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Problem 1

Let E be a Banach space and $T : E \to E^*$ such that for all $x \in E$,

$$\langle Tx, x \rangle \geq 0.$$

Show that T is a bounded operator.

SOLUTION: By the closed graph theorem, it suffices to see that $G(T) \subset E \times E^*$ is closed. Let $Tx_n \to y$ with $x_n \to x$. Evidently $x \in E$. It suffices to show that $Tx = y$, that is, for any $u \in E$, we have that

$$\langle Tx, u \rangle = \langle y, u \rangle.$$

Let $u \in E$. There exists some $\lambda \in \mathbb{R}$ such that $x\lambda = u$, where $\lambda \in \mathbb{R}$. By assumption,

$$\langle Tx_n - Tu, x_n - u \rangle \geq 0 \to \langle y - Tu, x - u \rangle \geq 0.$$

Thus,

$$\langle y, x - u \rangle \geq \langle Tu, x - u \rangle,$$

and so plugging in our definition of u, we see that

$$\langle y, \frac{u}{\lambda} - u \rangle \geq \langle T(\lambda x), \frac{u}{\lambda} - u \rangle,$$

and so

$$(\frac{1}{\lambda} - 1)\langle y, u \rangle \geq (\frac{1}{\lambda} - 1)\langle T(\lambda), u \rangle \implies \langle y, u \rangle \geq \lambda \langle Tx, u \rangle$$

Letting $\lambda \to 1$ we see that $\langle y, u \rangle \geq \langle Tx, u \rangle$ and letting $\lambda \to -1$, see the opposite inequality.

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Problem 2

Let E be Banach and $A : D(A) \subset E \to E^*$ be a densely defined unbounded operator.

(a) Suppose that there exists some C such that for all $u \in D(A)$, we have that

$$\langle Au, u\rangle \geq -C\|Au\|^2 \tag{1}$$

Show that $N(A) \subset N(A^*)$

Solution: Since

$$A : D(A) \subset E \to E^* \implies A^* : D(A^*) \subset E^{**} \to E^*$$

By that lecture Marr's did in class, it suffices to show that since $N(A^*) = R(T)^\perp$, we have that $N(A) \subset R(T)^\perp$. That is, we want to show that

$$\{u \in E \mid Au = 0\} \subset \{u \in E^{**} \ : \ \langle u, Av\rangle = 0 \ \forall v \in D(A)\}.$$

In non-reflexive spaces, we have that $E \subset E^{**}$, so this might be an equality in reflexive spaces. Anyways, let $u \in N(A)$, then $Au = 0$. Let $v \in D(A)$. Then for any $t \in \mathbb{R}$, we have that by the given inequality,

$$\langle A(u+tv), u+tv\rangle \geq -C\|A(u+tv)\|^2$$

Expanding the left side, we see that

$$\begin{aligned}\langle A(u+tv), u+tv\rangle &= \langle Au, u+tv\rangle + \langle tAv, u+tv\rangle \\ &= 0 + t\langle Av, u\rangle + t\langle Av, u\rangle + t^2\langle Av, v\rangle \\ &\geq -Ct^2\|Av\|^2\end{aligned}$$

Thus,

$$2t\langle Av, u\rangle + t^2(\langle Av, v\rangle + C\|Av\|^2) \geq 0.$$

This is a quadratic equation in terms of t, and thus

$$at^2 + bt \geq 0 \implies b^2 - 4ac \leq 0 \implies b^2 \leq 0,$$

but $b^2 = (2\langle Av, u\rangle)^2 \leq 0 \implies b = 0$ and so $\langle Av, u\rangle = 0$, and so $u \in R(A)^\perp$. ■

(b) Prove that the converse holds if A is closed and $R(A)$ is closed.

Solution: Since $N(A) \subset N(A^*)$ by assumption, then $N(A) \subset R(A)^\perp$, and so for any $u \in N(A)$, we have that $u \in R(A)^\perp$. We claim that it suffices to find some $v \in D(A)$ for any $u \in D(A)$ such that $Au = Av$. That is,

$$\langle Au, x\rangle = \langle Av, x\rangle, \quad \forall x \in D(A) \implies Au - Av = A(u-v) = 0 \implies u - v \in \ker A.$$

Thus, we see that by assumption, $u - v \in R(A)^{\perp}$, and so for any $x \in D(A)$

$$\langle u - v, Ax \rangle = 0.$$

In particular, letting $x = u$ we see that

$$\langle Au, u \rangle = \langle Au, v \rangle$$

Letting $x = v$, we see that

$$\langle Av, u \rangle = \langle Av, v \rangle.$$

Thus,

$$\langle Au, u \rangle = \langle Av, v \rangle \tag{2}$$

To find such a v, consider that since $R(A) \subset E^*$ is a closed subset of a Banach space, then $R(A)$ is a Banach space. Thus, A is an open mapping, i.e, there exists some $c > 0$ such that

$$B_c^{R(A)}(0) \subset A(B_1^{(D(A))}(0)).$$

Thus, for any $f \in R(A)$, with $\|f\| \leq c$, there exists some $u \in D(A)$ such that $Au = cf$, and so $Au' = f$, where $u' = \frac{u}{c} \in D(A)$ since $D(A)$ is densely defined linear subspace. Thus, $\|u'\| = \|\frac{u}{c}\| \leq \|f\| \implies \|u\| \leq c\|f\|$ Because this holds for any $f \in R(A)$ (you can just scale any f not in the $c-$ball) and A is surjective, then let $u \in D(A)$ with $Au = f$. But since $f \in R(A)$, then we know by the open mapping theorem there exists some $v \in D(A)$ with $Av = f$ and $-c\|Au\| \leq \|v\| \leq \|c\|\|Au\|$. Thus, we have found our $u - v \in N(A)$. Hence by (2), we have that since $|\langle f, u \rangle| \leq \|f\|\|u\|$, then

$$\langle Au, u \rangle \geq -c\|Au\|\|v\| \geq -c\|Au\|^2$$

■

Problem 3

Suppose X is a separable Banach space and $M \subset X$ is a closed subspace. Then X/M is separable.

SOLUTION: Let $\pi : X \to X/M$ be the canonical linear surjection and let (v_n) be a countably dense subset of X. We claim that $\pi(v_n)$ is a countably dense subset of X/M. Recall that for any since X is Banach, then X/M is Banach with respect to the norm

$$\|u\|_{X/M} = \inf_{m \in M} \|u - m\|_X.$$

Let $u \in X/M$, and $\epsilon > 0$. Since π is surjective, there exists some $v \in X$ such that $\pi(v) = x$, and thus there exists some $v_k \in (v_n)$ such that

$$\|v_k - v\| < \epsilon$$

Thus, we have that

$$\begin{aligned}
\|\pi(v_n) - u\|_{X/M} &= \|\pi(v_n) - \pi(v)\|_{X/M} \\
&= \|\pi(v_n - v)\|_{X/M} \\
&= \inf_{m \in M} \|v_n - v - m\|_X \\
&\leq \|v_n - v\|_X + \inf_{m \in M} \|m\|_X \\
&= \|v_n - v\| \\
&< \epsilon.
\end{aligned}$$

Then we have that $(\pi(v_n))$ is countably dense in X/M, as required. ■

Problem 4

Suppose that X is a Banach space, $M \subset X$ is closed and separable. If X/M is separable, then X is separable.

Solution: Let $(u_n) \subset M$ be a countably dense subset of M and let $([w_n]) \subset X/M$ be a countably dense subset of X/M and choose a representative $w_n \in X$ such that $[w_n] = \pi(w_n)$. Thus, let $u \in M$ and $[w] \in X/M$, then there exist $u_n \in (u_n)$, $[w_k] \in ([w_n])$ such that

$$\|u_n - u\|_X < \frac{\epsilon}{2} \tag{3}$$

$$\|[w_k] - [w]\|_{X/M} = \|\pi(w_k - w)\|_{X/M} = \inf_{m \in M} \|w_k - w - m\| < \frac{\epsilon}{2} \tag{4}$$

Consider $F = \{u_n + w_n \ : \ u_n \in (u_n), w_n \in (w_n)\}$, we claim that F is countably dense in X. The countability comes from the fact that

$$F = \bigcup_{k=1}^{\infty} \bigcup_{n=k}^{\infty} (u_n + w_k).$$

Let $x \in X$. Then for any $\epsilon > 0$, we have by the above that if $m \in M$ and u_n and w_k are chosen such that (3) and (4) are satisfied, then

$$\begin{aligned} \|x - (u_n + w_n)\|_X &\leq \|x - w_n - m\| + \|m - u_n\| \\ &< \epsilon \end{aligned}$$

Thus, we are done. ■

Reflections: This might be wrong, I didn't use the fact that X was Banach, but in the book the quotient space is only defined for Banach spaces, so maybe I did?