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# Problem 1

If $f \geq 0$ and $\int_E f dm = 0$, prove that $f(x) = 0$ almost everywhere on $E$.

SOLUTION: Suppose not. Define

$$X_n := \{x \in E \mid f(x) \geq \frac{1}{n}\}.$$

There exists some $n$ such that $m(X_n) > 0$. Since $\frac{1}{n} \leq f(x)$ for all $x \in X_n$ and $f \in \mathcal{L}(m)$ by the remarks in Problem 7, we get that by a few more results in Problem 7:

$$0 = \int_E f \geq \int_{X_n} f \geq \frac{1}{n} m(X_n) > 0,$$

which is a contradiction. ∎

## Problem 2

If $\int_A f dm = 0$ for every $A \in \mathcal{M}$ such that $A \subseteq E$ where $E \in \mathcal{M}$, then $f(x) = 0$ almost everywhere on $E$.

SOLUTION: Define

$$X_n^+ := \{x \in E \mid f(x) \geq \frac{1}{n}\}$$

as in the previous problem. Since $f$ is measurable, then by definition, $X_n^+ \subseteq E$ is measurable. Then by Problem 7

$$0 = \int_{X_n^+} f \geq \frac{1}{n} m(X_n^+),$$

and so $m(X_n^+) = 0$ for each $n$. We can write $X^+ = \bigcup_{\mathbb{N}} X_n^+$. Then

$$m(X^+) \leq \sum_{n=1}^{\infty} m(X_n) = 0,$$

and so $m(X^+) = 0$. Define

$$X_n^-; = \{x \in E \mid f(x) < \frac{1}{n}\}.$$

Again, $m(X_n^-) = 0$ for the same reason, and so if $X^- = \bigcup X_n^-$, then

$$m(X^-) \leq \sum_{n=1}^{\infty} m(X_n^-) = 0.$$

Thus,

$$\{x \in E \mid f(x) \neq 0\} = X^+ \sqcup X^-,$$

then

$$m(\{x \in E \mid f(x) \neq 0\}) = 0$$

■

## Problem 3

If $(f_n)$ is a sequence of measurable functions, prove that the set of points $x$ at which $(f_n(x))$ converges is measurable.

SOLUTION: Recall that since $f_n$ is measurable for each $n$, then $\{x \mid f_n(x) < \frac{1}{k}\}$ is measurable for each $n$ and for each $k$.

We claim that if $f_n(x) \to L_x$, then for any $\epsilon > 0$, for any $n > 0$,

$$\{x \mid |f_n(x) - L_x| < \epsilon\}$$

is measurable. That is, the set of $x$ such that $f_n(x)$ is close to a limit is measurable. We can check that this is measurable because

$$\{x \mid |f_n(x) - L_x| < \epsilon\} = \{x \mid f_n(x) > L_x - \epsilon\} \cap \{x \mid f_n(x) < L_x + \epsilon\},$$

where both are measurable since $f_n$ is measurable.

Note that if $f_n(x)$ is not Cauchy, then it does not converge. Thus, we must necessary have the condition that for all $\epsilon > 0$, $|f_n(x) - f_m(x)| < \epsilon$.

By all the reasons stated above, we can write the set of points for which $x$ converges as

$$\bigcap_{k=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{n=\geq k} \bigcap_{m \geq k} \{x \mid |f_n(x) - f_m(x)| < \frac{1}{k}\},$$

which is measurable since $\mathcal{M}$ is closed under countable intersections. ■

## Problem 4

Define $g : [0,1] \to \mathbb{R}$ and $f_n : [0,1] \to \mathbb{R}$ such that

$$g(x) := \begin{cases} 0, & x \in [0, \frac{1}{2}] \\ 1, & x \in (\frac{1}{2}, 1] \end{cases},$$

$$f_n(x) = \begin{cases} g(x), & n = 2k \\ g(1-x), & n = 2k+1 \end{cases}$$

Show that

$$\liminf_{n\to\infty} f_n(x) = 0$$

but

$$\int_0^1 f_n(x)dx = \frac{1}{2}$$

Solution: We have that

$$\lim_{n\to\infty} f_n(x) = \sup_n \inf_{k\geq n} f_n(x).$$

It is clear that for any $n$, for any $x \in [0, \frac{1}{2}]$

$$\inf_{k\geq n} f_n(x) = f_{2k}(x) = g(x) = 0$$

For any $x \in (\frac{1}{2}, 1]$, we have that

$$\inf_{k\geq n} f_n(x) = f_{2k+1} = g(1-x) = 0$$

and so

$$\sup_{n\to\infty} 0 = 0.$$

Thus,

$$\liminf_{n\to\infty} f_n(x) = 0$$

for any $x \in [0,1]$.

Meanwhile,

$$\begin{aligned} \int_0^1 f_n(x)dx &= \begin{cases} \int_0^1 g(x)dx, & n = 2k \\ \int_0^1 g(1-x), & n = 2k+1 \end{cases} \\ &= \begin{cases} \int_0^{\frac{1}{2}} g(x)dx + \int_{\frac{1}{2}}^1 g(x)dx \\ \int_0^{\frac{1}{2}} g(1-x)dx + \int_{\frac{1}{2}}^1 2g(1-x)dx \end{cases} \\ &= \begin{cases} \int_{\frac{1}{2}}^1 dx \\ \int_0^{\frac{1}{2}} dx \end{cases} \end{aligned}$$

$$= m([0, \frac{1}{2}])$$
$$= \frac{1}{2}$$

Comparing with (77), we indeed see that if $\liminf_{n\to\infty} f_n = f = 0$ as shown above, then

$$\int_0^1 f(x)dx = 0 \leq \frac{1}{2} = \liminf_{n\to\infty} \int_0^1 f_n(x)dx,$$

as per Fatou's lemma. ■

## Problem 5

Let

$$f_n(x) = \begin{cases} \frac{1}{n}, & |x| \leq n \\ 0, & |x| > n. \end{cases}$$

Then $f_n(x) \to 0$ uniformly on $\mathbb{R}$, but

$$\int_{\mathbb{R}} f_n dx = 2.$$

SOLUTION: Let $\epsilon > 0$ and $x \in \mathbb{R}$. Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Let $n \geq N$. If $|x| > n$, then

$$|f_n(x) - 0| = |0| < \epsilon.$$

If $|x| \leq n$, then

$$|f_n(x) - 0| = |\frac{1}{n}| < \epsilon.$$

Thus, $f_n \rightrightarrows 0$. However, for any $n \in \mathbb{N}$,

$$F_n = \int_{-n}^{n} f_n(x) = 2\int_0^n \frac{1}{n} dx = 2\frac{1}{n} m([0, n]) = 2$$

and so

$$\int_{\mathbb{R}} f_n dx = \lim_{n \to \infty} F_n = 2.$$

The dominated convergence theorem fails because for any function $g$ that dominates $f_n$, its integral is not finite over $\mathbb{R}$. ■

## Problem 6

Suppose $f : \mathbb{R}^n \to \mathbb{R}$. There exists a sequence $\varphi_n$ of simple functions such that $\varphi_n \to f$ pointwise. Moreover,

(a) If $f \geq 0$, then $\varphi_n \uparrow f$

(b) If $f$ is measurable, then $\varphi_n$ are all measurable.

SOLUTION: Let $f \geq 0$. Define

$$E_i^{(n)} := \{x \mid f(x) \in [\frac{i-1}{2^n}, \frac{i}{2^n})\} = f^{-1}\big([\frac{i-1}{2^n}, \frac{i}{2^n})\big), \quad F_n := \{x \mid f(x) \geq n\} = f^{-1}\big([n, \infty)\big).$$

For each $n \in \mathbb{N}$ and for each $i \in [n2^n]$, define

$$\varphi_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_i^{(n)}} + n\chi_{F_n}.$$

We claim that for any fixed $n \in \mathbb{N}$, the $E_i^{(n)}, F_n$ are mutually disjoint and

$$\left( \bigsqcup_{i=1}^{n2^n} E_i^{(n)} \right) \cup F_n = \mathbb{R}^n.$$

To see this, fix $n \in \mathbb{N}$. Clearly, since $f$ is a function, the sets are disjoint. For any $x \in \mathbb{R}^n$, we have that $f(x) > 0$. Thus, either $f(x) \geq n$, in which case $x \in F_n$, or $f(x) \leq n$, in which case there is some $i$ for which $f(x) \in [\frac{i-1}{2^n}, \frac{i}{2^n})$ and thus $x \in E_i^{(n)}$.

Let $\epsilon > 0$, let $x \in X$. There is some $N_1 \in \mathbb{N}$ such that $f(x) < N_1$. Thus, for all $n \geq N_1$, we have that there is some $j$ for which $x \in E_j^{(n)}$. Thus, $\chi_{E_j^{(n)}} = 1$ and by the mutual disjoint-ness of all the sets,

$$\chi_{E_i^{(n)}} = \chi_{F_n} = 0$$

Let $N_2 \in \mathbb{N}$ such that $\frac{1}{2^{N_2}} < \frac{\epsilon}{2}$. Thus, if $n \geq N_2$ and $x \in E_i^{(n)}$, then $f(x) \in [\frac{i-1}{2^n}, \frac{i}{2^n})$ and

$$|f(x) - \frac{i-1}{2^n}| \leq \frac{1}{2^n} \leq \frac{1}{2^{N_2}} < \frac{\epsilon}{2}$$

Then for any $n \geq N_x = \max\{N_1, N_2\}$, we have that

$$\begin{aligned} |\varphi_n(x) - f(x)| &= \left| \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_i^{(n)}}(x) + n\chi_{F_n}(x) - f(x) \right| \\ &= \left| \frac{i-1}{2^n} - f(x) \right| \\ &\leq \left| \frac{i-1}{2^n} - \frac{i}{2^n} \right| + \left| \frac{i}{2^n} - f(x) \right| \\ &< \epsilon \end{aligned}$$

Thus, $\varphi_n \to f$ pointwise.

Since $f \geq 0$, we can show that $\varphi_n \uparrow f$. It suffices to show that

(a) $0 \leq \varphi_n \leq f$ for all $n$;

(b) $\varphi_{n-1} \leq \varphi_n$ for all $n$.

To prove (a), first fix $n$ and let $x \in X$. We have two cases:

- If $f(x) \geq n$, then

$$\varphi_n(x) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_i^{(n)}}(x) + n\chi_{F_n}(x) = n \leq f(x)$$

- If $f(x) < n$, then there is some $j$ such that $\chi_{E_j^{(n)}} = 1$ and is 0 if $i \neq j$. Thus,

$$\varphi_n(x) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_i^{(n)}}(x) + n\chi_{F_n}(x) = \frac{j-1}{2^n} \leq f(x)$$

To prove (b), let $n \in \mathbb{N}$ and $x \in X$. We have a couple of cases:

- If $f(x) \geq n > n-1$, then

$$\varphi_n(x) = \chi_{F_n} = n \geq n-1 = \chi_{F_{n-1}}(x) = \varphi_{n-1}(x).$$

- If $n > f(x) \geq n-1$, then $f(x) \in [\frac{(j-1)}{2^n}, \frac{j}{2^n})$, for some $j$ such that $\frac{(j-1)}{2^n} \geq n-1$ and so

$$\varphi_n(x) = \frac{(j-1)}{2^n} \chi_{E_{(n-1)2^n}^{(n)}}(x) \geq n-1 = F_{n-1}(x) = \varphi_{n-1}(x)$$

- If $f(x) < n-1 < n$, then we claim that for any $i \in (n-1)2^{n-1}$,

$$E_i^{(n-1)} = E_{2i-1}^{(n)} \sqcup E_{2i}^{(n)}.$$

To see this,

$$\begin{aligned}
E_i^{(n-1)} &= [\frac{i-1}{2^{n-1}}, \frac{i}{n^{n-1}}) \\
&= [\frac{2(i-1)}{2^n}, \frac{2i}{2^n}) \\
&= [\frac{(2i-1)-1}{2^n}, \frac{2i}{2^n}) \\
&= [\frac{(2i-1)-1}{2^n}, \frac{2i-1}{2^n}) \sqcup [\frac{2i-1}{2^n}, \frac{2i}{2^n}) \\
&= E_{2i-1}^{(n)} \sqcup E_{2i}^{(n)}
\end{aligned}$$

Thus, we can see that

$$\chi_{E_i^{(n-1)}} = \chi_{E_{2i-1}^{(n)} \sqcup E_{2i}^{(n)}} = \chi_{E_{2i-1}^{(n)}} + \chi_{E_{2i}^{(n)}}$$

Thus, suppose $x \in E_i^{n-1}$, then either

- $x \in E_{2i-1}^{(n)}$, in which case:

$$\begin{aligned}\varphi_{n-1}(x) &= \frac{i-1}{2^{n-1}}\chi_{E_i^{(n-1)}}(x) \\ &= \frac{2(i-1)}{2^n}(\chi_{E_{2i-1}^{(n)}}(x) + \chi_{E_{2i}^{(n)}}(x)) \\ &= \frac{2(i-1)}{2^n}\chi_{E_{2i-1}^{(n)}}(x) \\ &= \varphi_n(x)\end{aligned}$$

- $x \in E_{2i}^{(n)}$, then in this case

$$\begin{aligned}\varphi_{n-1}(x) &= \frac{i-1}{2^{n-1}}\chi_{E_i^{(n-1)}}(x) \\ &= \frac{2(i-1)}{2^n}(\chi_{E_{2i-1}^{(n)}}(x) + \chi_{E_{2i}^{(n)}}(x)) \\ &= \frac{2(i-1)}{2^n}\chi_{E_{2i}^{(n)}}(x) \\ &\leq \frac{2i-1}{2^n}\chi_{E_{2i}^{(n)}}(x) \\ &= \varphi_n(x)\end{aligned}$$

Thus, $\varphi_n \uparrow f$ and we have proved (a)

Suppose now that $f \geq 0$ and $f$ is measurable. Then by the definition, $E_i^{(n)}$ and $F_n$ are measurable for any $n$ and any $i$. Thus, by a theorem done in class, since $\varphi_n$ is simple and made up of $E_i^{(n)}$ and $F_n$ measurable, then $\varphi_n$ is measurable for all $n$.

For the general case, decompose $f$ by $f = f^+ - f^-$ where both components are non-negative. By what we just showed, there exist $0 \leq \varphi_n^+ \uparrow f^+$ simple and $0 \leq \varphi_n^- \uparrow f^-$. We show in the following problem that the difference of simple functions is a simple function, so then

$$\varphi_n = \varphi_n^+ - \varphi_n^-$$

is simple. Moreover,

$$\lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} \varphi_n^+ - \lim_{n\to\infty} \varphi_n^- = f^+ - f^- = f,$$

and so $\varphi_n$ is a sequence of simple functions which converge pointwise to $f$. We have proved the main claim.

To prove (b), not that if $f$ is measurable, then $f^+$ and $f^-$ are non-negative and measurable, and so we have shown that $\varphi_n^+$ and $\varphi_n^-$ are measurable. The difference of measurable functions is measurable, and so $\varphi_n$ is measurable for all $n$. We have now proved (b).

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## Problem 7

(a) If $f$ is measurable and bounded on $E$ and if $m(E) < \infty$, then $f \in \mathcal{L}(m)$ on $E$.

(b) If $a \le f(x) \le b$ for $x \in E$ and $m(E) < \infty$, then

$$am(E) \le \int_E f dm \le bm(E)$$

(c) If $f, g \in \mathcal{L}(m)$ on $E$ and if $f(x) \le g(x)$ for $x \in E$, then

$$\int_E f dm \le \int_E g dm$$

(d) If $f \in \mathcal{L}(m)$ on $E$, then $cf \in \mathcal{L}(m)$ on $E$, and for every $c \in \mathbb{R}$,

$$\int_E cf dm = c \int_E f dm.$$

(e) If $m(E) = 0$ and $f$ is measurable, then

$$\int_E f dm = 0$$

(f) If $f \in \mathcal{L}(m)$ on $E$, $A \in \mathcal{M}$, and $A \subset E$, then $f \in \mathcal{L}(m)$ on $A$.

SOLUTION:

*Lemma 1.* Suppose $\varphi, \psi$ are measurable simple functions such that $0 \le \varphi \le \psi$. Then for any $E \in \mathcal{M}$,

$$\int_E \varphi \le \int_E \psi.$$

*Proof.* We claim that simple functions are closed under function addition. Consider that if $\varphi$ and $\psi$ are simple, then each has finite range. Thus, any linear combination of the two must also have finite range. Thus, $\psi - \varphi$ is a simple function. By a result proven in class,

$$\int_E \psi - \varphi = I_E(\psi - \varphi) = \sum_{k=1}^{K} c_k m(E_k \cap E).$$

Since $\psi - \varphi \ge 0$, then $c_k \ge 0$ for all $k$, and thus by linearity

$$0 \le \int_E \psi - \varphi = \int_E \psi - \int_E \varphi.$$

Thus, $\int_E \varphi \le \int_E \psi$. ∎

(c) Suppose $f \leq g$ where both are Lebesgue integrable. Assume first that $f \geq 0$. For any simple $0 \leq \varphi \leq f$

$$\int_E \varphi \leq \sup_{0 \leq \psi \leq g} \int_E \psi = \int_E g \implies \int_E f = \sup_{0 \leq \varphi \leq f} \int_E \varphi \leq \int_E g$$

$$f^+ - f^- \leq g^+ - g^-$$

and so $f^+ \leq g^+$ and $f^- \geq g^-$. Since both are nonnegative, then

$$\int_E f^+ dm \leq \int_E g^+ dm, \qquad \int_E f^- dm \geq \int_E g^- dm,$$

proving that (we exclude the $dm$ differential on each integral for readability):

$$\int_E f = \int_E f^+ - \int_E f^- \leq \int_E g^+ - \int_E g^- = \int_E g.$$

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(a) Suppose $f$ is measurable and bounded on $E$ and $m(E) < \infty$. First, assume $f \geq 0$. Since $f$ is bounded on $E$, there exists some $M > 0$ such that $f(x) \leq M$ for all $x \in E$. Let $\varphi$ be a simple measurable function such that $0 \leq \varphi \leq f$. By part (c) above, if we define $g = M\chi_E$, then $g$ is a simple measurable function and so since $\varphi, g \in \mathcal{L}(m)$ and $\varphi \leq g$ for all $n$, then for all $n$, we have by (c) that

$$\int_E \varphi \leq \int_E g \implies \sup_{0 \leq \varphi \leq f} \int_E \varphi \leq \int_E g \implies \int_E f \leq Mm(E \cap E) = Mm(E) < \infty.$$

Thus, $\int_E f < \infty$ and so $f \in \mathcal{L}(m)$. For the general case, we don't assume $f$ to be nonnegative. Then $f = f^+ - f^-$, where both $f^+$ and $f^-$ are measurable and nonnegative, and thus $f^+, f^- \in \mathcal{L}(m)$ by what we just proved, and so both integrals are finite, proving that $f \in \mathcal{L}(m)$.

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(b) Suppose $a \leq f(x) \leq b$ for all $x \in E$ and $m(E) < \infty$. By part (a), $f \in \mathcal{L}(m)$. By part (c), we get that

$$\int_E a \leq \int_E f \leq \int_E b.$$

Since $a$ and $b$ are simple functions, then $\int_E a = am(E)$ and $\int_E b = bm(E)$, and we get our result.

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(d) Suppose first that $f, c \geq 0$. Since $f \in \mathcal{L}(m)$, then $f$ is measurable and nonnegative. We claim that:

(1) If $\varphi$ is a simple measurable function, then $c\varphi$ is a simple measurable function. To see this, let $\varphi = \sum_{k=1}^n a_k \chi_{E_k}$. Then $c\varphi = \sum_{k=1}^n ca_k \chi_{E_k}$. Since each $E_k$ is measurable since $\varphi$ is measurable, then $c\varphi$ is a simple measurable function.

(2) If $\varphi$ is a simple measurable function, then

$$\int_E c\varphi = c\int_E \varphi.$$

First, we have that $c\varphi \in \mathcal{L}(m)$ since by the above claim it is a simple measurable function. Moreover,

$$\int_E c\varphi = \sum_{k=1}^n ca_k m(E\cap E_k) = c\sum_{k=1}^n a_k m(E\cap E_k) = c\int_E \varphi.$$

(3) If $0 \le \varphi \le f$ is simple, then $0 \le c\varphi \le cf$ is simple as well. The claim is pretty obvious by what we have showed.

By the above claims, we have that for any such simple $\varphi$, $c\varphi \in \mathcal{L}(m)$ and thus $cf \in \mathcal{L}(m)$ is measurable Moreover,

$$\int_E c\varphi = c\int_E \varphi \implies \sup_\varphi \int_E c\varphi = c\sup_\varphi \int_E \varphi \implies \int_E cf = c\int_E f.$$

Now suppose $f \ge 0$ and $c \le 0$. By what we just showed, we have that $-c \ge 0$ and so $-cf \in \mathcal{L}(m)$ with

$$\int_E -cf = -c\int_E f.$$

It suffices to show that

$$-\int_E f = \int_E -f.$$

To see this, it suffices to show it for simple functions. Let $\varphi$ be a simple measurable function, then $\varphi$ is bounded on $E$ and so $-\varphi$ is bounded on $E$. Then we get by (a) that $-\varphi \in \mathcal{L}(m)$. Then we claim that $\int_E -\varphi = -\int_E \varphi$. Consider that

$$\int_E -\varphi = I_E(-\varphi) = \sum_{k=1}^n -c_k m(E_k\cap E) = -\sum_{k=1}^n c_k m(E_k \cap E) = -I_E(\varphi) = -\int_E \varphi.$$

Thus, taking supremums over the simple functions yields the required result. Thus, since

$$\int_E -cf = -c\int_E f \implies c\int_E f = (-)(-c)\int_E f = -\int_E -cf = \int_E (-) - cf = \int_E cf$$

For any $f \in \mathcal{L}$, take $f = f^+ - f^-$ and $c \in \mathbb{R}$ Then by linearity and everything we have shown above:

$$c\int_E f = c(\int_E f^+ - \int_E f^+) = \int_E cf^+ - \int_E cf^- = \int_E c(f^+ - f^-) = \int_E cf$$

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(e) Suppose first that $f \geq 0$. for any simple $0 \leq \varphi \leq f$, we have that

$$I_E(\varphi) = \sum_{k=1}^{N} c_k m(E \cap E_k) \leq \sum_{k=1}^{N} c_k m(E) = 0.$$

And thus

$$\sup_{0 \leq \varphi \leq f} I_E(\varphi) = \int_E f = 0$$

For any $f$ measurable, take $f = f^+ - f^-$. Then by what we just showed, $\int_E f^+ = 0$ and $\int_E f^- = 0$.

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(f) We prove the result first for simple functions and then for non-negative functions. Let $\varphi = \sum_{k=1}^{N} c_k \chi_{E_k}$ be a simple non-negative measurable function on $E$ and $A \in \mathcal{M}$ with $A \subset E$. Then since for any $k$ we have that $E_k \cap A \subset E_k \cap E$, then

$$\int_A \varphi = I_A(\varphi) = \sum_{k=1}^{N} c_k m(E_k \cap A) \leq \sum_{k=1}^{N} c_k m(E_k \cap E) = I_E(\varphi) = \int_E \varphi < \infty,$$

and so $\int_A \varphi$ is finite and thus $\varphi \in \mathcal{L}(m)$ on $A$.

Let $f \geq 0$ be Lebesgue integrable. Then take some simple $0 \leq \varphi \leq f$. By the above case, we have that

$$\int_A \varphi \leq \int_E \varphi \implies \int_A f = \sup_{\varphi} \int_A \varphi \leq \sup_{\varphi} \int_E \varphi = \int_E f < \infty,$$

and so $\int_A f < \infty$ and thus $f \in \mathcal{L}(m)$ on $A$.

Let $f \in \mathcal{L}(m)$ on $E$. Then split it up into $f^+$ and $f^-$. Both are integrals over $A$ are finite by the above case, and so $f \in \mathcal{L}(m)$ on $A$. ■

## Problem 8

Suppose that $A, B \subseteq \mathbb{R}^n$ and $A \in \mathcal{M}(m)$ and $m^*(A \triangle B) = 0$. Show that $B \in \mathcal{M}(m)$ and determine $m(B)$.

SOLUTION:

*Lemma 2.* Let $E \subset \mathbb{R}^n$. If $m^*(E) = 0$, then $E \in \mathcal{M}$.

*Proof.* Let $\epsilon > 0$. Since $\emptyset \in \mathcal{E}$ and

$$m^*(E \triangle \emptyset) = m^*(E) = 0 < \epsilon,$$

then $\emptyset \to E$ in outer measure and so $E \in \mathcal{M}_{\mathcal{F}} \subset \mathcal{M}$. □

By sub-additivity, since

$$A \triangle B = (A \setminus B) \sqcup (B \setminus A) \implies 0 = m^*(A \triangle B) \geq \begin{cases} m^*(A \setminus B) \\ m^*(B \setminus A) \end{cases}$$

Thus, $m^*(A \setminus B) = m^*(B \setminus A) = 0$. By Lemma 3, $A \setminus B, B \setminus A, A \triangle B \in \mathcal{M}$. Since $\mathcal{M}$ is a ring and $A \in \mathcal{M}$, then

$$A \setminus (A \setminus B) = A \cap B \in \mathcal{M}. \tag{1}$$

Thus,

$$(A \cap B) \sqcup (B \setminus A) = B \in \mathcal{M}. \tag{2}$$

To see the second claim, we note that by (1) and by additivity of $m$ ::

$$m(A \cap B) = m(A) - m(A \setminus B) = m(A) - m^*(A \setminus B) = m(A).$$

By (2):

$$m(B) = m(A \cap B) + m(B \setminus A) = m(A) + m^*(B \setminus A) = m(A)$$

■