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Problem 1

Suppose $f : E \to \mathbb{R}$ is a linear functional with $\ker f$ is closed and E is a topological vector space. Then $\{x \in E \ : \ f(x) < 0\}$ is open and

$$\overline{\{x \in E \ : \ f(x) < 0\}} = \{x \in E \ : \ f(x) \leq 0\}$$

SOLUTION: If U is an open neighborhood of 0, then there exists some open $V \subset U$ such that

$$\lambda V \subset V, \qquad \forall\, |\lambda| < 1 \qquad (\star)$$

Since $\ker f$ is closed and $f \neq 0$, there exists some x and open $U \ni 0$ such that $x+U \subset (\ker f)^c$. Since U is an open neighborhood of 0, we can find some $V \subset U$ such that $\star$ above is satisfied. We claim that $f(V)$ is bounded. Suppose not, then we will show that $f(V) = \mathbb{R}$. There exists some $\{x_n\} \subset V$ such that $|f(x_n)| > n$ for all n. For $\lambda \in [0, 1]$, we have that $\lambda V \subseteq V$, and so $\lambda f(V) \subseteq f(V)$. If $c \in f(V)$, then for all $t \in [-c, c]$ $t \in f(V)$ Thus, $-f(V) \subseteq f(V)$, and given that $c \in f(V)$, we have that $-c \in f(V)$. Thus, $f(V) = \mathbb{R}$. Let $x \in V$. There exists some $y \in V$ such that $f(y) = -f(x)$, and so $f(x + y) = 0$, and so $x + y \in \ker f$. Thus, $\ker f = V$, and so $\ker f$ is open, which is a contradiction. Thus, $f(V)$ is bounded by some $c > 0$. We show in the next problem that this implies that f is continuous at 0. Since f is linear, we have that f is continuous everywhere. Since $(-\infty, 0)$ is open and f is continuous, then $f^{-1}((-\infty, 0))$ is open, proving the first claim.

For the second claim, suppose that $x \in f^{-1}((\infty, 0])$ and $(x_n) \in f^{-1}((-\infty, 0))$ with $x_n \to x$. This is convergence with respect to open sets. That is, for any open neighborhood of x, there exists some N such that for all $n \geq N$, we have that $x_n \in V$. If $x \in f^{-1}((-\infty, 0))$, then we can just take the sequence to be itself, so consider then $x \in f^{-1}(\{0\})$. But then since $x_n \to x$ and f is continuous, we have that $f(x_n) \to f(x) = 0$, and so $x \in \ker f$, but $\ker f$ is closed, and so $x \in f^{-1}(\{0\})$.

∎

Problem 2

Let E be a real topological vector space. Let $f : E \to \mathbb{R}$ be a linear functional and $p : E \to \mathbb{R}$ be a continuous function at 0 such that $f \leq p$, then f is continuous.

SOLUTION: Let $\epsilon > 0$. By the continuity at 0 of p, there exists some U open neighborhood of 0 such that for all $x \in U$, $|p(x) - p(0)| < \epsilon$. Since $f \leq p$, then $f(x) \leq p(0) + \epsilon$ for all $x \in U$. We have that $-p$ is continuous at 0, and so it must be the case that there exists some U' open neighborhood of 0 such that for all $x \in U$, $-p(x) \in (-p(0) - \epsilon, -p(0) + \epsilon)$. We have that $-p \leq -f$, and so $-p(0) - \epsilon < f(x)$ for all $x \in U'$. Take $V = U \cap U'$, then for all $x \in V$, $|f(x)| \leq c$ for some $c > 0$.

Let $W \subset \mathbb{R}$ be some open neighborhood containing 0. There exists some $\epsilon > 0$ such that if $x \in \mathbb{R}$ with $|x| < \epsilon$, then $x \in W$. Let

$$W' = \frac{\epsilon}{c} W \subset V$$

Evidently, W' is open. Moreover, $|f(x)| = \frac{\epsilon}{c}|f(x)| \leq \epsilon$, and so $f(x) \in W$. That is $f(W') \subset W$, and so $0 \in W' \subset f^{-1}(W)$. Thus, f is continuous at 0. Since f is linear, then f is continuous everywhere. ■

Problem 3

Let (x_n) be a sequence in X, where X is a Banach space. Then $x_n \rightharpoonup x$ if and only if $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\{f \in X^* : \langle f, x_n\rangle \to \langle f, x\rangle\}$ is dense in X^*.

SOLUTION: ($\implies$) Suppose $x_n \rightharpoonup x$. Then by definition, we have that for all $f \in X^*$, $\langle f, x_n\rangle \to \langle f, x\rangle$, and so $\{f \in X^* : \langle f, x_n\rangle \to \langle f, x\rangle\} = X^*$. By the uniform boundedness principle, we have that if $T_n f = \langle f, x_n\rangle$, then since $f_n \to f$, then $T_n f$ is bounded, and so there exists some $c > 0$ such that

$$\|T_n f\| < c\|f\|.$$

Thus,

$$\|x_n\| = \sup_{\|f=1\|} |\langle f, x_n\rangle| = \sup_{\|f=1\|} \|T_n f\| < c \sup_{\|f\|=1} \|f\| < c$$

and so $\|x_n\| < c$ for all n, and thus the set is bounded.

($\impliedby$) Let $f \in X^*$. Since the set $\{g \in X^* : \langle g, x_n\rangle \to \langle g, x\rangle\}$ is dense in X^*, then there exists some $g \in X^*$ such that $\|g - f\| < \epsilon$ and $\langle g, x_n\rangle \to \langle g, x\rangle$. Thus, we have that

$$\begin{aligned}
|\langle f, x_n\rangle - \langle f, x\rangle| &= |\langle f, x_n\rangle - \langle g, x_n\rangle + \langle g, x_n\rangle - \langle f, x\rangle| \\
&\leq |\langle f, x_n\rangle - \langle g, x_n\rangle| + |\langle g, x_n\rangle - \langle f, x\rangle| \\
&= |\langle f - g, x_n\rangle| + |\langle g, x_n\rangle - \langle g, x\rangle + \langle g, x\rangle - \langle f, x\rangle| \\
&\leq \|f - g\|\|x_n\| + |\langle g, x_n\rangle - \langle g, x\rangle| + |\langle g, x\rangle - \langle f, x\rangle| \\
&= \|f - g\|\|x_n\| + |\langle g, x_n\rangle - \langle g, x\rangle| + |\langle g - f, x\rangle| \\
&\leq \|f - g\|\|x_n\| + |\langle g, x_n\rangle - \langle g, x\rangle| + \|g - f\|\|x\| \\
&\to 0,
\end{aligned}$$

where the first and third terms use the fact that (x_n) is bounded (and x is bounded) and that f is ϵ close to f, and the second inequality uses the fact that $\langle g, x_n\rangle \to \langle g, x\rangle$. ■

Problem 4

Suppose $(x_n) \in X$ is Cauchy and X is a normed vector space. If $x_n \rightharpoonup 0$, then $x_n \to 0$.

Solution: We generalize and let $x_n \rightharpoonup x$. Thus, suppose $f \in E^*$, then for all $\epsilon > 0:$

$$\|\langle f, x_n \rangle - \langle f, x \rangle\| < \epsilon,$$

but we have that

$$\begin{aligned}
|\langle f, x_n \rangle - \langle f, x \rangle| &= |\langle f, x_n \rangle - \langle f, x_m \rangle + \langle f, x_m \rangle\| - \langle f, x \rangle| \\
&\leq |\langle f, x_n \rangle - \langle f, x_m \rangle| + |\langle f, x_m \rangle - \langle f, x \rangle| \\
&\leq \|f\|\|x_n - x_m\| + |\langle f, x_m \rangle - \langle f, x \rangle| \\
&< \|f\|\|x_n - x_m\| + \epsilon
\end{aligned}$$

Thus,

$$\|x_n - x\| = \sup_{\|f\|=1} |\langle f, x_n \rangle - \langle f, x \rangle| \leq \|x_n - x_m\| + \epsilon < 2\epsilon$$

■

Problem 5

Problem 6

Show that if X is infinite dimensional Banach Space, then $0 \in \overline{S_X}^{\sigma(X,X^*)}$.

SOLUTION: It suffices to show that

$$B_X = \overline{S_X}^{\sigma(X,X^*)}.$$

To show the first inclusion, let $x_0 \in B_X$. Let V be a weakly open neighborhood of x_0. We wish to show that $V \cap S_X \neq \emptyset$. Without loss of generality, we can assume

$$V = \{x \in E \,:\, |\langle f_i, x_0 \rangle| < \epsilon, \quad i \in [k]\}.$$

There exists some $y_0 \in E$ such that for all i, $\langle f_i, y_0 \rangle = 0$. If not, then $\ker F = 0$, where $F : E \to \mathbb{R}^k$ such that each component of F is f_i. Then F is injective and so $\dim E < \infty$. Thus, consider $g(t) = \|x_0 + ty_0\|$. Since $x_0 \in B_X$, then $g(0) \leq 1$ and $g(\infty) = \infty$. Since g is continuous, there exists some t_0 such that $g(t_0) = 1$, and so $x_0 t y_0 \in S_X$. Moreover, $g(t_0) \in V$ since for all i, we have that $\langle f_i, x_0 + ty_0 \rangle = \langle f_i, x_0 \rangle + t\langle f_i, y_0 \rangle = \langle f_i, x_0 \rangle < \epsilon$. Thus, $x_0 + yt_0 \in S_X$, and so $B_X \subset \overline{S_X}^{\sigma(X,X^*)}$. Since $S_X \subset B_X$, suffices to show that B_X is closed, which is inmediate since

$$B_X = \bigcap_{\|f\| \leq 1} \{x \in E \,:\, |\langle f, x \rangle| \leq 1\}.$$

Thus, $B_X = \overline{S_X}^{\sigma(X,X^*)}$, and since $0 \in B_X$, then we are done. ∎

Problem 7

In c_0, let $x_n = ne_n$. Show that $x_n \to 0$ pointwise but not weakly.

SOLUTION: Let $x_n = (x_n^{(k)}) = \left(\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ \vdots \end{pmatrix}, \dots \right)$ Let n be arbitrary. To show that $x_n^{(k)} \to 0$ as $k \to \infty$, consider that for all $k \geq n$, $x_n^{(k)} = 0$, and thus $x_n^{(k)} \to 0$.

Suppose that $x_n \rightharpoonup 0$. Then we have that for any $f \in (c_0)^*$, $f(x_n) \to f(0) = 0$. Since $(c_0)^* = \ell^1$, then for each $f \in (c_0)^*$, then there exists a unique $(a_{k,n}) \in \ell^1$ such that

$$\langle f, x_n \rangle = \sum_{k=1}^{\infty} a_{k,n} x_n^{(k)} \xrightarrow{n \to \infty} 0.$$

But we have that $x_n^{(k)} = \begin{cases} n, & n = k \\ 0, & n \neq k \end{cases}$, and so $\langle f, x_n \rangle = na_n$. Since $(c_0)^* = \ell^1$, we consider $a_k = \delta_{k,n}$, and thus $\langle f, x_n \rangle = n$. Note that $\delta_{k,n} \in \ell^1$ since $\sum_{k=1}^{\infty} \delta_{k,n} = 1 < \infty$. Thus, we have that $\langle f, x_n \rangle = n \to \infty$, which is a contradiction. ■

Problem 8

Show that there exists a sequence $(f_n) \in X^*$ for some normed linear space X such that $(f_n(x))$ is bounded for each $x \in X$ but $\|f_n\| \to \infty$.

SOLUTION: Let $X = c_0$, and thus $X^* = \ell^1$. Consider that for each $f \in (c_0)^*$, there exists a unique $a \in \ell^1$ such that

$$\langle f, x \rangle = \sum_{k=1}^{\infty} a_k x_k.$$

Consider the sequence

$$a_n = ne_n \implies a_n^{(k)} = n\chi_n$$

(χ_n being the indicator function) Let $x \in c_0$, then we have that

$$|\langle f_n, x \rangle| = |\sum_{k=1}^{\infty} a_n^{(k)} x_i| \leq \|a_n\|_{\ell^1} \|x\|_{c_0} = n\|x\|_{c_0}$$

Then we have that

$$\|f_n\|_{c_0^*} = \|a_n\|_{\ell^1} = \|ne_n\|_{\ell^1} = n \xrightarrow{n\to\infty} \infty$$

■

Problem 9

In c_0, there exists a sequence $f_n \in (c_0)^*$ such that $f_n \xrightarrow{*} 0$ and yet every convex combination h of the f_n has $\|h\| = 1$.

SOLUTION: Since $(c_0)^* = \ell^1$, consider a sequence $f_n \in (c_0)^*$ such that

$$\langle f_n, x\rangle = \sum_{i=1}^{\infty} e_n^{(i)} x_i = x_n,$$

where (e_n) is the canonical basis of ℓ^1. To show that $f_n \xrightarrow{*} 0$, let $x \in c_0$, and so there exists some N such that if $n \geq N$, we have that $x_n = 0$. Thus, for large n, we have that

$$\langle f_n, x\rangle = x_n \to 0.$$

Thus, $f_n \xrightarrow{*} 0$. Let $\lambda_i > 0$ and suppose $\sum_{i=1}^{k} \lambda_i = 1$. Then

$$\|h\| = \|\sum_{i=1}^{k} f_n \lambda_i\| = \sum_{i=1}^{k} \|f_i\||\lambda_i| = \sum_{i=1}^{k} \|e_i\|\lambda_i = \sum_{i=1}^{k} \lambda_i = 1.$$

■

Problem 10

Show that if T is bounded and injective from ℓ^1 to ℓ^2, then $T(\ell^1)$ is not closed in ℓ^2.

SOLUTION: Suppose $T(\ell^1)$ is closed in ℓ^2. Since ℓ^1 and ℓ^2 are both Banach and T is continuous bijection unto $T(\ell^1)$, then T is an isomorphism to the image of ℓ^1. Since ℓ^2 is reflexive and $T(\ell^1)$ is a closed linear subspace of ℓ^2, then $T(\ell^1)$ is reflexive. By the isomorphism, we must have that ℓ^1 is reflexive, which is of course not true. $\blacksquare$

Problem 11

Suppose E is a Banach space and let $A \subset E$ be weakly compact. Prove that A is bounded.

SOLUTION: We aim to show that for any $f \in E^*$, $f(A)$ is bounded. Suppose not, then we have that for all $n > 0$, there exist $x_n \in A$ with $\|f(x_n)\| > n$. Since $(x_n) \in A$ and A is weakly compact, there exists a subsequence $(x_{n_k}) \in A$ such that it weakly converges to some limit in A, that is, $x_{n_k} \rightharpoonup x \in A$. Thus, $f(x_{n_k}) \to f(x)$, and thus we have a contradiction since for n_k large enough, $|f(x_{n_k})| \leq n_k$. Thus, $f(A)$ is bounded. Since this is true for all $f \in E^*$, then the uniform boundedness principle (see problem 3) says that A is bounded. ■

Problem 12

Let E be Banach and suppose $(x_n) \in E$ with $x_n \rightharpoonup x$ in $\sigma(E, E^*)$. Define

$$\sigma_n = \frac{1}{n}(x_1 + \cdots + x_n).$$

Show that $\sigma_n \rightharpoonup x$ in $\sigma(E, E^*)$.

Solution: Suppose $f \in E^*$. Let $\epsilon > 0$. Since $x_n \rightharpoonup x$, we have that $f(x_n) \to f(x)$. Thus, there exists some N such that if $n \geq N$, then $\|f(x_n) - f(x)\| < \epsilon$. Therefore,

$$\begin{aligned}
\|\frac{1}{n-N-1}\sum_{N+1}^{n} f(x_i) - f(x)\| &= |\frac{1}{n-N-1}\sum_{N+1}^{n}(f(x_i) - f(x))\| \\
&\leq \frac{1}{n-N-1}\sum_{N+1}^{n}\|f(x_i) - f(x)\| \\
&< \epsilon.
\end{aligned}$$

Thus, we triangle on this equality till it ins:

$$\begin{aligned}
|f(\sigma_n) - f(x)| &\leq |f(\sigma_n) - \frac{1}{n-N-1}\sum_{N+1}^{n} f(x_i)| + |\frac{1}{n-N-1}\sum_{N+1}^{n} f(x_i) - f(x)| \\
&= |\frac{1}{n}\sum_{i=1}^{n} f(x_i) - \frac{1}{n-N-1}\sum_{N+1}^{n} f(x_i)| + \frac{\epsilon}{2}
\end{aligned}$$

The first term obviously goes to 0 for large n (do another triangle inequality) ■

Problem 13

Let E be Banach. Suppose $A \subset E$ is convex. Show that the strong closure of A is the same as the weak closure of A.

SOLUTION: Since every weakly closed set is strongly closed, then we have that

$$A \subset \overline{A}^{\sigma(E,E^*)} \implies \overline{A} \subset \overline{\overline{A}^{\sigma(E,E^*)}} = \overline{A}^{\sigma(E,E^*)}$$

$$A \subset \overline{A}^{\sigma(E,E^*)} \subset \overline{A}.$$

Since $\overline{A}$ is strongly closed and convex[a], then we know by a theorem in class that it is weakly closed. Thus,

$$A \subset \overline{A} \implies \overline{A}^{\sigma(E,E*)} \subset \overline{\overline{A}}^{\sigma(E,E^*)} = \overline{A},$$

and so we are done. ■

[a] We proved this in PSET 4

Problem 14

Let E be a Banach space and suppose $K \subset E$ is strongly compact. Suppose $(x_n) \in K$ such that $x_n \rightharpoonup x$. Then $x_n \to x$

SOLUTION: Suppose not. Thus, there exists some x_{n_k} subsequence such that $\|x_{n_k} - x\| \geq \epsilon$ for some $\epsilon > 0$. Since $(x_{n_k}) \in K$ and K is compact, we have that there exists some subsequence $x_{n_{k_j}} \to y$ where $y \in K$, and since strong convergence implies weak convergence, then $x_{n_{k_j}} \rightharpoonup y$. But since $X_{n_{k_J}}$ is a subsequence of a sequence converging to x, then it suffices to show that weak limits are unique and thus we must have that $y = x$, a contradiction!

To show that weak limits are unique, suppose not. ■

Problem 15

Let E and F be two Banach spaces. Let $T \in \mathcal{L}(E, F)$ so that $T^* \in \mathcal{L}(F^*, E^*)$ Prove that T^* is continuous from F^* (equipped with $\sigma(F^*, F)$) unto E^* (equipped with $\sigma(E^*, E)$).

Solution: Since $T^* : F^* \to E^*$, let's consider $T^* : F^*_* \to E^*_*$, where the underscore denotes that we are considering the weak $*$ topology. Let $\varphi_x : E^* \to \mathbb{R}$ such that $\varphi_x \circ T : F^*_* \to \mathbb{R}$ such that for all $x \in E$:

$$\varphi_x \circ T(v) = \langle T^* v, x \rangle = \langle v, Tx \rangle,$$

which is of course a linear functional on F, and is thus continuous in the weak * topology of F^*. ■

Problem 16

Let E be a Banach space. Let $(x_n) \in E$ and let

$$K_n = \overline{\mathrm{conv}\left(\bigcup_{i=n}^{\infty}\{x_i\}\right)}.$$

(a) If $x_n \rightharpoonup x$, then

$$\bigcap_{n=1}^{\infty} K_n = \{x\}$$

SOLUTION: Since K_n is convex and strongly closed, then K_n is weakly closed. Evidently, $x \in \overline{K_n}^{\sigma(E,E^*)}$ for all n, and thus since K_n is weakly closed, $x \in K_n$ for any n, and thus $x \in \bigcap K_n$.

Let V be some weak convex neighborhood of x. Since $x_n \rightharpoonup x$, we have that there exists some N such that for $n \geq N$, $K_n \subset \overline{V}$, and so $\bigcap K_n \subset \overline{V}$. Suppose $y \in \bigcap K_n$ with $y \neq x$. Suppose $r = \|y - x\|$. Let $B_{\frac{r}{2},\sigma(E,E^*)}(x)$ be a weakly open convex neighborhood of x. Then there exists some N such that for all $n \geq N$, we have that $x_n \in B_{\frac{r}{2},\sigma(E,E^*)}(x)$, and so $y \notin \overline{\mathrm{conv}\left(\bigcup_{i=n}^{\infty}\{x_i\}\right)}$, a contradiction to the fact that $y \in \bigcap K_n$. ■

(b) Assume that E is finite dimensional and $\bigcap_{n=1}^{\infty} K_n = \{x\}$. Prove that $x_n \to x$.

SOLUTION: Since E is finite dimensional, a $x_n \rightharpoonup x$ if and only if $x_n \to x$. Let V be a weakly open neighborhood of x. Consider $K_n' = K_n \cap V^c$. Since $\bigcap K_n = \{x\}$, then we must have that K_n is bounded for each n. To show this, suppose that for some N, we have that K_N is unbounded. This implies that $x_n \to \pm\infty$ for $n \geq N$, and thus K_n is unbounded for all $n \geq N$, and so $\bigcap K_n = \emptyset$. Thus, K_n is bounded. Since $K_n \subset E$ is closed and convex and E is reflexive (by finite dimensions), then K_n is compact in $\sigma(E^*, E)$, and since K_n' is a closed (V^c is closed) subset of K_n, then K_n' is compact. Since $\bigcap K_n' = \bigcap(K_n \cap V^c) = \bigcap K_n \cap V^c = \{x\} \cap V^c = \emptyset$. Since each K_n' is compact, and $K_n' \downarrow$ then we must necessarily have some N such that $K_N' = \emptyset$. Thus, $K_N \cap V^c = \emptyset$, and so $K_N \subset V$, and so for all $n \geq N$, $K_n \subset V$, and so $x_n \in V$. Thus, $x_n \rightharpoonup x$. ■

Problem 17

Let E be a Banach space.

(a) Let $(f_n) \in E^*$ such that for all $x \in E$, $\langle f_n, x\rangle$ converges to a limit. Prove that there exists some $f \in E^*$ such that $f \overset{*}{\rightharpoonup} f$.

> Solution: We want to show that there exists some $f \in E^*$ such that for all $x \in E$, $\langle f_n, x\rangle \to \langle f, x\rangle$. We know that $\|f_n\| > 0$ for large n for if not, then just take $f = 0$ and we are done.
>
> Let
> $$\langle f_n, x\rangle \to y_x.$$
> By the uniform boundedness principle, we have that $\sup_n \|f_n\| < \infty$. Call $A = \sup_n \|f_n\|$.
>
> Consider the sequence
> $$\hat{f}_n = \frac{f_n}{A} \implies \hat{f}_n \in B_{E^*}.$$
> Since B_{E^*} is compact, there exists some $\hat{f} \in B_{E^*}$ and some subsequence such that $\hat{f}_{n_k} \overset{*}{\rightharpoonup} f$. That is, for all $x \in E$,
> $$\langle \hat{f}_{n_k}, x\rangle \to \langle \hat{f}, x\rangle.$$
> Thus, we have that for n large
> $$\langle \hat{f}_{n_k}, x\rangle = \langle \frac{f_{n_k}}{A}, x\rangle = \langle \hat{f}, x\rangle \implies \langle f_{n_k}, x\rangle = \langle A\hat{f}, x\rangle.$$
> But we already know that $\langle f_n, x\rangle$ converges to a limit, and so the entire sequence must converge to that same limit, y_x, i.e,
> $$\langle f_n, x\rangle = \langle A\hat{f}, x\rangle.$$
> Thus, since limits are unique because $\sigma(E^*, E)$ is Hausdorff, then limits are unique, and thus $y_x = \langle Af, y_x\rangle$. Because this is true for all $x \in E$, we have that $f_{n_k} \overset{*}{\rightharpoonup} Af$. ■

(b) Assume that E is reflexive. Let (x_n) be a sequence in E such that for every $f \in E^*$, $\langle f, x_n\rangle$ converges to a limit. Prove that there exists some $x \in E$ such that $x_n \rightharpoonup x$ in $\sigma(E, E^*)$.

> Solution: It suffices to show that for all $f \in E^*$,
> $$\langle f, x_n\rangle \to \langle f, x\rangle.$$
> This proof would follow exactly as above, switching up the Es and the E^*. Define $T_n f = \langle f, x_n\rangle$. We know that $T_n f \to y_f$. Thus, by the uniform boundedness principle,

we know that $\sup_n \|T_n f\| = \sup_n \|\langle f, x_n\| < \infty$. Denote this by $A = \sup_n \|\langle f, x_n\rangle\| < \infty$. Define

$$\hat{T}_n = \frac{T_n}{A},$$

and thus

$$\hat{T}_n f = \frac{\langle f, x_n\rangle}{A} \leq 1 \qquad \forall n.$$

We know that B_{E^*} is compact in the weak $\star$ topology, but since E is reflexive, then we know that B_{E^*} is strongly compact. Thus, there exists some subsequence such that

$$\hat{T}_{n_k} \to \hat{T} \in B_{E^*},$$

and thus for any $f \in E^*$,

$$\frac{\langle f, x_{n_k}\rangle}{A} \to T(f) = \langle f, x\rangle \implies \langle f, x_{n_k}\rangle \to AT(f).$$

But we know that $\langle f, x_n\rangle \to y_f$, and thus $A\langle f, x\rangle = y_f$. ■

(c) Construct an example in a non-reflexive space E where the conclusion of 2 fails.

Solution: Consider $E = c_0$. Let

$$x_n = (1, 1, \dots, 1_{(n)}, 0, 0, \dots)$$

Let $f \in (c_0)^*$, then

$$f(x_n) = f\left(\sum_{i=1}^{\infty} x_i e_i\right) = f\left(\sum_{i=1}^{n} e_i\right) = \sum_{i=1}^{n} f(e_i).$$

We know that $(c_0)^* = \ell^1$, and thus $f(e_i) \in \ell^1$, and so

$$\lim_{n\to\infty} \langle f, x_n\rangle = \lim_{n\to\infty} \sum_{i=1}^{n} f(e_i) = \sum_{i=1}^{\infty} f(e_i) < \infty,$$

and so $\langle f, x_n\rangle$ converges to a limit, in particular, it converges to

$$\sum_{i=1}^{\infty} f(e_i) = f((1, 1, 1, \dots))$$

Thus,

$$x_n \rightharpoonup (1, 1, 1, \dots),$$

but $(1, 1, 1, \dots) \notin c_0$, which is a contradiction. ■