bTT

270 Spring 2025: Problem Set 1
Complex Variables
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Problem 1

A function $f : \mathbb{C} \to \mathbb{C}$ is said to satisfy the mean value property if and only for every $z_0 \in \mathbb{C}$ and $r > 0$, we have that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\cos(\theta) + ir\sin(\theta))d\theta.$$

Show that if $f(z) = z^2$, then f satisfies the mean value property.

SOLUTION: Note that since $f(z) = z^2$, we have that for any $z_0 \in \mathbb{C}$,

$$\begin{aligned} f(z_0 + r\cos(\theta) + ir\sin(\theta)) &= f(z_0 + re^{i\theta}) \\ &= z_0^2 + 2z_0 e^{i\theta} + r^2 e^{2i\theta} \end{aligned}$$

Lemma 1. For any $n \geq 0$, we have that

$$e^{i2\pi n} = 1$$

Proof.

$$e^{i2\pi n} = \cos(2\pi n) + i\sin(2\pi n) = 1$$

□

Evaluating the integral:

$$\begin{aligned} \int_0^{2\pi} (z_0^2 + 2z_0 e^{i\theta} + r^2 e^{2i\theta})d\theta &= z_0^2 \int_0^{2\pi} d\theta + 2z_0 \int_0^{2\pi} e^{i\theta} d\theta + r^2 \int_0^{2\pi} e^{2i\theta} d\theta \\ &= 2\pi z_0^2 + 2z_0 \frac{1}{i}[e^{i2\pi} - 1] + r^2 \frac{1}{2i}[e^{4i\pi} - 1] \\ &= 2\pi z_0^2 + 2z_0 \frac{1}{i}[1 - 1] + r^2 \frac{1}{2i}[1 - 1] \\ &= 2\pi z_0^2 \end{aligned}$$

Thus, dividing by 2π yields the result. ■

Problem 2

Let $f(z) = z^3$, and let

$$u(x,y) = \mathrm{Re}\{f(x+iy)\}, \quad v(x,y) = \mathrm{Im}\{f(x+iy)\}.$$

Prove that, for all $x, y \in \mathbb{R}$,

$$\triangle u(x,y) = \triangle v(x,y) = 0,$$

where $\triangle$ is the Laplacian operator

$$\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

SOLUTION: Let $x, y \in \mathbb{R}$, then

$$\begin{aligned} f(x+iy) &= (x+iy)^3 \\ &= (x^2 + 2ixy - y^2)(x+iy) \\ &= x^3 + 2ix^2y - xy^2 + ix^2y - 2xy^2 - iy^3 \\ &= x^3 - 3xy^2 + i(3x^2y - y^3) \end{aligned}$$

We then have that

$$u(x,y) = x^3 - 3xy^2, \qquad v(x,y) = 3x^2y - y^3.$$

Computing the Laplacian:

$$\begin{aligned} \triangle u(x,y) &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &= 6x + (-6x) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \triangle v(x,y) &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \\ &= 6y + (-6y) \\ &= 0 \end{aligned}$$

■

Problem 3

Suppose $\Omega \subset \mathbb{C}$ is an open connected set (a region) and $u : \Omega \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

Prove that u must be constant on Ω.

SOLUTION: We assume that $\Omega \neq \emptyset$ as otherwise the statement is vacuously true. Let $z_0 \in \Omega$.

Since Ω is connected, then it is polygonally connected. Let $z \in \Omega$. There exists some polygonal path γ that connects z and z_0. Since $\frac{\partial u}{\partial x} : \mathrm{Re}\{\Omega\} \to \mathbb{R}$ is constantly zero, then it is continuous. Similarly, $\frac{\partial u}{\partial y} : \mathrm{Im}\{\Omega\} \to \mathbb{R}$ is continuous. Thus,

$$D[u(z)] = \begin{bmatrix} \frac{\partial u(z)}{\partial x} & \frac{\partial u(z)}{\partial y} \end{bmatrix} : \Omega \to \mathbb{R}$$

(the total derivative) is continuous on Ω. Note that Du has a primitive, u on Ω, since $u' = Du$. Thus, we have that

$$\int_\gamma D[u(z)]dz = u(z_0) - u(z).$$

But we have that $D[u(z)] = 0$ along any path in Ω since all the components are 0, and thus the path integral along γ must be 0 and so

$$u(z_0) = u(z).$$

Because z was arbitrary, then u must be constant on Ω. ■

Problem 4

Suppose that $u : \overline{D_r(z_0)} \subseteq \mathbb{C} \to \mathbb{R}$ is a continuous function which attains its maximum at z_0. Suppose further that u satisfies

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + r\cos\theta + ir\sin\theta)d\theta.$$

Prove that if $z \in \overline{D_r(z_0)}$ such that $|z - z_0| = r$, then $u(z) = u(z_0)$.

SOLUTION:

Lemma 2. Let $S_r(z_0)$ denote the sphere of radius r around z_0. We claim that

$$S_r(z_0) = \{z_0 + re^{i\theta} \mid \theta \in [0, 2\pi)\}$$

Proof. Let $z \in S_r(z_0)$. Without much loss in generality from a translation, we can assume that $z \in S_r(0)$. Then $z = x+iy$. Letting $\theta = \tan^{-1}(\frac{y}{x})$, we see that $x = r\cos\theta$ and $y = r\sin\theta$. Thus, $x + iy = re^{i\theta}$.

Let $z \in \{z_0 + re^{i\theta} \mid \theta \in [0, 2\pi)\}$, then there exists some $\theta_0 \in [0, 2\pi)$ such that $z = z_0 + re^{i\theta}$. Thus,

$$|z - z_0| = |re^{i\theta}| = r|e^{i\theta}| = r,$$

and so $z_0 \in S_r(z_0)$. □

Suppose not. That is, there is some $z' \in S_r(z_0)$, such that $u(z') < u(z_0)$. Let $\epsilon = u(z_0) - u(z')$.

Then by continuity of u, there exists some $\delta > 0$ such that if $z \in \overline{D_r(z_0)}$ with $|z - z'| < \delta$, then $0 \leq |u(z) - u(z')| < \frac{\epsilon}{2}$. Take $X = S_r(z_0) \cap (z' - \delta, z' + \delta)$. That is, X are the points on the circle which are less than $u(z_0)$ by at least $\frac{\epsilon}{2}$. That is

$$u(z) < u(z_0) - \frac{\epsilon}{2} \quad \forall z \in X. \tag{1}$$

Then using (1) in the fourth line and Lemma 2 in the third line, we see that if $|X|$ denotes the area of X, then

$$\begin{aligned}
u(z_0) &= \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta \\
&= \frac{1}{2\pi}\int_{[0,2\pi]\setminus X} u(z_0 + re^{i\theta})d\theta + \frac{1}{2\pi}\int_X u(z_0 + re^{i\theta})d\theta \\
&= \frac{1}{2\pi}\int_{[0,2\pi]\setminus X} u(z_0 + re^{i\theta})d\theta + \frac{1}{2\pi}\int_X u(z)d\theta \\
&< \frac{1}{2\pi}\int_{[0,2\pi]\setminus X} u(z_0 + re^{i\theta})d\theta + \frac{1}{2\pi}\int_X (u(z_0) - \frac{\epsilon}{2})d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta - \frac{1}{2\pi}\int_X \frac{\epsilon}{2}d\theta
\end{aligned}$$

$$= u(z_0) - \frac{1}{2\pi}\epsilon|X|$$

But then $u(z_0) < u(z_0)$, a contradiction! Thus, we must have that $u(z) \geq u(z_0)$ for all $z \in S_r(z_0)$. However, since z_0 is a maximum on the disk, we have that $u(z) = u(z_0)$.

■

Problem 5

Suppose $u : \overline{\Omega} \to \mathbb{R}$ is a continuous function on the closure of a bounded region $\Omega \subset \mathbb{C}$. Suppose that u has the mean value property in Ω. That is, whenever $\overline{D_r(z_0)} \subseteq \Omega$,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

Prove that if u takes a maximum value inside of Ω, then u must be constant.

Solution: Since $\overline{\Omega}$ is bounded and closed, then it is compact. Since u is a continuous function over $\overline{\Omega}$, then it achieves its maximum at some $z_0 \in \overline{\Omega}$. Suppose that $z_0 \in \Omega$. Define

$$A := \{z \in \Omega \mid u(z) = u(z_0)\}.$$

We aim to show that A is clopen, since Ω is a region, then it is an open connected subset, and so the only clopen subsets it contains are itself and the emptyset. But since $z_0 \in A$, then $A \neq \emptyset$, and so if we show that A is clopen, then it must necessarily be Ω.

Let $z \in A$ and $\epsilon > 0$. By continuity of u, there exists some $\delta > 0$ such that if $|z - z'| < \delta$, then $|u(z) - u(z')| < \epsilon$. Since Ω is open, then there exists some $r' > 0$ such that if $|z - z'| < r'$, then $z' \in \Omega$. Take $R = \min\{\delta, r'\}$. Let $z' \in D_R(z)$. There is some $r > 0$ with $r \leq R$ such that $z' \in S_r(z')$. By the previous problem, since $u|_{D_r(z)} : D_r(z) \subseteq \overline{\Omega} \to \mathbb{R}$ attains its maximum at z and $|z' - z| = r$ and

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta,$$

then $u(z) = u(z')$. Therefore, $z' \in A$ and so we have found an $R > 0$ such that if $z' \in D_R(z)$, then $z' \in A$. Then A is open.

Let $(z_n) \in A$ with $z_n \to z$. We claim that $z \in A$. Since u is continuous, then $u(z_n) \to u(z)$, but since $u(z_n) = u(z_0)$ for all n since each $z_n \in A$, then $u(z) = u(z_0)$, and thus $z \in A$. Then we have that A is closed.

Thus, since A is both closed and open and since $A \neq \emptyset$, then $A = \Omega$. Thus, for all $z \in \Omega$, $u(z) = u(z_0)$. It remains to see that u is constant on $\partial\Omega$.

Let $z \in \partial\Omega$, then there exists some $(z_n) \in \Omega$ such that $z_n \to z$. By continuity of u, we have that $u(z_n) \to u(z)$, but since u is constant in Ω, we get that $u(z_n) = u(z_0)$, and thus $u(z) = u(z_0)$, and so u is indeed constant on the boundary as well. ■