Agustín Esteva
aesteva@uchicago.edu
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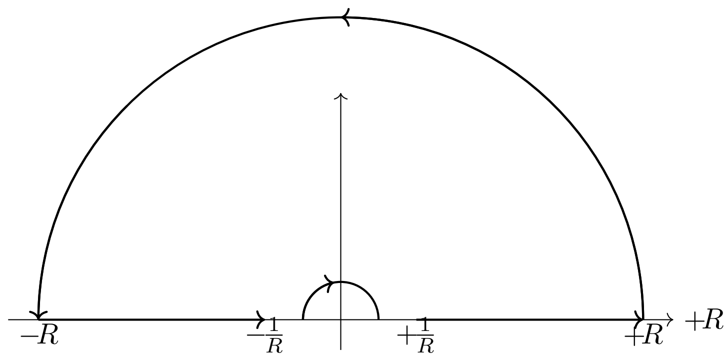
Problem 1

Consider the path Γ_R pictured below:



Prove that

$$\int_{\Gamma_R} (\frac{e^{iz}}{z} - \frac{1}{z})\, dz = 0.$$

Solution: Since Γ_R is closed, it suffices, by Cauchy's theorem, to show that $\frac{e^{iz}}{z} - \frac{1}{z}$ is entire on $D_{R+1}(0)$. Clearly, The function $\in H(D_{R+1}(0) \setminus \{0\})$. We will show that if we define

$$f(z) = \begin{cases} \frac{e^{iz}}{z} - \frac{1}{z}, & z \neq 0 \\ i, & z = 0 \end{cases},$$

then f is continuous at 0 since

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{1}{z} \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \frac{1}{z} = i.$$

Since we forgive singularities, we have that $f \in H(D_{R+1}(0))$. and so

$$\int_{\Gamma_R} (\frac{e^{iz}}{z} - \frac{1}{z})\, dz = \int_{\Gamma_R} f(z)\, dz = 0.$$

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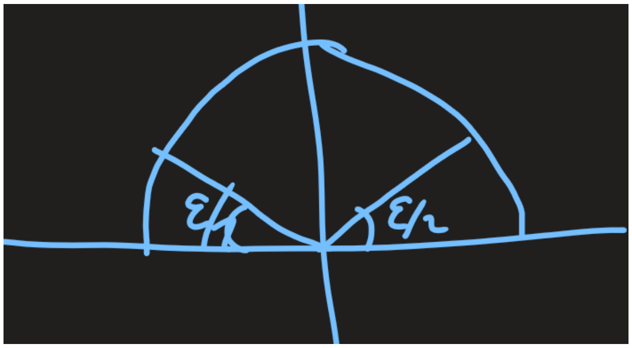
Problem 2

Suppose that $\gamma_R(\theta) = Re^{i\theta}, \quad \theta \in [0, \pi]$ is the upper semicricle arc. Show that

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z}\, dz = 0$$



Solution: Consider the paths in the picture above, where $\gamma_R^{(1)}, \gamma_R^{(3)}$ are the small pizza slices and $\gamma_R^{(2)}$ is the big pizza slice. Thus,

$$\begin{aligned}
\left|\int_{\gamma_R} \frac{e^{iz}}{z}\, dz\right| &\le \left|\int_{\gamma_R^{(2)}} \frac{e^{iz}}{z}\, dz\right| + 2\left|\int_{\gamma_R^{(1)}} \frac{e^{iz}}{z}\, dz\right| \\
&\le \text{arclength}\big[\gamma_R^{(2)}\big] \max_{z\in\gamma_R^{(2)}} (|\frac{e^{iz}|}{|z|}) + 2\cdot \text{arclength}\big[\gamma_R^{(1)}\big] \max_{z\in\gamma_R^{(1)}} (|\frac{e^{iz}|}{|z|}) \\
&\le \pi R \frac{1}{R} \max_{z\in\gamma_R^{(2)}} (|e^{iz}|) + \epsilon R \frac{1}{R} \max_{z\in\gamma_R^{(1)}} (|e^{iz}|) \\
&= \pi \max_{y\in\gamma_R^{(2)}} (e^{-y}) + \epsilon \max_{y\in\gamma_R^{(1)}} (e^{-y}) \\
&= \pi e^{-R\sin\left(\frac{\epsilon}{2}\right)} + \epsilon \\
&\to \epsilon
\end{aligned}$$

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Problem 3

Prove that if $\tilde{\gamma}_R = \frac{1}{R}e^{i\theta}$, $\theta \in [0, \pi]$, then

$$\lim_{R\to\infty} \int_{\tilde{\gamma}_R} (\frac{e^{iz}}{z} - \frac{1}{z})\, dz = 0$$

SOLUTION: Again, we estimate

$$\begin{aligned} \left| \int_{\tilde{\gamma}_R} (\frac{e^{iz}}{z} - \frac{1}{z})\, dz \right| &\leq \text{arclength}\left[\tilde{\gamma}_R\right] \max_{z\in\tilde{\gamma}_R} (\frac{|e^{iz}|}{|z|} - \frac{1}{|z|}) \\ &= \frac{\pi}{R} \max_{z=\frac{1}{R}} (z + \frac{1}{2}z^2 + \dots) \\ &\to 0 \end{aligned}$$

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Problem 4

Prove that $\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \pi$.

SOLUTION: By symmetry,

$$\int_{-R}^{R} \frac{\cos(x)}{x}\, dx = \int_{-R}^{R} \frac{1}{x}\, dx = 0.$$

Thus, by the previous part, we can say that in the limit,

$$\begin{aligned} i\int_{-R}^{R} \frac{\sin(x)}{x}\, dx &= \int_{-R}^{R} \frac{\cos(x) + i\sin(x)}{x} - \frac{1}{x}\, dx \\ &= \int_{-R}^{R} \frac{e^{ix}}{x} - \frac{1}{x}\, dx \\ &= \int_{\Gamma_R} \frac{e^{iz}}{z} - \frac{1}{z}\, dz - \int_{\gamma_R} \frac{e^{iz}}{z} - \frac{1}{z}\, dz \\ &= \int_{\gamma_R} \frac{1}{z}\, dz \\ &= \int_{0}^{\pi} \frac{1}{Re^{i\theta}} Rie^{i\theta} \\ &= i\pi, \end{aligned}$$

and thus

$$\int_{-R}^{R} \frac{\sin(x)}{x}\, dx = \pi$$

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Problem 5

Verify the Cauchy-Riemann equations for

$$f(z) = e^{z^2}$$

SOLUTION: $\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right)$ Note that

$$e^{z^2} = e^{2z} = e^{2x+2iy} = e^{2x}(\cos(2y) + i\sin(2y)) = e^{2x}\cos(2y) + ie^{2x}\sin(2y).$$

Thus

$$\begin{aligned}\frac{\partial u}{\partial x} &= 2e^{2x}\cos(2y) \\ &= e^{2x}(2\cos(2y)) \\ &= \frac{\partial v}{\partial y}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial v}{\partial x} &= 2e^{2x}\sin(2y) \\ &= -(-2e^{2x}\sin(2y)) \\ &= -\frac{\partial u}{\partial y}\end{aligned}$$

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Problem 6

Suppose that $a, b \in \mathbb{C}$ with $|a| < r < |b|$ and

$$\gamma(\theta) = re^{i\theta}, \quad \theta \in [0, 2\pi].$$

Evaluate

$$\int_\gamma \frac{1}{(z-a)(z-b)}\, dz.$$

SOLUTION: Consider the function

$$f(z) := \begin{cases} \frac{1}{(z-a)(z-b)} - \frac{1}{(z-a)(a-b)}, & z \neq a \\ \frac{-1}{(a-b)^2}, & z = a \end{cases}$$

Note that

$$\begin{aligned} \lim_{z \to a} f(z) &= \lim_{z \to a} \frac{1}{(z-a)(z-b)} - \frac{1}{(z-a)(a-b)} \\ &= \lim_{z \to a} \frac{1}{z-a} \left[\frac{1}{z-b} - \frac{1}{a-b} \right] \\ &= \lim_{z \to a} \frac{1}{z-a} \left[\frac{a-b-z+b}{(z-b)(b-a)} \right] \\ &= \lim_{z \to a} \frac{-1}{(z-b)(a-b)} \\ &= \frac{-1}{(a-b)^2} \end{aligned}$$

. Thus, f is continuous on $z = a$. Moreover, there exists some R such that $|r| < R < b$, and thus $f \in H(D_R(0))$ since it forgives $z = a$. Since γ is closed, we have by Cauchy that

$$\int_\gamma \frac{1}{(z-a)(z-b)} - \frac{1}{(z-a)(a-b)}\, dz = \int_\gamma f(z)\, dz = 0.$$

Thus, it suffices to calculate

$$\frac{1}{a-b} \int_\gamma \frac{1}{z-a}\, dz = \frac{1}{a-b} 2\pi i.$$

Thus,

$$\int_\gamma \frac{1}{(z-a)(z-b)}\, dz = \frac{2\pi i}{b-a}$$

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Problem 7

Suppose $f(z)$ is a nonconstant entire function. Prove that the range of f is dense in $\mathbb{C}$.

SOLUTION: Let $z_0 \in \mathbb{C}$. Suppose there exists some $r > 0$ such that $f(z) \notin B_r(z)$ for any $z \in \mathbb{C}$. Define

$$g(z) = \frac{1}{f(z) - z_0}.$$

Note that $g \in H(\mathbb{C})$. Note that

$$|g(z)| = \frac{1}{|f(z) - z_0|} < \frac{1}{r}.$$

Thus, $g(z)$ is constant and so $f(z) - z_0$ is constant and thus f is constant. A contradiction.

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