Instructor(s): Luis Silvestre

Agustin Esteva
aesteva@uchicago.edu
Due Date: 2024-11-11

Problem 1

Suppose that $f : M \to M$ and for all $x, y \in M$, if $x \neq y$ then $d(f(x), f(y)) < d(x, y)$. Such an f is a *weak contraction.*

(a) Is a weak contraction a contraction? (Proof or counterexample.)

Solution: **No.** Consider $f : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ such that $f(x) = x^2$.[a] f is a contraction because for any $x, y \in [0, \frac{1}{2}]$, we have that $|x + y| \leq 1$, and thus if $x \neq y$, then

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| < |x - y|.$$

Thus, f is a weak contraction. Suppose f is a contraction as well. Then there exists some $k < 1$ such that $d(f(x), f(y)) \leq kd(x, y)$. However, take $x = k$ and $y = \frac{1-k}{2}$, then we have that

$$|x + y| > k \implies |x + y||x - y| > k|x - y| \implies |f(x) - f(y)| > k|x - y|,$$

and thus f is not a contraction. ■

[a] Note that f does not need to be a surjection in order to be a contraction, which is good because $f([0, \frac{1}{2}]) = [0, \frac{1}{4}]$

(b) If M is compact is a weak contraction a contraction?

Solution: **No.** The above example works. ■

(c) If M is compact, prove that a weak contraction has a unique fixed point.

Solution: Since f is a contraction and $f : M \to M$, then we claim that $f(M) \subset M$. Since f is a contraction, we have that there exists some $\delta > 0$ such that if $d(x, y) < \frac{\delta}{2}$, then $d(f(x), f(y)) < \frac{\delta}{2}$. Cover M by δ balls. Then if $y \in B_\delta(x) \subset M$, we have that $d(f(x), f(y)) < \frac{\delta}{2}$, and thus $f(x)$ and $f(y)$ are in (possible another) δ ball of M, and so $f(M) \subset M$. Since f is continuous and M is compact, then $f(M)$ is compact. We can induct on this process and notice that

$$M \supset f(M) \supset f^2(M) \supset \dots$$

with each set compact. We now claim that if

$$X = M \cap \bigcap_{n \in \mathbb{N}} f^n(M),$$

then X is our set of fixed point. To see this, notice that each set is compact and nonempty, and thus X is compact and nonempty. We now wish to show that $f(X) = X$. One inclusion is easy. If $x \in f(X)$, then since $f(X) \subset X$ by the above logic, $x \in X$. Suppose now that $x \in X$. Thus, $x \in M \cap \bigcap_{n \in \mathbb{N}} f^n(M)$. Since $x \in f(M)$, then there exists some $m_1 \in M$ such that $f(m_1) = x$. Similarly, there exists some $m_2 \in M$ such that $f^2(m_2) = x$. Take the sequence

$$y_1 = m_1,\ y_2 = f(m_2),\ \ldots\ y_n = f^{n-1}(m_n).$$

We have by compactness of M that it has some convergent subsequence $(y_{n_k}) \to y_\infty$. We claim that $f(y_\infty) = x$. To see this, consider that since f is continuous, we have that $f(y_{n_k}) \to f(y_\infty)$. However, we by construction that

$$f(y_{n_k}) = f(f^{n_{k-1}}(m_{n_k})) = x \implies f(y_\infty) = x.$$

Moreover, we have that $y_\infty \in f^n(M)$ for every $n \in \mathbb{N}$ by closedeness, and thus

$$y_\infty \in X \implies f(y_\infty) \in f(X) \implies x \in f(X).$$

It suffices to show that $(X) = 0$. Suppose not, then $(X) > 0$. Thus, since X is compact, we have that there must exist $x_1, x_2 \in X$ such that $d(x_1, x_2) > 0$. However, we have proved that $f(x_1) = x_1$ and $f(x_2) = x_2$, and thus

$$d(f(x_1), f(x_2)) = d(x_1, x_2) > 0,$$

which is a contradiction to the fact that f is a contraction. Thus, $(X) = 0$ and thus X is a single point and thus we have that there exists a unique $x \in X$ such that $f(x) = x$.
■

Reflections: The following is a proof I am currently in the process of fixing, but have not figured out how:

Let $x_0 \in M$. Let $x_n = f^n(x_0)$, where $f^n(x_0) = (f \circ f \circ f \circ \cdots \circ f)(x_0)$, with f composite itself n times. Thus, since $f^n(x_0) \in M$ for any n, then by compactness, $(x_n) \in M$ has a convergent subsequence $x_{n_k} \to x_\infty$. We claim that x_∞ is a fixed point. To see this, notice that since (x_{n_k}) is convergent, then it is Cauchy, and thus for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $n_k, m_k \geq N_1$, we have $d(x_{n_k}, x_{m_k}) < \frac{\epsilon}{3}$. Since $x_{n_k} \to x_\infty$, then there exists some N_2 such that if $n_k \geq N_2$, then $d(x_{n_k}, x_\infty) < \frac{\epsilon}{3}$. Take

$N = \min\{N_1, N_2\}$, then we have if $x_{n_k} > N$,

$$\begin{aligned} d(x_\infty, f(x_\infty)) &\leq d(x_\infty, x_{n_k}) + d(x_{n_k}, f(x_{n_k})) + d(f(x_{n_k}), f(x_\infty)) \\ &< \frac{\epsilon}{3} + d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_k}, x_\infty) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}. \end{aligned}$$

The second term of the second inequality follows by definition of (x_{n_k}), and the last term of the second inequality follows from the fact that f is a contraction. Suppose f has another unique point at some $p \in M$, then $|f(p) - f(x_\infty)| = |p - x_\infty| \not< |p - x_\infty|$, and thus f is not a contraction.

Problem 2

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and its derivative satisfies $|f(x)| < 1$ for all $x \in \mathbb{R}$.

(a) Is f a contraction?

Solution: Consider $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} x - \arctan(x) \end{cases}$$

We claim without proof that $f'(x) = 1 - \frac{1}{x^2+1}$. Thus, we have that $f'(x) < 1$ for all $x \in \mathbb{R}$, but as $x \to \infty$, $f'(x) \to 1$. Suppose f is a contraction, then there exists some $k < 1$ such that if $x, y \in \mathbb{R}$, then

$$|f(x) - f(y)| \leq k|x - y| \implies \frac{|f(x) - f(y)|}{|x - y|} = |f'(\theta)| \leq k$$

for some $\theta \in (x, y)$. However, taking $x = 0$ and $y = k + 1$, then we have that

$$|f(x) - f(y)| = |k + 1 - \arctan(k + 1)|$$

■

(b) Is f a weak contraction?

Solution: **Yes.** Let $x, y \in \mathbb{R}$, then since f f is differentiable on (x, y) and continuous on $[x, y]$, there exists some $\theta \in (y, x)$ such that

$$|f(y) - f(x)| = f'(\theta)|x - y| < |x - y|$$

since $f'(\theta) < 1$. ■

(c) Does it have a fixed point?

Solution: **No.** ■

Problem 2

Give an example to show that the fixed-point in Brouwer's Theorem need not be unique.

SOLUTION: Let B^1 be the closed unit ball in $\mathbb{R}^1$, and let $f : B^1 \to B^1$ such that $f(x) = x$. Obviously, f is continuous. Every point in B^1 is a fixed point, and thus there is no uniqueness. ■

Problem 3

(a) Give an example of a function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that for each fixed x, then function $y \to f(x,y)$ is a continuous function of y, and for each fixed y, the function $x \to f(x,y)$ is a continuous function of x, but f is not continuous.

SOLUTION: Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Clearly, $x \to f(x,y)$ is continuous for all fixed $y \neq 0$. Take some sequence $(x_n, 0) \to (0,0)$. By examining the function, it is clear that $f(x_n, 0) = f(0,0) = 0$. Same for $y \to f(x,y)$. To prove that f is not continuous at $(0,0)$, take the sequence $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$. We want to show that $f(\frac{1}{n}, \frac{1}{n})$ does not converge to $f(0,0) = 0$. To see this, consider that

$$f(\frac{1}{n}, \frac{1}{n}) = \frac{\frac{1}{n}\frac{1}{n}}{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\frac{1}{n^2}}{\frac{2}{n^2}} = \frac{1}{2}.$$

■

(b) Suppose in addition that the set of functions

$$\mathcal{E} = \{x \to f(x,y) \;\; y \in [0,1]\}$$

is equicontinous. Prove that f is continuous.

SOLUTION: Let $(x_n, y_n) \to (x,y)$, where $(x_n) \in [0,1]$ and $(y_n) \in [0,1]$. We want to show that $f(x_n, y_n) \to f(x,y)$. Thus, it suffices to show that for any $\epsilon > 0$, we have n large such that

$$d(f(x_n, y_n), f(x,y)) < \epsilon.$$

Since $\mathcal{E}$ is equicontinuous, then for any $\epsilon > 0$, we have that there exists a $\delta > 0$ such that if $|x - t| < \delta$, then for any f such that f is a function that sends $x \to f(x,y)$ with y fixed, $|f(x,y) - f(t,y)| < \frac{\epsilon}{2}$, Since $x_n \to x$, then we have that for large n, $|x - x_n| < \delta$. Since for each fixed x, function $y \to f(x,y)$ is a continuous function of y, then we have that if $(y_n) \to y$, then $f(x, y_n) \to f(x,y)$. Thus, for large enough n, we have that $d(f(x,y_n), f(x,y)) < \frac{\epsilon}{2}$

$$d(f(x_n, y_n), f(x,y)) \leq d(f(x_n, y_n), f(x, y_n)) + d(f(x, y_n), f(x,y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

■

Problem 4

Let $T : V \to W$ be a linear transformation and let $p \in V$ be given. Prove that the following are equivalent.

(a) T is continuous at the origin.

(b) T is continuous at p.

(c) T is continuous at at least one point of V.

SOLUTION: Suppose T is continuous at the origin, then we claim that T is continuous. To see this, we will first show that $||T|| < \infty$. Let $\epsilon = 1$, then there exists a $\delta > 0$ such that if $u \in V$ and $|u| < \delta$, then

$$|T(u)| < 1.$$

Let $v \in V$ nonzero, then let $\lambda = \frac{\delta}{2|v|}$, and thus $u = \lambda v$. $|u| = \frac{\delta}{2}$ and due to the properties of linear transforms and norms, we have that:

$$\frac{|T(v)|}{|v|} = \frac{|T(\frac{u}{\lambda})|}{|\frac{u}{\lambda}|} = \frac{|T(u)|}{u} < \frac{1}{|u|} = \frac{2}{\delta}.$$

Thus, $||T|| < \infty$. Let $v, v' \in V$ with $|v - v'| < \frac{\epsilon}{||T||}$, then

$$|T(v) - T(v')| = |T(v - v')| \leq ||T|||v - v'| < \epsilon,$$

and thus T is uniformly continuous. Thus, we have b and c.

Suppose c, then T is continuous at some $u \in V$. Let $\epsilon > 0$, then get the $\delta > 0$ from the continuity of u. Thus, if $|v| < \delta$, then we let $v = u - (u + \frac{\delta}{2})$. Notice that we have that $|u - (u + \frac{\delta}{2})| = \frac{\delta}{2} < \delta$, and thus

$$|T(u) - T(u + \frac{\delta}{2})| < \epsilon.$$

Because T is a linear transform, we also have that

$$|T(u) - T(u + \frac{\delta}{2})| = |T(u - (u + \frac{\delta}{2}))| = |T(v)| < \epsilon.$$

Thus, we have that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $v < \delta$, then $|T(v)| < \epsilon$, and thus T is continuous at the origin. ∎

Problem 5

Let $\mathcal{L}$ be the vector space of continuous linear transformations from a normed space V to a normed space W. Show that the operator norm makes $\mathcal{L}$ a normed space.

SOLUTION: Suppose $T, T' \in \mathcal{L}$ and let $\lambda \in \mathbb{F}$. Note that $||T||$ is well defined since it is finite since f is continuous.

(a)
$$||T|| = \sup\{\frac{|T(v)|_W}{|v|_V}, v \neq 0.\}$$

Since $T : V \to W$, then $T(v) \in W$, and thus since W is a normed space, we have that $|T(v)|_W \geq 0$ for all $T(v) \in W$. Similarly, we have that $|v|_V \geq 0$ for all $v \in V$. Thus, $||T|| \geq 0$. Suppose T is the zero transformation, then $T(v) = 0$ for any $v \in V$. Thus, we have that

$$||T|| = \sup\{\frac{|T(v)|_W}{|v|_V}, v \neq 0.\} = \sup\{\frac{0}{|v|_V}, v \neq 0.\} = 0.$$

(b) Since $|T(v)|_W$ is a norm in W, then if λ is a scalar, we have that $|\lambda T(v)|_W = |\lambda||T(v)|_W$. Similarly for V.

$$||\lambda T|| = \sup_{v \in V}\{\frac{|\lambda T(v)|_W}{|v|_V}\ ;\ v \neq 0\} = \sup_{v \in V}\{\frac{|\lambda||T(v)|_W}{|v|_V}\ ;\ v \neq 0\} = |\lambda| \sup\{\frac{|T(v)|_W}{|v|_V}, v \neq 0.\}.$$

Thus, $||\lambda T|| = |\lambda|||T||$.

(c) Since W is a normed space, we have that $|T(v) + T'(v)|_W \leq |T(v)|_W + |T(v)|_W$.

$$\begin{aligned}
||T + T'|| &= \sup\{\frac{|T(v) + T'(v)|_W}{|v|_V}\ ;\ v \neq 0\} \\
&\leq \sup\{\frac{|T(v)| + |T'(v)|_W}{|v|_V}\ ;\ v \neq 0\} \\
&\leq \sup\{\frac{|T(v)|}{|v|_V}\ ;\ v \neq 0\} + \sup\{\frac{|T'(v)|_W}{|v|_V}\ ;\ v \neq 0.\} \\
&= ||T|| + ||T'||
\end{aligned}$$

The last inequality comes from the fact that $\sup(f(x)+g(x)) \leq \sup(f(x))+\sup(g(x))$.[a]

■

[a] Proved on PSET 5, but $f(x) \leq \sup f(x)$ and $g(x) \leq \sup g(x)$ imply that $f(x)+g(x) \leq \sup f(x) + \sup g(x)$ for all x.

Problem 6

Two norms $|\ |_1$ and $|\ |_2$ on a vector space are *comparable* if there are positive constants c and C such that for all nonzero vectors in V we have

$$c \leq \frac{|v|_1}{|v|_2} \leq C.$$

(a) Prove that comparability is an equivalence relation on norms.

SOLUTION: It will suffice to show the three properties of an equivalence relation. Let $|\ |_1, |\ |_2$ be norms on a vector field V, and let $v \in V$.

(i) (Reflexive) We want to show that $|\ |_1$ is comparable to itself. This is clear, since we have that

$$\frac{|v|_1}{|v|_1} = 1 \implies \frac{1}{2} \leq \frac{|v|_1}{|v|_1} \leq 2,$$

and thus $|\ |_1$ is comparable to itself.

(ii) (Symmetry) We want to show that if $|\ |_1$ is comparable to $|\ |_2$, then $|v|_2$ is comparable to $|v|_1$. By assumption, c and C are constants such that

$$c \leq \frac{|v|_1}{|v|_2} \leq C \implies \frac{1}{C} \leq \frac{|v|_2}{|v|_1} \leq \frac{1}{c}.$$

Since $\frac{1}{C}$ and $\frac{1}{c}$ are positive constants, then $|\ |_2$ is comparable to $|\ |_1$.

(iii) (Transitive) Suppose $|\ |_1$ is comparable to $|\ |_2$ and $|\ |_2$ is comparable to $|\ |_3$, then there exists positive constants c, C and c', C' such that

$$c \leq \frac{|v|_1}{|v|_2} \leq C, \qquad c' \leq \frac{|v|_2}{|v|_3} \leq C'.$$

Thus, we have that since everything is positive,

$$cc' \leq \frac{|v|_1}{|v|_2}\frac{|v|_2}{|v|_3} \leq CC' \implies cc' \leq \frac{|v|_1}{|v|_3} \leq CC',$$

and thus $|\ |_1$ is comparable to $|\ |_3$.

■

(b) Prove that any two norms on a finite-dimensional vector space are comparable.

SOLUTION: Let V be a finite dimensional vector space and $|\ |_1, |\ |_2$ be norms on V. Let $T : (V, |\ |_1) \to (V, |\ |_2)$ be the identity map.[a] By Corollary 4 on the book, we have that

T is continuous (and indeed, a homoemorphism), and thus by Theorem 2, $||T|| < \infty$. Thus, we have that

$$\sup_{v \in V} \frac{|T(v)|_2}{|v|_1} < \infty. \tag{1}$$

In particular, since we are dealing with the identity map, we have that there exists some positive C constant such that for all nonzero vectors $v \in V$,

$$\frac{|v|_2}{|v|_1} \leq C.$$

Now consider T^{-1}. This is also continuous because T is a homoeomorphism, and so $||T^{-1}1|| < \infty$. Thus, we have that

$$\sup_{v \in V} \frac{|T(v)|_1}{|v|_2} < \infty.$$

In particular, since we are dealing with the identity map, there exists some positive c constant such that for all nonzero vectors $v \in V$,

$$\frac{|v|_1}{|v|_2} \leq c. \tag{2}$$

Combining (1) and (2) we find that

$$\frac{1}{C} \leq \frac{|v|_1}{|v|_2} \leq c,$$

and thus $|\ |_1$ and $|\ |_2$ are comparable. ■

[a] T is a linear transform because $T(\alpha v + w) = \alpha v + w = \alpha T(v) + T(w)$.

(c) Consider the norms

$$|f|_{L^1} = \int_0^1 |f(t)| dt, \qquad |f|_{C^0} = \max\{f(t)\ :\ t \in [0,1]\},$$

defined on the infinite-dimensional vector space $C^0([0,1], \mathbb{R})$. Show that the norms are not comparable by finding functions $f \in C^0([0,1], \mathbb{R})$, whose integral norm is small but whose C^0 is 1.

SOLUTION: Suppose $|\ |_{L^1}$ and $|\ |_{C^0}$ are comparable, then there exists some positive c, C such that for any $f \in C^0([0,1], \mathbb{R})$,

$$c \leq \frac{\int_0^1 f(t) dt}{\max\{f(t)\ ;\ t \in [0,1]\}} \leq C.$$

Consider a sequence of functions $f_n : [0, 1] \to \mathbb{R}$.

$$f_n(x) = x^n.$$

Each f_n is continuous, and each achieves their maximum at $x = 0$ at $f(x) = 1$. However, as $n \to \infty$, we claim that $\int_0^1 |f_n(t)|dt \to 0$. To see this, use the FTC:

$$\left|\int_0^1 |t^n|dt\right| = \int_0^1 |t^n|dt = \int_0^1 t^n dt = \frac{1}{n+1} \to 0.$$

Thus, for any $c > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, we have that

$$|f_n|_{L^1} < c,$$

and thus we have a contradiction since for any $c > 0$, we have that for large n,

$$\frac{\int_0^1 f_n(t)dt}{\max\{f_n(t) \ : \ t \in [0, 1]\}} = \int_0^1 t^n dt = \frac{1}{n+1} < c.$$

■

Problem 7

Let $|\ | = |\ |_{C^0}$ be the supremum norm on C^0 as defined ini Problem 6. Define an integral transformation $T : C^0 \to C^0$ by

$$T : f \to \int_0^x f(t)dt.$$

(a) Show that T is linear, continuous, and find its norm.

SOLUTION: (i) (Linear) We want to show that if $f, g \in C^0$ and $\alpha \in \mathbb{R}$, then $T(\alpha f + g) = \alpha T(f) + T(g)$. Thus, we use the linearity of the integral:

$$\begin{aligned} T(\alpha f + g) &= \int_0^x \alpha f(t) + g(t)dt \\ &= \int_0^x \alpha f(t)dt + \int g(t)dt \\ &= \alpha \int_0^x f(t)dt + \int g(t)dt \\ &= \alpha T(f) + T(g). \end{aligned}$$

(ii) (Continuous) To show that T is continuous, then by Theorem 2, it will suffice to show that $||T|| < \infty$. Since f is continuous on $[0, 1]$, it achieves its maximum on it. Thus, for any $f \in C^0$, we have that

$$\left| \int_0^x f(t)dt \right| \leq \max\{f(t)\ :\ t \in [0, 1]\}.$$

Thus, for any $f \in C^0$, we have that

$$\begin{aligned} |T(f)| &= | \int_0^1 f(t)dt|_{C^0} \\ &\leq |\max\{f(t)\ :\ t \in [0, 1]\}|_{C^0} \\ &= \max\{f(t)\ :\ t \in [0, 1]\} \\ &= |f|_{C^0} \end{aligned}$$

Thus, for any $f \in C^0$:

$$\begin{aligned} \frac{|T(f)|_{C^0}}{|f|_{C^0}} &\leq \frac{|f|_{C^0}}{|f|_{C^0}} \\ &= 1 \end{aligned}$$

Thus, because this is true for any $f \in C^0$, we have that $||T|| < 1 < \infty$.

(iii) (Norm) We defined the usual operator norm on T :

$$||T|| = \sup_{f \in C^0} \frac{|T(f)|_{C^0}}{|f|_{C^0}}.$$

■

(b) Let $f_n(t) = \cos(nt)$, $n = 1, 2, \ldots$ What is $T(f_n)$?

SOLUTION: We use the fundamental theorem of calculus!

$$T(f_n) = \int_0^x \cos(nt)dt = \frac{1}{n}\sin(nx)$$

■

(c) Is the set of functions $K = \{f_n : n \in \mathbb{N}\}$ closed? Bounded? Compact?

SOLUTION: We shall check each condition.

(i) (Closed) Not closed since K has no limit points. Suppose it is closed though! Then $f_n(t) \to f$ with $f \in K$. Thus, we have that for large n,

$$|f_n - f|_{C^0} < \epsilon,$$

and thus using the reverse triangle, we have that

$$||f_n|_{C^0} - |f|_{C^0}| \le |f_n - f|_{C^0}, \epsilon.$$

Since $|f_n|_{C^0} = 1$, then we have that $|f|_{C^0} = 1$. Since $f_n(t) \to f$ and since T is continuous, we now have that $T(f_n) \to T(f)$. By work in the following section, we have that $T(f(n)) \to 0$, and thus by the same logic as above, $|T(f)|_{C^0} = 0$. However, this is a contradiction, since

$$T(f) = \int_0^1 f(t)dt$$

and $|f(t)|_{C^0} = 1$, which, since T is a linear transform, implies that T only sends the zero vector to the zero vector!

(ii) For any $n \in \mathbb{N}$, we have that

$$|\cos(nt)|_{C^0} = 1,$$

and thus f_n is uniformly bounded since for any $t \in [0,1]$, $n \in \mathbb{N}$, $f_n(t) \le 1$.

(iii) We claim that K is not compact. By Arzela-Ascoli, it suffices to show that K is not equicontinuous. Let $\epsilon = \frac{1}{2}$ and take $x = 0$ and $y = \frac{\pi}{2n}$ then for all $\delta > 0$, if n large, we have that $|x - y| < \delta$, but

$$|f_n(x) - f_n(y)|_{C^0} = |\cos(n0) - \cos\left(n\frac{\pi}{2n}\right)|_{C^0} = |1 - \cos\left(\frac{\pi}{2}\right)|_{C^0} = 1.$$

Thus, K is not equicontinuous, and thus not compact.

■

(d) Is $T(K)$ compact? How about its closure?

SOLUTION: We make heavy use of Arzela-Ascoli.

(i) $(T(K))$ We claim that $T(K)$ is not compact. To do this, it suffices by Arzela-Ascoli to show that it is not closed. Consider that $T(K) = \{\frac{1}{n}\sin(nx) \ : \ n \in \mathbb{N}\}$ by part (b). We claim that $z(x) = 0$ is a limit point of $T(k)$, but $z(x) \notin T(k)$. We claim that $T(f_n) \to z(x)$ uniformly. To see this, let $\epsilon > 0$, then for n large, we have that

$$|T(f_n(x)) - 0| = \left|\frac{1}{n}\sin(nx)\right| \leq \frac{1}{n} < \epsilon.$$

Evidently, we have that $z(x) \notin T(K)$ since $z(x) \neq \frac{1}{n}\sin(nx)$ for any $n \in \mathbb{N}$. Thus, $T(K)$ is not closed.

(ii) $(\overline{T(K)})$ By definition, $\overline{T(K)}$ is closed. Let $\overline{(T(f_n)) \in T(K)}$. Thus, $T(f_n) = \frac{1}{n}\sin(nx)$, which converges, and thus any subsequence of it converges to a function which is in the closure. Thus, the closure is compact. Evidently, since $T(K)$ is uniformly bounded, then $\overline{T(K)}$ is uniformly bounded. Thus, by Arzela-Ascoli, we have compactness.

■

Problem 8

Let $f : U \to \mathbb{R}^m$ be differentiable, $[p, q] \subset U \subset \mathbb{R}^n$, and ask whether the direct generalization of the one-dimensional Mean Value Theorem is true: Does there exist a point $\theta \in [p, q]$ such that

$$f(q) - f(p) = Df_\theta(q - p)? \tag{3}$$

(a) Take $n = 1$, $m = 2$, and examine the function $f(t) = (\cos(t), \sin(t))$ for $t \in [\pi, 2\pi]$. Take $p = \pi$ and $q = 2\pi$. Show that there is no $\theta \in [p, q]$ that satisfies (3).

SOLUTION: Suppose there does exist some $\theta \in [\pi, 2\pi]$ such that

$$f(2\pi) - f(\pi) = \begin{bmatrix}1 & 0\end{bmatrix} - \begin{bmatrix}-1 & 0\end{bmatrix} = \begin{bmatrix}2 & 0\end{bmatrix} = Df_\theta(\pi).$$

Since θ exists, we have that

$$Df_\theta = \begin{bmatrix}\frac{\partial f_1}{\partial \theta} & \frac{\partial f_2}{\partial \theta}\end{bmatrix} = \begin{bmatrix}-\sin(\theta) & \cos(\theta)\end{bmatrix},$$

thus, we have that

$$\begin{bmatrix}2 & 0\end{bmatrix} = \begin{bmatrix}-\pi\sin(\theta) & \pi\cos(\theta)\end{bmatrix} \implies \theta = \frac{3\pi}{2}.$$

However, since $\theta = \frac{3\pi}{2}$, then $-\pi\sin(\theta) = \pi \neq 2$, which is a contradiction. ■

(b) Assume the set of derivatives

$$(Df)_x \in \{\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m) \ : \ x \in [p, q]\}$$

is convex. Prove there exists $\theta \in [p, q]$ which satisfies (28).

SOLUTION: We use two facts from googling support plane:

(i) If X is compact convex and Y is closed convex and $X \cap Y = \emptyset$, there exists a hyperplane $H_{u,\alpha} = \{x \mid \alpha = \langle u, x\rangle\}$ such that for all $x \in X$ and $y \in Y$, we have that
$$\langle u, x\rangle < \alpha < \langle u, y\rangle.$$

(ii) If X is convex and non-singular and $x_0 \in$ rel. bd(X), then there exists a hyperplane $H_{u,\alpha}$ such that $x_0 \in H_{u,\alpha}$ and for all $x \in X$, $\langle u, x\rangle \leq \alpha$ and $X \not\subset H_{u,\alpha}$.

Define
$$\mathscr{A} := \{Df_x(q - p) \ : \ x \in [p, q]\}.$$

Note that $\mathscr{A}$ is convex since if $t \in [0, 1]$ we have that by the linearity of the derivative:

$$tDf_x(q - p) + (1 - t)Df_y(q - p) = [tDf_x + (1 - t)Df_y](q - p),$$

where the inside of the bracket is a convex combination of the derivatives, which are convex, and thus is a derivative itself. We claim that $f(p) - f(q) \in \mathscr{A}$. To see this, let $X = \{f(p) - f(q)\}$ and $Y = \overline{\mathscr{A}}$. Suppose $f(q) - f(p) \notin Y$, then we have that $X \cap Y = \emptyset$, and from fact (1) we have a hyperplane $H_{u,\alpha}$ such that

$$\langle u, x\rangle < \alpha < \langle u, Df_x(q-p)\rangle.$$

We now claim that if $U \subset \mathbb{R}^m$, then there exists some $z_u \in [p, q]$ such that

$$\langle u, f(q) - f(p)\rangle = \langle u, Df_{z_u}(q-p)\rangle.$$

Let $u \in \mathbb{R}^m$ and let

$$F_u(t) = \langle u, f\left(t(q) - (1-t)p\right)\rangle$$

and apply one-dimensional MVT and Leibniz product rule:

$$F_u(1) - F_u(0) = \langle u, f(q) - f(p)\rangle = F_u'(\theta) = \langle u, Df_{\theta q + (1-\theta)p}(q-p)\rangle = \langle u, D_{z_u}(q-p)\rangle.$$

Note here that $\theta \in [0, 1]$ and thus $z_u = \theta q + (1-\theta)p \in [p, q]$.

Thus, if $f(q) - f(p) \in \mathscr{A}$, then we are done. If it is not in $\mathscr{A}$, then either $f(q) - f(p) \in$ rel. bd.$(\mathscr{A})$ or $f(q) - f(p) \in \{rel.int\}(\mathscr{A})$. If the latter, then we are done since it is still in the closure. If the former, then by fact (ii), we have that there exists a hyperplane $H_{u,\alpha}$ such that $f(q) - f(p) \in H_{u,\alpha}$ and $\langle u, f(q) - f(p)\rangle = \alpha$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(t) = \langle u, f(tq + (1-t)p)\rangle - \langle u, f(q) - f(p)\rangle t.$$

F is differentiable, and thus using the product rule and the chain rule and the Leibniz product rule and fact (ii) and squeez theorem (jk!):

$$F'(t) = \langle u, D_{tq+(1-t)p}(q-p)\rangle - \langle u, f(q) - f(p)\rangle = \langle u, D_{z_u}(q-p)\rangle - \alpha \le 0.$$

Now, we basically win, since by fact (ii), we again have that $\mathscr{A} \not\subset H_{u,\alpha}$, then there exists some $t' \in [p, q]$ such that that since $F'(t) \le 0$ for all t and

$$F'(t') < 0 \implies F(1) < F(0).$$

However, by the very definition F, we have that

$$F(1) - F(0) = \langle u, f(p)\rangle - \langle u, f(q) - f(p)\rangle + \langle u, f(q)\rangle = 0.$$

A contradiction! Thus, $f(q) - f(p) \in \mathcal{A}$ and we are done. ■

Problem 9

Assume that U is a connected open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is differentiable everywhere on U. If $(Df)_p = 0$ for all $p \in U$, show that f is constant.

Solution: Let $p \in U$. Define:

$$A := \{x \ : \ f(x) = f(p)\}.$$

We want to show that A is equal to U. To do this, we prove that A is clopen and that $A \neq \emptyset$. The latter is obvious since $p \in A$. To prove that A is closed, consider that $A = f^{-1}\{f(p)\}$. Since f is differentiable on U, then it is continuous on U, and thus we have that since $\{f(p)\}$ is closed in $\mathbb{R}^m$ (since it is a single point), then $f^{-1}\{f(p)\}$ is closed in U. To prove that A is open, we must show that for any $a \in A$, there exists some $r > 0$ such that

$$B_r(a) \subset A.$$

Since U is open and $a \in U$, then there exists some $r' > 0$ such that

$$B_{r'}(a) \subset U.$$

Thus, let $b \in B_{\frac{r'}{2}}(a)$, then $b \in U$ and $[a, b] \subset U$. Thus, we have by the multivariate MVT that

$$|f(b) - f(a)| \leq M|b - a|, \quad M = \sup\{(Df)_x \; x \in [a, b]\} = 0.$$

Thus,

$$f(b) = f(a) = f(p) \implies b \in A.$$

Thus, A is open. Thus we have that A is clopen, and thus $A = U$. Thus, for all $x \in U$, $f(x) = f(p)$, and so f is constant on U. ∎