23500 Spring 2025: Problem Set 1   Agustín Esteva
Markov Chains, Martingales, and Brownian Motion   aesteva@uchicago.edu
Instructor(s): Stephen Yearwood   Due Date: 04/04/2025

Problem 1

A ball is in any one of n boxes and is in the ith box with probability P_i. If the ball is in box i, a search of that box will uncover it with probability α_i. Find the conditional probability that the ball is in box j, given that a search of box i did not uncover it.

SOLUTION: Let I_i denote the event that the ball is in box i, S_i denote the event of searching box i and finding the ball. Then

$$\mathbb{P}\{S_i \mid I_i\} = \alpha_i$$

$$\mathbb{P}\{I_i\} = P_i.$$

We use Bayes' rule and his law of total probability:

$$\begin{aligned}
\mathbb{P}\{I_j | S_i^c\} &= \frac{\mathbb{P}\{I_j \cap S_i^c\}}{\mathbb{P}\{S_i^c\}} \\
&= \frac{\mathbb{P}\{S_i^c | I_j\}\mathbb{P}\{I_j\}}{1 - \mathbb{P}\{S_i\}} \\
&= \frac{(1 - \mathbb{P}\{S_i | I_j\})\mathbb{P}\{I_j\}}{1 - \sum_{k=1}^{n} \mathbb{P}\{S_i | I_k\}\mathbb{P}\{I_k\}} \\
&= \boxed{\begin{cases} \frac{(1-\alpha_i)P_i}{1-\alpha_i P_i}, & i = j \\ \frac{P_j}{1-\alpha_i P_i}, & i \neq j \end{cases}}
\end{aligned}$$

The second to last equality comes from the fact that the probability of succesfully finding the ball in a box where the ball is not is obviously 0. ∎

Problem 2

Given some fixed integer $n \geq 1$, consider the set of all possible permutations of the list $(1, 2, \ldots, n)$. Suppose that one permutation $(i_1, i_2, \ldots, i_n)$ is chosen at random from the set, with $i_k \in \{1, 2, \ldots, n\}$ for every k. Assuming that all permutations are equally likely to occur, i.e., each permutation is chosen with probability $\frac{1}{n!}$, compute the probability p_m that exactly m of the numbers from the chosen permutation appear in their own positions (i.e., in the same positions in which they appear in the list $(1, 2, \ldots, n)$). What is the limit of these probabilities as n goes to infinity (m fixed)?

SOLUTION: The total number of possibilities is $n!$. To calculate the favorable outcomes, we first need to fix m items $\{1, 2, \ldots, n\}$. There are $\binom{n}{m}$ ways of doing this. Once the m numbers are fixed, we need to calculate the permutations of the $n - m$ elements of a set in which no element appears in its original position. This is called a derangement and is equal to

$$!(n - m) = (n - m)! \sum_{k=0}^{n-m} \frac{(-1)^k}{k!}$$

Thus, if we call $E_{m,n}$ the event that m numbers from a list of length n are in their exact positions,

$$\boxed{\mathbb{P}\{E_{m,n}\} = \frac{\binom{n}{m} \cdot !(n-m)}{n!} = \frac{\frac{n!}{(n-m)!m!} \cdot (n-m)! \sum_{k=0}^{n-m} \frac{(-1)^k}{k!}}{n!} = \frac{\sum_{k=0}^{n-m} \frac{(-1)^k}{k!}}{m!}}$$

Using the Taylor series expansion of

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

we see that

$$\boxed{\lim_{n \to \infty} \mathbb{P}\{E_{n,m}\} = \frac{1}{m!} e^{-1}}$$

■

Problem 3

Problem 5

A fair coin is tossed repeatedly with results $Y_0, Y_1, Y_2, \ldots$, where each Y_i is either 0 or 1 with probability $\frac{1}{2}$. For $n \geq 1$, let
$$X_n = Y_n + Y_{n-1}$$
be the number of 1's in the $(n-1)^{\text{th}}$ and n^{th} tosses. Is $\{X_n\}_{n \in \mathbb{N}}$ a Markov chain?

SOLUTION: No. Consider
$$\mathbb{P}\{X_3 = 2 \mid X_1 = 1, X_2 = 1\} = 0$$

$$\mathbb{P}\{X_3 = 2 \mid X_2 = 1\} = \frac{\mathbb{P}\{X_2 = 1 | X_3 = 2\}\mathbb{P}\{X_2 = 1\}}{\mathbb{P}\{X_2 = 1\}} = \frac{1}{2},$$

since if $X_3 = 2$, then necessarily, $Y_3 = 1$ and $Y_2 = 1$, and thus either $X_2 = 1$ or $X_2 = 2$. Thus, $\{X_n\}$ is **not** a Markov chain. ∎

Problem 6

Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step, we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n.

(a) Explain why $\{X_n\}$ is a (time-homogeneous) Markov chain.

> SOLUTION: Define
>
> $$L_n := \begin{cases} -1, & \text{white is picked from left urn on } n\text{th time} \\ 0 & \text{else} \end{cases}$$
>
> $$R_n := \begin{cases} 1, & \text{white is picked from right urn on } n\text{th time} \\ 0 & \text{else} \end{cases}$$
>
> Clearly, L_{n+1} depends only on X_n, since it is independent of the number of urns at any time before the present. Similarly, R_{n+1} is independent of any X_i with $i < n$. Since $X_{n+1} = X_n + (L_{n+1} + R_{n+1})$, and $(L_{n+1} + R_{n+1})$ are independent of any X_i with $i < n$, then for any $j \in \{0, 1, 2, 3, 4, 5\}$, for any $s_0, \ldots, s_n \in \{0, 1, 2, 3, 4, 5\}$
>
> $$\mathbb{P}\{X_{n+1} = j \mid X_n = s_n, X_{n-1} = s_{n-1}, \ldots, X_0 = s_0\} = \mathbb{P}\{X_n + R_{n+1} + L_{n+1} = j \mid X_n = s_n, X_{n-1} = s_{n-1},$$
> $$= \mathbb{P}\{X_n + R_{n+1} + L_{n+1} \mid X_n = s_n\}$$
>
> The process is time independent if and only if, for any $x, y \in S = \{0, 1, 2, 3, 4, 5\}$, we have $\mathbb{P}\{X_{n+1} = x \mid X_n = y\} = \mathbb{P}\{X_1 = x \mid X_0 = y\}$. From before,
>
> $$\begin{aligned} \mathbb{P}\{X_{n+1} = x \mid X_n = y\} &= \mathbb{P}\{X_n + L_{n+1} + R_{n+1} = x \mid X_n = y\} \\ &= \mathbb{P}\{L_{n+1} + R_{n+1} = x - y \mid X_n = y\} \\ &= \mathbb{P}\{L_1 + R_1 = x - y \mid X_0 = y\} \\ &= \mathbb{P}\{X_0 + L_1 + R_1 = x \mid X_0 = y\} \\ &= \mathbb{P}\{X_1 = x \mid X_0 = y\} \end{aligned}$$
>
> The key assumption being that L_{n+1} and R_{n+1} are independent from time n, which is obvious, as they depend only on the number of balls at time n.
>
> ∎

(b) Find the state space and transition matrix for this Markov chain.

> SOLUTION: Clearly,
> $$\boxed{S = \{0, 1, 2, 3, 4, 5\}},$$
> since at any point, there could be anywhere from 0 to 5 white balls in the left urn.

$$P = \begin{pmatrix} p(0,0) & p(0,1) & p(0,2) & p(0,3) & p(0,4) & p(0,5) \\ p(1,0) & p(1,1) & p(1,2) & p(1,3) & p(1,4) & p(1,5) \\ p(2,0) & p(2,1) & p(2,2) & p(2,3) & p(2,4) & p(2,5) \\ p(3,0) & p(3,1) & p(3,2) & p(3,3) & p(3,4) & p(3,5) \\ p(4,0) & p(4,1) & p(4,2) & p(4,3) & p(4,4) & p(4,5) \\ p(5,0) & p(5,1) & p(5,2) & p(5,3) & p(5,4) & p(5,5) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{25} & \frac{8}{25} & \frac{16}{25} & 0 & 0 & 0 \\ 0 & \frac{4}{25} & \frac{12}{25} & \frac{9}{25} & 0 & 0 \\ 0 & 0 & \frac{9}{25} & \frac{12}{25} & \frac{4}{25} & 0 \\ 0 & 0 & 0 & \frac{16}{25} & \frac{8}{25} & \frac{1}{25} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$\blacksquare$

Problem 7

An ant is sitting at vertex A of a regular tetrahedron. At the start of each minute, the ant (uniformly) randomly chooses one of the edges at the vertex it is currently sitting and crawls along that edge to an adjacent vertex. It takes one minute for the ant to crawl to the next vertex, at which point it chooses another edge (at random) and starts crawling again.

(a) What is the probability that, after 6 minutes, the ant is back at vertex A?

SOLUTION: Let the state space be

$$S = \{A, B, C, D\}$$

Let X_n represent the vertex the ant is at on the n-th minute. Evidently, $\{X_n\}$ is a time-homogenous Markov chain with transition matrix:

$$P = \begin{pmatrix} p(A,A) & p(A,B) & p(A,C) & p(A,D) \\ p(B,A) & p(B,B) & p(B,C) & p(B,D) \\ p(D,A) & p(C,B) & p(C,C) & p(C,D) \\ p(D,A) & p(D,B) & p(D,C) & p(D,D) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Thus, on the 6th minute, the transition probability will be

$$P^6 = \begin{pmatrix} \frac{61}{243} & \frac{182}{729} & \frac{182}{729} & \frac{182}{729} \\ \frac{182}{729} & \frac{61}{243} & \frac{182}{729} & \frac{182}{729} \\ \frac{182}{729} & \frac{182}{729} & \frac{61}{243} & \frac{182}{729} \\ \frac{182}{729} & \frac{182}{729} & \frac{182}{729} & \frac{61}{243} \end{pmatrix}$$

Thus, we have that

$$\boxed{p^6(A,A) = \frac{61}{243}}$$

is the probability that starting at A the ant will be back on A after 6 moves. ∎

(b) Do the same problem as in part (a), but this time the ant is crawling along the edges of a cube.

SOLUTION: Same scenario as before, but now

$$S = \{A, B, C, D, E, F, G, H\}$$

then

$$
P = \begin{pmatrix}
p(A,A) & p(A,B) & p(A,C) & p(A,D) & p(A,E) & p(A,F) & p(A,G) & p(A,H) \\
p(B,A) & p(B,B) & p(B,C) & p(B,D) & p(B,E) & p(B,F) & p(B,G) & p(B,H) \\
p(C,A) & p(C,B) & p(C,C) & p(C,D) & p(C,E) & p(C,F) & p(C,G) & p(C,H) \\
p(D,A) & p(D,B) & p(D,C) & p(D,D) & p(D,E) & p(D,F) & p(D,G) & p(D,H) \\
p(E,A) & p(E,B) & p(E,C) & p(E,D) & p(E,E) & p(E,F) & p(E,G) & p(E,H) \\
p(F,A) & p(F,B) & p(F,C) & p(F,D) & p(F,E) & p(F,F) & p(F,G) & p(F,H) \\
p(G,A) & p(G,B) & p(G,C) & p(G,D) & p(G,E) & p(G,F) & p(G,G) & p(G,H) \\
p(H,A) & p(H,B) & p(H,C) & p(H,D) & p(H,E) & p(H,F) & p(H,G) & p(H,H)
\end{pmatrix}
$$

$$
= \begin{pmatrix}
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
\frac{1}{3} & p(C,B) & p(C,C) & p(C,D) & p(C,E) & \frac{1}{3} & \frac{1}{3} & p(C,H) \\
\frac{1}{3} & p(D,B) & p(D,C) & p(D,D) & \frac{1}{3} & p(D,F) & \frac{1}{3} & p(D,H) \\
p(E,A) & \frac{1}{3} & p(E,C) & \frac{1}{3} & p(E,E) & p(E,F) & p(E,G) & \frac{1}{3} \\
p(F,A) & \frac{1}{3} & \frac{1}{3} & p(F,D) & p(F,E) & p(F,F) & p(F,G) & \frac{1}{3} \\
p(G,A) & p(G,B) & \frac{1}{3} & \frac{1}{3} & p(G,E) & p(G,F) & p(G,G) & \frac{1}{3} \\
p(H,A) & p(H,B) & p(H,C) & p(H,D) & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & p(H,H)
\end{pmatrix}
$$

$$
= \begin{pmatrix}
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\
\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\
0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0
\end{pmatrix}
$$

We use Wolfram and see that

$$
P^6 = \begin{pmatrix}
\frac{61}{243} & 0 & 0 & 0 & \frac{182}{729} & \frac{182}{729} & \frac{182}{729} & 0 \\
0 & \frac{61}{243} & \frac{182}{729} & \frac{182}{729} & 0 & 0 & 0 & \frac{182}{729} \\
0 & \frac{182}{729} & \frac{61}{243} & \frac{182}{729} & 0 & 0 & 0 & \frac{182}{729} \\
0 & \frac{182}{729} & \frac{182}{729} & \frac{61}{243} & 0 & 0 & 0 & \frac{182}{729} \\
\frac{182}{729} & 0 & 0 & 0 & \frac{61}{243} & \frac{182}{729} & \frac{182}{729} & 0 \\
\frac{182}{729} & 0 & 0 & 0 & \frac{182}{729} & \frac{61}{243} & \frac{182}{729} & 0 \\
\frac{182}{729} & 0 & 0 & 0 & \frac{182}{729} & \frac{182}{729} & \frac{61}{243} & 0 \\
0 & \frac{182}{729} & \frac{182}{729} & \frac{182}{729} & 0 & 0 & 0 & \frac{61}{243}
\end{pmatrix},
$$
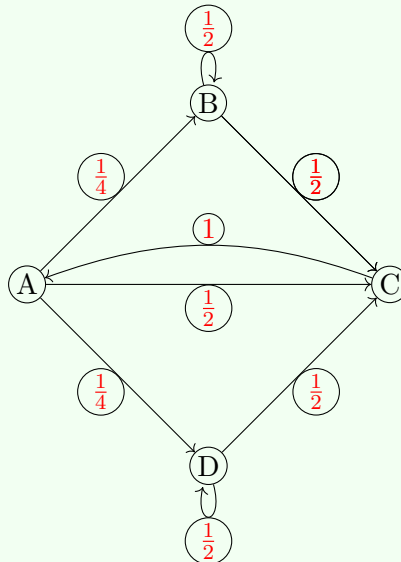
and thus

$$
\boxed{p^6(A,A) = \frac{61}{243}}
$$

■

Problem 8

Suppose $\{X_n\}$ is a Markov chain with state space $S = \{A, B, C, D\}$ and transition matrix

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(a) Draw a (directed) graph depicting the states and the corresponding transition probabilities (i.e., label each edge that you draw with the probability of traversing it). Explain why $\{X_n\}$ is irreducible.

> SOLUTION: Graph:
>
>
>
> $\{X_n\}$ is irreducible. For the following table, the first number is the smallest number of turns it takes for $p^n(i,j) > 0$.
>
> | | A | B | C | D |
> |---|---|---|---|---|
> | A | 3 w.p. $\geq \frac{1}{16}$ | 1 w.p. $\frac{1}{4}$ | 2 w.p. $\geq \frac{1}{8}$ | 1 w.p. $\frac{1}{4}$ |
> | B | 2 w.p. $\geq \frac{1}{2}$ | 1 w.p. $\frac{1}{2}$ | 1 w.p. $\frac{1}{2}$ | 3 w.p. $\geq \frac{1}{8}$ |
> | C | 1 w.p. 1 | 2 w.p. $\geq \frac{1}{4}$ | 3 w.p. $\geq \frac{1}{8}$ | 2 w.p. $\geq \frac{1}{4}$ |
> | D | 2 w.p. $\geq \frac{1}{4}$ | 2 w.p. $\geq \frac{1}{8}$ | 1 w.p. $\frac{1}{2}$ | 1 w.p. $\frac{1}{2}$ |
>
> Table 1: Irreducibility
>
> Thus, for every $i, j \in S$, there exists some $n, m > 0$ such that $p^n(i,j) > 0$ and $p^m(j,i) > 0$, as you can see in the table above. ∎

(b) Assume that the chain starts with $X_0 = B$. What is the probability that $X_3 = C$?

SOLUTION: Using Wolfram, we see that

$$P^3 = \begin{pmatrix} \frac{1}{4} & \frac{3}{16} & \frac{3}{8} & \frac{3}{16} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix}$$

and thus

$$\boxed{p^3(B,C) = \frac{3}{8}}$$

■

(c) Under the same assumption as in the previous part, what is the probability that $X_3 = C$, $X_4 = A$, and $X_6 = B$ all occur?

SOLUTION: We use the Markov property and the transition matrices above:

$$\begin{aligned} \mathbb{P}\{X_3 = C, X_4 = A, X_6 = B \mid X_0 = B\} &= \mathbb{P}\{X_4 = A, X_6 = B \mid X_3 = C\}\mathbb{P}\{X_3 = C \mid X_0 = \boxed{B}\} \\ &= \mathbb{P}\{X_6 = B \mid X_4 = A\}\mathbb{P}\{X_4 = A \mid X_3 = C\}\mathbb{P}\{X_3 = C \mid X_0 = B\} \\ &= p^3(B,C)p(C,A)p^2(A.B) \\ &= \frac{3}{64} \end{aligned}$$

We compute the first term above, $P(C, A) = 1$ is given by P and the last term is given by P^2 (not shown). ■

(d) Compute the invariant probability vector for this Markov chain.

SOLUTION: We need to find some $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ probability vector such that

$$\pi P = \pi \iff P^T \pi^T = \pi^T \iff (P^T - I)\pi^T = 0 \iff (4P^T - 4I)\pi^T = 0$$

We know that

$$P = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \implies 4P^T = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 2 & 0 & 2 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

Thus, we need to find $N(4P^T - 4I)$:

$$N(4P^T - 4I) = \begin{pmatrix} -4 & 0 & 4 & 0 \\ 1 & -2 & 0 & 0 \\ 2 & 2 & -4 & 2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$$

$$\simeq \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 1 & -2 & 1 \\ 1 & 0 & 0 & -2 \end{pmatrix}$$

$$\simeq \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$\simeq \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Call $\pi^T = x$ for simplification. We get the following four equations:

$$x_1 = 2x_4 \tag{1}$$

$$x_2 = x_4 \tag{2}$$

$$x_3 = 2x_4 \tag{3}$$

$$x_1 + x_2 + x_3 + x_4 = 1 \tag{4}$$

Solving:

$$x = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}$$

and thus

$$\boxed{\pi = (\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6})}$$

is our invariant probability vector.

$\blacksquare$

(e) Let J_n denote the probability that at time n the system has been in state C exactly the same number of times it has been in state A. Find

$$\lim_{n \to \infty} J_n.$$

Here again suppose that $X_0 = B$

SOLUTION: Define
$$N_C = |\{X_k = C \mid k \le N\}|$$
$$N_A = |\{X_k = A \mid k \le N\}|$$

Note that
$$N_A = \begin{cases} 1 + N_C, & X_N = C \\ N_C, & X_N \ne C \end{cases}$$

Thus,
$$J_N = \{N_C = N_A\} = \{X_N \ne C\}.$$

Thus,
$$\lim_{N \to \infty} \mathbb{P}\{J_N\} = \lim_{N \to \infty} \mathbb{P}\{X_N \ne C\} = 1 - \pi_C = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$$

∎

Problem 9

Let G be a finite, connected graph (i.e., any two vertices can be joined by a path of edges). Let X_n and Y_n be independent lazy random walks on G, meaning that at each step X_n stays put with probability $\frac{1}{2}$ and moves to a uniformly chosen neighboring vertex with probability $\frac{1}{2}$, and similarly for Y_n. In other words, the transition probabilities for $\{X_n\}$ (and also for $\{Y_n\}$) are given by

$$p(x,y) = \begin{cases} \frac{1}{2}, & \text{if } x = y \\ \frac{1}{2 \deg x}, & \text{if } x \neq y \text{ and } x \text{ and } y \text{ are joined by an edge} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that for any possible starting points $X_0 = x_0$ and $Y_0 = y_0$, with probability 1, there exists n such that $X_n = Y_n$.

> SOLUTION: Enumerate the vertices of $G(V, E)$ by $V(G) = \{1, 2, \ldots, N\}$. Consider the process $Z_n = (X_n, Y_n)$ taking place in the state space $V(G) \times V(G)$. We claim that $\{Z_n\}$ is a Markov process. First, notice that
>
> $$p_Z(v, v') = p_Z((x,y),(x',y')) = \mathbb{P}\{X_1 = x', Y_1' = y' \mid X_0 = x, Y_0 = y = \begin{cases} p(x,x')p(y,y'). & x \neq y \\ p(x,x'), & x = y \end{cases}\}$$
>
> Thus, $\{Z_n\}$ acts like $\{X_n\}$ once the chains couple up. We claim that if $\tau = \min\{n \geq 0 \mid X_n = Y_n\} = \min\{n \geq 0 \mid Z_n = (z,z).z \in V(G)\}$, then
>
> $$\mathbb{P}\{\tau < \infty \mid Z_0 = (x_0, y_0)\} = 1.$$
>
> To do this, it suffices to show that Z_n is an irreducible and recurrent Markov chain, since they hit all the states in S with probability 1.
>
> (Markov Chain) Let $v_0, v_1, \ldots, v_n \in V(G) \times V(G)$. then by definition,
>
> $$\mathbb{P}\{Z_n = v_n \mid Z_0 = v_0, \ldots, Z_{n-1} = v_{n-1}\} = \mathbb{P}\{(X_n, Y_n) = (x_n, y_n) \mid (X_0, Y_0) = (x_0, y_0), \ldots, (X_{n-1}, Y_{n-1}) = (x_{n-1}$$
> $$= \mathbb{P}\{X_n = x_n \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}\} \mathbb{P}\{Y_n = u_n \mid Y_0 = y_0, \ldots, Y_n$$
> $$= \mathbb{P}\{X_n = x_n \mid X_{n-1} = x_{n-1}\} \mathbb{P}\{Y_n = u_n \mid Y_{n-1} = y_{n-1}\}$$
> $$= \mathbb{P}\{(X_n, Y_n) = (x_n, y_n) \mid (X_{n-1}, Y_{n-1}) = (x_{n-1}, y_{n-1})\}$$
> $$= \mathbb{P}\{Z_n = v_n \mid Z_{n-1} = v_{n-1}\}$$
>
> (Irreducible) It suffices to show that for any $v, w \in S \times S$, there exists some $n > 0$ such that $p^n(v, w) > 0$. But if $v = (x, y)$ and $w = (x', y')$, then since X_n is irreducible (since the graph is connected), there exists some $n_x > 0$ such that $p^{n_x}(x, x') > 0$ and similarly $p^{n_y}(y, y') > 0$. Let $n = \max\{n_x, n_y\}$, then
>
> $$p^n((x,y),(x',y')) = p^n(x,x')p^n(y,y') \geq p^{n_x}(x,x')p^{n_y}(y,y') > 0.$$
>
> Since $S \times S$ is irreducible, then it is recurrent. Note the importance that X_n and Y_n are lazy, since this guarantees that
> $$p^n(v, v) > 0.$$

Thus, by a theorem done in class, since $S \times S$ is finite and $\{Z_n\}$ is recurrent and $(x, x) \in S \times S$, then with probability one, $\mathbb{P}\{Z_n = (x, x)\}$. Thus, $X_n = x$ and $Y_n = x$ for some n with probability one.

Here is another proof which works in the degenerate case when $G(V, E)$ is a circle. Consider the process $\{Z_n\}$, which is defined as

$$Z_n := \min \# \text{ edges between } X_n \text{ and } Y_n$$

Since $G(V, E)$ is finite and connected, the state space of Z_n is $S_Z = \{0, 1, \ldots, K\}$, where $K \leq \frac{N(N-1)}{2}$. We claim that $\{Z_n\}$ is a Markov chain with a single communication class that is recurrent and aperiodic.

To see that $\{Z_n\}$ is a Markov chain, note that Z_{n+1} depends only on X_n and Y_n. By independence, $\{X_n\}$ is independent of Y_{n-1}. Similarly, Y_n is independent of X_{n-1}. Since both are Markov chains, then X_n is independent of X_{n-1} and Y_n is independent of Y_{n-1}. Thus, since X_n and Y_n completely determine Z_n, then Z_{n+1} depends only on Z_n and is independent of Z_{n-1}.

TO see that $\{Z_n\}$ has one communication class, we suppose otherwise. That is, for some $i, j \in S$, and for all $n > 0$, $p^n(i, j) = 0$. Without loss of generality, suppose $i < j$ and let $k = j - i$ We know that $p^n(i, j) \geq p^{n-1}(i, j-1)p(i, j-1) \geq \cdots \geq p^{n-k}(i, i)p(i, i+1) \cdots p(i, j-1)$. It suffices to show that each terms is positive. We know that, for any i, $p(i, i+1) > 0$ since there is a chance X_n stays at some $x \in S$ and Y_n leaves, resulting in an edge of 1. Similarly, $p^{n-k}(i, i) \geq \frac{1}{2^{n-k+1}}$. Thus, there exists some $n \in \mathbb{N}$ such that $p^n(i, j) > 0$ for all i, j. This also shows that $\{Z_n\}$ is recurrent.

$\{Z_n\}$ is aperiodic since $p(i, i) > 0$.

By a theorem in class, we have that for any $s \in S$

$$\lim_{n \to \infty} \mathbb{P}\{Z_n = 0 \text{ i.o. } \mid s\} = 1$$

Thus, with probability 1, there exists some $M > 0$ large such that $Z_n = 0$ and thus $X_M = Y_M$. ∎

(b) Give an example of a graph G for which the conclusion of part (a) fails if X_n and Y_n are ordinary random walks rather than lazy random walks (i.e., X_{n+1} is never equal to X_n and Y_{n+1} is never equal to Y_n.

SOLUTION: Consider the following graph G:

①————①————②
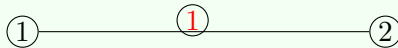
And consider when $X_0 = 1$ and $Y_0 = y$. Then since $X_n = S \setminus \{Y_n\}$, the Markov chains will never be the same. ∎

Problem 10

Let P be irreducible and aperiodic,