}{1+u^2+v^2} &= \frac{-1+u'^2+v'^2}{1+u'^2+v'^2} \end{aligned}$$

If we call the denominators d and d', respectively, then we get that

$$u = \frac{u'd}{d'}, \quad v = \frac{v'd}{d'}, \quad \frac{-1+u^2+v^2}{d} = \frac{-1+u'^2+v'^2}{d'}.$$

Three equations, three unknowns. Oh boy. From the third equation:

$$\frac{-2+d}{d} = \frac{-2+d'}{d'} \implies d = d'.$$

The rest follows from here, as $u = u'$ and $v = v'$. Thus, we have a nice injection.

Let $s \in S^2(0,0,1)$. We can express s as (x,y,z). Thus, we get that

$$z = \frac{-1+u^2+v^2}{1+u^2+v^2} \implies z(1+u^2+v^2) = -1+u^2+v^2 \implies u^2+v^2 = \frac{1+z}{1-z}.$$

Thus, we get that

$$x = \frac{2u}{1+\frac{1+z}{1-z}} \implies u = \frac{x + x\frac{1+z}{1-z}}{2}, \qquad v = \frac{y + y\frac{1+z}{1-z}}{2}.$$

It is left up to the reader to double check that using these choices of u and v yields (x,y,z), but note that the reader should use the fact that $x^2+y^2+z^2 = 1$. $\square$

Example 1.205.

(c) Using the parametrization γ, compute the surface area of S^2.

Proof. We compute using the flux form of Φ_F, where F is just the parametrization.

$$\begin{aligned}
\int_S^2 \Phi_F &= \int_{S^2} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \\
&= \int_{\gamma[\mathbb{R}^2]} dy \wedge dz - dx \wedge dz + dx \wedge dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \left(P_{\gamma(u,v)} D_1\gamma(u,v), D_2(\gamma(u,v))\right) dudv \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \left(P_{\gamma(u,v)} \begin{bmatrix} \frac{2(1+v^2-u^2)}{(1+u^2+v^2)^2} \\ \frac{-4uv}{(1+u^2+v^2)^2} \\ \frac{2u}{(1+u^2+v^2)^2} \end{bmatrix}, \begin{bmatrix} \frac{-4uv}{(1+u^2+v^2)^2} \\ \frac{2(1+u^2-v^2)}{(1+u^2+v^2)^2} \\ \frac{2v}{(1+u^2+v^2)^2} \end{bmatrix}\right) dudv \\
&= \int_{\mathbb{R}^2} \frac{2u}{1+u^2+v^2} \det \begin{bmatrix} \frac{-4uv}{(1+u^2+v^2)^2} & \frac{2(1+u^2-v^2)}{(1+u^2+v^2)^2} \\ \frac{2u}{(1+u^2+v^2)^2} & \frac{2v}{(1+u^2+v^2)^2} \end{bmatrix} dudv \\
&\quad - \int_{\mathbb{R}^2} \frac{2v}{1+u^2+v^2} \det \begin{bmatrix} \frac{2(1+v^2-u^2)}{(1+u^2+v^2)^2} & \frac{-4uv}{(1+u^2+v^2)^2} \\ \frac{2u}{(1+u^2+v^2)^2} & \frac{2v}{(1+u^2+v^2)^2} \end{bmatrix} dudv \\
&\quad + \int_{\mathbb{R}^2} \frac{-1+u^2+v^2}{1+u^2+v^2} \det \begin{bmatrix} \frac{2(1+v^2-u^2)}{(1+u^2+v^2)^2} & \frac{-4uv}{(1+u^2+v^2)^2} \\ \frac{-4uv}{(1+u^2+v^2)^2} & \frac{2(1+u^2-v^2)}{(1+u^2+v^2)^2} \end{bmatrix} dudv \\
&= \int_{\mathbb{R}^2} \frac{4}{(1+u^2+v^2)^2} dudv
\end{aligned}$$

Using the change of variable

$$g(u,v) = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix},$$

then we get that using the nifty change of variable formula:

$$\int_{\mathbb{R}^2} \frac{4}{(1+u^2+v^2)^2} du \wedge dv = \int_0^{2\pi}\int_0^{\infty} \frac{4}{(1+u^2+v^2)^2} du \wedge dv \left(|P_{\begin{pmatrix} r\cos(\theta) \\ r\sin(\theta)\end{pmatrix}}, \begin{pmatrix} \cos(\theta) \\ \sin(\theta)\end{pmatrix}, \begin{pmatrix} -r\sin(\theta) \\ r\cos(\theta)\end{pmatrix} \right) dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} \frac{4r}{(1+r^2)^2} dr$$

$$= 4\pi$$

□

(d) Compute the surface area of S^2 again, now using the parametrization $\beta : [0, 2\pi) \times [0, \pi] \to \mathbb{R}^3$ given by

$$\beta(\theta, \phi) = \big(\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\big).$$

Proof. Using the same thing:

$$\int_{S^2} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy = \int_{\beta} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \left(P_{\beta(\theta,\phi)}, D_1\beta, D_2\beta\right)$$

$$= \int_0^{2\pi}\int_0^{\pi} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \left(P_\beta \begin{bmatrix} -\sin\theta\sin\phi \\ \cos\theta\sin\phi \\ 0 \end{bmatrix} \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix} \right) d\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \cos\theta\sin\phi \det \begin{bmatrix} \cos\theta\sin\phi & \sin\theta\cos\phi \\ 0 & -\sin\phi \end{bmatrix} - \sin\theta\sin\phi \det \begin{bmatrix} -\sin\theta\sin\phi & \cos\theta\cos\phi \\ 0 & -\sin\phi \end{bmatrix} +$$

$$+ \cos\phi \det \begin{bmatrix} -\sin\theta\sin\phi & \cos\theta\cos\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \end{bmatrix} d\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} (-\cos^2\theta\sin^3\phi - \sin^2\theta\sin^3\phi - \sin\phi\cos^2\phi) d\theta d\phi$$

$$= -\int_0^{2\pi}\int_0^{\pi} \sin^3\phi + \sin\phi\cos^2\phi d\theta d\phi$$

$$= -\int_0^{2\pi}\int_0^{\pi} \sin\phi d\theta d\phi$$

$$= 4\pi$$

□

PSET 8: Problem 3

Example 1.206.

(a) Compute the surface integral

$$\iint_{S_r} F \cdot n \, dA,$$

where F is the vector field

$$F(x, y, z) = \frac{\vec{r}}{|r|^3} = \left(\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right).$$

Here S_r is the sphere of radius r centered at the origin.

Proof. We have that

$$\int\int_{S_r} F \cdot ndA = \int\int_{S_r} \Phi_F,$$

where Φ_F is the flux form of F. Using Stoke's Theorem/Divergence Theorem, we have that since S_r is the boundary of the open ball $B_r(0)$, then

$$\int\int_{S_r} \Phi_F = \int_{B_r(0)} d\Phi_F = \int_{B_r(0)} M_{\nabla \cdot F} = \int_{B_r(0)} \nabla \cdot F dx \wedge dy \wedge dz.$$

We can parameterize $B_r(0)$ by

$$\gamma : [0, 2\pi] \times [0, \pi] \times (0, r) \to \mathbb{R}^3$$
$$\gamma(\theta, \phi, r) = \begin{pmatrix} r \cos\theta \sin\phi \\ r \sin\theta \sin\phi \\ r \cos\phi \end{pmatrix}$$

Thus, we have that

$$D_1\gamma = \begin{bmatrix} -r \sin\theta \sin\phi \\ r \cos\theta \sin\phi \\ 0 \end{bmatrix}, \quad D_2\gamma = \begin{bmatrix} r \cos\theta \cos\phi \\ r \sin\theta \cos\phi \\ -r \sin\phi \end{bmatrix}, \quad D_3\gamma = \begin{bmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ \cos\phi \end{bmatrix}.$$

Moreover, we have that

$$\nabla \cdot F = \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - 3x^2(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^3} + \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - 3y^2(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^3} + \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - 3z^2(x^2 +}{(x^2 + y^2 + z^2)^3}$$
$$= 0, \qquad (x, y, z) \neq 0$$

Since the divergence is not well defined over all the sphere, we cannot resort to using the Divergence Theorem. Thus, we must resort to applying computing the surface integral directly (the second equality I put for how the differential form would have worked, but it is much too tedious to compute) using a change of variable with the parameterization of the sphere given in the previous problem.

$$\begin{aligned}
\int\int_{S_r} \Phi_F &= \int\int_{\beta} F_1 dy \wedge dz - F_2 dx \wedge dz + F_3 dx \wedge dy \left(P_\beta \begin{bmatrix} -\sin\theta\sin\phi \\ \cos\theta\sin\phi \\ 0 \end{bmatrix} \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix} \right) d\theta d\phi \\
&= \int_0^{2\pi}\int_0^{\pi} \frac{\hat{r}}{r^2} \cdot \hat{r} \det \begin{bmatrix} -r\sin\theta\sin\phi & r\cos\theta\cos\phi & \cos\theta\sin\phi \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi & \sin\theta\sin\phi \\ 0 & -r\sin\phi & \cos\phi \end{bmatrix} d\theta d\phi \\
&= \int_0^{2\pi}\int_0^{\pi} \frac{1}{r^2} \cdot r^2 \sin\theta d\theta d\phi \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\theta d\theta d\phi \\
&= 4\pi
\end{aligned}$$

□

Example 1.207.

(b) Compute div$F$ on $\mathbb{R}^3 \setminus \{0\}$.

Proof. As shown above, $\nabla \cdot F = 0$. □

(c) Let Ω be some arbitrary bounded open set in $\mathbb{R}^3$ that contains the origin and has a smooth boundary. Compute

$$\int_{\partial\Omega} F \cdot n \, \mathrm{d}A.$$

Proof. Since Ω is open, there exists some $r > 0$ such that $B_r(0) \subset \Omega$. Thus, we compute the surface integral by splitting Ω into $B_r(0)$ and $\Omega \setminus B_r(0)$:

$$\int_{\partial\Omega} F \cdot n \, \mathrm{d}A = \int_{S_r} F \cdot n \mathrm{d}A + \int_{\partial(\Omega\setminus B_r(0))} F \cdot n \mathrm{d}A.$$

The first term we have computed in part A, and the second we use the fact that we can now in fact use the divergence theorem with $\nabla \cdot F = 0$. Thus, we are simply left with

$$\int_{\partial\Omega} F \cdot n \, \mathrm{d}A = 4\pi.$$

□

Reflections: This motivates us saying that div$F = 4\pi\delta$ (where δ is the "Dirac delta").

PSET 8: Problem 4

Example 1.208. For a C^2 function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$, we define the Laplacian as

$$\Delta f = \operatorname{div}(\nabla f).$$

Let Ω be any open set inside U with a piecewise smooth boundary. We write $\partial\Omega$ to denote the boundary of Ω and n is the unit normal vector pointing outwards. We write $\mathrm{d}A$ to denote the differential of area on $\partial\Omega$ and $\partial_n u$ is the directional derivative in the direction n. Prove the following two identities.

$$\int_\Omega |\nabla u|^2 + u\Delta u \,\mathrm{d}x = \int_{\partial\Omega} u\partial_n u \,\mathrm{d}A,$$
$$\int_\Omega u\Delta v - v\Delta u \,\mathrm{d}x = \int_{\partial\Omega} u\partial_n v - v\partial_n u \,\mathrm{d}A.$$

Proof. We assume $u : U \subset \mathbb{R}^2 \to \mathbb{R}$ and same for v. We have by the Divergence theorem that

$$\int_\Omega (\nabla \cdot F) d^3x = \int_{d\Omega} F \cdot n dA.$$

Recall that

$$\partial_n u = \nabla u \cdot n$$

Thus, if we let $F = u\nabla u$, then

$$F \cdot n = u\nabla u \cdot n = u\partial_n u.$$

On the left hand side, we claim that

$$\nabla \cdot F = \nabla \cdot (u\nabla u) = \nabla u \cdot \nabla u + u\nabla \cdot \nabla u = |\nabla u|^2 + u\nabla^2 u.$$

The second equality is all we need to show, and so it suffices to show that

$$\nabla \cdot (uA) = \nabla u \cdot A + u(\nabla \cdot A).$$

We only need to prove it for our case, so we let

$$A = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \implies uA = \begin{bmatrix} uv_1 \\ uv_2 \end{bmatrix} \implies \nabla \cdot (uA) = D_1 uv_1 + D_2 uv_2 = \left(\frac{\partial u}{\partial x} v_1 + \frac{\partial v_1}{\partial x} u\right) + \left(\frac{\partial u}{\partial y} v_2 + \frac{\partial v_2}{\partial y} u\right)$$

Thus, we have that

$$\nabla \cdot (uA) = \left(\frac{\partial u}{\partial x} v_1 + \frac{\partial u}{\partial y} v_2\right) + u\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right)$$

On the RHS, we have that

$$\nabla u \cdot A = \begin{bmatrix} D_1 u \\ D_2 u \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left(\frac{\partial u}{\partial x} v_1 + \frac{\partial u}{\partial y} v_2\right)$$
$$\nabla \cdot A = D_1 v_1 + D_2 v_2 = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$$

For the latter identity, we again make use of the divergence theorem, letting

$$F = u\nabla v - v\nabla u \implies F \cdot n = (u\nabla v - v\nabla u) \cdot n = u\partial_n v - v\partial_n u.$$

Thus, we have that by Gauss that

$$\int_{d\Omega} F \cdot n dA = \int_\Omega (\nabla \cdot F) d^3x,$$

so it suffices to show that

$$\nabla \cdot (u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u.$$

We have that

$$\nabla \cdot (u\nabla v - v\nabla u) = \nabla \cdot u\nabla v - \nabla \cdot v\nabla u.$$

By the logic in the previous problem, we have that

$$\nabla \cdot u\nabla v = \nabla u \cdot \nabla v + u(\nabla \cdot \nabla v) = \nabla u \cdot \nabla v + u\nabla^2 v, \quad \nabla \cdot v\nabla u = \nabla v \cdot \nabla u + v(\nabla \cdot \nabla u) = \nabla v \cdot \nabla u + v\nabla^2 u$$

Thus, we have that the difference between the two quantities above gives

$$u\nabla^2 v - v\nabla^2 u,$$

as desired. $\square$

PSET 8: Problem 5

Example 1.209. Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function. Write f as

$$f(x, y) = u(x, y)e_1 + v(x, y)e_2 = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.$$

Assume that for all $p \in U$ the derivative of f at p (which we write Df_p) is a scalar matrix (a multiple of the identity). In other words, we have

$$Df_p = \lambda(p)\mathrm{I} = \begin{bmatrix} \frac{\partial u(x,y)}{\partial x} = \lambda(p) & \frac{\partial u(x,y)}{\partial y} = 0 \\ \frac{\partial v(x,y)}{\partial x} = 0 & \frac{\partial v(x,y)}{\partial y} = \lambda(p) \end{bmatrix}$$

where $\lambda : U \to \mathbb{R}$ is some continuous strictly- positive function on U. Let γ be a simple closed curve in U which bounds a region entirely contained in U. Prove that

$$\int_\gamma u\, dx + u\, dy = \int_\gamma -v\, dx + v\, dy.$$

Proof. We use Green's theorem, which states that if W_f is the one-form of f, then if S is the surface bounded by γ :

$$\int_\gamma W_f = \int_S dW_f \tag{16}$$

Let

$$W_{f_1} = udx + udy, \qquad W_{f_2} = -vdx + vdy.$$

Thus. we claim that that:

$$dW_{f_1} = (D_1u - D_2u)dx \wedge dy = (\lambda(p) - 0)dx \wedge dy; \tag{17}$$
$$dW_{f_2} = (D_1v + D_2v)dx \wedge dy = (0 + \lambda(p))dx \wedge dy. \tag{18}$$

This first equality can be shown as follows. Suppose $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$, then $W_F = F_1dx + F_2$. Then by properties of the exterior derivative and the wedge product:

$$\begin{aligned} dW_F &= d(F_1dx + F_2dy) \\ &= (D_1F_1dx + D_2F_1dy) \wedge dx + (D_1F_2dx + D_2F_2dy) \wedge dy \\ &= (D_1F_2 - D_2F_1)dx \wedge dy. \end{aligned}$$

Thus, from (1), (2), and (3), we get:

$$\int_\gamma udx + udy = \int_\gamma W_{f_1} = \int_S dW_{f_1} = \int_S \lambda(p)dx \wedge dy = \int_S dW_{f_2} = \int_\gamma W_{f_2} = \int_\gamma -vdx + vdy.$$

$\square$

PSET 8: Problem 6

Example 1.210. Find an open set $\Omega \subset \mathbb{R}^2$ and a smooth vector field $F : \Omega \to \mathbb{R}^2$ such that the set

$$\left\{ \int_C F \cdot \tau \, \mathrm{d}s : C \text{ is a closed loop contained in } \Omega \right\}$$

is dense in $\mathbb{R}$.

Proof. We have by work shown in class that if $F_1 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$, then $F \cdot \tau ds = W_F = F_1 dx + F_2 dy$. Thus, Green states that if C is the curve around some region $S \subset \mathbb{R}^2$, then

$$\int_C F_1 dx + F_2 dy = \int_C W_F = \int_S dW_F = \int_S (D_2 F_1 - D_1 F_2) dx \wedge dy.$$

Thus, it suffices to find some F and some Ω such that if $S_\alpha \subset \Omega$, then

$$\{ \int_{S_\alpha} (D_2 F_1 - D_1 F_2) dx \wedge dy \}$$

is dense in $\mathbb{R}$. We let

$$F = \begin{bmatrix} -y \\ x \end{bmatrix}$$

and $\Omega = \mathbb{R}^2$. Then we have that if S_r is square of side length r, then

$$\int_{S_r} (D_2 F_1 - D_1 F_2) dx \wedge dy = \int_0^r \int_0^r (-1-1) dx \wedge dy = -2 \int_0^r \int_0^r dx = -2r^2.$$

Since we are ranging over all of $\mathbb{R}$ and we are able to change the orientation of S_r and thus change the sign of the integral. $\square$

Reflections: If F were not smooth for the sake of me not reading the problem, then We let

$$F = \begin{bmatrix} \frac{3}{2y^2} \\ x \end{bmatrix}$$

and $\Omega = B_{10000}(0)$. Then we have that if S_r is square of side length r, then

$$\int_{S_r} (D_2 F_1 - D_1 F_2) dx \wedge dy = \int_0^r \int_0^r (\frac{-3}{y^3} - 1) dx \wedge dy = (\frac{1}{r^2} - r) \int_0^r dx = \frac{1}{r} - r^2.$$

We claim that $\{\frac{1}{r} - r^2\}$ is dense in $\mathbb{R}$. This is because r can range over $(-2, 2)$ since then we would have $S_r \subset B_{10000}(0)$. From this, it is obvious that if $(a, b) \subset \mathbb{R}$, then there exists some $r \in (-2, 2)$ such that $\frac{1}{r} - r^2 \in (a, b)$.

PSET 8: Problem 7

Example 1.211. This exercise is asking you to verify the uniqueness of solutions of an ODE without assuming that F is Lipchitz, but assuming something else in exchange. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field so that for every closed curve C in $\mathbb{R}^2$, we have

$$\int_C F^\perp \cdot \tau \, \mathrm{d}s = 0.$$

Assume further that $F(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. Here $v^\perp$ denotes the ninety degree rotation of the vector v. Thus, $(x, y)^\perp := (-y, x)$.

(a) If F is C^1, prove that $\mathrm{div}F = 0$.

Proof. We use Green's Theorem. Let $F(x,y) = \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \end{bmatrix}$, then we have that

$$F^\perp(x,y) = \begin{bmatrix} -F_2(x,y) \\ F_1(x,y) \end{bmatrix} \tag{19}$$

Then we have that that if S is the region bounded by C, then

$$\int_C F^\perp \cdot \tau ds = \int_C W_{F^\perp} = \int_C -F_2(x,y)dx + F_1(x,y)dy = \int_S \left(-D_2F_2 - D_1F_1\right) dx \wedge dy = 0.$$

Thus, because we have this for any (closed) curve C, then

$$D_1F_1 + D_2F_2 = 0,$$

but this expression is exactly $\nabla \cdot F$. $\square$

(b) Without assuming that F is C^1 (or even Lipchitz), prove that the ODE

$$x'(t) = F(x(t))$$

has at most one solution on any time interval $t \in (-\delta, \delta)$.

I know two different proofs of this fact. When I was reviewing them, I realized that in both there is an elegant idea to prove that if we have two solutions $x(t)$ and $y(t)$, they must parametrize the same curve on $\mathbb{R}^2$. However, we are then left with the nontrivial task of analyzing if they could potentially be two different parametrizations of the same curve. Let us keep it simple and focus on the first part only. That is, let us prove that any two solutions of the ODE give us the same curve on $\mathbb{R}^2$. That would get you full score.

Assuming that there are two solutions of the ODE that trace different curves on $\mathbb{R}^2$, these curves must split somewhere. If we look at their last point in common, we would have two solutions of the ODE going to different paths from there. It is easy to see that the two curves must be tangent at any contact point. Can you find something that goes wrong in the last intersection point, leading to a contradiction? Alternatively, you may construct a clever curve on $\mathbb{R}^2$ using some theorem that we learned earlier in this class and then verify that any solution to the ODE must stay within this curve.

Proof. Suppose that we dislike using x and so we have that

$$\gamma'(t) = F(\gamma(t)) \tag{20}$$

where $\gamma : \mathbb{R} \to \mathbb{R}^2$. Since

$$\int_C F^\perp \cdot \tau ds = 0,$$

then we have that there exists some scalar function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\nabla f(x,y) = F^\perp(x,y) = \begin{bmatrix} -F_2(x,y) \\ F_1(x,y) \end{bmatrix} \neq 0 \qquad \forall (x,y) \in \mathbb{R}^2$$

by the assumption that $F(x,y) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \forall (x,y) \in \mathbb{R}^2$. Thus, we use the fundamental theorem of line integrals (seen in class) that if $t_0, t_1 \in C$, where C is the curve defined by γ (not necessarily closed) then

$$\int_C F^\perp \cdot \tau ds = \int_C \nabla f \cdot \tau ds = f(\gamma(t_1)) - f(\gamma(t_0)).$$

However, we also have that by (5) and a u-substitution:

$$\int_C F^\perp \cdot \tau ds = \int_{\gamma[t_0,t_1]} F^\perp \cdot \tau ds = \int_{t_0}^{t_1} F^\perp(\gamma) \cdot \gamma'(t) dt = \int_{t_0}^{t_1} F^\perp(\gamma) \cdot F(\gamma) dt = 0$$

since $F \cdot F^\perp = 0$ by orthagonality. Thus, we have that $f(\gamma(t))$ is constant for any $t \in (-\delta, \delta)$. Without loss of generality, we call $f(\gamma(t)) = 0$ for all $t \in (-\delta, \delta)$. Thus, we have that γ lies within the level set 0 of f. If we show that the level sets of f are curves, then we will know that if γ and σ are two solutions to (5), then they must lie on the same curve and indeed be the same path.

For the next step, we will assume with loss of generality that F is continuous, as I do not know how to solve the problem without this. Thus we have that f is C^1 since $F^\perp$ is continuous and $\boldsymbol{\nabla} f = F^\perp$. Thus, if we let $z_0 = (x_0, y_0) \in \mathbb{R}^2$ with $f(x_0, y_0) = f(z_0) = 0$ for some $c \in \mathbb{R}$ (we do this without loss of generality), then by the Implicit Function Thm, there exists some open $V_1 \subset \mathbb{R}$ and $V_2 \subset \mathbb{R}$ such that $x_0 \in V_1$ and $y_0 \in V_2$ and a map $g \in C^1(V_1, V_2)$ such that

$$f(t, g(t)) = 0, \qquad \forall t \in V_1.$$

Thus, locally within $t \in (-\delta, \delta)$ out level set is given by

$$L(t) = \begin{pmatrix} t \\ g(t) \end{pmatrix} \implies L((-\delta, \delta)) \subset \mathbb{R}^2 \qquad \text{is a curve,}$$

and so we know that any solution to (5) lies locally within L, implying uniqueness as before. $\square$

PSET 8: Problem 8

Example 1.212. Find a differential form ω (of any degree and dimension) so that $\omega \wedge \omega \neq 0$.

Proof. Consider the $0-$ differential form f, where $f = 1$ is a function. Note that $f \wedge f$ is still a zero form, since it does not take in any vectors, and thus we have that $f \wedge f = 1$ still. $\square$

PSET 8: Problem 9

Example 1.213. In any dimension, a 1-form is associated to a vector field F. In particular, in 3D it takes the form

$$W_F = F_1 \mathrm{d}x + F_2 \mathrm{d}y + F_3 \mathrm{d}z.$$

In 3D, 2-forms are also associated to a vector field F by the following identification

$$\Phi_F = F_1 \mathrm{d}y \wedge \mathrm{d}z + F_2 \mathrm{d}z \wedge \mathrm{d}x + F_3 \mathrm{d}x \wedge \mathrm{d}y.$$

We say that $G = \boldsymbol{\nabla} \times F$ if $\mathrm{d}W_F = \Phi_G$.

Example 1.214.

(a) Compute an explicit formula for $\boldsymbol{\nabla} \times F$ in terms of the components of F and their partial derivatives.

Proof. Using the properties of the exterior derivative, we get that

$$\begin{aligned} dW_F &= (D_1F_1dx + D_2F_1dy + D_3F_1dz)dx + (D_1F_2dx + D_2F_2dy + D_3F_2dz)dy \\ &+ (D_1F_3dx + D_2F_3dy + D_3F_3dz)dz \\ &= -D_2F_1dx \wedge dy + D_3F_1dz \wedge dx + \\ &+ D_1F_2dx \wedge dy - D_3F_2dy \wedge dz \\ &- D_1F_3dz \wedge dx + D_2F_3dy \wedge dz \\ &= (D_2F_3 - D_3F_2)dy \wedge dz + (D_3F_1 - D_1F_3)dz \wedge dx + (D_1F_2 - D_2F_1)dx \wedge dy \\ &= G_1 \mathrm{d}y \wedge \mathrm{d}z + G_2 \mathrm{d}z \wedge \mathrm{d}x + G_3 \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

where

$$G = \nabla \times F = \begin{bmatrix} D_2F_3 - D_3F_2 \\ D_3F_1 - D_1F_3 \\ D_1F_2 - D_2F_1 \end{bmatrix}$$

□

Example 1.215.

(b) For any C^1 scalar function $p : \mathbb{R}^3 \to \mathbb{R}$, prove that $dp = W_{\nabla p}$.

Proof. We have that

$$dp = D_1pdx + D_2pdy + D_3pdz$$

Similarly, we have byb definition that since

$$\nabla p = \begin{bmatrix} D_1p \\ D_2p \\ D_3p \end{bmatrix},$$

then

$$W_{\nabla p} = D_1pdx + D_2pdy + D_3pdz.$$

□

Example 1.216.

(c) For any C^1 vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, prove that $d\Phi_F = (\nabla \cdot F)dx \wedge dy \wedge dz$.

Proof. We have that since $dx \wedge dx = 0$, then:

$$\begin{aligned}
\Phi_F &= F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy \implies \\
d\Phi_F &= (D_1F_1dx + D_2F_2dy + D_3F_3dz) \wedge dy \wedge dz + dF_2 \wedge dz \wedge dx + dF_3 \wedge dx \wedge dy \\
&= D_1F_1dx \wedge dy \wedge dz + D_2F_2dy \wedge dz \wedge dx + D_3F_3dz \wedge dx \wedge dy \\
&= (D_1F_1 + D_2F_2 + D_3F_3)dx \wedge dy \wedge dz \\
&= (\nabla \cdot F)dx \wedge dy \wedge dz
\end{aligned}$$

Where the second to last equality comes from the fact that we only needed even permutations to get the desired quantity. □

Example 1.217.

(d) For any C^2 scalar function $f : \mathbb{R}^3 \to \mathbb{R}$, prove that $\boldsymbol{\nabla} \times \nabla f = 0$.

Proof. Consider that since f is a zero form, then

$$df = D_1 f dx + D_2 f dy + D_3 f dz$$

is a one form. Note that we have by (b) that

$$df = W_{\nabla f}.$$

By a theorem in class, we have that the two form

$$ddf = 0,$$

and by (a) we have that

$$ddf = dW_{\nabla f} = \Phi_{\nabla \times \nabla f} = 0.$$

Note that this happens if and only if $\nabla \times \nabla f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. □

Example 1.218.

(e) For any C^2 vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, prove that $\nabla \cdot (\boldsymbol{\nabla} \times F) = 0$.

Proof. Consider the one form defined by F, which is

$$W_F = F_1 dx + F_2 dy + F_3 dz.$$

We have by (a) that

$$dW_F = \Phi_{\nabla \times F},$$

and thus by the same logic as before we have that by (c):

$$ddW_F = 0 \implies ddW_F = d\Phi_{\nabla \times F} = (\nabla \cdot (\nabla \times F))dx \wedge dy \wedge dz = 0.$$

Thus, we have that $\nabla \cdot (\nabla \times F) = 0$. □

2 Spring 2020