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Due Date: 5-22-2025

Problem 1

Suppose $f \in H(O)$, where $O \subseteq \mathbb{C}$ is an open connected region. If there is some $z_0 \in O$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in O$, then f is constant on O.

SOLUTION: Define
$$A = \{z \in O \mid |f(z)| = |f(z_0)|\}.$$

It suffices to show that A is clopen in O. Note that $A \neq \emptyset$ since $z_0 \in A$.

Let $z \in A$. Since O is open, there exists some $R > 0$ such that $\overline{D_R(z)} \subseteq O$. Let $z' \in D_R(z)$, then consider the circle $C_r(z)$ where $r := |z - z'|$. As a consequence of the Cauchy integral equation, we have seen in class that

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

By PSET 1 problem 4, we have that since $|f|$ achieves its max at z and $|f|$ is continuous since f is holomorphic, then $|f(z)| = |f(z')|$, and so $z' \in A$.

Let $z \in O$ such that $(z_n) \in A$ such that $z_n \to z$. Then since $z_n \in A$, then $f(z_n) = f(z_0)$, and so by continuity, $f(z) = f(z_0)$, and thus $z \in A$.

Since O is connected and A is a nonempty clopen set, we are dnoe.

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Problem 2

Suppose $f \in H(O)$ where $O \subseteq \mathbb{C}$ is an open connected region. If $|f|$ is constant, then f is constant.

SOLUTION: Let $f = u + iv$. Then since $|f|$ is constant, we have that

$$|f| = |u + iv| = \sqrt{u^2 + v^2} \implies u^2 + v^2 \equiv C.$$

Thus, differentiating the above with respect to x and then y,

$$\frac{\partial u^2}{\partial x} + \frac{\partial v^2}{\partial x} = 0$$

$$\frac{\partial u^2}{\partial y} + \frac{\partial v^2}{\partial y} = 0$$

Thus,

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} u + \frac{\partial v}{\partial x} v \\ \frac{\partial u}{\partial y} u + \frac{\partial v}{\partial y} v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

Consider that using the Riemann-Cauchy equations, we find that the above implies that every component in the Jacobian is zero.

$$\begin{aligned} \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \\ &= (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 \\ &= |\nabla u|^2 \\ &= 0 \end{aligned}$$

but also using similar logic,

$$\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} = |\nabla v|^2 = 0$$

This then implies that $\nabla u = \nabla v = 0$, showing that f is constant.

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Problem 3

Give another proof of the fundamental theorem of algebra.

Solution: Let $P(z)$ be a non-constant polynomial that doesn't vanish. Since $P(0) \neq 0$, we know that $f(z) := \frac{1}{|P(0)|} \neq 0$. Let $\epsilon > 0$ such that $\frac{1}{|P(0)|} > \epsilon$.

We know that as $z \to \infty$, $|P(z)| \to \infty$. Thus, $\frac{1}{|P(z)|} \to 0$ as $z \to \infty$. Thus, there exists some $R > 0$ such that for $z \notin \overline{D_R(0)}$, $\frac{1}{|P(z)|} < \frac{\epsilon}{2}$. Since $\frac{1}{|P(z)|}$ is continuous and $\overline{D_R(0)}$ is compact, then $\frac{1}{|P(z)|}$ achieves its maximum on some $z_0 \in \overline{D_R(0)}$. Necessarily, $\frac{1}{|P(z_0)|} \geq \epsilon > \frac{1}{|P(z)|}$ for any $z \notin \overline{D_R(0)}$. Thus, $\frac{1}{|P(z)|}$ attains its maximum on $\mathbb{C}$. This is a contradiction by the Maximum Modulus Principle, and thus $\frac{1}{|P(z)|}$ is constant. By Problem 2, this implies that $\frac{1}{P(z)}$ is constant. Hence, $P(z)$ is constant, which is a contradiction. ■

Problem 4

Solve the following using residue theorem

(a) What is

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

Solution: We use $\gamma_R = \gamma_1 \circ \gamma_{\text{arc}}$ to be the upper semircle of radius R about the origin. We note that if $f(z) = \frac{1}{1+z^2}$, then f has a pole at $z = i$. We have shown in a previous PSET that $\operatorname{Ind}_\gamma(i) = 1$, and thus

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}_i f(z).$$

We see that since $\frac{1}{z+i}$ is a perfectly analytic function about $z = i$, we can write

$$\begin{aligned} f(z) &= \frac{1}{(z-i)(z+i)} \\ &= \frac{1}{z-i}(a_0 + a_1(z-i) + \dots) \end{aligned}$$

and it becomes clear that the residue is $a_0 = \frac{1}{2i}$. Hence

$$\begin{aligned} \pi &= \int_{\gamma_R} f(z)\,dz \\ &= \int_{\gamma_1} f(z)\,dz + \int_{\gamma_{\text{arc}}} f(z)\,dz \\ &= \int_{-R}^{R} \frac{1}{1+x^2}\,dx + \int_{\gamma_{\text{arc}}} f(z)\,dz \\ &\to \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx \end{aligned}$$

$$\left| \int_{\gamma_{\text{arc}}} f(z)\,dz \right| \le \pi R \frac{1}{1+R^2} \to 0$$

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(b) What is

$$\int_{C_r(0)} \frac{1}{(z-a)(z-b)}$$

where $|a| \le r \le |b|$.

Solution: The function $f(z)$ only has a single pole about $z = a$, and we have seen in a previous PSET that this residue is simply $\frac{1}{a-b}$. Similarly, the winding number of

$z = a$ is 1 since it is in the bounded region of $C_r(0)$ which we have shown in a previous PSET is constantly one. Thus, we use the residue theorem

$$\int_{C_r(0)} \frac{1}{(z-a)(z-b)}\, dz = 2\pi i \operatorname{Res}_{z=a} f(z) = \frac{2\pi i}{a-b}$$

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(c) What is

$$\int_0^\infty \frac{1}{1+x^3}\, dx.$$

Solution: Let

$$f(z) = \frac{1}{1+z^3}$$

Let $\gamma_1(t) = t$ for $t \in [0, R]$ be the straight line. Then

$$\int_{\gamma_1} f(z)\, dz = \int_0^R f(\gamma(t)\gamma'(t)\, dt = \int_0^R \frac{1}{1+t^3}\, dt$$

Let $\gamma_2(t) = -t$ for $t \in [0, R]$ be the straight line in the opposite direction. Then

$$\int_{\gamma_2} f(z)\, dz = -\int_0^R \frac{1}{1-t^3}\, dt$$

This doesn't work! Let $\omega^3 = 1$ and let

$$\gamma_2(t) = t\omega$$

Then

$$\int_{\gamma_2} f(z)\, dz = \int_0^R f(\gamma_2(t))\gamma_2'(t)\, dt = \int_0^R \frac{1}{1+t^3}\omega\, dt = \omega \int_0^R \frac{1}{1+t^3}\, dt$$

We know that

$$\omega = e^{\frac{2}{3}\pi i}.$$

We also know that f has a single pole in the closed path $\gamma_R := \gamma_1 \circ \gamma_{arc} \circ -\gamma_2$. To see this, we need to find z such that

$$z^3 = -1$$

We know that $z = -e^{\frac{2\pi i}{3}\cdot 0}, -e^{\frac{2\pi i}{3}}, -e^{\frac{4\pi i}{3}} = z_1, z_2, z_3$. The only pole within our arc is $-e^{\frac{4\pi i}{3}}$. Thus,

$$\int_{\gamma_R} f(z)\, dz = 2\pi i \operatorname{Res}_{-e^{\frac{4\pi i}{3}}} f(z)\, dz$$

To find this residue, note the Laurent expansion

$$\frac{1}{1+z^3} = \frac{1}{(z-z_1)(z-z_2)(z-z_3)} = \frac{1}{z-z_3}(a_0 + a_1(z-z_3) + \cdots)$$

since $\frac{1}{(z-z_1)(z-z_2)}$ is a perfectly analytical function about z_3, and thus $a_0 = \frac{1}{(z_3-z_1)(z_3-z_2)} = \frac{1}{(-e^{\frac{4\pi i}{3}}-(-1))(-e^{\frac{4\pi i}{3}}-(-e^{\frac{2\pi i}{3}}))} = \frac{1}{(-e^{\frac{4\pi i}{3}}+1)(-e^{\frac{4\pi i}{3}}+e^{\frac{2\pi i}{3}})} = \frac{1}{e^{\frac{8\pi i}{3}}-e^{\frac{4\pi i}{3}}-e^{\frac{6\pi i}{3}}+e^{\frac{2\pi i}{3}}} = \frac{1}{-1+2e^{\frac{2\pi i}{3}}-e^{\frac{4\pi i}{3}}} = -\frac{1}{6} - \frac{i}{2\sqrt{3}}$

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}_{-e^{\frac{4\pi i}{3}}} f(z)\,dz = 2\pi i\, a_0 = -\frac{\pi i}{3} + \frac{\pi}{\sqrt{3}}$$

To see that the arc portion of the integral goes to zero at infinity, we estimate the size of the integral, noting that the angles $-z_1, -z_2 = \omega, -z_3$ cut the circle into three parts.

$$\left|\int_{\gamma_{\text{arc}}} f(z)\right| \leq \frac{2}{3}\pi R \max_{z\in\gamma_{\text{arc}}} |\frac{1}{1+z^3}| \propto \frac{1}{R^2} \to 0.$$

Thus,

$$\begin{aligned}
\frac{2\pi i}{3} e^{\frac{2}{3}\pi i} &= \int_{\gamma_R} f(z)\,dz \\
&= \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz + \int_{\gamma_{\text{arc}}} f(z)\,dz \\
&\to \int_0^R \frac{1}{1+x^3}\,dx - \omega \int_0^R \frac{1}{1+t^3}\,dt \\
&= \int_0^R \frac{1}{1+x^3}\,dx \left(1 - e^{\frac{2}{3}\pi i}\right) \\
&= \left(1 - e^{\frac{2}{3}\pi i}\right) \int_0^R \frac{1}{1+x^3}\,dx
\end{aligned}$$

Dividing both sides yields that

$$\int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi}{3\sqrt{3}}.$$

Note that in this proof, we used the assumption that $\operatorname{Ind}_{\gamma_R}(z_k) = 1$. To prove this, note that you can add a curve until $\gamma_R \circ \gamma_R' = C_R$, and we know that $\operatorname{Ind}_{C_R}(z_k) = 1$. But z_k is in the unbounded portion of γ_R', implying that $\operatorname{Ind}_{\gamma_R'}(z_k) = 0$. ■

Problem 5

Use the Residue theorem to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

SOLUTION: We use a lot of facts from class:

(a) $\cot z$ is $2\pi-$periodic

(b) $\frac{1}{z^2}\cot z$ is bounded on $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus D_r(0)$ for small $r > 0$.

(c) $\operatorname{Res}_{n\pi} \cot z = 1$ for any $n \in \mathbb{Z}$, and $n\pi$ is a simple pole.

We need one further fact which was utilized without proof in class:

Lemma 1. Suppose f has a simple pole at $z = z_0$. Then if $g \in H(D_r(z_0))$ for $r > 0$, we have that

$$\operatorname{Res}_{z_0} fg = g(z_0) \operatorname{Res}_{z_0} f.$$

Proof. Since f has a simple pole at z_0, we know that $\operatorname{Res}_{z_0} f = a_{-1}$, where

$$f(z) = a_{-1}(z - z_0)^{-1} + a_0 + a_1(z - z_0) + \cdots$$

hence

$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots$$

and thus in the limit,

$$\lim_{z \to z_0} f(z) = \operatorname{Res}_{z_0} f$$

We claim that fg has a simple pole at z_0. To see this, we know that g is analytic about z_0, so we can express it as

$$\begin{aligned}(fg)(z) &= (a_{-1}(z - z_0)^{-1} + a_0 + a_1(z - z_0) + \cdots)(b_0 + b_1(z - z_0) + \cdots) \\ &= a_{-1}b_0(z - z_0)^{-1} + a_{-1}b_1 + \cdots\end{aligned}$$

Since the Laurent series only has a zero of order 1 at z_0, we know that fg has a simple pole at $z = z_0$. Thus, we have shown that

$$\operatorname{Res}_{z_0} fg = \lim_{z \to z_0} (z - z_0)f(z)g(z) = \lim_{z \to z_0} [(z - z_0)f(z)] \lim_{z \to z_0} g(z) = g(z_0) \operatorname{Res}_{z_0} f$$

$\square$

First, we claim that $\frac{1}{z^4}\cot z$ is bounded on $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus D_r(0)$ for small $r > 0$. We have that $\frac{1}{z^2}\cot z$ is bounded on this compact set, and since $\frac{1}{z^2}$ is continuous on this set, then it is bounded as well. Hence $\frac{1}{z^2}\frac{1}{z^2}\cot z$ is bounded on this set by some M. We aim to calculate $\text{Res}_0 f(z)$, where $f(z) = \frac{1}{z^4}\cot z$. To do this, we note from class that

$$\frac{1}{z^4}\cot z = \frac{1}{z^5}\left[(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots)(1 + (\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots) + (\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots)^2 + \cdots)\right]$$

Hence, we are looking for powers of z^4 in the bracketed term. Considering first the 1 term in the first sum, we see that using the distributive property, we have the following coefficients of z^4

$$1 \mapsto -\frac{1}{5!} + (\frac{1}{3!})^2$$
$$-\frac{z^2}{2!} \mapsto -\frac{1}{2!}\frac{1}{3!}$$
$$\frac{z^4}{4!} \mapsto \frac{1}{4!}$$

and so $a_0 = -\frac{1}{120} + \frac{1}{36} - \frac{1}{12} + \frac{1}{24} = -\frac{1}{45}$ is the residual at 0 For $n\pi$ where $n \in \mathbb{Z} \setminus \{0\}$. We claim that the residual of f is $\frac{1}{n^4\pi^4}$. To see this, we note that $\frac{1}{z^4}$ does not have a pole at $z = n\pi$ and thus

$$\text{Res}_{n\pi} \frac{1}{z^4}\cot z = \frac{1}{(n\pi)^4}\text{Res}_{n\pi}\cot z = \frac{1}{n^4\pi^4}.$$

Here we used our Lemma 1. Thus, we see that if $\mathcal{D} = \{D_r(n\pi)\}_{n\in\mathbb{Z}}$, then integrating $f(z)$ in $\mathbb{C} \setminus \mathcal{D}$ over the curve $C_{(n+\frac{1}{2})\pi}(0)$, we estimate the integral (using the fact that its bounded) by

$$\left|\int_{C_{(n+\frac{1}{2})\pi}(0)} f(z)\,dz\right| \leq 2\pi(n+\frac{1}{2})\pi \max_{z\in C_{(n+\frac{1}{2})\pi}(0)} |\frac{\cot z}{z^4}| \leq 2\pi(n+\frac{1}{2})\pi\frac{M}{(n\pi)^4}| \to 0.$$

Using the Cauchy Residue Theorem,

$$\int_{C_{(n+\frac{1}{2})\pi}(0)} f(z)\,dz = 2\pi i\sum_{z_k\in\text{Res}}\text{Res}_{z_k} f = 2\pi i\left(-\frac{1}{45} + \sum_{|k|\leq n, k\neq 0}\frac{1}{k^4\pi^4}\right)$$

Thus, we see that in the limit,

$$\begin{aligned} 0 &= -\frac{1}{45} + \sum_{n\neq 0}\frac{1}{n^4\pi^4} \\ &= -\frac{1}{45} + \frac{2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} \end{aligned}$$

Rearranging, we see that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

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