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Problem 1

Let E be a vector space of dimension n and let $(e_i)_{1\leq i\leq n}$ be a basis of E. That is, given $x \in E$, we can write

$$x = \sum_{i=1}^{n} x_i e_i,$$

with $x_i \in \mathbb{R}$. Given $f \in E^*$, we can set $f_i = f(e_i)$. Recall that we define the duality map $F : E \to E^*$ as

$$F(x) = \{f \in E^* \ ; \ \|f\| = \|x\| \ \text{ and } \ \langle f, x\rangle = \|x\|^2\}$$

(a) Consider on E the norm

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

(i) Compute explicitly, in terms of the f_i's, the dual norm $\|f\|_{E^*}$ of $f \in E^*$.

Solution:

$$\begin{aligned}
\|f\|_{E^*} &= \sup_{x\in E, \|x\|=1} |\langle f, x\rangle| \\
&= \sup_{x\in E, \|x\|=1} |\langle f, \sum_{i=1}^{n} x_i e_i\rangle| \\
&= \sup_{x\in E, \|x\|=1} |\sum_{i=1}^{n} x_i \langle f, e_i\rangle| \\
&= \sup_{x\in E, \|x\|=1} |\sum_{i=1}^{n} x_i f_i| \\
&\leq \sup_{x\in E, \|x\|=1} |\sum_{i=1}^{n} x_i \max_{1\leq i\leq n} f(e_i)| \\
&= \max_{1\leq i\leq n} |f(e_i)| |\sum_{i=1}^{n} x_i| \\
&\leq \max_{i\in[n]} |f_i|
\end{aligned}$$

Suppose $\max_{i\in[n]} |f_i| = f(e_j)$ for some $j \in [n]$. then we have that if $x \in E$ such that $x_i = 0$ and $x_j = 1$, then $\|x\| = 1$ and

$$f(e_j) = f_j = |\sum_{i=1}^{n} x_i f_i| \leq \sup_{x\in E, \|x\|=1} |\sum_{i=1}^{n} x_i f_i| = \|f\|_{E^*}.$$

Thus, we have that $\max_{i\in[n]} f_i = \|f\|_{E^*}$ ■

(ii) Determine explicitly the set $F(x)$ (duality map) for every $x \in E$.

Solution: We claim that,

$$F(x) = \{f\ ;\ f(e_i) = \text{sign}(x_i)\|x\|\}.$$

Evidently,

$$\|f\| = \max_{i\in[n]} |f_i| = \max_{i\in[n]} \|x\| = \|x\|$$

and

$$\langle f, x\rangle = \sum_{i=1}^{n} x_i f_i = \sum_{i=1}^{n} |x_i|\|x\| = \|x\| \sum_{i=1}^{n} |x_i| = \|x\|^2$$

■

(b) Same questions but where E is provided with the norm

$$\|x\|_\infty = \max_{1\leq i\leq n} |x_i|.$$

Solution: From the previous problem, we have that

$$\begin{aligned}\|f\|_{E^*} &= \sup_{x\in E, \|x\|=1} |\sum_{i=1}^{n} x_i f_i| \\ &\leq \sup_{x\in E, \|x\|=1} \sum_{i=1}^{n} |x_i||f_i| \\ &= \sum_{i=1}^{n} |f_i|\end{aligned}$$

Conversely, we have that if $x \in E$ with $x_i = \text{sign}(f_i)$ for all $i \in [n]$, then we have that

$$\sum_{i=1}^{n} |f_i| = \sum_{i=1}^{n} |x_i f_i| = |\sum_{i=1}^{n} x_i f_i| \leq \sup_{x\in E, \|x\|=1} |\sum_{i=1}^{n} x_i f_i| = \|f\|_{E^*}$$

■

SOLUTION: We claim that $f \in F(x)$ if

$$f_i = 0 \qquad \text{if } |x_i| \neq \max_{k\in[n]} |x_k|$$

$$\sum_{i=1}^{n} |f_i| = \max_{k\in n} |x_i| \qquad \text{if } |x_i| \neq \max_{k\in[n]} |x_k|$$

Thus, we obviously have that

$$\|f\|_{E^*} = \|x\|_\infty$$

and sure enough, we have that

$$\langle f, x\rangle = \sum_{i=1}^{n} x_i f_i = \sum_{x_i=\|x\|_\infty} x_i^2 = \|x\|_\infty^2$$

∎

(c) Same questions but where E is provided with the norm

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2},$$

and more generally with the norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad \text{where } p \in (1,\infty).$$

SOLUTION: Let $q \in (1,\infty)$ such that $\frac{1}{p} = \frac{1}{q} = 1$. We use holder's inequality from last PSET

$$\begin{aligned}
\|f\|_{E^*} &= \sup_{x\in E, \|x\|=1} |\sum_{i=1}^{n} x_i f_i| \\
&= \sup_{x\in E, \|x\|=1} \sum_{i=1}^{n} |x_i||f_i| \\
&= \sup_{x\in E, \|x\|=1} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |f_i|^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{i=1}^{n} |f_i|^q\right)^{\frac{1}{q}}
\end{aligned}$$

∎

Solution: $f_i \in F(x)$ if

$$\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n |f_i|^q\right)^{\frac{1}{q}}$$

and

$$\sum_{i=1}^n |f_i||x_i| = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{2}{p}}.$$

Using Hölder's inequality on the last statement we find that applying the first equality,

$$\sum_{i=1}^n |f_i||x_i| \le \left(\sum_{i=1}^n |f_i|^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{2}{p}},$$

an so

$$\langle f, x\rangle = \|f\|_q \|x\|_p.$$

As proved in a previous PSET, an equality like this implies that

$$\sum_{i=1}^n |\hat{x_i}||\hat{f_i}| = 1,$$

where

$$\hat{x_i} = \frac{x_i}{\|x\|_p}, \qquad \hat{f_i} = \frac{f_i}{\|f\|_q}.$$

Reversing the argument from the previous PSET, we get that

$$f_i^q = Cx_i^p \implies f_i = C^{\frac{1}{q}} x_i^{\frac{p}{q}}.$$

Using the second line, we find that

$$\sum_{i=1}^n |f_i||x_i| = \sum_{i=1}^n |C^{\frac{1}{q}} x_i^{\frac{p}{q}}||x_i^{\frac{1}{p}}| = C^{\frac{1}{q}} \sum_{i=1}^n |x_i^{p-1}||x_i| = C^{\frac{1}{q}} \sum_{i=1}^n |x_i|^p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{2}{p}}.$$

Thus,

$$C^{\frac{1}{q}} = (\|x\|_p)^{2-p} \implies f_i = (\|x\|_p)^{2-p} |x_i|^{p-1}$$

We claim that

$$F(x) =$$

■

Problem 2

Consider the space $E = c_0$ (sequences tending to zero) with its usual norm (see Section 11.3). For every element $u = (u_1, u_2, u_3, \ldots)$ in E define

$$f(u) = \sum_{n=1}^{\infty} \frac{1}{2^n} u_n.$$

(a) Check that f is a continuous linear functional on E and compute $\|f\|_{E^*}$.

SOLUTION: Obviously, $f : E \to \mathbb{R}$.

Let $u, w \in E$ and $\lambda \in \mathbb{R}$. Then we have that

$$f(\lambda u + w) = \sum_{n=1}^{\infty} \frac{1}{2^n}(\lambda u_n + w_n) = \lambda \sum_{n=1}^{\infty} \frac{1}{2^n} u_n + \sum_{n=1}^{\infty} \frac{1}{2^n} w_n = \lambda f(u) + f(w)$$

Let $u \in E$. Then

$$\|f(u)\|_{\mathbb{R}} = |\sum_{n=1}^{\infty} \frac{1}{2^n} u_n| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} |u_n| \leq \|u\| \sum_{n=1}^{\infty} \frac{1}{2^n} = \|u\| \tag{1}$$

and thus f is bounded linear functional. Thus, f is continuous, and so $f \in E^*$. Using (1), we see that

$$\begin{aligned}\|f\|_{E^*} &= \sup_{u \in E, \|u\|=1} \|f(u)\| \\ &= \sup_{u \in E, \|u\|=1} \|f(u)\| \\ &= \sup_{u \in E, \|u\|=1} \|u\| \\ &= 1.\end{aligned}$$

■

(b) Can one find some $u \in E$ such that $\|u\| = 1$ and $f(u) = \|f\|_{E^*}$?

SOLUTION: No, but suppose we can. That is, there exists some $u \in E$ with $\|u\| = 1$ such that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} u_n = 1.$$

We claim that for all n, $u_n = 1$. Suppose not, that there exists some i such that $u_n < 1$.

Then

$$f(u) = \sum_{n=1}^{\infty} \frac{1}{2^n} u_n = \sum_{n \neq i} \frac{1}{2^n} + \frac{1}{2^i} u_n < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Thus, $f(u) < 1$. ■

Problem 3

Let E be a normed vector space and let $H \subset E$ be a hyperplane. Let $V \subset E$ be an affine subspace such that $H \subset V$.

(a) Prove that either $V = H$ or $V = E$.

SOLUTION: Since V is affine, there exists some linear subspace $U \subset E$ such that

$$V = x_0 + U.$$

Suppose $V \neq H$. That is, there exists some $v \in V$ such that $f(v) \neq \alpha$. Let $x \in E$. Either $f(x) = \alpha$, in which case $x \in H \subset V$ and we are done, or $f(x) \neq \alpha$. We wish to show that $x = x_0 + u$ for some $u \in U$.

Suppose $f(v) = f(x)$. Then $f(v-x) = 0$ and so if $z \in H$, we have that $f(v-x+h) = \alpha$, and so $v - x + h \in H$ and so $v - x + h \in V$. Consider now that $(v - x + z) \in V$, $v \in V$ and $z \in V$, then there exist u_i for $i = \{1, 2, 3\}$ such that

$$x = -(x - y + z) + x + z = -(x_0 + u_1) + (x_0 + u_2) + (x_0 + u_3) = x_0 + (u_2 + u_3 - u_1).$$

Since U is a linear subspace, $(u_2 + u_3 - u_1) \in U$ and thus $x \in V$.

Suppose $f(v) \neq f(x)$, and now consider

$$g(t) = tf(v) + (1-t)f(x)$$

to be the line intersecting $f(v)$ and $f(x)$. We claim that for some t, $g(t) = \alpha$. Indeed, after some algebra, one can confirm that if

$$\tau = \frac{f(x) - \alpha}{f(x) - f(v)}, \implies g(\tau) = \alpha.$$

Thus, $g(\tau) = \tau f(v) - (1-\tau)f(x) \in H$. Moreover, we have that

$$\alpha = \tau f(v) + (1-\tau)f(x) = f(v\tau) + f((1-\tau)x) = f(v\tau + (1-\tau)x),$$

and so

$$v\tau + (1-\tau)x =\in H \subset V. \implies v\tau + (1-\tau)x = u_1 + x_0, \quad u_1 \in U.$$

Since $v \in V$, we let $v = u_2 + x_0$, where $u_2 \in U$. Thus, we get that

$$(1-\tau)x = (1-\tau)x_0 + u_1 - \tau u_2,$$

and so $(1-\tau)x$, dividng by $(1-\tau)$ and using the fact that U is a subspace gives that $x \in V$.

In either case, we have that $E \subset V \subset E$, and thus $E = V$ whenever $H \neq V$. ■

(b) Prove that either H is closed or dense in E.

Solution: Let $x \in E$ such that $f(x) = \alpha$. Let $v \in \ker f$, then $f(x+v) = f(x)+f(v) = \alpha$, and so $v + x \in H$, and so $\ker f + x \subset H$. We claim that $\ker f$ is a subspace of E, and thus $\ker f + x$ is an affine subspace. Let $z \in H$. Then

$$f(z) = \alpha = f(x) \implies f(z-x) = 0 \implies z-x \in \ker f \implies z \in \ker f+x \implies H \subset \ker f+x.$$

Thus, we have that $\ker f + x = H$, and so $H \subset \overline{\ker f} + x = V \subset E$. Thus, we have from part (a) that either $V = H$, in which case H is closed since V is closed, or $V = E$. Suppose the latter, then $\overline{\ker f} + x = E$, and so since $H = \ker f + x$, we have that H is dense in E by definition. ■

Problem 4

Let E be an n.v.s. with norm $\|\cdot\|$. Let $C \subset E$ be an open convex set such that $0 \in C$. Let p denote the gauge of C (see Lemma 1.2).

(a) Assuming C is symmetric (i.e., $-C = C$) and C is bounded, prove that p is a norm which is equivalent to $\|\cdot\|$.

SOLUTION: Recall that

$$p(x) := \inf\{\alpha \; ; \; \frac{1}{\alpha}x \in C\}$$

It was shown in class that p was a Minkowski functional. That is,

$$p(\lambda v) = |\lambda| p(v), \qquad p(v+w) \leq p(v) + p(w),$$

thus, it remains to show that $p(v) = 0$ if and only if $v = 0$.

Suppose $v = 0$, then we know $v \in C$, moreover, for any $\alpha > 0$, we have that $\frac{1}{\alpha}v = 0 \in C$, in particular, as $\alpha \to 0$, we have that $\frac{1}{\alpha}v = 0$, and thus $p(v) = 0$. On the other hand, suppose $p(v) = 0$, then for any $n \in \mathbb{N}$, there exists an $\alpha > 0$ such that $0 < \alpha < \frac{1}{n}$ and $\frac{1}{\alpha}v \in C$. However, we have that $nv < \frac{1}{\alpha}v \in C$, and thus $nv \in C$ for all C, which implies C is unbounded. Thus, $v = 0$ and so p is a norm.

We will show that for some $K \in \mathbb{R}$, for any $v \in E$

$$\|v\| \leq K p(v).$$

Since C is bounded, then for any $v \in E$, we have that $\|v\| < K$. That is, if $c \in C$, then $\|c\| < K$. Thus, let $v \in E$, then by definition of $p(v)$: $\frac{v}{p(v)} \in C$[a], which implies that $\|v\| \leq K\|p(v)\|$

■

[a] This is not necessarily true, since $p(v)$ could be 0, but we make this statement informally and note that one could talk about this using arbitrarily close α. Either way, the next line is correct.

(b) Let $E = C([0,1];\mathbb{R})$ with its usual norm

$$\|u\| = \max_{t \in [0,1]} |u(t)|.$$

Let

$$C = \left\{ u \in E; \int_0^1 |u(t)|^2 \, dt < 1 \right\}.$$

Check that C is convex and symmetric and that $0 \in C$. Is C bounded in E? Compute the gauge p of C and show that p is a norm on E. Is p equivalent to $\|\cdot\|$?

Solution: Let $f, g \in C$. Let $t \in [0,1]$, we need to check that

$$tf + (1-t)g \in C.$$

Consider that since we are working with real valued functions, $|u(t)|^2 = (u(t))^2$

$$\begin{aligned}
\int_0^1 |tf(x) - (1-t)g(x)|^2 dx &= \int_0^1 (tf(x) - (1-t)g(x))^2 dx \\
&= \int_0^1 t^2 f^2(x) dx - \int_0^1 2t(1-t) f(x) g(x) dx + \int_0^1 (1-t)^2 g^2(x) dx \\
&= t^2 \int_0^1 f^2(x) dx - 2t(1-t) \int_0^1 f(x) g(x) dx + (1-t)^2 \int_0^1 g^2(x) dx \\
&< t^2 - 2t(1-t) \int_0^1 f(x) g(x) dx + (1-t)^2
\end{aligned}$$

We claim that $\int_0^1 f(x)g(x)dx < 1$, which we prove using Holder's inequality

$$\int_0^1 f(x)g(x) \leq \left(\int_0^1 f(x)dx\right)^{\frac{1}{2}} \left(\int_0^1 g(x)dx\right)^{\frac{1}{2}} < 1.$$

Thus, we have that since $t \in [0,1]$:

$$\int_0^1 |tf(x) - (1-t)g(x)|^2 < t^2 - 2t(1-t) + (1-t)^2 = (t-(1-t))^2 = (2t-1)^2 < 1,$$

and so C is convex.

Let $f \in C$. symmetry comes straight out of the fact that $(-f)^2 = f^2$ and thus they integrate to less than one.

Consider the function $f_n(x) = \frac{\sqrt{n}}{1+nx}$, then we have that

$$\int_0^1 \frac{\sqrt{n}}{1+nx} dx = \sqrt{n} \int_0^1 \frac{1}{1+nx} dx = \sqrt{n} \int_1^{1+n} \frac{1}{u} du = \frac{\log(1+n)}{\sqrt{n}} < 1.$$

To see the final inequality, consider that for $x > 1$, we have that $4x < 1 + 2x + x^2$, which implies that $\frac{1}{1+x} < \frac{1}{2\sqrt{x}}$, thus, we gave that $(\log(1+x))' < (\sqrt{x})'$, for all $x > 1$, so it suffices to show that $\log(2) < \sqrt{1}$, which you can just look up, i don't care. So then we have that $\frac{\log(n+1)}{\sqrt{n}} < 1$ for $n > 1$. Thus, $f_n \in C$ for all n. However, we have that

$$\|f_n\| = \frac{\sqrt{n}}{1} = \sqrt{n},$$

which is certainly not bounded, and so p is not equivalent to $\|\cdot\|$, since $\|f\| \geq M$ for any $M \in \mathbb{R}$. ■

Problem 5

Let E be a finite-dimensional normed space. Let $C \subset E$ be a nonempty convex set such that $0 \notin C$. We claim that there always exists some hyperplane that separates C and $\{0\}$.

> [Note that every hyperplane is closed (why?). The main point in this exercise is that no additional assumption on C is required.]

SOLUTION: We see that every hyperplane is closed because if $H = [f = \alpha]$, where $f : E \to \mathbb{R}$ is a linear functional, then the fact that E is finite dimensional implies that f is finite. ∎

(i) Let $(x_n)_{n\geq 1}$ be a countable subset of C that is dense in C (why does it exist?). For every n let

$$C_n = \operatorname{conv}\{x_1, x_2, \ldots, x_n\} = \left\{ x = \sum_{i=1}^{n} t_i x_i;\ t_i \geq 0\ \forall i \text{ and } \sum_{i=1}^{n} t_i = 1 \right\}.$$

Check that C_n is compact and that $\bigcup_{n=1}^{\infty} C_n$ is dense in C.

SOLUTION: Define $\varphi : \mathbb{R}^n \to C_n$ such that if $y \in \mathbb{R}^n$, then

$$\varphi(y) = \varphi \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} y_i x_i.$$

Define

$$X = \{t = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix};\ t_i \geq 0, \quad \sum_{i=1}^{n} t_i = 1\},$$

then we have that $c^{(n)} \in C_n$ if and only if $\varphi^{-1}(c^{(n)}) \in X$. That is, $\varphi(X) = C_n$. Equip X with the $\|y\|_1 = \sum_{i=1}^{n} y_i$ norm. We see that $X \subset \mathbb{R}^n$, and thus it suffices to show that X is closed. Let $t_m = \begin{pmatrix} t_1^{(m)} \\ t_2^{(m)} \\ \vdots \\ t_n^{(m)} \end{pmatrix} \to \tilde{t} = \begin{pmatrix} \tilde{t_1} \\ \tilde{t_2} \\ \vdots \\ \tilde{t_n} \end{pmatrix}$ where $(t_m) \in X$. Then we have (by the previous PSET) that $t_i^{(m)} \to \tilde{t_i}$ at the same rate. That is, there exists an $N \in \mathbb{N}$ such that if $m \geq N$, then $|\tilde{t_i} - t_i{}^{(m)}| < \frac{\epsilon}{2^n}$, and so

$$|1 - \sum_{i=1}^{n} \tilde{t_i}| \leq |1 - \sum_{i=1}^{n} t_i^{(m)}| + |\sum_{i=1}^{n} t_i^{(m)} - \sum_{i=1}^{n} \tilde{t_i}| < \epsilon.$$

Thus, X is closed and bounded (bounded by $B_2(0)$), and so X is compact. Since φ is continuous, (linear function between finite dimensional spaces), we have that $\varphi(X) = C_n$ is compact.

Evidently, we have that $C_n \subset C_{n+1}$. Let $x \in C$ and $\epsilon > 0$. We claim that there is some $c \in \bigcup C_n$ such that $c \in B_\epsilon(x)$. Since (x_n) is dense in C, then there is some $x_k \in (x_n)$ such that $x_k \in B_\epsilon(x)$. Consider now that if $t_i = 0$ except for $t_k = 1$, then $\sum_{i=1}^k t_i = 1$ and

$$c = x_k = \sum_{i=1}^{k} t_i x_i \in C_k \in \bigcup_{n=1}^{\infty} C_n$$

■

(ii) Prove that there is some $f_n \in E^\star$ such that

$$\|f_n\| = 1 \text{ and } \langle f_n, x\rangle \geq 0 \ \forall x \in C_n.$$

Solution: We have that C_n compact and $\{0\}$ is closed, and they are disjoint. By the Hahn-Banach separation theorem (second geometric form), there exists a closed hyperplane $H_n = [f_n = \alpha_n]$ that strictly separates C_n and $\{0\}$. That is, for all $x \in C_n$,

$$0 = f_n(0) \leq \alpha_n - \epsilon < \alpha_n + \epsilon \leq f_n(x) = \langle f_n, x\rangle.$$

Thus, we can take

$$\varphi_n(x) = \frac{f_n(x)}{\|f_n(x)\|},$$

and so we have that $\|\varphi_n(x)\| = 1$ and $\langle \varphi_n, x\rangle \geq 0$. ■

(iii) Deduce that there is some $f \in E^\star$ such that

$$\|f\| = 1 \text{ and } \langle f, x\rangle \geq 0 \ \forall x \in C.$$

Conclude.

Solution: Take

$$\lim_{n\to\infty} \varphi_n(x) := \varphi(x)$$

Take some $x \in C$. Let $\delta > 0$ from continuity. By the density of the (x_n), there exists some k such that $x_k \in B_{\frac{\delta}{2}}(x)$. Thus, we have that since $C_n \subset C_{n+1}$, then by the definition of φ, we have that $\langle \varphi, x_k\rangle \geq 0$ and $\|\varphi\| = 1$. Thus, by continuity, we have that $\|\varphi(x)\| \geq 0$ and $\|\varphi\| = 1$ for all $x \in C$. ■

(iv) Let $A, B \subset E$ be nonempty disjoint convex sets. Prove that there exists some hyperplane H that separates A and B.

Solution: Take $C = A - B$. Since A and B are disjoint, we have that $0 \notin C$. Evidently, C is nonempty. We just have to show that C is convex. Let $x, y \in C$ and let

$$f(t) = tx+(1-t)y = t(a_1-b_1)+(1-t)(a_2-b_2) = ta_1+(1-t)a_2-(tb_1+(1-t)b_2) = a-b \in C,$$

by convexity. By the previous parts, there exists some $f \in E^*$ such that $\|f\| = 1$ and

$$0 \leq \langle f, C\rangle = \langle f, A - B\rangle = \langle f, A\rangle - \langle f, B\rangle.$$

Thus, for all $a \in A$, $b \in B$, we have that

$$\langle f, b\rangle \leq \langle f, a\rangle,$$

and so there exists a hyperplane separating the two. Specifically, fix a constant such that

$$\langle f, b\rangle \leq \alpha \leq \langle f, a\rangle$$

and take the hyperplane $H = [f = \alpha]$. ■

Problem 6

Let $E = \ell^1$ (see Section 11.3) and consider the two sets

$$X = \{x = (x_n)_{n\geq 1} \in E; \quad x_{2n} = 0 \quad \forall n \geq 1\}$$

and

$$Y = \left\{ y = (y_n)_{n\geq 1} \in E; \quad y_{2n} = \frac{1}{2^n} y_{2n-1} \quad \forall n \geq 1 \right\}.$$

(i) Check that X and Y are closed linear spaces and that $\overline{X+Y} = E$.

Solution: Let

$$\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_i^{(1)} \\ \vdots \end{pmatrix}, \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \\ \vdots \\ x_i^{(2)} \\ \vdots \end{pmatrix}, \dots, \begin{pmatrix} x_1^{(n)} \\ x_2^{(n)} \\ \vdots \\ x_i^{(n)} \\ \vdots \end{pmatrix}, \dots = x_i^{(n)} \to (x_i) = x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \end{pmatrix}$$

where $(x_i^{(n)}) \in X$. Thus, we have that for all $\epsilon > 0$, for all i, for n large, we have

$$\|x_{(i)}^n - x\|_{\ell^1} = \sum_{i=1}^{\infty} |x_i^{(n)} - x_i| < \epsilon.$$

Thus, we have that

$$\sum_{i=2k}^{\infty} |x_i^{(n)} - x_i| = \sum_{i=2k}^{\infty} |x_i| < \epsilon.$$

Because this is true for all $\epsilon > 0$, then it must be that $x_i = 0$ for all $i = 2k$ for $k \in \mathbb{N}$, and so $x \in X$.

Let

$$\begin{pmatrix} y_1^{(1)} \\ y_2^{(1)} \\ \vdots \\ y_i^{(1)} \\ \vdots \end{pmatrix}, \begin{pmatrix} y_1^{(2)} \\ y_2^{(2)} \\ \vdots \\ y_i^{(2)} \\ \vdots \end{pmatrix}, \dots, \begin{pmatrix} y_1^{(n)} \\ y_2^{(n)} \\ \vdots \\ y_i^{(n)} \\ \vdots \end{pmatrix}, \dots = y_i^{(n)} \to (y_i) = y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \end{pmatrix}$$

where $(y_i^{(n)}) \in Y$. Thus, we have that for all $\epsilon > 0$, for all i, for n large, we have

$$\|y_{(i)}^n - y\|_{\ell^1} = \sum_{i=1}^{\infty} |y_i^{(n)} - y_i| < \epsilon.$$

and thus for all i :

$$|y_i^{(n)} - y_i| \leq \|y_i^{(n)} - y\|_{\ell^1} < \epsilon,$$

and so

$$y_i^{(n)} = y_i$$

for large n, and thus

$$y_{2i}^{(n)} = \frac{1}{2^{2i}} y_{2i-1}^{(n)} \xrightarrow{n\to\infty} \frac{1}{2^{2i}} y_{2i-1},$$

and so we are done, since this shows that $y \in Y$.

Evidently, $\overline{X+Y} \subset E$. Let $e \in E$. Then we define

$$x_{2n} = 0, \quad y_{2n} = e_{2n}$$

$$x_{2n-1} = e_{2n-1} - 2^{2n} e^{2n}, \quad y_{2n-1} = 2^{2n} e_{2n},$$

then obviously, $x \in X$ and $y \in Y$ and this sequence converges to $e \in E$. Thus, $E \subset \overline{X+Y}$

■

(ii) Let $c \in E$ be defined by

$$c_n = \begin{cases} 0, & \forall n \geq 1, \quad n \text{ odd}, \\ \frac{1}{2^n}, & \forall n \geq 1, \quad n \text{ even}. \end{cases}$$

Check that $c \notin X + Y$.

Solution: Suppose it is! Then we would have (by logic from above), that

$$x_{2n} = 0, \qquad x_{2n-1} = \frac{1}{2^{2n-1}} - 1$$

$$y_{2n} = \frac{1}{2^{2n}} \qquad y_{2n-1} = 2^{2n} \frac{1}{2^{2n}} = 1.$$

Thus, $c = x + y$, But then $y \notin \ell^1$ ■

(iii) Set $Z = X - c$ and check that $Y \cap Z = \emptyset$. Does there exist a closed hyperplane in E that separates Y and Z? Compare with Theorem 1.7 and Exercise 1.9.

Solution: Suppose not, then let $\chi \in Y \cap Z$. Since $\chi \in Y$, there exists some sequence $y \in Y$ such that $\chi = y$. Similarly, there exists some $z \in Z$ such that $\chi = z$. Since $Z = X - c$, then there exists some $x \in X$ such that $\chi = x - c$, and thus $y = x - c$, and so $c = x + (-y)$, which is a contradiction to what we have seen above, since $-y \in Y$.

Suppose we could separate Y and Z with some closed hyperplane $H = [f = \alpha]$. Then there exists some α such that for all $y \in Y$, $z \in Z$, we have that

$$f(y) < \alpha < f(z) = f(x - c) = f(x) - f(c) \implies f(Y) < \alpha < f(Z).$$

By the lemma below, we have that this implies that since Y and X are linear subspaces (easy to show), then $f(Y) = f(X) = 0$. Thus, we have that $Y, X \subset \ker f$. Moreover, we also know that by the strict separation, we have that $f(c) < 0$. We get that $X + Y \subset \ker f$, and since f is continuous, then $\overline{X + Y} \subset \ker f$, and so $E \subset \ker f$. Thus, $f \equiv 0$ on all E, which is a contradiction to the fact that $f(c) < 0$. ■

Reflections: We claim that if W is subspace and $f : W \to \mathbb{R}$ is a linear functional. Then either $f \equiv 0$ or $f(W) = \mathbb{R}$. Suppose $f \not\equiv 0$, then there exists some $w \in W$ such that $f(w) = a > 0$. Then for any $x \in \mathbb{R}$, we have that $\frac{x}{a} f(w) = \frac{x}{a} a$, and thus since $f(\frac{x}{a} w) = x$ and $\frac{x}{a} q \in W$, then we are done.

Problem 7

Let E be a normed vector space and let $f \in E^*$ with $f \neq 0$. Let $M = [f = 0]$.

(i) Determine $M^\perp$.

SOLUTION: We recall that

$$M^\perp = \{\varphi \in E^* \ ; \ \langle \varphi, x \rangle = 0 \ \forall x \in M\}.$$

We claim that

$$M^\perp = \{\sum_{i=1}^{n} \lambda_i f(x) \ ; \ x \in M\},$$

where f is the functional from the hyperplane.

Let $g \in \{\sum_{i=1}^{n} \lambda_i f(x) \ ; \ x \in M\}$, then $g(x) = \lambda_1 f(x) + \cdots \lambda_n f(x) = 0$.

Let $g \in M^\perp$, then for all $x \in M$, we have that $g(x) = 0$. Suppose that $g \notin \operatorname{span}\{f\}$, then we have that f and g are linearly independent, and thus

$$a_1 f + a_2 g = 0$$

is satisfied if and only if $a_1, a_2 = 0$. This is obviously not true, since $f, g = 0$ and so the equation is satisfied for any scalars, and so the vectors are linearly dependent, and so $g \in \operatorname{span}(f)$. ■

(ii) Prove that for every $x \in E$,

$$\operatorname{dist}(x, M) = \inf_{y \in M} \|x - y\| = \frac{|\langle f, x \rangle|}{\|f\|}.$$

[Find a direct method or use Example 3 in Section 1.4.]

SOLUTION: We have that for $y \in M$, $\langle f, y \rangle = 0$, and so

$$|\langle f, x \rangle| = |\langle f, x - y \rangle| \leq \|f\| \|x - y\| \implies \frac{|\langle f, x \rangle|}{\|f\|} \leq \|x - y\|$$

Thus,

$$\frac{|\langle f, x \rangle|}{\|f\|} \leq \operatorname{dist}(x, M).$$

For the other direction, take $v \in E \setminus M$.[a] Then

$$\|f\| = \sup_{v} \frac{|f(v)|}{|v|}.$$

Note that we have that

$$d(x, M) \leq \|x - (x - \frac{f(x)}{f(u)} u)\| = \|\frac{f(x)}{f(u)}(u)\| \leq \frac{\|f\|}{|f(u)|} \|u\|,$$

and so using the definition of the operator norm above, we are done. ■

[a] We know $v \notin M$ since $f(v) = 0$ if it were, and thus we can ignore it for the operator norm

(iii) Assume now that

$$E = \{u \in C([0,1];\mathbb{R}); \quad u(0) = 0\}$$

and that

$$\langle f, u \rangle = \int_0^1 u(t)dt, \quad u \in E.$$

Prove that

$$\text{dist}(u, M) = \left| \int_0^1 u(t)dt \right| \quad \forall u \in E.$$

Show that

$$\inf_{v \in M} \|u - v\|$$

is never achieved for any $u \in E \setminus M$.

Solution: Consider that

$$\|f\| = \sup_{\|u\|=1} \langle |f(u)\rangle| = \sup_{\|u\|=1} |\int_0^1 u(t)dt| = |\int_0^1 dt| = 1.$$

Thus, by part (ii),

$$\text{dist}(u, M) = \inf_{u \in M} \|x - u\| = \frac{|\langle f, u \rangle|}{\|f\|} = |\langle f, u \rangle| = \left| \int_0^1 u(t)dt \right|.$$

Suppose $u \in E \setminus M$, then

$$|\int_0^1 u(t)dt)| = 0,$$

then $\int_0^1 u(t)dt = 0$, and so $\langle f, u \rangle = 0$, and so $u \in M$. ■

Problem 8

Let X be a normed vector space. Assume that for some $x, y \in X$, we have that

$$\|x + y\| = \|x\| + \|y\|.$$

Show that for every $\alpha, \beta > 0$ we have that

$$\|\alpha x + \beta y\| = \alpha\|x\| + \beta\|y\|$$

SOLUTION: Without loss of generality, suppose $\alpha \geq \beta$. By the triangle inequality, we have that

$$\|\alpha x + \beta y\| \leq \alpha\|x\| + \beta\|y\|,$$

so it suffices to show that

$$\|\alpha x + \beta y\| \geq \alpha\|x\| + \beta\|y\|.$$

Consider that by the reverse triangle inequality,

$$\begin{aligned}
\|\alpha x + \beta y\| &= \|\alpha(x + y) + (\beta - \alpha)y\| \\
&\geq \|\alpha(x + y)\| - \|(\beta - \alpha)y\| \\
&= \alpha\|x\| + \alpha\|y\| + (\beta - \alpha)\|y\| \\
&= \alpha\|x\| + \beta\|y\|
\end{aligned}$$

. ■

Problem 9

Let $1 \leq p \leq q < \infty$. Then $\|x\|_q \leq \|x\|_p$.

SOLUTION: Consider the case when $\|x\|_p = 1$. Then we have that necessarily, $|x_i| \leq 1$ for all i, and thus $|x_i|^q \leq |x_i|$ for all i. Thus,

$$\sum_{i=1}^{\infty} |x_i|^q \leq \sum_{i=1}^{\infty} |x_i| = 1 \implies \|x\|_q \leq 1.$$

For the general case, take

$$y_k = \frac{x_k}{\|x\|_p},$$

then $\|y\|_p = 1$ and thus

$$\|y\|_q \leq \|y\|_p = 1,$$

but we have that

$$\|y\|_q = \frac{\|x\|_q}{\|x\|_p} \leq 1,$$

and so we are done. ■

Problem 10

Show that

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$$

SOLUTION: Consider that for any n, we have that by taking limits

$$\sum_{k=1}^{\infty} |a_k|^p \geq |a_n|^p \implies \|a\|_\infty \leq \|a\|_p.$$

On the other hand, we have that since $a \in \ell^p$, then

$$\sum_{k=1}^{\infty} |a_k|^p < \infty,$$

Thus, there must exist some C such that

$$\sum_{k=1}^{\infty} |\frac{a_k}{A}|^p \leq C \implies \left(\sum_{k=1}^{\infty} |\frac{a_k}{A}|^p\right)^{\frac{1}{p}} = C^{\frac{1}{p}}.$$

By homogeneity, we have that

$$\|a\|_p \leq C^{\frac{1}{p}} \|a\|_\infty.$$

Since for large p, we have that $|C^{\frac{1}{p}} - 1| < \epsilon$, then

$$\|a\|_p \leq C^{\frac{1}{p}} \|a\|_\infty \to \|a\|_\infty$$

Thus, as $p \to \infty$, we have that $\|a\|_p = \|a\|_\infty$. ■

Problem 11

Show that a normed space X is Banach if and only if $\sum x_n$ converges in X whenever $\|x_n\| < \frac{1}{2^n}$ for every n.

SOLUTION: Suppose X is Banach. Suppose $\|x_n\| < \frac{1}{2^n}$, then

$$\sum_{n=1}^{\infty} \|x_n\| < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

and thus since X is Banach and the series is absolutely summable, then it is summable, and thus

$$\sum_{n=1}^{\infty} x_n < \infty.$$

Let $(x_n) \in X$ be Cauchy. Thus, for all $n, m \geq N_k$, we have that

$$\|x_n - x_m\| < \frac{1}{2^k}.$$

Let $n_K = N_1 + N_2 + \cdots + N_k$. Obviously, $n_{k+1} \geq n_k \geq N_K$ and thus our subsequence (x_{n_k}) is Cauchy by

$$\|x_{n_k} - x_{n_{k+1}}\| < \frac{1}{2^k}.$$

Thus, we have by assumption that our series is summable or that

$$\sum_{k=1}^{K} x_{n_k} - x_{n_{k+1}} = x_{n_K} - x_{n_1} \xrightarrow{n\to\infty} x - x_{n_1}$$

as $K \to \infty$, and thus our Cauchy subsequence converges to a limit. Thus, it suffices to show that if (a_n) is Cauchy and it has a convergent subsequence, then (x_n) must converge. Observe that since (x_n) is Cauchy, then there exists some $N_1 \in \mathbb{N}$ such that $n > N_1$, then $d(a_n, a_{n_k}) < \frac{\epsilon}{2}$ (since n_k is implicitly greater than n). Since (x_{n_k}) converges to x, then there exists some N_2, such that if $n > N_2$, we have that $d(x_{n_k}, x) < \frac{\epsilon}{2}$. Thus if $N = \max\{N_1, N_2\}$ and $n \geq N$,

$$d(x_n, x) \leq d(x_n, a_{x_k}) + d(x_{n_k}, x) \leq \epsilon.$$

■

Problem 12

Let X be a normed vector space obtained from

$$c_0 = \{(x_n) \; ; \; \lim_{n\to\infty} x_n = 0\}$$

equipped with the norm $\|x\|_0 = \sum_{n=1}^{\infty} \frac{1}{2^k}|x_k|$. Show that X is not Banach.

SOLUTION: Consider the following sequence in c_0 :

$$(x_n^{(i)}) = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & \cdots \\ 0 & 0 & 1 & \cdots \end{pmatrix},$$

where the columns are the sequences in $(x_n^{(i)})$. Let $\epsilon > 0$, we can find a K such that $\frac{1}{2^K} < \epsilon$ and thus if $n, m \geq K$, then

$$\|x_n^{(i)} - x_m^{(i)}\|_0 \leq \sum_{n+1}^{m} \frac{1}{2^k} < \frac{1}{2^n} < \frac{1}{2^K},$$

and thus $(x_n^{(i)})$ is Cauchy. Evidently, we have that

$$(x_n^{(i)}) \to (1, 1, 1, \dots),$$

which is not in c_0. ■

Problem 13

Let X be a normed vector space and suppose $C \subset X$ is nonempty and convex. Then $\text{int}(C)$ and $\overline{C}$ are both convex.

SOLUTION: Suppose not. Let $x, y \in \text{int}(C)$ and let

$$z = tx + (1-t)y$$

such that $z \notin \text{int}(C)$. Thus, for all $\epsilon > 0$, $B_\epsilon(z) \not\subset C$. Specifically, let z_0 be such a point. Since $x, y \in \text{int}(C)$, then there exist $\epsilon > 0$ such that

$$B_\epsilon(x) \subset \text{int}(C) \qquad C_\epsilon(y) \subset \text{int}(C). \tag{2}$$

Consider now that

$$f(t) = t(x - (z - z_0)) + (1-t)(y - (z - z_0)) = z_0,$$

and so it suffices to show that $z - (z - z_0)$ and $y - (z - z_0)$ are in C, since this would imply that there is a convex combination of them that are not in C. Since $z_0 \in B_\epsilon(z)$, we have that

$$\|z - z_0\| < \frac{\epsilon}{2}.$$

Thus, by (2),

$$\|x - (z - z_0)\| \leq \|x\| + \|z - z_0\| < \|x\| + \frac{\epsilon}{2} \implies (x - (z - z_0)) \in C.$$

Similarly, $y - (z - z_0) \in C$. ■

SOLUTION: Let $x, y \in \overline{C}$ and suppose there exists

$$z = tx - (1-t)y \notin \overline{C}$$

Thus, there exists some $\epsilon > 0$ such that $B_\epsilon(z)$ contains no points of C. Specifically, let $z_0 \in B_\epsilon(z)$ such that $z_0 \notin C$. Again, we do the exact same thing as above to conclude that we can express z_0 as convex combinations of $x_0 \in B_\epsilon(x)$ and $y_0 \in B_\epsilon(y)$, where $x_0 \in C$ and $y_0 \in C$, and thus C is not convex. ■

Problem 14

Let X be a normed vector space. If $A \subset X$ is open, then $\text{conv}(A)$ is open. Is the convex hull of closed sets in $\mathbb{R}^n$ closed?

Solution: Recall that

$$\text{conv}(A) = \{\sum_{i=1}^{n} t_i a_i,\ a_i \in A\ \forall i,\ t_i \geq 0\ \forall i,\ \sum_{i=1}^{n} t_i = 1\}.$$

Let $a_0 \in \text{conv(A)}$, then

$$a_0 = \sum_{i=1}^{n} t_i a_i.$$

Since each $a_i \in A$, then there exists some $r_i > 0$ such that

$$B_{r_i}(a_i) \subset U.$$

Let $r = \min_{i \in [n]}\{r_i\}$. We can then conclude that

$$\sum_{i=1}^{n} t_i B_r(a_i) = \sum_{i=1}^{n}(t_i a_i + t_i B_r(0)) = B_r(0) + \sum_{i=1}^{n} t_i a_i = B_r(a_0),$$

and thus any $a \in B_r(a_0)$ can be written as a convex combination of $a_i' \in A$.

The convex hull of closed sets in $\mathbb{R}^n$ is not necessarily closed. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that $f(x) = \frac{1}{x^2}$. Then consider the graph of f as

$$G(f) = \{(x, f(x))\}.$$

Obviously, $G(f)$ is closed. It is not hard to see that the convex hull is simply the upper plane, which is not closed. ■

Problem 15

Suppose X, Y are nonempty normed vector spaces and $T \in \mathcal{L}(X, Y)$, then

$$\|T\| = \sup_{\|x\|=1} \{\|T(x)\}\} = \sup_{\|x\| \leq 1} \{\|T(x)\}\}$$

SOLUTION: Consider that

$$\|Tx\| \leq \|T\|\|x\| \leq \|T\|,$$

and thus

$$\sup_{\|x\| \leq 1} \{\|T(x)\|\} \leq \sup_{\|x\|=1} \{\|T(x)\|\}$$

The other direction is obvious, since we can just take $\|x| = 1$ and thus $\|x\| \leq 1$ and so

$$\sup_{\|x\|=1} \{\|T(x)\|\} \leq \sup_{\|x\| \leq 1} \{\|T(x)\|\}$$

■

Problem 16

Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. If there exists $\delta > 0$ such that $\|T(x)\| \geq \delta\|x\|$ for all $x \in X$, then $T(X)$ is closed in Y.

SOLUTION: Suppose that $(w_n) \in T(X)$ with $(w_n) \to w$. We want to show that there exists some $v \in X$ such that $T(v) = w$. Because $(w_n) \in T(X)$ for all n, then there exists (v_n) such that $T(v_i) = w_i$. Since $(w_n) \to w$, then (w_n) is Cauchy. There exists some N such that if $n, m \geq N$, then $\|w_n - w_m\| = \|T(v_n) - T(v_m)\| < \epsilon\delta$.

$$\delta\|v_n - v_m\| \leq \|T(v_n - T_m)\| = \|T(v_n) - T(v_m)\| = \|v_n - v_m\| < \epsilon\delta,$$

and thus

$$\|v_n - v_m\| < \epsilon.$$

Since $(v_n) \in X$ is Cauchy and X is Banach, we have that there exists some $v \in X$ such that $v_n \to v$. Since $T \in \mathcal{L}(V, W)$, then T is continuous, and thus

$$T(v_n) \to T(v) = w \in Y.$$

■

Problem 17

Show that the norm $\|x\| = \sum_{i=1}^{\infty} 2^{-i}|x_i|$ in ℓ^2 is not equivalent to the $\|\cdot\|_2$ norm.

Solution: Let $(x_n) = \frac{1}{\sqrt{n}}$. Since (x_n) is monotonic and bounded, then by the claim in the reflection, we have that $\|x\| = \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{1}{\sqrt{n}} < \infty$. However, in the other norm, we have that

$$\|x\|_2 = \left(\sum_{n=1}^{\infty} |\frac{1}{\sqrt{n}}|^2\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} \frac{1}{n}\right)^{\frac{1}{2}} = \infty$$

since the harmonic series diverges. Thus, the norms are not equivalent. ■

Reflections: If $\sum a_n$ converges and (b_n) is monotonic and bounded, prove that $\sum a_n b_n$ converges. We claim that if $(c_n) \to 0$ is monotone decreasing, then $\sum a_k c_n$ converges. By Luis' hint during office hours, we have that:

$$\begin{aligned}\sum_{k=1}^{n-1} A_k(c_k - c_{c+1}) &= A_1(c_1 - c_2) + A_2(c_2 - c_3) + \dots \\ &= \sum_{k=1}^{n-1} a_k c_k - A_{n-1}c_n,\end{aligned}$$

which can be proved by induction. We have that since A_n converges, then it is bounded by some $|A_k| \leq A$. Thus, we note that since $c_k - c_{k+1} \geq 0$ for all k, then

$$\begin{aligned}\left|\sum_{k=m+1}^{n} a_k c_k\right| &= \left|\sum_{k=1}^{n-1} A_k(c_k - c_{k+1}) + A_{n-1}c_n - (\sum_{k=1}^{m} A_k(c_k - c_{k+1}) + A_m c_{m+1})\right| \\ &= \left|\sum_{k=m+1}^{n-1} A_k(c_k - c_{k+1}) + A_{n-1}c_n + A_m c_{m+1}\right| \\ &\leq \left|\sum_{k=m+1}^{n-1} A(c_k - c_{k+1}) + Ac_n + Ac_{m+1}\right| \\ &\leq A\left(\sum_{k=m+1}^{n-1} (c_k - c_{k+1}) + c_n + c_{m+1}\right) \\ &= A\left(2c_{m+1}\right) \\ &< 2A\frac{\epsilon}{2A}.\end{aligned}$$

The last inequality comes from the fact that $c_n \to 0$. Suppose that b_n is monotone increasing and bounded. Thus, it converges to some b. Let $c_n = b - b_n$. Then we have that $c_n \to 0$ and

c_n is monotonically decreasing. Thus, by the work above, we have that

$$\sum_{k=1}^{n} a_k(b - b_k) = b\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_k b_k$$

converges. Since $b\sum_{k=1}^{n} a_k$ converges, then $\sum_{k=1}^{n} a_k b_k$ converges. Suppose b_n is monotone decreasing and bounded, then $b_n - b \to 0$ and $c_n = b_n - b$ is monotone decreasing. Thus, we can use the same strategy as above to show that $\sum a_k b_k$ converges.

<3 love you (thanks gaya)

Problem 17

Let E and F be two Banach spaces and let (T_n) be a sequence in $\mathcal{L}(E, F)$. Assume that for every $x \in E$, $T_n x$ converges as $n \to \infty$ to a limit denoted by Tx. Show that if $x_n \to x$ in E, then $T_n x_n \to Tx$ in F.

Solution: Since $T_n x \to Tx$ for every x, we have that there exists some N_1 such that if $n \geq N_1$, then

$$\|T_n x - Tx\| < \frac{\epsilon}{2}.$$

By the uniform boundedness theorem, we have that

$$\|T_n(x_n - x)\| \leq c\|x_n - x\|,$$

so there exists some N_2 such that if $n \geq N_2$, we have that because $x_n \to x$, then

$$\|x_n - x\| < \frac{\epsilon}{2c}.$$

Take the max between the N_1 and N_2, then for n larger,

$$\begin{aligned}
\|T_n x_n - Tx\| &\leq \|T_n x_n - T_n x\| + \|T_n x - Tx\| \\
&\leq \|T_n(x_n - x)\| + \|T_n x - Tx\| \\
&< c\|x_n - x\| + \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

■

Problem 18

Let E and F be two Banach spaces and let $a : E \times F \to \mathbb{R}$ be a bilinear form satisfying:

$$(i) \text{ for each fixed } x \in E, \text{ the map } y \mapsto a(x,y) \text{ is continuous;}$$
$$(ii) \text{ for each fixed } y \in F, \text{ the map } x \mapsto a(x,y) \text{ is continuous.}$$

Prove that there exists a constant $C \geq 0$ such that

$$|a(x,y)| \leq C\|x\|\|y\| \quad \forall x \in E, \quad \forall y \in F.$$

SOLUTION: We introduce new notation (because the author believes a is stupid) and let $T(x,y)$ be the bilinear map denoted by a in the problem. It suffices to show that

$$\sup_{\substack{\|x\|\leq 1 \\ \|y\|\leq 1}} |T(x,y)| = \|T\| < \infty,$$

since this shows that T is bounded in both arguments. Fix $x \in E$. We have that $\langle T, (x,y)\rangle = T_x(y)$ is continuous, that is:

$$\sup_{\|y\|\leq 1} |T_x| < \infty.$$

We will show that if $\overline{B_E}$ is the closed unit ball in E, then $y \mapsto T(\overline{B_E}, y)$ is bounded. Obviously, $T(\overline{B_E}, y) \subset F^*$. By corollary 2.5, it suffices to show that for every $y \in F$, the set

$$T_y(B_E)$$

is bounded, which comes from the fact that for fixed y, T is continuous, and thus

$$|T_y(B_E)| < \infty,$$

and so by the corollary $|T(B_E, y)| < \infty$. In particular, we have that for all $\|y\| \leq 1$, $|T(B_E, B_F)| < \infty$, which is equivalent to saying that

$$\|T\| < \infty.$$

■

Problem 19

Let $\alpha = (\alpha_n)$ be a given sequence of real numbers and let $1 \leq p \leq \infty$. Assume that $\sum |\alpha_n||x_n| < \infty$ for every element $x = (x_n)$ in ℓ^p (the space ℓ^p is defined in Section 11.3).

Prove that $\alpha \in \ell^{p'}$.

SOLUTION: Define

$$\langle \alpha_n x \rangle = \sum_{i=1}^{n} |\alpha_i||x_i|.$$

We see that by assumption, each α_n is a bounded linear map and that

$$\langle \alpha_n, x \rangle \to \langle \alpha, x \rangle = \sum_{i=1}^{\infty} |a_i||x_i| < \infty.$$

and so $\alpha \in \mathcal{L}(\ell^p, \mathbb{R})$. It remains to show that $\alpha \in \ell^{p'}$, that is, there exists some C such that

$$\|\alpha\|_{\ell^{p'}} = \left(\sum_{n=1}^{\infty} |\alpha_n|^{p'} \right)^{\frac{1}{p'}} \leq C.$$

By the uniform boundedness principle, there exists some c such that since

$$\|\alpha_n x\|_{\mathbb{R}} = |\alpha_n x| = \sum_{i=1}^{n} |\alpha_i||x_i| \leq c\|x\|_{\ell^p} = c \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}},$$

for all n, then

$$\sum_{i=1}^{\infty} |\alpha_i||x_i| \leq c\|x\|_{\ell^p},$$

and so α is a bounded linear map.

Choose x such that $x_i = \frac{|\alpha_i|^{p'}}{|a_i|}$. Thus, we find that

$$\sum_{i=1}^{\infty} |\alpha_i||x_i| = \sum_{i=1}^{\infty} |\alpha_i||\frac{|\alpha_i|^{p'}}{|a_i|}| = \sum_{i=1}^{\infty} |\alpha_i|^{p'} < \infty,$$

and thus $\alpha \in \ell^{p'}$

■

Problem 20

Let E and F be two Banach spaces and let $T \in \mathcal{L}(E, F)$ be surjective.

1. Let M be any subset of E. Prove that $T(M)$ is closed in F if and only if $M + N(T)$ is closed in E.

SOLUTION: Suppose $T(M)$ is closed in F. Let $(x_n) \in M + N(T)$, such that $x_n \to x$.

then $x_n = m_n + n_n$ [a] Thus, we have that $T(x_n) = T(m_n + T(n_n)) = T(m_n)$. Since $T(M)$ is closed we have that the graph of T is closed and so $(x_n, T(x_n)) \to (x, y)$. Thus $T(x_n) \to y$ and $y \in T(M)$ and thus $Tx = y$. In other words, we have that $M + N(T) = T^{-1}(T(M))$ is closed since by the closed graph theorem, T is continuous.

Suppose $M + N(T)$ is closed. Then we have that $E \setminus (M + N(T))$ is open. Thus, we have by the open mapping theorem that

$$T(E \setminus (M + N(T))) = T(E) \setminus T(M + N(T)) = F \setminus (T(M) + T(N(T))) = F \setminus T(M)$$

is open, and thus $T(M)$ is closed. ■

[a] I would be sorry about notation but then I remember how I am not sorry about notation

2. Deduce that if M is a closed vector space in E and $\dim N(T) < \infty$, then $T(M)$ is closed.

SOLUTION: It suffices to show that $M + N(T)$ is closed. Take a sequence $(x_n) \in M + N(T)$ such that $x_n = m_n + \eta_n$ where $(m_n) \in M$ and $(\eta_n) \in N(T)$ and $(x_n) \to x$. We want to show $m_n \to m \in M$, and that $\eta_n \to \eta \in N(T)$.

Suppose $M \cap N(T) = \{0\}$. Define $\eta'_n = \frac{\eta_n}{\|\eta_n\|}$. Then $\eta' = (\eta'_n)$ is in the closed unit ball in $N(T)$ since $\|\eta'\| = 1$, and so it has a convergent subsequence $\eta'_{n_k} \to \eta' \in N(T)$ since $N(T)$ is finite dimensional[a]. Suppose $\|\eta_{n_k}\| \to \infty$, then

$$m'_{n_k} = x'_{n_k} - \eta'_{n_k} \to -\eta'.$$

Since M is closed, we have that $-\eta' \in M$, and so $\eta' \in M$, and thus we must have that $\eta' = 0$, which is a contradiction to the fact that $\|\eta_{n_k}\| \to \infty$. Thus, by scaling, $\eta_{n_k} \to \eta \in N(T)$, and so

$$m_{n_k} = x_{n_k} - \eta_{n_0} \to x - \eta \in M,$$

and thus $x \in M + N(T)$.

For the general case, let α be the basis of $M \cap N(T)$. Let β be a basis of $N(T)$ such that $\alpha \setminus \{0\} \subset \beta$. We see that

$$M + N(T) = M + N(T) \setminus \{\text{span}(\alpha)\}$$

and that $M \cap N(T) \setminus \{\alpha\} = \{0\}$, and so $M + N(T) \setminus \{\text{span}(\alpha)\}$ is closed, and so $M + N(T)$ is closed. ■

[a] The closed unit ball is compact if and only if the space is finite dimensional is a big theorem from last year

Problem 21

Let E be a Banach space, $F = \ell^1$, and let $T \in \mathcal{L}(E, F)$ be surjective. Prove that there exists $S \in \mathcal{L}(F, E)$ such that $T \circ S = I_F$, i.e., S has a right inverse of T.

[**Hint:** Do not apply Theorem 2.12; try to define S explicitly using the canonical basis of ℓ^1.]

Solution: T is an open map by the open mapping theorem, and thus

$$cB_0^{\ell^1}(1) \subset T(B_0^E(1)).$$

Thus, if $(e_n) = (0, 0, \ldots, 1, 0, 0, \ldots)$, where 1 is in the nth spot is the canonical ℓ^1 basis, then since $e_n \in B_0^{\ell^1}(1)$, then there exists some $v_n \in B_0^E(\frac{1}{c})$ such that $e_n = T(v_n)$. Define $S : \ell^1 \to E$ such that if $y = (y_1, y_2, \ldots) \in \ell^1$, then

$$S(y) = \sum_{n=1}^{\infty} v_n y_n.$$

Then we have that

$$T(S(y)) = T \sum_{n=1}^{\infty} v_n y_n = \sum_{n=1}^{\infty} T(v_n) y_n = \sum_{n=1}^{\infty} e_n y_n = y.$$

We might need to justify the fact that linear transforms are linear under an infinite sum, but if we really wanted to, we would just define S_n as a limit series up to S and take limits then, which would be fine since S_n is continuous. ■

Problem 22

Let E be a Banach space. Let G and L be two closed subspaces of E. Assume that there exists a constant C such that

$$\operatorname{dist}(x, G \cap L) \leq C \operatorname{dist}(x, L), \quad \forall x \in G.$$

Prove that G + L is closed.

SOLUTION: (From answer key) Let $T : G \to E \setminus L$ such that $Tx = \pi x$, where $\pi : E \to E/L$ is the canonical surjection. Thus, we have that

$$\operatorname{dist}(x, \ker T) = \operatorname{dist}(x, G \cap L) \leq C\operatorname{dist}(x, L) = C\|Tx\|.$$

Thus, we have that $\pi(G)$ is closed, and so

$$\pi^{-1}(\pi(G)) = G \cap L$$

is closed. ■

Problem 23

Let $E = C([0,1])$ with its usual norm. Consider the operator $A : D(A) \subset E \to E$ defined by

$$D(A) = C^1([0,1]) \quad \text{and} \quad (Au)(t) = u'(t).$$

(a) Check that $\overline{D(A)} \neq E$.

SOLUTION: Obviously, we have that $D(A) \subset E$, and thus

$$\overline{D(A)} \subset \overline{E}.$$

Under the sup norm, we know that $C^1([]0,1])$ is closed. Thus, we have that

$$\overline{D(A)} \subset E.$$

Let $f \in E$. By the Stone-Weirstrass theorem, for all n, there exists a polynomial $P_n \in C^1([0,1])$ such that

$$\|P_n - f\| \leq \frac{1}{n},$$

and so $P_n \to f$, and thus $f \in LP(D(A))$, and so $f \in \overline{D(A)}$. Thus, we have that

$$E \subset \overline{D(A)}.$$

■

(b) Is A closed?

SOLUTION: Ee know that a linear operator T is closed if and only if its graph is closed. Thus, consider if $f_n \to f$ with $(f_n, f_n') \to (f, g)$. We have that

$$f_n = \int f_n' \to f = \int g \implies f' = g,$$

and so $g = Af$. To see that $f \in D(A)$, we see that g must be continuous since it is the uniform limit of continuous functions, and thus f' is continuous (and evidently f is differentiable). ■

(c) Consider the operator $B : D(B) \subset E \to E$ defined by

$$D(B) = C^2([0,1]) \quad \text{and} \quad (Bu)(t) = u'(t).$$

Is B closed?

SOLUTION: No, it is not. Let $(f_n) \in C^2([0,1])$ be a sequence of functions converging to $f(x) = x|x|$. Then $f_n' \to g$, where $g = |x| = f'$. However, f is obviously not in C^2, and so we are done. ■

Problem 24

Let E and F be two Banach spaces. Let $T \in \mathcal{L}(E, F)$ and let $A : D(A) \subset E \to F$ be an unbounded operator that is densely defined and closed. Consider the operator $B : D(B) \subset E \to F$ defined by

$$D(B) = D(A), \quad B = A + T.$$

(a) Prove that B is closed.

SOLUTION: Let $(x_n) \in D(B)$ such that $x_n \to x$ and $B(x_n) \to y$. First, we know that

$$B(x_n) = (A + T)(x_n) = A(x_n) + T(x_n).$$

Since $(x_n) \in D(B)$ and $D(A) = D(B)$, then $(x_n) \in D(A)$, and so A being closed implies that 1) $x \in D(A) = D(B)$ and that $A(x_n) \to A(x)$. Since T is continuous, we have that $T(x_n) \to T(x)$. Thus,

$$B(x_n) \to A(x) + T(x) = Bx,$$

and so we have proved that $y = Bx$ and that $x \in D(B)$. ■

(b) Prove that $D(B^\star) = D(A^\star)$ and $B^\star = A^\star + T^\star$.

SOLUTION: Let $c_A, c_B > 0$. By definition, we have that

$$D(A^*) = \{f \in F^* \ |\langle f, Ax\rangle| \leq c_A\|x\| \quad \forall x \in D(A)\}$$

$$D(B^*) = \{f \in F^* \ |\langle f, Bx\rangle| \leq c_B\|x\| \quad \forall x \in D(B)\}$$

Let $f \in D(A^*)$, then for all $x \in D(A) = D(B)$, we have that

$$|\langle f, Bx\rangle| = |\langle f, (A+T)x\rangle| = |\langle f, Ax+Tx\rangle| \leq |\langle f, Ax\rangle| + |\langle f, Tx\rangle|, \qquad \forall\, x \ni D(B) = D(A)$$

The first term is bounded by $c_A\|x\|$. As for the second, we know that T is a bounded linear operator and that $f \in F^*$,, and thus there is some $c_T, c_f > 0$ such that for all $x \in E$,

$$\|\langle f, Tx\rangle\| \leq \|f\|\|Tx\| \leq \|f\|c_T\|x\| \leq c_F c_T\|x\|.$$

Thus, we have that $f \in D(B^*)$.

Let $f \in D(B^*)$. Then there exists some c_B such that for all

$$|\langle f, Bx\rangle| \leq c_B\|x\|.$$

Thus, we have that

$$|\langle f, Bx\rangle| = |\langle f, Ax\rangle| + |\langle f, Tx\rangle| \leq c_B\|x\|.$$

In particular, we have that

$$\|\langle f, Ax\rangle\| \leq c_B\|x\|, \qquad \forall\, x \in D(A) = D(B).$$

and so $f \in D(A^*)$.

Thus, we have that $D(A^*) = D(B^*)$. To show that $B^* = A^* + T^*$, then since for all $x \in D(B)$, $f \in D(B^*)$

$$\langle f, Bx \rangle = \langle B^* f, x \rangle,$$

then by the previous parts, it will suffice to show that

$$\langle f, Bx \rangle = \langle (A^* + T^*) f, x \rangle.$$

We use bilinearity and the definition of the adjoints:

$$\langle (A^* + T^*) f, x \rangle = \langle A^* f + T^* f, x \rangle = \langle A^* f, x \rangle + \langle T^* f, x \rangle = \langle f, Ax \rangle + \langle f, Tx \rangle = \langle f, (A+T)x \rangle = \langle f, Bx \rangle.$$

If you have any issues with the forall $x \in D(A)$ or $x \in E$ definition of the adjoint, consider that $D(A) = D(B) \subset E$ and $D(A^*) = D(B^*) \subset F^*$. ■

Problem 25

The purpose of this exercise is to construct an unbounded operator $A : D(A) \subset E \to E$ that is densely defined, closed, and such that $\overline{D(A^\star)} \neq E^\star$.

Let $E = \ell^1$, so that $E^\star = \ell^\infty$. Consider the operator $A : D(A) \subset E \to E$ defined by

$$D(A) = \left\{u = (u_n) \in \ell^1;\ (nu_n) \in \ell^1\right\} \text{ and } Au = (nu_n).$$

1. Check that A is densely defined and closed.

SOLUTION: Let $x = (x_n) \in \ell^1$ and $\epsilon > 0$. Since $x \in \ell^1$, we have that

$$\sum_{n=1}^{\infty} x_n < \infty \implies \sum_{n=N_\epsilon}^{\infty} x_n < \epsilon$$

for some N_ϵ. Take

$$u = (u_n) = \begin{cases} x_n, & n \leq N \\ 0, & n > N \end{cases}.$$

Evidently, we have that $u \in \ell^1$ and $(nu_n) \in \ell^1$. Moreover, we have that

$$\sum_{n=1}^{\infty} |u_n - x_n| = \sum_{N+1}^{\infty} |x_n| < \epsilon,$$

and so we have that for any $B_\epsilon(x)$, there is some $u \in D(A)$ such that $u \in B_\epsilon(x)$, implying that $\overline{D(A)} = \ell^1$.

Let $(x_n) \to x$ with $(x_n) \in D(A)$ and suppose $A(x_n) \to y$. We want to show that $y = A(x)$, where $x \in D(A)$. For the latter, fix $\epsilon = 1$. Then we have that there exists some N such that $\|x_n - x\| < 1$, and so since $x_N \in \ell^1$:

$$\|x\| \leq \|x - x_N\| + \|x_N\|| < 1 + \|x_N\| < \infty.$$

Thus, $x \in \ell^1$. To show that $(nx_n) \in \ell^1$, we let $(kx_n^{(k)}) = x'_n$ and $(kx^{(k)}) = x'$. We have that $x'_n \in \ell'$ by assumption, and

$$\|x'\|_1 \leq \|x' - x'_n\| + \|x'_n\||,$$

so it remains to bound the first term. For all $\epsilon > 0$, there exists some $k > 0$ so that $\frac{1}{k2^k} < \epsilon$. Thus, for n large enough, we have that that for all k,

$$|x_n^{(k)} - x^{(k)}| \leq \|x_n - x\| < \frac{1}{k2^k}$$

and thus

$$\|x'_n - x'\| = \sum_{k=1}^{\infty} |kx_n^{(k)} - kx^k| = 1,$$

and so we are done, since this also shows that $(nx_n) = y$. ■

2. Determine $D(A^\star)$, $A^\star$, and $\overline{D(A^\star)}$.

Solution: We have that

$$D(A^*) = \{x \in \ell^\infty \; ; \; \langle x, Au \rangle \leq c\|u\|, \quad \forall\, u \in \ell^1\} = \{x \in \ell^\infty \; ; \; \langle x, (nu_n) \rangle \leq c\|u\|, \quad \forall\, u \in \ell^1\}$$
$$= \{x \in \ell^\infty \; ; \; (nx_n) \in \ell^\infty\}$$

Using the fact that if $x \in \ell^\infty$ and $u \in \ell^1$, then

$$\langle x, Au \rangle = \langle x, (nu_n) \rangle = \sum_{n=1}^{\infty} nx_n u_n = \langle A^* x, u \rangle. \tag{3}$$

And clearly, we define $A^* : D(A^*) \subset \ell^\infty \to \ell^\infty$ such that if $x \in \ell^\infty$, then

$$A^*(x) = (nx_n),$$

which satisfies (3). Since for $x \in D(A^*)$ we need that

$$\limsup_{n\to\infty} (nx_n) < \infty,$$

then we must have that $x_n \to 0$. Thus, we claim that $c_0 = \overline{D(A^*)}$. One inclusion is clear as previously stated. Let $y \in c_0$. Then we have that $y_n \to 0$ as $n \to \infty$. We claim that there exists some sequence $(x_n) \in D(A^*)$ such that $x_n \to y$. To see this, just let

$$x_1 = (0, y_2, y_3, \dots),$$

which is evidently in ℓ^∞ and so is (nx_n). Then we have that

$$\|x_n - y\| = \sup_n |y_n - x_n| = y_n \to 0.$$

■

Problem 27

Let $E = \ell^1$, so that $E^\star = \ell^\infty$. Consider the operator $T \in \mathcal{L}(E, E)$ defined by

$$Tu = \left(\frac{1}{n}u_n\right)_{n\geq 1} \quad \text{for every } u = (u_n)_{n\geq 1} \text{ in } \ell^1.$$

Determine $N(T)$, $N(T)^\perp$, $T^\star$, $R(T^\star)$, and $\overline{R(T^\star)}$. Compare with Corollary 2.18.

Solution: Obviously,

$$N(T) = \{0\},$$

since the only map to zero is the zero vector. We have that

$$N(T)^\perp = \{x \in \ell^\infty \; ; \; \langle y, x\rangle = 0, \quad \forall\, y \in N(T)\}.$$

Since $\{0\} = N(T)$, we have that for any $x \in \ell^\infty$,

$$\langle 0, x\rangle = 0 \implies N(T)^\perp = \ell^\infty.$$

Let $y \in D(T^*)$, (evidently, we have that

$$D(T^*) = \{y \in \ell^\infty \; ; \; (\frac{1}{n}y_n) \in \ell^1\}$$

), then

$$\begin{aligned}
\langle y, Tx\rangle &= \langle y, (\frac{1}{n}x_n)\rangle \\
&= \sum_{n=1}^{\infty} |y_n||\frac{1}{n}x_n| \\
&= \sum_{n=1}^{\infty} |\frac{1}{n}y_n||x_n| \\
&= \langle(\frac{1}{n}y_n), x\rangle \\
&= \langle T^*y, x\rangle,
\end{aligned}$$

an so $T^* : D(T^*) \subset \ell^\infty \to \ell^\infty$ is defined by

$$T^*(y) = (\frac{1}{n}y_n).$$

This is also the range, $R(T^*)$, all the sequences that can be expressed as $\frac{1}{n}y_n$.

It is clear that $\overline{R(T^*)} = c_0$ from similar work as in the previous problem. ■