23500 Spring 2025: Problem Set 5 Agustín Esteva
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Problem 1 (5 points)

Show that if X and Y are random variables such that $\mathbb{E}[Y \mid X] = \mathbb{E}[Y]$, then it holds that

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y],$$

but the reverse implication does not hold.

SOLUTION: We use the law of total expectation to note that

$$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY \mid X]].$$

X is trivially $X-$measurable, so then it acts as a constant

$$\mathbb{E}[\mathbb{E}[XY \mid X]] = \mathbb{E}[X\mathbb{E}[Y \mid X]] = \mathbb{E}[X\mathbb{E}[Y]].$$

$\mathbb{E}[Y]$ is just a constant, not a random variable, and so

$$\mathbb{E}[X\mathbb{E}[Y]] = \mathbb{E}[Y]\mathbb{E}[X],$$

as desired.

Let $S = \{-2, -1, 1, 2\}$ with

$$\mathbb{P}\{X = -1\} = \mathbb{P}\{X = 1\} = \mathbb{P}\{X = -2\} = \mathbb{P}\{X = 2\} = \frac{1}{4}.$$

Define

$$Y := X^2.$$

Then $\mathbb{E}[X^2] = \frac{10}{4}$

$$\mathbb{E}[XY] = \mathbb{E}[X^3] = 0$$

but

$$\mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[X]\mathbb{E}[X^2] = 0 \cdot 1.$$

But $\mathbb{E}[X^2 \mid X] = X^2$ since X^2 is $X-$measurable. But $X^2 \neq \frac{5}{2}$. ■

Problem 2 (10 points)

Suppose $X \sim \mathrm{Poi}(\lambda)$.

(a) Compute the expected value of X given its parity (i.e., find $\mathbb{E}[X \mid X \text{ is odd}]$ and $\mathbb{E}[X \mid X \text{ is even}]$).

Solution: Since X takes values in $\mathbb{N}_0$, then definition of conditional expectation,

$$\begin{aligned}
\mathbb{E}[X \mid X \text{ odd}] &= \frac{\displaystyle\sum_{n=0}^{\infty} n\mathbb{P}\{X = n, X \text{ odd}\}}{\displaystyle\sum_{n=0}^{\infty} \mathbb{P}\{X = n, X \text{ odd}\}} \\
&= \frac{\displaystyle\sum_{n=0}^{\infty} n\mathbb{P}\{X \text{ odd } \mid X = n\}\mathbb{P}\{X = n\}}{\displaystyle\sum_{n=0}^{\infty} \mathbb{P}\{X \text{ odd } \mid X = n\}\mathbb{P}\{X = n\}} \\
&= \frac{\displaystyle\sum_{n=0}^{\infty} (2n+1)\mathbb{P}\{X = 2n+1\}}{\displaystyle\sum_{n=0}^{\infty} \mathbb{P}\{X = 2n+1\}} \\
&= \frac{\displaystyle\sum_{n=0}^{\infty} (2n+1)\frac{e^{-\lambda}\lambda^{2n+1}}{(2n+1)!}}{\displaystyle\sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^{2n+1}}{(2n+1)!}} \\
&= \frac{\displaystyle\sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{(2n)!}}{\displaystyle\sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{(2n+1)!}} \\
&= \frac{\lambda \cosh \lambda}{\sinh \lambda} \\
&= \lambda \coth \lambda
\end{aligned}$$

Using similar logic, one can see that

$$\mathbb{E}[X \mid X \text{ even}] = \lambda \tanh \lambda$$

■

(b) Suppose we buy X raffle tickets, each of which has a chance $p \in (0, 1)$ of winning independently of others. Let Y be the number of prizes given out. Compute $\mathbb{E}[Y \mid X]$ and $\mathbb{E}[Y]$.

SOLUTION: $Y \mid X = k$ is binomial with probability of success p and X trials. Thus, $\mathbb{E}[Y \mid X = k] = kp$, and so $\mathbb{E}[Y \mid X] = pX$. We then use the law of total expectation to note that

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]] = \mathbb{E}[pX] = p\mathbb{E}[X] = p\lambda.$$

$\blacksquare$

Problem 3 (10 points)

Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{P}\{X_i = 1\} = \mathbb{P}\{X_i = -1\} = \frac{1}{2}$. Let $S_0 = 0$, and $S_n = X_1 + X_2 + \cdots + X_n$ define a simple symmetric random walk on $\mathbb{Z}$. As shown in class, S_n is a martingale with respect to $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

(a) Find a deterministic sequence $a_n \in \mathbb{R}$ such that $M_n := S_n^3 + a_n S_n$ is a martingale with respect to $\mathcal{F}_n$.

SOLUTION: Using linearity and a few other facts, we see that

$$\begin{aligned}
\mathbb{E}[M_n \mid \mathcal{F}_{n-1}] &= \mathbb{E}[S_n^3 + a_n S_n \mid \mathcal{F}_{n-1}] \\
&= \mathbb{E}[(S_{n-1} + X_n)^3 + a_n S_n \mid \mathcal{F}_{n-1}] \\
&= \mathbb{E}[S_{n-1}^3 + 3S_{n-1}^2 X_n + 3S_{n-1}X_n^2 + X_n^3 + a_n S_n \mid \mathcal{F}_{n-1}] \\
&= S_{n-1}^3 + 3S_{n-1}^2 \mathbb{E}[X_n \mid \mathcal{F}_{n-1}] + 3S_{n-1}\mathbb{E}[X_n^2 \mid \mathcal{F}_{n-1}] + \mathbb{E}[X_n^3 \mid \mathcal{F}_{n-1}] + a_n S_{n-1} \\
&= S_{n-1}^3 + 3S_{n-1}^2 \mathbb{E}[X_n] + 3S_{n-1}\mathbb{E}[X_n^2] + \mathbb{E}[X_n^3] + a_n S_{n-1} \\
&= S_{n-1}^3 + 3S_{n-1} + a_n S_{n-1}
\end{aligned}$$

and so M_n is a martingale if and only if

$$S_{n-1}^3 + 3S_{n-1} + a_n S_{n-1} = M_{n-1} = S_{n-1}^3 + a_{n-1}S_{n-1}$$

and thus

$$a_n = a_{n-1} - 3 \implies \boxed{a_n = a_0 + (-3n)}.$$

We showed that this satisfies the martingale condition. Since the sequence is deterministic and S_n is a martingale, then any deterministic function of S_n is $\mathcal{F}_n$ measurable, and thus M_n is $\mathcal{F}_n$ measurable. Moreover,

$$\mathbb{E}[|M_n|] = \mathbb{E}[|S_n|^3] + a_0\mathbb{E}[|S_n|] - 3n\mathbb{E}[|S_n|] \leq n^3 + a_0 n - 3n^2 < \infty$$

Thus, M_n is a martingale.

∎

(b) Find deterministic sequences $b_n, c_n \in \mathbb{R}$ such that $Z_n := S_n^4 + b_n S_n^2 + c_n$ is a martingale with respect to $\mathcal{F}_n$.

SOLUTION: We see that in order to satisfy the martingale property,

$$\begin{aligned}
\mathbb{E}[S_n^4 + b_n S_n^2 + c_n \mid \mathcal{F}_{n-1}] &= \mathbb{E}[(S_{n-1} + X_n)^4 \mid \mathcal{F}_{n-1}] + b_n\mathbb{E}[S_n^2 \mid \mathcal{F}_{n-1}] + c_n \\
&= \mathbb{E}[S_{n-1}^4 + cX_n + 6S_{n-1}X_n^2 + cX_n^3 + X_n^4 \mid \mathcal{F}_{n-1}] \\
&\quad\quad + b_n\mathbb{E}[(S_n^2 - n) + n \mid \mathcal{F}_{n-1}] + c_n \\
&= S_{n-1}^4 + 6S_{n-1}^2 + 1 + b_n(S_{n-1}^2 - (n-1)) + nb_n + c_n
\end{aligned}$$

$$
\begin{aligned}
&= S_{n-1}^4 + S_{n-1}^2(6 + b_n) + 1 + b_n + c_n \\
&= S_{n-1}^4 + b_{n-1}S_{n-1}^2 + c_{n-1}
\end{aligned}
$$

Thus, $6 + b_n = b_{n-1}$ and $1 + b_n + c_n = c_{n-1}$, implying that

$$b_n = b_0 - 6n$$

$$c_n = c_{n-1} - 1 - b_0 + 6n = c_0 - b_0 n + 3n^2 - n$$

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Problem 4 (20 points)

Let $\{X_n\}$ be a biased random walk on the integers with probability $p \in (0, 1/2)$ to move to the right and probability $1 - p \in (1/2, 1)$ to move to the left.

(a) Show that $M_n = \left(\frac{1-p}{p}\right)^{X_n}$ is a martingale with respect to $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$.

SOLUTION: Without loss of generality, assume that $X_0 = 0$. Since M_n depends only on X_i for $i \leq n$, then clearly M_n is $\mathcal{F}_n$ measurable.

We can bound $|X_n|$ by n since that is the furthest it can get in n steps. Thus, since $\frac{1-p}{p} > 1$, we have that

$$\mathbb{E}[|M_n|] = \mathbb{E}\left[\left|\left(\frac{1-p}{p}\right)^{X_n}\right|\right] = \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{|X_n|}\right] \leq \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{n}\right] < \infty$$

Finally, we have that since we can write $X_n = \sum_{i=1}^n \xi_i$, where ξ_i are i.i.d. such that

$$\mathbb{P}\{\xi_i = 1\} = p, \quad \mathbb{P}\{\xi_i = -1\} = 1 - p.$$

Then

$$\begin{aligned}
\mathbb{E}[M_n \mid \mathcal{F}_{n-1}] &= \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{X_n} \mid \mathcal{F}_{n-1}\right] \\
&= \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{X_{n-1}} \left(\frac{1-p}{p}\right)^{\xi_n} \mid \mathcal{F}_{n-1}\right] \\
&= \left(\frac{1-p}{p}\right)^{X_{n-1}} \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{\xi_n}\right] \\
&= \left(\frac{1-p}{p}\right)^{X_{n-1}} (p\left(\frac{1-p}{p}\right)^{1} + (1-p)\left(\frac{1-p}{p}\right)^{-1}) \\
&= \left(\frac{1-p}{p}\right)^{X_{n-1}} \\
&= M_{n-1}
\end{aligned}$$

■

(b) Use the optional stopping theorem to compute, for any $x \in \{0, \ldots, N\}$, the probability that the walk reaches 0 before N if $X_0 = x$.

SOLUTION: Define

$$\tau := \min\{n \geq 0 : X_n \in \{0, N\} \mid X_0 = x\}$$

be the first time X_n reaches 0 or N given that it begins at $X_0 = x$. Assuming we can use the OST, we have that

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0] = \left(\frac{1-p}{p}\right)^x$$

and thus if we call p_L the probability we 'lose' (reach 0) and $p_W = 1-p_L$ the probability we 'win' (reach N), we see that

$$\left(\frac{1-p}{p}\right)^x = \mathbb{E}[M_\tau] = p_L(1) + p_W\left(\frac{1-p}{p}\right)^N \implies p_W = \frac{1-\left(\frac{1-p}{p}\right)^x}{1-\left(\frac{1-p}{p}\right)^N},$$

and $p_L = 1 - p_W$.

Thus, it suffices to notice that M_n satisfies the conditions for the OST:

(a) The state $\{1, 2, \ldots, N-1\}$ is transient, and thus since τ is the first time we leave the state, then a result from Markov chains states that

$$\mathbb{P}\{\tau < \infty\} = 1$$

(b) We can bound the expectation by the fact that $|X_\tau| \leq N$ and thus

$$\mathbb{E}[|M_\tau|] \leq \left(\frac{1-p}{p}\right)^N < \infty$$

(c) We have by a result in class that for transient random walks,

$$\mathbb{E}[M_n \mathbb{1}_{\tau > n}] \leq (\frac{1-p}{p})^n e^{-cn} \to 0.$$

■

(c) Show that $\widetilde{M}_n = X_n + (1-2p)n$ is a martingale with respect to $\mathcal{F}_n$.

Solution: We assume WLOG that $X_0 = 0$. $\widetilde{M}_n$ is clearly $\mathcal{F}_n$ measurable.

Again, we bound $|X_n|$ by n and so

$$\mathbb{E}[|\widetilde{M}_n|] \leq n + (1-2p)n < \infty$$

$$\begin{aligned}\mathbb{E}[\widetilde{M}_n \mid \mathcal{F}_{n-1}] &= \mathbb{E}[X_n + (1-2p)n \mid \mathcal{F}_{n-1}] \\ &= \mathbb{E}[X_n \mid \mathcal{F}_{n-1}] + (1-2p)n\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}[X_n \mid X_{n-1}] + (1-2p)n \\
&= p(X_{n-1}+1) + (1-p)(X_{n-1}-1) + (1-2p)n \\
&= pX_{n-1} + (1-p)X_{n-1} + p - (1-p) + (1-2p)n \\
&= X_{n-1} - (1-2p) + (1-2p)n \\
&= X_{n-1} + (1-2p)(n-1) \\
&= \widetilde{M}_{n-1}
\end{aligned}
$$

■

(d) Use the optional stopping theorem to compute, for any $x \in \{0, \ldots, N\}$, the expectation of the first time that $X_n \in \{0, N\}$ if $X_0 = x$.

Solution: Define

$$\tau := \min\{n \geq 0 : X_n \in \{0, N\} \mid X_0 = x\}$$

be the first time X_n reaches 0 or N given that it begins at $X_0 = x$. Assuming we can use the OST, we have that

$$\mathbb{E}[\widetilde{M}_\tau] = \mathbb{E}[\widetilde{M}_0] = x$$

and

$$\mathbb{E}[\widetilde{M}_\tau] = \mathbb{E}[X_\tau + (1-2p)\tau] = \mathbb{E}[X_\tau] + (1-2p)\mathbb{E}[\tau] = p_L(0) + p_W(N) + (1-2p)\mathbb{E}[\tau].$$

Thus,

$$\mathbb{E}[\tau] = \frac{x - Np_W}{1-2p},$$

where p_W was derived in part (b). Thus, it suffices to show that we satisfy the conditions of the OST.

(a) $\mathbb{P}\{\tau < \infty\}$ for the same reason as in part (b)

(b) We bound the expectation by the same reason as in b, and using the fact that by transience, $\mathbb{E}[\tau] < \infty$

$$\mathbb{E}[|\tilde{M}_\tau|] \leq \mathbb{E}[|X_\tau|] + (1-2p)\mathbb{E}[\tau] = N + (1-2p)\mathbb{E}[\tau] < \infty$$

(c) For the same reason as in part (b), we see that

$$\mathbb{E}[|M_n| \mathbb{1}_{\tau > n}] \leq (N + (1-2p)n)e^{-cn} \to 0,$$

where e^{-cn} is the probability that X_n still has not left the class $\{1, 2, 3, \ldots, N-1\}$.

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Problem 5 (10 points)

Let $\{M_n\}_{n\geq 0}$ be a martingale. Suppose that $M_0 = 0$ and $\mathbb{P}[|M_n| \leq 1] = 1$ for every $n \geq 1$.

(a) Let τ be a stopping time for $\{M_n\}_{n\geq 0}$ such that $\mathbb{P}[\tau < \infty] = 1$. Explain why $\mathbb{E}[M_\tau] = 0$.

SOLUTION: Since M_n is almost surely bounded, then (this computation is mostly for me, as the result is pretty clear, but it gave me good intuition)

$$\begin{aligned}
\mathbb{E}[|M_n|\mathbb{1}_{\tau>n}] &= \mathbb{E}[\mathbb{E}[|M_n|\mathbb{1}_{\tau>n} \mid |M_n|]] \\
&= \mathbb{E}[|M_n|\mathbb{1}_{\tau>n} \mid |M_n| > 1]\mathbb{P}\{|M_n| > 1\} + \mathbb{E}[|M_n|\mathbb{1}_{\tau>n} \mid |M_n| \leq 1]\mathbb{P}\{|M_n| \leq 1\}] \\
&= \mathbb{E}[|M_n|\mathbb{1}_{\tau>n} \mid |M_n| \leq 1] \\
&\leq \mathbb{E}[\mathbb{1}_{\tau>n}] \\
&= \mathbb{P}\{\tau > n\} \\
&= 1 - \mathbb{P}\{\tau \leq n\} \\
&\to 1 - \mathbb{P}\{\tau < \infty\} \\
&= 0
\end{aligned}$$

Also we have that since $\tau = n$ for some $n \in \mathbb{N}$,

$$\mathbb{E}[|M_\tau|] \leq 1.$$

Thus, we can apply the optional stopping theorem and say that

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0] = 0$$

■

(b) Show that for each $r \in (0, 1]$,

$$\mathbb{P}[M_n \leq r, \forall n \geq 0] > 0.$$

SOLUTION: Suppose not, that for some $r \in (0, 1]$, we have that

$$\mathbb{P}\{M_n \leq r,\ \forall n \geq 0\} = 0.$$

Let $\tau := \min\{n \geq 0 : M_n > r\}$. By our contradiction, we have that $\mathbb{P}\{\tau < \infty\} = 1$. By the optional stopping theorem, we have that

$$0 = \mathbb{E}[M_\tau],$$

but by definition,

$$\mathbb{E}[M_\tau] > r\mathbb{P}\{\tau < \infty\} = r,$$

which is a contradiction. ■

Problem 6 (10 points)

Let X_n be a Markov chain on the two-dimensional integer lattice $\mathbb{Z}^2$ with the following transition probabilities:

$$\mathbb{P}(X_{n+1} = (i \pm 1, j) \mid X_n = (i,j)) = \frac{1}{8}, \quad \mathbb{P}(X_{n+1} = (i, j \pm 1) \mid X_n = (i,j)) = \frac{1}{8},$$
$$\mathbb{P}(X_{n+1} = (i \pm 1, j \pm 1) \mid X_n = (i,j)) = \frac{1}{8}.$$

(a) Prove that $M_n := |X_n|^2 - \frac{3}{2}n$ is a martingale with respect to the natural filtration of the process. (We denote by $|x|$ the Euclidean norm of $x \in \mathbb{Z}^2$.)

Solution: We assume WLOG that $X_0 = (0,0)$. It is clear that M_n is $\mathcal{F}_n$ measurable. We can bound the expectation by the fact that $|X_n|^2 \leq 2n^2$ (since the farthest X_n can travel is diagonally all the way, which is $n\sqrt{2}$ distance from the origin)

$$\mathbb{E}[|M_n|] \leq \mathbb{E}[|X_n|^2 + \frac{3}{2}n] = \mathbb{E}[|X_n|^2] + \frac{3}{2} = 2n^2 + \frac{3}{2} < \infty$$

For the martingale property, we note that $X_n = \sum_{i=1}^n \xi_i$, where ξ_i is the $8-$sided die that determines what the next step of the random walk is. Then

$$\begin{aligned}
\mathbb{E}[M_n \mid \mathcal{F}_{n-1}] &= \mathbb{E}[|X_n|^2 \mid \mathcal{F}_{n-1}] - \frac{3}{2}n \\
&= \mathbb{E}[|X_n - X_{n-1} + X_{n-1}|^2 \mid \mathcal{F}_n] - \frac{3}{2}n \\
&= \mathbb{E}[|\xi_n + X_{n-1}|^2 \mid \mathcal{F}_n] - \frac{3}{2}n \\
&= \mathbb{E}[|\xi_n|^2 + |X_{n-1}|^2 + 2\langle \xi_n, X_{n-1}\rangle \mid \mathcal{F}_{n-1}] - \frac{3}{2}n \\
&= \mathbb{E}[|\xi_n|^2] + |X_{n-1}|^2 + 2\mathbb{E}[\langle \xi_n, X_{n-1}\rangle \mid \mathcal{F}_{n-1]}] - \frac{3}{2}n \\
&= |X_{n-1}|^2 + \frac{3}{2} - \frac{3}{2}n + 2\mathbb{E}[\langle \xi_n, X_{n-1}\rangle \mid \mathcal{F}_{n-1]}] \\
&= |X_{n-1}|^2 - \frac{3}{2}(n-1) + 2\mathbb{E}[\langle \xi_n, X_{n-1}\rangle \mid \mathcal{F}_{n-1]}].
\end{aligned}$$

Moreover, we note that by linearity and symmetry, we have that

$$\mathbb{E}[\langle \xi_n, X_{n-1}\rangle \mid \mathcal{F}_{n-1]}] = \langle \mathbb{E}[\xi_n \mid \mathcal{F}_{n-1]}], \mathbb{E}[X_{n-1} \mid \mathcal{F}_{n-1]}]\rangle = \langle \mathbb{E}[\xi_n], X_{n-1}\rangle = \langle 0, X_{n-1}\rangle = 0,$$

and so we are done. ■

(b) For $R \in \mathbb{R}_+$, define the stopping time

$$T_R := \inf\{n \geq 0 : |X_n|^2 \geq R^2\}.$$

Give sharp lower and upper bounds for $\mathbb{E}[T_R \mid X_0 = (0,0)]$.

Solution: We apply the OST to M_n, and thus

$$\mathbb{E}[M_{T_R}] = \mathbb{E}[M_0] = 0,$$

but we also have that

$$\mathbb{E}[M_{T_R}] = (\mathbb{E}[|X_{T_R}|^2] - \frac{3}{2}\mathbb{E}[T_R])\mathbb{P}\{T_R < \infty\} = (\mathbb{E}[|X_{T_R}|^2] - \frac{3}{2}\mathbb{E}[T_R]).$$

We know first off that $|X_{T_R}|^2 \geq R^2$. But we can bound it from above by $(R+\sqrt{2})^2$, since the martingale can be at most one diagonal step from R^2. Thus,

$$(R+\sqrt{2})^2 - \frac{3}{2}\mathbb{E}[T_R]) \geq 0 = (\mathbb{E}[|X_{T_R}|^2] - \frac{3}{2}\mathbb{E}[T_R]) = 0 \geq R^2 - \frac{3}{2}\mathbb{E}[T_R])$$

Thus,

$$\frac{2}{3}R^2 \leq \mathbb{E}[T_R] \leq \frac{2}{3}(R+\sqrt{2})^2.$$

It remains to be seen that we can actually apply the OST to M_n. To do this, recall that X_n is null recurrent. Consider the state space $S = \{(x,y) \in \mathbb{R}^2 \mid |(x,y)| < R\}$. Suppose that X_n remains in this circle S. Then X_n is recurrent within the circle, and so $\mathbb{P}\{X_n = (0,0)\text{i.o.} \mid X_0 = (0.0)\} = 1$, implying that X_n is positive recurrent. Thus, with probability 1, X_n will leave the circle, and we have the fact that the probability the X_n is still within the circle after time n is bounded above by e^{-cn}.

- By the above discussion, $\mathbb{P}\{T_R < \infty\}$
- We easily bound

  $$\mathbb{E}[|M_\tau|] \leq (R+\sqrt{2})^2 + \frac{3}{2}\mathbb{E}[\tau],$$

  where $\mathbb{E}[\tau] < \infty$ since X_n is null recurrent, and hence $p^n((0,0),(x,y)) \to 0$ for any $|(x,y)| < R$, implying that we must leave the circle at some point almost surely.
- Consider the state

  $$\mathbb{E}[|M_\tau|\mathbb{1}_{\tau > n}] \leq (R^2 + \frac{3}{2}n)e^{-cn} \to 0$$

■

Problem 7 (15 points)

Let G be a connected graph. We allow G to be infinite, but we assume that every vertex of G has finite degree. Let $\{X_n\}_{n\geq 0}$ be the simple random walk on G. A function $f : V(G) \to \mathbb{R}$ is called harmonic at a vertex $x \in V(G)$ if

$$\frac{1}{\deg x}\sum_{y\sim x} f(y) = f(x),$$

where $\deg x$ denotes the number of neighbors of x, and $y \sim x$ means there is an edge from y to x.

(a) Fix $x_0 \in V(G)$ and assume that $X_0 = x_0$. Show that if f is harmonic, then $\{f(X_n)\}_{n\geq 0}$ is a martingale with respect to $\sigma(X_1, \ldots, X_n)$.

SOLUTION: We claim that $f(X_n)$ is $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ measurable. To see this, note that clearly, X_n is $\mathcal{F}_n$ measurable, and so $f(X_n)$ since it's value depends only on information about X_n, since this will tell you the value of the neighbors of X_n.

Since $f : V(G) \to \mathbb{R}$ and $X_n \in V(G)$, then $f(X_n) < \infty$ almost surely. Thus, $|f(X_n)| < M \in \mathbb{R}$, and thus except possibly for a set of measure zero, we have that since $\mathbb{P}\{X\} = 1$ (X is the whole space), then

$$\mathbb{E}[|f(X_n)|] = \int_X |f(X)| d\mathbb{P} \leq \int_X M d\mathbb{P} = M < \infty,$$

To show the martingale property, we note that X_n is a Markov chain, and thus so we apply the Markov property to compute:

$$\begin{aligned}
\mathbb{E}[f(X_n) \mid \mathcal{F}_{n-1}] &= \mathbb{E}[f(X_n) = x_n \mid X_{n-1} = x_{n-1}] \\
&= \sum_{x_n \sim x_{n-1}} p(x_{n-1}, x_n) f(x_n) \\
&= \sum_{x_n \sim x_{n-1}} \frac{1}{\deg x_{n-1}} f(x_n) \\
&= \frac{1}{\deg x_{n-1}} \sum_{x_n \sim x_{n-1}} f(x_n) \\
&= f(X_{n-1})
\end{aligned}$$

■

(b) Show using the martingale convergence theorem that if $\{X_n\}_{n\geq 0}$ is recurrent, then every non-negative harmonic function on G is constant.

SOLUTION: Since X_n is recurrent. Since $f(X_n)$ is a martingale and $f(X_n) \geq 0$ a.s., then for any $n \geq 0$, we have that

$$\mathbb{E}[|f(X_n)|] = \mathbb{E}[f(X_n)] = \mathbb{E}[f(X_0)] = f(x_0) < \infty$$

by definition of f. Thus, we can apply the MCT. With probability 1, there exists some $X_\infty \in V(G)$ such that

$$\lim_{n\to\infty} f(X_n) = f(X_\infty).$$

Since X_n is recurrent, then X_n visits state $x_0 \in V(G)$ infinitely many times. Consider the subsequence $X_{n_k^1} = x_0$. Since subsequences converge to the same value as the parent sequence, then $f(X_{n_k^1}) \to f(X_\infty)$, but we know that for all k, $f(X_{n_k^1}) = f(x_0)$, and so $f(X_\infty) = f(x_0)$. Consider now the general subsequence $X_{n_k^i} = x_i$. We know that $f(X_{n_k^i}) \to f(X_\infty)$, and so $f(x_i) = f(X_\infty)$. Because this holds for any $x_i \in V(G)$, then f is constant on G. ■

(c) Show that if $\{X_n\}_{n\in\mathbb{N}}$ is transient (in which case $V(G)$ is infinite), then for any vertex $x_0 \in V(G)$ there is a non-constant function on G which takes values in $[0,1]$ and is harmonic at every vertex of G except for x_0.

Solution: Let $x_0 \in V(G)$. Define $\tau_i := \min\{n \geq 0 : X_n = x_0 \mid X_0 = x_i\}$. Then define

$$f(x) = \mathbb{P}\{\tau_x < \infty\}.$$

Clearly, $f(x) \in [0,1]$. Let $x \in V(G)$ such that $x \neq x_0$. Then if $y_1, \dots, y_n$ are the neighbors of x, we have that using the law of total probability and the Markov property

$$\begin{aligned}
f(x) &= \mathbb{P}\{\tau_x < \infty\} \\
&= \sum_{i=1}^{n} \mathbb{P}\{\tau_x < \infty \mid X_{n+1} = y_i\}\mathbb{P}\{X_{n+1} = y_i\} \\
&= \sum_{i=1}^{n} \mathbb{P}\{\tau_{y_i} < \infty\}\frac{1}{\deg x} \\
&= \frac{1}{\deg x}\sum_{i=1}^{n} f(y_i) \\
&= \frac{1}{\deg x}\sum_{y\sim x} f(y)
\end{aligned}$$

Hence, f is harmonic away from x_0. To see that it is not harmonic at x_0, note that $f(x_0) = 1$ by definition. So if it were harmonic at x_0, then $f(y) = 1$ for all $y \sim x$ since 1 is the maximum of f. Inducting, we see that $f(x_i) = 1$ for all $x_i \in V(G)$, implying that

$$\mathbb{P}\{\tau_{x_i} < \infty\} = 1$$

for any x_i, and thus X_n is recurrent, a contradiction. Thus, $f(x_0)$ is not harmonic.

To see that f is non-constant, then again, note that if it were, since $f(x_0) = 1$, then $f(x_i) = 1$ for all $x_i \in V(G)$, again contradicting transience. ■

Problem 8 (Optional)

We model a sequence of gamblings as follows. Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with $\mathbb{P}\{\xi_n = +1\} = p$, $\mathbb{P}\{\xi_n = -1\} = q$, where $p = 1 - q > \frac{1}{2}$. Define the entropy of this distribution by

$$\alpha = p \ln\left(\frac{p}{1/2}\right) + q \ln\left(\frac{q}{1/2}\right) = p \ln p + q \ln q + \ln 2.$$

A gambler starts playing with initial fortune $Y_0 > 0$, and her fortune after round n is

$$Y_n = Y_{n-1} + C_n \xi_n,$$

where C_n is the amount she bets in this round. The bet C_n may depend on the values $\xi_1, \xi_2, \ldots, \xi_{n-1}$, and satisfies $0 \le C_n < Y_{n-1}$.

The expected rate of winnings up to time n is

$$r_n := \mathbb{E}\left[\ln\left(\frac{Y_n}{Y_0}\right)\right],$$

which the gambler wishes to maximize.

(a) Prove that no matter what strategy C the gambler chooses,

$$X_n := \ln Y_n - n\alpha$$

is a supermartingale (i.e., $\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] \le X_{n-1}$), hence her expected average winning rate $r_n/n \le \alpha$.

(b) Find a gambling strategy that makes the above X_n a martingale, thus achieving the expected average winning rate α.