20510 Spring 2025: Problem Set 6 Agustín Esteva
Accelerated Analysis in $\mathbb{R}^n$ III aesteva@uchicago.edu
Instructor(s): Don Stull Due Date: 05-07-2025

Problem 1

Let $\omega = x_2^2\,dx_1 + x_1\,dx_2$ be a 1-form and let $\Phi(u) = (\cos u, \sin u)$ for $0 \leq u \leq 1$.

Compute

$$\int_\Phi \omega.$$

SOLUTION: We have that

$$f_1 = x_2^2, \quad f_2 = x_1, \quad \Phi'(u) = \begin{pmatrix} -\sin u & \cos u \end{pmatrix}$$

Thus, using a few trig trick like $\cos^2(x) = 1 + \cos(2x)$, we find that

$$\begin{aligned}
\int_\Phi \omega &= \int_0^1 f_1(\Phi(u))\Phi_1'(u) + f_2(\Phi(u))\Phi_2'(u)\,du \\
&= \int_0^1 \sin^2(u)(-\sin(u))\,du + \int_0^1 \cos(u)\cos(u)\,du \\
&= \frac{1}{4}\sin(2) + \cos(1) - \frac{1}{3}\cos^3(1) - \frac{1}{6}
\end{aligned}$$

■

Problem 2

Let $\omega = y\,dz \wedge dx$ be a 2-form in $\mathbb{R}^3$, and $\Phi : [0,1]^2 \to \mathbb{R}^3$ be the 2-surface defined by

$$\Phi(\phi, \theta) = (\sin(\pi\phi)\cos(2\pi\theta), \sin(\pi\phi)\sin(2\pi\theta), \cos(\pi\phi)).$$

Compute

$$\int_\Phi \omega.$$

Solution: The Jacobian of $\Phi(\phi, \theta)$ is given by

$$J(\Phi) = \begin{pmatrix} \pi\cos(\pi\phi)\cos(2\pi\theta) & -2\pi\sin(\pi\phi)\sin(2\pi\theta) \\ \pi\cos(\pi\phi)\sin(2\pi\theta) & 2\pi\sin(\pi\phi)\cos(2\pi\theta) \\ -\pi\sin(\pi\phi) & 0 \end{pmatrix}$$

Thus,

$$\begin{aligned}
\int_\Phi \omega &= \int_0^1 \left(\int_0^1 \sin(\pi\phi)\sin(2\pi\theta) dz \wedge dx J(\Phi(\phi,\theta))\, d\phi \right) d\theta \\
&= \int_0^1 \left(\int_0^1 \sin(\pi\phi)\sin(2\pi\theta) \det \begin{pmatrix} -\pi\sin(\pi\phi) & 0 \\ \pi\cos(\pi\phi)\cos(2\pi\theta) & -2\pi\sin(\pi\phi)\sin(2\pi\theta) \end{pmatrix} d\phi \right) d\theta \\
&= \int_0^1 \left(\int_0^1 \sin(\pi\phi)\sin(2\pi\theta) 2\pi^2 \sin^2(\pi\phi)\sin(2\pi\theta)\, d\phi \right) d\theta \\
&= 2\pi^2 \int_0^1 \sin^2(2\pi\theta) \left(\int_0^1 \sin^3(\pi\phi)\, d\phi \right) d\theta \\
&= \frac{4\pi}{3}
\end{aligned}$$

■

Problem 3

Let ω be the 1-form given by

$$\omega = \frac{-y\,dx + x\,dy}{x^2+y^2},$$

which is defined on $\mathbb{R}^2 \setminus \{(0,0)\}$.

(a) Show that $d\omega = 0$.

SOLUTION: Define

$$r = x^2 + y^2 \implies dr = 2x\,dx + 2y\,dy$$

We compute the exterior derivative of ω using its linearity

$$\begin{aligned}
d\omega &= d\left(\frac{-y\,dx + x\,dy}{x^2+y^2}\right)\\
&= -d\left(\frac{y}{x^2+y^2}dx\right) + d\left(\frac{x}{x^2+y^2}dy\right)\\
&:= -d\left(\frac{y}{r}dx\right) + d\left(\frac{x}{r}dy\right)\\
&= -\frac{dy\,r - y\,dr}{r^2}\wedge dx + \frac{dx\,r - x\,dr}{r^2}\wedge dy\\
&= \frac{-(r)\,dy\wedge dx + (2xy\,dx\wedge dx + 2y^2\,dy\wedge dx) + (r)\,dx\wedge dy - (2x^2\,dx\wedge dy + 2xy\,dy\wedge dy)}{r^2}\\
&= \frac{(r)\,dx\wedge dy - 2y^2\,dx\wedge dy + (r)\,dx\wedge dy - 2x^2\,dx\wedge dy}{r^2}\\
&= \frac{2(r)\,dx\wedge dy - 2(r)\,dx\wedge dy}{r^2}\\
&= 0
\end{aligned}$$

■

(b) Let $\gamma_k : [0,1] \to \mathbb{R}^2$ be the curve $\gamma_k(t) = (\cos(2k\pi t), \sin(2k\pi t))$. Compute

$$\int_{\gamma_k} \omega.$$

What is the "meaning" of this integral?

SOLUTION:

$$\gamma_k'(t) = \begin{pmatrix} -2k\pi\sin(2k\pi t) & 2k\pi\cos(2k\pi t) \end{pmatrix}$$

And so

$$\int_{\gamma_k}\omega = \int_0^1 \frac{2k\pi\sin^2(2k\pi t)}{\cos^2(2k\pi t)+\sin^2(2k\pi t)}dt + \int_0^1 \frac{2k\pi\cos^2(2k\pi t)}{\cos^2(2k\pi t)+\sin^2(2k\pi t)}dt$$

$$
\begin{aligned}
&= \int_0^1 2k\pi \, dt \\
&= 2k\pi
\end{aligned}
$$

γ_k is a circular path about the origin that circles k times. ω measures the rotation, so the integral is the total angle the path winded about the origin. ■

Problem 4

The k-form $\omega_k = dx_1 \wedge \cdots \wedge dx_k$ is called the volume form in $\mathbb{R}^k$.

(a) Define a 2-surface $\Phi : [0,1]^2 \to \mathbb{R}^2$ so that

$$\int_\Phi \omega_k$$

explains this terminology.

SOLUTION: Consider $\Phi : [0,1] \times [0,1] \to \mathbb{R}^2$ defined by

$$\Phi(x_1, x_2) = \begin{pmatrix} x_1 \cos(2\pi x_2) & x_1 \sin(2\pi x_2) \end{pmatrix}$$

to be the unit disk 2-surface. Then

$$\begin{aligned}
\int_\Phi \omega_k &= \int_0^1 \int_0^1 dx_1 \wedge dx_2 \begin{pmatrix} \cos(2\pi x_2) & -2\pi x_1 \sin(2\pi x_2) \\ \sin(2\pi x_2) & 2\pi x_1 \cos(2\pi x_2) \end{pmatrix} dx_1\, dx_2 \\
&= \int_0^1 \int_0^1 2\pi x_1 \cos^2(2\pi x_2) + 2\pi x_1 \sin^2(2\pi x_2)\, dx_1\, dx_2 \\
&= \int_0^1 \int_0^1 2\pi x_1 dx_1\, dx_2 \\
&= \pi
\end{aligned}$$

Thus, the volume of the unit disk is π. Yay! ■

(b) Define a 3-surface $\Phi : [0,1]^3 \to \mathbb{R}^3$ so that

$$\int_\Phi \omega_k$$

explains this terminology.

SOLUTION: Consider $\Phi : [0,1]^3 \to \mathbb{R}^3$ be the 2-surface unit ball defined by

$$\Phi(x_1, x_2, x_3) = (x_3 \sin(\pi x_1) \cos(2\pi x_2),\ x_3 \sin(\pi x_1) \sin(2\pi x_2),\ x_3 \cos(\pi x_1))$$

Then

$$\begin{aligned}
\int_\Phi \omega_k &= \int_0^1 \int_0^1 \int_0^1 dx_1 \wedge dx_2 \wedge dx_3 \begin{pmatrix} \pi x_3 \cos \pi x_1 \cos 2\pi x_2 & -2\pi x_3 \sin \pi x_1 \sin 2\pi x_2 & \sin \pi x_1 \cos 2\pi x_2 \\ \pi x_3 \cos \pi x_1 \sin 2\pi x_2 & 2\pi x_3 \sin \pi x_1 \cos 2\pi x_2 & \sin \pi x_1 \sin 2\pi x_2 \\ -\pi x_3 \sin \pi x_1 & 0 & \cos \pi x_1 \end{pmatrix} \\
&= \int_{[0,1]^3} (-\pi x_3 \sin(\pi x_1)) \begin{vmatrix} -2\pi x_3 \sin(\pi x_1) \sin(2\pi x_2) & \sin(\pi x_1) \cos(2\pi x_2) \\ 2\pi x_3 \sin(\pi x_1) \cos(2\pi x_2) & \sin(\pi x_1) \sin(2\pi x_2) \end{vmatrix} \\
&\quad - \cos(\pi x_1) \begin{vmatrix} \pi x_3 \cos(\pi x_1) \cos(2\pi x_2) & -2\pi x_3 \sin(\pi x_1) \sin(2\pi x_2) \\ \pi x_3 \cos(\pi x_1) \sin(2\pi x_2) & 2\pi x_3 \sin(\pi x_1) \cos(2\pi x_2) \end{vmatrix} dx_1 dx_2 dx_3
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi^2 \int_{[0,1]^3} x_3^2 \sin \pi x_1 dx_1 dx_2 dx_3 \\
&= \frac{2}{3}\pi^2 \frac{2}{\pi} \\
&= \frac{4}{3}\pi
\end{aligned}
$$

Which is the volume of a sphere. Yay! ■

In parts (a) and (b) choose non-trivial surfaces (i.e., not just the identity function). Please compute both integrals.

Problem 5

In this exercise, you will see that differential forms are closely related to vector fields. Let $F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a C^1 vector field in $\mathbb{R}^3$, i.e., $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$ are C^1 functions. We can define differential forms from a vector field as follows:

- $\omega_F^1 = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$
- $\omega_F^2 = P(x,y,z)\,dy \wedge dz + Q(x,y,z)\,dz \wedge dx + R(x,y,z)\,dx \wedge dy$

Recall the definitions:

- $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
- $(\nabla \times F)(x,y,z) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$
- $(\nabla \cdot F)(x,y,z) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

(a) Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^2 function and $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^2 vector field. Show that:

(i) $df = \omega_{\nabla f}^1$

SOLUTION: This one is a straight up definition by writing $P = D_1 f$, $Q = D_2 f$ and $R = D_3 f$, then

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \omega_{\nabla f}^1 \end{aligned}$$

$\blacksquare$

(ii) $d\omega_F^1 = \omega_{\nabla \times F}^2$

SOLUTION:

$$\begin{aligned} d\omega_F^1 &= d\left(P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz\right) \\ &= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz\right) \wedge dy \\ &\quad + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz\right) \wedge dz \\ &= \frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial P}{\partial z}dz \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy + \frac{\partial Q}{\partial z}dz \wedge dy + \frac{\partial R}{\partial x}dx \wedge dz + \frac{\partial R}{\partial y}dy \wedge dz \\ &= -\frac{\partial P}{\partial y}dx \wedge dy + \frac{\partial P}{\partial z}dz \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy - \frac{\partial Q}{\partial z}dy \wedge dz - \frac{\partial R}{\partial x}dz \wedge dx + \frac{\partial R}{\partial y}dy \wedge dz \end{aligned}$$

$$= (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})dy \wedge dz + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})dz \wedge dx + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy$$
$$= \omega^2_{\nabla \times F}$$

■

(iii) $d\omega^2_F = (\nabla \cdot F)\, dx \wedge dy \wedge dz$

SOLUTION:

$$\begin{aligned}
dw^2_F &= d\left(P(x,y,z)\, dy \wedge dz + Q(x,y,z)\, dz \wedge dx + R(x,y,z)\, dx \wedge dy\right) \\
&= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) \wedge dy \wedge dz + d(Q\, dz \wedge dx) + d(R\, dx \wedge dy) \\
&= \frac{\partial P}{\partial x}dx \wedge dy \wedge dz + d(Q\, dz \wedge dx) + d(R\, dx \wedge dy) \\
&= \cdots \\
&= \frac{\partial P}{\partial x}dx \wedge dy \wedge dz + \frac{\partial Q}{dy}dy \wedge dz \wedge dx + \frac{\partial R}{\partial z}dz \wedge dx \wedge dy \\
&= \frac{\partial P}{\partial x}dx \wedge dy \wedge dz + \frac{\partial Q}{dy}dx \wedge dy \wedge dz + \frac{\partial R}{\partial z}dx \wedge dy \wedge dz \\
&= (\nabla \cdot F)dx \wedge dy \wedge dz
\end{aligned}$$

Where the second to last inequality holds because the wedges were interchanged an even number of times. ■

(b) Use $d^2\omega = 0$ to prove that $\nabla \times (\nabla f) = 0$ and $\nabla \cdot (\nabla \times F) = 0$.

SOLUTION: Consider that since f is a zero form, then

$$df = D_1 f dx + D_2 f dy + D_3 f dz$$

is a one form. Note that we have by (a, i) that

$$df = \omega^1_{\nabla f}.$$

By a theorem in class, we have that the two form

$$ddf = 0,$$

and by (a, ii) we have that

$$ddf = d\omega^1_{\nabla f} = \omega^2_{\nabla \times \nabla f} = 0.$$

Note that this happens if and only if $\nabla \times \nabla f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Consider the one form defined by F, which is

$$\omega_F^1 = F_1 dx + F_2 dy + F_3 dz.$$

We have by (a) that

$$d\omega_F^1 = \omega_{\nabla \times F}^2,$$

and thus by the same logic as before we have that by (a, iii):

$$dd\omega_F^1 = 0 \implies dd\omega_F^1 = d\omega_{\nabla \times F}^2 = (\nabla \cdot (\nabla \times F))dx \wedge dy \wedge dz = 0.$$

Thus, we have that $\nabla \cdot (\nabla \times F) = 0$. ■

Problem 6

Let H be the parallelogram in $\mathbb{R}^2$ whose vertices are $(1,1),(3,2),(4,5),(2,4)$. Find the affine map T which sends

$$T((0,0)) = (1,1) \tag{1}$$
$$T((1,0)) = (3,2) \tag{2}$$
$$T((0,1)) = (2,4) \tag{3}$$

Show that $\mathcal{J}(T) = 5$. Use T to convert the integral

$$\alpha = \int_H e^{x-y}\,dx\,dy$$

into an integral over I^2 and thus compute α

SOLUTION: In order to find the affine map $T : \mathbb{R}^2 \to \mathbb{R}^2$, we need to find a linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ and $(v_1, v_2) \in \mathbb{R}^2$ such that for any $(x,y) \in \mathbb{R}^2$

$$T((x,y)) = L((x,y)) + (v_1, v_2).$$

(1) tells us that $v_1 = 1$ and $v_2 = 1$. Without loss of generality, we take

$$L(x,y) = A(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

Giving us two equations, each stemming from (2) and (3) respectively:

$$T((1,0)) = L((1,0)) + (1,1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+1 \\ c+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

and so $c = 1$ and $a = 2$. From (3)

$$T((0,1)) = \begin{pmatrix} 1 & b \\ 4 & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} b+1 \\ d+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix},$$

and so $b = 1$ and $d = 3$. Thusm

$$T((x,y)) = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2x+y+1 \\ x+3y+1 \end{pmatrix}.$$

To calculate the Jacobian, we need to take a few partials:

$$\mathcal{J}(T) = \det\begin{pmatrix} D_1A_1 & D_2A_1 \\ D_2A_1 & D_2A_2 \end{pmatrix} = \det\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = 6 - 1 = 5,$$

as required.

Since $T((1,1)) = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, we have that $T(I^2) = H$. Thus, we use the change of variables formula which states that since e^{x-y} is integrable and T is a C^1 diffeomorphism since it is linear, then

$$\begin{aligned}
\alpha &= \int_H e^{x-y}\,dx\,dy \\
&= \int_{I^2} e^{2x+y+1-x-3y-1}\,dx\,dy \\
&= \int_0^1 \int_0^1 e^{x-2y} |\mathcal{J}(T)|\,dx\,dy \\
&= 5\int_0^1 e^x\,dx \int_0^1 e^{-2y}\,dy \\
&= 5(e-1)(-\frac{1}{2e^2} + \frac{1}{2}) \\
&= \frac{5}{2}(e-1)(1-e^{-2})
\end{aligned}$$

■

Problem 7

Let I^k be the set of all $\mathbf{u} = (u_1, \ldots, u_k) \in \mathbb{R}^k$ with $0 \leq u_i \leq 1$ for all i. Let Q^k be the set of all $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ with $x_i \geq 0$ and $\sum x_i \leq 1$. Define $\mathbf{x} = T(\mathbf{u})$ by

$$x_1 = u_1$$

$$x_2 = (1 - u_1)u_2$$

$$\cdots$$

$$x_k = (1 - u_1) \cdots (1 - u_{k-1})u_k$$

Show that

$$\sum_{i=1}^{k} x_i = 1 - \prod_{i=1}^{k} (1 - u_i).$$

Show that T maps I^k onto Q^k and that T is 1-1 in the interior of I^k. Show that its inverse S is defined in the interior of Q^k by $u_1 = x_1$ and

$$u_i = \frac{x_i}{1 - x_1 - \cdots - x_{i-1}} \tag{4}$$

Show that

$$\mathcal{J}_T(u) = (1 - u_1)^{k-1}(1 - u_2)^{k-2} \cdots (1 - u_{k-1})$$

and that

$$\mathcal{J}_S(x) = [(1 - x_1)(1 - x_1 - x_2) \cdots (1 - x_1 - \cdots - x_{k-1})]^{-1}$$

Solution: We induct on k. Suppose $k = 1$, then clearly this relation ship is true. Now suppose the relation holds for $k = n - 1$. Then

$$\begin{aligned}
\sum_{i=1}^{n} x_i = \sum_{i=1}^{n-1} x_i + x_n \\
&= 1 - \prod_{i=1}^{n-1} (1 - u_i) + x_n \\
&= 1 - \prod_{i=1}^{n-1} (1 - u_i) + u_n \prod_{i=1}^{n-1} (1 - u_i) \\
&= 1 - (1 - u_1)(1 - u_2)(1 - u_3) \cdots (1 - u_{n-1}) + u_n(1 - u_1)(1 - u_2)(1 - u_3) \cdots (1 - u_{n-1}) \\
&= 1 - \prod_{i=1}^{n} (1 - u_i)
\end{aligned}$$

To show that $T : I^k \to Q^k$ is surjective, let $\mathbf{x} \in Q^k$. Then $x_i \geq 0$ and $\sum x_i \leq 1$. Suppose $\sum_{i=1}^{m} x_i < 1$ for $m \leq k$. Then we can use the formula provided up to the largest such m^* and set the rest of the u_i equal to 0. To see that this results in a point in the unit square, not that

$u_1 = x_1$, and since $x_1 \leq 1$, then $u_1 \in [0,1]$. For $m^* \geq i \geq 2$, we claim that $u_i = \frac{x_i}{1-x_1-\cdots-x_{i-1}}$. To see this, note that if $m^* > 2$, then

$$u_2 = \frac{x_2}{1-x_1} \implies x_2 = u_2(1-x_1) = u_2(1-u_2).$$

Suppose $x_2 > 1 - x_1 \implies x_1 + x_2 > 1$, which is a contradiction, and so $u_2 \in [0,1]$. We can easily induct on the rest of the $i \leq m^*$. For $i \geq m^* + 1$, we let $u_i = 0$. Note that then our definition of our map is satisfied since $\sum x_i = 1 - \prod(1-u_i)$. Thus, T is surjective.

Suppose now $T(\mathbf{x}) = T(\mathbf{x}')$, where $\mathbf{x}, \mathbf{x}' \in \operatorname{int}(I^k)$ and thus $x_i, x_i' < 1$ for all i. We induct on $n \leq m^*$. By definition, $x_1 = u_1 = x_1'$. Suppose the following that for general n, $x_n = x_n'$, Then by construction and since we can divide by the following denominator since we are in the interior of I^k, then

$$x_{n+1} = u_{n+1} \prod_{i=1}^{n}(1-u_i) = u_{n+1} \prod_{i=1}^{n}(1-u_i') \implies u_{n+1} = \frac{x_{n+1}}{\prod_{i=1}^{n}(1-u_i')} = u_{n+1}',$$

and thus $x_{n+1} = x_{n+1}'$.

We claim that the Jacobian is a triangular matrix. To see this, note that

$$D_i T_j(x) = \frac{\partial}{\partial u_i} u_j \prod_{k=1}^{j-1}(1-u_k),$$

and so if $i < j$, then u_i is somewhere in the product, resulting in a nonzero partial. If $i > j$, then u_i has not appeared in the expression for u_j, resulting in a 0 derivative. If $i = j$, then we are differentiating linear equations with coefficients $\prod_{k=1}^{j-1}(1-u_k)$. The determinant of an upper triangular matrix is the product of the diagonals, which is

$$J = 1(1-u_1)(1-u_1)(1-u_2)\cdots(1-u_1)(1-u_2)\cdots(1-u_{k-1}) = (1-u_1)^{k-1}\cdots(1-u_{k-1}).$$

Similar reasoning can be used to show that J_S is a lower triangular matrix with entries 1, $\frac{1}{1-x_1}$, $\frac{1}{1-x_1-x_2}$, and the result follows. ■

Problem 8

If ω and λ are k and m forms, prove that

$$\omega \wedge \lambda = (-1)^{km}(\lambda \wedge \omega)$$

SOLUTION: We can write

$$\omega = \sum_{0 \le i_1 \le i_2 \le \cdots \le i_k} a_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum_I a_I dx_I$$

$$\lambda = \sum_{0 \le j_1 \le j_2 \le \cdots \le j_m} b_{j_1,\ldots,j_m} dx_{j_1} \wedge \cdots \wedge dx_{j_k} = \sum_J b_J dx_J$$

Thus, we have that

$$\omega \wedge \lambda = \sum_{I,J} b_I c_J (dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots dx_{j_m}) = \sum_{[I,J]} (-1)^\alpha d_{[I,J]}(dx_{[I,J]}).$$

Thus, we claim it suffices to show it for $[I,J] = (i_1, i_2, \ldots, i_k, i_{k+1}, \ldots, i_{k+m})$. Thus, we have that

$$\begin{aligned}
\omega \wedge \lambda &= \sum_{[I,J]} d_{[I,J]}(dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_{k+m}}) \\
&= \sum_{[I,J]} d_{[I,J]}(-1)^k (dx_{i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{i_{k+2}} \wedge \cdots \wedge dx_{i_{k+m}}) \\
&= \sum_{[I,J]} d_{[I,J]}(-1)^k(-1)^k (dx_{i_k} \wedge dx_{i_{k+2}} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{i_{k+3}} \wedge \cdots \wedge dx_{i_{k+m}}) \\
&\cdots \\
&= \sum_{[I,J]} d_{[I,J]}((-1)^k)^m (dx_{i_{k+1}} \wedge dx_{i_{k+2}} \wedge \cdots \wedge dx_{i_{k+m}} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \\
&= (-1)^{km} \sum_{[I,J]} d_{[I,J]} dx_{[J,I]} \\
&= (-1)^{km}(\lambda \wedge \omega)
\end{aligned}$$

Now for a general ω and λ that are not in standard presentation, we have that

$$\begin{aligned}
\omega \wedge \lambda &= \sum_{I,J} b_I c_J (dx_I \wedge dx_J) \\
&= \sum_{[I,J]} d_{[I,J]}(-1)^\alpha (dx_{[I,J]}) \\
&= \sum_{[I,J]} d_{[I,J]}(-1)^{\alpha+km} dx_{[J,I]}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{km} \sum_{I,J} c_J b_I \, dx_J dx_I \\
&= (-1)^{km} (\lambda \wedge \omega)
\end{aligned}
$$

■