Agustín Esteva
aesteva@uchicago.edu
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Problem 1

Consider the queuing model as discussed in class (section 1.2 of the week 3 notes on canvas).

(a) For the transient case (i.e., when $q < p$) compute

$$\alpha(x) := \mathbb{P}\{\text{starting at } x \text{ the queue ever reaches state } 0\}.$$

Solution: Let $x \geq 0$, We use the law of total probability and the Markov property to compute:

$$\begin{aligned}
\alpha(x) &= \alpha(x-1)q(1-p) + \alpha(x)(pq + (1-q)(1-p)) + \alpha(x+1)p(1-q) \\
&= \alpha(x-1)q(1-p) + \alpha(x)\big(1 - p(1-q) - q(1-p)\big) + \alpha(x+1)p(1-p) \\
&= \alpha(x-1)q(1-p) + \alpha(x) - \alpha(x)\big(p(1-q) + q(1-p)\big) + \alpha(x+1)p(1-q) \\
&= \alpha(x-1)\frac{q(1-p)}{p(1-q)+q(1-q)} + \alpha(x+1)\frac{p(1-q)}{p(1-q)+q(1-p)} \\
&= \alpha(x-1)A + \alpha(x+1)B
\end{aligned}$$

So then after some algebra and using the general formula that

$$\alpha_{\pm} = \frac{1 \pm \sqrt{1-4AB}}{2A} \implies \alpha \in \{1, \frac{q(1-p)}{p(1-q)}\}$$

Thus,

$$\alpha(x) = \lambda_1 + \lambda_2 \left(\frac{q(1-p)}{p(1-q)}\right)^x$$

We have two boundary conditions:

$$\alpha(0) = 1, \qquad \lim_{n\to\infty} \alpha(n) = 0$$

From the first, we see that $\lambda_1 + \lambda_2 = 1$. From the second, we see that since $q < p$, then

$$q < p \iff q - qp < p - qp \iff q(1-p) < p(1-q) \iff \frac{q(1-p)}{p(1-q)} < 1 \implies \left(\frac{q(1-p)}{p(1-q)}\right)^n \to 0,$$

and so $\lambda_1 = 0$. Thus, $\lambda_2 = 1$ and so

$$\boxed{\alpha(x) = \left(\frac{q(1-p)}{p(1-q)}\right)^x}$$

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(b) For which values of p, q is the chain null/positive recurrent? In the positive recurrent case, give the stationary distribution.

Solution: The chain is positive recurrent if and only if a stationary distribution exists, so it suffices to find a condition for which the stationary distribution exists. A stationary distribution must satisfy

$$\pi_0 = (1-p)\pi_0 + q(1-p)\pi_1$$

$$\pi_1 = p\pi_0 + \big(pq + (1-p)(1-q)\big)\pi_1 + q(1-p)\pi_2$$

$$\pi_n = p(1-q)\pi_{n-1} + \big(pq + (1-p)(1-q)\big)\pi_n + q(1-p)\pi_{n+1}, \quad n \geq 2$$

We have already solved the recursive relation.

$$\pi_n = \lambda_1 + \lambda_2 \left(\frac{p(1-q)}{q(1-p)}\right)^n$$

Solving for the constants, we see that

$$1 = \sum_{n=0}^{\infty} \pi_n = \sum_{n=1}^{\infty} \lambda_1 + \lambda_2 \left(\frac{p(1-q)}{q(1-p)}\right)^n \implies \lambda_1 = 0.$$

Thus, we see that

$$\lambda_2 \sum_{n=0}^{\infty} \left(\frac{p(1-q)}{q(1-p)}\right)^n < \infty \iff p < q.$$

Thus, the chain is null recurrent if, and only if, $p = q$. It is positive recurrent if $p < q$. From the above, we see that if $p < q$, then the series is geometric and thus

$$1 = \lambda_2 \sum_{n=0}^{\infty} \left(\frac{p(1-q)}{q(1-p)}\right)^n = \frac{\lambda_2}{1 - (\frac{p(1-q)}{q(1-p)})} \implies \lambda_2 = 1 - \frac{p(1-q)}{q(1-p)}.$$

Thus,

$$\boxed{\pi_n = (1 - \frac{p(1-q)}{q(1-p)}) \left(\frac{p(1-q)}{q(1-p)}\right)^n}$$

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(c) What is the average length of the queue in equilibrium (i.e., the long-run average length of the queue)?

Solution: Clearly, if $q > p$, then the average length is infinity. If $q < p$, then we see that

$$\mathbb{E}[\pi] = \sum_{n=0}^{\infty} n\pi_n$$

$$= (1 - \frac{p(1-q)}{q(1-p)}) \sum_{n=0}^{\infty} n \left(\frac{p(1-q)}{q(1-p)}\right)^n$$

$$= (1 - \frac{p(1-q)}{q(1-p)}) \left(\frac{\frac{p(1-q)}{q(1-p)}}{(1 - \frac{p(1-q)}{q(1-p)})^2}\right)$$

$$= \frac{\frac{p(1-q)}{q(1-p)}}{1 - \frac{p(1-q)}{q(1-p)}}$$

$$= \boxed{\frac{p(1-q)}{q-p}}$$

Finally, if $q = p$, then we claim that the average length is also infinite. By the previous problem, this is the case when the queue is null recurrent. Suppose not, that as $n \to \infty$, the queue reaches an equilibrium. That is,

$$\lim_{n\to\infty} X_n = x.$$

But then if we define

$$T_x = \min\{n \geq 1 \mid X_n = x \mid X_0 = x\},$$

then clearly, $\mathbb{P}\{T_x < \infty\} = 1$, and so

$$\mathbb{E}[T_x \mid X_0 = x] < \infty.$$

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Problem 2

Consider a Markov chain $\{X_n\}$ with state space $S = \{0, 1, 2, \ldots\}$. A sequence of positive numbers $p_1, p_2, \ldots$ is given such that

$$\sum_{i=1}^{\infty} p_i = 1.$$

The transition probabilities are defined as follows:

$$p(x, x-1) = 1, \quad \text{for } x > 0,$$

$$p(0, x) = p_x, \quad \text{for } x > 0.$$

That is, whenever the chain reaches state 0, it jumps to a new state $x > 0$ with probability p_x. From any state $x > 0$, it moves deterministically to $x - 1$ in the next step. This chain is recurrent because it keeps returning to state 0.

We want to determine the conditions necessary on the p_x so that this is positive recurrent.

Solution: Let $x \in S$, then define

$$T := \min\{n \geq 1 : X_n = 0 \mid X_0 = 0\}.$$

Using the law of total expectation, we find that

$$\mathbb{E}[T] = 2p_1 + 3p_2 + \cdots = \sum_{x=1}^{\infty} (x+1) p_x = 1 + \sum_{x=0}^{\infty} x p_x.$$

Thus, the chain is positive recurrent if, and only if,

$$\boxed{\sum_{x=0}^{\infty} x p_x < \infty.}$$

In this case, we have that

$$\boxed{\pi_0 = \frac{1}{\mathbb{E}[T]} = \frac{1}{1 + \sum_{x=1}^{\infty} x p_x}.}$$

A stationary distribution must satisfy

$$\pi_{n+1} = \pi_n - p_n \pi_0 = (\pi_{n-1} - p_{n-1}\pi_0) - p_n \pi_0 = \pi_0(1 - p_1 - p_2 - \cdots - p_n) = \pi_0(1 - \sum_{x=1}^{n} p_x)$$

Thus,

$$\boxed{\pi_n = \frac{1 - \sum_{x=1}^{n-1} p_x}{1 + \sum_{x=1}^{\infty} x p_x}, \quad n \geq 1}$$

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Problem 3 (Optional)

A *diagonal lattice path* is a "curve" in the plane made up of line segments that go from a point (i, j) to either $(i+1, j+1)$ (an up step) or $(i+1, j-1)$ (a down step). A *Dyck path of length* $2n$ is a diagonal lattice path from $(0,0)$ to $(2n,0)$ that does not go below the x-axis.

(a) Prove that the diagonal lattice paths from $(0,0)$ to $(2n,0)$ that go below the x-axis are in bijection with the diagonal lattice paths from $(0,0)$ to $(2n,-2)$. *(Hint: Given a path P from $(0,0)$ to $(0,2n)$ that goes below the x-axis, consider the first edge e that crosses $y=0$. Switch the directions of every edge after e, i.e., an up edge becomes down, and a down edge becomes up.)*

(b) Show that the number of Dyck paths from $(0,0)$ to $(2n,0)$ is given by

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

The quantity C_n is called the n^{th} *Catalan number*, and appears very frequently in enumerative combinatorics.

(c) Let $\{X_n\}$ be a simple random walk on $\mathbb{Z}$ starting at 0, and let $T := \min\{n \geq 1 : X_n = 0\}$.

   (i) Let $E_k := \{T = 2k\}$ be the event that the walk first returns to 0 at time $2k$. Use the previous parts to find $\mathbb{P}\{E_k\}$ in terms of k.

   (ii) Use Stirling's approximation to show that $E[T] = \infty$.

Problem 4

For each of the following Markov chains, determine whether the chain is positive recurrent, null recurrent, or transient. In the positive recurrent case, find the stationary distribution.

(a) For $x \in \mathbb{Z}$ with $x \geq 0$, $p(x, 0) = (x+1)/(x+2)$ and $p(x, x+1) = 1/(x+2)$ ($p(x, y) = 0$ for all other y).

SOLUTION: We claim that this process is positive recurrent. It suffices to find a stationary distribution. The stationary distribution must satisfy

$$\pi_0 = \sum_{n=0}^{\infty} \frac{n+1}{n+2}\pi_n, \quad \pi_n = \frac{1}{(n+1)!}\pi_0, \quad \sum_{n=0}^{\infty} \pi_n = 1.$$

From the second and third equations, we see that

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} \pi_n \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\pi_0 \\ &= \pi_0(e-1) \end{aligned}$$

and so $\pi_0 = (e-2)!$. Thus,

$$\boxed{\pi_n = \frac{1}{(n+1)!}\frac{1}{(e-1)}, \text{ positive recurrent}}$$

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(b) For $x \in \mathbb{Z}$ with $x \geq 0$, $p(x, 0) = 1/(x+2)^2$ and $p(x, x+1) = 1 - 1/(x+2)^2$ ($p(x, y) = 0$ for all other y).

SOLUTION: Consider that if

$$T = \min\{n > 0 : X_n = 0 \mid X_0 = 0\},$$

then

$$\begin{aligned} \mathbb{P}\{T = \infty\} &= \lim_{n\to\infty} \mathbb{P}\{T > n\} \\ &= \lim_{n\to\infty} \prod_{k=0}^{n-1}(1 - \frac{1}{(k+2)^2}) \\ &= L \\ &\implies \ln(L) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \ln\left(\prod_{k=0}^{n-1}(1 - \frac{1}{(k+2)^2})\right) \\
&= \lim_{n\to\infty} \sum_{k=0}^{n-1} \ln\left(1 - \frac{1}{(k+2)^2}\right) \\
&\sim \sum_{k=0}^{\infty} \ln\left(\frac{(k+2)^2 - 1}{(k+2)^2}\right) \\
&= \sum_{k=0}^{\infty} \ln\big(k^2 + 4k + 3\big) - \ln\big((k+2)^2\big) \\
&= \sum_{k=0}^{\infty} \ln(k+3) + \ln(k+1) - 2\ln(k+2) \\
&= \sum_{n=3}^{\infty} \ln(n) + \sum_{n=1}^{\infty} \ln(n) - 2\sum_{n=2}^{\infty} \ln(n) \\
&= \ln(1) - \ln(2) \\
&= \ln\left(\frac{1}{2}\right) \\
&\implies L = \frac{1}{2}.
\end{aligned}
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Thus,

$$
\boxed{\mathbb{P}\{T = \infty\} = \frac{1}{2} > 0, \text{ transient}}
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Problem 5

Consider the Markov chain with state space $S = \{0, 1, 2, \cdots\}$ with transition probabilities

$$p(0,0) = \frac{2}{3}, \quad p(0,1) = \frac{1}{3},$$

$$p(x, x-1) = \frac{2}{3}, \quad p(x, x+1) = \frac{1}{3}, \quad x > 0.$$

(a) Show that this is positive recurrent by giving the invariant probability.

SOLUTION: A stationary probability satisfies the recursive relation

$$\pi_n = \frac{1}{3}\pi_{n-1} + \frac{2}{3}\pi_{n+1}.$$

Then

$$\alpha = \frac{1 \pm \sqrt{1 - 4(\frac{1}{3} \cdot \frac{2}{3})}}{\frac{4}{3}} \in \{1, \frac{1}{2}\}.$$

Thus,

$$\pi_n = \lambda_1 + \lambda_2 \frac{1}{2^n}.$$

We have that

$$1 = \sum_{n=0}^{\infty} \pi_n = \sum_{n=0}^{\infty} \lambda_1 + \lambda_2 \sum_{n=0}^{\infty} \frac{1}{2^n} \implies \lambda_1 = 0, \lambda_2 = \frac{1}{2}.$$

Thus,

$$\boxed{\pi_n = \frac{1}{2^{n+1}}}.$$

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(b) For $x > 0$, let E_x denote the expected number of steps in the chain until it reaches the origin assuming that $X_0 = x$. Find E_1. *(Hint: first consider the expected return time starting at the origin and write E_1 in terms of this.)*

SOLUTION: From above, since $\pi_0 = \frac{1}{2}$, then

$$E_0 = 2.$$

However, we can also write

$$E_0 = \mathbb{E}[n > 0 : X_n = 0 \mid X_0 = x] = \mathbb{E}[\mathbb{E}[n > 0 : X_n = 0 \mid X_1]] = (1+0)\frac{2}{3} + (1+E_1)\frac{1}{3} = 1 + \frac{1}{3}E_1.$$

Thus,

$$1 + \frac{1}{3}E_1 = 2 \implies \boxed{E_1 = 3}.$$

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(c) Find E_x for all $x > 0$.

SOLUTION: We claim that $\boxed{E_x = 3x}$ for all $x > 0$. To see this, note that by the law of total expectation, they satisfy the recursive relation

$$E_x = 1 + \frac{2}{3}E_{x-1} + \frac{1}{3}E_{x+1} \tag{1}$$

We induct. For $n = 1$, we have that by the previous part,

$$E_1 = 3 = 3(1).$$

Suppose (1) holds for a general n. Then

$$E_n = 1 + \frac{2}{3}E_{n-1} + \frac{1}{3}E_{n+1} \implies 3n - 1 - \frac{2}{3}3(n-1) = \frac{1}{3}E_{n+1}.$$

Solving,

$$E_{n+1} = 9n - 3 - 6n + 6 = 3n + 3 = 3(n+1).$$

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(d) Suppose we modify the chain so that

$$p(0,1) = \frac{1}{4}, \quad p(0,2) = \frac{1}{4}, \quad p(0,3) = \frac{1}{4}, \quad p(0,4) = \frac{1}{4}.$$

The transitions for $x > 0$ are the same as before. Let π denote the invariant probability for this new chain. Find $\pi(0)$.

SOLUTION: For $x \geq 1$, the transition probability is unaffected. Thus, E_x remains unchaged for every $x \neq 0$. We know that using the law of total expectation:

$$E_0 = \frac{1}{4}E_1 + \frac{1}{4}E_2 + \frac{1}{4}E_3 + \frac{1}{4}E_4 + 1 = \frac{1}{4}3 + \frac{1}{4}6 + \frac{1}{4}9 + \frac{1}{4}12 + 1 = 8.5.$$

Thus,

$$\mathbb{E}[n : X_n = 0 \mid X_0 = 0] = 8.5 \implies \boxed{\pi(0) = \frac{1}{8.5} = \frac{2}{17}}.$$

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(e) Find $\pi(1)$ for this new chain.

SOLUTION: From the transition probabilities we have that

$$\pi_0 = \frac{2}{3}\pi_1 \implies \boxed{\pi_1 = \frac{3}{2}\frac{2}{17} = \frac{3}{17}}$$

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Problem 6

Let $\{Y_j\}_{j\in\mathbb{N}}$ be independent, identically distributed integer-valued random variables which are not identically equal to zero. For a given value of $X_0 \in \mathbb{Z}$, let $X_n = X_0 + \sum_{j=1}^n Y_j$ for each $n \geq 1$. We view $\{X_n\}$ as a Markov chain taking values in $\mathbb{Z}$. Show that $\{X_n\}$ does not have a stationary distribution. Conclude that $\{X_n\}$ is either null recurrent or transient, not positive recurrent. *(Hint: assume for contradiction that there is a stationary distribution π, and look at a value of $n \in \mathbb{Z}$ such that $\pi(n)$ is maximal).*

SOLUTION: Let π be a stationary distribution. Let $N \in \mathbb{Z}$ such that $\pi_N \geq \pi_n$ for all $n \in \mathbb{Z}$. Note that such a π_N must exist. To show this, suppose it doesn't exist. Consider π_{n_0}. Either $\pi_{n_0} = 0$ or $\pi_{n_0} > 0$. If the former, we know that π_{n_0} is not maximal, and so there exists some n_1 such that $\pi_{n_1} > \pi_{n_0}$. Thus, take $\pi_{n_0} > 0$ without loss of generality. Let $\epsilon = \frac{\pi_{n_0}}{2}$. There exists some n_1 such that $\pi_{n_1} > \pi_{n_0}$. Since π_{n_1} isn't maximal, there exists some $\pi_{n_2} > \pi_{n_1} > \epsilon$. Thus, since $\pi_x \geq 0$ for all $x \in \mathbb{Z}$, we have that

$$1 = \sum_{x\in\mathbb{Z}} \pi_x \geq \sum_{i=1}^{\infty} \pi_{n_i} > \sum_{i=1}^{\infty} \epsilon = \infty,$$

which is clearly a contradiction. Thus, we can let π_N be maximal.

By definition,

$$\begin{aligned}
\pi_N &= \sum_{x\in\mathbb{Z}} \pi_x p(x, N) \\
&= \sum_{x\in\mathbb{Z}} \pi_x \mathbb{P}\{X_1 = N \mid X_0 = x\} \\
&= \sum_{x\in\mathbb{Z}} \pi_x \mathbb{P}\{X_0 + Y_1 = N \mid X_0 = x\} \\
&= \sum_{x\in\mathbb{Z}} \pi_x \mathbb{P}\{Y_1 = N - x\} \\
&\leq \pi_N \sum_{x\in\mathbb{Z}} \mathbb{P}\{Y_1 = N - x\} \\
&= \pi_N
\end{aligned}$$

We claim that

$$\pi_x = \pi_N \quad \text{whenever} \quad \mathbb{P}\{Y_1 = N - x\} > 0. \tag{2}$$

Suppose not. Let x' such that $\pi_{x'} < \pi_N$ and $\mathbb{P}\{Y_1 = N - x'\} > 0$. But then

$$\pi_N = \sum_{x\in\mathbb{Z}} \pi_x \mathbb{P}\{Y_1 = N - x\} < \sum_{x\in\mathbb{Z}} \pi_N \mathbb{P}\{Y_1 = N - x\} = \pi_N.$$

We claim now that $\pi_N > 0$. Suppose now, that $\pi_N = 0$, but then $\pi_x \leq \pi_N = 0$ for all $x \in N$ and thus

$$\sum_{x\in\mathbb{Z}} \pi_x = 0,$$

which is a contradiction.

Suppose that for all $x \in \mathbb{Z}$, $\pi_x < \pi_N$. Then by what we showed above, $\mathbb{P}\{Y_1 = N - x\} = 0$. Thus,

$$\pi_N = \sum_{x\in\mathbb{Z}} \pi_x \mathbb{P}\{Y_1 = N - x\} \implies \mathbb{P}\{Y_1 = N - N\} = 1,$$

which contradicts the fact that Y_1 is not identically 0. Let $\pi_{i_1} = \pi_N$. We claim that $\pi_{2N-i} = \pi_N$.

$$\pi_{i_1} = \pi_N \mathbb{P}\{i_1 - N\} + \pi_{i_1}\mathbb{P}\{i_1 - i_1\} \implies \mathbb{P}\{i_1 - N\} > 0.$$

But then since $i - N = N - (2N - i_1)$, we by (2) that $\pi_{2N-i_1} = \pi_N$. Call $i_2 := 2N - i_1$. Using this process, we find $\{i_1, i_2, \dots\}$ such that

$$\pi_{i_n} = \pi_N, \quad \forall\, n > 0.$$

Thus,

$$1 = \sum_{x\in\mathbb{Z}} \pi_x \geq \sum_{n=1}^{\infty} \pi_{i_n} = \sum_{n=1}^{\infty} \pi_n = \infty,$$

which is a contradiction.

Thus, there does not exist a stationary distribution, implying that this Markov chain is either null recurrent or transient. ∎

Problem 7 (Optional)

In this exercise, we will establish Stirling's formula. Let $X_1, X_2, \ldots$ be independent Poisson random variables with mean 1 and let $Y_n = X_1 + \cdots + X_n$, which is a Poisson random variable with mean n. Let

$$p(n,k) = \mathbb{P}\{Y_n = k\} = e^{-n}\frac{n^k}{k!}.$$

(a) Use the central limit theorem to show that if $a > 0$,

$$\lim_{n\to\infty} \sum_{n \le k < n + a\sqrt{n}} p(n,k) = \int_0^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

Solution: We have that

$$\sum_{n \le k \le n + a\sqrt{n}} p(n,k) = \mathbb{P}\{n \le Y_n \le n + a\sqrt{n}\} = \mathbb{P}\{\frac{Y_n - n}{\sqrt{n}} \le a\}.$$

By the CLT, since $\mathbb{E}[Y_n] = n$ and $\mathbb{V}[Y_n] = n$, then in the limit,

$$\frac{Y_n - n}{\sqrt{n}} \sim N(0,1),$$

and so

$$\lim_{n\to\infty} \mathbb{P}\{\frac{Y_n - n}{\sqrt{n}} \le a\} = \int_0^a \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

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(b) Show that if $a > 0$, n is a positive integer, and $n \le k < n + a\sqrt{n}$, then

$$e^{-a^2} p(n,n) \le p(n,k) \le p(n,n).$$

Solution: Since $p(n,k) = e^{-n}\frac{n^k}{k!}$, we have that since $n \le k$, then

$$p(n,n) = e^{-n}\frac{n^k}{n!} \ge p(n,k)$$

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(c) Use (a) and (b) to conclude that

$$p(n,n) \sim \frac{1}{\sqrt{2\pi n}}.$$

Stirling's formula follows immediately.