Agustín Esteva
aesteva@uchicago.edu
Due Date: 05-21-2025

Problem 1

Which of the following forms on $\mathbb{R}^3$ are closed?

(a) $\omega = x\,dx \wedge dy \wedge dz$

SOLUTION:

$$\begin{aligned}
d\omega &= d\left(x\,dx \wedge dy \wedge dz\right) \\
&= \left(\frac{\partial}{\partial x} x\,dx + \frac{\partial}{\partial y} x\,dy + \frac{\partial}{\partial z} x\,dz\right) \wedge dx \wedge dy \wedge dz \\
&= 1\,dx \wedge\, dx \wedge dy \wedge dz \\
&= (dx \wedge dx) \wedge dy \wedge dz \\
&= 0
\end{aligned}$$

■

(b) $\omega = z\,dy \wedge dx + x\,dy \wedge dz$

SOLUTION:

$$\begin{aligned}
d\omega &= d\left(z\,dy \wedge dx + x\,dy \wedge dz\right) \\
&= d\left(z\,dy \wedge dx\right) + d\left(x\,dy \wedge dz\right) \\
&= \left(\frac{\partial}{\partial x} z\,dx + \frac{\partial}{\partial y} z\,dy + \frac{\partial}{\partial z} z\,dz\right) \wedge dy \wedge dx + \left(\frac{\partial}{\partial x} x\,dx + \frac{\partial}{\partial y} x\,dy + \frac{\partial}{\partial z} x\,dz\right) \wedge dy \wedge dz \\
&= dz \wedge dy \wedge dx + dx \wedge dy \wedge dz \\
&= (-1)^3\,dx \wedge dy \wedge dz + dx \wedge dy \wedge dz \\
&= 0
\end{aligned}$$

■

(c) $\omega = x\,dx + y\,dy$

SOLUTION:

$$d\omega = d\left(x\,dx\right) + d(y\,dy)$$

$$
\begin{aligned}
&= \left(\frac{\partial}{\partial x} x\, dx + \frac{\partial}{\partial y} x\, dy + \frac{\partial}{\partial z} x\, dz\right) \wedge dx + \left(\frac{\partial}{\partial x} y\, dx + \frac{\partial}{\partial y} y\, dy + \frac{\partial}{\partial z} y\, dz\right) \wedge dy \\
&= dx \wedge dx + dy \wedge dy \\
&= 0
\end{aligned}
$$

■

Problem 2

Show that every k-form on $\mathbb{R}^k$ is closed.

Solution: Consider first the case when

$$\omega = f\, dx_1 \wedge \cdots dx_k,$$

Then

$$\begin{aligned} d\omega &= \left(\sum_{i=1}^{k} \frac{\partial f}{\partial x_i} \wedge dx_i\right) \wedge dx_1 \cdots \wedge dx_k \\ &= \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i} \wedge dx_i \wedge dx_1 \cdots \wedge dx_k\right) \\ &= \sum_{i=1}^{k} 0 \end{aligned}$$

by the repeated index in every sum. Now consider

$$\omega = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} = (-1)^{\alpha} f\, dx_1 \wedge \cdots \wedge dx_k,$$

we get that by the first case, since $(-1)^{\alpha}$ is a constant[a]

$$d\omega = (-1)^{\alpha} d\left(f\, dx_1 \wedge \cdots \wedge dx_k\right) = (-1)^{\alpha}(0) = 0$$

For the general case, let

$$\omega = \sum_I f_I\, dx_I,$$

where the $I = \sigma\{1, 2, \ldots, k\}$ are permutations of $\{1, 2, \ldots, k\}$. Then by definition,

$$d\omega = \sum_I (df_I) \wedge dx_I = \sum_I 0 = 0$$

■

[a] Let ω be a $k-$form and $c > 0$, then $d(c\omega) = cd(\omega)$. To see this, note that we can pull out the c constant out of every partial in the sum

Problem 3

In $\mathbb{R}^4$, consider the following 2-form

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4.$$

Compute $\omega \wedge \omega$ and $\omega \wedge \omega \wedge \omega$. Find a 1-form η such that $d\eta = \omega$. This 1-form is called the Liouville form.

SOLUTION:

$$\begin{aligned}
\omega \wedge \omega &= (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \\
&= (dx_1 \wedge dx_2) \wedge (dx_1 \wedge dx_2) + 2\,(dx_1 \wedge dx_2) \wedge (dx_3 \wedge dx_4) + (dx_3 \wedge dx_4) \wedge (dx_3 \wedge dx_4) \\
&= 2\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4
\end{aligned}$$

Using this,

$$\begin{aligned}
\omega \wedge \omega \wedge \omega &= (\omega \wedge \omega) \wedge \omega \\
&= (2\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \\
&= 2(dx_1 \wedge dx_2) \wedge dx_3 \wedge dx_4 \wedge (dx_1 \wedge dx_2) + 2\,dx_1 \wedge dx_2 \wedge (dx_3 \wedge dx_4) \wedge (dx_3 \wedge dx_4) \\
&= 0 + 0 \\
&= 0
\end{aligned}$$

Consider

$$\eta = x_1\,dx_2 + x_3\,dx_4.$$

Then

$$\begin{aligned}
d\eta &= \left(\frac{\partial}{\partial x_1} x_1 dx_1 + \frac{\partial}{\partial x_1} x_1 dx_2 + \frac{\partial}{\partial x_1} x_1 dx_3 + \frac{\partial}{\partial x_1} x_1 dx_4\right) \wedge dx_2 \\
&\quad + \left(\frac{\partial}{\partial x_1} x_3 dx_1 + \frac{\partial}{\partial x_2} x_3 dx_2 + \frac{\partial}{\partial x_3} x_3 dx_3 + \frac{\partial}{\partial x_4} x_3 dx_4\right) \wedge dx_4 \\
&= dx_1 \wedge dx_w + dx_3 \wedge dx_4
\end{aligned}$$

■

Problem 4

Suppose $\sigma = [p_0, \ldots, p_k]$ is an oriented affine $k-$simplex. Show that $\partial^2\sigma = 0$.

Solution: Let

$$\sigma_i = [p_0, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k]$$

and

$$\sigma_{ij} = [p_0, \ldots, p_{j-1}, p_{j+1}, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k].$$

That is, σ_{ij} is the oriented affine $k-2$simplex with the i and j coordinate removed. Without loss of generality up to a sign, suppose k is odd. Then by definition

$$\partial\sigma = \sigma_0 - \sigma_1 + \cdots - \sigma_k.$$

Since $\partial\sigma$ is a oriented affine $k-1$ chain, then by definition

$$\begin{aligned}\partial(\partial\sigma) &= \partial\sigma_0 - \partial\sigma_1 + \cdots - \partial\sigma_k \\ &= (\sigma_{01} - \sigma_{02} + \cdots + \sigma_{0k}) - (\sigma_{10} - \sigma_{12} + \cdots + \sigma_{1k}) + \cdots - (\sigma_{k0} - \sigma_{k1} + \cdots + \sigma_{k(k-1)})\end{aligned}$$

It is clear that for any i, j where $i \neq j$, $\sigma_{ij} = \sigma_{ji}$. Thus, we see that the sum above is zero. ■

Problem 5

Define $J^2 = \tau_1 + \tau_2$, where

$$\tau_1 = [\mathbf{0}, e_1, e_1 + e_2] \qquad \tau_2 = -[\mathbf{0}, e_2, e_2 + e_1].$$

Explain why it is reasonable to call J^2 the positively oriented unit square in $\mathbb{R}^2$. What is ∂J^2?

SOLUTION: We know that τ_1 is the triangle with edges $(0,0), (1,0), (1,1)$ with the orientation that

$$(0,0) \to (1,0) \to (1,1) \to (0,0).$$

Meanwhile, τ_2 is the triangle with edges $(0,0), (0,1), (1,1)$ such that

$$(0,0) \leftarrow (0,1) \leftarrow (1,1) \leftarrow (0,0).$$

Thus, the diagonal lines 'cancel out' and we are left with the unit square. By definition,

$$\begin{aligned}
\partial J^2 &= \partial \tau_1 + \partial \tau_2 \\
&= ([e_1, e_1 + e_2] - [0, e_1 + e_2] + [0, e_1]) - ([e_2, e_2 + e_1] - [0, e_2 + e_1] + [0, e_2]) \\
&= [e_1, e_1 + e_2] + [0, e_1] - [e_2, e_2 + e_1] - [0, e_2]
\end{aligned}$$

$\blacksquare$

Consider the oriented affine 3−simplex

$$\sigma_1 = [0, e_1, e_1 + e_2, e_1 + e_2 + e_3]$$

has determinant 1 when regarded as a linear transform and is thus positively oriented.

SOLUTION: σ_1 when regarded as a linear transform is simply the identity mapping and therefore has determinant 1. ■

Problem 6

State the conditions under which the formula

$$\int_{\Phi} f\, d\omega = \int_{d\Phi} f\, \omega - \int_{\Phi} (df) \wedge \omega$$

Solution: Let $E \subseteq \mathbb{R}^n$ be open and $F \subseteq \mathbb{R}^m$ be open. Let $f \in C^1(E, \mathbb{R})$ and $\omega \in \Lambda^{k-1}(F)$ be of class C^1. Suppose $\Phi \in C^2(E, F)$ is a $k-$chain. Since $f\omega = f \wedge \omega \in \Lambda^{k-1}(F)$, then we apply Stokes' theorem to find that

$$\int_{\partial\Phi} f\omega = \int_{\Phi} d(f\omega) = \int_{\Phi} (df \wedge \omega + f \wedge d\omega) = \int_{\Phi} df \wedge \omega + \int_{\Phi} f \wedge d\omega,$$

and so the conclusion follows from subtracting. Recall the integration by parts formula. Let $f, g \in C^1([a, b], \mathbb{R})$, then

$$\int_a^b (f'g)(x)\, dx = [(fg)(b) - (fg)(a)] - \int_a^b (fg')(x)\, dx.$$

Letting $\Phi = [a, b]$ be the oriented $1-$simplex, we have that $\partial\Phi = [b] - [a]$. Letting $\omega = g$, we see that

$$d\omega = g' \qquad df = f'.$$

Thus,

$$\int_{\partial\Phi} fg = \int_{[b]-[a]} fg = \int_{[b]} fg - \int_{[a]} fg = (fg)(b) - (fg)(b).$$

Letting $\Delta : [a, b] \to [a, b]$ be the identity, we use the definition of the integral over forms to see that

$$\int_{\Phi} f\, d\omega = \int_{[a,b]} fg' = \int_{\Delta([a,b])} fg' = \int_a^b (fg')(\Delta(x))\Delta'(x)\, dx = \int_a^b (fg')(x)\, dx.$$

Similarly,

$$\int_{\Phi} (df) \wedge \omega = \int_a^b f'g.$$

Thus, we have recovered the integration by parts formula. ■

Problem 7

Define the annulus to be the 2−surface

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta) \qquad a \le r \le b, \theta \in [0, 2\pi].$$

Let $\omega = x^3\,dy$. Show that Stokes theorem holds by computing both sides explicitly.

SOLUTION: Note that by the standard change of variables,

$$\int_\Phi d\omega = \int_{[a,b]\times[0,2\pi]} (d\omega)_\Phi.$$

We have that

$$d\omega = 3x^2\,dx \wedge dy.$$

Pulling back,

$$dt_1 = \cos\theta\,dr - r\sin\theta\,d\theta$$
$$dt_2 = \sin\theta\,dr + r\cos\theta\,d\theta$$

and thus

$$\begin{aligned}
(d\omega)_\Phi &= 3x^2(T(x_1))\,dt_1 \wedge dt_2 \\
&= 3r^2\cos^2\theta\,[(\cos\theta\,dr - r\sin\theta\,d\theta) \wedge (\sin\theta\,dr + r\cos\theta\,d\theta)] \\
&= 3r^2\cos^2\theta[r\cos^2\theta\,dr \wedge d\theta - r\sin^2\theta\,d\theta \wedge dr] \\
&= 3r^3\cos^2\theta\,dr \wedge d\theta
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_\Phi d\omega &= \int_{\Delta([a,b]\times[0,2\pi])} 3r^3\cos^2\theta\,dr \wedge d\theta \\
&= \int_0^{2\pi}\int_a^b 3(\Delta_r(u))^3\cos^2(\Delta_\theta(u)\,\Delta'(u)\,du \\
&= \int_0^{2\pi}\int_a^b 3r^4\cos^2(\theta)\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{3}{4}(b^4 - a^4)\cos^2\theta\,d\theta \\
&= \frac{3\pi}{4}(b^4 - a^4)
\end{aligned}$$

On the other hand, since $\Phi = T \circ J^2$, where $T(r,\theta) = (r\sin\theta, r\cos\theta)$ and $J^2 = \tau_1 + \tau_2$, where

$$\tau_1 = [(a,0),(b,0),(b,2\pi)] \qquad \tau = -[(a,0),(0,2\pi),(b,2\pi)].$$

We've computed that

$$\partial J^2 = [(b,0),(b,2\pi)] + [(a,0),(b,0)] - [(a,2\pi),(b,2\pi)] - [(a,0),(a,2\pi)]$$

and so

$$T \circ \partial J^2 = [b \sin\theta, b\cos\theta] + [0, r] - [0, r] - [a \sin\theta, a\cos\theta].$$

Define $R(\theta) = (b \sin\theta, b\cos\theta)$ and $S(\theta) = (a\sin\theta, a\cos\theta)$.

$$\begin{aligned}
\int_{\partial\Phi} \omega &= \int_{[b\sin\theta, b\cos\theta]} x^3\,dy - \int_{[a\sin\theta, a\cos\theta]} x^3\,dy \\
&= \int_{R[0,2\pi]} x^3\,dy - \int_{S[0,2\pi]} x^3\,dy \\
&= \int_{\Delta[0,2\pi]} (x^3\,dy)_R - \int_{\Delta[0,2\pi]} (x^3\,dy)_S \\
&= \int_{\Delta[0,2\pi]} (-b^4 \sin^4\theta)\,d\theta - \int_{\Delta[0,2\pi]} (-a^4 \sin^4\theta)\,d\theta \\
&= (a^4 - b^4) \int_0^{2\pi} \sin^4\theta\,d\theta \\
&= (a^4 - b^4)\frac{3\pi}{4}
\end{aligned}$$

■