UChicago Complex Analysis Notes: 27000

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Contents

1 Lectures

1.1 Tuesday, Mar 25: $\mathbb{C}$ as a field

Define the Complex numbers:

$$\mathbb{C} := \{(x, y) \mid x, y \in \mathbb{R}\} = \mathbb{R}^2$$

as a field:

$$+ : \mathbb{C} \to \mathbb{C}; \quad (a, b) + (c, d) = (a + c, b + d)$$

$$\times : \mathbb{C} \to \mathbb{C}; \quad (a, b) \times (c, d) = (ac - bd, ad + bc)$$

such that for any $z = (a, b) \in \mathbb{C}$, then

$$z = a + ib,$$

where,

$$a = (a, 0) = \mathrm{Re}\{z\}, \qquad i = (0, 1), \qquad b = (0, b) = \mathrm{Im}\{z\}$$

Remark 1. Why is $\mathbb{C}$ not an ordered field? Suppose it is, then there exists some $\mathscr{P} \subseteq \mathbb{C}$ of positive elements that is closed under addition and multiplication and that satisfies the trichotomy such that for any $z \in \mathscr{P}$, exactly one of the following holds: $z = 0, z \in \mathscr{P}, -z \in \mathscr{P}$.

Lemma 1. If $z \neq 0$, then $z^2 \in \mathscr{P}$.

Proof. If $z \in \mathscr{P}$, then since $\mathscr{P}$ is closed under multiplication, $z \times z \in \mathscr{P}$. If $z \notin \mathscr{P}$, then $-z \in \mathscr{P}$, and thus $(-z)(-z) \in \mathscr{P}$. But then $-z = (-1)(z)$, and so

$$(-z)(-z) = (-1)(-1)(z \times z) = z^2 \in \mathscr{P}.$$

$\square$

Consider that since 1 is a square, then $1 \in \mathscr{P}$. Moreover, since -1 is a square (in $\mathbb{C}$!), then $-1 \in \mathscr{P}$.

Definition 1. Let $z \in \mathbb{C}$ such that $z = a + ib$. We say that $\overline{z}$ is the **complex conjugate** of z if

$$z = a - ib.$$

That is, we reflect z over the real axis by flipping the sign of the imaginary line.

Remark 2.

$$z \cdot \overline{z} = (a + ib)(a - ib) = (a^2 + b^2, 0) = a^2 + b^2 = |z^2|,$$

by the norm defined below in (1). Suppose that $z \neq 0$, then

$$z \cdot \frac{\overline{z}}{|z|} = \frac{|z|^2}{|z|^2} = 1.$$

We have found the inverse of any nonzero $z \in \mathbb{C}$.

Remark 3. It is easy to show the following:

$$\overline{zw} = \overline{z} \cdot \overline{w}, \qquad \overline{z + w} = \overline{z} + \overline{w}, \qquad |zw|^2 = |z|^2 |w|^2.$$

Proposition 1. $\mathbb{C}$ is Banach under the norm,

$$\|z\| = \|(a, b)\| = \sqrt{a^2 + b^2} \tag{1}$$

Proof. We will first show that the norm satisfies the triangle inequality. It suffices to show that

$$|z+w| \leq |z| + |w|,$$

and thus we will show that

$$|z+w|^2 \leq (|z| + |w|)^2.$$

We have by the above remarks that

$$|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w}) = (z\overline{z}+\overline{z}w+\overline{w}z+w\overline{w}) = (|z|^2+|w|^2+z\overline{w}+\overline{z\overline{w}}) = |z|^2+|w|^2+2\,\mathrm{Re}\{z\overline{w}\}.$$

Thus, we want to show that $\mathrm{Re}\{z\overline{w}\} \leq |z||w|$, which comes from the fact that

$$\mathrm{Re}\{z\overline{w}\} \leq |z||\overline{w}| = |z||w|.$$

$\square$

1.2 Thursday, Mar 27: The Topology of $\mathbb{C}$

Theorem 1. $\mathbb{C}$ is complete.

This theorem is in the sense of Cauchy convergence, since the least upper bound property is meaningless in a non-ordered field, such as $\mathbb{C}$.

Remark 4. We denote that disk of radius r around z_0 as

$$D(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$$

Definition 2. We say that $O \subset \mathbb{C}$ is **open** if and only if for all $z_0 \in O$, there exists some $\epsilon > 0$ such that $D(z_0, r) \subset O$.

It is easy to show that disks are open with the triangle inequality.

Definition 3. Let O be open. We say that O is **connected** if, whenever $O = O_1 \cup O_2$, where O_1, O_2 are open, disjoint, then at least one of the O_1, O_2 are empty.

Remark 5. How do we make the real valued $f(x) = x$ continuously differentiable on $[0, 1]$? The endpoints present a problem! We say that f is differentiable at 0 or at 1 if the one sided limit exists and agrees with the derivative near the endpoint.

Definition 4. Let O be an open set. A **path** is a function $\gamma : [a, b] \to O$ such that γ is piecewise continuously differentiable. That is, there exists some finite partition $a = t_0 < t_1 < \cdots < t_n = b$ such that γ is continuously differentiable on each closed interval (t_{k-1}, t_k) and

$$\lim_{t \to t_{k-1}^+} \gamma'(t) = \gamma'_+(t_{k-1}) = \lim_{h \to 0^+} \frac{\gamma(t_{k-1} + h) - \gamma(t_{k-1})}{h}$$

and similarly for t_k.

Theorem 2. If O is pathwise connected, then it is connected.

Proof. Suppose O is disconnected, then $O = O_1 \sqcup O_2$ and take $[a, b] = \gamma^{-1}(O_1) \sqcup \gamma^{-1}(O_2)$ as a disconnection of the interval, a contradiction! $\square$

Definition 5. Let O be open. We say that a **polygonal path** inside of O is a path made up of only horizontal and vertical lines.

Theorem 3. Suppose O is connected, then O is polygonally connected.

Proof. Let $z_0 \in O$. We will show that

$$A = \{z \mid \exists \text{ polygonal path between } z_0 \text{ and } z\}$$

is the same as O. We will first show that O is open. Let $z \in A$, since O is open, there exists some $r > 0$ such that $D(z, r) \subseteq O$. We will show that for any $z' \in D(z, r)$, $z' \in A$. It suffices to show that disks are polygonally connected.

Since disks are convex, there exists a straight line connecting z, z'. We claim that the imaginary component of this line is in the disk, which is because of the Pythagorean identity (or because $\text{Im}\{r\} \leq |r|$.) By convexity, the straight line (i.e, the real component) between the imaginary line and z' is in the disk. Thus, we can polygonally connect z and z'.

Thus, A is open. Let B be the set of points in O that cannot be polygonally connected. Evidently, $A \cap B = \emptyset$. By dichotomy, $A \cup B = O$. Evidently, $z_0 \in A$. Clearly, B is open, and so B must be empty and $A = O$. $\square$

Example 1.1. Let $t \in [0,1]$.

(a) The straight path from z_2 to z_1:
$$\gamma(t) = tz_1 + (1-t)z_2$$

(b) The straight path from z_1 to z_2 is
$$\gamma(t) = (1-t)z_1 + tz_2$$

(c) The circle centered at z_0 of radius r counterclockwise.
$$\gamma(\theta) = z_0 + r\cos\theta + ir\sin\theta = z_0 + re^{i\theta}, \qquad \theta \in [0, 2\pi)$$

Definition 6. Suppose $O \subseteq \mathbb{C}$ be open, and let $f : O \to \mathbb{C}$. We say that f is **continuous** at $z_0 \in O$ if
$$\lim_{z \to z_0} f(z) = f(z_0), \quad z \in O.$$

We say that f is differentiable at $z_0 \in O$ if
$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0), \quad h \in \mathbb{C}$$

Continuity in $\mathbb{C}$ is identical to continuity in $\mathbb{R}^2$, since they are identical as metric spaces. But dividing in $\mathbb{R}^2$ makes no sense, so the differentiability is completely different!

Example 1.2. (a) Suppose $f(z) = z^2$, then
$$\frac{(z+h)^2 - z^2}{h} = 2z + h \to 2z = f'(z)$$

(b) Suppose $f(z) = \overline{z}$. Then at the origin,
$$\lim_{h \to 0} \frac{\overline{(0+h)} - \overline{0}}{h} = \lim_{h \to 0} \frac{\overline{h}}{h},$$

but the looking at a purely real component, the limit approached 1. Looking at a purely imaginary components, the limit approaches -1. Oops!

1.3 Tuesday, Apr 1: The Complex Integral

Suppose $f : [a, b] \to \mathbb{C}$. Then, if the limit exists,

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$

All the normal quotient, product, and chain rules apply.

Definition 7. We define the **integral** of $f : [a, b] \to \mathbb{C}$, where $f(t) = u(t) + iv(t)$, then

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$$

Proposition 2. (Linearity) Suppose $f, g : [a, b] \to \mathbb{C}$, then

$$\int_a^b f + g = \int_a^b f + \int_a^b g$$

(Scalar Linearity) Suppose $c \in \mathbb{C}$, then

$$\int_a^b cf = c \int_a^b f$$

(Triangle Inequality) Suppose $f : [a, b] \to \mathbb{C}$, where f is continuous. Then

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Proof. (Linearity) We have that

$$\begin{aligned}
\int_a^b f + g &= \int_a^b [(u_f + u_g) + i(v_f + v_g)] \\
&= \int_a^b (u_f + u_g) + i \int_a^b (v_f + v_g) \\
&= (\int_a^b u_f + i \int_a^b v_f) + (\int_a^b u_g + i \int_a^b v_g) \\
&= \int_a^b f + \int_a^b g
\end{aligned}$$

(Scalar Linearity) Suppose $c \in \mathbb{R}$, then

$$\int_a^b cf = \int_a^b cu + i \int_a^b cv = c(\int_a^b f)$$

For $i \in \mathbb{C}$, we have that

$$\begin{aligned}
\int_a^b if &= \int_a^b iu - v \\
&= \int_a^b -(v - iu) \\
&= -\int_a^b v - iu \\
&= i\left(\int_a^b u + i\int_a^b v\right) \\
&= i\int_a^b f
\end{aligned}$$

Let $z = \alpha + i\beta \in \mathbb{C}$. Then

$$\int_a^b zf = \int_a^b [\alpha f + i\beta f] = \int_a^b \alpha f + \int_a^b i\beta f = \alpha\int_a^b f + i\beta\int_a^b f = z\int_a^b f$$

(Triangle inequality). Given any $z \in \mathbb{C}$, there exists some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $\alpha z = |z|$. To see this for $z \neq 0$, note that $\alpha = \frac{|z|}{z}$. Thus, there exists some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that

$$\begin{aligned}
\left|\int_a^b f\right| &= \alpha\int_a^b f = \int_a^b \alpha f \\
&= \int_a^b \alpha u + i\int_a^b \alpha v = \int_a^b \alpha u \\
&= \int_a^b au + i\int_a^b bu = \int_a^b \operatorname{Re}\{\alpha f\} \\
&\leq \int_a^b |\operatorname{Re}\{\alpha f\}| \leq \int_a^b |\alpha f| = \int_a^b |f|
\end{aligned}$$

□

Theorem 4. Suppose $f : [a, b] \to \mathbb{C}$ and f' is continuous on $[a, b]$. Then

$$\int_a^b f' = f(b) - f(a)$$

Let $f : O \to \mathbb{C}$ be continuous, where $O \subset \mathbb{C}$ is open. Suppose $\gamma : [a, b] \to O$ is a path in O. Partition γ into $\{z_k\}$ such that $z_k = \gamma(t_k)$. Then we estimate the integral along the curve by

$$\lim_{\max|z_k - z_{k-1}|}\sum_{k=1}^n f(z_k)[z_k - z_{k-1}] = \lim_{\max|t_k - t_{k-1}|}\sum_{k=1}^n f(\gamma(t_k))\frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}}(t_k - t_{k-1}) = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Definition 8. Define the quantities as in the above remark, then the **line integral** of f over γ is

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Proposition 3. (Linearity)

$$\int_\gamma f+g=\int_\gamma f+\int_\gamma g$$

(Estimation)

$$\int_\gamma f(z)dz \le \max_{t\in[a,b]} |f(\gamma(t))| \text{ length}(\gamma)$$

where

$$\text{length}(\gamma)=\lim_{\max|t_k-t_{k-1}|}\sum_{k=1}^{n}|\gamma(t_k)-\gamma(t_{k-1})|=\int_a^b|\gamma'(t)|dt$$

Proof. (Estimation)

$$\left|\int_\gamma f(z)dz\right|=\left|\int_a^b f(\gamma(t))\gamma'(t)dt\right|\le\int_a^b|f(\gamma(t))\gamma'(t)|dt\le\max_{t\in[a,b]}|f(\gamma(t))|\int_a^b|\gamma'(t)|dt=\max_{t\in[a,b]}|f(\gamma(t))| \text{ length}(\gamma)$$

□

Example 1.3. (a)

$$\gamma(t)=(1-t)z_1+tz_2, \qquad t\in[0,1]$$

is the straight line from z_1 to z_2. Then intuitively:

$$\text{length}(\gamma)=|z_2-z_1|$$

By definition:

$$\text{length}(\gamma)=\int_0^1|\gamma'(t)|dt=\int_0^1|z_2-z_1|dt=|z_2-z_1|$$

(b)

$$\gamma(\theta)=z_0+r[\cos(\theta)+i\sin(\theta)], \qquad \theta\in[0,2\pi]$$

is the circle of radius r centered at z_0 with counterclockwise orientation.

$$\text{length}(\gamma)=\int_0^{2\pi}\gamma'=r\int_0^{2\pi}|\sin(\theta)+i\cos(\theta)|=2\pi|r|$$

1.4 Thursday, Apr 3: Properties of the Integral

Theorem 5. Suppose (f_n) are continuous functions on open $O \subseteq \mathbb{C}$. Assume that for some path γ in O, (a path being in O means that $\gamma([a,b]) \subseteq O$). Assume that $f_n(z) \to f$ uniformly on $\{\gamma(t) \mid t \in [a,b]\}$. Then

$$\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz$$

Proof. Since $f_n \to f$ uniformly, $\sup_z |f_n(z) - f(z)| < \frac{\epsilon}{\text{length}(\gamma)+1}$ for large n.

$$\begin{aligned}\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)\right| &= \left|\int_\gamma f_n(z) - f(z)\right| \\ &\leq \sup_{t\in[a,b]} |f_n(\gamma(t) - f(\gamma(t)))|\text{length}(\gamma) \\ &< \epsilon\end{aligned}$$

□

Theorem 6. Suppose f is continuous on open $O \subseteq \mathbb{C}$ and $f : O \to \mathbb{C}$. Let γ be a path in O. If f has a primitive on O (i.e. there is some $F(t)$ in O such that $F'(z) = f(z)$ for all $z \in O$), then

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

With these conditions, these immediately implies path independence, as long as the paths start and end at the same time, and that this also implies that a line integral yields 0 along a closed loop.

Proof. Suppose γ' is continuous. Then

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b F'(\gamma(t))\gamma'(t)dt = \int_a^b (F\circ\gamma)'(t)dt = F(\gamma(b)) - F(\gamma(a))$$

For the general case, we only know that γ' is piecewise continuous. Partition $[a,b]$ into $a = t_0 < t_1 < \cdots < t_n = b$ such that γ' is continuous on each subinterval $\{[t_{i-1}, t_i]\}_{i\in[n]}$. Then we use the degenerate case above for each subinterval

$$\int_\gamma f(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)dt = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} F'(\gamma(t))\gamma'(t) = \sum_{i=1}^n [F(\gamma(t_i)) - F(\gamma(t_{i-1}))] = F(\gamma(b)) - F(\gamma(a))$$

□

What happens when we have the same path, γ, but the parametrization of γ changes? That is, what if there is another parameterization of the same path?

Theorem 7. Suppose $\gamma : [a,b] \to O$, where $O \subseteq \mathbb{C}$ is open, and suppose $f : O \to \mathbb{C}$ is continuous. Suppose further that $\phi : [c,d] \to [a,b]$ such that $\tilde{\gamma}(t) = \gamma(\phi(t))$ for $t \in [c,d]$. If ϕ' is continuous and ϕ is a bijection and $\phi' > 0$ (ϕ and γ have the same orientation), then

$$\int_{\tilde{\gamma}} f(z)dz = \int_\gamma f(z)dz$$

Proof. Recall the standard change of variables formula from analysis, if

$$\int_{[c,d]} f(\phi(x))|\phi'(x)|dx = \int_{\phi([c,d])} f(y)dy$$

By definition

$$\begin{aligned}
\int_{\tilde{\gamma}} f(z)dz &= \int_c^d f(\tilde{\gamma}(t))\tilde{\gamma}'(t)dt \\
&= \int_c^d f(\gamma(\phi(t)))\tilde{\gamma}'(t)dt \\
&= \int_c^d f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)dt \\
&= \int_c^d f(\gamma(\phi(t)))\gamma'(\phi(t))|\phi'(t)|dt \\
&= \int_a^b f(\gamma(s))\gamma'(s)ds \\
&= \int_\gamma f(z)dz
\end{aligned}$$

If $\phi' < 0$, then we must compensate where we put in $|\phi'(t)|$ by putting in a negative sign in front of the integral, and thus when the orientation of the parameterization is opposite, the integrals are opposite. ☐

Remark 6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}}, & x > 0 \\ 0, & x \leq 0 \end{cases} .$$

It is not hard to show that, although $f(x)$ is smooth (infinitely differentiable), it is not analytical! That is, there does not exist a converging power series at every $x \in \mathbb{R}$ that equals $f(x)$. The converse clearly holds for $\mathbb{R}$.

We will show that if $O \subset \mathbb{C}$ is open, and there is a holomorphic function on O, then if there is a disk around any point $z_0 \in O$, then the function is analytical!

Proposition 4. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$. Then there exists some $\rho \in [0, \infty]$ called the **radius of convergence**, such that if $|z| < \rho$, then the series absolutely converges. If $|z| > \rho$, then $\{a_n z^n\}$ is unbounded.

Example 1.4.

- Consider

  $$f(z) = \sum_{n=1}^{\infty} z^n,$$

  then $\rho = 1$ since it is a geometric series if $z < 1$ and obviously diverges if $|z| \geq 1$. When $|z| = 1$, try to see for yourself why $z = i$ diverges.

- The classic

  $$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!},$$

  which converges everywhere! That is. $\rho = \infty$.

1.5 Tuesday, Apr 8: Power Series

We recalled Theorem 6 and 7 from the previous class.

Remark 7. Consider the following change of variable:

$$\varphi : [a, b] \to [a, b], \quad \varphi(t) = a + b - t,$$

then note that φ makes γ run in reverse

Remark 8. A general power series, centered at some $z_0 \in O \subset \mathbb{C}$, is defined to be the formal sum

$$\sum_{n=1}^{\infty} a_n (z - z_0)^n.$$

We will usually just take $z_0 = 0$.

Proposition 5. For any power series $\sum_{n=1}^{\infty} a_n z^n$, the radius of convergence is

$$\rho = \sup\{|z| \mid (a_n z^n) \text{ is bounded}\}$$

Proof. If $|z_0| > \rho$, then $(a_n z_0^n)$ is unbounded, and thus the series cannot converge, since the terms do not go to zero.

If $|z_0| < \rho$, then let $\epsilon > 0$ such that $|z_0| = \rho - \epsilon$. Consider that since $|a_n|(\rho - \frac{\epsilon}{2})^n$ is bounded by some M since it is less than ρ.

$$\begin{aligned}\sum_{n=1}^{\infty} |a_n||z_0|^n &= \sum_{n=1}^{\infty} (|a_n|(\rho - \frac{\epsilon}{2}))^n \frac{|z_0|^n}{(\rho - \frac{\epsilon}{2})^n} \\ &\leq M \sum_{n=1}^{\infty} \left(\frac{|z_0|}{(\rho - \frac{\epsilon}{2})} \right)^n \\ &< \infty,\end{aligned}$$

The series converges geometrically since $\frac{|z_0|}{(\rho - \frac{\epsilon}{2})} < 1$. □

Lemma 1. We claim that

$$\frac{w^n - w^n}{z - w} = z^{n-1} + z^{n-2} w + \dots w^{n-1}$$

Proof. Note that

$$(z - w) \left(z^{n-1} + z^{n-2} w + \dots w^{n-1} \right) = z^n - w^n$$

□

Theorem 8. If $|z| < \rho$, and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then f is differentiable at z and

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

Proof. Let $|z_0| = \rho - \epsilon$ The difference quotient is given by

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{\sum_{n=1}^{\infty} a_n(z_0+h)^n - \sum_{n=1}^{\infty} a_n z_0^n}{h}$$

Note since the terms $a_n z_0^{n-1}$ are geometric, then the series

$$\sum_{n=1}^{\infty} n a_n z_0^{n-1}$$

converges (one can check this with the ratio test). Thus, consider that

$$\begin{aligned}
&\left|\frac{\sum_{n=0}^{\infty} a_n(z_0+h)^n - \sum_{n=1}^{\infty} a_n z_0^n}{h} - \sum_{n=0}^{\infty} n a_n z_0^{n-1}\right| = \\
&= \left|\sum_{n=0}^{N} a_n \frac{(z_0+h)^n - z_0^n}{h} + \sum_{N+1}^{\infty} \frac{(z_0+h)^n - z_0^n}{h} - \sum_{n=0}^{\infty} n a_n z_0^{n-1} - \sum_{N+1}^{\infty} n a_n z_0^{n-1}\right| \\
&\leq \left|\sum_{n=0}^{N} a_n \frac{(z_0+h)^n - z_0^n}{h} - n a_n z_0^{n-1}\right| + \left|\sum_{N+1}^{\infty} a_n \frac{(z_0+h)^n - z_0^n}{h}\right| + \left|n a_n z_0^{n-1}\right| \\
&\xrightarrow{h\to 0} \left|\sum_{n=0}^{N} n a_n z_0^{n-1} - n a_n z_0^{n-1}\right| + \left|\sum_{N+1}^{\infty} a_n \frac{(z_0+h)^n - z_0^n}{h}\right| + \left|n a_n z_0^{n-1}\right| \\
&= \left|\sum_{N+1}^{\infty} a_n \frac{(z_0+h)^n - z_0^n}{h}\right| + \left|\sum_{N+1}^{\infty} n a_n z_0^{n-1}\right|
\end{aligned}$$

Since $\sum_{N+1}^{\infty} n a_n z_0^{n-1}$ converges geometrically, then the sum goes to zero as $N \to \infty$. Using Lemma 1, we note that if $h < \frac{\epsilon}{2}$ and $|z_0| = \rho - \epsilon$, then

$$|z_0 + h| \leq |z_0| + |h| < \rho - \frac{\epsilon}{2}$$

$$\left|\sum_{N+1}^{\infty} a_n \frac{(z_0+h)^n - z_0^n}{h}\right| = \sum_{N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{(z_0+h)-z_0}\right| \leq \sum_{N+1}^{\infty} |a_n| \sum_{k=0}^{n-1} |(z_0+h)|^k |z_0|^{n-1-k} < \sum_{N+1}^{\infty} n|a_n|(\rho - \frac{\epsilon}{2})^{n-1} \xrightarrow[N\to\infty]{} 0$$

since it converges geometrically again □

Here is a much cleaner proof.

Proof. We know that if $f_n \rightrightarrows f$, and $f_n' \rightrightarrows g$, then if f is sufficiently smooth, $f' = g$. Call $F_N(z) = \sum_{n=1}^{N} a_n z^n$. By Proposition 5, $F_N \rightrightarrows F$ for some F. We claim that $F_N' \rightrightarrows G$ for some G. To see this, it suffices to show that F_N' is uniformly convergent. Consider that

$$F_N' = \sum_{n=1}^{N} n z^{n-1} a_n.$$

Thus,

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{n a_n}} = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{a_n}}.$$

Hence, the radius of convergence is the same as F_N, and thus we know that $F_N' \rightrightarrows G$ for some G. Hence, $G = F'$, but we know that

$$F = \sum_{n=1}^{\infty} z^n a_n \qquad G = G' = \sum_{n=1}^{\infty} n z^{n-1} a_n.$$

□

Corollary 1. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is infinitely differentiable for all z such that $|z| < \rho$. Moreover,

$$a_n = \frac{1}{n!} f^{(n)}(0)$$

Proof. By Theorem 8, we have that for all $|z| < \rho$

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Note that $f'(z)$ has a bigger radius of convergence than $f(z)$ since $n a_n z^{n-1}$ is still bounded and is still geometrically 'small.' Thus, we apply Theorem 8 again to $f'(z)$, then

$$f''(z) = \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2}.$$

For the second claim,

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$$

Then $f(0) = a_0$. Also, $f'(z) = a_1 + \ldots n a_n z^{n-1}$ and so $f'(0) = a_1$. Also,

$$f''(z) = 2a_2 z + \cdots + n(n-1) z^{n-2} + \cdots ,$$

and so $f''(0) = 2a_2$. Inducting yields the result. □

1.6 Thursday, Apr 10: The Complex Exponential

Remark 9. We begin with defining some famous functions in $\mathbb{C}$:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

Proposition 6. Suppose that if the functions are as defined in Remark 9, then

$$e^{iz} = \cos z + i \sin z \tag{2}$$

Proof. This is in the Homework for this week. □

Lemma 2. If $\Omega \subseteq \mathbb{C}$ is a region (a connected open set), then if $f : \Omega \to \mathbb{R}$ is holomorphic with $f'(z) = 0$ for all $z \in \Omega$, then f is constant.

Proof. If $f'(z) = 0$, then 0 has f as a primitive. Thus, using the fundamental theorem of path integrals (Theorem 6), then for any path γ on Ω, if z_1, z_2 are the endpoints of γ, then

$$f(z_2) - f(z_1) = \int_\gamma 0 dz = 0 \implies f(z_2) = f(z_1).$$

□

Lemma 3. Let $z \in \mathbb{C}$. Then $e^z \neq 0$.

Proof. Consider that for any $z \in \mathbb{C}$, we have that

$$\frac{\partial}{\partial z} e^z e^{-z} = e^z e^{-z} - e^z e^{-z} = 0.$$

By Lemma 2, we have that $e^z e^{-z}$ is constant. Thus, for any $z \in \mathbb{C}$,

$$e^z e^{-z} = e^0 e^{-0} = 1 \cdot 1 = 1.$$

But then $e^z \neq 0$ and $e^{-z} \neq 0$. □

Proposition 7. Let $z, w \in \mathbb{C}$, then

$$e^{z+w} = e^z e^w$$

Proof. Fix $w \in \mathbb{C}$, then $e^{z+w} : z \mapsto e^{z+w}$. We claim that

$$f(z) = \frac{e^{z+w}}{e^z e^w} = 1.$$

Note that by Lemma 3, we are able to divide by $e^z e^w$. Differentiating with respect to only z

$$\frac{\partial f}{\partial z} = (\frac{e^{z+w}}{e^z e^w})' = \frac{e^z e^w (e^{z+w})' - e^{z+w}(e^z e^w)'}{(e^z e^w)^2} = \frac{e^z e^w (e^{z+w}) - e^{z+w}(e^z e^w)}{(e^z e^w)^2} = 0.$$

By Lemma 2, $f(z)$ is a constant. Then

$$f(0) = \frac{e^w}{e^w} = 1.$$

□

Proposition 8. For any $z \in \mathbb{C}$, e^z is periodic with period $2\pi i$. That is, $e^{z+2\pi i} = e^z$.

Proof. By Proposition 7, $e^{z+2\pi i} = e^z e^{2\pi i}$, so it suffices to show that $e^{2\pi i} = 1$. But by Proposition 6,

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + 0 = 1.$$

$\square$

Proposition 9. If $e^{z+w} = e^w$ for some z, w, then $z = 0 + 2n\pi i$ for any $n \in \mathbb{N}$.

Proof. By Proposition 7, $e^{z+w} = e^z e^w$, so it suffices to show that $e^z = 1$. Call $z = x + iy$. Then

$$e^{x+iy} = 1 \implies |e^{x+iy}| = |1|,$$

but since $|e^{iy}| = \cos^2 x + \sin^2 y = 1$, then $|e^{x+iy}| = |e^x||e^{iy}| = e^x$. Thus, $e^x = 1$, and so $x = 0$. Thus, $z = iy$. But

$$e^{iy} = 1 \implies \cos y + i \sin y = 1 \implies \cos y = 1, \sin y = 0 \implies y = 2\pi n$$

By Proposition 8, $y = 2\pi n$ for some $n \geq 0$. $\square$

Definition 9. We define $\log : \mathbb{C} \to \mathbb{R}$ by

$$e^{\log z} = z$$

Proposition 10. The derivative of $\log z$ is $\frac{1}{z}$.

Proof.

$$\frac{\partial}{\partial z} e^{\log z} = (z)' = 1$$

Thus,

$$z = (e^{\log z})' z = z \log z.$$

$\square$

Remark 10. Let $\gamma(\theta) = \cos\theta + i \sin\theta = e^{i\theta}$. Then

$$0 \neq \int_\gamma \frac{1}{z} dz.$$

Proof.

$$\int_\gamma \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{\gamma(\theta)} \gamma'(\theta) d\theta = \int_0^{2\pi} \frac{1}{e^{i\theta}} i e^{i\theta} d\theta = 2\pi i \neq 0.$$

$\square$

What is going on here? γ is a closed curve! Well, $\frac{1}{z}$ does not have a primitive on $\mathbb{C} \setminus \{0\}$.

Proposition 11. We claim that e^z is bijective from $\mathrm{Im}\{z\} \in (-\pi, \pi) =: S$ to $\mathbb{C} \setminus \{\text{negative real axis}\}$

Proof. Suppose $z_1, z_2 \in S$ such that $e^{z_1} = e^{z_2}$. We have that

$$e^{x_1 + iy_1} = e^{x_2 + iy_2} \implies |e^{x_1}|e^{iy_1}| = |e^{x_2}|e^{iy_2}|.$$

But $|e^{iy_1}| = 1$ and $|e^{iy_2}| = 1$, and so

$$e^{x_1} = e^{x_2} \implies x_1 = x_2.$$

Then dividing by e^{x_1}, we have that

$$e^{iy_1} = e^{iy_2} \implies \frac{e^{iy_1}}{e^{iy_2}} = e^{i(y_1-y_2)} = 1,$$

so then since $y_1 - y_2 \in (0, 2\pi)$, then

$$\cos(y_1 - y_2) + i\sin(y_1 - y_2) = 1 \implies \cos(y_1 - y_2) = 1, i\sin(y_1 - y_2) = 0 \implies y_1 - y_2 = 0 \implies y_1 = y_2.$$

Thus, $z_1 = z_2$ and so the function is injective. $\square$

1.7 Tuesday, Apr 15: Log and the Winding Number

Recall the final proposition from last class that stated that e^z is bijective from $\text{Im}\{z\} \in (-\pi, \pi) =: S$ to $\mathbb{C} \setminus \{\text{negative real axis}\}$. We showed it was injective last class.

Proof. We claim that e^z is surjective as well. Let $w \in \mathbb{C} \setminus \{\text{negative real axis}\}$. Since $0 \in S$, then $|w| > 0$, and so $\frac{w}{|w|}$ is on the unit circle. Thus, there exists some θ such that

$$\frac{w}{|w|} = \cos\theta + i\sin\theta = e^{i\theta} \implies w = |w|e^{i\theta}.$$

We claim that $\log w = \log|w| + i\theta$. Using proposition 7, we see that

$$e^{\log|w|+i\theta} = e^{\log|w|}e^{i\theta} = |w|e^{i\theta},$$

and since $\theta \in S$, we are done. $\square$

Remark 11. Since there is a bijection from $S \to \mathbb{C} \setminus \{\text{negative real axis}\}$, we define $\text{Log}(z)$ to be the inverse function of S. Note that by definition

$$\left(e^{\text{Log}z}\right)' = z' = 1.$$

But by the chain rule,

$$\left(e^{\text{Log}z}\right)' = z \cdot \text{Log}z'.$$

Thus,

$$\text{Log}z' = \frac{1}{z}.$$

Remark 12. We try to derive the power series of $\text{Log}(1-z)$. Let $|z| < 1$. Then

$$\text{Re}\{1-z\} = 1 - \text{Re}\{z\} \geq 1 - |z| > 0.$$

Thus, $1-z$ is in the right half plane of $\mathbb{C}$. Thus, Log is well defined for $1-z$ when $|z| < 1$. To get the power series of $\text{Log}(1-z)$, we will want to find the coefficients of the Taylor series by Corollary 1. By the chain rule,

$$[\text{Log}(1-z)]' = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n.$$

Then one can see by integrating both sides that

$$\text{Log}(1-z) = -\sum_{n=1}^{\infty} \frac{1}{n} z^n + C.$$

Plugging in $z = 0$, we see that $C = 0$.

Definition 10. Let γ be a closed path on $\mathbb{C}$. Suppose $z_0 \notin \gamma$. Then we define the **winding number** of γ at z_0 to be the number of times γ winds around z_0. That is,

$$\text{Ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z_0} d\zeta.$$

Suppose γ, is a closed path. We call the **complement of** γ to be γ^c, the set of points not bounded by γ.

Example 1.5. Informally! Consider $\gamma = e^{i\theta}$, for $\theta \in [0, 2\pi]$ to be the counterclockwise path around 0. It should be the case that

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta}d\zeta = 1.$$

We have showed this in Remark 10. Moreover, we know by a question on our PSET that if Ω is open and connected and Ind has a derivative of 0 within Ω, then Ind is constant. Let $z \neq 1$. Then

$$\begin{aligned}\frac{\partial}{\partial z}\left(\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}d\zeta\right) &= \frac{1}{2\pi i}\int_\gamma \frac{\partial}{\partial z}\frac{1}{\zeta - z}d\zeta\\ &= \frac{1}{2\pi i}\int_\gamma -\frac{1}{(\zeta - z)^2}d\zeta\\ &= 0\end{aligned}$$

Since the function has a primitive. Thus, it suffices to show that the first equality is valid. That is,

$$\begin{aligned}\left|\frac{1}{h}\left[\int_\gamma \frac{1}{\zeta-(z+h)}d\zeta - \int_\gamma \frac{1}{\zeta - z}d\zeta\right] - \int_\gamma \frac{1}{(\zeta-z)^2}\right| &= \left|\frac{1}{h}\left[\int_\gamma \frac{h}{(\zeta-z)(\zeta-(z+h))}d\zeta\right] - \int_\gamma \frac{1}{(\zeta-z)^2}d\zeta\right|\\ &= \left|\int_\gamma \frac{1}{(\zeta-z)(\zeta-(z+h))}d\zeta - \int_\gamma \frac{1}{(\zeta-z)^2}d\zeta\right|\\ &= h\left|\int_\gamma \frac{1}{(\zeta-z)^2(\zeta-(z+h))}d\zeta\right|\\ &\leq h\max_{t\in[0,2\pi]}\left|\frac{1}{(\gamma(t)-z)^2(\gamma(t)-(z+h))}\right|2\pi\\ &\to 0\end{aligned}$$

So it suffices to show that the denominator is bounded below. Since γ is closed, then it is compact. Thus, we are able to bound the denominator below since the distance is achieved and positive for any $z < 1$.

Theorem 9. For any γ in an open connected subset of γ^c, the winding number is constant.

1.8 Thursday, Apr 17: Cauchy's Theorem

Theorem 10. (Cauchy-Goursat) Suppose $O \subseteq \mathbb{C}$ is open, and $\triangle \subseteq O$ is a solid closed triangle. Then for $f \in H(O)$, we have that

$$\int_{\partial\triangle} f(z)dz = 0.$$

Proof. Take $d_0 = \operatorname{diam}(\triangle)$ and $\ell = \operatorname{length}(\partial\triangle)$. Take the midpoints of the triangle, and we get four sub triangles with specific orientation, $\triangle_i$, for $i \in \{1, 2, 3, 4\}$ as follows:

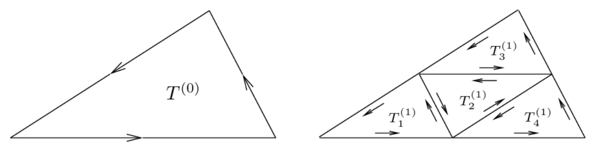

Figure 1: Goursat's Triangle

(Note that $T^{(i)} = \triangle_i$ in the picture.) It is clear that

$$\int_{\partial\triangle} f(z)dz = \sum_{k=1}^{4} \int_{\partial\triangle_k} f(z)dz.$$

By the triangle inequality,

$$\left|\int_{\partial\triangle} f(z)dz\right| \le \sum_{k=1}^{4} \left|\int_{\partial\triangle_k} f(z)dz\right| \le 4\left|\int_{\partial\triangle_{(1)}} f(z)dz\right|$$

We know that

$$d_1 = \operatorname{diam}(\triangle_{(1)}) \le \frac{d_0}{2} \qquad \ell_1 = \operatorname{length}(\partial\triangle_{(1)}) \le \frac{\ell}{2}.$$

Split up $\triangle_{(1)}$ into four sub-triangles with the same process as before. We can induct on this process to find a sequence such that

$$\triangle_{(1)} \supset \triangle_{(2)} \supset \cdots, \qquad \left|\int_{\partial\triangle} f(z)dz\right| \le 4^n \left|\int_{\partial\triangle_{(n)}} f(z)dz\right|, \qquad d_n \le \frac{d_0}{2^n} \qquad \ell_n \le \frac{\ell_0}{2^n}$$

Since $\mathbb{C}$ is complete, we can use Cantor's nested set theorem to find some $z_\infty \in \bigcap \triangle_{(n)}$. Let $\epsilon > 0$. Since f is holomorphic at z_∞, we have that for z close to z_∞,

if we call R to be an error function such that $R(z) \to 0$ as $z \to z_\infty$,

$$f'(z_\infty) = \frac{f(z) - f(z_\infty)}{z - z_\infty} + R(z) \iff f(z) = f'(z_\infty)(z - z_\infty) - R(z)(z - z_\infty) + f(z_\infty).$$

We can use the fact that $f(z_\infty)$ and $f'(z_\infty)(\zeta - z_0)$ are linear functions with primitives to calculate

$$\begin{aligned}\int_{\partial\triangle_{(n)}} f(z)dz &= \int_{\partial\triangle_{(n)}} [f(z_\infty) + f'(z_\infty)(z - z_\infty)] + R(z)(z - z_\infty)d\zeta \\ &= \int_{\partial\triangle_{(n)}} R(z)(z - z_\infty)\, dz\end{aligned}$$

Hence, since z lies on $\partial\triangle_{(n)}$, then

$$\left|\int_{\partial\triangle_{(n)}} f(z)dz\right| \le \frac{\ell_0}{2^n}\frac{d_0}{2^n}\epsilon'.$$

Thus, choosing $\epsilon' = \frac{\epsilon}{d_0,\ell_0}$

$$\left|\int_{\partial\triangle} f(z)dz\right| < \epsilon$$

□

Theorem 11. (Cauchy) Suppose $O \subseteq \mathbb{C}$ is an open, convex set, and $f \in H(O)$. Then for every closed path γ in O,

$$\int_\gamma f(z)dz = 0$$

Proof. It suffices to show that f has a primitive. Fix $z_0 \in O$. Let $z \in O$. Let $[z_0, z]$ be the straight line from z_0 to z. Define

$$F(z) = \int_{[z_0,z]} f(\zeta)d\zeta.$$

To see that $F' = f$, consider that $F(z)$ integrates from $[z_0, z]$, $F(z+h)$ integrates from $[z_0, z+h]$, and we claim that $F(z+h) - F(z)$ integrates from $[z, z+h]$. Note that if we show this, then we formed a triangle, $\triangle$. To show this, consider that by Goursat,

$$\int_{\partial\triangle} f(\zeta)d\zeta = \int_{[z_0,z]} f(\zeta)d\zeta + \int_{[z,z+h]} f(\zeta)d\zeta + \int_{[z+h,z_0]} f(\zeta)d\zeta = 0,$$

and thus

$$\int_{[z_0,z]} f(\zeta)d\zeta + \int_{[z,z+h]} f(\zeta)d\zeta - \int_{[z_0,z+h]} f(\zeta)d\zeta = 0 \implies F(z+h) - F(z) = \int_{z,z+h} f(\zeta)d\zeta$$

Thus, since f is continuous at z, then for any $\epsilon > 0$, if $|\zeta - z| < \delta$, then $|f(\zeta) - f(z)| < \epsilon$. Take $h < \frac{1}{2}\delta$, then

$$\begin{aligned}
F(z+h) - F(z) &= \int_{[z,z+h]} f(\zeta)d\zeta \\
\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \left|\frac{1}{h}\int_{[z,z+h]} f(\zeta)d\zeta - \frac{1}{h}\int_{[z,z+h]} f(z)\right| \\
&= \frac{1}{|h|}\left|\int_{z,z+h} f(\zeta) - f(z)d\zeta\right| \\
&\le \frac{1}{|h|}\text{length}(z, z+h) \max_{\zeta\in[z,z+h]} |f(\zeta) - f(z)| \\
&= \max_{\zeta\in[z,z+h]} |f(\zeta) - f(z)| \\
&< \epsilon
\end{aligned}$$

□

1.9 Tuesday, Apr 22: Goursat's Theorem and the Cauchy Integral Formula

Theorem 12. (Goursat) Let $O \subseteq \mathbb{C}$ is an open set and $f \in H(O \setminus \{z\})$, where f is merely continuous at z. Then if $\triangle \subseteq \mathbb{C}$ is closed,

$$\int_{\triangle} f(\zeta) d\zeta = 0.$$

Proof. We first sketch the proof. If $z \notin \triangle$, then by Cauchy-Goursat's Theorem, we are done. Thus, assume $z \in \triangle$. Make three triangles, $\triangle_1, \triangle_2, \triangle_3$, where $\triangle_i \subseteq \triangle$ and $\bigcap \triangle_k = z$, and the orientation of the triangles cancel out on the inside.

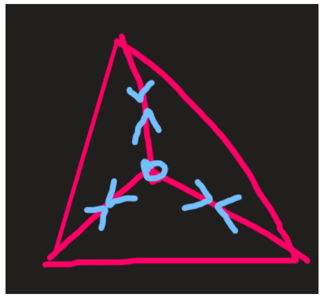

Figure 2: z is the middle point

If z_0 is on the vertex, we can make another three triangles, as follows:

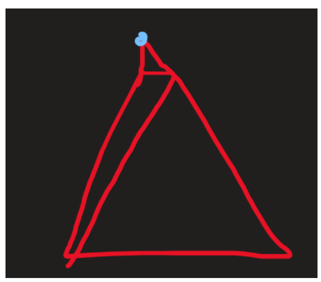

Figure 3: z_0 is the blue point

We can make the top triangle as small as we wish. Thus, we can bound the line integral using Proposition 3, namely we can make it be 0. □

Theorem 13. (Cauchy Integral Formula) Suppose $O \subseteq \mathbb{C}$ is open and $\overline{D_r(z_0)} \subseteq O$. Suppose that $f \in H(O)$. Then if $z \in D_r(z_0)$, we have that

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where $C_r(z_0)$ is the circle of radius r centered at z_0.

Example 1.6. We have that $C_r(z_0)(\theta) = z_0 + re^{i\theta}$

Proof. Define $F(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z}, & \zeta \neq z \\ f'(z), & \zeta = z \end{cases}$. It is not hard to show F is continuous. For all $\zeta \neq z$, F is holomorphic. Goursat theorem tells us we can forgive F for its blunder at z, and thus Cauchy's Theorem,

$$\int_{C_r(z_0)} F(\zeta)d\zeta = 0 \implies \frac{1}{2\pi i}\int_{C_r(z_0)} F(\zeta)d\zeta.$$

For small enough r, we have that

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)-f(z)}{\zeta - z}d\zeta = 0.$$

Thus,

$$\begin{aligned}
0 &= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)-f(z)}{\zeta - z}d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(z)}{\zeta - z}d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{f(z)}{2\pi i}\int_{C_r(z_0)} \frac{1}{\zeta - z}d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z}d\zeta - f(z)\mathrm{Ind}_{C_r(z_0)}(z) \\
&= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z}d\zeta - f(z)
\end{aligned}$$

□

Remark 13. With the same assumptions as the above, we have that

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z}d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z_0 - (z - z_0)}d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)}\frac{1}{1 - \frac{z-z_0}{\zeta - z_0}}d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)}\frac{1}{1 - a}d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\zeta - z_0)^n}d\zeta
\end{aligned}$$

Recall the Weierstrass M-test. If $G_N(s) = \sum_{k=1}^N g_k(s)$, where $|g_k(s)| \le M_k$ for all k, and $\sum_{k=1}^\infty M_k < \infty$, then G_n converges uniformly. We have that

$$|G_n(s) - G_m(s)| = |\sum_{m+1}^n g_k| \le |\sum_{m+1}^n |g_k| \le \sum_{m+1}^n M_k < \epsilon,$$

and thus G_n is a Cauchy sequence of real numbers. Thus, by completeness, we have that $\|G(s) - G_n(s)\|_{\sup(s)} < \epsilon$. Thus, since the F_N converge uniformly to F_∞, then we have that

$$\lim_{n\to\infty}\int F_n = \int F,$$

and thus

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(\zeta)}{(\zeta-z_0)}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\zeta-z_0)^n}d\zeta &= \lim_{N\to\infty}\frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(\zeta)}{(\zeta-z_0)}\sum_{n=0}^{N}\frac{(z-z_0)^n}{(\zeta-z_0)^n}d\zeta \\
&= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(\zeta)}{(\zeta-z_0)}\frac{(z-z_0)^n}{(\zeta-z_0)^n}d\zeta \\
&= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\right](z-z_0)^n d\zeta
\end{aligned}
$$

Thus, $f(z)$ is equal to a convergent power series inside of a disk. In particular, $f(z)$ is infinitely differentiable since power series are infinitely differentiable!!!!! (See Corollary 1).

We summarize this remark with the following theorem.

Theorem 14. Suppose $\overline{D_r(z_0)} \subseteq O$, where O is open and $f \in H(O)$. Then f is infinitely differentiable.

Remark 14. Consider the special case of the Cauchy integral formula:

$$
\begin{aligned}
f(z_0) &= \frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(\zeta)}{\zeta-z_0}d\zeta \\
&= \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{z_0+re^{i\theta}-z_0}ire^{i\theta}d\theta \\
&= \frac{1}{2\pi i}\int_0^{2\pi}f(z_0+re^{i\theta})d\theta
\end{aligned}
$$

We recognize this as the mean value property from PSET 1.

Corollary 2.

Proof. Since $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, where $a_n = \frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta$. By a previous theorem, we know also that

$$
a_n = \frac{f^{(n)}(z_0)}{n!}
$$

□

1.10 Thursday, Apr 24: Louisville's Theorem

Today he proved everything he did last class again, so you can look at last classes note's for that.

Theorem 15. (Louisville) If $f \in H(\mathbb{C})$ and $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f is constant.

Proof. By Corollary 2 in the previous class, we have that

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta,$$

and thus

$$|f'(z_0)| \leq |\frac{1}{2\pi i}| \text{length}(C_r(z_0)) \max_{\zeta \in C_z(r)} \left| \frac{f(\zeta)}{(\zeta - z_0)^2} \right| \leq \frac{1}{2\pi} 2\pi r \frac{M}{r^2} = \frac{M}{r}.$$

But since f is holomorphic over all of $\mathbb{C}$, we can take $r \to \infty$, and thus $|f'(z_0)| \leq 0$, and so because z_0 was arbitrary in $\mathbb{C}$, f is constant. $\square$

Remark 15. Let P be a non-constant polynomial Suppose that $P(z) \neq 0$ for any z, then as $z \to \infty$,

$$|P(z)| \to \infty.$$

Then $\frac{1}{|P(z)|} \to 0$ and since $\frac{1}{P(z)}$ is bounded and holomorphic, then it is constant, and thus $P(z)$ is constant. Thus, we have the fundamental theorem of algebra.

1.11 Tuesday, Apr 29: Fundamental Theorem of Algebra

Remark 16. Recall that

$$f(z) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

Then

$$a_n = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = \frac{1}{n!} f^{(n)}(z_0)$$

Thus, for $n = 1$, we have that

$$f'(z_0) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta,$$

and so the rate of change is dependent on the size of the function, hence why Louisville's theorem makes sense.

Theorem 16. Suppose f is an entire function that is bounded. Then f is constant.

Proof. Since f is bounded, we have that $|f(\zeta)| \leq M$ for all $\zeta \in M$. By the remark above, we know that

$$\begin{aligned}
|f'(z_0)| &\leq \frac{1}{2\pi}\big|\text{arclength}\big[C_r(z_0)\big]\big| \max_{\zeta \in C_r(z_0)} \Big|\frac{f(\zeta)}{(\zeta - z_0)^2}\Big| \\
&= \frac{1}{2\pi} 2\pi r \max_{\zeta \in C_r(z_0)} \frac{|f(\zeta)|}{(\zeta - z_0)^2} \\
&\leq \frac{M}{2\pi} 2\pi r \max_{\zeta \in C_r(z_0)} \frac{1}{(\zeta - z_0)^2} \\
&= \frac{M}{2\pi} 2\pi r \frac{1}{r^2} \\
&= \frac{M}{r}
\end{aligned}$$

But $f \in H(O)$, and so we can let $r \to \infty$, implying that $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. □

Theorem 17. (Fundamental Theorem of Algebra) If $P(z)$ is a polynomial with degree of more than zero, then $P(z_0) = 0$ for some $z_0 \in \mathbb{C}$.

Proof. Suppose $P(z) \neq 0$ for all $z \in \mathbb{C}$. Let

$$f(z) = \frac{1}{P(z)}.$$

Note that $f \in H(\mathbb{C})$.

Note that

$$|P(z)| = |a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0| = |z^n| \left|(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n})\right| \xrightarrow[z \to \infty]{} \infty$$

Thus,, we have that

$$f(z) = \frac{1}{|P(z)|} \to 0.$$

Thus, there exists some closed $D_r(0)$ such that for all $z \notin D_r(0)$, we have that $|f(z)| < \epsilon$. Since f is continuous on the compact disk, we have that f is bounded by the extreme value theorem. Thus, $|f(z)| < M$ for all $z \in \mathbb{C}$. Thus, by Louisville's theorem, $f(z) = \frac{1}{P(z)}$ is constant, and thus $P(z)$ is constant, which is a contradiction. □

Definition 11. Let $f \in H(O)$ and $z \in O$. The **Laurent Series** of f is the series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Example 1.7. Consider $C = D_1(0) \setminus \{0\}$ to be the punctured disk and consider $f(z) = \frac{1}{z}$.

Definition 12. Let $\gamma : [0,1] \to O$ be a path. We define a **homotopy** in $O \subset \mathbb{C}$ to be a function $\Gamma(t,s) : [0,1] \times [0,1] \to O$ such that $\gamma_s(t) = \Gamma(t,s)$ and

(a) Γ is continuous from $[0,1] \times [0,1]$.

(b) For all $s \in [0,1]$, $\Gamma(0,s) = \gamma_s(0) = z_1$ and $\Gamma(1,s) = \gamma_s(1) = z_2$.

(c) For each $s \in [0,1]$, $\gamma_s(t)$ is a path and for each $t \in [0,1]$, the map $s \mapsto \gamma_s(t)$ is piecewise continuously differentiable as a function of s.

We will next show the invariancy of line integrals about homotopies, and use that for the Laurent series.

1.12 Tuesday, May 6: Homotopic Defomations

Let A be an annulus around z_0. Clearly, if $f \in H(A)$, then we cannot express $f(z)$ as a convergent power series, since then f would be holomorphic within the disk. The following few lectures will talk about how useful Laurent series are in expressing functions like this. First we discuss homotopies in detail.

Definition 13. A **homotopic deformation** of a path γ_0 connecting z_1 to z_2 to a path γ_1 also connecting z_1 to z_2 is a function $\Gamma(t,s) : [0,1]^2 \to O$ such that

(a) Γ is continuous from $[0,1] \times [0,1]$.

(b) For all $s \in [0,1]$, $\Gamma(0,s) = \gamma_s(0) = z_1$ and $\Gamma(1,s) = \gamma_s(1) = z_2$.

(c) For each $s \in [0,1]$, $\gamma_s(t)$ is a path and for each $t \in [0,1]$, the map $s \mapsto \gamma_s(t)$ is piecewise continuously differentiable as a function of s.

Intuitively, s measures how far along the deformation has gone on, the line from bottom to top is a path, and all these paths are continuous.

Remark 17. Since $\Gamma([0,1] \times [0,1])$ is compact since Γ is continuous, and since O^c is closed, then for every $w \in O^c$, there exists some $\epsilon > 0$ such that

$$|\Gamma(s,t) - w| = |\gamma_s(t), w| \geq \epsilon > 0.$$

Equivalently,

$$D_{\frac{\epsilon}{2}}(\Gamma(s_0,t_0)) \subseteq O, \quad \forall (s_0,t_0) \in [0,1]^2$$

Moreover, since Γ is continuous and the domain is compact, then Γ is uniformly convergent. Thus, take $\delta > 0$ such that if $d((s_1,t_1),(s_2,t_2)) < \delta$, then $d(\Gamma(s_1,t_1),\Gamma(s_2,t_2)) < \frac{\epsilon}{2}$.

Partition $[0,1]^2$ into squares, S_k, of equal diameter which is less that δ. Thus, $\Gamma(S_k) \subseteq D_{\frac{\epsilon}{2}}(z)$ for some $z \in O$. Consider the path, γ_{S_k}, formed by traversing each S_k by the four time intervals (the sides of the square). Then $\gamma_{S_k} \subseteq D_{\frac{\epsilon}{2}}(z)$, and so by Cauchy's theorem, since $f \in H(O)$, we have that

$$\int_{\gamma_{S_k}} f(z)\, dz = 0$$

Then

$$\sum_k \int_{\gamma_{S_k}} f(z)\, dz = 0,$$

but since we can make it so the orientations cancel each other out, then you end up with the path over the boundary of $[0,1]^2$. Since the left and right vertical boundary lines are simply constant z_1 and z_2, then

$$\int_{\gamma_0} f(z)\, dz - \int_{\gamma_1} f(z)\, dz = \sum_k \int_{\gamma_{S_k}} f(z)\, dz = 0$$

and so

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz$$

Theorem 18. Let $O \subset \mathbb{C}$ be open, and suppose that γ_1 and γ_2 are homotopically deformable paths. Then if $f \in H(O)$,

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz$$

Proof. The proof follows from Remark 17 ∎

Definition 14. We say that a region $O \subset \mathbb{C}$ is **simply connected** if every closed path γ is homotopically deformable to a point path

Theorem 19. Suppose $O \subseteq \mathbb{C}$ is simply connected and $f \in H(O)$. Then for any path γ in O,

$$\int_\gamma f(z)\, dz = 0$$

Proof. We can apply Theorem 18 to establish the equivalence between a point path and our path γ. ∎

Theorem 20. Let A be an annulus. Let $f \in H(A)$ and fix $z \in A$ between γ_i and γ_0, where γ_i is a circle around the inner disk of A and γ_0 is a large circle around γ_i. Then

$$\int_{\gamma_i} f(z)\, dz = \int_{\gamma_0} f(z)\, dz.$$

Proof. Consider the function

$$F(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z}, & \zeta \neq z \\ f'(z) \end{cases}.$$

Then F is holomorphic and so

$$\frac{1}{2\pi i}\int_{\gamma_i} F(\zeta) d\zeta = \frac{1}{2\pi i}\int_{\gamma_i} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z)\int_{\gamma_i} \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z)\int_{\gamma_0} \frac{d\zeta}{\zeta - z}$$

But clearly,

$$\begin{aligned} \frac{1}{2\pi i}\int_{\gamma_i} \frac{f(\zeta)}{\zeta - z}\, d\zeta &= \frac{1}{2\pi i}\int_{\gamma_i} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z)\int_{\gamma_i} \frac{d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z)\int_{\gamma_0} \frac{d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z) \end{aligned}$$

and thus

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\int_{\gamma_i} \frac{f(\zeta)}{z - \zeta} d\zeta$$

∎

1.13 Thursday, May 8: Riemann's Theorem and the Weierstrass Casserati Theorem

We begin by redoing the derivation of the formula for the Laurent series from last class.

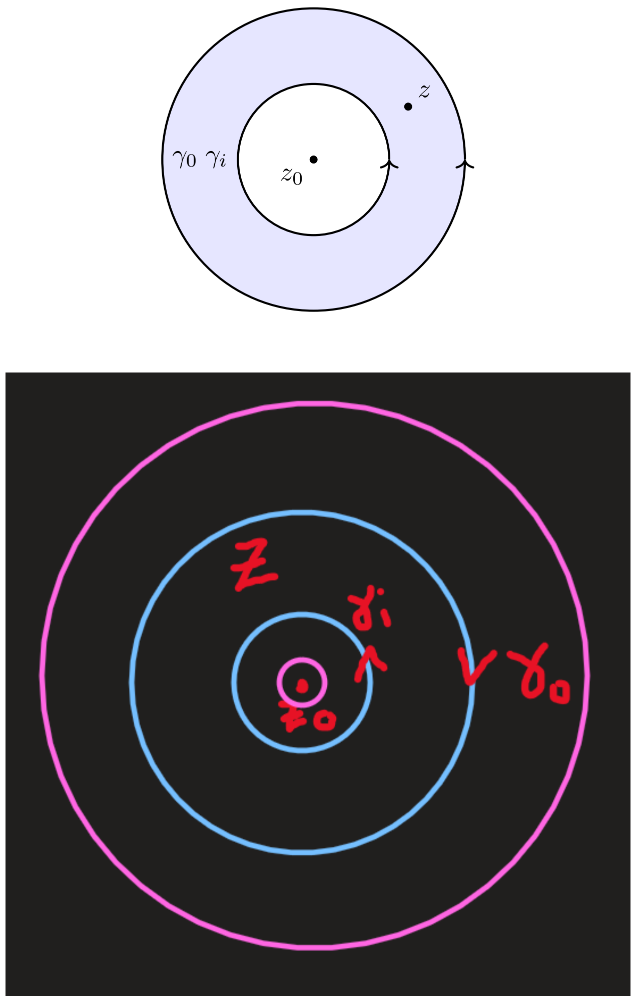

Figure 4: The Annulus and the Paths

Call

$$F(\zeta) = \begin{cases} \frac{f(\zeta)-z}{\zeta - z}, & \zeta \neq z \\ f'(z), & \zeta = z \end{cases}$$

to be a holomorphic function. While γ_0 and γ_1 are not homotopically equivalent, then can be made that by a single line going out and in of the inner and out circle, but those line integrals cancel out. Thus, by Theorem 18,

$$\int_{\gamma_1} F(\zeta)\, d\zeta = \int_{\gamma_0} F(\zeta)\, d\zeta$$

$$\frac{1}{2\pi i} \int_{\gamma_1} F(\zeta)\, d\zeta = \frac{1}{2\pi i} \int_{\gamma_0} F(\zeta)\, d\zeta$$

$$\frac{1}{2\pi i}\int_{\gamma_i}\frac{f(\zeta)}{\zeta-z}\,d\zeta - f(z)\int_{\gamma_i}\frac{d\zeta}{\zeta-z} = \frac{1}{2\pi i}\int_{\gamma_0}\frac{f(\zeta)}{\zeta-z}\,d\zeta - f(z)\int_{\gamma_0}\frac{d\zeta}{\zeta-z}$$

The winding number of the left side is 0, while the winding number of the right side is 1. Hence,

$$\frac{1}{2\pi i}\int_{\gamma_i}\frac{f(\zeta)}{\zeta-z}\,d\zeta = \frac{1}{2\pi i}\int_{\gamma_0}\frac{f(\zeta)}{\zeta-z}\,d\zeta - f(z)$$

And we then compute:

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\gamma_0}\frac{f(\zeta)}{\zeta-z}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_i}\frac{f(\zeta)}{\zeta-z}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\gamma_0}\frac{f(\zeta)}{\zeta-z}\,d\zeta + \frac{1}{2\pi i}\int_{\gamma_i}\frac{f(\zeta)}{z-\zeta}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\gamma_0}\frac{f(\zeta)}{(\zeta-z_0)-(z-z_0)}\,d\zeta + \frac{1}{2\pi i}\int_{\gamma_i}\frac{f(\zeta)}{(z-z_0)-(\zeta-z_0)}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\gamma_0}\frac{1}{1-\frac{(z-z_0)}{(\zeta-z_0)}}\frac{f(\zeta)}{\zeta-z_0}\,d\zeta + \frac{1}{2\pi i}\int_{\gamma_i}\frac{1}{1-\frac{\zeta-z_0}{z-z_0}}\frac{f(\zeta)}{z-z_0}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\gamma_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n\frac{f(\zeta)}{\zeta-z_0}\,d\zeta + \frac{1}{2\pi i}\int_{\gamma_i}\sum_{n=0}^{\infty}\left(\frac{\zeta-z_0}{z-z_0}\right)^n\frac{f(\zeta)}{z-z_0}\,d\zeta \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\gamma_0}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n + \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\gamma_i}(\zeta-z_0)^n f(\zeta)(z-z_0)^{-(n+1)} \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\gamma_0}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n + \sum_{n=-\infty}^{-1}\frac{1}{2\pi i}\int_{\gamma_i}(\zeta-z_0)^{-(n+1)}f(\zeta)(z-z_0)^n \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\gamma_0}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n + \sum_{n=-\infty}^{-1}\frac{1}{2\pi i}\int_{\gamma_i}\frac{f(\zeta)}{(\zeta-z_0)^{-(n+1)}}(z-z_0)^n \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n + \sum_{-\infty}^{-1}\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{(\zeta-z_0)^{(n+1)}}(z-z_0)^n \\
&= \sum_{n=-\infty}^{\infty}\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n
\end{aligned}$$

Theorem 21. (Riemann) Suppose $f \in H(D_\epsilon(z_0)\setminus\{z_0\})$ and $|f(z)| \le M$ for all $z \in D_\epsilon(z_0)\setminus\{z_0\}$. Then we can define $f(z_0)$ such that $f \in H(D_\epsilon(z_0))$.

Proof. Let $r > 0$, and let γ_r be a circle of radius r about z_0.

$$|a_{-n}| = \left|\frac{1}{2\pi i}\int_{\gamma_r} f(\zeta)(\zeta-z_0)^n\right| \le rMr^n = Mr^{n+1}$$

Thus, $a_{-n} = 0$ for any $n > 0$. Thus, as $r \to 0$, we have that $f(z_0)$ is an absolutely convergent power series. ☐

Theorem 22. (Weierstrass-Casserati) Suppose $f \in H(D_\epsilon(z_0)\setminus\{z_0\})$. Then either:

(a) f has a removable singularity at z_0;

(b) f has a pole at z_0, i.e,

$$\lim_{z\to z_0}|f(z)| = \infty;$$

(c) f has an essential isolated singularity if it is neither (a) or (b). Then the image of f when we consider the domain $D_\delta(z_0) \setminus \{z_0\}$ for $0 < \delta < \epsilon$ is dense in $\mathbb{C}$.

Proof. If f has no negative terms in the Laurent series, then f can be expressed as a convergent power series. If f is bounded, then it clearly a removable singularity. If f has at most finitely many negative powers in the Laurent expansion, then for z such that $0 < |z - z_0| < \epsilon$, take n_k to be the highest negative power

$$
\begin{aligned}
|f(z)| &= \frac{a_{n_k}}{(z-z_0)^{n_k}} + \frac{a_{n_{k=1}}}{(z-z_0)^{n_{k-1}}} + \cdots + \sum_{n=0}^{\infty} a_n (z-z_0)^n \\
&= \frac{1}{(z-z_0)^{n_k}} \left(a_{n_k} + a_{n_{k-1}}(z-z_0)^{m_k} + \cdots + a_{n_1}(z-z_0)^{m_1}\right) + \sum_{n=0}^{\infty} a_n (z-z_0)^n \\
&\to \infty
\end{aligned}
$$

□

1.14 Tuesday, May 13: Poles and Introducing the Residue Theorem

Continuing the proof from last class, where we were dealing with essential isolated singularities. Before we do so, consider the following example:

Example 1.8. Consider the function $f(z) = e^{\frac{1}{z}}$. Then our puncture disk is $\mathbb{C} \setminus \{0\}$. The expansion is

$$e^{\frac{1}{z}} = 1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}(\frac{1}{z})^2 + \cdots$$

Clearly, as $z \to 0$, the function blows up, and so we can see why this essential singularity theorem could be true.

Proof. Suppose now f has infinitely many negative powers in the Laurent expansion of z_0. But suppose, for the sake of contradiction, that there exists some $\epsilon > 0$ such that the value set of $f(z)$ for $z \in D_\epsilon(z)$ that is not dense in $\mathbb{C}$. Thus, there is some open disk in $\mathbb{C}$ that contains no such $f(z)$. Thus, $f(z) \notin D_\delta(\alpha)$ for some $\delta > 0$ and $\alpha \in \mathbb{C}$. Consider now

$$g(z) = \frac{1}{f(z) - \alpha}$$

for $D_\epsilon(z_0) \setminus \{z_0\}$. We know that g is bounded since the denominator is bounded below by δ. Hence, by Riemann's Theorem (21), we can define $g(z_0)$ such that $g(z_0) \in H(D_\epsilon(z_0))$. If $g(z_0) \neq 0$, then $f(z) - \alpha$ is holomorphic at z_0, contradicting the fact that z_0 is an essential singularity. If $g(z_0) = 0$, then f has a pole at z_0, another contradiction. □

Definition 15. Suppose $f \in H(D_\epsilon(z_0))$ for some $z_0 \in \mathbb{C}$. We say that f has a **zero of order** N at z_0 if

$$f(z) = \sum_{n=N} a_n (z - z_0)^n$$

Remark 18. Clearly,

$$f(z_0) = f'(z_0) = \cdots = f^{(N-1)}(z_0) = 0$$

and $f^{(N)}(z_0) \neq 0$.

Definition 16. Suppose $f \in H(D_\epsilon(z_0) \setminus \{z_0\})$ and f has a **pole of at** z_0 **of order** N if the negative powers of the Laurent series are both finite and the smallest negative power is of order N.

Definition 17. Let $O \subseteq \mathbb{C}$ be an open set and suppose $f : O \setminus S \to \mathbb{C}$, where S has no limit points in O and $f \in H(O \setminus S)$. If at each point of S, f has either a removable singularity or a pole, then f is **meromorphic.**

Remark 19. In this class, $S = \{z_0, z_1, \ldots, z_n\}$ is finite.

Definition 18. Suppose f has an isolated singularity at z_0, and the Laurent expansion at z_0 is

$$f(z_0) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n,$$

then the **residue** of f at z_0 is

$$\operatorname{Res}_{z_0} f = a_{-1}$$

Theorem 23. (Residue Theorem) Let f be meromorphic in a simply connected region O with poles at $z_0, \ldots, z_N$. Let γ be any closed path in O not passing through any of the poles. Then

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=0}^{N} \operatorname{Ind}_\gamma(z_k) \operatorname{Res}_{z_k} f$$

1.15 Thursday, May 15: The Residue Theorem

Theorem 24. (Residue Theorem) Suppose O is a simply connected region and f is meromorphic in O. Let γ be a closed path not passing through any of the n poles. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Ind}_\gamma(z_k)\,\mathrm{Res}_{z_k}(f)$$

Proof. If f has a pole at z_k, then by the Caseratti-Weierstrass theorem,

$$f(z) = \sum_{j=1}^{N} \frac{a_j}{(z-z_k)^j} + \text{power series in } (z-z_k).$$

Let

$$P_k = \sum_{j=1}^{N} \frac{a_j}{(z-z_k)^j}$$

be the principal part of the Laurent expansion.

Thus, there is some $r_k < 0$ such that if $z \in D_{r_k}(z_k)$, then z is not a residual of f. We claim that

$$f(z) - \sum_{k=1}^{n} P_k(z) := g(z) \in H(O).$$

Consider z_1. We claim that $f(z)-P_1(z)$ has a removable singularity at z_1. One can see this because at $D_1(z_1)$, $f(z) - P_1(z)$ can be expressed as a power series with a removable singularity at z_1. Note that $\sum_{k=2}^{n} P_k(z)$ does not interfere with this because of how we defined r_1. Thus, $g(z) \in H(O)$ by Riemann's theorem. Hence, we use Cauchy's theorem

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_\gamma \left(f(z) - \sum P_k(z)\right) dx + \int_\gamma P_k(z)\,dz \\
&= \int_\gamma \sum P_k(z)\,dz \\
&= \sum_{k=1}^{n} \int_\gamma P_k(z)\,dz \\
&= 2\pi i \sum_{k=1}^{n} \frac{1}{2\pi i} \int_\gamma \sum_{j=1}^{N_k} \frac{a_j}{(z-z_k)^j}\,dz \\
&= 2\pi i \sum_{k=1}^{n} \sum_{j=1}^{N_k} \frac{a_j}{2\pi i} \int_\gamma \sum_{j=1}^{N_k} \frac{1}{(z-z_k)^j}\,dz \\
&= 2\pi i \sum_{k=1}^{n} a_1^{(k)} \frac{1}{2\pi i} \int_\gamma \frac{1}{(z-z_k)}\,dz \\
&= 2\pi i \sum_{k=1}^{n} \frac{1}{2\pi i} \mathrm{Res}_{z_k}(f) \mathrm{Ind}_\gamma(z_k)
\end{aligned}$$

□

Example 1.9. Consider some function $f(z)$ with poles at $n\pi$ on the real axis, where $n \in \mathbb{Z}$. Let $R > 0$ such that $|n\pi| < R$ for all $n \in \mathbb{Z}$. The consider the disk $D_R(0)$. We want this function $f(z)$ have residuals such that

$$\mathrm{Res}_{n\pi}(f) = \frac{1}{(n\pi)^2}.$$

Consider now $\sin^{-1} z$. The Laurent expansion about 0 is

$$\frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots} = \frac{1}{z}\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots}$$

Near $z = 0$, the second term is holomorphic, and thus it has a power series. Thus,

$$\frac{1}{\sin z} = \frac{1}{z}(a_0 + a_1 z + a_2 z^2 + \cdots).$$

It suffices to find a_0, which is clearly 1. Now consider

$$\mathrm{Res}_{n\pi}(\frac{1}{z^2(\sin z)})$$

We have that about $n\pi$,

$$(\frac{1}{z^2(\sin z)}) = \frac{1}{z^2}\left[\frac{1}{z - n\pi} + \text{ power series}\right]$$

But $\frac{1}{z^2}$ is perfectly holomorphic about $n\pi$ for $n \neq 0$, and so it is analytic, and thus

$$\frac{1}{z^2(\sin z)} = (b_0 + b_1(z - n\pi) + \cdots)\left[\frac{1}{z - n\pi} + \text{power series}\right]$$

so it suffices to find b_0, which is clearly $\frac{1}{(n\pi)^2}$. Observe that as $z \to \infty$,

$$\left|\frac{\cos z}{\sin z}\right| = \left|\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}\right| \to 1.$$

Theorem 25.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Proof. We claim that $\cot z$ is π periodic. Consider that

$$\begin{aligned}
\cot z &= \frac{\cos z}{\sin z} \\
&= \frac{\frac{e^{iz}+e^{-iz}}{2}}{\frac{e^{iz}-e^{-iz}}{2i}} \\
&= i\frac{e^{i(w+\pi)}+e^{-i(w+\pi)}}{e^{i(w+\pi)} - e^{i(w+\pi)}} \\
&= i\frac{e^{i(w)e^{\pi}}+e^{-i(w)e^{\pi}}}{e^{i(w)}e^{\pi} - e^{i(w)}e^{\pi}} \\
&= \cot z
\end{aligned}$$

We now claim that $\cot z \frac{1}{z^2}$ has a pole exactly at $n\pi$ for $n \in \mathbb{Z}$. We can expand $\cot z$ at 0 by

$$\begin{aligned}
\cot z &= \frac{\cos z}{\sin z} \\
&= \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots}{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots} \\
&= \frac{1}{z}\left[\frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots}\right] \\
&= \frac{1}{z}(a_0 + a_1 z + \cdots)
\end{aligned}$$

And so

$$_0 \cot(z) = a_0 = 1$$

Now consider for $n\pi \neq 0$,

$$\frac{1}{z^2}\cot z = \frac{1}{z^2}\left[\frac{1}{z - n\pi} + \text{power series}\right]$$

so it suffices to find a_1. We will find next class that $a_1 = \frac{1}{n^2\pi^2}$. □

1.16 Tuesday, May 20: $\sum \frac{1}{n^2}$

Final: Thursday, May 29. 10am. Ryerson 358. Continuing from the previous class:

Proof. We have shown that $\cot z$ is $\pi-$periodic and that it has poles at exactly multiples of π and that $\text{Res}_{n\pi} \cot z = 1$. We also showed that $\text{Res}_{n\pi}(\frac{\cot z}{z^2}) = \frac{1}{n^2\pi^2}$ for $n \neq 0$. It remains to show that if we remove $\mathscr{D} = \{D_r(n\pi)\}_{n\in\mathbb{Z}}$ for r small, then $\frac{\cot z}{z^2}$ is bounded in $\mathscr{D}^c$. By periodicity, it suffices to see that it is bounded in the strip $[-\frac{\pi}{2}, \frac{\pi}{2}]$. To see this, first note that $|e^{iz}| = e^{-y}$ and $|e^{-iz}| = e^{y}$

$$\begin{aligned}
|\cot(z)| &= |\cot(x+iy)| \\
&= \left|\frac{\cos(x+iy)}{\sin(x+iy)}\right| \\
&= \left|\frac{\frac{e^{iz}+e^{-iz}}{2}}{\frac{e^{iz}-e^{-iz}}{2}}\right| \\
&= \frac{|e^{iz}+e^{-iz}|}{|e^{iz}-e^{-iz}|} \\
&= |\frac{e^{iz}e^{-2y}+e^{ix}}{e^{ix}e^{-2y}-e^{ix}}| \\
&\xrightarrow[y\to\infty]{} |\frac{e^{ix}}{e^{ix}}| \\
&= 1
\end{aligned}$$

Thus, our function is bounded as $y \to \infty$ If we remove the disk $D_r(0)$, then our function is a continuous function on a compact set, and is thus bounded in x. Consider $n = 5$. Recall the Laurent expansion to be

$$\frac{\cot z}{z^2} = \frac{1}{z^2}\left[\frac{1}{(z-5\pi)^2} + a_0 + a_1(z-5\pi) + \dots\right] \implies \text{Res}_{5\pi} f = \frac{1}{25\pi^2}$$

When $z = 0$,

$$\frac{\cot z}{z^2} = \frac{1}{z^2}\left[\frac{1}{z^2} + a_0 + a_1 z + \dots\right]$$

and so $\text{Res}_0 \frac{\cot z}{z^2} = a_1$. We see that

$$\begin{aligned}
\frac{1}{z^2}\frac{\cos z}{\sin z} &= \frac{1}{z^2}\frac{1-\frac{z^2}{2!}+\cdots}{z(1-\frac{z^2}{3!}+\cdots)} \\
&= \frac{1}{z^3}(1-\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots)\left(\frac{1}{1-(\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots)}\right) \\
&= \frac{1}{z^3}\left[(1-\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots)(1+(\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots)+(\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots)^2+\cdots)\right]
\end{aligned}$$

$$\begin{aligned}
1 &\mapsto \frac{1}{3!} = \frac{1}{6} \\
-\frac{z^2}{2!} &\mapsto -\frac{1}{2}
\end{aligned}$$

Where the last two lines are due to the powers of z^2 in the brackets from distributing 1 in the first term to the second and then distributing $\frac{-z^2}{2!}$ into the second term. Thus, adding up the coefficients of z^2, we see that $a_1 = -\frac{1}{3}$.

Consider $C_R(0)$ to be the circle of radius R about the origin and remove $\mathscr{D}$ from this circle. Then since we have shown $\cot Z$ is bounded by some M,

$$|\int_{C_{(n+\frac{1}{2})\pi}(0)} \frac{\cot z}{z^2}\,dz| \leq 2\pi((n+\frac{1}{2})\pi)\frac{M}{((n+\frac{1}{2})\pi)^2} \to 0$$

By the Residue Theorem,

$$\int_{C_{(n+\frac{1}{2})\pi}} \frac{\cot z}{z}\,dz = 2\pi i \sum_{z_k \text{ residues}} \mathrm{Res}_{z_k f} = 2\pi i(-\frac{1}{3} + \sum_{k=-n,k\neq 0}^{n} \frac{1}{n^2\pi^2}) \to 0$$

Thus, rearranging in the limit,

$$\frac{1}{3} = 2\sum_{k=1}^{\infty} \frac{1}{n^2\pi^2} \implies \sum_{k=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$\square$

Remark 20. Consider the expression $z^n = w$. It turns out there are n such z. Consider the roots of unity $z^n = 1$. To find the n such roots, consider that

$$z^n = 1 \iff n\log z = \log 1 = 0 \implies e^{n\log z} = e^0 = 1.$$

By a theorem done in class, this implies that $\log z = 2\pi i\frac{k}{n}$ for $k \in \mathbb{N}$. and thus

$$z = e^{2\pi i\frac{k}{n}}.$$

But by periodicity of e, only $k = 0, 1, \ldots, n-1$ are distinct. We claim that the nth root of w has the form

$$w = e^{\frac{1}{n}} e^{i\frac{\rho}{n}} (n\text{th roots of unity})$$

1.17 Thursday, May 22: The Argument Principle

Theorem 26. (Argument Principle) Suppose $O \subseteq \mathbb{C}$ is open and let $\overline{D_r(z_0)} \subseteq O$. Let f be meromorphic on O. Suppose f has no poles on $\gamma = C_r(z_0)$ and it doesn't vanish on γ. Then the winding number

$$\mathrm{Ind}_{f\circ\gamma}(0) = \# \text{ zeros of } f \text{ inside } D_r(z_0) - \# \text{ poles inside } D_r(z_0)$$

where the zeros are counted according to multiplicity

Proof. Recall that if f has a zero of order n at z_0, then you can write

$$f(z) = a_n(z-z_0)^n + \cdots = (z-z_0)^n(a_n + a_{n+1}(z-z_0) + \dots) = (z-z_0)^n g(z).$$

where $g(z)$ is analytic and is never 0 around z_0. If f has a pole of order n at z_0, then the most negative power to appear has coefficient a_{-n} and

$$f(z) = (z-z_0)^n \left[a_{-n} + a_{-n_{k+1}}(z-z_0) + \dots\right] = (z-z_0)^n g(z),$$

where $g(z)$ is analytic and is never 0 around z_0.

Lemma 4. If $f(z), g(z) \in H(D_r(z_0) \setminus \{z_0\})$ never vanish in $D_r(z_0) \setminus \{z_0\}$, then in $D_r(z_0) \setminus \{z_0\}$

$$\frac{(fg)'(z)}{(fg)(z)} = (\frac{f'}{f})(z) + (\frac{g'}{g})(z).$$

Proof. Using the product rule, $(fg)' = f'g + g'f$, and the result follows by dividing by fg. □

Lemma 5. Suppose that f is holomorphic, never zero in $D_r(z_0) \setminus \{z_0\}$. Then $\frac{f'(z)}{f(z)}$ has a simple pole at z_0 in two cases:

(a) f has a zero of order n at z_0, in which case

$$\mathrm{Res}_{z_0}(\frac{f'}{f}) = n$$

(b) If f has a pole of order n at z_0, then

$$\mathrm{Res}_{z_0}(\frac{f'}{f}) = -n$$

Proof. If f has a zero of order n at z_0, we can write

$$f(z) = (z-z_0)^n g(z),$$

where $g(z) \in H(D_r(z_0))$ and $g(z_0) \neq 0$ Then using our beautiful lemma 4,

$$\frac{f'(z)}{f(z)} = \frac{((z-z_0)^n)'}{(z-z_0)} + \frac{g'(z)}{g(z)} = \frac{n}{(z-z_0)} + \frac{g'(z)}{g(z)} = \frac{n}{(z-z_0)} + (a_0 + a_1(z-z_0) + \dots).$$

Hence, the residue is clearly n.

If f has a pole at order n at z_0, we can run it back and it is clear that the residue is $-n$. □

Consider that by Lemma 5, we can apply the residue theorem to

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{z_k \text{ are zeros or poles}} \operatorname{Res}_{z_k} \frac{f'}{f}$$

and by our Lemma 5 we get our result. To conclude, we compute

$$\operatorname{Ind}_{f\circ\gamma}(0) = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{z}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(f\circ\gamma)'(\theta)}{f(\gamma(\theta))}\,d\theta = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(\gamma(\theta))\,d\theta}{f(\gamma(\theta))}\gamma'(\theta)\,d\theta = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$$

□

Definition 19. Let $f \in H(D_r(z_0))$. We say that f has an **a-point** of order n at z_0 if

$$f(z) - a$$

has a zero of order n at z_0.

Corollary 3. Suppose $f \in H(O)$, and $\overline{D_r(z_0)} \subseteq O$. If $f(z) \neq a$ on $\gamma = C_r(z_0)$, then the number of a-points inside γ (counted according to multiplicity) is equal to $\operatorname{Ind}_{f\circ\gamma}(a)$

Proof. Let $g(z) = f(z) - a$. Then g has no zeros on γ. By the argument principle,

$$\#\text{ a points of } f = \#\text{ zeros of } g \text{ inside } \gamma = \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a}\,dz = \operatorname{Ind}_{f\circ\gamma}(a).$$

□

This theorem was not done in class, but it is necessary for the open mapping theorem.

Theorem 27. (Rouche's Theorem) Suppose that $f, g \in H(O)$, where O is an open set containing $\overline{D_r(z_0)}$. If $|f(z)| > |g(z)|$ for any $z \in C_r(z_0)$, then f and $f+g$ have the same number of zeros inside of $C_r(z_0)$.

Proof. Define

$$f_t(z) = f + tg(z)$$

for all $t \in [0,1]$ such that $f_0 = f$ and $f_1 = f+g$. Let n_t be the number of zeros of f_t counted with multiplicity. Since $|f(z)| > |g(z)|$ on $C_r(z_0)$, then $f_t(z) \neq 0$ for any $z \in C_r(z_0)$, and thus by the argument principle,

$$n_t = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f_t'(z)}{f_t(z)}\,dz.$$

We know that n_t is continuous from the fact that both $f_t'(z)$ and $f_t(z) \neq 0$ are continuous. Hence, n_t must be constant, since otherwise, the continuity implies that n_t takes on some non-integer values. Hence, $n_0 = n_1$, and we conclude. □

Theorem 28. (Open Mapping Theorem) Suppose $f \in H(\Omega)$ is non-constant, where Ω is a region. Then $f(\Omega)$ is open.

Proof. Let $a \in f(\Omega)$ such that $a = f(z_0)$. Define

$$F(z) = f(z) - a.$$

We know that F has an isolated zero at z_0. Let $r > 0$ such that F doesn't vanish anywhere on $\overline{D_r(z_0)} \setminus \{z_0\}$. Since $|F|$ is continuous on the compact $C_r(z_0) =: C$, $|F|$ achieves its minimum. Let

$$\rho = \min_{z\in C}|F(z)| = \min_{z\in C}|f(z)-a| > 0.$$

Now we consider the disk $\overline{D_\rho(a)}$. Let $w \in D_\rho(a)$. It suffices to show that $w \in f(\Omega)$. Define

$$g(z) = (f(z) - a) + (a - w) =: F(z) + H(z) = f(z) - w.$$

To use Rouche's theorem, it suffices to see that $|F(z)| > |H(z)|$ for $z \in C$. Let $z \in C$, then

$$|F(z)| = |f(z) - a| = \rho > |w - a| = |H(z)|.$$

Thus, $f(z)-w$ has the same number of zeros as $f(z)-a$. Namely, there is some $z_1 \in \Omega$ such that $f(z_1)-w = 0$, and thus $f(z_1) = w$. Hence, $w \in f(\Omega)$. $\square$

Theorem 29. (Maximum Modulus Principle) Let Ω be a region and suppose $f \in H(\Omega)$. If f is non-constant, then f cannot attain its maximum in Ω.

Proof. Suppose there exists some z_0 such that $|f(z_0)| \geq f(z)$ for all $z \in H(O)$. Since Ω is open, then $f(\Omega)$ is open by the open mapping theorem. Thus, there is some $r > 0$ such that $D_r(f(z_0)) \subset f(\Omega)$, and thus there is some z_1 such that $f(z_1) \in D_r(f(z_0))$ and $|f(z_1)| \geq |f(z_0)|$. Which is a contradiction. $\square$

Theorem 30. (Fundamental Theorem of Algebra)

Theorem 31. Suppose $P(z)$ is never zero. Then $\frac{1}{P(z)} \in H(\Omega)$. We know that $|\frac{1}{P(z)}| \to 0$ for $z \to \infty$. For z large, say larger than some $R > 0$, we know that $\frac{1}{P(z)} \leq \frac{1}{P(0)}$. Take $\overline{D_R(0)}$. Since $\frac{1}{P(z)}$ is continuous and $\overline{D_R(0)}$ is compact, then $\frac{1}{P(z)}$ achieves its maximum on $\overline{D_R(0)}$. But by the maximum modulus principle, this implies that $\frac{1}{P(z)}$ is constant, and thus $P(z)$ is constant, which is a contradiction!