20510 Spring 2025: Problem Set 4
Accelerated Analysis in $\mathbb{R}^n$ III
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Problem 1

Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative measurable function such that $0 \leq f(x) \leq 1$ for every $x \in \mathbb{R}$. Suppose that $\int_{\mathbb{R}} f\,dm < \infty$. Prove that

$$\lim_{k\to\infty} \int_{\mathbb{R}} (f(x))^k \,dm = m(f^{-1}(1)).$$

Solution: Define

$$A := \{x \in \mathbb{R} \mid f(x) = 1\}.$$

A is measurable since f is measurable. Define the measurable simple function

$$g = f\chi_A = \begin{cases} 1, & f(x) = 1 \\ 0, & f(x) < 1 \end{cases}.$$

Define the sequence $f_k(x) = (f(x))^k$.

Let $x \in \mathbb{R}$. If $x \in A$, then $f_k(x) = 1 = g(x)$ for each k. If $x \notin A$, then $0 < f(x) < 1$, and thus $f_k(x) \to 0 = g(x)$. Thus, $f \to g$ pointwise. Moreover, $|f_k(x)| \leq 1$, for all x, and so by the dominated convergence theorem,

$$\lim_{k\to\infty} \int_{\mathbb{R}} f_n dm = \int_{\mathbb{R}} g dm = m(A) = m(f^{-1}(\{1\})$$

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Problem 2

Let

$$f(x) = \frac{1}{\sqrt{x}} \quad \text{for } x \in (0,1], \quad f(0) = 0.$$

Prove that

$$\int_0^1 f\, dm = 2.$$

(Hint: Use the Monotone Convergence Theorem.)

SOLUTION: Define the sequence of functions

$$f_n(x) = f(x)\chi_{[\frac{1}{n},1]}.$$

We know that f_n is Riemann integrable on $[\frac{1}{n}, 1]$ for each n, and so

$$\int_{[0,1]} f_n\, dm = \int_{\frac{1}{n}}^1 f\, dm = \int_{\frac{1}{n}}^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}\Big|_{\frac{1}{n}}^1 = 2 - \sqrt{\frac{1}{n}} \to 2.$$

But by the monotone convergence theorem, we have that since $f_n \uparrow f$ pointwise, then

$$\int_{[0,1]} f\, dm = \lim_{n\to\infty} \int_{[0,1]} f_n$$

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Problem 3

Let

$$f(x) = \frac{1}{1+x^2}.$$

Prove that

$$\int_{\mathbb{R}} f\, dm = \pi.$$

Solution: Define $f_n := f(x)\chi_{[-n,n]}$. Then $f_n \uparrow f$. By the MCT:

$$\int_{\mathbb{R}} f\, dm = \lim_{n\to\infty} \int_{\mathbb{R}} f_n\, dm = \lim_{n\to\infty} \int_{[-n,n]} f\, dm = \lim_{n\to\infty} \int_{[-n,n]} \frac{1}{1+x^2}\, dm = \lim_{n\to\infty} \int_{[-n,n]} \frac{1}{1+x^2}\, dx.$$

We compute the Riemann integral:

$$\int_{[-n,n]} \frac{1}{1+x^2}\, dx = \arctan(\theta)\Big|_{-n}^{n} = 2\arctan(n) \to 2\frac{\pi}{2} = \pi.$$

$\blacksquare$

Problem 4

Suppose $f \in L^1$ on E. Prove that, for every $\epsilon > 0$, there exists a simple function g such that

$$\int_E |f - g| \, dm < \epsilon.$$

Prove that, if $f \in L^1$ on $E \subseteq \mathbb{R}$, then, for every $\epsilon > 0$, there exists a step function s such that

$$\int_E |f - s| \, dm < \epsilon.$$

Recall that a step function s is a simple function such that, for every $c \in \operatorname{range}(s)$, $s^{-1}(c)$ is an interval.

SOLUTION: Let $\epsilon > 0$. Since $f \in \mathcal{L}^1(E)$, we have that $\int_E |f| < \infty$. Let $A = \{x \in E \mid f(x) \geq 0\}$ and $B = \{x \in E \mid f(x) < 0\}$. Then

$$\infty > \int_E |f(x)| = \int_A |f(x)| + \int_B |f(x)| = \int_E f^+ + \int_E f^-.$$

Thus, $\int_E f^+, \int_E f^- < \infty$. By definition, there exists a simple function φ^+, φ^- such that

$$I_E(\varphi^+) > \int_E f^+ - \frac{\epsilon}{2}, \quad I_E(\varphi^-) > \int_E f^- - \frac{\epsilon}{2}.$$

Thus,

$$\int_E f^+ - \varphi^+ < \frac{\epsilon}{2}, \quad \int_E f^- - \varphi^- < \frac{\epsilon}{2}.$$

Define $\varphi = \varphi^+ - \varphi^-$. Then

$$\begin{aligned} \int_E |f - \varphi| &= \int_A |f - \varphi| + \int_B |f - \varphi| \\ &= \int_A f^+ - \varphi^+ + \int_B f^- - \varphi^- \\ &< \epsilon. \end{aligned}$$

For the second part, it suffices to show that we can approximate $f = \chi_E$ with step functions and then use dominated convergence theorem. ■

Problem 5

Prove that, if $f : \mathbb{R} \to \mathbb{R}$, and $f \in \mathcal{L}$, then

$$\lim_{n\to\infty} \int_0^{2\pi} f(x)\cos(nx)\,dm = 0.$$

(Hint: First prove this when f is simple.)

SOLUTION: By the previous problem, we can approximate f by step functions. Suppose first f is a step function. That is, we can write

$$f = \sum_{k=1}^{N} c_k \chi_{R_k},$$

where each R_k is a disjoint interval. Then for any n, we have that

$$\begin{aligned}
\int_0^{2\pi} f(x)\cos(nx)\,dm &= \int_0^{2\pi} \sum_{k=1}^{N} c_k \chi_{R_k} \cos(nx)\,dm \\
&= \sum_{k=1}^{N} c_k \int_{R_k} \cos(nx)\,dm \\
&= \sum_{k=1}^{N} c_k \left[\frac{1}{n}\sin(nx)\right]_{a_k}^{b_k} \\
&\le 2Nc_{(N)}\frac{1}{n} \\
&\to 0,
\end{aligned}$$

where $c_{(N)} = \max\{c_1, \ldots, c_N\}$.

For a general f, we use the previous problem. There exists a simple function g such that

$$\int_0^{2\pi} |f-g|\,dm < \epsilon.$$

Then

$$\left|\int_0^{2\pi} (f-g)\cos(nx)\,dm\right| \le \int_0^{2\pi} |f-g||\cos(nx)| \le \int_0^{2\pi} |f-g| < \epsilon.$$

And so because this is true for any ϵ, then

$$\begin{aligned}
\epsilon &> \left|\int_0^{2\pi} (f-g)\cos(nx)\,dm\right| \\
&\ge \int_0^{2\pi} (f-g)\cos(nx) \\
&= \int_0^{2\pi} f\cos(nx)\,dm - \int_0^{2\pi} g\cos(nx)\,dm,
\end{aligned}$$

and so

$$\int_0^{2\pi} g\cos(nx)\,dm = \int_0^{2\pi} f\cos(nx)\,dm.$$

Thus, by what we just showed, as $n \to \infty$, we have that

$$\int_0^{2\pi} f\cos(nx)\,dm \to 0.$$

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Problem 6

Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is measurable and nonnegative, and, for every $n, k \in \mathbb{N}$, define

$$E_{n,k} = \{x \mid \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\}.$$

Show that, as $n \to \infty$,

$$\sum_{k=1}^{\infty} \frac{k}{2^n} m(E_{n,k}) \to \int_{\mathbb{R}^d} f \, dm.$$

Solution: Define the simple function $f_n(x) = \sum_{k=1}^{\infty} \frac{k}{2^n} \chi_{E_{n,k}}$. Note that $f_n \uparrow f$. By the MCT, we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \, dm = \int_{\mathbb{R}^d} f \, dm,$$

where

$$\int_{\mathbb{R}^d} f_n \, dm = \sum_{k=1}^{\infty} \frac{k}{2^n} m(E_{n,k} \cap \mathbb{R}^d) = \sum_{k=1}^{\infty} \frac{k}{2^n} m(E_{n,k})$$

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