270 Spring 2025: Problem Set 5 Agustín Esteva
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Problem 1

Prove that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Solution: If we let

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx,$$

then if we let $r = x^2 + y^2$ and $\theta = \tan\frac{y}{x}$ we get the change of variables

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dxdy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\,dr\,d\theta \\
&= \pi \int_{0}^{\infty} e^{-u}\,du \\
&= \pi \Big[- e^{-u} + e^{-u} \Big]_0^{\infty} \\
&= \pi,
\end{aligned}$$

and so $I = \sqrt{\pi}$

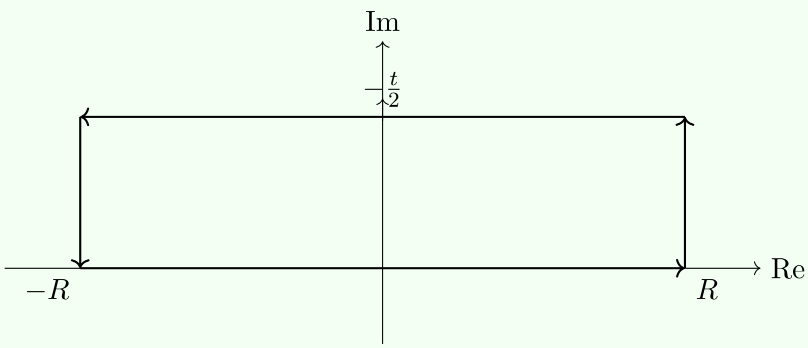
$\blacksquare$

Problem 2

Let $t \in \mathbb{R}$, what is
$$\int_{-\infty}^{\infty} e^{-x^2} e^{itx}\, dx.$$

Solution: We compute:
$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2} e^{itx}\, dx &= \int_{-\infty}^{\infty} e^{-x^2+itx}\, dx \\
&= \int_{-\infty}^{\infty} e^{-(x^2-itx)}\, dx \\
&= \int_{-\infty}^{\infty} e^{-(x^2-itx-\frac{t^2}{4})-\frac{t^2}{4}}\, dx \\
&= \int_{-\infty}^{\infty} e^{-(x-\frac{it}{2})^2-\frac{t^2}{4}}\, dx \\
&= e^{-\frac{t^2}{4}} \int_{-\infty}^{\infty} e^{-(x-\frac{it}{2})^2}\, dx
\end{aligned}$$

Let γ_R be the the following path:



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Since e^{-z^2} is holomorphic and γ_R is closed, we know that
$$\int_{\gamma_R} e^{-z^2}\, dz = 0.$$

Moreover, if let let $U_\gamma, D_\gamma, L_\gamma, R_\gamma$ be the obvious choices for the segments of the paths, we also have that
$$\begin{aligned}
0 &= \int_{\gamma_R} e^{-z^2}\, dz \\
&= \int_{U_\gamma} e^{-z^2}\, dz + \int_{L_\gamma} e^{-(x-\frac{t}{2})^2}\, dx + \int_{D_\gamma} e^{-z^2}\, dz + \int_{R_\gamma} e^{-x^2}\, dx \\
&= \int_0^{-\frac{t}{2}} e^{-(R+y)^2}\, dy - \int_{-R}^{R} e^{-(x-\frac{t}{2})^2}\, dx + \int_{-\frac{t}{2}}^{0} e^{-(-R+y)^2}\, dy + \int_{-R}^{R} e^{-x^2}\, dx
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{-R}^{R} e^{-(x-\frac{t}{2})^2}\,dx + \int_{-R}^{R} e^{-x^2}\,dx \\
&= -\int_{-R}^{R} e^{-(x-\frac{t}{2})^2}\,dx + \sqrt{\pi}
\end{aligned}
$$

Thus, our integral computes to

$$\boxed{\sqrt{\pi}e^{-\frac{t^2}{4}}}$$

■

Problem 3

Let $n \in \mathbb{N}$. Show that there are exactly n nth roots of unity. What are they?

SOLUTION: As a consequence of the fundamental theorem of algebra, the polynomial $P(z) = z^n - 1$ has exactly n roots. Thus, there exist n different $z \in \mathbb{C}$ such that $z^n = 1$. To characterize such z, we have that

$$z^n = 1 \iff n\text{Log}(z) = 0 \iff e^{n\text{Log}(z)} = 1 \iff n\text{Log}(z) = 2\pi ik \geq 0 \iff z = e^{\frac{2\pi ik}{n}}.$$

For only $k \in \{0, 1, 2, \ldots, n-1,\}$ we have by periodicity that $\frac{2\pi ik}{n}$ takes on distinct values and so the nth roots of unity are given by

$$\boxed{\{e^0, e^{\frac{2\pi i}{n}}, \ldots, e^{\frac{2\pi i(n-1)}{n}}\}}$$

■

Problem 4

Find the following residues:

(a) Let $a, b \in \mathbb{C}$ with $a \neq b$, what is

$$\mathrm{Res}_a \left(\frac{1}{z-a} \frac{1}{z-b} \right).$$

Solution: We calculate the power series of $g(z) = \frac{1}{z-b}$ around a to be

$$g(z) = \frac{1}{a-b} + \sum_{n=1}^{\infty} a_n (z-a)^n$$

$$f(z) = \frac{1}{z-a} g(z) = \frac{1}{z-a} \left(\frac{1}{a-b} + \sum_{n=1}^{\infty} a_n (a-z)^n \right),$$

and so

$$a_{-1} = \boxed{\frac{1}{a-b}},$$

which is the residue. ■

(b)

$$\mathrm{Res}_i \left(\frac{1}{z} e^{iz} \right)$$

Solution: The expansion of f around i has no negative powers since $f(z)$ is holomorphic in a disk not containing $z = 0$, and so $f(z)$ is analytic around $z = i$, implying that

$$a_{-1} = \boxed{0}$$

is our residue. ■

(c)

$$\mathrm{Res}_0 \left(\frac{1}{z} e^{iz} \right)$$

Solution: We have that

$$\frac{1}{z} e^{iz} = \frac{1}{z} \left(1 + \frac{1}{1!}(iz) + \frac{1}{2!}(iz)^2 + \cdots + \right),$$

and so

$$a_{-1} = \boxed{1}$$

which is the residue. ■

(d)

$$\text{Res}_1(\frac{1}{z^3-1}$$

Solution: We can factor and consider expanding about $z = 1$

$$f(z) = \frac{1}{z^3-1} = \frac{1}{z-1}\frac{1}{z^2+1+z} =: \frac{1}{z-1}g(z) = \frac{1}{z-1}(b_0 + b_1 z + \cdots)$$

where the last equality comes due to $g(z)$ being perfectly analytic/holomorphic about $z = 1$. Thus, we look for

$$b_0 = g(1) = \boxed{\frac{1}{3}},$$

which is a_{-1} in the Laurent series, and thus our residual. ■

Problem 5

Are complex polynomials dense in the set of continuous complex functions $f(z) : \overline{D_1(z)} \to \mathbb{C}$?

SOLUTION: **No**, we need the added assumption that f is holomorphic.

Suppose we could approximate $f(z)$ though! Then

$$P_n \rightrightarrows f$$

for polynomials P_n. Let $K \subseteq \overline{D_1(z)}$ be a compact subset. It should be clear that the convergence is uniform on K as well. Since $P_n \in H(\overline{D_1(z)})$ for all n, then by Problem 4 on PSET 3, $f \in H(\overline{D_1(z)})$. Thus, it suffices to show there exists a function merely continuous on the unit disk. Take

$$f(z) = \overline{z}.$$

We have shown that f is not differentiable anywhere. To check continuity, note that for $|z - w| < \epsilon$

$$|f(z) - f(w)| = |\overline{z} - \overline{w}| = |\overline{z - w}| = |z - w| < \epsilon$$

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