207 Winter 2025: Problem Set 3
Honors Analysis in $\mathbb{R}^n$ II
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Problem 1

Prove Hölder's inequality. Suppose that $n \in \mathbb{N}$ and $\frac{1}{q} + \frac{1}{p} = 1$ with $1 \leq p < \infty$. Let $(a_k), (b_k) \in \mathbb{R}$ with $1 \leq k \leq n$, then

$$\sum_{k=1}^{\infty} [a_k b_k] \leq \left(\sum_{k=1}^{\infty} [a_k]^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} [b_k]^q \right)^{\frac{1}{q}}$$

Solution: Consider the degenerate case when

$$\sum_{k=1}^{n} [a_k]^p = \sum_{k=1}^{n} [a_k]^q = 1 \tag{1}$$

By properties of log, it is obvious that

$$\log(a_k b_k) = \frac{1}{p} \log([a_k]^p) + \frac{1}{q} \log([b_k]^q).$$

By the convexity of log (I talk more about this later), we have that

$$\frac{1}{p} \log([a_k]^p) + \frac{1}{q} \log([b_k]^q) \leq \log\left(\frac{1}{p} [a_k]^p + \frac{1}{q} [b_k]^q \right).$$

By monotonocity of log, we thus have that

$$a_k b_k \leq \frac{1}{p} [a_k]^p + \frac{1}{q} [b_k]^q \implies \sum_{k=1}^{n} [a_k b_k] \leq \sum_{k=1}^{n} \left[\frac{1}{p} [a_k]^p + \frac{1}{q} [b_k]^q \right] = \frac{1}{p} \sum_{k=1}^{n} [a_k]^p + \frac{1}{q} \sum_{k=1}^{n} [b_k]^q = 1.$$

Thus, we get the result that when (1) holds, we have that

$$\sum_{k=1}^{n} [a_k b_k] \leq 1. \tag{2}$$

Consider now any (a_k) and (b_k). Consider

$$c_k = \frac{a_k}{\left(\sum_{k=1}^{n} [a_k]^p\right)^{\frac{1}{p}}}, \implies \sum_{k=1}^{n} [c_k]^p = \frac{\sum_{k=1}^{n} [a_k]^p}{\sum_{k=1}^{n} [a_k]} = 1.$$

$$d_k = \frac{b_k}{\left(\sum_{k=1}^{n} [b_k]^q\right)^{\frac{1}{q}}} \implies \sum_{k=1}^{n} [b_k]^q = 1.$$

Thus, we see that c_k and d_k satisfy (1) and thus use (2), which yields

$$\sum_{k=1}^{n}[c_k d_k] = \sum_{k=1}^{n}\left[\frac{a_k}{(\sum_{k=1}^{n}[a_k]^p)^{\frac{1}{p}}}\right]\left[\frac{b_k}{(\sum_{k=1}^{n}[b_k]^q)^{\frac{1}{q}}}\right] \leq 1.$$

Multipying by the denominator (and taking $n \to \infty$) yields Hölder's Inequality.

For the case when $p = \infty$, we will show that

$$\lim_{p\to\infty} \|a\|_p = \|a\|_\infty = A$$

Consider that for any n, we have that by taking limits

$$\sum_{k=1}^{\infty} |a_k|^p \geq |a_n|^p \implies \|a\|_\infty \leq \|a\|_p.$$

On the other hand, we have that since $a \in \ell^p$, then

$$\sum_{k=1}^{\infty} |a_k|^p < \infty,$$

Thus, there must exist some C such that

$$\sum_{k=1}^{\infty} |\frac{a_k}{A}|^p \leq C \implies \left(\sum_{k=1}^{\infty} |\frac{a_k}{A}|^p\right)^{\frac{1}{p}} = C^{\frac{1}{p}}.$$

By homogeneity, we have that

$$\|a\|_p \leq C^{\frac{1}{p}}\|a\|_\infty.$$

Since for large p, we have that $|C^{\frac{1}{p}} - 1| < \epsilon$, then

$$\|a\|_p \leq C^{\frac{1}{p}}\|a\|_\infty \to \|a\|_\infty$$

Thus, as $p \to \infty$, we have that $\|a\|_p = \|a\|_\infty$. Moreover, as $p \to \infty$, we have that $q \to 1$, and thus

$$\left(\sum_{k=1}^{\infty} |b_k|^q\right)^{\frac{1}{q}} = \sum_{k=1}^{\infty} |b_k|.$$

so it suffices to show that

$$\sum_{k=1}^{n} a_k b_k \leq \|a\|_\infty \|b\|_1,$$

which is obvious, since

$$a_k b_k \leq \|a\|_\infty b_k \implies \sum_{k=1}^{\infty} a_k b_k \leq \|a\|_\infty \sum_{k=1}^{\infty} b_k.$$

■

REFLECTIONS: Here we use the fact that if f is convex, then for all x, y and for all $0 < \alpha < 1$, we have that

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y).$$

To see that $f(x) = \log(x)$ is convex, take the second derivative:

$$f'(x) = \frac{1}{x}, \qquad f''(x) = \frac{-1}{x^2} < 0 \; \forall x \in \mathbb{R} \setminus 0$$

Problem 2

Prove Minkowski's Inequality. Suppose that $(a_k), (b_k) \in \mathbb{R}$ and that $n \in \mathbb{N}$ and $1 \leq k \leq n$. If $1 \leq p < \infty$, then

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}}$$

Solution: By the triangle inequality, we have that

$$|a_k + b_k|^p = |a_k + b_k||a_k + b_k|^{p-1} \leq |a_k||a_k + b_k|^{p-1} + |b_k||a_k + b_k|^{p-1},$$

and thus

$$\sum_{k=1}^{n} |a_k + b_k|^p \leq \sum_{k=1}^{n} |a_k||a_k + b_k|^{p-1} + \sum_{k=1}^{n} |b_k||a_k + b_k|^{p-1}. \tag{3}$$

Applying Hölder's inequality to both terms using $\frac{1}{p}$ and $\frac{p-1}{p}$:

$$\sum_{k=1}^{n} |a_k||a_k + b_k|^{p-1} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{\frac{p-1}{p}}$$

$$\sum_{k=1}^{n} |b_k||a_k + b_k|^{p-1} \leq \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{\frac{p-1}{p}}.$$

Putting those back into (3):

$$\begin{aligned}\sum_{k=1}^{n} |a_k + b_k|^p &\leq \left(\left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}}\right) \left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{\frac{p-1}{p}} \\ &= \left(\left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}}\right) \frac{\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)}{\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{\frac{1}{p}}}\end{aligned}$$

Multiplying both sides by

$$\left(\frac{\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)}{\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{\frac{1}{p}}}\right)^{-1}$$

yields Minkowski's inequality.

For the case when $p = \infty$, we have seen that

$$\|a + b\|_p = \|a + b\|_\infty,$$

and thus since (we will prove this in a later problem)

$$\sup_n |a_n + b_n| \leq \sup_n |a_n| + \sup_n |b_n|,$$

then

$$\|a+b\|_\infty \leq \|a\|_\infty + \|b\|_\infty,$$

and so the inequality does hold. ■

Problem 3

Show that ℓ^p is a Banach Space with the $p-$norm from above when $p \in [1, \infty)$ and when $p = \infty$, show it is a Banach space with $\|a\|_\infty = \sup_n |a_n|$

Solution: For the following, let $a = (a_n) = (a_1, a_2, \dots) \in \ell^p$. Starting when $p \in [1, \infty)$, we have that if $a_n \neq (0, 0, 0, \dots)$, then there exists some $a_i \in (a_n)$ such that $|a_i| > 0$, and thus

$$\|a_n\| = \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{\frac{1}{p}} > 0.$$

By the absolute value, we obviously have that $\|a_n\| = 0$ if and only if $a_n = (0, 0, \dots)$.

Let $\lambda \in \mathbb{R}$.

$$\|\lambda a\| = \left(\sum_{n=1}^{\infty} |\lambda a_n|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |\lambda|^p |a_n|^p\right)^{\frac{1}{p}} = \left(|\lambda|^p \sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} = |\lambda| \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} = |\lambda| \|a\|$$

Using problem 2 (Minkowski's inequality), we have inmediately that

$$\|a + b\| = \left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}} = \|a\| + \|b\|$$

To show it is complete, take a Cauchy sequence $(a_n) = (a_n^{(i)})_{n \in \mathbb{N}} \in \ell^p$. That is,

$$(a_n^{(i)})_{n \in \mathbb{N}} = (a_1^{(i)}, a_2^{(i)}, \dots)_{i \in \mathbb{N}}$$

such that for all $\epsilon > 0$, there exists an N such that if $n, m \geq N$, then

$$\|a_n - a_m\| = \left(\sum_{i=1}^{\infty} |a_n^{(i)} - a_m^{(i)}|^p\right)^{\frac{1}{p}} < \epsilon$$

We claim that for each fixed i, $(a_n^{(i)})$ converges to some $a^{(i)}$. To see this, we claim that $(a_n^{(i)})$ is/are Cauchy. Take $i = 1$, then for any $n, m \geq N$, we have that

$$\left(\sum_{i=1}^{\infty} |a_n^{(i)} - a_m^{(i)}|^p\right)^{\frac{1}{p}} = \left(|a_n^{(1)} - a_m^{(1)}|^p + \sum_{i=2}^{\infty} |a_n^{(i)} - a_m^{(i)}|^p\right)^{\frac{1}{p}} < \epsilon,$$

and thus

$$|a_n^{(1)} - a_m^{(1)}| < \left(\epsilon^p - \sum_{i=2}^{n} |a_n^{(i)} - a_m^{(i)}|^p\right)^{\frac{1}{p}} \to 0.$$

Thus, we have that for $n, m \geq N$ (notice how this choice of has not at any point depended on our choice of $i = 1$), then

$$|a_n^{(1)} - a_m^{(1)}| < \epsilon,$$

and thus $(a_n^{(1)})$ is a Cauchy sequence of reals. Thus, it converges, $(a_{n,1}) \to a^{(1)}$. Because there was nothing special about $i = 1$, we have that for all $i \in \mathbb{N}$, there exists a limit $a^{(i)} \in \mathbb{R}$ such that for each fixed i,

$$a_n^{(i)} \to a^{(i)}.$$

Let $(A_n) = (a^{(1)}, a^{(2)}, \dots) = (a^{(i)})_{i\in\mathbb{N}}$. We claim that $(a_n^{(i)})_{n\in\mathbb{N}} \to (A_n)$. Let $\epsilon > 0$. Since each $a_n^{(i)} \to a^{(i)}$ (in a 'uniform' sense, as we have a single N controlling the convergence of all i, as seen by the indifference of choosing N in the $i = 1$ calculation above) we let $n \geq N$ such that

$$|a_{n,i} - a^{(i)}| < \frac{\epsilon}{2^{\frac{i}{p}}},$$

then

$$\begin{aligned}
\|(a_n) - A_n\| &= \left(\sum_{i=1}^{\infty} |a_{n,i} - a^{(i)}|^p\right)^{\frac{1}{p}} \\
&< \left(\sum_{i=1}^{\infty} \frac{\epsilon^p}{2^i}\right)^{\frac{1}{p}} \\
&= (\epsilon^p)^{\frac{1}{p}} \\
&= \epsilon.
\end{aligned}$$

In the case when $p = \infty$, I think that the positive definiteness of $\|a\|$ is obvious.

Homogeneity is also kinda obvious, noting that

$$\|\lambda a\| = \sup_{n\in\mathbb{N}} |\lambda a_n| = |\lambda| \sup_{n\in\mathbb{N}} |a_n| = \lambda \|a\|$$

Triangle inequality follows from triangle inequality since for any n :

$$|a_n + b_n| \leq |a_n| + |b_n| \implies \sup_{n\in\mathbb{N}} |a_n + b_n| \leq |a_n| + |b_n| \leq \sup_{n\in\mathbb{N}} |a_n| + \sup_{n\in\mathbb{N}} |b_n|.$$

Thus, we have that

$$\|a + b\| = \sup_{n\in\mathbb{N}} |a_n + b_n| \leq \sup_{n\in\mathbb{N}} |a_n| + \sup_{n\in\mathbb{N}} |b_n| = \|a\| + \|b\|$$

Let $(a_n) = (a_n^{(i)})_{n\in\mathbb{N}} = (a_{1,i}, a_{2,i}, \dots) \in \ell^\infty$ be Cauchy. That is, for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n, m \geq N$, then

$$\|a_n^{(i)} - a_m^{(i)}\| = \sup_{i\in\mathbb{N}} |a_n^{(i)} - a_m^{(i)}| < \epsilon.$$

Fix i. Then we have that there exists an $N \in \mathbb{N}$ such that if $n, m \geq N$, then

$$|a_n^{(i)} - a_m^{(i)}| \leq \sup_{i \in \mathbb{N}} |a_n^{(i)} - a_m^{(i)}| < \epsilon,$$

which implies that for fixed i, $(a_n^{(i)})_{n \in \mathbb{N}}$ is Cauchy sequence of reals, and thus converges to something. For each i, let

$$a_n^{(i)} \to a^{(i)}.$$

Then we have that if $(A_n) = (a^{(1)}, a^{(2)}, \dots)$, then for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$|a_n^{(i)} - a^{(i)}| < \epsilon \qquad \forall \epsilon \implies \|a_n^{(i)} - A_n\| = \sup_{i \in \mathbb{N}} |a_{n,i} - a^{(i)}| \leq \epsilon.$$

It suffices to show that $(A_n) \in \ell^p$. To see this, it will suffice to see that $\|(A_n)\|_p < \infty$ for any $p \in [1, \infty]$. Using Minkowski's inequality (which we have seen in valid for all $p \geq 1$,) we see that for large enough n,

$$\|(A_n)\|_p \leq \|(A_n) - a_n^{(i)}\|_p + \|a_n^{(i)}\|_p < \epsilon + \|a_n^{(i)}\|_p.$$

Since for each n, we have that $a_n^{(i)} \in \ell^p$, then $\|a_n^{(i)}\|_p < \infty$, and thus $\|(A_n))\|_p < \infty$, and thus $(A_n) \in \ell^p$. $\blacksquare$

Problem 4

Prove that $c_0 = \{(a_n)_{n\in\mathbb{N}} \in l^\infty : \lim_{n\to\infty} a_n = 0\}$ is a Banach space with norm $\|a\|_\infty$.

Solution: We claim that it suffices to show that c_0 is closed. Why? Let $W \subset V$, where V is Banach and W is closed. The $(x_n) \in W$ be Cauchy, then it inherits from the subspace vector space to the super space Banach space, and thus (x_n) is Cauchy in W, implying that (x_n) converges to some $x_n \to x$. Since W is closed, we have that $x \in W$, and thus W is Banach.

Let $(a_n^i)_{n\in\mathbb{N}}^{i\in\mathbb{N}} = (a_1^i, a_2^2, \dots) \in c_0$ such that $(a_n^i) \to a^i = (a^{(1)}, a^{(2)}, \dots)$. We want to show that $a^i \in c_0$, so it suffices that $\lim\limits_{i\to\infty} a^i = 0$. Since $(a_n^i) \to a^i$, we have that there exists an $N_1 \in \mathbb{N}$ such that if $m \geq N_1$, then

$$\|a_m^i - a^i\| < \frac{\epsilon}{2}.$$

Since $(a_n^i) \in c_0$, then there exists an N_2 such that if $m \geq N_2$, then

$$\|a_m^i\| < \frac{\epsilon}{2}.$$

Thus, take $N = \max\{N_1, N_2\}$, and we have that if $m \geq N$, then

$$\|a^i\| \leq \|a^i - a_m^i\| + |a_m^i| < \epsilon$$

Thus, $a^i \in c_0$ and so c_0 is closed and so c_0 is Banach. ■

Problem 5

Let $p \in [1, \infty)$ and $1/p + 1/q = 1$. Show that, for any $b \in l^q$, the map $F_b : l^p \to \mathbb{R}$ given by

$$F_b(a) = \sum_{k=1}^{\infty} a_k b_k$$

is a bounded linear functional on ℓ^p.

Solution: For the following, we let $b \in \ell^q$ be arbitrary, noting that

$$\|b\|_q = C < \infty$$

by definition of it being the the ℓ^q space.

Evidently, F_b is a functional. To show it is linear, let $a, c \in \ell^p$ and $\lambda \in \mathbb{R}$. We use the linearity of the sum:

$$F_b(\lambda a + c) = \sum_{k=1}^{\infty} (\lambda a_k + c_k) b_k = \sum_{k=1}^{\infty} \lambda a_k b_k + c_k b_k = \lambda \sum_{k=1}^{\infty} a_k b_k + \sum_{k=1}^{\infty} c_k b_k = \lambda \lambda F_b(a) + F_b(c).$$

TO show it is bounded, we use Hölder's inequality. Let $a \in \ell^p$, then

$$\begin{aligned} |F_b(a)| &= |\sum_{k=1}^{\infty} a_k b_k| \\ &\leq \sum_{k=1}^{\infty} |a_k b_k| \\ &\leq \left(\sum [a_k]^p\right)^{\frac{1}{p}} \left(\sum [b_k]^q\right)^{\frac{1}{q}} \\ &= \|a\|_p \|b\|_q \\ &= C\|a\|_p \end{aligned}$$

■