Agustín Esteva
aesteva@uchicago.edu

Due Date: March 27, 2025

Problem 1

Define

$$c_0 := \{(a_n) \in \ell^\infty \ : \ a_n \to 0\}$$

Show that $c_0^* = \ell^1$.

Solution: Let $x \in c_0$. We represent x via a Schauder basis:

$$x = \sum_{n=1}^\infty x_n e_n,$$

where $e_n = \delta_{ni}$ (kronecker-delta) and $x_n \in \mathbb{R}$. Evidently, this representation is unique for each x. Thus, we have that for any $f \in c_0^*$,

$$\langle f, x\rangle = \langle f, \sum_{n=1}^\infty x_n e_n\rangle = \sum_{n=1}^\infty x_n f(e_n).$$

We define $f(e_n) = a_n$. Thus,

$$f(x) = \sum_{n=1}^\infty x_n a_n.$$

Uniqueness comes from the uniqueness of the Schauder basis. We note that

$$|f(x)| \leq \|f\|_{c_0^*}\|x\|_\infty \tag{1}$$

We will next show that $(a_n) \in \ell^1$. Define

$$x_i = \begin{cases} \text{sign}(a_n), & n \leq N \\ 0, & n > N \end{cases}.$$

Evidently, $x \in c_0$ with $\|x\|_\infty = 1$ We use (1) and see that

$$\sum_{n=1}^N |a_n| = |f(x)| = \left|\sum_{n=1}^\infty x_n a_n\right| \leq \|f\|_{c_0^*}\|x\|_\infty = \|f\|_{c_0^*} \tag{2}$$

Thus, because $f \in c_0^*$ then $\|f\| < \infty$ and because $x \in c_0$. Thus, because (2) holds for any N, we see that $a \in \ell^1$. To see that $f \to a$ is an isometry, we need to show that $\|f\|_{c_0^*} = \|a\|_1$. We use Hölder's inequality for one side:

$$|f(x)| = \sum_{n=1}^\infty x_n a_n \leq \|x\|_\infty \|a\|_1 \implies \sup_{\|x\|_\infty = 1} |f(x)| = \|f\|_{c_0^*} \leq \|a\|_1.$$

We use (2) to directly show that $\|a\|_1 \leq \|f\|_{c_0^*}$.

Finally, we see that f is bounded. f linear is obvious from the definition. $\blacksquare$

Problem 2

Show that $(\ell^1)^* = \ell^\infty$.

Solution: Let $x \in \ell^1$. We represent x via a Schauder basis:

$$x = \sum_{n=1}^{\infty} x_n e_n$$

as in the problem above. Thus, for any $f \in (\ell^1)^*$:

$$\langle f, x \rangle = \langle f, \sum_{n=1}^{\infty} x_n e_n \rangle = \sum_{n=1}^{\infty} x_n f(e_n).$$

We define $f(e_n) = a_n$. Thus,

$$f(x) = \sum_{n=1}^{\infty} x_n a_n.$$

Uniqueness comes from the uniqueness of the Schauder basis. We note that

$$|f(x)| \leq \|f\|_{c_0^*} \|x\|_1$$

To show that $a = (a_n) \in \ell^\infty$, simply notice that

$$|a_n| = |f(e_n)| \leq \|f\| \|e_n\|_1 = \|f\| \implies \|a\|_\infty = \sup_n |a_n| \leq \|f\| \tag{3}$$

Thus, $a \in \ell^\infty$.

To see that $f \mapsto a$ is an isometry, we use Hölder's inequality:

$$|f(x)| = \left| \sum_{n=1}^{\infty} x_n a_n \right| \leq \|x\|_1 \|a\|_\infty \implies \|f\| = \sup_{\|x\|_1 = 1} |f(x)| \leq \|a\|_\infty.$$

We use (3) to see the other side of the inequality. ■

Problem 3

Suppose $p \in (1, \infty)$ and $q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $(\ell^p)^* = \ell^q$.

Solution: Let $x \in \ell^p$. We represent x via a Schauder basis:

$$x = \sum_{n=1}^{\infty} x_n e_n$$

as in the problem above. Let $f \in (\ell^p)^*$. Then

$$\langle f, x \rangle = \langle f, \sum_{n=1}^{\infty} x_n e_n \rangle = \sum_{n=1}^{\infty} x_n f(e_n).$$

We define $f(e_n) = a_n$. Thus,

$$f(x) = \sum_{n=1}^{\infty} x_n a_n,$$

and so

$$|f(x)| \leq \|f\| \|x\|_p.$$

To show that $a \in \ell^q$, we define

$$x_n = \begin{cases} \frac{|a_n|^q}{a_n}, & a_n \neq 0, n \leq N \\ 0, & a_n = 0, n > N \end{cases}$$

Thus, we see that

$$|f(x)| = \left| \sum_{n=1}^{\infty} x_n a_n \right| = \sum_{n=1}^{N} |a_n|^q \leq \|f\| \|x\|_p = \|f\| \left(\sum_{n=1}^{\infty} |a_n|^{(q-1)p} \right)^{\frac{1}{p}} = \|f\| \left(\sum_{n=1}^{\infty} |a_n|^q \right)^{\frac{1}{p}},$$

and so taking $N \to \infty$ and dividing by the very last term, we see that $a \in \ell^q$ since $\|a\|_q \leq \|f\|$.

We use Hölder again:

$$|f(x)| \leq \sum_{n=1}^{\infty} |x_n||a_n| \leq \|x\|_p \|a\|_q \implies \|f\| = \sup_{\|x\|_p = 1} \leq \|a\|_q.$$

■

Problem 4

Let C be a convex symmetric subset of a Banach space X. Assume that the linear functional f on X is continuous at 0 when restricted to C. Show that $f|_C$ is uniformly continuous.

Solution: Since $f|_C$ is continuous at 0, then for all $\epsilon > 0$, there exists some $\delta > 0$ such that if $v \in C$ with $\|v\| < \delta$, then $\|f(v)\| < \epsilon$. Take $x, y \in C$ with $\|x - y\| < \delta$. Since $y \in C$ and C is symmetric, then $-y \in C$. Take $t = \frac{1}{2}$, then by convexity,

$$g(t) := tx + (1-t)(-y) \implies g(\frac{1}{2}) = \frac{1}{2}x - \frac{1}{2}y = \frac{1}{2}(x-y) \in C.$$

Thus,

$$\|\frac{1}{2}(x-y)\| < \delta \implies \|f(\frac{1}{2}(x-y))\| = \frac{1}{2}\|f(x) - f(y)\| < \epsilon,$$

and so f is uniformly continuous on C. ■

Problem 5

Let f be a linear functional on a Banach space X. Suppose $f \not\equiv 0$, then the following are equivalent:

(a) f is continuous.

(b) f is continuous at 0.

(c) $f^{-1}(\{0\})$ is closed.

SOLUTION: (i $\mapsto$ ii) is obvious.

(ii $\mapsto$ iii) Let $(x_n) \in f^{-1}(0)$ with $x_n \to x$. We want to show that $f(x) = 0$. Since f is continuous at 0, then for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $v \in X$ with $\|v\| < \delta$, then $\|f(v)\| < \epsilon$.

For n large, we have that $\|x_n - x\| < \delta$, and thus since $f(x_n) = 0$:

$$\|f(x)\| = \|f(x) - f(x_n)\| = \|f(x - x_n)\| < \epsilon,$$

and thus $f(x) = 0$, and so $x \in f^{-1}(0)$.

(iii $\mapsto$ i) Suppose that for each n, there exists some $x_n \in X$ with $x_n \neq 0$ such that

$$\|f(x_n)\| \geq n\|x\_n|,$$

that is X is not bounded. Let

$$z_n ;= x_1 - f(x_1)\frac{x_n}{f(x_n)}.$$

Thus, $f(z_n) = 0$, and so $z_n \in f^{-1}(\{0\})$. Moreover, $z_n \to x_1$ since

$$\|z_n - x_1\| = \|x_1 - f(x_1)\frac{x_n}{f(x_n)} - x_1\| = \|f(x_1)\frac{x_n}{f(x_n)}\| = |f(x_1)|\frac{\|x_n\|}{|f(x_n)|} \leq |f(x_1)|\frac{1}{n}$$

Thus, $z_n \to x_1$ but $f(x_1) \geq |x_1| > 0$ and so $x_1 \notin f^{-1}(\{0\})$, and thus $f^{-1}(\{0\})$ is not closed, a contradiction. Thus, f is bounded and so f is continuous. ■

Problem 6

Suppose X is finite dimensional Banach space and C is a convex subset that is dense in X. Then $C = X$.

Solution: Suppose not. Then let $x_0 \in C \setminus C$. Since C and $\{x_0\}$ are convex, disjoint, and nonempty, then the finite dimensional Hahn-Banach states that there exists some closed hyperplane $H = [f = \alpha]$ such that

$$f(C) \leq \alpha \leq f(x_0).$$

Let $z_0 = x_0 + \frac{x_0}{f(x_0)} \in X$. Then

$$f(C) \leq \alpha \leq f(x_0) < f(z_0) = f(x_0) + 1. \tag{4}$$

Let $\epsilon = \frac{f(z_0)}{2}$. By continuity of f, there exists some $\delta > 0$ such that if $\|x - z_0\| < \delta$, we have that $\|f(x) - f(z_0)\| < \frac{f(z_0)}{2}$. Since C is dense in X, we can find some $c \in C$ with $\|c - z_0\| < \delta$, a contradiction to (4). ■

Problem 7

Suppose $C \subset X$ with X Banach and $f : C \to \mathbb{R}$ Lipshitz. Show that f can be extended to all of X.

SOLUTION: Define $F : X \to \mathbb{R}$ such that

$$F(x) := \inf_{c \in C} [f(c) + L\|x - c\|] .$$

If $c_0 \in C$, then

$$F(c_0) = \inf_{c \in C} [f(c) + L\|c_0 - c\|] \leq f(c_0) + L\|c_0 - c_0\| = f(c_0).$$

We also have that for any $c \in C$:

$$f(c_0) - f(c) \leq L\|c_0 - c\| \implies f(c_0) \leq f(c) + L\|x_0 - c\| \implies f(c_0) \leq F(x).$$

We see that $F|_C = f$.

Suppose $x, y \in X$. By definition, for all $\epsilon > 0$, there exists some $c_x \in C$ such that

$$F(x) \geq f(c_x) + L\|x - c_x\| - \epsilon.$$

Since $c_x \in C$, then

$$F(y) \leq f(c_x) + L\|y - c_x\|.$$

Using the reverse triangle inequality:

$$\begin{aligned} F(y) - F(x) &\leq f(c_x) + L\|y - c_x\| - f(c_x) - L\|x - c_x\| + \epsilon \\ &\leq L\|y - c_x\| - L\|c_x - x\| + \epsilon \\ &\leq L\|y - x\| + \epsilon \end{aligned}$$

Because this is true for all $\epsilon > 0$, we see that $F(y) - F(x) \leq L\|y - x\|$.

For all $\epsilon > 0$, there exists some $c_y \in C$ such that

$$F(y) \geq f(c_y) + L\|y - c_y\| - \epsilon.$$

Since $c_y \in C$, we have that

$$F(x) \leq f(c_y) + L\|x - c_y\|.$$

Using the reverse triangle inequality:

$$\begin{aligned} F(x) - F(y) &\leq f(c_y) + L\|x - c_y\| - [f(c_y) + L\|y - c_y\| - \epsilon] \\ &= f(c_y) + L\|x - c_y\| - f(c_y) - L\|y - c_y\| + \epsilon \\ &= L(\|x - c_y\| - \|c_y - y\|) + \epsilon \\ &\leq L(\|x - y\|) + \epsilon, \end{aligned}$$

and so $F(x) - F(y) \leq L\|x - y\|$. Putting it together, we see that

$$|F(x) - F(y)| \leq L\|x - y\|$$

■

Problem 8

Let $X = \mathbb{R}^2$ with

$$\|x\| = (|x_1|^4 + |x_2|^4)^{\frac{1}{4}}.$$

Calculate directly the dual norm on X^*.

Solution: Let $x \in \mathbb{R}^2$.

$$f(x) = f(x_1 e_1 + x_2 e_2) = x_1 f(e_1) + x_2 f(e_2).$$

Thus, using Hölder's inequality:

$$\begin{aligned} |f(x)| &= |x_1 f(e_1) + x_2 f(e_2)| \\ &\leq |x_1||f(e_1)| + |x_2||f(e_2)| \\ &\leq (|x_1|^4 + |x_2|^4)^{\frac{1}{4}} (|f(e_1)|^{\frac{4}{3}} + |f(e_2)|^{\frac{4}{3}})^{\frac{3}{4}}. \end{aligned}$$

Thus,

$$\sup_{\|x\|=1} |f(x)| \leq (|f(e_1)|^{\frac{4}{3}} + |f(e_2)|^{\frac{4}{3}})^{\frac{3}{4}}.$$

Let $x \in \mathbb{R}^2$ with

$$x_n = \begin{cases} \frac{|f_n|^{\frac{4}{3}}}{f_n}, & f_n \neq 0 \\ 0, & f_n = 0 \end{cases}.$$

Thus, we see that

$$|f(x)| = \left| \sum_{n=1}^{2} x_n f_n \right| = \sum_{n=1}^{2} |f_n|^{\frac{4}{3}} \leq \|f\| \|x\| = \|f\| \left(\sum_{n=1}^{2} |x_n|^4 \right)^{\frac{1}{4}} = \|f\| \left(\sum_{n=1}^{2} |f_n|^{\frac{4}{3}} \right)^{\frac{1}{4}}.$$

Dividing by the very last term, we see that

$$\left(\sum_{n=1}^{2} |f_n|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \|f\|.$$

Thus,

$$\|f\| = \left(\sum_{n=1}^{2} |f_n|^{\frac{4}{3}} \right)^{\frac{3}{4}}$$

■

Problem 9

Let X, Y be Banach spaces and suppose $T \in \mathcal{L}(X, Y)$. Show the following are equivalent:

(a) $T(X)$ is closed.

(b) T is an open mapping from X unto $T(X)$

(c) There exists some $M > 0$ such that for all $y \in T(X)$, there exists some $x \in T^{-1}(y)$ such that $\|x\|_X \leq M\|y\|_Y$.

SOLUTION: (i $\mapsto$ ii) Suppose $T(X)$ is closed. Since $T(X) \subset Y$ and Y is Banach, then $T(X)$ is Banach. Thus, T is surjective unto $T(X)$, and so by the open mapping theorem, T is an open map from X unto $T(X)$.

(ii $\mapsto$ iii) Suppose T is an open map from X unto $T(X)$. Thus, there exists some $c > 0$ such that $B_c^F(0) \subset T(B_E)$. Let $y \in T(X)$. Then $\frac{1}{c}y \in B_c^F(0)$, and so $\frac{1}{c}y \in T(B_E)$. Thus, there exists some $x \in B_E$ such that $T(x) = \frac{1}{c}y$. That is, $x = T^{-1}(\frac{1}{c}y)$. Thus,

$$\|x\| = \|T^{-1}\frac{1}{c}y\| \leq \frac{1}{c}\|T\|\|y\|_Y.$$

Let $M = \frac{1}{c}\|T^{-1}\|$. It suffices to show that $T^{-1} : T(X) \to Y$ is bounded, but this comes from the fact that $T^{-1}(B_c^F(0)) \subset B_E$.

(iii $\mapsto$ i) Define $\pi : E \to E/\ker T$ to be the canonical surjection. We know that

$$\|\pi x\| = \text{dist}\|x - N(T)\|,$$

and that $T = \tilde{T} \circ \pi$, where $\tilde{T} : E/\ker T \to F$ is a bijection from $E/\ker T$ to $R(T)$ with $R(T) = R(\tilde{T})$ Moreover, $\|T\| = \|\tilde{T}\|$. Since $\tilde{T}$ is bijective and it is known that $E/\ker T$ is a Banach Space, then $R(\tilde{T}) \subset F$ is closed if and only if $R(\tilde{T})$ is a Banach space if and only if (by a corollary of the open mapping theorem) $\tilde{T}^{-1}$ is continuous if and only if there exists some $M > 0$ such that for all $y \in R(T)$, we have that

$$\|\tilde{T}^{-1}y\| = \|x\| \leq M\|y\|.$$

Without using the open mapping theorem, we have that $\tilde{T}^{-1}$ is continuous since $\tilde{T}$ is an isomorphism. Thus, we see that if $y_n \to y$ with $(y_n) \in T(X)$, then (y_n) is Cauchy and so for large enough n, we get that if $x_n = \tilde{T}^{-1}y_n$

$$\|x_n - x_m\| \leq M\|T(x_n - x_m)\| = M\|Tx_n - Tx_m\| = M\|y_n - y_m\| < \epsilon,$$

and so $(x_n) \in X$ is Cauchy and thus converges to some x via the completeness of X, and we claim that $Tx = y$:

$$\|Tx - y\| \leq \|Tx - Tx_n\| + \|Tx_n - y\| \leq \|T\|\|x - x_n\| + \|y_n - y\| < \epsilon.$$

■

Problem 10

Let X, Y be Banach and suppose $T \in \mathcal{L}(X, Y)$. Show that if T maps closed sets in X unto closed sets on Y, then $T(X)$ is closed in Y.

Solution: By the previous problem, it suffices to show that T is an open mapping from X unto $T(X)$. By class, it suffices to show that there exists some $c > 0$ such that $B_c^F(0) \subset T(B_E)$. We will show this result for $\tilde{T}$. That is, $T = \tilde{T} \circ \pi$, where $\tilde{T} : X/\ker T \to Y$ is a bijection from $E/\ker T$ to $R(T)$ with $R(T) = R(\tilde{T})$ Moreover, $\|T\| = \|\tilde{T}\|$ and $\tilde{T}$ is bijective.

To show that $\tilde{T}$ maps closed sets (in $X/\ker T$ with the norm $\|[x]\|_{X/\ker T} = \text{dist}(x - \ker T)$) to closed sets in Y, let K be closed in $X/\ker T$. Let $(y_n) \in \tilde{T}(K)$ with $y_n \to y$. Since $\tilde{T}$ is bijective, then the inverse exists and is continuous, and thus there exists $(x_n) \in X \setminus \ker T$ such that $x_n = y_n$ and $\|x\| = \|\tilde{T}^{-1} y_n\| \leq \|T^{-1}\| \|y_n\| \leq C\|y_n\|$. Thus, since (y_n) is Cauchy, then for large n, m :

$$\|x_n - x_m\| \leq C\|y_n - y_m\| < \epsilon,$$

and thus (x_n) is Cauchy. We wish to show that $x_n \to x \in K$, so it suffices to notice that K is closed and $X/\ker T$ is Banach, and thus K is Banach. Of course, it remains to check that $X/\ker T$ is actually Banach:

Let $M \subset X$ be closed and suppose $\pi : X \to X/M$ is the natural surjection. Then let $(\pi(x_n)) \in X/M$ be Cauchy. We can assume by passing unto a subsequence that

$$\|\pi(x_{n_{k+1}}) - \pi(x_{n_k})\|_{X/M} < \frac{1}{2^k}$$

By the definition of the norm, there exists some $(m_{n_k}) \in M$ such that

$$\|x_{n_{k+1}} - x_{n_k} - m_{n_k}\|_X < \frac{1}{2^k}.$$

Without any issue, we write $m_{n_k} = u_{n_{k+1}} - u_{n_k}$ and $(u_{n_1} = 0)$ to see that $x_{n_k} - u_{n_k}$ is Cauchy in X, and thus converges to a limit in X, and so $\pi(x_{n_k} - u_{n_k}) = \pi(x_{n_k})$ also converges to a limit, and since the sequence is Cauchy, then the entire sequence converges to a limit.

Thus, we find that since $\ker T$ is closed, then $X/\ker T$ is Banach, and so we have found that $x \in K$. To see that $Tx = y$, notice that for large n :

$$\|Tx - y\| \leq \|Tx - Tx_n\| + \|Tx_n - y\| \leq \|T\|\|x - x_n\| + \|y_n - y\| < \epsilon.$$

Thus, $\tilde{T}$ maps closed maps into closed maps.

Suppose $\tilde{T}$ is not an open mapping. Then for all $n > 0$, there exists some $y_n \in R(\tilde{T})$ with $\|y_n\| \leq \frac{1}{n}$ such that $y_n \notin \tilde{T}(B_{x/\ker T})$. By the bijectivity of $\tilde{T}$, there exist $x_n \in X/\ker T$ such that $x_n = \tilde{T}^{-1} y_n$ and $\|x_n\| \geq 1$. Since $\tilde{T}^{-1}$ is continuous, then $\|x_n\| \leq C\|y_n\|$ for all n, and thus if

$$\hat{y}_n = \frac{y_n}{\|x_n\|}$$

and $\hat{x}_n = \tilde{T}^{-1}\hat{y}_n$, then $\hat{x}_n \in \overline{B_{X/\ker T}}$ for any n since $\|\hat{x}_n\| = 1$. Notice how we have that $\|\hat{y}_n\| < \frac{1}{n}$, and so $y_n \to y$, where $(\hat{y}_n) \in \tilde{T}(\overline{B_{X/\ker T}})$. Since $\tilde{T}$ is a closed mapping, then $\hat{y} \in \tilde{T}(B_{X/\ker T})$, and since $\|\hat{y}_n\| < \frac{1}{n}$, $\hat{y} = 0$. There exists some $\hat{x} \in B_{X/\ker T}$ such that $\tilde{T}\hat{x} = \hat{y}$. To see that $\hat{x}_n \to x$, consider that

$$\|\hat{x}_n - \hat{x}\| = \|T^{-1}(\hat{y}_n - \hat{y})\| \le C\|\hat{y}_n - \hat{y}\| < \epsilon.$$

We have that $\hat{x} \in B_{X/\ker T}$ and in fact, $\|\hat{x}\| = 1$. However,

$$T\hat{x}_n = \hat{y}_n \to \hat{y} = 0$$

But

$$T\hat{x}_n \to T\hat{x} \neq 0,$$

a contradiction!

Thus, $\tilde{T}$ is an open mapping unto $R(\tilde{T})$. By the previous problem, $R(\tilde{T})$ is closed. Finally, we use that $R(\tilde{T}) = R(T)$ and we conclude.

■

Problem 11

Let $T \in \mathcal{L}(X, Y)$. Prove the following:

(a) $\ker(T) = R(T^*)^\perp$

(b) $\ker(T^*) = R(T)^\perp$

(c) $\overline{R(T)} = \ker(T^*)^\perp$

(d) $\overline{R(T^*)} \subset \ker(T)^\perp$

Solution: (a) We begin by noting the definitions:

$$\ker(T) = \{x \in X \ : \ Tx = 0\}$$

$$R(T^*)^\perp = \{x \in X \ : \ \langle T^* v, x\rangle = 0, \ \forall\, v \in Y^*\}.$$

Let $x \in \ker T$, then $Tx = 0$, and so

$$\langle T^* v, x\rangle = \langle v, Tx\rangle = \langle v, 0\rangle = 0,$$

and so $x \in R(T^*)^\perp$. Thus $\ker T \subset R(T^*)^\perp$.

Let $x \in R(T^*)^\perp$ and suppose $x \notin \ker(T)$. Thus, $(x, 0) \notin G(T)$. T is continuous, and so $G(T)$ is closed. Thus, we use Hahn-Banach. There exist $(f, g) \in E^* \times F^*$ such that

$$\langle f, x\rangle + \langle g, Tx\rangle < \alpha < \langle f, u\rangle + \langle g, 0\rangle = \langle f, u\rangle, \quad \forall x \in D(T).$$

Because $G(T)$ is a linear subspace, then

$$\langle f, x\rangle + \langle g, Tx\rangle = 0 \tag{5}$$

and thus $|\langle g, Tx\rangle| \leq \|f\|\|x\|$, and so $g \in D(T^*)$. Using (5), we see that

$$\langle f + T^* g, x\rangle = 0, \quad \forall x \in D(T) \implies f = T^*(g).$$

Thus, $\langle f, u\rangle = \langle T^* g, u\rangle > 0$, but $u \in R(T^*)^\perp$, and so $\langle T^* g, u\rangle = 0$. A contradiction! Thus, we get (a).

To see (b), we let $v \in \ker(T^*)$. Thus, $T^* v = 0$, and so for any $x \in D(T)$, we have that

$$0 = \langle T^* v, x\rangle = \langle v, Tx\rangle,$$

and so $v \in R(T)^\perp$.

Let $v \in R(T)^\perp$. Then for all $x \in D(T)$, $\langle v, Tx\rangle = 0$. Thus, $|\langle T^* v, x| \leq 0\|x\|$. If $v \in D(T^*)$, then $0 = \langle v, Ax\rangle = \langle T^* v, x\rangle = 0$ for all $x \in D(T)$ and so $T * v = 0$ and so $v \in \ker(T^*.)$ This shows (b).

For (c), it suffices to see that $(M^\perp)^\perp = \overline{M}$, where $M \subset X$ is a linear subspace.

$$M^\perp = \{f \in Y^* \ : \ \langle f, x\rangle = 0, \ \forall\, x \in X\} \subset Y^*.$$

Thus,

$$(M^\perp)^\perp = \{x \in X \ : \ \langle x, f \rangle = 0, \quad \forall\, f \in M^\perp\} \subset X$$

Evidently, $M \subset (M^\perp)^\perp$. $(M^\perp)^\perp$ is closed, and thus $\overline{M} \subset (M^\perp)^\perp$.

Suppose there is some $x_0 \in \overline{M}$ such that $x \notin (M^\perp)^\perp$. We use Hahn Banach to separate $\{x_0\}$ and $\overline{M}$. There exists some closed $f = [H = \alpha]$ such that

$$f(\overline{M}) < \alpha < f(x_0), \tag{6}$$

and thus $f(M) = 0$, and so for all $x \in M$, we have that $\langle f, x \rangle = 0$, and so $f \in M^\perp$ and thus $\langle f, x_0 \rangle = 0$, a contradiction to (6). Thus, since $R(T) \subset Y$ is a linear subspace, then $(R(T)^\perp)^\perp = \overline{R(T)}$, and by (c), we see that $\overline{R(T)} = \ker(T^*)^\perp$.

Suppose that $M \subset E^*$. We want to show that $\overline{M} \subset (M^\perp)^\perp$. This is clear from the definition and the above. Thus the result follows directly from (a). ■

Problem 12

Let X, Y be Banach with $T \in \mathcal{L}(X, Y)$. Show that T maps X unto a dense set if and only if T^* is injective.

SOLUTION: T^* is injective if and only if $\ker T^* = \{0\}$ if and only if $\ker(T^*)^\perp = Y$ if and only if $\overline{R(T)} = Y$. ■