Agustín Esteva
aesteva@uchicago.edu

Due Date: 27/02/25

Problem 1

Let $(u_n) \in H$ and $(t_n) \in (0, \infty)$ such that

$$(t_n u_n - t_m u_m, u_n - u_m) \leq 0,$$

(a) Suppose (t_n) is nondecreasing. Prove that (u_n) converges to a limit.

SOLUTION: Consider that

$$|u_n - u_m|^2 = |u_n|^2 + |u_m|^2 - 2(u_n, u_m).$$

Using bilinearity and properties of the inner product, we have that by plugging the above at a convenient spot:

$$\begin{aligned}
(t_n u_n - t_m u_m, u_n - u_m) &= (t_n u_n, u_n) - (t_n u_n, u_m) - (t_m u_m, u_n) + (t_m u_m, u_m) \\
&= t_n |u_n|^2 - t_n(u_n, u_m) - t_m(u_n, u_m) + t_m |v_m|^2 \\
&= t_n |u_n|^2 - (t_n + t_m)(u_n, u_m) + t_m |v_m|^2 \\
&= t_n |u_n|^2 - \frac{1}{2}(t_n + t_m)(-|u_n - u_m|^2 + |u_n|^2 + |u_m|^2) + t_m |v_m|^2 \\
&= \frac{1}{2}(t_n - t_m)|u_n|^2 + \frac{1}{2}(t_n + t_m)(|u_n - u_m|^2) - \frac{1}{2}(t_n - t_m)|u_m|^2 \\
&= \frac{1}{2}(t_n - t_m)(|u_n|^2 - |u_m|^2) + \frac{1}{2}(t_n + t_m)(|u_n - u_m|^2) \\
&\leq 0
\end{aligned}$$

The first term must necessarily be negative since the second is positive, and so

$$(t_n - t_m)(|u_n|^2 - |u_m|^2) \leq 0 \tag{1}$$

Since (t_n) is nondecreasing, then for $n > m$, we have that $t_n - t_m \geq 0$, and thus by (1) we have that

$$|u_n|^2 - |u_m|^2 \leq 0 \implies |u_n| \leq |u_m|,$$

and so $|u_n|$ is nonincreasing. Moreover, we have that

$$\frac{1}{2}(t_n + t_m)(|u_n - u_m|^2) \leq -\frac{1}{2}(t_n - t_m)(|u_n|^2 - |u_m|^2) = \frac{1}{2}(t_n - t_m)(|u_m|^2 - |u_n|^2) \leq \frac{1}{2} t_n (|u_m|^2 - |u_n|^2),$$

and so since $t_n \leq t_n + t_m$, we have that

$$|u_n - u_m|^2 \leq |u_m|^2 - |u_n|^2.$$

Letting $n \to \infty$, we see that since $|u_n|$ is nonincreasing and bounded below, that $|u_n| \to L$, and so $|u_n|^2 \to L^2$ Thus, taking letting $m \to \infty$ and letting $n = m+1$, we see that

$$|u_n - u_m|^2 \leq |u_m|^2 - |u_n|^2 \to L^2 - L^2 = 0,$$

and so $|u_n - u_m|$ is Cauchy. Thus, since $(u_n) \in H$, we have that (u_n) converges. ■

Assume that (t_n) is non increasing. Then either:

(i) $|u_n| \to \infty$

(ii) or (u_n) converges

If $t_n \to t > 0$, show that (u_n) converges.

SOLUTION: Let $n > m$, then by (1) we have that since $t_n - t_m \leq 0$, then $(|u_n|^2 - |u_m|^2) \geq 0 \implies |u_n|^2 \leq |u_m|^2$, and so $|u_n|$ is non decreasing. From the above calculations, we have that

$$|u_n - u_m|^2 \leq |u_n|^2 - |u_m|^2.$$

Thus, if $|u_n| \to L \leq \infty$, then by the above reasoning, (u_n) converges. If $|u_n| \to \infty$, then evidently $(u_n) \to \infty$.

Suppose $t_n \to t$. As per the hint, we let $v_n = t_n u_n$ and so $s_n = \frac{1}{t_n}$, then

$$(s_n v_n - s_m v_m, v_n - v_m) \leq 0,$$

then by the work above, v_n converges and so u_n converges. ■

Problem 2

Let $K \subset H$ be a closed convex set and let $f \in H$. If $u = P_K f$, then show that for any $v \in K$

$$|v-u|^2 \leq |v-f|^2 - |f-u|^2,$$

deduce that

$$|v-u| \leq |v-f|$$

and give a geometric interpretation.

SOLUTION: Consider that

$$\begin{aligned}
|v-u|^2 &= |v-f+f-u|^2 \\
&= (v-f+f-u, v-f+f-u) \\
&= (v-f, v-f) + 2(v-f, f-u) + (f-u, f-u) \\
&= |v-f|^2 + |f-u|^2 + 2(v-f, f-u) \\
&= |u-f|^2 + |v-f|^2 + 2(f-u, v-f) \\
&= |u-f|^2 + |v-f|^2 + 2(f-u, v-u+u-f) \\
&= |u-f|^2 + |v-f|^2 + 2(f-u, v-u) - 2(f-u, f-u) \\
&= |u-f|^2 + |v-f|^2 + 2(f-u, v-u) - 2|f-u|^2 \\
&= |v-f|^2 - |f-u|^2 + 2(f-u, v-u) \\
&\leq |v-f|^2 - |f-u|^2
\end{aligned}$$

Suppose not for the second part, then

$$|v-u| > |v-f| \implies |v-u|^2 > |v-f|^2 \implies |v-u|^2 > |v-f|^2 - |f-u|^2,$$

which is a contradiction. A geometric interpretation:

$\blacksquare$

Problem 3

(a) Let (K_n) be a nonincreasing sequence of closed convex sets in H such that $\bigcap_n K_n \neq \emptyset$. Prove that for every $f \in H$ the sequence $u_n = P_{K_n} f$ converges (strongly) to a limit and identify the limit.

SOLUTION: There really is only one natural candidate for the limit. We claim that

$$u_n \to u, \quad u = P_{K_\infty} f.$$

Where $K_\infty = \bigcap_{n=1}^\infty K_n$. We know that K_∞ is is closed since it is the intersection of closed sets, and we know it is convex since the intersection of nested convex sets is convex. We have that since

$$K_1 \supset K_2 \cdots,$$

then since $\bigcap K_n \subset K_n$ for any n, $u \in K_n$ for all n. Let $d_n = |f - u_n|$. Since $K_{n+1} \subset K_n$, and each K_n is closed, then

$$d_n = \inf_{v \in K_n} |f - v| \leq \inf_{v \in K_{n+1}} |f - v| = d_{n+1} \leq \cdots \leq \inf_{v \in K_\infty} |f - v| = d.$$

Thus, d_n is monotonic increasing and bounded above, and thus converges to some $d_n \to L$.

Apply the parallelogram law to $a = f - u_n$ and $b = f - u_m$, then

$$\left|\frac{f - u_n + f - u_m}{2}\right|^2 + \left|\frac{u_n - u_m}{2}\right|^2 = \frac{1}{4}|(f-u_n)+(f-u_m)|^2 + \frac{1}{4}|u_n - u_m|^2 = \frac{1}{4}(d_n + d_m)^2 + \frac{1}{4}|u_n - u_m|^2$$

is equal to

$$\frac{1}{2}(|d_n|^2 + |d_m|^2).$$

Thus, we have that since $d_n \geq 0$ for any n, then

$$|u_n - u_m|^2 = 2|d_n|^2 + 2|d_m|^2 - (d_n + d_m)^2 = |d_n|^2 + 2d_n d_m + |d_m|^2 = (d_m - d_n)^2.$$

Thus, since (d_n) converges, then it is Cauchy and thus

$$|u_n - u_m| < d_m - d_n \to 0,$$

and so (u_n) is Cauchy in a Hilbert space and thus converges to some u. We claim that $u \in K_\infty$. Since $(u_n) \in K_1$ for all n and K_1 is closed, we have that $u \in K_1$, similarly, since $(u_n) \in K_2$ for all n except for possible u_1, then $u \in K_2$. Because this holds for all n, then $u \in K_\infty$. Similarly, we have that $|f - u_1| \leq |f - v|$ for all $v \in K_\infty$, and $|f - u_2| \leq |f - v|$ for all $v \in K_\infty$, and taking the limit we see that $|f - u| \leq |f - v|$ for all $v \in K_\infty$. ∎

(b) Let (K_n) be a nondecreasing sequence of nonempty closed convex sets in H. Prove that for every $f \in H$ the sequence $u_n = P_{K_n} f$ converges (strongly) to a limit and identify the limit.

SOLUTION: Since $K_1 \subset K_2 \cdots$, then either $\bigcup K_n = H$ or $\bigcup K_n = K_\infty \neq H$. Suppose the first case, then $f \in H$, and so $f \in K_n$ for some n, and thus $P_{K_n} f = f$ Since $K_n \subset K_m$ for all $m \geq n$, then $P_{K_m} = f$ for all $m \geq n$, and thus $u_n \to f$.

Consider the second case now, then either $f \in K_\infty$, in which case we revert back to the first case, or $f \notin K_\infty$. If the latter, then consider that $d_n = P_{K_n} f$ is a decreasing sequence bounded below by 0, and thus converges to a limit. By the reasoning above, we have that (u_n) converges to some u. We claim that $u = P_{\overline{K_\infty}} f$, where $\overline{K_\infty}$ is obviously closed. To see that $\overline{K_\infty}$ convex, it suffices to notice that $\bigcup K_n$ is convex (since the closure of a convex set is convex). Since $u_n \in K_m$ for all $m > n$, which implies $u \in \overline{K_\infty}$. Moreover, let $v \in \overline{K_\infty}$, then either $v \in K_n$ for some n or $v \in LP(\bigcup K_n)$. Suppose the former, then

$$|f - u| \leq |f - u_n| \leq |f - v| \quad \forall v \in K_n.$$

Now suppose the latter, then there exist some $(v_n) \in \bigcup K_n$ such that $v_n \to v$. But we have that for any n,

$$|f - u| \leq |f - v_n| \implies |f - u| \leq \liminf_{n \to \infty} |f - v_n| = |f - v|,$$

the conclusion from both of these cases is that while we first showed that $u_n \to u$, now this shows that $u = P_{\overline{K_\infty}} f$ ■

(c) Let $\varphi : H \to \mathbb{R}$ be a continuous function that is bounded from below and let K_n be as above in part (b). Prove that the sequence $\alpha_n = \inf_{K_n} \varphi$ converges and identify the limit.

SOLUTION: Consider that $\alpha_n = \inf\{\varphi(x) \mid x \in K_n\}$. Thus, if $n < m$, then $K_n \subseteq K_m$, and thus $\alpha_n \geq \alpha_m$. Thus, (α_n) is non-increasing and bounded below, and so we let

$$\alpha_n \to \alpha_\infty,$$

and claim that

$$\alpha_\infty = \inf_{\overline{\bigcup_{n=1}^\infty K_n}} \varphi.$$

Let $u \in \overline{\bigcup K_n}$, then by part (b), we have that $u_n = P_{K_n} u \to u$. Thus, since $\alpha_n(x) \leq \varphi(x)$ for any $x \in K_n$, then since $u_n \in K_n$, we have that $\alpha_n(u_n) \leq \varphi(u_n)$. Since both are continuous, we have that $\alpha_n(u) \leq \varphi(u)$ for all n, and so $\alpha_\infty(u) \leq \varphi(u)$. Because this holds for any $u \in K_\infty$, then

$$\alpha_\infty \leq \inf_{\overline{\bigcup K_n}} \varphi$$

Since for any n, we have that $K_n \subset \overline{\bigcup_{i=1}^\infty} K_i$, then $\alpha_n \geq \alpha_\infty$, and so

$$\alpha_\infty \geq \inf_{\overline{\bigcup_{n=1}^\infty K_n}} \varphi.$$

■

Problem 4

Let $F : H \to \mathbb{R}$ be convex and C^1. Let $K \subset H$ be convex and let $u \in H$. Show the following are equivalent:

(i) $F(u) \leq F(v), \qquad \forall v \in K$

(ii) $(F'(u), v-u) \geq 0 \qquad \forall v \in K.$

SOLUTION: $(a \mapsto b)$ Suppose $F(u) \leq F(v)$, then if we let $v' = tu + (1-t)v$, where $v \in K$, we have that since K is convex, $v' \in K$, and thus

$$F(u) \leq F(v') = F((1-t)u + tv),$$

and so

$$0 \leq \frac{F(u + t(u-v)) - F(u)}{t} \xrightarrow{t \to 0} F'_{u-v}(u) = (F'(u), u-v)$$

$(b \mapsto a)$ We claim that a a continuously differentiable function is convex if and only if its graph lies above all its tangents. That is, since $u \in H$, then[a]

$$F(v) \geq F(u) + F'(u) \cdot (v-u).$$

By assumption, we have that

$$F(v) - F(u) \geq (F'(u), (v-u)) \geq 0 \implies F(v) \geq F(u)$$

■

[a]MVT

Problem 5

Let $G \subset H$ be a linear subspace of a Hilbert space H; G is equipped with the norm of H. Let F be a Banach space. Let $S : G \to F$ be a bounded linear operator. Prove that there exists a bounded linear operator $T : H \to F$ that extends S and such that

$$\|T\|_{\mathcal{L}(H,F)} = \|S\|_{\mathcal{L}(G,F)}.$$

SOLUTION: Since $\overline{G}$ is a closed linear subspace, then $P_{\overline{G}} : H \to \overline{G}$ is a continuous function since for any $f_1, f_2 \in H$, we have that

$$\|P_{\overline{G}} f_2 - P_{\overline{G}} f_2\| \leq \|f_1 - f_2\|.$$

Define $\overline{S} : \overline{G} \to F$ as an extension of S such that if $v \in \overline{G} \setminus G$ and $(v_n) \in G$ with $v_n \to v$, then

$$\overline{S}(v) = \lim_{n\to\infty} S(v_n).$$

If $v \in G$, then let $\overline{S}(v) = S(v)$. To show that $\overline{S}$ is continuous, let $s_1, s_2 \in \overline{G}$, such that $\|s_1 - s_2\| \leq \epsilon$, then for n large enough, we have that if $s_n^1 \to s_1$ and $s_n^2 \to s_2$, then by continuity:

$$\|s_n^1 - s_n^2\| \leq \|s_n^1 - s_1\| + \|s_1 - s_2\| + \|s_2 - s_n^2\| < \delta \implies \|S(s_n^1) - S(s_n^2)\| < \frac{\epsilon}{3}$$

Since $S(s_n^1) \to \overline{S}(s_1)$ and $S(s_n^2) \to \overline{S}(s_2)$, then

$$\|\overline{S}(s_1) - \overline{S}(s_2)\| \leq \|\overline{S}(s_1) - S(s_n^1)\| + \|S(s_n^1) - S(s_n^2)\| + \|S(s_n^2 - \overline{S}(s_n^2))\| < \epsilon,$$

and thus $\overline{S}$ is continuous. $\overline{S}$ is clearly linear by the linearity of limits. Since the composition of continuous functions is continuous, then

$$T = \overline{S} \circ P_{\overline{G}} : H \to F$$

extends S and is a bounded linear operator.

Consider that since $T = \overline{S} \circ P_{\overline{G}}$, then since for any $\|x\| = 1$, we have that $\|P_{\overline{G}}\| = 1$

$$\|T\| \leq \|\overline{S}\|\|P_{\overline{G}}\| = \|S\|\|P_{\overline{G}}\| \leq \|S\|.$$

The other inequality is clear since T is just an extension of S. ∎

Problem 6

Let $M, N \subset H$ be two closed linear subspaces. Assume that $(u, v) = 0$ for all $u \in M$, and $v \in N$. Show that $M + N$ is closed.

Solution: Suppose $f \in H$ with $(u_k) \in M + N$ such that $u_k \to f$. Since $(u_k) \in M + N$, then there exist $m_k \in M$ and $n_k \in N$ such that $m_k + n_k = u_k \to f$. Thus, for any k, we have that

$$\|m_k + n_k\|^2 = \|m_k\|^2 + \|n_k\|^2 + 2(m_k, n_k) = \|m_k\|^2 + \|n_k\|^2 \geq \|m_k\|^2 \geq 0.$$

That is,

$$\|m_k + n_k\| \geq \|m_k\| \geq 0,$$

and so m_k is bounded (since $m_k + n_k$ converges and is thus bounded). Thus, there exists some convergent subsequence $m_{k_j} \to m$. Since m is closed, we have that $m \in M$.

Consider now that

$$n_{k_j} = u_{k_j} - m_{k_j} \to f - m,$$

where again, $f - m \in N$ by the closedness of N. Thus, $f = f - m + m \in M + N$, and we are done. ■

Problem 7

Let $C \subset H$ be a nonempty closed convex set and suppose $T : C \to C$ is a non-linear contraction such that for any $u, v \in C$,

$$|Tu - Tv| \leq |u - v|$$

(a) Let $(u_n) \in C$ such that $u_n \rightharpoonup u$ and $(u_n - Tu_n) \to f$. Prove that

$$u - Tu = f$$

SOLUTION: Let $g : C \to H$ such that $g(v) = v - Tv$. Since T is continuous (since it is a contraction), then g is continuous. Since C is convex and strongly closed, then C is weakly closed, and so $u_n \rightharpoonup u \in C$.

$$\begin{aligned}
\|u - u_n\|^2 &\geq \|Tu - Tu_n\|^2 \\
&= \|(g(u_n) - g(u) - u_n + u\|^2 \\
&= \|(g(u_n) - g(u)) + (u - u_n)\|^2 \\
&= \|g(u_n) - g(u)\|^2 + \|u - u_n\|^2 + 2(g(u_n) - g(u), (u - u_n))
\end{aligned}$$

and so

$$0 \geq \|g(u_n) - g(u)\|^2 + 2(g(u_n) - g(u), (u - u_n))$$

Consider $\varphi : C \to \mathbb{R}$ defined by

$$\varphi(v) = (g(v) - g(u), u - v).$$

Then $\varphi \in C^*$ since the inner product is bilinear and thus $\varphi(u_n) \to \varphi(u)$, and so

$$2\varphi(u_n) = (g(u_n) - g(u), (u - u_n)) \to (g(u) - g(u), (u - u)) = \varphi(u) = 0.$$

Thus, for large enough n, we have that $\|g(u_n) - g(u)\|^2 \leq 0$, and so $g(u_n) = g(u)$, but this then implies that

$$u_n - Tu_n = u - Tu, \qquad n \text{ large}$$

■

(b) If C is bounded with $T(C) \subset C$, then T has a fixed point.

SOLUTION: As per the hint, fix $a \in C$ and consider $T_\epsilon : C \to C$

$$T_\epsilon(u) = (1 - \epsilon)Tu + \epsilon a.$$

Consider that for any $u, v \in C$,

$$|T_\epsilon u - T_\epsilon v| = |(1 - \epsilon)Tu + \epsilon a - (1 - \epsilon)Tv + \epsilon a| = (1 - \epsilon)|Tu - Tv| \leq (1 - \epsilon)|u - v|.$$

Thus, T_ϵ is a contraction and C is Banach (closed subset of a Hilbert space), and Banach contraction principle tells us that for all $1 > \epsilon > 0$, T_ϵ has a fixed point at some p_ϵ. Thus,

$$p_\epsilon = (1-\epsilon)Tp_\epsilon + \epsilon a$$

Consider letting $\epsilon = \frac{1}{n}$ for each n, then

$$p_{\frac{1}{n}} = (1-\frac{1}{n})Tp_{\frac{1}{n}} + \frac{1}{n}a = Tp_{\frac{1}{n}} - \frac{1}{n}Tp_{\frac{1}{n}} + \frac{1}{n}a,$$

and so as $n \to \infty$,

$$p_{\frac{1}{n}} - Tp_{\frac{1}{n}} \to 0.$$

Since $(p_{\frac{1}{n}}) \in C$ and C is bounded and closed and convex, then we claim that C is compact in the weak topology. Consider that since H is reflexive, then B_E is weakly compact, and thus there exists some K such that $C \subset KB_E$, where KB_E is weakly compact. Since C is convex and strongly closed, then it is weakly closed, and thus C is weakly compact. Thus, there exists some subsequence $(p_{\frac{1}{n_k}}) \rightharpoonup p_0$, where $p_0 \in C$. Note that we still have that

then by part 1, we have that $p_{\frac{1}{n_k}} - Tp_{\frac{1}{n_k}} \to 0$. By part 1, we have that

$$p_0 - Tp_0 = 0 \implies p_0 = Tp_0.$$

■

Problem 8

Let $D \subset H$ be a subset such that the linear space spanned by D is dense in H. Let $(E_n)_{n\geq 1}$ be a sequence of closed subspaces in H that are mutually orthogonal. Assume that

$$\sum_{n=1}^{\infty} |P_{E_n} u|^2 = |u|^2 \quad \forall u \in D \tag{2}$$

Prove that H is the Hilbert sum of the E_n's.

SOLUTION: Let $E = \operatorname{span} \bigcup E_n$.

We know that for any $v \in H$, if $v_k = P_{E_k} v$, then if $S_n = \sum_{k=1}^{n} P_{E_k}$, we have that

$$S_n \to S = P_{\overline{E}} v.$$

Thus, by Parseval's identity, we have that

$$\sum_{k=1}^{\infty} |v_k|^2 = |P_{\overline{E}} v|^2 \tag{3}$$

Now let $u \in D$, combining (2) and (3), we see that $|u|^2 = |P_{\overline{E}} u|^2$. First, we note that if M is a closed subspace of a Hilbert space H and $f \in H$, then

$$f = P_M f + P_{M^\perp} f.$$

To see this, consider that $M \cap M^\perp = \{0\}$ and $M + M^\perp = H$, then

$$f = P_M f + (f - P_M f) = P_M f + P_{M^\perp} f.$$

Thus, since $u \in D \subset H$ and $\overline{E}$ is a closed subspace, then by orthogonality

$$u = P_{\overline{E}} u + P_{\overline{E}^\perp} u \implies |u|^2 = |P_{\overline{E}} u|^2 + |P_{\overline{E}} u|^2 + 2|(P_{\overline{E}} u, P_{\overline{E}^\perp} u)| = |P_{\overline{E}} u|^2 + |P_{\overline{E}^\perp} u|^2$$

From our conclusion above, we see that

$$0 = |P_{\overline{E}^\perp} u|^2 \implies P_{\overline{E}^\perp} u = 0.$$

Because this holds for any $u \in D$, and D is dense in H, then $\overline{E}^\perp = \{0\}$, and so $\overline{E} = H$. $\blacksquare$

Problem 9

(a) Suppose H is separable. Let $V \subset H$ be a linear subspace that is dense in H, then V contains an orthonormal basis of H.

SOLUTION: Since H is separable and $V \subset H$, then V is separable. Let (v_n) be a countably dense subset of V, and let $F_k = \text{span}\{v_1, \dots, v_k\}$. F_k is finite and $\bigcup F_k$ is dense in V. Since F_1 is finite, then for any $x \in F_1$, $x = x_1 e_1$, where $\|e_1\| = 1$. Thus, take e_1. If $F_2 \neq F_1$, then let $u_2 \in F_2 \setminus F_1$, and define

$$e_2 = \frac{u_2 - (u_2, e_1)e_1}{\|u_2 - (u_2, e_1)e_1\|},$$

and so

$$(e_1, e_2) = (e_1, \frac{u_2 - (u_2, e_1)e_1}{\|u_2 - (u_2, e_1)e_1\|}) = \frac{1}{\|u_2 - (u_2, e_1)e_1\|}[(e_1, u_2) - (u_2, e_1)(e_1, e_1)] = 0.$$

Continue this process, then (e_n) is a an orthonormal basis of V, and since $\overline{V} = H$, then

$$\overline{\text{span}\{e_1, \dots\}} = V \implies \overline{\text{span}\{e_1, \dots\}} = H,$$

and so (e_n) is an orthonormal basis of H. ■

(b) Let (e_n) be an orthonormal sequence in H such that $(e_i, e_j) = \delta_{ij}$. Prove there exists some orthonormal basis of H that contains $\bigcup_{i=1}^{\infty} e_n$.

SOLUTION: Consider $E_k = \text{span}\{e_1, \dots, e_k\}$. Define F_k as above, then if

$$\overline{E} = \overline{\bigcup_{k=1}^{\infty} E_k} \neq \bigcup F_k = F,$$

we let $w_k \in F \setminus \overline{E}$, then let k be such that $w \in F_k$ but $w \notin F_{k-1}$.

Define e_1' such that

$$e_1' = \frac{w_k - \sum_{j=1}^{k-1}(w_k, e_j)e_j}{\|w_k - \sum_{j=1}^{k-1}(w_k, e_j)e_j\|}.$$

Then for any $e_i \in E$, we have that by the same reasoning as the first problem, $(e_i, e_1') = 0$. Then redefine $E = (e_n) \cup e_1'$ which is an orthonormal sequence in H. Continue this procedure until $\overline{E} = F$, and then E is an orthonormal basis of H containing $\bigcup e_n$. ■