UChicago Measure and Integration Notes

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Contents

1 Lectures

1.1 Tuesday, Jan 21: Measurable Sets

Definition 1. We say that $E \subset \mathbb{R}^d$ is *measurable* if, given $\epsilon > 0$, there exists an open set O such that $E \subset O$ and

$$m_*(O - E) < \epsilon$$

This is almost equivalent to saying that

$$m_*(O) \leq m_*(E) + \epsilon$$

Proposition 1. The following are properties of measurable sets:

(a) If O is open, then O is measurable (they cover themselves)

(b) If $m_*(E) = 0$, then E is measurable.

(c) A countable union of measurable sets is measurable.

(d) Closed sets are measurable

(e) Complements are measurable

Proposition 2. A countable intersection of measurable sets is measurable.

Proof. Let

$$E = \bigcap E_j,$$

where E_j is measurable for all j. Then we get that by DeMorgan's law:

$$\bigcap E_j = \left(\bigcup E_j\right)^c,$$

so we are done by the previous proposition. □

Theorem 1. Suppose $E_1, E_2, \ldots$ are measurable and mutually disjoint, then

$$m\left(\bigcup E_j\right) = \sum_j m(E_j).$$

Proof. We claim that if E is measurable, and $\epsilon > 0$, then there exists some closed $F \subset E$ such that $m(E - F) < \epsilon$. To see this, consider that E^c is measurable, and thus there exists an open set O such that $m_*(O - E^c) < \epsilon$. We let $F = O^c$, and then $F \subset E$, and F is closed. Then we have that $O - E^c = O \cap E = F \cap E$. We know that $m(O \cap E) < \epsilon$. Note that we have that

$$E - F = E \cap F^c = E \cap O,$$

and thus $m(E - F) < \epsilon$.

Assume E_j are bounded for all j. By the previous claim, there exists a closed $F_j \subset E_j$ such that

$$m(E_j - F_j) < \frac{\epsilon}{2^j}$$

for all j. Note that by boundedness of E_j and by closedness of F_j, the F_j are compact and disjoint. Fix $N \in \mathbb{N}$, then we have that

$$m\left(\bigcup_1^N F_j\right) = \sum_1^N m(F_j)$$

Let $E = \bigcup E_j$, then we have that $\bigcup F_j \subset E_j$. Thus, we have that

$$m(E) \geq m(\bigcup_1^N E_j) = \sum_1^N m(E_j).$$

Thus, we get that

$$m(E) \geq \sum_{j=1}^N m(F_j) \geq \sum_{j=1}^N m(E_j) - \frac{\epsilon}{2^j} \geq \sum_{j=1}^N m(E_j) - \epsilon.$$

Letting $N \to \infty$, we get that

$$m(E) \geq \sum_{j=1}^\infty m(E_j).$$

By subadditivity, we also have that

$$m(E) \leq \sum_{j=1}^\infty m(E_j).$$

For the general case, find cubes $Q_k \subset Q_{k+1}$ such that $\bigcup Q_k = \mathbb{R}^d$. Then let $E_{j,k} = E_j \cap (Q_k - Q_{k-1})$. Then we have that $E = \bigcup_{k,j} E_{k,j}$, where the $E_{k,j}$ are disjoint and bounded. Then we have by the work done above that

$$m(E_j) = \sum_j \sum_k m(E_{k,j}) = \sum_j m(E_j)$$

$\square$

Corollary 1. Suppose $\{E_j\}$ is a countable collection of measurable subsets of $\mathbb{R}^d$. Assume further that $E_j \subset E_{j+1}$, and that $E = \bigcup_j E_j$. Then we have that

$$m(E) = \lim_{n\to\infty} m(E_n).$$

If, on the other hand, the E_j decrease to E, that is, $E = \bigcap_j E_j$, then if $m(E_k) < \infty$, then

$$m(E) = \lim_{n\to\infty} m(E_n)$$

Proof. For the first result, let $G_1 = E_1$, $G_2 = E_2 - E_1$, and so on until

$$G_n = E_n - E_{n-1}.$$

Obviously, the G_k are measurable and disjoint and $E = \bigcup_k G_k$. Then

$$m(E) = \sum_{k=1}^\infty m(G_k) = \lim_{n\to\infty} \sum_{k=1}^n m(G_k) = m\left(\lim_{n\to\infty} \bigcup_{k=1}^N G_k\right) = m\left(\lim_{n\to\infty} E_n\right)$$

For the second result, we assume without loss of generality that $E_1 < \infty$, and we set $G_1 = E_1 - E_2$, $G_2 = E_2 - E_3$. Again, the G_k are mutually disjoint and measurable and

$$E_1 = E \cup \bigcup_{k=1}^\infty G_k.$$

Thus, we get that (letting $N \to \infty$)

$$m(E_1) = m(E) + \sum_{k=1}^{N} m(G_k) = m(E) + \sum_{k=1}^{\infty} m(G_k) - m(G_{k+1}) = m(E) + m(E_1) - m(E_N).$$

$\square$

Theorem 2. Suppose $E \subset \mathbb{R}^d$, E is measurable. Then for all $\epsilon > 0$:

(a) There exists an open O with $E \subset O$ such that $m(O - E) < \epsilon$.

(b) There exists a closed F with $F \subset E$ such that $m(E - F) < \epsilon$.

(c) If $m(E) < \infty$, then there exists a compact set K such that $K \subset F$ and $m(E - K) < \epsilon$.

(d) If $m(E) < \infty$, then there exists $F = \bigcup Q_j$ such that $m(E \triangle F) < \epsilon$. (note that the symmetric difference refers to the points belonging to only one of the sets).

Proof. (i) and (ii) have already been proved. For (iii), pick F closed such that $F \subset E$ and $m(E - F) < \frac{\epsilon}{2}$. Let $K_m = F \cap \overline{B_m(0)}$. Evidently, each K_m is compact and we have that

$$E - K_m \downarrow E - F.$$

We know by the previous corollary that

$$m(E - F) = \lim_{m \to \infty} (E - K_m),$$

and so for m large, $m(E - K_m) < \epsilon$.

For (iv), we pick closed $\{Q_j\}$ such that $E \subset \bigcup Q_j$ and $\sum Q_j \leq m(E) + \frac{\epsilon}{2}$. Thus, the series is convergent, and there exists some large N such that $\sum_{N+1}^{\infty} |Q_j| \leq \frac{\epsilon}{2}$. Now we let $F = \bigcup_{j=1}^{N} Q_j$. We thus get that

$$\begin{aligned} m(E \triangle F) =& m(E - F) + m(F - E) \\ &= \sum_{N+1}^{\infty} m(Q_j) + (\sum m(Q_j) - m(E)) \\ &< \epsilon. \end{aligned}$$

$\square$

1.2 Thursday, Jan 23: Constructing a Non-Measurable Set and Measurable Functions

Lemma 1. Suppose $a \in \mathbb{R}$, then $m^*(E+a) = m^*(E)$.

Proof. Note that $E + a = \{x + a \; ; \; x \in E\}$. Pick $E \subset^\infty \bigcup Q_i$ such that $\sum |Q_i| \leq m^*(E) + \epsilon$, then

$$E \subseteq \bigcup_{i=1}^{\infty} (Q_i + a) \implies m^*(E+a) \geq \sum |Q_i + a| + \epsilon.$$

For the reverse direction, consider that

$$m^*(E+a) \leq m^*((E+a) - a) =^* (E).$$

□

Theorem 3. (Axiom of Choice) If $\mathscr{E} = \{S_\alpha \; : \; \alpha \in \mathscr{A}, S_\alpha \neq \emptyset \; \forall \alpha \in \mathscr{A}\}$, then there exists a function $f : \mathscr{E} \to \bigcup S_\alpha$ such that $f(S_\alpha) \in S_\alpha$.

Define $\sim$ from $[0,1]$ such that $x \sim y$ if $x - y \in \mathbb{Q}$. Thus,

$$[0,1] = \bigcup_{x \in [0,1]} [x].$$

As an example,

$$[0] = \mathbb{Q} \cap [0,1], \qquad [\frac{1}{\sqrt{2}}] = \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} + \frac{1}{4}, \dots\}$$

Using the Axiom of Choice, pick $\alpha \in [x]$ such that if $[x] \neq [y]$, then $\alpha_x \neq \alpha_y$. Define

$$N := \{\alpha_{[x]}\}.$$

Theorem 4. N is not measurable.

Proof. Suppose it is. Define $\{r_k\} = \mathbb{Q} \cap [-1,1]$, and define $N_k = N + r_k$. We claim that $N_k \cap N_j = \emptyset$ when $k \neq j$. To see this, let x be in the intersection, then

$$x = \alpha_1 + r_k = \alpha_2 + r_j \implies \alpha_1 - \alpha_2 = r_j - r_k \in \mathbb{Q} \implies \alpha_1 = \alpha_2 \implies r_j = r_k,$$

contradiction the fact that $k \neq j$. We further claim that

$$[0,1] \subseteq \bigcup N_k \subseteq [-1,2].$$

The second inclusion is by construction. Let $x \in [0,1]$, then there exists some $\alpha \in [x]$, and thus

$$\alpha \in N \implies x - a \in \mathbb{Q} \implies x = \alpha + r_k \implies x \in N_k$$

By the first claim, we have that by translation invariance, that

$$m(\bigcup N_k) = \sum m(N_k) = \sum m(N) = \{0, \infty\}.$$

But then

$$1 = m([0,1]) \leq m(\bigcup N_k) \leq m([-1,2]) = 3,$$

which is a contradiction. □

Definition 2. A collection of sets $S \subset P(X)$ is a $\sigma-$**algebra** if:

(a) $X \in S$;

(b) If $\{E_i\} \in S$, then $\bigcup E_i \in S$.

(c) If $E \in S$, then $E^c \in S$.

Example 1.1. Examples of $\sigma-$algebras:

(a) $S = \{\emptyset, X\}$

(b) Set of measurable sets.

(c) $2^{\mathbb{R}}$

Lemma 2. Suppose $\mathscr{F}_\alpha$, $\alpha \in \mathscr{A}$ is a collection of $\sigma-$algebras on X, then

$$\mathscr{F} = \bigcap_\alpha \mathscr{F}_\alpha$$

is a $\sigma-$algebra.

Proof. Since $X \in \mathscr{F}_\alpha$ for all α, then $X \in \mathscr{F}$. Suppose $\{E_i\} \in \mathscr{F}$, then for all α, $\{E_i\} \in \mathscr{F}_\alpha$, and thus

$$\bigcup E_i \in \mathscr{F}_\alpha$$

for all α, and because this holds for all α, then we are done. The proof for the third property is the same as the previous one. □

Definition 3. If $\mathscr{E} \subseteq P(X)$, then we define

$$\sigma(\mathscr{E}) := \bigcap_{\mathscr{E} \subseteq \mathscr{F}_\alpha} \mathscr{F}_\alpha$$

Definition 4. The **Borel** $\sigma-$**algebra**, $\mathscr{B}(\mathbb{R}^n)$, is defined to be $\sigma(O)$, where $O \subseteq \mathbb{R}^n$ are open.

Proposition 3. $\mathscr{B}$ satisfies the following properties:

(a) All open sets are in the $\mathscr{B}$

(b) All closed sets are in $\mathscr{B}$

(c) If $E \in \mathscr{B}$, then E is measurable.

Definition 5. A set is G_δ if $E = \bigcap O_i$, where O_i is open.

A set is F_σ if $E = \bigcup F_i$, where F_i is closed.

Theorem 5. $A \subseteq \mathbb{R}^n$ is measurable if and only there exists a $G \in G_\delta$ set such that $A \subseteq G$ and $m^*(G - A) = 0$. The symmetric claim holds for F_σ sets as well.

Proof. ($\Longrightarrow$) If A is measurable, then there exists $O_n \supset A$ open such that $m^*(O_n - A) < \frac{1}{n}$. We let $G = \bigcap O_n$.

($\Longleftarrow$) Since null sets are measurable, then since $A = G - (G - A)$, so A is measurable. □

Remark 1. Let $E\mathbb{R}^d$, then

$$\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}.$$

It would make sense for $\chi_E(x)$ to be measurable when E is measurable!

Definition 6. A **simple function** is defined to be

$$f(x) = \sum_{k=1}^{n} a_k \chi_{E_k}(x), \qquad m(E) < \infty.$$

Definition 7. A **step function** is defined to be

$$f = \sum_{k=1}^{n} a_k \chi_{R_k}, \qquad R_k \text{ rect}$$

Definition 8. Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$. We say f is **finite value** if $f(x) \neq \{\pm\infty\}$ for all $x \in \mathbb{R}^d$

Definition 9. We say f is **measurable** if for all $a \in \mathbb{R}$, $f^{-1}([-\infty, a))$ is measurable. In other words,

$$m(\{x \mid f(x) < a\}) < \infty.$$

Theorem 6. f is measurable if and only if for all $a \in \mathbb{R}$, $\{x \mid f(x) \leq a\}$ is measurable.

Proof. We can write

$$\{f \leq a\} = \bigcap \{f < a + \frac{1}{k}\},$$

and since the countable intersection of measurable sets is measurable, we are done. For the other direction, consider that

$$\{f < a\} = \bigcup \{f \leq a - \frac{1}{k}\}.$$

□

1.3 Tuesday, Jan 28: Measurable Functions

Proposition 4. If f is finite valued, then f is measurable if and only if $f^{-1}(O)$ is measurable for all O open if and only if $f^{-1}(F)$ is measurable for all F closed.

This It follows immediately that

Proposition 5. If f is continuous on $\mathbb{R}^d$, then f is measurable. Moreover, if f is measurable, finite valued, and Φ is continuous on $\mathbb{R}$, then $\Phi \circ f$ is measurable.

Proof. We prove the second statement. Since Φ is continuous, we have that $O_a = \Phi^{-1}((-\infty, a))$ is open. Since f is measurable, we have that $f^{-1}(O_a)$ is measurable, and thus $(\Phi \circ f)((-\infty, a)) = f^{-1}(\Phi^{-1}((-\infty, a)))$ □

Proposition 6. Suppose $\{f_n\}$ is a sequence of measurable functions. Then $\sup_n f_n(x)$ and $\inf_n f_n(x)$, $\limsup_n f_n(x)$ and $\liminf_n f_n(x)$ are all measurable.

Proof. Consider that

$$\{\sup_n f_n(x) > a\} = \bigcup_n \{f_n(x) > a\}$$

We also have that $-\inf_n(-f_n(x)) = \sup_n f_n(x)$ and that

$$\limsup_n f_n(x) = \inf_n(\sup_k \geq n f_k(x))$$

□

As a quick consequence, we have that if the f_n are measurable, then $\lim_{n\to\infty} f_n(x) = f(x)$ is measurable.

Proposition 7. (a) Suppose that f is measurable, then f^k is measurable.

(b) If f and g are measurable, then $f + g$ and fg are measurable (given that f and g are finite valued).

Proof. The first is a simple consequence of Proposition 5. We claim that

$$\{f + g > a\} = \bigcup_{r\in\mathbb{R}} (f > a - r) \cap \{g > r\}.,$$

which proves the second. Consider now that

$$fg = \frac{1}{4}\left[(f + g)^2 - (f - g)^2\right]$$

□

Definition 10. We say that f and g are equal **almost everywhere**, or equal **a.e.** if $m\{f(x) \neq g(x)\} = 0$.

Proposition 8. If f is measurable and $f = g$ a.e, then g is measurable.

Remark 2. If f_n are measurable and $\lim_{n\to\infty} f_n(x) = g(x)$ a.e, then g is measurable.

Theorem 7. Suppose $f \geq 0$ and measurable on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions (φ_k) such that

$$\varphi_k(x) \leq \varphi_{k+1}(x), \qquad \lim_{k\to\infty} \varphi_k(x) = f(x). \qquad \forall x$$

Proof. For each k, subdivide the values of f which fall in $[0, k]$. Partition $[0, k]$ into subintervals $[\frac{j-1}{2^k}, \frac{j}{2^k}]$. Define

$$\tilde{\varphi_k}(x) = \begin{cases} \frac{j-1}{2^k}, & f(x) \in [\frac{j-1}{2^k}, \frac{j}{2^k}] \\ k, & f(x) \geq k \end{cases},$$

Obviously, $\tilde{\phi_k}$ is measurable and $\tilde{\varphi_k} \leq \tilde{_{k+1}}$.

When $f(x) = \infty$, we have that $\tilde{\varphi_k}(x) = k$ for any k, so then $\varphi_k \uparrow f$. On the other hand, suppose $f(x) < \infty$, then there exists some k_0 such that for all $k > k_0$, $f(x) < k$, and

$$0 \leq f(x) - \tilde{\varphi_k} \leq \frac{1}{2^k} \implies \tilde{\varphi_k} \to f.$$

Define

$$\varphi_k(x) = \tilde{\varphi_k}(x) \cdot \chi_{B_k(0)}.$$

Because $\chi_{B_k(0)} \leq \chi_{B_{k+1}(0)}$, we have that $\varphi_k(x) \leq \varphi_{k+1}(x)$.

Let x be fixed, and choose k such that $|x| < k$, then $\varphi_k(x) = \tilde{\varphi_k}(x)$. $\square$

1.4 Thursday, Jan 30: Constructing the Lebesgue Integral

Remark 3. To handle negative functions, then consider the general case when f is measurable. Then let

$$f_+ = \max(f(x), 0), \qquad f_- = \max(-f(x), 0).$$

Both are nonnegative, and we have that

$$f(x) = f_+(x) - f_-(x).$$

Thus, when $f \leq 0$, we have that

$$f_+(x) = 0, \qquad f_-(x) = -f(x).$$

Theorem 8. Suppose f is measurable. Then there exists a sequence $(\varphi_k) \to f$ such that φ_k are simple functions with $|\varphi_k| \leq |\varphi_{k+1}|$. In particular, we have that

$$|\varphi_k(x)| \leq |f(x)|, \qquad \forall k, x$$

Proof. From Theorem 7, we can pick a sequence $(\varphi_k^{(i)})$, where $i = \{1, 2\}$ such that

$$0 \leq \varphi_{k+1}^{(1)}(x) \leq \varphi_{k+1}^{(1)}, \varphi_k^{(1)} \uparrow f_+(x),$$

and similarly for $\varphi_k^{(2)} \uparrow f(x)$.

$$\varphi_k := \varphi_k^1 - \varphi_k^2 \to f(x).$$

It remains to be seen that

$$|\varphi_k(x)| = \varphi_k^1 + \varphi_k^2.$$

If $f_+ = 0$, then $\varphi_k^{(1)} = 0$, and so

$$\varphi_k(x) = -\varphi_k^{(2)},$$

and so our claim is proven.

If $f_- = 0$, then $\varphi_k^{(2)} = 0$, and so

$$\varphi_k(x) = \varphi_k^{(1)},$$

and so our claim is proven.

When $f_+(x) > 0$, then $f_-(x) = 0$, and so $\varphi - k^{(2)} = 0$, and $|f(x)| = f(x)$ and thus our claim is proven. Similarly for when $f_-(x) < 0$. □

Theorem 9. Suppose f is measurable. Then there exists a sequence of step functions $(\psi_k(x))$ such that

$$\psi_k(x) \to f(x) \qquad a.e.$$

Proof. Suppose that

$$f(x) = \chi_E(x),$$

where E is a measurable set of finite measure, then aply the previous theorem. Let $\epsilon > 0$. There exist cubes Q_j such that $E \subset \bigcup_{j=1}^N Q_j$ and

$$m(\Delta \bigcup_{j=1}^{N} Q_j) < \epsilon.$$

Thus, there exist almost disjoint rectangles $\tilde{R_1}, \ldots, \tilde{R_n}$ such that

$$\bigcup_{j=1}^{N} Q_j = \bigcup_{j=1}^{M} \tilde{R_j},$$

then find $R_j \subset \tilde{R_j}$. We have that R_j are disjoint and

$$m(E \Delta \bigcup_{j=1}^{M} R_j) \leq 2\epsilon.$$

Thus, except for a set of measure $\leq 2\epsilon$, we have that

$$f(x) = \sum_{j=1}^{M} \chi_{R_j}(x)$$

For all $k \geq 1$, there exists a step funcntion $\psi_k(x)$ such that if

$$E_k = \{x \,:\, f(x) \neq \psi_k(x)\},$$

then $m(E_k) \leq \frac{1}{2^k}$. Thus,

$$F_K = \bigcup_{j=K+1}^{\infty} E_j, \qquad F = \bigcap_{k=1}^{\infty} F_K.$$

By the Borel-Cantelli lemma, we have that $m(F) = 0$. To see that $\psi_k \to f$ for $x \in F^c$ consider that

$$F^c = \bigcup_{j=1}^{\infty} \bigcap_{k \geq j} F_k^c.$$

Thus, if $x \in F^c$, then there exists a j such that than $x \in \bigcap_{k \geq j} F_k^c$, and then $\psi_k(x) = f(x)$. $\square$

1.5 Tuesday, Feb 4: Littlewood's Three Principles

(a) Every measurable set is a finite union of intervals. (For intuition, look at Theorem 2.d)

(b) Every measurable function is continuous almost everywhere. (Lusin)

(c) Every pointwise convergence sequence of measurable functions is uniformly convergence. (Egorov)

We begin with (c).

Theorem 10. (Egorov) Let $\{f_k(x)\}$ be a sequence on measurable functions on a measurable set E such that $m(E) < \infty$. Assume that $f_n(x) \to f$ almost everywhere. Then for all $\epsilon > 0$, there exists $A_\epsilon \subset E$ such that A_ϵ is closed and $m(E \setminus A_\epsilon) < \epsilon$. Moreover, $f_n(x) \to f(x)$ uniformly for $x \in A_\epsilon$.

Proof. Without loss of generality, we suppose

$$f_n(x) \to f(x) \qquad \forall\, x \in E.$$

For each $k, n \in \mathbb{N}$, define

$$E_k^n = \{x \in E \; ; \; |f_j(x) - f(x)| < \frac{1}{n}, \quad \forall j \geq k\}.$$

Obviously, we have that

$$E_k^n \subset E_{k+1}^n, \qquad E_k^n \uparrow E \implies E = \bigcup_{k=1}^{\infty} E_k^n = \lim_{n \to \infty} E_k^n,$$

and so

$$\lim_{t \to \infty} m(E_k^n) = m(E).$$

There exists some k_m such that $m(E) - m(E^{k}{}_{k_m}^{n}) = m(E \setminus E_{k_m}^n) < \frac{1}{2^n}$. Thus,

$$|f_j(x) - f(x)| < \frac{1}{2^n} \qquad j \geq k_m, \; x \in E_{n_m}^n.$$

Let $\epsilon > 0$. Choose N such that

$$\sum_N^\infty \frac{1}{2^n} < \frac{\epsilon}{2},$$

and so if we define

$$\tilde{A}_\epsilon = \bigcap_{m \geq N} E_{k_m}^n.$$

Thus,

$$m(E \setminus \tilde{A}_\epsilon) = m(E \cap \bigcup_{m \geq N} (E_{k_m}^n)^c) = m(\bigcup E \cap (E_{k_m}^n)^c) = m(\bigcup E \setminus E_{k_m}^n) \leq \sum_N^\infty \frac{1}{2^n} < \frac{\epsilon}{2}$$

Let $\delta > 0$ and let $\frac{1}{n} < \delta$, then for any $x \in \tilde{A}_\epsilon$, we have that $x \in E_{k_m}^n$, and thus if $j \geq n$, we have that

$$|f_j(x) - f(x)| < \frac{1}{n} < \delta,$$

and so f is uniformly continuous on $\tilde{A}_\epsilon$.

By Theorem 2, there exists some closed $A_\epsilon \subset \tilde{A}_\epsilon$ such that $m(\tilde{A}_\epsilon \setminus A_\epsilon) < \frac{\epsilon}{2}$. $\square$

Theorem 11. (Luzin) Suppose f is measurable and finite valued on E, where E is of finite measure. For all $\epsilon > 0$, there exists a closed F_ϵ such that $m(F \setminus F_\epsilon) < \epsilon$ and $f|_{F_\epsilon}$ is continuous.

Proof. Let (f_n) be a sequence of step functions such that $f_n \to f$ almost everywhere. Find $E_n \subset E$ such that $m(E_n) < \frac{1}{2^n}$ and f_n are continuous on E_n^c (since we would be taking only the value 1 or 0 on this set). By Egorov, there exists a closed set $A_{\frac{\epsilon}{3}}$ such that $f_n \to f$ uniformly on $A_{\frac{\epsilon}{3}}$ and $m(A_{\frac{\epsilon}{3}}) < \frac{\epsilon}{3}$. Let

$$F := A_{\frac{\epsilon}{3}} \setminus \bigcup_{n \geq N} E_n = A_{\frac{\epsilon}{3}} \cap \bigcup_{n \geq N} E_n^c.$$

Thus, we have that

$$m(E \setminus F') = m(E \cap (\bigcup_{n \geq N} A_{\frac{\epsilon}{3}} \cap E_n^c)^c) = m(E \cap \bigcap_{n \geq N} A_{\frac{\epsilon}{3}}^c \cup E_n) \leq m(E \setminus A_{\frac{\epsilon}{3}}) + m(E \setminus E \bigcap_{m \geq n} E_n) < \frac{2\epsilon}{3}.$$

We know that $f_n \to f$ uniformly on F', and since each f_n is continuous on F', then f is continuous on F'. There exists a closed $F_\epsilon \subset F'$ closed such that $m(F' \setminus F_\epsilon) < \frac{\epsilon}{3}$ $\square$

1.6 Tuesday, Feb 11: The Lebesgue Integral

We will build the integral in four steps

(a) Simple functions

(b) Bounded functions

(c) Supported on sets with final measure

(d) Non-negative functions

Then we will generalize.

Remark 4. Let $\varphi(x)$ be simple, then

$$\varphi(x) := \sum_{i=k}^{n} a_k \chi_{E_k}(x).$$

We run into a problem!

$$0 = \chi_E(x) - \chi_E(x),$$

so if we define the integral as simply the $\sum a_k \times m(E_k)$, we run into a uniqueness problem.

Definition 11. The **canonical form** of φ, a simple function, is such that φ takes finitely many different values on disjoint sets.

Remark 5. (Existence) Suppose φ takes the values $c_1, \ldots, c_m$ which are all distinct (we throw out repeats). Let

$$F_k := \{x \ ; \ \varphi(x) = c_k\}.$$

Note first that F_k is measurable. Then note that for $k \neq k'$, then $F_k \cap F_{k'} = \emptyset$. Evidently

$$\varphi(x) = \sum_{i=1}^{M} c_k \chi_{F_k}(x).$$

Becase $m(E_j) < \infty$ for each j, and $F_k \subset \bigcup_{j=1}^{N} E_j$, and thus $m(F_k) < \infty$.

Definition 12. Let $\varphi(x) = \sum_{k=1}^{M} c_k \chi_{F_k}$ be the canonical representation of a simple function φ, then the **Lebesgue integral** of φ is

$$\int \varphi = \sum_{k=1}^{M} c_k m(F_k)$$

Remark 6. If E is measurable, then $\chi_E(\varphi)$ is simple. Then by definition,

$$\int_E \varphi = \int \varphi \cdot \chi_E$$

Proposition 9. The following hold

(a) If $\varphi = \sum_{i=1}^{N} a_k \chi_{E_k}$ is any representation of φ as a simple function, then

$$\int \varphi = \sum_{k=1}^{N} a_k m(E_k).$$

(b) If φ and ψ are simple, and $a, b \in \mathbb{R}$, then

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi$$

(c) Suppose E, F are disjoint measurable sets with finite measure, then

$$\int_{E \cup F} \varphi = \int_E \varphi + \int_F \varphi$$

(d) Suppose $\varphi \leq \psi$ where both are simple. Then

$$\int \varphi \leq \int \psi$$

(e) Let φ be a simple function, then $|\varphi|$ is a simple function and

$$\left| \int \varphi \right| \leq \int |\varphi|$$

Proof. (a) Consider first the case when

$$\varphi(x) = \sum_{k=1}^{N} a_k \chi_{E_k},$$

such that E_k are mutually disjoint but the a_k are not necessarily distinct. For all $a \neq 0$ such that $a \in \{a_k\}$, we define $E'_a = \bigcup E_{k_a}$, such that $a_k = a$. Evidently, E'_a are disjoint and measurable. Moreover, $m(E'_a) = \sum m(E_{k_a})$. Thus, $\phi = \sum a \chi_{E'_a}$, which is the canonical representationof φ. Thus, we find that

$$\int \varphi = \sum a m(E'_a) = \sum a \sum m(E_{k_a}) = \sum a_k m(E_k)$$

Now for the general case, suppose $\varphi = \sum a_k \chi_{E_k}$. Refine

$$\bigcup_{k=1}^{N} E_k = \bigcup_{j=1}^{m} E_j^*,$$

where the E_j^* are mutually disjoint, and for each k,

$$E_k = \bigcup E_k^* \cdot E_j^* \subset E_k.$$

For each j, we let $a_j^* = \sum a_k$, where $E_k \supset E_j^*$. Thus, we have decomposed it to the first case.

(b) Obvious

(c) Obvious with $\chi_{E \cup F} = \chi_E + \chi_F$

(d) Obvious with $\varphi \geq 0$ case.

(d) Obvious from triangle inequality of summations of the canonical representation ◻

1.7 Thursday, Feb 18: The Lebesgue Integral, Bounded Convergence Theorem, Fatou

I missed the class where the integral was constructed with (b) bounded functions and (d) None-negative functions. Srry.

Theorem 12. (Bounded Convergence Theorem) Suppose (f_n) is a sequence of measurable functions, all uniformly bounded by some $M > 0$, all suppose on some finitely measurable E, with $f_n(x) \to f(x)$ almost everywhere. Then

$$\lim_{n\to\infty} \int f_n = \int f$$

The proof pretty much uses Ergorov's Theorem (Look at Tuesday, Feb 4: Littlewood's Three Principles)

Theorem 13. Suppose f is Riemann integrable on $[a, b]$, then f is measurable and

$$\int_{[a,b]}^{\mathscr{R}} f = \int_{[a,b]}^{\mathscr{L}} f$$

Proof. By definition of the Riemann integral, f is bounded by some $M > 0$. There exist $(\varphi_k), (\psi_k)$ step functions uniformly bounded by M, and the φ_k are increasing and the ψ_k are decreasing. Here, φ_k and ψ_k are the infemum and supremum over the sub intervals of partitions.

$$\lim_{k\to\infty} \int_{[a,b]}^{\mathscr{R}} \varphi_k = \lim_{k\to\infty} \int_{[a,b]}^{\mathscr{R}\psi=\int_{[,]}^{\mathscr{R}}.}$$

For step functions, we obviously have that since the Riemann integral of the step function is the Lebesgue integral of the step function, then

$$\lim_{k\to\infty} \int_{[a,b]}^{\mathscr{R}} \varphi_k = \lim_{k\to\infty} \int_{[a,b]}^{\mathscr{L}} \varphi_k, \qquad \lim_{k\to\infty} \int_{[a,b]}^{\mathscr{R}} \psi_k = \lim_{k\to\infty} \int_{[a,b]}^{\mathscr{L}} \psi_k. \tag{1}$$

Since both sequences are monotonic and bounded, we define

$$\tilde{\varphi} := \lim_{k\to\infty} \varphi_k, \qquad \tilde{\psi} := \lim_{k\to\infty} \psi_k,$$

and clearly,

$$\tilde{\varphi} \leq f \leq \tilde{\psi}.$$

Limits of measurable functions are measurable, and the bounded convergence theorem yields that

$$\lim_{k\to\infty} \int_{[a,b]}^{\mathscr{L}} \varphi_k = \int_{[a,b]}^{\mathscr{L}} \widetilde{\varphi}$$

$$\lim_{k\to\infty} \int_{[a,b]}^{\mathscr{L}} \psi_k = \int_{[a,b]}^{\mathscr{L}} \widetilde{\psi}$$

Using, (1), we see that

$$\int_{[a,b]}^{\mathscr{L}} \widetilde{\varphi} = \int_{[a,b]}^{\mathscr{L}} \widetilde{\psi} = \int_{[a,b]}^{\mathscr{L}} f,$$

and so

$$\int_{[a,b]}^{\mathscr{L}} (\widetilde{\psi} - \widetilde{\varphi}) = 0.$$

Since $\widetilde{\psi} \geq \widetilde{\varphi}$, then by a classic result, we have that $\widetilde{\psi} = \widetilde{\varphi}$ a.e., and so they also equal f a.e. Moreover, $\psi_k \to f$ a.e., and so by the BCT,

$$\lim_{k\to\infty} \int_{[a,b]}^{\mathscr{L}} \psi_k = \int_{[a,b]}^{\mathscr{L}} f,$$

and we are done by (1) and the results above. □

If $f \geq 0$, then recall we construct its Lebesgue integral by

$$\int f = \sup \int g,$$

where $0 \leq g \leq f$, g are bounded by M, and $\sup g$ has finite measure.

Proposition 10. If f is integrable, then $f(x) < \infty$ a.e.

Proof. Let $E_k = \{x \ : \ f(x) > k\}$, then $k\chi_{E_k} \leq k$, and so

$$km(E_k) \leq \int f < \infty,$$

and so $m(E_k) \to 0$ by moving the k to the other side. Thus, since $E_k \downarrow E_\infty$, then

$$m(E_\infty) = 0.$$

□

Proposition 11. If $\int f = 0$, and $f \geq 0$, then $f = 0$ a.e.

Proof. If g is bounded w support of finite measure, then

$$0 \leq \int g \leq f,$$

but $g = 0$, and so $\sup g = f = 0$. Define

$$\widetilde{E}_k = \{x \ : \ f(x) > \frac{1}{k}\},$$

and thus

$$\frac{1}{k} m(\tilde{E}_k) \leq \int f = 0,$$

and so $m(\widetilde{E}_k) = 0$, and so

$$\{x \ : \ f(x) > 0\} = \bigcup_k \widetilde{E}_k \implies m(x \ : \ f(x) > 0) = 0.$$

□

Lemma 3. (Fatou's Lemma) Suppose (f_n) is non-negative and measurable, and $f(x) = \lim_{n\to\infty} f_n(x)$ a.e. Then

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

Proof. Let $0 \le g \le f$, with g bounded and has finitely measured support E. Let $g_n = \min(g, f_n)$. Evidently, g_n is bounded and support of g_n is contained in E. Notice that $g_n \to g$ a.e. Using the bounded convergence theorem, we have that

$$\lim_{n\to\infty} \int g_n = \int g$$

$$\int g_n \le \int f_n \implies \int g \le \liminf_{n\to\infty} \int f_n,$$

and so taking the supremum over all such g yields the result. □

Corollary 2. (Monotone Convergence Theorem) Suppose $f_n \ge 0$, f_n is measurable with $0 \le f_n \le f$ such that $f_n \to f$ a.e. Then

$$\lim_{n\to\infty} \int f_n = \int f$$

Proof. We have that

$$f_n \le f \implies \int f_n \le \int f \implies \limsup_{n\to\infty} \int f_n \le \int f.$$

By Fatou's Lemma, we have that

$$\int f \le \liminf_{n\to\infty} \int f_n.$$

Thus, the limit exists and it equals the integral of the limit. □

Corollary 3. Consider $\sum_{k\ge 0} a_k(x)$, with $a_k \ge 0$ and measurable. Then

$$\int \sum a_k(x) = \sum \int a_k$$

and if

$$\sum^{\infty} \int a_k < \infty,$$

then $\sum a_k(x)$ converges a.e.

Proof. Let $f_n(x) = \sum_{k=1}^n a_k(x)$ and $f(x) = \sum^{\infty} a_k(x)$. Then $f_n(x) \ge 0$, and $f_n \uparrow f$. By the monotone convergence theorem, we have that

$$\int \sum_{k=1}^{\infty} a_k(x) = \lim_{n\to\infty} \int f_n(x) = \lim_{n\to\infty} \int \sum_{k=1}^{n} a_k(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \int a_k(x) = \sum_{k=1}^{\infty} \int a_k(x).$$

For the second claim, we have that if $\sum \int a_k < \infty$, then $\int \sum a_k < \infty$, and so $\sum a_k < \infty$ by Proposition 10. □

Lemma 4. (Borel-Cantelli) Suppose (E_k). Define

$$E := \{x \ : \ x \in \bigcup E_k \text{ i.o.}\}.$$

That is,

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k\ge m}^{\infty} E_k.$$

If $m(E_k) < \infty$, then $m(E) = 0$.

Proof. Let $a_k(x) = \chi_{E_k}$, since $x \in E$, then

$$\sum a_k(x) = \infty \iff \sum m(E_k) = \sum \int a_k < \infty,$$

and so $\sum a_k < \infty$ a.e, and so we are done. $\square$

We will begin our final stage of constructing the integral. If f is real valued, then f is Lebesgue integrable if $|f|$ is.

Definition 13. Suppose f is integrable. Then

$$f^+ = \max(0, f), \qquad f^- = \max(0, -f).$$

Thus,

$$|f| = f^+ - f^-.$$

Moreover, we say that f is **Lebesgue Integrable** if f^+ and f^- are integrable. If f is integrable, then

$$\int f = \int f^+ - \int f^-$$

Proposition 12. Suppose $f = f_1 - f_2$, where $f_i \geq 0$ is measurable and integrable. Then

$$\int f = \int f_1 - \int f_2.$$

Proof. We have that $f = f^+ - f^-$, and so

$$f^+ - f^- = f_1 - f_2.$$

Thus,

$$f^+ + f_2 = f_1 + f^- \implies \int f^+ + \int f_2 = \int f_1 + \int f^- \implies \int f = \int f^+ - \int f^- = \int f^1 - \int f^2$$

$\square$

Proposition 13. If f is integrable, then $|f(x)| < \infty$ a.e.

Proposition 14. The integral of Lebesgue measurable functions is linear, additive, monotonic, and satisfies the triangle inequality.

Proposition 15. Let f be integrable on $\mathbb{R}^d$, $\epsilon > 0$. Then

- there exists a set B of finite measure such that

$$\int_{B^c} |f| < \epsilon$$

- (absolute continuity) There exists a $\delta > 0$ such that if $m(E) \leq \delta$, then $\int_E |f| \leq \epsilon$.

Proof. Without loss of generality, let $f \geq 0$. Let

$$B_N := \{\|x\| < N\}, \quad f_N(x) = f(x)\chi_{B_N}(x).$$

Evidently, $f_N \geq 0$ and $f_N \uparrow f$. Thus, we use the monotone convergence theorem:

$$\int f = \lim_{N\to\infty} \int f_N = \lim_{N\to\infty} \int f(x)\chi_{B_N} = \lim_{N\to\infty} \int_{B_N} f(x).$$

Thus, for large N :

$$|\int f - \int_{B_N} f| = |\int_{B_N^c} f| < \epsilon$$

For (ii), let

$$E_N := \{x \in :\ f(x) < N\}, \qquad f_N = \chi_{E_N} f.$$

Then $f_N \geq 0$, $f_N \uparrow f$. Let $\epsilon > 0$. Again, by the monotone convergence therem, for large $n \geq N$,

$$\int f - f_n < \frac{\epsilon}{2}.$$

Let $\delta > 0$ such that $N\delta \leq \frac{\epsilon}{2}$. Thus, if $m(E) < \delta$, then

$$\int_E f = \int_E f - f_N + \int_E f_N \leq \frac{\epsilon}{2} + \int_{E_N \cap E} f < \frac{\epsilon}{2} + m(E_N)N < \epsilon$$

□

Theorem 14. (Dominated Convergence Theorem) Suppose (f_N) is measurable, $f_n(x) \to f(x)$ a.e. If $|f_x(x)| \leq g(x)$, where g is integrable, then

$$\int |f_n - f| \to 0 \implies \int f_n \to \int f$$

Proof. Let $E_N = \{x\ :\ |x| \leq N, |g(x)| \leq N\}$. Let $\epsilon > 0$. By the previous proposition, letting $g_n = g\chi_{E_n}$ there exists some N such that $\int_{E_N^c} g < \epsilon$.

By the bounded convergence theorem, since $f_n\chi_{E_n} \leq g$ for all f, $E_N \subset B_N$ (finite measure), then

$$\int_{E_n} f_n \to f.$$

Thus, we find that

$$\begin{aligned}
\int |f_n - f| &= \int_{E_N} |f_n - f| + \int_{E_N^c} |f_n - f| \\
&< \epsilon + 2\int_{E_N^c} g \\
&< \epsilon
\end{aligned}$$

□

1.8 Tuesday, Feb 25: The Completeness of L^1

Definition 14. A **complex valued function** is a function such that $f(x) = u(x)+iv(x)$, where $u(x), v(x) \in \mathbb{R}$ for all x.

Remark 7. We note that

$$|f(x)| = (u^2(x) + v^2(x))^{\frac{1}{2}},$$

and f is measurable if and only if u, v are measurable. We say that f is integrable if and only if $|f(x)|$ is integrable. Moreover, f is integrable on E measurable if $\chi_E f$ is integrable.

Proposition 16. If $a \in \mathbb{C}$ and f is integrable, then af is integrable.

Proof. We can express a as $a = \alpha + i\beta$ and so

$$af = (\alpha + i\beta)(u + iv) = (au - \beta v) + i(\alpha v + \beta u).$$

It remains to check that $|af| = |a||f|$. □

Definition 15. We say that

$$L^1 := \{f \ : \ |f| \text{ is integrable}\}$$

and L^1 is equipped with

$$\|f\|_{L^1} = \int |f|$$

Proposition 17. Suppose $f, g \in L^1(\mathbb{R}^d)$, then

(a) $\|af\|_{L^1} = |a|\|f\|_{L^1}$

(b) $\|f + g\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1}$

(c) $\|f\|_{L^1} = 0$ if and only if $f \equiv 0$

(d) $d(f, g) = \|f - g\|_{L^1}$

Thus, L^1 is a normed space.

Theorem 15. (Riesz-Fischer) L^1 is complete.

Proof. Suppose $(f_n) \in L^1$ is Cauchy, then for large $n, m > N_k$ we have that

$$\|f_n - f_m\|_{L^1} = \int |f_n(x) - f_m(x)| dx < \frac{1}{2^k}.$$

Thus, if there exists a subsequence where $n_k = \sum N_K$ such that

$$\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k}.$$

Thus, let

$$f(x) := f_{n_1}(x) + \sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x).$$

To show this f is well defined, first we have that

$$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_k+1}(x) - f_{n_k}(x)|.$$

First, $g \geq 0$ and

$$\int |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_k+1}(x) - f_{n_k}(x)| dx = \int f_{n_1}(x) + \sum_{k=1}^{\infty} \int |f_{n_k+1}(x) - f_{n_k}(x)| \leq \int |f_{n_1}| + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty,$$

and so $g \in L^1$. By the dominated convergence theorem, we have that $f_j \in L^1$, where

$$|f_j(x)| := \left| f_{n_1}(x) + \sum_{k=1}^{j} f_{n_{k+1}}(x) - f_{n_k}(x) \right|,$$

and since $f_j \to f$, and $|f_j| \in L^1$, then by the dominated convergence theorem,

$$\int |f_j| \leq \int g \implies \int |f| \leq \int g.$$

Thus, $f \in L^1$, and so $|f| < \infty$ almost everywhere. Consider that by telescoping the sum, we have that

$$f_j(x) = f_{n_{j+1}},$$

and so $f_{n_{k+1}}(x) \to f$ a.e. Thus, since our subsequence converges to f and (f_n) is Cauchy, then the series converges to f pointwise almost everywhere. By Egorov's theorem, we are done. $\square$

Definition 16. A family $\mathscr{F}$ is dense in L^1 if for any $\epsilon > 0$ and for any $f \in L^1$, there exists some $g \in \mathscr{F}$ such that $\|f - g\| < \epsilon$.

Theorem 16. The following families $\mathscr{F}$ are dense in L^1 :

(a) Simple functions

(b) Step functions

(c) Continuous functions with compact support

1.9 Thursday Feb 27: Invariance Properties

For a function on $\mathbb{R}^d$, $h \in \mathbb{R}^d$, we denote $f_h(x) = f(x-h)$.

Proposition 18. Suppose f is integrable. Then f_h is integrable and

$$\int f_h = \int f$$

Proof. Suppose in the degenerate case that $f = \chi_E$, where E is measurable. Then of course, $f_h = \chi_{E_h}$ where E_h is E translated by h, and so f is integrable since $m(E+h) = m(E)$. Thus we know this is true for simple functions, and we can approximate all L^1 functions by simple functions and so we conclude. ☐

The following two proofs are similar

Proposition 19. Suppose f is integrable, then if $\delta > 0$, then $f_\delta(x) = f(\delta x)$ is integrable. Moreover,

$$\int f_\delta = \delta^d \int f$$

Proposition 20. Suppose $f(x)$ is integrable, then $f(-x)$ is integrable and

$$\int f(x) = \int f(-x)$$

Proposition 21. Suppose $f \in L^1$, then $\|f_h - f\|_{L^1} \xrightarrow{h\to 0} 0$.

Proof. If g is continuous w compact support, then g is uniformly continuous. Thus,

$$\int |g_h(x) - g(x)| dx \xrightarrow{|h|\to 0} 0$$

Thus, if $f \in L^1$, then

$$\|f_h - f\| \leq \|f_h - g_h\| + \|g_h - g\| + \|g - f\| = 2\|f - g\| + \|g_h - g\|,$$

where the first term on the RHS is less than ϵ since $C([a,b])$ is dense in L^1. ☐

Theorem 17. (Fubini) Let $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$, where $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Suppose f is a function on $\mathbb{R}^d$, then the slice of f corresponding on $\mathbb{R}^{d_1}$ is (we hold y constant)

$$f^y(x) := f(x,y)$$

and similarly for the slice for the y's. Then

(a) f^y is integrable on $\mathbb{R}^{d_1}$

(b) $F_y(x) = \int_{\mathbb{R}^{d_1}} f^y(x)$ is integrable on $\mathbb{R}^{d_2}$.

(c)

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y(x) \right) = \int_{\mathbb{R}^d} f$$

Proof. (Step 1) Define

$$\mathscr{F} := \{\text{integrable functions on } \mathbb{R}^d \text{ such that } a, b, c \text{ hold}\}$$

Let $\{f_n\} \subset \mathscr{F}$. For all $k \in [n]$, there exists some $A_k \subset \mathbb{R}^{d_2}$ such that $m(A_k) = 0$ and f_k^y is integrable on $\mathbb{R}^d$, where $y \notin A_k$. Let $A = \bigcup_{k=1}^N A_k$. Then $m(A) = 0$ and $y \in A^c$, and f_k^y is measurable and integrable for all $k \leq n$. By linearity, we have that $\operatorname{span} \bigcup f_n \subset \mathscr{F}$.

(Step 2) Suppose $\{f_n\} \subset \mathscr{F}$ and either $f_n \uparrow f$ or $f_n \downarrow f$. We claim that either $f \in \mathscr{F}$. Consider $-f_n$, then we can assume WLOG that $f_n \uparrow f$. Now consider $f_n - f_1 \geq 0$, then WLOG, $f_n \geq 0$. Let $A = \bigcup A_n$. the $m(A) = 0$ and $y \notin A$ and f_n^y is integrable on $\mathbb{R}^{d_1}$. Thus, by monotone convergence theorem on x,

$$g_n(y) = \int_{\mathbb{R}^{d_1}} f_n^y(x) \uparrow g(y) = \int_{\mathbb{R}^{d_1}} f^y(x) dx$$

By the monotone convergence in y, we have that

$$\int f_k(x, y) dx dy = \int g_k(y) \uparrow \int g(y) = \int \left(\int f(x, y) dx \right) dy$$

(Step 3) Suppose $f = \chi_E$, $E = G_\delta$, then $f \in \mathscr{F}$ (if $m(E) < \infty$.)

(a) $E_1 = Q_1 \times Q_2$, where $Q_1 \subset \mathbb{R}^{d_1}$ and $Q_2 \subset \mathbb{R}^{d_2}$ are both open. Then

$$\chi_E(x, y) = \chi_{Q_1 \times Q_2}(x, y) = \chi_{Q_1}(x) \chi_{Q_2}(y),$$

and so for each y, and so for each y, $\chi_E(x, y)$ is measurable in x, and

$$\int \chi_E(x, y) dx = \chi_{Q_2}(y) \int \chi_{Q_1}(x) dx = \chi_{Q_2}(y) |Q_1|$$

and

$$\int \left(\int \chi_E(x, y) dx \right) dy = |Q_2||Q_1| = m(E),$$

and so $\chi_E \in \mathscr{F}$.

(b) Suppose $E \subset \partial Q$, where $Q \subset \mathbb{R}^d$ is a closed cube. Evidently, $m(E) = 0$, and so $\int \chi_E(x, y) dx dy = 0$. For almost every y (except for bottom and top lines), E^y has measure 0 on $\mathbb{R}^{d_1}$, and so $\int \chi_E^y(x) dx = 0$ almost everywhere, and thus $\int \left(\int \chi_E^y(x) dx \right) dy = 0$, and thus $\chi_E \in \mathscr{F}$.

(c) Suppose $E = \bigcup_{k=1}^K Q_k$, where Q_k are closed cubes with disjoint interiors. Let $\operatorname{int} Q_k = \tilde{Q}_k$. And let χ_E be a finite linear combination of $\chi_{\tilde{Q}_k}$ and χ_{A_k}, where $A_k \subset \partial Q_k$. That is,

$$\chi_E = \sum a_k \left(\chi_{\tilde{Q}_k} + \chi_{A_k} \right),$$

and thus by previous steps we know that $\chi_E \in \mathscr{F}$.

(d) Suppose E is open and has finite measure. Then

$$E = \bigcup_{k=1}^{\infty} Q_k,$$

Q_k closed cubes with disjoint interior. Let $f_k = \sum_{j=1}^k \chi_{Q_j} \in \mathscr{F}$ from stage 3. But $f_k \uparrow \chi_E$, and so then by step 2 we are done.

(e) Suppose E is G_δ of finite measure. Since $E \subset G_\delta$, then there exists $(\tilde{O}_n)$ such that

$$E = \bigcap \tilde{O}_k$$

E is measurable, and thus $E \subset O$, where O is open and measurable, and so take

$$O_k = O \cap \bigcap_{n=1}^{k} \tilde{O}_k,$$

and we see that (O_k) is non increasing with

$$E = \bigcap O_k.$$

Let $f_k = \chi_{O_k}$, then $f_k \in \mathscr{F}$ and $\chi_E \in L^1$ and $f_k \downarrow \chi_E$, and thus $\chi_E \in \mathscr{F}$.

(Step 4) Suppose $E \subset \mathbb{R}^d$ with $m(E) = 0$. There exists some $G \in G_\delta$ such that $E \subset G$ with $m(G) = 0$. But by the previous step, $\chi_G \in \mathscr{F}$ with

$$\int \left(\int \chi_G(x, y) dx \right) dy = 0.$$

Thus, for almost every y,

$$\int \chi_G(x, y) dx = 0 = \int_{\mathbb{R}^{d_1}} \chi_G^y x dx.$$

But $E^y \subset G^y$, and so $m_*(E^y) \leq m_*(G^y) = 0$, and thus $m(E^y) = 0$ and $\chi_E^y(x)$ is measurable with integral 0 and thus $\chi_E \in \mathscr{F}$.

(Step 5) Suppose $f = \chi_E$ with $m(E) < \infty$, then $E = G \cup Z$, where $G \in G_\delta$ with $m(G) < \infty$ and $m(Z) = 0$. Then $\chi_E \in \mathscr{F}$.

(Step 6) Let $f \in L^1$. WLOG, $f \geq 0$. There exist (φ_n) simple functions such that $\varphi_n \uparrow f$, where $\varphi \in \mathscr{F}$ by like step 1 or something. Thus, $f \in \mathscr{F}$. □

Theorem 18. Assume $f(x, y) \geq 0$ is measurable on $\mathbb{R}^d$. Then for almost every $y \in \mathbb{R}^{d_2}$,

- f^y is measurable on $\mathbb{R}^{d_1}$
- $F^y = \int f^y(x) dx$ is measurable on $\mathbb{R}^{d_2}$
- $\int F^y = \int f(x, y) dx dy$.