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Problem 1

Let $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^\ell$. Suppose $T \in C^1(E, F)$. Let $\omega \in \Lambda^k(F)$ and $\lambda \in \Lambda^m(F)$. Prove that

$$(\omega \wedge \lambda)_T = \omega_T \wedge \lambda_T$$

Solution: We remark that by definition, $(dx_i)_T = dt_i$ Suppose $r = 1$, then by definition

$$(dx_1)_T = (dt_1).$$

Suppose that we preserve order in the pullback when $r = n$, i.e,

$$(dx_{i_1} \wedge \cdots \wedge dx_{i_n})_T = (dx_{i_1})_T \wedge \cdots \wedge (dx_{i_n})_T = dt_{i_1} \wedge \cdots \wedge dt_{i_n}$$

Now for $r = n + 1$, we have that (if we denote dx_I to be the standard presentation of the form), then if α is the number of permutations necessary to make $i_1, \dots, i_{n+1}$ into the standard presentation $I' = i'_1, \dots, i'_{n+1}$

$$\begin{aligned}
(dx_{i_1} \wedge \cdots \wedge dx_{i_n} \wedge dx_{i_{n+1}})_T &= (-1^\alpha dx_{I'})_T \\
&= (-1)^\alpha (dx_{I'})_T \\
&= (-1)^\alpha \left(dt_{i'_1} \wedge \cdots \wedge dt_{i'_{n+1}}\right) \\
&= dt_{i_1} \wedge \cdots \wedge dt_{i_n} \wedge dt_{i_{n+1}} \\
&= (dt_{i_1} \wedge \cdots \wedge dt_{i_n}) \wedge dt_{i_{n+1}} \\
&= (dt_{i_1} \wedge \cdots \wedge dt_{i_n}) \wedge (dx_{i_{n+1}})_T \\
&= (dx_{i_1} \wedge \cdots \wedge dx_{i_n})_T \wedge (dx_{i_{n+1}})_T \\
&= (dx_{i_1})_T \wedge \cdots \wedge (dx_{i_n})_T \wedge (dx_{i_{n+1}})_T
\end{aligned}$$

Consider first the case when

$$\omega = f\, dx_I = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \qquad \lambda = g\, dx_J = g\, dx_{j_1} \wedge \cdots \wedge dx_{j_m}.$$

Then by the lemma,

$$\begin{aligned}
(\omega \wedge \lambda)_T &= (fg\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_m})_T \\
&= fg(T(\mathbf{x}))\, dt_{i_1} \wedge \cdots \wedge dt_{i_k} \wedge dt_{j_1} \wedge \cdots \wedge dt_{j_m} \\
&= (f(T(\mathbf{x}))\, dt_{i_1} \wedge \cdots \wedge dt_{i_k}) \wedge (g(T(\mathbf{x}))\, dt_{j_1} \wedge \cdots \wedge dt_{j_m}) \\
&= (f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k})_T \wedge (g\, dx_{j_1} \wedge \cdots \wedge dx_{j_m})_T \\
&= \omega_T \wedge \lambda_T
\end{aligned}$$

Now consider general ω and λ. We can express

$$\omega = \sum_I f_I dx_I, \qquad \lambda = \sum_J g_J dx_J.$$

We have that using part (a) of the Theorem,

$$\begin{aligned}
\omega_T \wedge \lambda_T &= (\sum_I f_I dx_I)_T \wedge (\sum_J g_J dx_J)_T \\
&= \sum_I (f_I dx_I)_T \wedge \sum_J (g_J dx_J)_T \\
&= \sum_{I,J} (f_I dx_I)_T \wedge (g_J dx_J)_T \\
&= \sum_{I,J} (f_I g_J dx_I \wedge dx_J)_T \\
&= (\omega \wedge \lambda)_T
\end{aligned}$$

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Problem 2

Let ω be a 1-form on $\mathbb{R}^n$ and let $\gamma : [a,b] \to \mathbb{R}^n$ be a C^1 curve. Let $\Delta : [a,b] \to [a,b]$ be the identity function (which is a curve in $\mathbb{R}^1$). Prove that

$$\int_\gamma \omega = \int_\Delta \omega_\gamma.$$

Do not just apply Theorem 10.24 or 10.25, please give a direct proof.

SOLUTION: Suppose first $\omega = \sum_I f_I dx_I$ Then by definition of integrating $k-$forms,

$$\begin{aligned}
\int_\gamma \omega &= \int_a^b \sum_I f_I(\gamma(x)) dy_I(\gamma_1'(x), \ldots, \gamma_n'(x))\, dx \\
&= \int_a^b \sum_I f_I(\gamma(x))\gamma_I'(x)\, dx \\
&= \int_a^b \sum_I f_I(\gamma(x)) dt_I\, dx \\
&= \int_a^b \omega_\gamma(x)\, dx \\
&= \int_a^b \omega_\gamma(\Delta(x))\Delta'(x)\, dx \\
&= \int_\Delta \omega_\gamma.
\end{aligned}$$

For general $\omega = \sum_I f_I dx_I$, we use the linearity of the integral to conclude. ∎

Problem 3

Define the forms

$$\begin{aligned}
\omega_1 &= x\,dx - y\,dy \\
\omega_2 &= z\,dx \wedge dy + x\,dy \wedge dz \\
\omega_3 &= z\,dy.
\end{aligned}$$

Compute $\omega_1 \wedge \omega_2$, $\omega_1 \wedge \omega_3$ and $\omega_2 \wedge \omega_3$. Write all forms in standard presentation.

SOLUTION:

- To compute $\omega_1 \wedge \omega_2$, we see that

$$\begin{aligned}
\omega_1 \wedge \omega_2 &= (x\,dx - y\,dy) \wedge (z\,dx \wedge dy + x\,dy \wedge dz) \\
&= xz\,dx \wedge dx \wedge dy + x^2\,dx \wedge dy \wedge dz - yz\,dy \wedge dx \wedge dy - yx\,dy \wedge dy \wedge dz \\
&= \boxed{x^2\,dx \wedge dy \wedge dz}
\end{aligned}$$

- For $\omega_1 \wedge \omega_3$, we compute

$$\begin{aligned}
\omega_1 \wedge \omega_3 &= (x\,dx - y\,dy) \wedge z\,dy \\
&= xz\,dx \wedge dy - yz\,dy \wedge dy \\
&= \boxed{xz\,dx \wedge dy}
\end{aligned}$$

- For $\omega_2 \wedge \omega_3$, we compute

$$\begin{aligned}
\omega_2 \wedge \omega_4 &= (z\,dx \wedge dy + x\,dy \wedge dz) \wedge z\,dy \\
&= z^2\,dx \wedge dy \wedge dy + xy\,dy \wedge dz \wedge dy \\
&= \boxed{0}
\end{aligned}$$

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Problem 4

Let $\omega = xy\,dx \wedge dz + z\,dx \wedge dy$ be a 2-form in $\mathbb{R}^3$. Compute $d\omega$.

SOLUTION: Computing,

$$\begin{aligned} d(\omega) &= d\left(xy\,dx \wedge dz + z\,dx \wedge dy\right) \\ &= y\,dx \wedge dx \wedge dz + 0 + x\,dy \wedge dx \wedge dz + 0 + \ dz \wedge dx \wedge dy \\ &= -x\,dx \wedge dy \wedge dz + dx \wedge dy \wedge dz \\ &= (1-x)\,dx \wedge dy \wedge dz \end{aligned}$$

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Problem 5

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by $T(x, y, z) = (xy, xz, yz)$. Find the following forms:

(a) $(dx)_T$, $(dy)_T$ and $(dz)_T$.

(b) $(dx \wedge dy)_T$

(c) $(dx \wedge dy \wedge dz)_T$.

Write all forms in standard presentation.

Solution: We consider

$$J = \begin{pmatrix} y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix}$$

and we can immediately see

$$dt_1 = ydx + xdy$$

$$dt_2 = zdx + xdz$$

$$dt_3 = zdy + ydz$$

(a) We have by definition

$$(dx)_T = dt_1 = ydx + xdy$$

$$(dy)_T = dt_2 = zdx + xdz$$

$$(dz)_T = dt_3 = zdy + ydz$$

(b) The pullback is distributive, so

$$(dx \wedge dy)_T = (dx)_T \wedge (dy)_T = (y\,dx + x\,dy) \wedge (z\,dx + x\,dz) = -xz\,dx \wedge dy + xy\,dx \wedge dz + x^2\,dy \wedge dz$$

(c) Similarly to (b) but just more annoying

$$\begin{aligned} (dx \wedge dy \wedge dz)_T &= (dx \wedge dy)_T \wedge (dz)_T \\ &= (-xz\,dx \wedge dy + xy\,dx \wedge dz + x^2\,dy \wedge dz) \wedge z\,dy + y\,dz \\ &= -xyz\,dx \wedge dy \wedge dz - xyz\,dx \wedge dy \wedge dz \\ &= -2xyz\,dx \wedge dy \wedge dz \end{aligned}$$

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Problem 6

Let $T(r,\theta,\phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$ (this function gives the spherical coordinates of $\mathbb{R}^3$). Calculate ω_T for each of the following forms ω:

dx, dy, dz, $dx\wedge dy$, $dx\wedge dz$, $dy\wedge dz$, $dx\wedge dy\wedge dz$.

SOLUTION: Holy moly. It is a lot easier if we just look at the Jacobian, which is given by

$$J_T = \begin{pmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix}$$

$$\begin{aligned}
(dx)_T &= d(T_x) = \cos\theta\sin\phi\, dr - r\sin\theta\sin\phi\, d\theta + r\cos\theta\cos\phi\, d\phi \\
(dy)_T &= d(T_y) = \sin\theta\sin\phi\, d\theta + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\theta \\
(dz)_T &= d(T_z) = \cos\phi\, dr - r\sin\phi\, d\phi
\end{aligned}$$

We do not show our work for the following, but we make a lot of use of the fact that wedge products are zero when indices are shared.

$$\begin{aligned}
(dx\wedge dy)_T &= (dx)_T \wedge (dy)_T = \\
&= r\sin^2\phi dr\wedge d\theta - r^2\sin\phi\cos\phi d\theta\wedge d\phi \\
(dx\wedge dz)_T &= (dx)_T \wedge (dz)_T \\
&= (r\sin\theta\sin\phi\cos\phi) dr\wedge d\theta - r\cos\theta dr\wedge d\phi + r^2\sin\theta\sin^2\phi d\theta\wedge d\phi \\
(dy\wedge dz)_T &= (dy)_T \wedge (dz)_T \\
&= -r\cos\theta\sin\phi\cos\phi dr\wedge d\theta - r\sin\theta dr\wedge d\phi - r^2\cos\theta\sin^2\phi d\theta\wedge d\phi \\
(dx\wedge dy\wedge dz)_T &= (dx\wedge dy)_T \wedge dz = -r^2\sin\phi dr\wedge d\theta\wedge d\phi
\end{aligned}$$

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Problem 7

Consider the 2-form $dx \wedge dy$ in $\mathbb{R}^2$. Find all linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\omega_T = \omega$.

Solution: Since $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, then

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

We see the Jacobian is given by

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and thus

$$\begin{aligned} (dx \wedge dy)_T &= d(T_x) \wedge d(T_y) \\ &= (a\,dx + b\,dy) \wedge (c\,dx + d\,dy) \\ &= (ad)\,dx \wedge dy + (cb)\,dy \wedge dx \\ &= (ad - bc)\,dx \wedge dy \\ &= dx \wedge dy \end{aligned}$$

Thus, the only linear map T is one such that $ad - bc = 1$. ■

Problem 8

Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^1, and let df be the 1-form which is the derivative of the 0-form f. For any curve $\gamma : [a, b] \to \mathbb{R}^n$, prove that

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

SOLUTION: By definition, we have that

$$\begin{aligned}
\int_\gamma df &= \int_a^b df(\gamma(u))J(u)\,du \\
&= \int_a^b \sum_{j=1}^n (D_j f)(\gamma(u))dx_j J(u)\,du \\
&= \int_a^b \sum_{j=1}^n (D_j f(\gamma(u)))(\gamma_j'(u))\,du \\
&= \int_a^b \langle \nabla f(\gamma(u)), \gamma'(u) \rangle\,du \\
&= \int_a^b (f(\gamma(u)))'\,du \\
&= f(\gamma(b)) - f(\gamma(a))
\end{aligned}$$

Where the normal FTC was used in the last step. ■

Problem 9

Let ω be the 1-form on $\mathbb{R}^2 \setminus \{0\}$ given by

$$\omega = \frac{y\,dx - x\,dy}{x^2 + 4y^2}$$

Let $\gamma : [0,1] \to \mathbb{R}^2$ be the curve defined by $\gamma(t) = (2\cos(2\pi t), \sin(2\pi t))$.

(a) Compute $d\omega$.

SOLUTION:

$$\begin{aligned}
d\omega &= d\left(\frac{y}{x^2+4y^2}\,dx - \frac{x}{x^2+4y^2}\,dy\right) \\
&= d\left(\frac{y}{x^2+4y^2}\,dx\right) - d\left(\frac{x}{x^2+4y^2}\,dy\right) \\
&= \frac{(x^2+4y^2) - (y(8y))}{(x^2+4y^2)^2} dy \wedge dx - \frac{(x^2+4y^2) - (x(2x))}{(x^2+4y^2)^2}\,dx \wedge dy \\
&= \frac{-x^2+4y^2}{(x^2+4y^2)^2} dx \wedge dy - \frac{-x^2+4y^2}{(x^2+4y^2)^2}\,dx \wedge dy \\
&= 0
\end{aligned}$$

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(b) Compute $\int_\gamma \omega$.

SOLUTION: Notice that

$$\gamma'(x) = \begin{pmatrix} -4\pi\sin(2\pi t) & 2\pi\cos(2\pi t) \end{pmatrix}$$

By definition,

$$\begin{aligned}
\int_\gamma \omega &= \int_0^1 \omega(\gamma(t))\gamma'(t)\,dt \\
&= \int_0^1 \frac{\sin(2\pi t))}{4\cos^2(2\pi t)^2 + 4\sin^2(2\pi t)}(-4\pi\sin(2\pi t)) - \int_0^1 \frac{2\cos(2\pi t)}{4\cos^2(2\pi t)^2 + 4\sin^2(2\pi t)}(2\pi\cos(2\pi t)) \\
&= \int_0^1 -\pi\sin^2(2\pi t)\,dt - \int_0^1 \pi\cos^2(2\pi t)\,dt \\
&= -\pi
\end{aligned}$$

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(c) Is ω closed? Is it exact? (Hint: For exactness, use the previous Problem.)

Solution: Part (a) shows that ω is closed.

Suppose ω is exact. Since ω is a one form, then there exists some $f \in C^1(\mathbb{R}^n, \mathbb{R})$ such that $df = \omega$. Since $(2,0) = \gamma(1) = \gamma(0) = (2,0)$, we have by Problem 8 that

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)) = 0.$$

But by (b) we have that

$$\int_\gamma df = \int_\gamma \omega \neq \pi.$$

Thus, ω cannot be exact. $\blacksquare$