23500 Spring 2025: Problem Set 6 Agustín Esteva
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Problem 1

Let $\{X_n\}$ be a branching process started with a single individual, so that $X_0 = 1$ and X_n is the number of individuals in generation n. Let $\{p_k\}_{k\geq 0}$ be the offspring distribution. Assume $p_0 > 0$ and let $\mu = \sum_{k=0}^{\infty} kp_k$ be the mean of the offspring distribution.

(a) Show that $M_n = \mu^{-n}X_n$ is a martingale with respect to $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$.

SOLUTION: Note that M_n is only well defined when $p_0 < 1$, so we will assume this.

M_n is trivially $\mathcal{F}_n$ measurable.

Since $X_n \geq 0$ for any n, we have that by a result in class about branching processes,

$$\mathbb{E}[|M_n|] = \frac{1}{\mu^n}\mathbb{E}[|X_n|] = \frac{1}{\mu^n}\mathbb{E}[X_n] = \frac{1}{\mu^n}\mu^n\mathbb{E}[X_0] = 1$$

For the Martingale property, we let ξ_i be the number of offspring produced by individual i. We know that ξ_i is independent of X_n and we also infer that $X_n = \sum_{i=1}^{X_n}\xi_i = \sum_{i=1}^{X_n}\xi = X_n\xi$. Thus, since X_{n-1} is $\mathcal{F}_{n-1}$ measurable

$$\begin{aligned}
\mathbb{E}[M_n \mid \mathcal{F}_{n-1}] &= \frac{1}{\mu^n}\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] \\
&= \frac{1}{\mu^n}\mathbb{E}[\sum_{i=1}^{X_n}\xi_i \mid \mathcal{F}_{n-1}] \\
&= \frac{1}{\mu^n}\sum_{i=1}^{X_n}\mathbb{E}[\xi_i \mid \mathcal{F}_{n-1}] \\
&= \frac{1}{\mu^n}\sum_{i=1}^{X_n}\mathbb{E}[\xi_i] \\
&= \frac{1}{\mu^n}X_{n-1}\mathbb{E}[\xi] \\
&= \frac{1}{\mu^n}X_{n-1}\mu \\
&= M_{n-1}
\end{aligned}$$

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(b) Suppose that $\mu = 1$. For each $K \in \mathbb{N}$, use the optional stopping theorem applied to the stopping time

$$T_K = \min\{n \geq 1 : X_n = 0 \text{ or } X_n \geq K\}$$

to show that the probability that the population reaches at least K individuals before going extinct is at most $1/K$.

SOLUTION: Assumption of the OST:

- Since $\mu = 1$ and $p_0 \neq 0$, we have by a result in class that with probability 1, the population will go extinct. Thus, $X_n = 0$ for some large n, and so

$$\mathbb{P}\{T_K < \infty\} = 1.$$

- Since X_n is non-negative for all n and $\mu = 1$,

$$\mathbb{E}[|M_{T_K}|] = \mathbb{E}[M_{T_K}] = \mu^{-n}\mathbb{E}[X_{T_K}] \leq \mathbb{E}\left[\sum_{i=1}^{R} \xi_i\right] = \sum_{i=1}^{R} \mathbb{E}[\xi_i] = R\mu = R$$

- For $T_k \geq n$, we have that $0 < X_n < K$ and so

$$\mathbb{E}[M_n \mathbb{1}_{T_K \geq n}] \leq K\mathbb{P}\{T_K \ \geq n\} \to 0$$

  since the population must go extinct at some point.

By the OST, we have that

$$\mathbb{E}[M_{T_k}] = \mathbb{E}[M_0] = 1.$$

But since $\mu = 1$,

$$1 = \mathbb{E}[M_{T_K}] \geq 0P_L + KP_W \implies P_W \leq \frac{1}{K},$$

where P_W is the probability that we get to K before 0 and $P_L = 1 - P_W$ is the probability we get to 0 before we get to K. ■

(c) Use part (b) to show that the extinction probability is 1 if $\mu = 1$.

SOLUTION: Since $P_L = 1 - P_W$, where

$$P_L = \mathbb{P}\{X_n = 0 \text{ before } X_n = K\} = a = 1 - P_W \geq 1 - \frac{1}{K},$$

then as $K \to \infty$,

$$1 \geq P_L \geq 1 \implies a = 1.$$

Where the implication comes since $a = 1 \iff \exists n : X_n = 0 \iff P_L = 1$. ■

Problem 2

Let $\{M_n\}$ be a martingale with respect to its natural filtration $\{F_n\}$, and let τ be a stopping time for M_n with $\mathbb{E}[\tau] < \infty$. Suppose that there exists a constant $K > 0$ such that $|M_{n+1} - M_n| \leq K$ for all n. Show that $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$.

Solution: It suffices to show we can apply the OST.

- Since $\mathbb{E}[\tau] < \infty$ and

  $$\mathbb{E}[\tau] = \sum_{n=1}^{\infty} \mathbb{P}\{\tau \geq n\} < \infty.$$

  Since the series converges, then we necessarily have that $\sum_m^\infty \mathbb{P}\{\tau \geq n\} < \epsilon$ for large enough m. Since the terms are all positive, we have that $\mathbb{P}\{\tau \geq n\} < \epsilon$ and thus $\mathbb{P}\{\tau \geq n\} \to 0$, and so $\mathbb{P}\{\tau = \infty\} = 0$[a]

- Using the triangle inequality, we have that

  $$\begin{aligned}\mathbb{E}[|M_\tau|] &= \mathbb{E}[|M_\tau - M_{\tau-1} + M_{\tau-1} - M_{\tau-2} + \ldots M_0|] \\ &\leq \mathbb{E}[|M_\tau - M_{\tau-1}| + |M_{\tau-1} - M_{\tau-2}| + \cdots + |M_1 - M_0|] \\ &\leq K\mathbb{E}[\tau] < \infty\end{aligned}$$

- By the first bullet point and similar logic to the second one to bound M_n with nK, we have that

  $$\mathbb{E}[M_n \mathbb{1}_{\tau \geq n}] \leq nK\mathbb{P}\{\tau \geq n\} \to 0$$

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[a] For another proof of this, consider that by the Markov inequality,

$$\mathbb{P}\{\tau \geq n\} \leq \frac{\mathbb{E}[\tau]}{n} < \frac{C}{n} \to 0.$$

Problem 3 (Optional)

Let N be a fixed positive integer, and let A be an arbitrary alphabet of size N, which we view as a collection of characters $\{s_i\}_{i\in\{1,\ldots,N\}}$. A string of length $l \in \mathbb{N}$ is a concatenation of elements of A, written as

$$s_{j_1}s_{j_2}\ldots s_{j_l},$$

where the indices j_k may or may not be distinct. Suppose each character s_i has probability p_i of being selected, and let S be an arbitrary (finite) string. We wish to compute the expected time until the string S is first observed, if we repeatedly sample according to the probabilities p_i. To that end, let $\{X_n\}_{n\in\mathbb{N}}$ denote the characters sampled up to time n, and let

$$T := \min\{n \geq 0 : X_{n-|S|+1}X_{n-|S|+2}\ldots X_n = S\}.$$

Let $L_n(S)$ be the first n (leftmost) characters of S, and let $R_n(S)$ be the last n (rightmost) characters of S. Show that

$$\mathbb{E}[T] = \sum_{i=1}^{|S|}\left(\prod_{j=1}^{i} p_j\right)^{-1} \mathbb{1}_{\{R_i(S)=L_i(S)\}}.$$

Give the analogous formula in the case of uniform sampling, and give a condition for a string S to maximize this expected time.

Problem 4

Suppose $X_1, X_2, \dots$ are independent random variables with distribution

$$\mathbb{P}(X_j = 3) = 1 - \mathbb{P}\left(X_j = \frac{1}{3}\right) = \frac{1}{4}.$$

Let $M_0 = 1$ and for $n \geq 1$,

$$M_n = \prod_{j=1}^{n} X_j.$$

(a) Show that M_n is a martingale with respect to $F_n = \sigma(X_1, \dots, X_n)$.

Solution: Note that since $X_1, \dots, X_n \sim$ i.i.d, then

$$\mathbb{E}[X_j] = 3\frac{1}{4} + \frac{1}{3}\frac{3}{4} = 1, \qquad \forall j = 1, 2, \dots$$

- We see that M_n is $\mathcal{F}_n-$measurable.
- Since the X_is are independent, we can distribute the expectation over the product and see that since everything is non-negative,

$$\begin{aligned}
\mathbb{E}[|M_n|] &= \mathbb{E}[M_n] \\
&= \mathbb{E}\left[\prod_{j=1}^{n} X_j\right] \\
&= \prod_{j=1}^{n} \mathbb{E}[X_j] \\
&= (\mathbb{E}[X])^n \\
&= 1
\end{aligned}$$

  where we use the fact that $\mathbb{E}[X_j] = \mathbb{E}[X] = 1$ for any j.
- For the Martingale property,

$$\begin{aligned}
\mathbb{E}[M_n \mid \mathcal{F}_{n-1}] &= \mathbb{E}\left[\prod_{j=1}^{n-1} X_j \cdot X_n \mid \mathcal{F}_{n-1}\right] \\
&= \prod_{j=1}^{n-1} X_j \cdot \mathbb{E}[X_n \mid \mathcal{F}_{n-1]}] \\
&= M_{n-1}\mathbb{E}[X_n] \\
&= M_{n-1},
\end{aligned}$$

  where we use the fact that X_i are $\mathcal{F}_{n-1}$ measurable for $i \leq n-1$ and that X_n is independent of X_i for $i < n$.

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(b) Use the optional stopping theorem to show that the probability that the value of M_n ever gets as high as 3^6 equals 3^{-6}.

SOLUTION: Define $T_m = \min\{j : M_j = 3^6 \text{ or } M_j = 3^{-m}\}$. We see that T_m is a stopping time for M_n.

- We can reach 3^6 in 6 steps, and so

$$\mathbb{P}\{T_m \geq 6\} \leq \frac{1}{4^6} \implies \mathbb{P}\{T_m \geq 6(2k)\} \leq \frac{1}{4^{6(2k)}} \implies \mathbb{P}\{T_m \geq k\} \leq \frac{1}{4^k},$$

  where we are bounding the probability by the event where M_n keeps bouncing between 1 and $\frac{1}{3}$ $2k$ times until it decides to go 6 times to 3^6 and so as $k \to \infty$, we find that

$$\mathbb{P}\{T_m \geq mk\} \to 0 = \mathbb{P}\{T_m = \infty\}.$$

- We see that by the previous bullet point

$$\mathbb{E}[T_m] = \sum_{k=1}^{\infty} \mathbb{P}\{T_m \geq k\} \leq \sum_{k=1}^{\infty} \frac{1}{4^k} < \infty$$

- We can bound

$$\mathbb{E}[M_n \mathbb{1}_{T_m \geq n}] \leq 3^6 \mathbb{P}\{T_m \geq n\} \to 0$$

Thus, we use the optional the optional stopping theorem that states that

$$\mathbb{E}[M_{T_m}] = \mathbb{E}[M_0] = 1.$$

But

$$1 = \mathbb{E}[M_{T_m}] = 3^6 P_{3^6} + 3^{-m} P_{3^{-m}},$$

where

$$P_{3^6} = \mathbb{P}\{M_n = 3^6 \text{ before } M_n = 3^{-m}\}, \quad P_{3^m} = 1 - P_{3^6}$$

Taking $m \to \infty$, we see that

$$P_{3^6} = \frac{1}{3^6}.$$

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(c) Show that there exists M_∞ such that, with probability one, $M_n \to M_\infty$.

SOLUTION: It suffices to show that M_n satisfies the conditions of the MCT. We showed in the first step that $\mathbb{E}[|M_n|] = 1$, and so we are done since then the $|M_n| \leq 1$ uniformly. ■

(d) Does there exist a $C < \infty$ such that for all n, $\mathbb{E}[M_n^2] \leq C$?

Solution: **No.** Consider first that

$$\mathbb{E}[X_j^2] = 9\frac{1}{4} + \frac{1}{9}\frac{3}{4} = \frac{7}{3}, \qquad \forall j = 1, 2, \ldots$$

$$\begin{aligned}
\mathbb{E}[M_n^2] &= \mathbb{E}\left[\prod_{j=1}^{n} X_j^2\right] \\
&= \prod_{j=1}^{n} \mathbb{E}[X] \\
&= \mathbb{E}[X]^n \\
&= \frac{7}{3}^n \\
&\to \infty
\end{aligned}$$

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Problem 5

Define random variables $\{X_n\}$ recursively by $X_0 = 1$ and for $n \geq 1$, X_n is sampled uniformly from $(0, X_{n-1})$.

(a) Show that $M_n := 2^n X_n$ is a martingale.

SOLUTION:

- It is left as an exercise to the grader that M_n is $\mathcal{F}_n-$measurable
- We claim that
$$\mathbb{E}[X_n] = \frac{1}{2^n}.$$
For $n = 1$, we have that $X_1 \sim U([0, X_0]) = U([0, 1])$, and so $\mathbb{E}[X_1] = \frac{1}{2}$. Suppose this holds for a general $n = k$, that $X_k \sim (0, X_{k-1})$ and $\mathbb{E}[X_k] = \frac{1}{2^k}$. For $n = k+1$, we see that
$$\mathbb{E}[X_{k+1}] = \frac{\mathbb{E}[X_k]}{2} = \frac{1}{2^{k+1}}$$
Thus,
$$\mathbb{E}[|M_n|] = 2^n \mathbb{E}[X_n] = 1.$$
- For the martingale property,
$$\begin{aligned}
\mathbb{E}[M_n \mid \mathcal{F}_{n-1}] &= \mathbb{E}[2^n X_n \mid \mathcal{F}_{n-1}] \\
&= 2^n \mathbb{E}[X_n \mid X_{n-1}] \\
&= 2^n \frac{X_{n-1}}{n} \\
&= 2^{n-1} X_{n-1} \\
&= M_{n-1}
\end{aligned}$$

■

(b) Show that there exists M_∞ such that, with probability one, $M_n \to M_\infty$.

SOLUTION: We showed above that $\mathbb{E}[M_n] = 1$ for all n, and so by the MCT we are done. ■

(c) Find M_∞. (Hint: Consider $\log M_n$.)

SOLUTION: We claim that $X_n \mid \mathcal{F}_{n-1} \sim U_n U_{n-1} \cdots U_1$, where $U_1, \ldots, U_n \sim \text{Uniform}(0, 1)$. To see this, we can induct. For the $n+1$th case, we use the fact that $aU(0, 1) = U(0, a)$

$$X_n \sim U(0, X_{n-1}) = X_{n-1} U(0, 1) = U_{n+1} U_n \cdots U_1.$$

Thus.

$$M_n = 2^n X_n = 2^n \prod_{k=1}^{n} U.$$

Hence,

$$\log M_n = n \log 2 + \sum_{k=1}^{n} \log U,$$

where $\tilde{U} \sim \text{Uniform}(0, 2^n)$ We have that

$$\frac{\log M_n}{n} = \log 2 + \frac{\sum_{k=1}^{n} \log U}{n} \to \log 2 + \mathbb{E}[\log U$$

Where we can compute

$$\mathbb{E}[\log U] = \int_0^1 \log x \, dx = -1$$

and thus

$$\frac{\log M_n}{n} \to -c$$

for some $c \in \mathbb{R}$ and so $\log M_n \to -\infty$, implying that $M_n \to 0 = M_\infty$.

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Problem 6

Suppose X is a standard normal random variable, i.e., $X \sim \mathcal{N}(0,1)$.

(a) Let $\Phi(x) := \mathbb{P}(X \leq x)$. Compute $\mathbb{E}[X\Phi(X)]$.

Solution: Brute forcing our way thru: If we let

$$u = \Phi(x) \implies du = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$dv = x\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \implies v = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\begin{aligned}
\mathbb{E}[X\Phi(X)] &= \int_{-\infty}^{\infty} x f_X(x)\Phi(x)\,dx \\
&= \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\Phi(x)\,dx \\
&= [-\Phi(x)\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2}\,dx
\end{aligned}$$

If we let

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx,$$

then if we let $r = x^2 + y^2$ and $\theta = \tan\frac{y}{x}$ we get the change of variables

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dxdy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\,dr\,d\theta \\
&= \pi\int_{0}^{\infty} e^{-u}\,du \\
&= \pi\Big[-e^{-u} + e^{-u}\Big]_0^{\infty} \\
&= \pi,
\end{aligned}$$

and so $\mathbb{E}[X\Phi(X)] = \frac{1}{2\sqrt{\pi}}$ ∎

(b) Let $Y = |X| + X$. Compute $\mathbb{E}[Y^3]$.

Solution: We use the moment generating function to note that

$$\mathbb{E}[Y^3] = M_Y^{(3)}(0),$$

where

$$M_Y(t) := \mathbb{E}[e^{tY}] = \sum_{n=0}^{\infty} \frac{t^n \mathbb{E}[Y^n]}{n!}$$

and so since Y is continuous.

$$M_Y(t) = \mathbb{E}[e^{tY}] = \int_{-\infty}^{\infty} e^{ty} f_Y(y)\,dy.$$

To compute $f_Y(y)$, we first consider that (if $y \geq 0$)

$$\begin{aligned} F_Y(y) &= \mathbb{P}\{Y \leq y\} \\ &= \mathbb{P}\{|X| + X \leq y\} \\ &= \mathbb{P}\{2X \leq y \mid X \geq 0\}\frac{1}{2} + \mathbb{P}\{0 \leq y\}\frac{1}{2} \\ &= \frac{1}{2}\int_0^{\frac{y}{2}} f_X(x)\,dx + \frac{1}{2} \end{aligned}$$

and so

$$f_Y(y) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{y^2}{8}}$$

Computing,

$$\begin{aligned} \implies M_Y(t) &= \mathbb{E}[e^{tY}] \\ &= \int_{-\infty}^{\infty} e^{ty} \frac{1}{2\sqrt{2\pi}} e^{-\frac{y^2}{8}}\,dy \\ &= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ty - \frac{y^2}{8}}\,dy \\ &= \frac{1}{2\sqrt{2\pi}} e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{8}(y-4t)^2}\,du \\ &= \frac{1}{2\sqrt{2\pi}} e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{u}{\sqrt{8}}\right)^2}\,du \\ &= \frac{1}{\sqrt{\pi}} e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-s^2}\,ds \\ &= e^{\frac{t^2}{2}} \end{aligned}$$

Thus, taking the third derivative,

$$M_Y^{(3)}(0) = 0$$

∎

(c) Show that $\mathrm{Var}(\sin X) > \mathrm{Var}(\cos X)$.

SOLUTION: Since $[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, it suffices to find the first and second moments of $Y = \sin X$ and $Z = \cos X$. We can compute

$$\begin{aligned}\mathbb{E}[Z] &= \mathbb{E}[\cos X] \\ &= \int_{-\infty}^{\infty} \cos x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos x e^{-\frac{x^2}{2}}\, dx\end{aligned}$$

Letting

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(tx) e^{-\frac{x^2}{2}},$$

we find that by some integration by parts,

$$\begin{aligned}G'(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -x \sin(tx) e^{-\frac{x^2}{2}}\, dx \\ &= -te^{-\frac{t^2}{2}}\end{aligned}$$

We are interested in $G(1)$. which Feynman tells us is given by

$$G(1) = \int_{-\infty}^{1} G'(t)$$

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Problem 7

Consider the infinite series

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}, \qquad S(2k) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k}}.$$

(a) Show that $\zeta(2k) = \frac{2^{2k}}{2^{2k}-1} S(2k)$.

SOLUTION: Computing,

$$\begin{aligned}\zeta(2k) &= \sum_{n \text{ odd}} \frac{1}{n^{2k}} + \sum_{n \text{ even}} \frac{1}{n^{2k}} \\ &= \sum_{n=0} \frac{1}{(2n+1)^{2k}} + \sum_{n=0} \frac{1}{(2n)^{2k}} \\ &= S(2k) + \frac{1}{2^{2k}} \zeta(2k)\end{aligned}$$

and so rearranging

$$\zeta(2k) = \frac{1}{1 - \frac{1}{2^{2k}}} S(2k) = \frac{2^{2k}}{2^{2k}-1} S(2k)$$

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(b) Suppose X and Y are continuous, non-negative, independent random variables with densities $f_X(x)$ and $f_Y(y)$. Let $Z = \frac{Y}{X}$. Show that the density of Z is given by

$$f_Z(z) = \int_0^{\infty} x f_Y(zx) f_X(x)\, dx.$$

SOLUTION: If we let $f(x, y)$ be the joint density of X and Y, then

$$\begin{aligned}F_Z(z) &= \mathbb{P}\{Z \leq z\} \\ &= \mathbb{P}\{\frac{Y}{X} \leq z\} \\ &= \int_0^{\infty} \left(\int_0^{xz} f_X(x, y)\, dy \right) dx \\ &= \int_0^{\infty} \left(\int_0^{z} x f(x, ux)\, du \right) dx \\ &= \int_0^{z} \left(\int_0^{\infty} x f(x, ux)\, dx \right) du\end{aligned}$$

Differentiation, and using the fact that $f_{X,Y}(u,v) = f_X(u)f_Y(v)$ by independence,

$$f_Z(z) = \int_0^\infty x f(x, zx)\, dx = \int_0^\infty x f_X(x) f_Y(xz)\, dx$$

∎

(c) Assume that the random variables X and Y obey the Cauchy distribution, i.e.,

$$f_X(x) = \frac{2}{\pi(1+x^2)}, \qquad x \geq 0.$$

Show that

$$f_Z(z) = \frac{4\log(z)}{\pi^2(z^2-1)}.$$

(d) By consider $\mathbb{P}\{Y \leq X\}$, Show that

$$\int_0^1 \frac{\log z}{z^2-1}\, dz = \frac{\pi^2}{8}.$$

SOLUTION: Since Y and X are i.i.d, then

$$\mathbb{P}\{Y < X\} = \frac{1}{2}.$$

But we also have that

$$\mathbb{P}\{Y < X\} = \mathbb{P}\{Z < 1\} = F_Z(1) = \int_0^1 f_Z(z)\, dz = \frac{4}{\pi^2}\int_0^1 \frac{\log z}{z^2-1}.$$

Putting these equations together, we see that

$$\frac{\log z}{z^2-1} = \frac{\pi^2}{8}$$

∎

(e) Use the previous part to deduce that $S(2) = \frac{\pi^2}{8}$.

SOLUTION: Since $z < 1$, we have a geometric series, we have that

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} = \sum_{n \text{ odd}} z^n + \sum_{n \text{ even}} z^n$$

But

$$\sum_{n \text{ odd}} z^n = z + z^3 + z^5 + \cdots = z \sum_{n \text{ even}} z^n$$

so then

$$\frac{1}{1-z} = \sum_{n \text{ even}} z^n + z \sum_{n \text{ even}} z^n = (1+z) \sum_{n \text{ even}} z^n$$

Rearranging:

$$\sum_{n \text{ even}} z^n = \frac{1}{z^2-1} = -\frac{1}{1-z^2}.$$

So then

$$\begin{aligned}
\int_0^1 \log z \frac{1}{z^2-1}\, dz &= -\int_0^1 \log z \sum_{n \text{ even}}^{\infty} z^n \\
&= -\sum_{n \text{ even}} \int_0^1 \log(z) z^n \\
&= -\sum_{n \text{ even}} -\frac{1}{(n+1)^2} \\
&= \sum_{n \text{ even}} \frac{1}{(n+1)^2} \\
&= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \\
&= S(2)
\end{aligned}$$

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(f) Conclude that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution: Using part (a), we see that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \zeta(2) \\
&= \frac{2^2}{2^2-1} S(2) \\
&= \frac{4}{3}\frac{\pi^2}{8} \\
&= \frac{\pi^2}{6}
\end{aligned}$$

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