UChicago Accelerated Analysis III Notes: 20510

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Academic Year 2024-2025

Contents

1 Lectures

1.1 Monday, Mar 24: Motivation for the Lebesgue Measure

Definition 1. A family of sets $\mathscr{A}$ is called a **ring** if it is closed under finite unions and under set complements.

Definition 2. A ring is called a $\sigma-$**ring** if it is closed under countable unions.

Remark 1. I will most often refer to $\sigma-$rings as $\sigma-$algebra. The only difference if $\mathscr{F}$ is a $\sigma-$algebra, then if $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$. Meanwhile, if $A \in \mathscr{A}$, then A^c might not necessarily be in the ring. Instead, if $B \in \mathscr{A}$, then $B \setminus A \in \mathscr{A}$.

Using DeMorgan's Law, a $\mathscr{F}$ is closed under countable intersections as well.

Definition 3. A **set function** ϕ on an algebra $\mathscr{F}$ satisfies that for all $A \in \mathscr{F}$, $\phi(A) \in \mathbb{R} \cup \{\pm\infty\}$ (but not both at the same time).

Definition 4. A set function ϕ is **additive** if for all A, B disjoint, we have that

$$\phi(A \sqcup B) = \phi(A) + \phi(B).$$

Definition 5. We say that ϕ is **countably additive** if for any $A_1, A_2, \ldots$ mutually disjoint, we have that

$$\phi\left(\bigsqcup_{n=1}^{\infty} A_i\right) = \sum_{n=1}^{\infty} \phi(A_i)$$

Remark 2. Let ϕ be an additive set function on $\mathscr{F}$. Then

(a) $\phi(\emptyset) = 0$. Let $A \in \mathscr{F}$ with $\phi(A) < \infty$. Then $A = A \sqcup \emptyset$ and so $\phi(A) = \phi(A) + \phi(\emptyset)$.

(b) Let $N < \infty$, then for $A_1, \ldots, A_N$ mutually disjoint,

$$\phi\left(\bigsqcup_{n=1}^{N} A_i\right) = \phi\left(A_1 \sqcup \bigsqcup_{n=2}^{N} A_i\right) = \phi(A_1) + \phi\left(\bigsqcup_{n=2}^{N} A_i\right) = \cdots = \sum_{n=1}^{N} \phi(A_i).$$

(c) We have that for any $A, B \in \mathscr{F}$,

$$\phi(A \cup B) + \phi(A \cap B) = \phi(A) + \phi(B)$$

(d) If ϕ is positive real valued, then if $A \subset B$, then

$$\phi(A) \leq \phi(B).$$

Notice that

$$\phi(B) = \phi(A \sqcup (B \setminus A)) = \phi(A) + \phi(B \setminus A) \geq \phi(A).$$

(e) If $A \subseteq B$ and $\phi(B) < \infty$, then

$$\phi(B \setminus A) = \phi(B) - \phi(A).$$

Theorem 1. Let ϕ be a countably additive set function on $\mathscr{F}$. Suppose $(A_n) \in \mathscr{F}$ such that $A_1 \subseteq A_2 \subseteq \cdots$ and

$$\bigcup_{n=1}^{\infty} A_i = A.$$

Then $\phi(A_n) \to \phi(A)$.

Proof. Define

$$B_1 := A_1, \quad B_2 := A_2 \setminus A_1, \quad B_3 := A_3 \setminus A_2, \cdots$$

Then (B_n) is a collection of disjoint sets such that $\bigsqcup_{n=1}^{\infty} B_i = A$. Then since ϕ is countably additive, we have that

$$\phi(A_n) = \phi(\bigsqcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} \phi(B_i) \to \sum_{i=1}^{\infty} \phi(B_i) = \phi(\bigsqcup_{i=1}^{\infty} B_i) = \phi(A).$$

$\square$

Definition 6. An **interval** $I = \{a_i, b_i\}_{i=1}^{n} \subseteq \mathbb{R}^n$ is a set of points $x = (x_1, \ldots, x_n)$ such

$$a_i \leq x_i \leq b_i,$$

where the $\leq$ can be replaced with $<$.

Definition 7. We say that A is **elementary** if A is a union of finitely many intervals.

Remark 3. We call the set of elementary sets $\mathscr{E}$.

Definition 8. Suppose I is an interval of $\mathbb{R}^n$. Then the **volume** of I is

$$\mathrm{Vol}(I) = \prod_{i=1}^{n} (b_i - a_i).$$

Remark 4. Let $A \in \mathscr{E}$. Then $A = \bigcup_{i=1}^{n} I_i$. Then

$$\mathrm{Vol}(A) = \sum_{i=1}^{n} \mathrm{Vol}(I_i)$$

1.2 Wednesday, Mar 26: The Lebesgue Outer Measure

Remark 5. (a) $\mathscr{E}$ is a ring, but not a $\sigma-$ring.

(b) If $A \in \mathscr{E}$, then A can be decomposed into a finite union of disjoint intervals.

(c) If $A \in \mathscr{E}$, then $\text{Vol}(A)$ is well defined.

Definition 9. A non-negative set function on $\mathscr{E}$ is called **regular** if for all $A \in \mathscr{E}$, for all $\epsilon > 0$, there exists open $O \in \mathscr{E}$ and closed $F \in \mathscr{E}$ such that $F \subseteq A \subseteq O$ and

$$\phi(G) \leq \phi(A) + \epsilon, \quad \phi(A) \leq \phi(F) + \epsilon.$$

Note that Vol is regular.

Definition 10. The **Lebesgue Outer Measure** of $E \subseteq \mathbb{R}^n$ is defined by

$$m^*(E) = \inf\left(\sum_{n=1}^{\infty} \text{Vol}(A_i)\right),$$

where the infemum is taken over all the countable open covers of E.

Remark 6. Let $E \in \mathbb{R}^n$. Then

(a) $m^*(E)$ is well defined.

(b) If $E_1 \subseteq E_2 \subseteq \mathbb{R}^n$. Then
$$m^*(E_1) \leq m^*(E_2).$$

(c) The outer measure is non-negative.

Theorem 2. If $A \in \mathscr{E}$, then $\text{Vol}(A) = m^*(A)$. Moreover, if $E = \bigcup_{i=1}^{\infty} E_i$, then $m^*(E) \leq \sum_{i=1}^{\infty} m^*(E_i)$

Proof. Let $\epsilon > 0$. Since Vol is regular, then there exists some open $O \supseteq A$ such that $\text{Vol}(O) \leq \text{Vol}(A) + \epsilon$. Since O is an open cover, we have that $m^*(A) \leq \text{Vol}(O) \leq \text{Vol}(A) + \epsilon$. Thus, $m^*(A) \leq \text{Vol}(A) + \epsilon$. Let $F \subseteq A$ closed such that $\text{Vol}(A) \leq \text{Vol}(F) + \frac{\epsilon}{2}$. Then since F is closed and bounded in $\mathbb{R}^n$, it is compact. Let $\{A_n\}_{n=1}^{\infty}$ be an open cover of A such that

$$\sum_{n=1}^{\infty} \text{Vol}(A_i) \leq m^*(A) + \frac{\epsilon}{2}$$

Then there exists $\{A_n\}_{n=1}^{N}$ finite open cover of F by compactness. By the finite sub-additivity of F, we have that

$$\text{Vol}(A) \leq \text{Vol}(F) + \frac{\epsilon}{2} \leq \sum_{n=1}^{N} \text{Vol}(A_n) + \frac{\epsilon}{2} \leq \sum_{n=1}^{\infty} \text{Vol}(A_n) + \frac{\epsilon}{2} \leq m^*(A) + \epsilon.$$

Thus, $\text{Vol}(A) \leq m^*(A)$.

Suppose $m^*(E_n) < \infty$ for all n. Let $\epsilon > 0$. For each n, there exists a countable open cover such that

$$\sum_{i=1}^{\infty} \text{Vol}(A_i^{(n)}) \leq m^*(E_n) + \frac{\epsilon}{2^n}.$$

Since $E \subseteq \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} A_i^{(n)}$, then

$$m^*(E) \leq \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \text{Vol}(A_i^{(n)}) \leq \sum_{n=1}^{\infty} m^*(E_n) + \frac{\epsilon}{2^n} \leq \sum_{n=1}^{\infty} m^*(E_n) + \epsilon$$

$\square$

1.3 Friday, Mar 28: The Lebesgue Measure

Definition 11. Let $A, B \subseteq \mathbb{R}^n$. The **symmetric difference** of A and B is

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

Definition 12. The **distance** between A and B is defined as

$$d(A, B) = m^*(A \triangle B).$$

Definition 13. Let $(A_n) \in \mathbb{R}^n$. We say that A_n **converges in (outer) measure** if $d(A_n, A) \to 0$.

Definition 14. If there exists a sequence $(A_n) \in \mathscr{E}$ such that $A_n \to A$, then A is **finitely-measurable**. We say that $A \in \mathscr{M}_F(m)$.

Definition 15. We say that $E \in \mathscr{M}(m)$ if

$$E = \bigcup_{n=1}^{\infty} A_n,$$

where each $A_n \in \mathscr{M}_F(m)$.

Theorem 3. (Caratheodory) $\mathscr{M}$ is a $\sigma-$family, and m^* is countably additive on $\mathscr{M}$.

Definition 16. The **Lebesgue Measure** is the set function

$$m : \mathscr{M}(m) \to [0, \infty], \quad m(A) = m^*(A).$$

Remark 7. As a small review we recap our set-functions so far:

Set Function	Domain	Properties
Vol	$\mathscr{E}$	Non-negative, Finitely Additive, Regular
m^*	$\mathbb{R}^n$	Non-negative, Countably-sub-additive, $m\|_{\mathscr{E}} = \text{Vol}$
m	$\mathscr{M}$	Non-negative, Countably-additive, $m\|_{\mathscr{M}} = m^*$

Table 1: Set Functions

Example 1.1. (a) If $A \in \mathscr{E}$, then $A \in \mathscr{M}$.

(b) Since $\mathbb{R}^d = \bigcup_{n=1}^{\infty} [-n, n]^d$, then $\mathbb{R}^d \in \mathscr{M}$.

(c) If $A \in \mathscr{M}$, then $A^c \in \mathscr{M}$.

(d) For all $x \in \mathbb{R}^n$, $x \in \mathscr{M}$. This is because

$$x = \bigcap_{n=1}^{\infty} (x - \frac{1}{n}, x + \frac{1}{n}).$$

Moreover, $m(\{x\}) = 0$.

(e) $m(\mathbb{Q}) = 0$.

1.4 Monday, Mar 31: Measurable Functions

Definition 17. We say that $f : \mathbb{R}^n \to \mathbb{R}$ is **(Lebesgue) measurable** if, for every $a \in \mathbb{R}^n$, $\{x \in \mathbb{R}^n \mid f(x) > a\}$ is measurable.

Remark 8. If f is continuous, then $f^{-1}\big((a, \infty)\big)$ is open, and thus measurable. Then f is measurable.

Proposition 1. Equivalently, f is measurable if the following are measurable:

- $\{x \in \mathbb{R}^n \mid f(x) > a\}$
- $\{x \in \mathbb{R}^n \mid f(x) \geq a\}$
- $\{x \in \mathbb{R}^n \mid f(x) < a\}$
- $\{x \in \mathbb{R}^n \mid f(x) \leq a\}$

Proof. Suppose f is measurable. We can write

$$\{x \in \mathbb{R}^n \mid f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x \in \mathbb{R}^n \mid f(x) > a + \frac{1}{n}\}.$$

$$\{x \in \mathbb{R}^n \mid f(x) \leq a\} = \{x \in \mathbb{R}^n \mid f(x) > a\}^c$$
$$\{x \in \mathbb{R}^n \mid f(x) < a\} = \{x \in \mathbb{R}^n \mid f(x) \geq a\}^c.$$

By Caratheodory's theorem (Theorem 3), we are done. □

Theorem 4. Suppose f is measurable. Then $|f|$ is measurable.

Proof. Let $a \in \mathbb{R}^n$. Then we can write

$$\{x \in \mathbb{R}^n \mid |f(x)| < a\} = \{x \in \mathbb{R}^n \mid -a < f(x) < a\} = \{x \in \mathbb{R}^n \mid f(x) < a\} \cap \{x \in \mathbb{R}^n \mid f(x) > -a\}.$$

□

Theorem 5. Suppose (f_n) are measurable and $g = \sup f_n$ and $h = \limsup f_n$. Then g and h are measurable.

Proof. Let $a \in \mathbb{R}^n$. Then

$$\{x \in \mathbb{R}^n \mid g(x) > a\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R}^n \mid f_n(x) > a\},$$

and

$$\{x \in \mathbb{R}^n \mid h(x) > a\} = \bigcap_{n=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in \mathbb{R}^n \mid f_m(x) > a\}.$$

□

Corollary 1. (a) If f, g are measurable, then so are $\max\{f, g\}$ and $\min\{f, g\}$.

(b) Suppose f is measurable. We can write $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. Both f^+ and f^- are measurable.

Theorem 6. Suppose $f, g : \mathbb{R}^n t\o\mathbb{R}$ are measurable. If $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then $h(x) = F(f(x), g(x))$. Then h is measurable.

Thus, $f + g$, fg, and all the rest are measurable if the components are measurable.

Definition 18. A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a **simple function** if $R(\varphi) < \infty$.

Remark 9. Equivalently, if φ is simple, then

$$\varphi = \sum_{k=1}^{n} c_k \chi_{E_k},$$

where $R(\varphi) = \{c_1, \ldots, c_n\}$ and $E_k = \{x \in \mathbb{R}^n \mid \varphi(x) = c_k\}$.

Note that E is measurable iff χ_E is measurable.

Corollary 2. Suppose is a simple function. Then φ is measurable if and only if each E_k is measurable

1.5 Wednesday, Apr 2: The Lebesgue Integral

Theorem 7. Suppose $f : \mathbb{R}^n \to \mathbb{R}$. There exists a sequence of (φ_n) simple functions such that $\varphi_n \to f$ pointwise. Moreover, if $f \geq 0$, then one can choose the sequence such that $0 \leq \varphi_n \uparrow f$. If f is measurable, one can choose the φ_n to be measurable.

The proof can be found in the second PSET.

Definition 19. Suppose φ is a simple non-negative measurable function. Let $E \in \mathscr{M}$. We define the **integral of a simple function** to be

$$I_E(\varphi) = \sum_{k=1}^{n} c_k m(E_k \cap E).$$

Definition 20. Suppose $f \geq 0$ is measurable. If $E \in \mathscr{M}$, the define the **Lebesgue Integral** to be

$$\int_E f\, dm = \sup_{0 \leq \varphi \leq f,\ \varphi \text{ simple}} I_E(\varphi).$$

Definition 21. Suppose f is measurable. We define the **Lebesgue integral of** f over $E \in \mathscr{E}$ to be

$$\int_E f\, dm = \int_E f^+\, dm - \int_E f^-\, dm.$$

If either is finite, then we write that $f \in \mathscr{L}$ and say that f is **Lebesgue integrable**.

Remark 10.

- The Lebesgue integral is well defined.
- The Lebesgue integral can be infinity.
- If φ is non-negative and simple and measurable, then $\int_E \varphi = I_E(\varphi)$.

1.6 Friday, Apr 4: Properties of the Lebesgue Integral

Remark 11. Let f be measurable.

(a) If $a \leq f(x) \leq b$ for all $x \in E \in \mathscr{M}$, then

$$a\, m(E) \leq \int_E f\, dm \leq b\, m(E).$$

(b) Suppose f is bounded and $E \in \mathscr{M}$ with $m(E) < \infty$. Then $f \in \mathscr{L}(E)$.

(c) If $f, g \in \mathscr{L}(E)$ and $f \leq g$ on E, then

$$\int_E f\, dm \leq \int_E g\, dm$$

(d) If $f \in \mathscr{L}(E)$ and $c \in \mathbb{R}^n$, then $cf \in \mathscr{L}(E)$ and

$$\int_E cf\, dm = c \int_E f\, dm.$$

(e) If $m(E) = 0$, then

$$\int_E f\, dm = 0.$$

(f) If $f \in \mathscr{L}(A)$ and $A \in \mathscr{M}$ and $E \subseteq A$, then $f \in \mathscr{L}(E)$.

(g) If $f \in \mathscr{R}([a, b])$, then $f \in \mathscr{L}([a, b])$ and the Riemann integrals and Lebesgue integrals are equivalent.

Theorem 8. Suppose $f \geq 0$ is measurable. For all $A \in \mathscr{M}$, define the set function

$$\phi(A) = \int_A f\, dm.$$

Then ϕ is countably additive.

Proof. Suppose $(A_n) \in \mathscr{M}$.

- Suppose f is a characteristic function. That is, $f = \chi_E$ for some $E \in \mathscr{M}$. Then

$$\phi(\bigsqcup_{n=1}^{\infty} A_n) = \int_{\bigsqcup_{n=1}^{\infty} A_n} f = m(E \cap \bigsqcup_{n=1}^{\infty} A_n) = m(\bigsqcup_{n=1}^{\infty} E \cap A_n) = \sum_{n=1}^{\infty} m(E \cap A_n) = \sum_{n=1}^{\infty} \phi(A_n)$$

- Suppose f is simple function. That is, $f = \sum_{k=1}^{N} c_k E_k$. Then

$$\begin{aligned}\phi(\bigsqcup_{n=1}^{\infty} A_n) &= \int_{\bigsqcup A_n} f\,dm \\ &= \int_{\bigsqcup A_n} \sum_{k=1}^{N} c_k \chi_{E_k}\,dm \\ &= \sum_{k=1}^{N} c_k \phi(\bigsqcup_{n=1}^{\infty} \chi_{E_k}) \\ &= \sum_{k=1}^{N} c_k \sum_{n=1}^{\infty} \phi(\chi_{E_k}) \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{N} c_k m(E_k \cap A_n) \\ &= \sum_{n=1}^{\infty} \int_{A_n} f \\ &= \sum_{n=1}^{\infty} \phi(A_n)\end{aligned}$$

- Suppose $f \geq 0$. Let $0 \leq \varphi \leq f$ be measurable. Then we know that

$$\int_{\bigsqcup A_n} \varphi \leq \sum_{n=1}^{\infty} \int_{A_n} \varphi\,dm \leq \sum_{n=1}^{\infty} \int_{A_n} f\,dm = \sum_{n=1}^{\infty} \phi(A_n).$$

  Let $\epsilon > 0$. By definition, there exists a simple function φ such that

$$\int_{A_n} \varphi \geq \int_{A_n} f - \frac{\epsilon}{2^n}.$$

  Then

$$\phi(\bigsqcup_{n=1}^{k} A_n) \geq \int_{\bigsqcup_{n=1}^{k} A_n} \varphi\,dm \geq \sum_{n=1}^{k} \phi(A_n) - \frac{\epsilon}{2^n} \to \sum_{n=1}^{\infty} \phi(A_n) - \epsilon.$$

$\square$

Corollary 3. Suppose $A, B \in \mathscr{M}$ with $B \subseteq A$ such that $m(A \setminus B) = 0$. Then for all $f \in \mathscr{L}(A)$, we have that

$$\int_A f\,dm = \int_B f\,dm$$

Proof. Write $A = B \sqcup A \setminus B$ and conclude using the previous theorem and remark 11. $\square$

1.7 Monday, Apr 7: Lebesgue's Monotone Convergence Theorem

Theorem 9. Suppose $f \in \mathscr{L}(E)$. Then $|f| \in \mathscr{L}(E)$ and

$$\left|\int_E f\, dm\right| \leq \int_E |f|\, dm$$

Proof. Let

$$A := \{x \in E \mid f(x) \geq 0\}, \quad B := \{x \in E \mid f(x) < 0\}.$$

Then $E = A \sqcup B$ and so

$$\int_E |f|\, dm = \int_A |f|\, dm + \int_B |f|\, dm = \int_A f^+\, dm + \int_B f^-\, dm < \infty.$$

Thus, $|f| \in \mathscr{L}(E)$. Since $f \leq |f|$ and $-f \leq |f|$, we have that

$$\int_E f\, dm \leq \int_E |f|\, dm, \quad -\int_E f\, dm \leq \int_E |f|\, dm.$$

□

Theorem 10. (Monotone Convergence Theorem) Suppose f_n is a sequence of non-negative measurable functions with $f_1(x) \leq f_2(x) \leq \cdots$ for al x and with

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all x. Then $\int f_n\, dm = \int f\, dm$

Proof. $\int f_n$ is an increasing sequence of real numbers. Denote the limit by L Note that $L \leq \int f$. Let $0 \leq \varphi = \sum_{k=1}^N c_k \chi_{E_k} \leq f$. Let $c \in (0,1)$ and define

$$A_n := \{x \in E \mid f_n(x) \geq c\varphi(x)\}.$$

Note that $A_n \uparrow E$ since $c \in (0,1)$. Thus,

$$L \geq \int_E f_n \geq \int_{A_n} f_n \geq c \int_{A_n} \varphi = \sum_{k=1}^N c\, c_k m(E_k \cap A_n) \xrightarrow[n\to\infty]{} \sum_{k=1}^N c\, c_k m(E_k \cap A) = c\int_E \varphi.$$

Thus, since c is arbitrary, $L \geq \int_E \varphi$. Taking the supremum over all such φ, we get that $L \geq \int_E f$. □

Theorem 11. If f, g are non-negative and measurable, then if $E \in \mathscr{M}$ with $m(E) < \infty$, we have that

$$\int_E (f+g)\, dm = \int_E f\, dm + \int_E g\, dm.$$

Proof. If f and g are simple functions, the result is obvious. Let $f, g \geq 0$. By Theorem 7, there exist f_n, g_n simple, non-negative, and measurable such that $f_n \uparrow f$ and $g_n \uparrow g$. Then by the monotone convergence theorem, since $(f_n + g_n) \uparrow (f+g)$

$$\int_E (f+g) = \lim_{n\to\infty} \int_E (f_n + g_n) = \lim_{n\to\infty} (\int_E f_n + \int_E g_n) = \int_E f + \int_E g.$$

Suppose now f, g are integrable. Then by the above

$$\int_E |f+g| \leq \int_E |f| + |g| = \int_E |f| + \int_E |g| < \infty,$$

and thus $|f+g|$ is integrable and so $f+g$ is integrable. Write

$$(f+g) = (f+g)^+ - (f+g)^- = f^+ + g^+ - f^- - g^-,$$

then

$$(f+g)^+ + f^- + g^- = f^+ + g^+ + (f+g)^-.$$

Thus, we use the above to show that

$$\int_E (f+g)^+ + \int_E f^- + \int_E g^- = \int_E f^+ + \int_E g^+ + \int_E (f+g)^-.$$

After rearranging we achieve out solution. $\square$

1.8 Wednesday, Apr 9: Fatou's Lemma and Dominated Convergence Theorem

Theorem 12. (Fatou's Lemma) Suppose (f_n) is a sequence of non-negative measurable functions. Then if $E \in \mathscr{M}$,

$$\int_E \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int_E f_n.$$

Proof. Define $f := \liminf_{n\to\infty} f_n$. Define $g_n := \inf_{k\ge n} f_k$. We have that g_n is measurable for each n, and $g_n \le f_n$. Thus, for each n, we have that

$$\int_E g_n \le \int_E f_n \implies \int_E g_n \le \inf_{k\ge n} \int f_k$$

By the MCT, since $g_n \uparrow f$, we have that

$$\lim_{n\to\infty} \int_E g_n = \int_E f \le \liminf_{n\to\infty} \int_E f_n$$

□

Theorem 13. (Dominated Convergence Theorem) Suppose (f_n) are measurable such that $f_n \to f$ pointwise and $|f_n| \le g$ for all n. Then if $g \in \mathscr{L}$, we have that

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. Note that $f_n + g \ge 0$, and thus by Fatou's Lemma, we have that

$$\int_E f + g \le \liminf_{n\to\infty} \int_E f_n + g \implies \int_E f \le \liminf_{n\to\infty} \int_E f_n.$$

Similarly, we have that $g - f_n \ge 0$, and so by Fatou's lemma,

$$\int_E g - f_n \le \liminf_{n\to\infty} \int_E g - f_n \implies -\int_E f_n \le -\limsup_{n\to\infty} \int_E f_n.$$

□

1.9 Friday, Apr 11: A Non-Measurable Set

Theorem 14. There exists some set $V \subseteq \mathbb{R}$ that is not measurable.

Proof. Define the equivalence relation $x \sim y$ if $x - y \in \mathbb{Q}$, where $x, y \in [0, 1]$ For each equivalence class, choose a representative using the axiom of choice. Let V be the collection of such elements. Assume V is measurable. Let $a \in (V + q) \cap (V + q')$ where $q, q' \in \mathbb{Q}$. Then $a = x + q = x' + q'$, for $x \in [x]$ and $x' \in [x']$ and so $a - x = q$ and $a - x' = q'$. Then $x - x' = (a - x') - (a - x) = q' - q \in \mathbb{Q}$, and so $x \in [x']$, which is a contradiction to the way we chose the representatives. Note that $m(V + q) = m(V)$ and

$$[0, 1] \subseteq \bigcup_{q \in [-1,1] \bigcap \mathbb{Q}} (V + q)$$

and thus

$$1 \leq \sum_{q \in [-1,1] \bigcap \mathbb{Q}} m(V) \implies m(V) > 0.$$

But we know that

$$\bigcup_{q \in [-1,1] \bigcap \mathbb{Q}} (V + q) \subseteq [-1, 2] \implies \sum_{q \in [-1,1] \bigcap \mathbb{Q}} m(V) \leq 3 \implies m(V) = 0$$

A contradiction!

□

1.10 Monday, Apr 14: Fourier Series

Let $f : \mathbb{R} \to \mathbb{C}$. If $f = u + iv$, then recall we define the integral to be

$$\int f = \int u + i \int v.$$

For the discussion on Fourier series, we suppose f is defined on intervals of length 2π and is $2\pi-$periodic that are Riemann integrable. We denote these functions to be in $\mathscr{R}$.

Definition 22. A **trigonometric polynomial** is a function f defined by

$$f(x) = \sum_{-N}^{N} c_n e^{inx},$$

where $a_n, b_n, c_n \in \mathbb{C}$.

Remark 12. On PSET 5, we show that we can alternatively and equivalently take

$$f(x) = S_N(f) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n i \sin(nx)$$

Remark 13. We work on $\mathscr{R}$ with the following inner product. Let $f, g \in \mathscr{R}$, then

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx.$$

In this space, $\{e^{inx}\}_{n \geq 0}$ is an orthonormal basis. That is,

$$(e_n, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-inx} = 1$$

$$(e_n, e_m) = 0$$

Definition 23. Let $f \in \mathscr{R}$. We define the **Fourier Coefficients** of f to be

$$\hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

Definition 24. The **Fourier series**, $S(f)$ of f is defined to be

$$f(x) \sim \sum_{-\infty}^{\infty} \hat{f}(n) e_n$$

Remark 14. Suppose f is a trigonometric polynomial. Then

$$\begin{aligned}
\hat{f}(m) &= (f, e_n) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx}\, dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{-N}^{N} c_n e^{inx} e^{-imx}\, dx \\
&= \sum_{-N}^{N} c_n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-imx}\, dx \\
&= c_m
\end{aligned}$$

Hence, if f is a trigonometric polynomial, then

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}(n) e_n = \sum_{-N}^{N} \hat{f}(n) e_n = S_N(f)$$

Example 1.2. Consider $f(x) = x$ on $[-\pi, \pi]$. Then the Fourier coefficients of f are

$$\begin{aligned} \hat{f}(n) = (f, e_n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} = \frac{1}{2\pi} \left[\left[\frac{-x}{in} e^{-inx} \right]_{-\pi}^{\pi} + \frac{1}{in} \int_{-\pi}^{\pi} e^{-inx} \, dx \right] \\ &= \frac{(-1)^{n+1}}{in} \end{aligned}$$

Hence,

$$S(f) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(nx)}{n}$$

1.11 Wednesday, Apr 16: Properties of Fourier Series

Theorem 15. Suppose $f \in \mathscr{R}$ on $[-\pi, \pi]$ and is $2\pi-$periodic. If $\hat{f}(n) = 0$ for all n, then $f(x) = 0$ whenever f is continuous at x.

Corollary 4. Suppose $f \in \mathscr{R}$ on $[-\pi, \pi]$ and is $2\pi-$periodic. If f is continuous and $\hat{f}(n) = 0$ for all n, then $f(x) \equiv 0$.

Corollary 5. If $f, g \in \mathscr{R}$ are continuous, $2\pi-$periodic, and $\hat{f}(n) = \hat{g}(n)$ for all n, then $f \equiv g$

Proof. Consider that it suffices to see that $f - g \equiv 0$, which by the previous corollary, implies that it suffices to to show that $\widehat{f-g}(n) = 0$. We know that $\hat{f}(n) - \hat{g}(n) = 0$, so it suffices to see that the Fourier coefficients are linear in some sense. This follows by the linearity of the inner product:

$$\widehat{f-g}(n) = (f - g, e_n) = (f, e_n) - (g, e_n) = \hat{f}(n) - \hat{g}(n) = 0$$

$\square$

Corollary 6. Suppose f is continuous, $2\pi-$periodic and $S_N(f)$ converges absolutely. Then $S_N(f) \rightrightarrows f$.

Proof. Since $S_N(f)$ converges absolutely, then

$$\sum_{-N}^{N} |(f, e_n)e_n| = \sum_{-N}^{N} |\hat{f}(n)| < \infty.$$

From this, we can define

$$g(x) := S(f).$$

By the above corollary, it suffices to note that $\hat{g}(n) = \hat{f}(n)$. We compute:

$$\hat{g}(m) = (g, e_m) = \left(\sum_{-\infty}^{\infty} (f, e_n)e_n, e_m \right) = \sum_{-\infty}^{\infty} (f, e_n)(e_n, e_m) = (f, e_n) = \hat{f}(n)$$

$\square$

Lemma 1. Suppose $f \in C^2$ and is $2\pi-$periodic. Then there exists some $c > 0$ such that for large enough $|n|$,

$$\hat{f}(n) \leq \frac{c}{|n|^2}$$

The proof is an exercise is integration by parts.

1.12 Friday, Apr 18: Inner Product Spaces

Theorem 16. Let $f \in \mathscr{R}$ be $2\pi-$periodic. Then

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}|f - S_N(f)|^2 = 0$$

Definition 25. Let V be a vector space. An **inner product** $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is that satisfies:

(a) $(x, y) = \overline{(y, x)}$

(b) $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$.

(c) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$

Remark 15. We can induce a norm on an inner product space quite easily:

$$\|x\| = \sqrt{(x, x)}.$$

The opposite is not always true. For the vector space we have been working on, recall that

$$(f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g}(x)\,dx$$

Thus,

$$\|f\| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx\right)^{\frac{1}{2}}$$

Hence, we can interpret Theorem 16 to mean convergence in this norm.

Definition 26. We say that $x, y \in V$ are **orthogonal** and write $x \perp y$ if $(x, y) = 0$.

Proposition 2. Let V be an inner product space. Then for any $x, y, z \in V$

(a) (C-S Inequality). $|(x, y)| \leq \|x\|\|y\|$

(b) (Pythagoras). If $(x \perp y)$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$

(c) (Triangle Inequality) $\|x - y\| \leq \|x - z\| + \|z - y\| \iff \|x + y\| \leq \|x\| + \|y\|$

Remark 16. A useful property of $S_N(f)$ is that

$$|S_N(f)|^2 = (S_N(f), S_N(f)) = (\sum_{-N}^{N}(f, e_n)e_n, \sum_{-N}^{N}(f, e_m), e_m) = \sum_{n=-N}^{N}(f, e_n)\left(e_n, \sum_{m=-N}^{N}(f, e_m)e_m\right) = \sum_{-N}^{N}|(f, e_n)|^2$$

Proposition 3. For all $|m| \leq N$, we have that $(f - S_N(f)) \perp e_n$.

Proof. Computing,

$$(f - S_N(f), e_n) = (f, e_n) - (S_N(f), e_n) = 0$$

$\square$

1.13 Monday, Apr 21: Parseval's Theorem

Remark 17. Recall that:

- We defined $\mathscr{R}$ is the set of $2\pi-$periodic (Riemann) integrable functions
- The inner product on $\mathscr{R}$ was defined as

$$(f,g) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx$$

- If we called $e_n = e^{inx}$, then $(f, e_n) = \hat{f}(n)$, and the $\{e_n\}$ are orthonormal, so

$$(f - S_N(f)) \perp e_m, \quad \forall\, |m| \leq N.$$

Corollary 7. For every sequence $\{c_n\}_{-N}^{N}$, we have that

$$(f - S_n(f)) \perp \sum_{-N}^{N} c_n e_n.$$

Remark 18. The consequences of Remark 17 and Corollary 7 are

(a) We can write $f = f - S_N(f) + S_N(f)$ and thus

$$\|f\|^2 = \|f - S_N(f)\|^2 + \|S_N(f)\|^2$$

By orthogonality,

$$\|S_N(f)\|^2 = \sum_{-N}^{N} \|\hat{f}(n)e_n\|^2 = \sum_{-N}^{N} \|\hat{f}(n)\|^2.$$

Plugging into the above, we see that

$$\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{-N}^{N} \|\hat{f}(n)\|^2$$

Lemma 1. (Best approximation) If $f \in \mathscr{R}$, then

$$\|f - S_N(f)\| \leq \|f - \sum_{-N}^{N} c_n e_n\|$$

for any complex numbers $\{c_n\}_{-N}^{N}$.

Theorem 17. If $f \in \mathscr{R}$, then

$$\lim_{N\to\infty} \int_0^{2\pi} |f - S_N(f)|^2\, dx = 0.$$

That is, in the mean squared norm,

$$S_N(f) \to f.$$

Proof. Let $f \in \mathscr{R}$ be continuous. Let $\epsilon > 0$, by the Stone-Weierstrass Theorem, there exists some trigonometric polynomial P such that for all $x \in [0, 2\pi]$, we have

$$|f(x) - P(x)| < \epsilon.$$

We know that

$$\|f-P\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)-P(x)|\,dx\right)^{\frac{1}{2}} < \left(\frac{1}{2\pi}\int_0^{2\pi} \epsilon^2\,dx\right)^{\frac{1}{2}} = \epsilon.$$

Suppose $P = \sum_{-M}^{M} c_n e_n$. By the best approximation lemma, for all $N \geq M$, we have that

$$\|f - S_N(f)\| \leq \|f-P\| < \epsilon.$$

For a general $f \in \mathscr{R}$, we let $\epsilon > 0$. Since continuous functions are dense in the set of Riemann integrable functions, then exists some continuous function g such that

$$\|g(x)\|_{\sup} \leq \|f(x)\|_{\sup} = B$$

and

$$\int_0^{2\pi} |f(x)-g(x)|\,dx < \epsilon^2.$$

Note that by definition,

$$\|f-g\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)-g(x)|^2\,dx\right)^{\frac{1}{2}} \leq \left(\frac{B}{\pi}\epsilon^2\right)^{\frac{1}{2}} = \sqrt{\frac{B}{\pi}}\,\epsilon.$$

We use part 1 to see that

$$\|f - S_N(f)\| \leq \|f-g\| + \|g - S_N(f)\| < \epsilon$$

□

Corollary 8. (Parseval's Identity) If $f \in \mathscr{R}$, then

$$\sum_{-\infty}^{\infty} |\hat{f}(n)|^2 = \|f\|^2$$

Proof. We use Remark 13, which states that (Bessel's Inequality)

$$\|f\|^2 \geq \sum_{-N}^{N} \|\hat{f}(n)\|^2.$$

By the previous theorem we have that for all $\epsilon > 0$,

$$\|f - S_N(f)\| < \epsilon$$

and so

$$\sum_{-N}^{N} \|\hat{f}(n)\|^2 \geq \|f\|^2 - \epsilon$$

□

Corollary 9. (Riemann-Lebesgue) If $f \in \mathscr{R}$, then

$$\lim_{n\to\infty} |\hat{f}(n)| = 0.$$

Wednesday, Apr 23: Midterm

Theorem 18. (Borel Cantelli) Suppose $\{A_n\}_{n\geq 0}$ is a sequence of measurable sets. Let

$$A = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_n.$$

Show that if $\sum_{n=1}^{\infty} m(A_n) < \infty$, then $m(A) = 0$.

Proof. Note that A is measurable since it is the countable intersection of measurable sets. Let $\epsilon > 0$.Since $\sum_{n=1}^{\infty} m(A_n) < \infty$, then then there is some large m such that $\sum_{m=1}^{\infty} m(A_n) < \infty$. Since

$$A \subset \bigcup_{m}^{\infty} A_n,$$

then using the countable (sub)-additivity of measure

$$m(A) \leq m(\bigcup_{m=1}^{\infty} A_n) \leq \sum_{m=1}^{\infty} m(A_n) < \epsilon.$$

□

Theorem 19. (Markov's Inequality) Let $E \subseteq \mathbb{R}^n$ be measurable, and suppose $f : E \to \mathbb{R}$ is non-negative and measurable. Show that for all $c > 0$,

$$m(\{x \in E \mid f(x) \geq c\}) \leq \frac{1}{c}\int_E f.$$

Proof. Let $C = \{x \in E \mid f(x) \geq c\}$. Define

$$g := f\chi_C.$$

We know that $g \leq f$ on E and thus

$$\int_E f \geq \int_E g = \int_C f \geq \int_C c = c\, m(C).$$

The result follows from dividing by c. □

Theorem 20. Let $f : [0, 1] \to \mathbb{R}$ be integrable. Prove that

$$\int_{[0,1]} x^k f(x)\, dm \to 0$$

Proof. Let $f_k = x^k f$. Define

$$g(x) = \begin{cases} f(x) & x = 1 \\ 0 & x = 0 \end{cases}$$

We have that $f_k \to g$ since $x \in [0, 1]$ and $|f_k| \leq f$ since $x \in [0, 1]$. We use DCT to show that

$$\int_{[0,1]} f_k \to \int_{[0,1]} g = \int_{[0,1]\setminus\{1\}} 0 = 0$$

□

Theorem 21. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative integrable function. Prove that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if A is measurable with $m(A) < \delta$, then

$$\int_A f < \epsilon$$

Proof. First we show this for simple functions. Let φ be a simple non-negative measurable function . Consider that

$$\int_A \varphi = \sum_{k=1}^{N} c_k m(A_k \cap A) \leq \sum_{k=1}^{N} c_k m(A) \leq N\, c_{(N)} m(A).$$

Letting $\delta = \frac{\epsilon}{c_{(N)} N}$ yields the result.

By definition, there is some φ such that

$$\int_A \varphi \geq \int_A f - \frac{\epsilon}{2}.$$

Hence,

$$\int_A f \leq \int_A \varphi + \frac{\epsilon}{2}.$$

Choose A such that the above holds for this choice of φ, then the result follows. $\square$

1.14 Friday, Apr 25: Introduction to Differential Forms

Remark 19. (Recall) Let $f : E \to \mathbb{R}$ and $E \subseteq \mathbb{R}^n$ be open. Then the partial $D_1 f, \dots, D_n f$ are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}$. If the partials are all differentiable, then the second order derivatives of f are defined by

$$D_{ij} f = D_i D_j f, \quad i, j \in \{1, 2, \dots, n\}.$$

If these second order partials are continuous in E, then we say f is C^2.

Theorem 22. (Symmetry of the Hessian) If $f \in C^2(E)$, then $D_{ij} f = D_{ji} f$.

Suppose now $f : E \to \mathbb{R}^n$ and f is differentiable for some $x \in E$. The determinant of the linear operator $(Df)(x)$ is called the **Jacobian of** f **at** x and is defined by

$$J_f(x) = \det((Df)(x)) = \frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)}, \quad f(x_i) = y_i.$$

Definition 27. Let $k \in \mathbb{N}$. A **k-cell** in $\mathbb{R}^k$ is the set of points $I^k \ni x = (x_1, \dots, x_k)$ such that $a_i \le x \le b_i$ for all $i = 1, 2, \dots, k$.

Note that a $k-$cell is simply a closed rectangle in $\mathbb{R}^k$.

Remark 20. Suppose I^k is a $k-$cell in $\mathbb{R}^k$, $f : I^k \to \mathbb{R}$ is continuous. For every $j \le k$, let I^j be the restriction of I^k to the first j components. Define

$$g_k : I^k \to \mathbb{R}, \quad g_k := f$$

$$g_{k-1} : I^{k-1} \to \mathbb{R}, \quad g_{k-1}(x_1, \dots, x_{k-1}) := \int_{a_k}^{b_k} g_k(x_1, \dots, x_k) dx_k.$$

Since g_k is uniformly continuous on I^k, we know that g_{k-1} is (unif) continuous on I^k. Define

$$g_{k-2} : I^{k-2} \to \mathbb{R}, \quad g_{k-2}(x_1, \dots, x_{k-2}) := \int_{a_{k-1}}^{b_{k-1}} g_k(x_1, \dots, x_{k-1}) dx_{k-1}.$$

Again, g_{k-2} is (unif) continuous. We can repeat this process until $k = 0$, at which point we arrive at a number

$$g_0 := \int_{a_1}^{b_1} g_1(x_1) dx_1.$$

Definition 28. Using the notation from the above remark, we say that g_0 **is the iterated integral of** f **over** I^k, and write

$$\int_{I^k} f(x) dx = g_0$$

Example 1.3. Let $I^2 = [1, 2] \times [0, 1]$. Define $f : I^2 \to \mathbb{R}$ as $f(x_1, x_2) = 2x_1 x_2^2$. Then

$$g_1(x_1) = \int_0^1 g_2(x_1, x_2) dx_2 = \int_0^1 2x_1 x_2^2 dx_2 = 2x_1 \int_0^1 x_2^2 dx_2 = \frac{2}{3} x_1$$

$$g_0 = \int_1^2 g_1(x_1) dx_1 = \int_1^2 \frac{2}{3} x^1 dx_1 = 1$$

Clearly,

$$\int_{I^2} f dx = \int_1^2 \left(\int_0^1 2x_1 x_2^2 \right) dx_1$$

Proposition 4. The iterated integral is the same as the multivariate integral. That is,

$$\int_{I^k} f dx,$$

The result is important, and shows the order of the iterated integral does not matter by Fubini's theorem.

Definition 29. Suppose $f : \mathbb{R}^k \to \mathbb{R}$. The **support** of f defined to be

$$\text{supp}(f) = \overline{\{x \in \mathbb{R}^k \mid f(x) \neq 0\}}.$$

Definition 30. If $f : \mathbb{R}^k \to \mathbb{R}$ is continuous with compact support. Let I^k be a $k-$cell containing supp(f). Then we define

$$\int_{\mathbb{R}^k} f dx := \int_{I^k} f dx.$$

Theorem 23. (Change of Variables) Let $T \in C^1(E, \mathbb{R}^n)$ be bijective and has nonzero Jacobian everywhere on E, where $E \subseteq \mathbb{R}^n$ is open. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $\mathbb{R}^n$ and has compact support and contained in $T(E)$, then

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} f(T(x))|J_T(x)|dx$$

1.15 Monday, Apr 28: Differential 1−forms

Remark 21. A 1−*form* in $\mathbb{R}^n$ is:

(a) any object which can be integrated on any curve in $\mathbb{R}^n$.

(b) A rule assigning a real number to every orientated line segment in $\mathbb{R}^n$ in a suitable way

Definition 31. Let $p \in \mathbb{R}^n$. The **tangent space** to $\mathbb{R}^n$ at p is the set $T_p\mathbb{R}^n : \{(p, v) \mid v \in \mathbb{R}^n\}$

Remark 22. If α is a 1−form and $p \in \mathbb{R}^n$, then we write α_p to denote the restriction of α to $T_p\mathbb{R}^n$. Thus, $\alpha_p(v)$ is the value α assigns to the oriented line segment from p to $p + v$.

Moreover, we require that α_p is a linear functional for all $p \in \mathbb{R}^n$. That is

(a) $\alpha_p(tv) = t\alpha_p(v)$.

(b) $\alpha_p(v + w) = \alpha_p(v) + \alpha_p(w)$

With differential forms, we denote that projection maps in $\mathbb{R}^n$ by $dx_1, \ldots, dx_m$, where

$$dx_1(v) = dx_i(v_1, \ldots, v_n) = v_i$$

These form a basis for the set of linear functional, and so for any 1−form α, its restriction α_p can be written as

$$\alpha_p = A_1dx_1 + A_2dx_2 + \cdots + A_ndx_n = A_1(p)dx_1 + \cdots + A_n(p)dx_n,$$

where the $A_i(p)$ must be sufficiently continuous w.r.t. p.

Definition 32. A **differential 1−form** α on $\mathbb{R}^n$ is a map from every tangent vector (p, v) in $\mathbb{R}^n$ which can be expressed in the form

$$\alpha = f_1dx_1 + f_2dx_2 + \cdots + f_ndx_n,$$

where $f_i \in C^2(\mathbb{R}^n, \mathbb{R})$

Example 1.4. Suppose $\alpha = y\,dx + dz = y\,dx_1 + dx_3$ on $\mathbb{R}^3$ with $p = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $v = \begin{pmatrix}4\\5\\6\end{pmatrix}$ and

$$\alpha_p(v) = f_1(p)dx_1(v) + f_2(p)dx_2(v) + f_2(p)dx_3(v) = 2 \cdot 4 + 0 + 1 \cdot 6 = 14$$

Remark 23. A **curve** (1-surface) in $\mathbb{R}^n$ is a C^1−mapping $\gamma : [a, b] \to \mathbb{R}^n$.

Example 1.5. Suppose we want to integrate a 2−form $\alpha = f_1dx_1 + f_2dx_2$ over a curve, $\gamma : [0, 1] \to \mathbb{R}^2$. Partition $[0, 1]$ by $0 = t_0 < t_1 < t_2 < t_3 = 1$. Define

$$L_i := \gamma'(t_{i-1})(t_i - t_{i-1}).$$

By Taylor's Theorem, $L_i \approx \gamma(t_i) - \gamma(t_{i-1})$

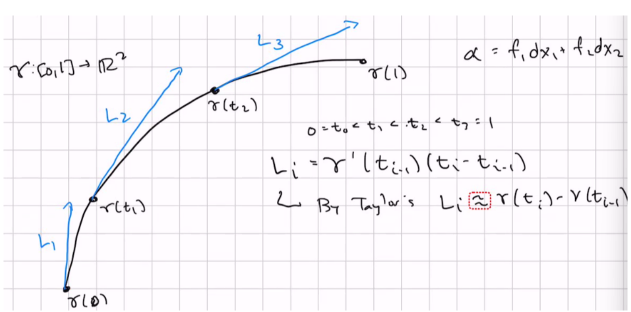


Figure 1: Visualizing L_i

Then for k large,

$$\begin{aligned}
\sum_{i=1}^{k} \alpha(L_i) &= \sum_{i=1}^{k} \alpha_{\gamma(t_{i-1})}\big(\gamma'(t_{i-1})(t_i - t_{i-1})\big) \\
&= \sum_{i=1}^{k} f_1(\gamma(t_{i-1}))\gamma_1'(t_{i-1})(t_i - t_{i-1}) + f_2(\gamma(t_{i-1}))\gamma_2'(t_{i-1})(t_i - t_{i-1}) \\
&\to \int_0^1 (f_1(\gamma(t)\gamma_1'(t) + f_2(\gamma(t))\gamma_2'(t))\, dt
\end{aligned}$$

Definition 33. Let $\alpha = f_1 dx_1 + \cdots + f_n dx_n$ be a $1-$form in $\mathbb{R}^n$, let $\gamma : [a,b]\mathbb{R}^n$ be C^1. Then we define the integral of α over γ to be

$$\int_\gamma \alpha = \int_a^b \left(f_1(\gamma(t))\gamma_1'(t) + \cdots + f_n(\gamma(t))\gamma_n'(t)\right)\, dt$$

Example 1.6. Let $\alpha = x^2 dx_1 + dx_2$ on $\mathbb{R}^2$ and $\gamma : [0,1] \to \mathbb{R}^2$ be $\gamma(t, t^2)$. Then

$$\gamma_1'(t) = 1, \quad \gamma_2'(t) = 2t.$$

Thus,

$$\begin{aligned}
\int_\gamma \alpha &= \int_0^1 \left(f_1(\gamma(t))\gamma_1'(t) + f_2(\gamma(t))\gamma_t'(t)\right) dt \\
&= \int_0^1 \left(t^2 + t^2 \cdot 2t\right) dt \\
&= \int_0^1 t^2\, dt + 2\int_0^1 t^3\, dt \\
&= \frac{4}{3}
\end{aligned}$$

1.16 Wednesday, Apr 30: Differential 2−Forms

A 2−surface is a C^1−map $\gamma : I^2 \to \mathbb{R}^n$.

Remark 24. Informally a 2−*form* is

(a) an object which can be integrated over any 2−surface.

(b) a rule which assigns a real number to every orientated parallelogram in $\mathbb{R}^n$ in a suitable way.

Note that we specify any orientate parallelogram in $\mathbb{R}^n$ based at some $p \in \mathbb{R}^n$ by giving an ordered pair (v, w). A 2−form, ω, should satisfy:

(a) (Bilinear) $\omega_p(tv_1, v_2) = t\omega_p(v_1, v_2) = \omega_p(v_1, tv_2)$.

(b) (Bilinear)

$$\omega_p(v_1, v_2 + v_3) = \omega_p(v_1, v_2) + \omega_p(v_1, v_3)$$

and

$$\omega_p(v_1 + v_2, v_3) = \omega_p(v_1, v_2 + v_3)$$

(c) (Asymmetric)

$$\omega_p(v_1, v_2) = -\omega_p(v_2, v_1)$$

Note that (c) implies that $\omega_p(v, v) = 0$.

Definition 34. For any $v, w \in \mathbb{R}^n$, we denote a **basic** 2−**form** by

$$(dx_i \wedge dx_j)(v, w) = \det \begin{pmatrix} v_i & w_i \\ v_j & w_j \end{pmatrix}$$

Intuitively, a 2−form is the orientated area of the parallelogram's shadow on the (i, j) plane!

Remark 25. If ω_p satisfies (a,b,c), then ω_p can be expressed as

$$\omega_p = \sum_{i,j} A_{i,j}(p)(dx_i \wedge dx_j)$$

Definition 35. A **differential 2-form** in $\mathbb{R}^n$ is a rule assigning a real number to each oriented parallelogram in $\mathbb{R}^n$ that can be written as

$$\omega = \sum_{i,j} f_{i,j}(dx_1 \wedge dx_j),$$

where $f_{i,j} \in C^2(\mathbb{R}^n, \mathbb{R})$. Thus, for any $p \in \mathbb{R}^n$, $v, w \in \mathbb{R}^n$,

$$\omega_p(v, w) = \sum_{i,h} f_{i,j}(p)(dx_i \wedge dx_j)(v, w)$$

Example 1.7. Let ω be a two form in $\mathbb{R}^3$. Then

$$\begin{aligned}\omega &= f_{1,1}(dx_1 \wedge dx_1) + f_{1,2}(dx_1 \wedge dx_2) + f_{2,1}(dx_2 \wedge dx_1) + f_{2,2}(fx_2 \wedge dx_2) \\ &= f_{1,2}(dx_1 \wedge dx_2) + f_{2,1}(dx_2 \wedge dx_1) \\ &= (f_{1,2} - f_{2,1})(dx_1 \wedge dx_2)\end{aligned}$$

Thus, any ω 2−form in $\mathbb{R}^2$ can be written as $\omega = f(dx_1 \wedge dx_2)$.

Example 1.8. Let ω be a two form in $\mathbb{R}^3$, then

$$\omega = f_1(dx_1 \wedge dx_2) + f_2(dx_1 \wedge dx_3) + f_3(dx_2 \wedge dx_3)$$

Definition 36. Let γ be a C^1 $2-$surface in $\mathbb{R}^3$, and ω be a $2-$form in $\mathbb{R}^3$. Then the integral of ω over γ is

$$\begin{aligned}
\int_\gamma \omega &= \int_{I^2} \omega_{\gamma(z)}(\frac{\partial\gamma}{\partial x_1}(z), \frac{\partial\gamma}{\partial x_2}(z))\, dz \\
&= \int_{I^2} \omega_{\gamma(z)} \left(\begin{pmatrix} D_1\gamma_1(z) \\ D_1\gamma_2(z) \\ D_1\gamma_3(z) \end{pmatrix}, \begin{pmatrix} D_2\gamma_1(z) \\ D_2\gamma_2(z) \\ D_2\gamma_3(z) \end{pmatrix} \right) dz \\
&= \int_{I^2} f_1(\gamma(z))(dx_1 \wedge dx_2) \left(\frac{\partial\gamma}{\partial x_1}(z), \frac{\partial\gamma}{\partial x_2}(z) \right) + f_2(\gamma(z))(dx_1 \wedge dx_3) \left(\frac{\partial\gamma}{\partial x_1}(z), \frac{\partial\gamma}{\partial x_2}(z) \right) + f_3(\gamma(z))(dx_2 \wedge dx_3) \left(\frac{\partial\gamma}{\partial x_1}(z), \right. \\
&= \int_{I^2} f_1(\gamma(z)) \det \begin{pmatrix} D_1\gamma_1(z) & D_2\gamma_1(z) \\ D_1\gamma_2(z) & D_2\gamma_2(z) \end{pmatrix} + \ldots\, dz
\end{aligned}$$

1.17 Friday, May 2: Differential $k-$Forms

> God grant me the serenity to accept the things I cannot change, courage to change the things I can, and wisdom to know the difference.

Definition 37. A $k-$surface in $\mathbb{R}^n$ is a C^1-map $\phi : D \to \mathbb{R}^n$ where D is a $k-$cell

Remark 26. A $k-$form on $\mathbb{R}^n$ is a rule which assigns a real number to every orientated $k-$dimensional parallelepiped in $\mathbb{R}^n$ in a suitable way.

Note that we specify any orientated parallelepiped in $\mathbb{R}^n$ based at some $p \in \mathbb{R}^n$ by giving an ordered list $(v_1, \ldots, v_k) \in T_p(\mathbb{R}^n)$ A $k-$form, ω, should satisfy:

(a) ($k-$tensor)

$$\omega_p(v_1, \ldots, tv_i, \ldots, v_k) = t\omega_p(v_1, \ldots, v_k)$$

(b) ($k-$tensor)

$$\omega_p(v_1, \ldots, v_i + w, \ldots, v_k) = \omega_p(v_1, \ldots, v_k) + \omega_p(v_1, \ldots, w, \ldots, v_k)$$

(c) (Asymmetric)

$$\omega_p(v_1, \ldots, v_i, \ldots, v_j \ldots, v_k) = -\omega_p(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

Definition 38. A **multi-index** of length k in $\mathbb{R}^n$ is a list $I = (i_1, \ldots, i_k)$ such that $i_j \in [1, n] \cap \mathbb{N}$.

Definition 39. Let $I = (i_1, \ldots, i_k)$ be a multi-index in $\mathbb{R}^n$. For any $v, w \in \mathbb{R}^n$, we denote a **basic $k-$form** by

$$dx_I(v^1, \ldots, v^k) = (dx_{i_1} \wedge \cdots \wedge dx_{i_k})(v^1, \ldots, v^k) = \det \begin{pmatrix} v^1_{i_1} & v^2_{i_1} & \cdots & v^k_{i_1} \\ v^1_{i_2} & v^2_{i_2} & \cdots & v^k_{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ v^1_{i_k} & v^2_{i_k} & \cdots & v^k_{i_k} \end{pmatrix}$$

Remark 27. (a) If I contains a repeated index, then $dx_I(v^1, \ldots, v^k) = 0$ since the columns are not linearly independent.

(b) If J is attained by swapping a single pair of indices in I, then

$$dx_I(v^1, \ldots, v^k) = -dx_J(v^1, \ldots, v^k)$$

Definition 40. A **differential $k-$form** in $\mathbb{R}^n$ is a rule assigning a real number to each oriented parallelepiped in $\mathbb{R}^n$ that can be written as

$$\omega = \sum_I f_I dx_I,$$

where the sum is taken over all the multi-index I of length k in $\mathbb{R}^n$ and each $f_I : \mathbb{R}^n \to \mathbb{R}$ is C^2. Thus, if $p \in \mathbb{R}^n$, $v_1, \ldots, v_k \in \mathbb{R}^n$, then

$$\omega_p(v_1, \ldots, v_k) = \sum_I f_I(p) dx_I(v_1, \ldots, v_k).$$

Definition 41. Let $\phi : D \to \mathbb{R}^n$ be a $k-$surface in $\mathbb{R}^n$, and let ω be a $k-$form. Then the **integral** of ω over ϕ is

$$\begin{aligned} \int_\phi \omega &= \int_D \omega_{\phi(z)} \begin{pmatrix} \frac{\partial \phi}{\partial u_1}(u) & \cdots & \frac{\partial \phi}{\partial u_k}(u) \end{pmatrix} du \\ &= \int_D \sum_I f_I(\phi(u)) dx_I \begin{pmatrix} \frac{\partial \phi}{\partial u_1}(u) & \cdots & \frac{\partial \phi}{\partial u_k}(u) \end{pmatrix} du \\ &= \int_D \sum_I f_I(\phi(u)) J(\phi(u))\, du \end{aligned}$$

Example 1.9. Let $\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$ be a $2-$form in $\mathbb{R}^3$. Define the $3-$surface to be $\phi : [0,3] \times [0,2\pi] \to \mathbb{R}^3$ such that $\phi(r,\theta) = (r\cos\theta, r\sin\theta, 5)$ to be a disk of radius r sitting at $z = 5$. Let $I_1 = \{2,3\}$, $I_2 = \{1,3\}$ and $I_3 = \{1,2\}$. Then

$$f_{I_1}(x,y,z) = x, \quad f_{I_2}(x,y,z) = -y, \quad f_{I_3}(x,y,z) = z.$$

Then

$$\frac{\partial \phi}{\partial r} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \qquad \frac{\partial \phi}{\partial \theta} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix}$$

Then

$$\begin{aligned} \int_\phi \omega &= \int_D f_{I_1}\phi(r,\theta)(dy \wedge dz)(\frac{\partial \phi}{\partial r}, \frac{\partial \phi}{\partial \theta}) + \ldots \, du \\ &= \int_D r\cos\theta \cdot 0 + (-r\sin\theta) \cdot 0 + 5r \, du \\ &= \int_0^3 \left(\int_0^{2\pi} 5r d\theta \right) dr \\ &= 45\pi \end{aligned}$$

1.18 Monday, May 5: The Wedge Product

Definition 42. Suppose $I = (i_1, \dots, i_k)$ is a multi-index such that $i_1 < i_2 < \cdots < i_k$. We call I an **increasing multi-index** and say that

$$dx_I = dx_{i_1} \wedge \cdots dx_{i_k}$$

is a **basic $k-$form in** $\mathbb{R}^n$

Remark 28. The basic $k-$forms form a basis for the $k-$forms. That is, if ω is a $k-$form in $\mathbb{R}^n$, then

$$\omega = \sum_I b_I dx_I.$$

We note that the space of $k-$forms in $\mathbb{R}^n$ is a vector space and denote it by $\Delta^k(\mathbb{R}^n)$. The space has dimension $\binom{n}{k}$.

Example 1.10. Consider the $k-$form $dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_3$. Then

$$\omega = -dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

Definition 43. For any $\omega = \sum_I a_I dx_I$, we can convert each I into an increasing multi-index J such that $\omega = \sum_J b_J dx_J$. We call this the **standard presentation of** J.

Definition 44. Suppose $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$ are increasing multindex. Then the **wedge product** of the corresponding forms is the $(p+q)$ form equal to

$$dx_I \wedge dx_J := dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{jq}.$$

Remark 29. If I and J have no element in common (if they have one in common the wedge product is clearly 0), we denote the increasing $(p+q)-$index obtained by arranging the elements of $I \cup J$ in increasing order by $[I, J]$. Thus,

$$dx_I \wedge dx_J = (-1)^\alpha dx_{[I,J]},$$

where α is the number of swaps needed to arrange the union in increasing order.

Example 1.11. Suppose $\omega \in \Delta^p(\mathbb{R}^n)$ and $\lambda \in \Delta^q(\mathbb{R}^n)$. Then

$$\omega = \sum_I b_I dx_I, \quad \lambda = \sum_J c_J dx_J$$

and the wedge product is the $(p+q)-$form in $\mathbb{R}^n$ such that

$$\omega \wedge \lambda = \sum_{I,J} b_I c_J (x_I \wedge dx_J)$$

Remark 30. Suppose $\omega_1, \omega_2, \lambda$ are all forms in $\mathbb{R}^n$, then

(a) (Distribution)

$$(\omega_1 + \omega_2) \wedge \lambda = \omega_1 \wedge \lambda + \omega_2 \wedge \lambda$$

(b) (Distribution)

$$\omega \wedge (\lambda_1 + \lambda_2) = \omega \wedge \lambda_1 + \omega \wedge \lambda_2$$

$$(\omega \wedge \lambda) \wedge \sigma = \omega \wedge (\lambda \wedge \sigma)$$

Definition 45. A 0**-form** in $\mathbb{R}^n$ is a $C^1(\mathbb{R}^n)$ function.

Remark 31. The wedge product of a 0 form with some $\omega \in \Delta^k(\mathbb{R}^n)$ is

$$f\omega := \omega \wedge f = f \wedge \omega =: \omega f = \sum_I f b_I dx_I$$

Remark 32. Informally, the differential operator assigns a $(k+1)-$form $d\omega$ to each $\omega \in \Delta^k(\mathbb{R}^n)$. Suppose $f : E \to \mathbb{R}$ is a $0-$form where $f \in \Delta^0(\mathbb{R}^n)$. Then

$$df = D_1 f dx_1 + \cdots + D_n f dx_n.$$

Suppose $\omega \in \Delta^k(\mathbb{R}^n)$ such that $\omega = \sum_I b_I dx_I$. Then

$$d\omega = \sum_I (db_I) \wedge dx_I$$

Example 1.12. Let $\omega = xz\, dx + y^2 dz \in \Delta^2(\mathbb{R}^3)$. Then

$$d\omega = (z\, dx \wedge dx + 0 + \ x\, dz \wedge dx) + (0 + 2y\, dy \wedge dz + 0) = -x\, dx \wedge dz + 2y\, dy \wedge dz.$$

Moreover, it can be seen that

$$d(d\omega) = 0$$

Example 1.13. Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the projection map of the ith component. Then

$$d\pi_i = D_1 \pi_1 dx_1 + \cdots + D_n \pi_n dx_n = dx_i.$$

1.19 Wednesday, May 7: Properties of the Exterior Derivative

Theorem 24. (a) (Graded Product Rule) Let $\omega \in \Delta^k(\mathbb{R}^n)$ and let $\lambda \in \Delta^m(\mathbb{R}^m)$ both of class C^1. Then

$$d(\omega \wedge \lambda) = d\omega \wedge \lambda + (-1)^k(\omega \wedge d\lambda)$$

(b) Let $\omega \in \Delta^k(\mathbb{R}^n)$ of class C^2, then $d(d\omega) = 0$.

Proof. (a) By the distributive properties of the exterior derivative, it is enough to show this for $\omega = f dx_I$ and $\lambda = dx_J$, where $f, g \in C^1$. Then recall that $\omega \wedge \lambda = fg\, dx_I \wedge dx_J$. Recall as well that $d(fg) = g\,df + f\,dg$

If I and J have any common indices, then the result trivially holds. Thus, take I disjoint from J. Computing, noting the fifth equality comes from an identity in the homework,

$$\begin{aligned} d(\omega \wedge \lambda) &= d\left(fg\, dx_I \wedge dx_J\right) \\ &= (-1)^\alpha d(fg\, dx_{[I,J]}) \\ &= (-1)^\alpha \left(f\, dg + g\, df\right) \wedge dx_{[I,J]} \\ &= (f\, dg + g\, df) \wedge dx_I \wedge dx_J \\ &= (-1)^k f \wedge dx_I \wedge dg \wedge dx_J + df \wedge dx_I \wedge g \wedge\ dx_J \\ &= (-1)^k f dx_I \wedge dg \wedge dx_J + df \wedge dx_I \wedge g\, dx_J \\ &= (-1)^k(\omega d\lambda) + (d\omega \wedge \lambda) \end{aligned}$$

(b) We note that $d(dx_I) = 0$ since by (a) we have $d(1 \cdot dx_I) = d(1) \wedge dx_I = 0$. Thus, let $f \in C^2(\mathbb{R}^n)$. We have that

$$d^2(f) = d\left(\sum_{i=1}^n (D_i(f))dx_i\right) = d\left(D_1(f)dx_1 + \cdots + D_n(f)dx_n\right) = \sum_{i=1}^n \left(\sum_{j=1}^n D_{ij}(f)dx_j\right) \wedge dx_i = 0.$$

Thus, we see that by part (a),

$$d(f \wedge dx_I) = df \wedge dx_I + (-1) f d(dx_I) = 0$$

□

Definition 46. We say that $\omega \in \Delta^k(\mathbb{R}^n)$ is **closed** if $d\omega = 0$. We say that ω is **exact** if there exists some $\alpha \in \Delta^{k-1}(\mathbb{R}^n)$ such that $d\alpha = \omega$

Remark 33. Suppose $\omega \in \Delta^k(\mathbb{R}^n)$

(a) Since $d(d(\omega)) = 0$, then $d\omega$ is an exact form. Thus, every exact form is closed.

(b) Not every closed form is exact. This is shown in PSET 6 question 3.

1.20 Friday, May 9: Pullbacks

For this class, we let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$ be open, and $T : E \to F$ be C^1. Let $\omega \in \Delta^k(F)$, such that

$$\omega = \sum_I f_I(y)\, dy_I$$

If $T(x) = (t_1(x), \dots, t_m(x)) = (y_1, \dots, y_m) = \mathbf{y}$, then

$$dt_i = \sum_{j=1}^{n} (D_j t_i)\, dx_j$$

is a one form on E. Thus, T transforms ω by pulling it back to E.

Definition 47. The **pullback form** of ω is

$$\omega_T(x) = \sum_I f_I(T(x))\, dt_I$$

Example 1.14. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the identity mapping. Let $\omega = \sum_I f_I\, dx_I$. Then $t_i(x) = x_i$,

$$dt_i = D_1 x_i\, dx_1 + \cdots + D_n x_i\, dx_n = dx_i,$$

and so

$$\omega_T(x) = \sum_I f_I(x)\, dx_I$$

Example 1.15. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$(x_1, x_2) \mapsto (x_2, x_1^2, x_1 + x_2) = (t_1, t_2, t_3).$$

Let $\omega \in \Delta^2(\mathbb{R}^3)$ such that

$$\omega(y_1, y_2, y_3) = y_1\, dy_2 \wedge dy_3.$$

Computing, we see that

$$dt_1 = dx_2, \quad dt_2 = 2x_1\, dx_1, \quad dt_3 = dx_1 + dx_2,$$

and so

$$\begin{aligned}
\omega_T(x_1, x_2) &= f_{2,3}(T(x_1, x_2))\, (dt_2 \wedge dt_3) \\
&= f_{2,3}(x_2, x_1^2, x_1 + x_2)\, (2x_1\, dx_1) \wedge (dx_1 + dx_2) \\
&= (2x_1 x_2)\, (dx_1 \wedge dx_1 + dx_1 \wedge dx_2) \\
&= 2x_1 x_2\, dx_1 \wedge dx_2
\end{aligned}$$

Lemma 2. Let $f \in C^1(F, \mathbb{R})$. Then if we call $f_T = f \circ T$,

$$d(f_T) = (df)_T.$$

Proof. Follow your nose and use chain rule

$$\begin{aligned} d(f_T) &= \sum_{j=1}^{n} D_j f_T \, dx_j \\ &= \sum_{j=1}^{n} D_j (f \circ T) dx_j \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} (D_i f_i)(T) \cdot D_j t_i \, dx_j \\ &= \sum_{i=1}^{m} (D_i f)(T) dt_i \\ &= (df)_T \end{aligned}$$

□

Theorem 25. Let $\omega \in \Delta^k(F)$ and $\lambda \in \Delta^l(F)$. If $T : E \to F$ is C^1, then

(a) if $k = l$, $(\omega + \lambda)_T = \omega_T + \lambda_T$

(b) $(\omega \wedge \lambda)_T = \omega_T \wedge \lambda_T$

(c) $d(\omega_T) = (d\omega)_T$ if $T \in C^2(E, F)$

Proof. (i) Let $\omega = \sum_I f_I dy_I$ and $\lambda = \sum_I g_I dy_I$. Then

$$\begin{aligned} (\omega + \lambda)_T &= \left(\sum_I (f_I + g_I) \, dy_I \right)_T \\ &= \sum_I (f_I + g_I)(T) \, dt_I \\ &= \omega_T + \lambda_T \end{aligned}$$

The proof for (ii) is on PSET 7.

(iii) Suppose $T \in C^2(E, F)$. First consider the case when

$$\omega = dy_{i_1} \wedge \cdots \wedge dy_{i_k}, \quad \omega_T = dt_{i_1} \wedge \cdots \wedge dt_{i_k}.$$

We use the graded product rule to easily conclude that

$$d\omega = 0 = d(\omega_T),$$

and so $(d\omega)_T = 0$. For the general case, use the previous lemma. □

1.21 Monday, May 12: Change of Variables

For this class, we let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$ be open.

Theorem 26. If $T \in C^1(E, F)$, and $S \in C^1(F, G)$, where $G \subseteq \mathbb{R}^\ell$, and $\omega \in \Delta^k(G)$. Then

$$(\omega_S)_T = \omega_{S \circ T}$$

Remark 34. As a remark, note that $\omega_S \in \Delta^k(G)$ and $(\omega_S)_T, \omega_{S\circ T} \in \Delta^k(E)$.

Theorem 27. Suppose $\omega \in \Delta^k(E)$, and ϕ is a $k-$surface in E with parameter domain $D \subseteq \mathbb{R}^k$. If Δ is the trivial $k-$surface, $\Delta : D \to \mathbb{R}^k$, where $\Delta(u) = u$, then

$$\int_\phi \omega = \int_\Delta \omega_\phi$$

Proof. It suffices to show this for the case when

$$\omega = f\, dx_I = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Let $\phi_i, \ldots, \phi_n$ denote the components of ϕ. Then

$$\omega_\phi = \sum_I f_I(\phi)\, d\phi_I = f(\phi)\, d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}.$$

It suffices to show that

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = J(u)\, du_1 \wedge \cdots \wedge du_k, \tag{1}$$

where $J(u) = \frac{\partial(x_1, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)}$. Assuming (1),

$$\begin{aligned}\int_\Delta \omega_\phi &= \int_\Delta f(\phi)\, d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} \\ &= \int_\Delta f(\phi) J(u)\, du_1 \wedge \cdots \wedge du_k \\ &= \int_D f(\phi(u)) J(u) du \\ &= \int_\phi \omega.\end{aligned}$$

To prove (1), let $[A]$ be the $k \times k$ matrix with entries

$$\alpha(p, q) = (D_q \phi_p)(u)$$

for all $p, q = 1, \ldots, k$. Note that $\det A = J(u)$. Since

$$d\phi_{i_p} = \sum_q \alpha(p, q)\, du_q$$

Thus,

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = \sum \alpha(1, q_1) \cdots \alpha(k, q_k)\, du_{q_1} \wedge \cdots \wedge du_{q_k},$$

where the sum ranges over all the $q_1, \ldots, q_k \in \{1, \ldots, k\}$. In order to get the wedge product into standard presentation, we rearrange $du_{q_1} \wedge \cdots \wedge du_{q_k}$ to get

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = (\det A)\, du_1 \wedge \cdots \wedge du_k = J(u)\, du_1 \wedge \cdots \wedge du_k$$

$\square$

Example 1.16. Consider when $k = 2$.

$$A = \begin{bmatrix} \alpha(1,1) & \alpha(1,2) \\ \alpha(2,1) & \alpha()2,2 \end{bmatrix}$$

Then

$$\begin{aligned} d\phi_{i_1} \wedge d\phi_{i_2} &= \alpha(1,1)\alpha(2,1)\, du_1 \wedge du_1 + \alpha(1,1)\alpha(2,2)\, du_1 \wedge du_2 \\ &+ \alpha(1,2)\alpha(2,1)\, du_2 \wedge du_1 + \alpha(2,2)\alpha(1,2)\, du_2 \wedge du_2 \\ &= (\alpha(1,1)\alpha(2,2) - \alpha(1,2)\alpha(2,1))\, du_1 \wedge du_2 \\ &= \det(A)\, du_1 \wedge du_2 \end{aligned}$$

Theorem 28. (Change of Variables) Suppose $T \in C^1(E, F)$ and ϕ is a $k-$surface in E. If $\omega \in \Delta^k(F)$, then

$$\int_{T\circ\phi} \omega = \int_\phi \omega_T$$

Proof. Let D be the parameter domain of ϕ (and thus of $T \circ \phi$). Let $\Delta : D \to D$ be the identity map on D such that $\Delta(u) = u$. Then by Theorem 21 and Theorem 20 and Theorem 21 again,

$$\int_{T\circ\phi} \omega = \int_\Delta \omega_{T\circ\phi} = \int_\Delta (\omega_T)_\phi = \int_\phi \omega_T$$

□

Definition 48. A map $f : X \to Y$, where X, Y are vector spaces, is called **affine** if $f - f(0)$ is linear.

Remark 35. In other words, f is affine if

$$f(x) = f(0) + Ax, \quad A : X \to Y \text{ is linear.}$$

An affine map from $\mathbb{R}^k \to \mathbb{R}^n$ is determined by $f(0)$ and its value for each $f(e_i)$.

Definition 49. The $k-$**simplex** in $\mathbb{R}^k$ is $Q^k \subseteq \mathbb{R}^k$ such that

$$Q^k := \{x = (x_1, \ldots, x_k) \mid x_i \geq 0, \sum_{i=1}^k x_i \leq 1\}$$

Remark 36. The one simplex is $[0, 1]$. The two-simplex is the right triangle with endpoints $(0, 0), (0, 1), (1, 0)$.

1.22 Wednesday, May 14: Oriented Simplexes

Definition 50. Let $p_0, p_1, \ldots, p_k \in \mathbb{R}^n$. The **oriented affine** $k-$**simplex** $\sigma = [p_0, \ldots, p_k]$ is the $k-$surface in $\mathbb{R}^n$ with parameter domain Q^k given by the affine map

$$\sigma(\alpha_1 e_1, \ldots, \alpha_k e_k) = p_0 + \sum_1^k \alpha_i(p_i - p_0)$$

Remark 37. For all $u \in Q^k$, we can write $\sigma(u) = p_0 + Au$, where $A \in L(\mathbb{R}^k, \mathbb{R}^n)$ such that $Ae_i = p_i - p_0$.

For intuition, σ is a map from the endpoints of Q^k to the points $p_0, \ldots, p_k$.

σ is called oriented to emphasize that the order of the points $p_0, \ldots, p_k$ matters. To illustrate this, consider $\overline{\sigma} = [p_{i_0}, \ldots, p_{i_k}]$, where $\{i_0, \ldots, i_k\}$ is a permutation of $\{0, 1, \ldots, k\}$. Then

$$\overline{\sigma} = s(i_0, \ldots, i_k)\sigma, \quad s(\vec{i_j}) = (-1)^\alpha, \alpha \text{ is min \# swaps needed to permute } i_0, \ldots, i_k \text{ to } 0, \ldots, k$$

Suppose $\overline{\sigma} = \varepsilon\sigma$, where $\varepsilon = \pm 1$. If $\epsilon = 1$, we say that $\overline{\sigma}$ and σ have the same orientation. Otherwise, we say that they have opposite orientations.

Definition 51. An **oriented** 0**-simplex** is a point $p_0 \in \mathbb{R}^n$ with a sign attached. We write $\sigma = \pm p_0$. If f is a $0-$form, $\sigma = \varepsilon p_0$, where $\varepsilon = \pm 1$, then

$$\int_\sigma f = \varepsilon f(p_0).$$

Theorem 29. If σ is an oriented $k-$simplex in $E \subseteq \mathbb{R}^n$ open and if $\overline{\sigma} = \varepsilon\sigma$, $\varepsilon = \pm 1$, then for all $\omega \in \Lambda^k(E)$ is given by

$$\int_\sigma \omega = \varepsilon \int_{\overline{\sigma}} \omega$$

Definition 52. An **affine** $k-$**chain** Γ in an open set $E \subseteq \mathbb{R}^n$ is a collection of finitely many oriented affine $k-$simplexes in E, denoted by

$$\sigma_1, \ldots, \sigma_r$$

Note that Γ may not be distinct, there might be multiples of the same simplex many times within the same chain.

Definition 53. If Γ is an affine $k-$chain is $E \subseteq \mathbb{R}^n$ and $\omega \in \Lambda^k(E)$, then

$$\int_\Gamma \omega = \sum_{i=1}^r \int_{\sigma_i} \omega$$

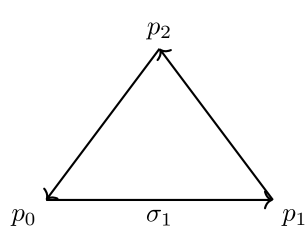
Remark 38. We will often abuse notation and write formally

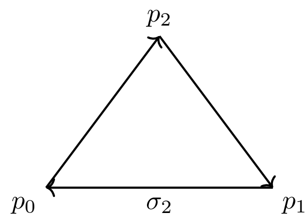
$$\Gamma = \sigma_1 + \cdots + \sigma_r = \sum_1^r \sigma_i.$$

Example 1.17. Consider

$$\sigma_1 = [p_0, p_1, p_2], \quad \sigma_2 = [p_1, p_0, p_2],$$

where $\sigma_1 = -\sigma_2$.





If $\Gamma = \sigma_1 + \sigma_2$, then if $\omega \in \Lambda^2(E)$, we have that by Theorem 23,

$$\int_\Gamma \omega = \int_{\sigma_1} \omega + \int_{\sigma_2} \omega = \int_{\sigma_1} \omega - \int_{\sigma_1} \omega = 0.$$

Thus, we abuse notation again and write

$$\Gamma = 0$$

Moreover, we note that the boundary of σ_1 is simply three lines (a triangle), making

$$\partial\sigma_1 = \sigma_1' + \sigma_2' + \sigma_3',$$

where σ_i' are all oriented affine 1−simplexes.

Definition 54. For $k \geq 1$, the boundary of an orientated affine $k-$simplex $\sigma = [p_0, \ldots, p_k]$ is the affine $(k-1)$ chain

$$\partial\sigma = \sum_{j=0}^{k} (-1)^k [p_0, \ldots, p_{j-1}, p_{j+1}, p_k]$$

Example 1.18. Consider $\sigma = [p_0, p_1, p_2]$ to be the filled in triangle. By definition

$$\partial\sigma = [p_1, p_2] - [p_0, p_2] + [p_0, p_1] = [p_1, p_2] + [p_2, p_0] + [p_0, p_1]$$

Example 1.19. Consider the tetrahedron $\sigma = [p_0, \ldots, p_3]$. Intuitively, the boundary of the tetrahedron are the faces of the triangles. Formally,

$$\partial\sigma = [p_1, p_2, p_3] - [p_0, p_2, p_3] + [p_0, p_1, p_2] - [p_0, p_1, p_2]$$

1.23 Friday, May 16: Introducing Stokes Theorem

Definition 55. Let $T \in C^2(E, F)$. Let σ be an oriented affine $k-$simplex in E. The map $\phi : T \circ \sigma$ is a k =surface in F. We call ϕ an **oriented** $k-$**simplex.**

Definition 56. A finite collection Ψ of oriented $k-$simplex, $\{\phi_1, \ldots, \phi_r\}$ is called a $k-$**chain** of class C^2 in F.

Formally, we denote $\Psi = \sum \phi_i$

Definition 57. If $\omega \in \Lambda^k(F)$, we define

$$\int_\Psi \omega = \sum_{i=1}^r \int_{\phi_i} \omega$$

If $\Gamma = \sum \sigma_i$ and $\phi_i = T \circ \sigma_i$, we formally write $\Psi = T \circ \Gamma$.

Definition 58. The **boundary of an oriented** $k-$**simplex** $\phi = T \circ \sigma$ is the $k-1$ chain defined by

$$\partial\phi = T \circ \partial\sigma.$$

Note that if $\phi \in C^2(E, F)$, then so is $\partial\phi$.

Definition 59. The boundary of a $k-$chain $\Psi = \sum \phi_i$ is the $k-1$chain denoted

$$\partial T = \sum \partial\phi_i$$

Theorem 30. (Stokes) If $\Psi \in C^2(E, F)$ is a $k-$chain, $\omega \in \Lambda^{k-1}(F)$ of class C^1, then

$$\int_\Psi d\omega = \int_{\partial\Psi} \omega$$

The proof is deferred till next class.

Remark 39. A few consequences of the FTC:

(a) (FTC)

When $k = m = 1$, then $\omega = f \in C^1(E, F)$. Since $m = 1$, then $F =\subseteq \mathbb{R}$. It suffices the case when

$$\Psi = \sigma = [a, b].$$

Thus, the boundary is a $0-$chain of oriented points:

$$\partial\sigma = [b] - [a].$$

Thus, by definition

$$\int_{\partial\sigma} f = \int_{[b]} f - \int_{[a]} f = f(b) - f(a).$$

Thus,

$$f(b) - f(a) = \int_\Psi d\omega = \int_\sigma df = \int_a^b df(x)\, dx$$

(b) (Green's Thm) is the case when $k = m = 2$. To see this, consider a smooth vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$. We can write $F = F_1 + F_2$, where F_1 and F_2 are the x and y components of F. Green's Theorem states that if D is a surface in $\mathbb{R}^2$ bounded by a curve C, then

$$\int_C F_1\, dx + F_2\, dy = -\int_D \left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right) dx\, dy.$$

Let

$$\omega = F_1\, dx + +F_2\, dy \in \Lambda^1(\mathbb{R}^2).$$

Then Stokes' theorem states that

$$\int_C F_1 dx + F_2 dy = \int_{\partial D} \omega = \int_D d\omega = \int_D \frac{\partial F_1}{\partial y}\, dy \wedge dx + \frac{\partial F_2}{\partial x}\, dx \wedge dy = \int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

(c) (Divergence) is the case when $k = m = 3$. Consider a smooth vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, writing $F = F_1 + F_2 + F_3$. Let $\omega_F^2 = F_1\, dx \wedge dy + F_2\, dz \wedge dx + F_3\, dy \wedge dz$. We have shown that $d\omega_F^2 = (\nabla \cdot F) dx \wedge dy \wedge dz = (\boldsymbol{\nabla} \cdot F) dV$. Thus, we have by Stokes' Theorem that

$$\int_\Phi (\boldsymbol{\nabla} \cdot F) dV = \int_\Phi d\omega_F^2 = \int_{\partial\Phi} \omega_F^2 = \int_{\partial\Phi} F \cdot n$$

(d) (OG Stokes) is the case when $k = 2$ and $m = 3$.

Theorem 31. (Baby Stokes) Let $E \subset \mathbb{R}^k$ containing Q^k. Let $\sigma = [0, e_1, \ldots, e_k]$. Let $\lambda \in \Lambda^{k-1}(E)$ of class C^1. Then

$$\int_\sigma d\lambda = \int_{\partial\sigma} \lambda.$$

The proof is again differed to next class.

Proposition 5. To prove Stokes' Theorem, it suffices to prove Baby Stokes.

Proof. It suffices to prove stokes with $\Psi = \phi$ by the linearity of addition. Moreover, it suffices to prove Stokes when $\Psi = \phi$ and $\phi = \sigma$, where σ is an affine $k-$simplex. To show this, we suppose $\phi = T\sigma$. Then using Theorem 22 (Change of Var)

$$\int_\Psi d\omega = \int_{T\sigma} d\omega = \int_\sigma (d\omega)_T = \int_\sigma d(\omega_T)$$

Supposing Baby stokes,

$$\int_\sigma d(\omega_T) = \int_{\partial\sigma} \omega_T = \int_{T \circ \partial\sigma} \omega = \int_{\partial\psi} \omega.$$

It remains to show that is suffices to show that Stokes holds when $\sigma = [0, e_1, \ldots, e_k]$. Let $\Psi = T\sigma$, where T is affine. Then

$$\int_\Psi d\omega = \int_{T\circ\sigma} d\omega = \int_\sigma d(\omega_T) = \int_{\partial\sigma} \omega_T = \int_{T\circ\partial\sigma} \omega = \int_{\partial\Psi} \omega$$

$\square$

1.24 Monday, May 19: Stokes Theorem

We prove Baby Stokes (Theorem 25)

Proof. If $k = 1$, the conclusion follows by FTC. Fix r such that $1 \leq r \leq k$, and let $f \in C^1(E)$. It suffices to prove the conclusion when

$$\lambda = f dx_1 \wedge \dots dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k.$$

By definition,

$$\partial\sigma = [e_1, \dots, e_k] + \sum_{j=1}^{\infty} (-1)^i \tau_i,$$

where

$$\tau_i = [0, e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_k].$$

For convenience, we let

$$\tau_0 := [e_r, e_1, \dots, e_{r-1}, e_{r+1}, \dots, e_k] = (-1)^{r-1}[e_1, \dots, e_k].$$

Let $u \in \mathbb{Q}^{k-1}$ and let $x = (x_1, \dots, x_k) = \tau_0(u)$. Then the jth component of x is given by

$$x_j = \begin{cases} u_j & 1 \leq j \leq r-1 \\ 1 - (u_1 + \cdots + u_k) & j = r \\ u_{j-1} & r+1 \leq j \leq k \end{cases}$$

Let $x = \tau_i(u)$, where $i \neq 0$. Then

$$x_j = \begin{cases} u_j & 1 \leq j \leq i-1 \\ 0 & j = i \\ u_{j-1} & i+1 \leq j \leq k \end{cases}$$

Let J_i be the Jacobian of the map $u_1, \dots, u_{k-1} \mapsto (x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_k)$ induced by τ_i. Then if $i = 0$, $J_0 = 1$ since the map is the identity map. If $i = r$, then $J_r = 1$. For any other i, $J_i = 0$. Then by definition,

$$\begin{aligned}
\int_{\partial\sigma} \lambda &= (-1)^{r-1} \int_{\tau_0} \lambda + (-1)^r \int_{\tau_r} \lambda \\
&= (-1)^{r-1} \left[\int_{\tau_0} \lambda - \int_{\tau_r} \lambda \right] \\
&= (-1)^{r-1} \left[\int_{Q^{k-1}} f(\tau_0(u))\, du - \int_{Q^{k-1}} f(\tau_r(u))\, du \right] \\
&= (-1)^{r-1} \int_{Q^{k-1}} [f(\tau_0(u)) - f(\tau_r(u))]\, du
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d\lambda &= \left(\sum_{i}^{k} D_i f\, dx_k \right) \wedge dx_1 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k \\
&= D_r f\, dx_r \wedge dx_1 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k \\
&= (-1)^{r-1} D_r f\, dx_1 \wedge \cdots \wedge dx_k
\end{aligned}$$

Then by definition,

$$\begin{aligned}
\int_\sigma d\lambda &= (-1)^{r-1}\int_{Q^k} D_r f(x)\,dx \\
&= (-1)^{r-1}\int_{Q^{k-1}}\left(\int_0^{1-x_r-x_{r-1}+\cdots x_1} D_r f(x)\,dx_r\right)dx_1 dx_1\cdots dx_{r-1}dx_{r+1}\cdots dx_k \\
&= (-1)^{r-1}\int_{Q^{k-1}} f(1-(\sum_1^r x_i)) - f(0)\,du \\
&= (-1)^{r-1}\int_{Q^{k-1}} f(\tau_0(u)) - f(\tau_r(u))\,du
\end{aligned}$$

□

By proposition 3, we have proved Stokes theorem.

1.25 Wednesday, May 21: Baire Category

Definition 60. Let X be a metric space with metric d. Let $E \subseteq X$.

(a) The **interior** of E is
$$E^\circ = \bigcup G_n,$$
where $G_n \subseteq E$ are open.

(b) The **closure** of E is
$$\overline{E} = \bigcap F_n,$$
where $F_n \supset E$ are closed.

(c) We say that E is **dense** in X if
$$\overline{E} = X.$$

(d) We say that E is **nowhere dense** if
$$(\overline{E})^\circ = \emptyset$$

(e) We say that E is of **first category** or **meager** if
$$E = \bigcup E_n,$$
where E_n are nowhere dense sets (e.g, the rationals)

(f) If E is not of first category, then E is **second category**.

(g) If E^c is of first category, then E is said to be a **residual** or **generic** set.

Theorem 32. (Baire Category) If X is a complete metric, then X is of second category. That is, if $X = \bigcup F_n$, where each F_n is closed, then at least one of the F_n is not nowhere dense.

Proof. Suppose X is not of second category. Then $X = \bigcup F_n$, where F_n are nowhere dense. Without loss of generality, take F_n to be closed. We will show that there exists some $x \in X$ such that $x \notin \bigcup F_n$. Since F_1 closed and nowhere dense, then $F_1 \neq X$ and there is some $r_1 > 0$ and $x_1 \in X \setminus F_1$ such that $\overline{B_{r_1}(x_1)} \subseteq F_1^c$. Since F_2 is nowhere dense, then $B_{r_1}(x_1) \not\subseteq F_2$. Let $x_2 \in B_{r_1}(x_1) \setminus F_2$. There is some $0 < r_2 < \frac{r_1}{2}$ such that $\overline{B_{r_2}(x_2)} \subseteq B_{r_1}(x_1)$ and $\overline{B_{r_2}(x_2)} \subseteq F_2^c$. We obtain a sequence of balls with
$$B_1 \supset B_2 \supset \cdots$$
and $r_1, r_2, \ldots$, such that $r_n \to 0$ and $F_n \cap \overline{B}_n = 0$. The sequence $\{x_n\}$ is clearly Cauchy and thus converges to some $x_\infty \in X$. But since $x_\infty \in \bigcap B_n$, $x_\infty \notin F_n$ for all $n \in \mathbb{N}$. Thus, $x_\infty \notin \bigcup F_n$, a contradiction! Hence, X is of second category. □

Corollary 10. If X is complete, then a residual set of X is dense.

Proof. Let E be residual, and suppose E is not dense. Thus, there exists some $\overline{E} \neq X$. Thus, there exists some $r > 0$ such that $B = B_r(x) \subset E^c$ where $x \in E^c$. We know that E^c is of first category. I.e,
$$E^c = \bigcup_{n=1}^{\infty} F_n,$$
where F_n are nowhere dense. Thus,
$$\overline{B} = \bigcup_{1}^{\infty} (F_n \cap \overline{B}).$$
But $\overline{B}$ is a complete metric space, contradiction BCT. □

For the following, we let $X = C([0,1], \mathbb{R})$ be equipped with the sup metric, i.e.

$$d(f,g) = \|f - g\| = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

We have shown (X, d) to be complete.

Theorem 33. The set of functions $f \in X$ that are nowhere differentiable is residual.

Proof. Let $D = \{f \in X \mid f'(x) \text{ exists for some } x \in [0,1]\}$. It suffices to show that D is of first category. That is, it suffices to show that

$$D = \bigcup_{n=1}^{\infty} D_n,$$

where the D_n are nowhere dense. Consider defining

$$D_n = \{f \in X \mid \exists x^* \in [0,1] : |f(x) - f(x^*)| \leq n|x - x^*| \forall x \in [0,1]\}.$$

We know that

$$D \subseteq \bigcup_{n=1}^{\infty} D_n$$

Lemma 3. D_n is closed.

Proof. Let $(f_k) \in D_n$ such that $f_k \to f$. It suffices to show that $f \in D_n$. Since $f_k \in D_n$, then for each f_k, there exists some $x_k^* \in [0,1]$ such that $|f_k(x) - f_k(x_k^*)| \leq n|x - x_k^*|$ for all $x \in [0,1]$. Consider that (x_k^*) is a sequence of real numbers, and thus has a subsequence converging to some x^*. Using the triangle inequality, we have that for large enough k_j,

$$\begin{aligned}
|f(x) - f(x^*)| &\leq |f(x) - f_{k_j}(x)| + |f_{k_j}(x) - f_{k_j}(x^*)| + |f_{k_j}(x^*) - f(x^*)| \\
&\leq \|f - f_{k_j}\| + |f_{k_j}(x) - f_{k_j}(x^*)| + \|f - f_{k_j}\| \\
&\leq \epsilon + |f_{k_j}(x) - f_{k_j}(x^*)| \\
&\leq \epsilon + |f_{k_j}(x) - f_{k_j}(x_{k_j}^*)| + |f_{k_j}(x_{k_j}^*) - f_{k_j}(x^*)| \\
&\leq \epsilon + n|x - x_{k_j}^*| + n|x_{k_j}^* - x^*| \\
&\leq \epsilon + n|x - x_{k_j}^*| + \epsilon \\
&\leq 2\epsilon + n(|x - x^*| + |x^* - x_{k_j}^*|) \\
&\leq 3\epsilon + n|x - x^*|
\end{aligned}$$

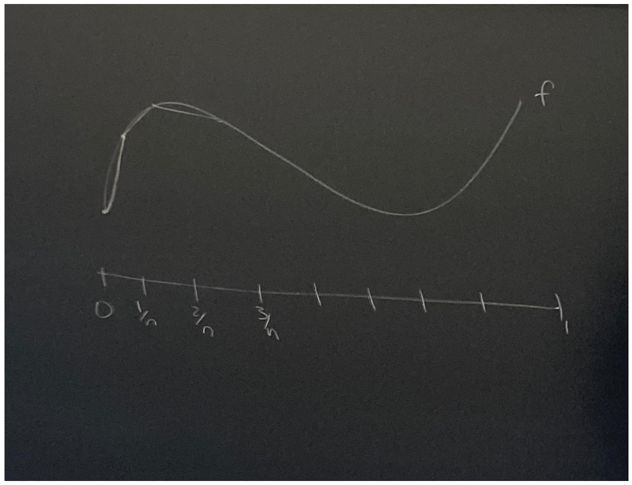
Taking $\epsilon \to 0$, we see that $f \in D_n$. ☐

Let $\mathscr{P} \subseteq X$ be the set of piecewise linear functions in $C([0,1])$. Let $\mathscr{P}_m \subseteq \mathscr{P}$ be the piecewise continuous functions in $\mathscr{P}$ such that the if β is a slope of any line segment in $f \in \mathscr{P}_m$, then $|\beta| \geq M$. Note that if $M > N$, then $\mathscr{P}_M \cap D_N = \emptyset$.

Lemma 4. $\mathscr{P}_M$ is dense in X for all $M \geq 0$.

Proof. Let $\epsilon > 0$ and let $f \in X$. We will first show that $\mathscr{P}$ is dense in X. Since f is uniformly continuous, there exists $\delta > 0$ such that for all $x, y \in [0,1]$ with $d(x,y) < \delta$, we have that $d(f(x), f(y)) < \epsilon$. Let $N \in \mathbb{N}$ such that $n > \frac{1}{\delta}$. Define g to be the piecewise linear function such that for all $k = 0, \ldots, n-1$, we have

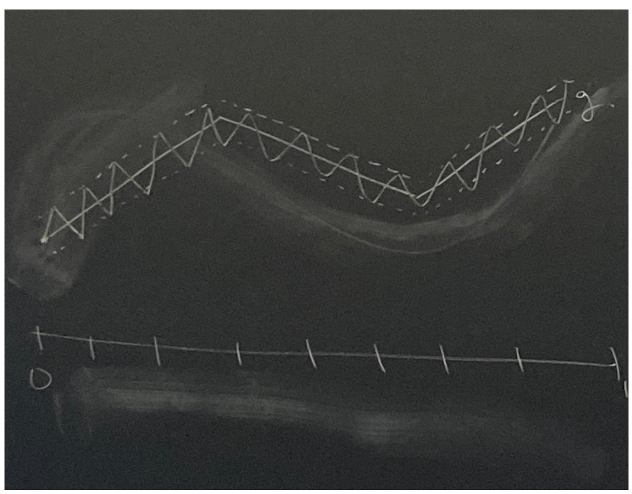
$$g(\frac{k}{n}) = f(\frac{k}{n}) \qquad g(\frac{k+1}{n}) = f(\frac{k+1}{n}).$$



By definition, $d(f,g) < \epsilon$. Now, we will show that there is some $h \in \mathscr{P}_M$ such that $d(g,h) < \epsilon$. For each interval $[\frac{k}{n}, \frac{k+1}{n}]$ such that the slope of g on this segment is between $(-M, M)$, we construct h as follows. For $k = 0$, let

$$\varphi_\epsilon(x) = g(x) + \epsilon \qquad \psi_\epsilon(x) = g(x) - \epsilon$$

Starting at $g(0)$, travel the line segment of slope M until we intersect φ_ϵ. Then travel the line segment of slope $-M$ until we intersect ψ_ϵ. Repeat to obtain $\psi_\epsilon \leq h \leq \varphi_\epsilon$ on $[0, \frac{1}{n}]$. Then repeat on $[\frac{1}{n}, \frac{2}{n}]$ starting at $h(\frac{1}{n})$.



Thus, $\mathscr{P}_M$ are dense in X. $\square$

By Lemma 2, we know that $D_N^\circ = \emptyset$ for all $N > 0$, since for $M > 0$, there is some $h \in \mathscr{P}_M$ such that $d(f,h) < \epsilon$, but $\mathscr{P}_M$ and D_N are disjoint, and so there is no open ball containing only $f \in D_N$. By Lemma 1, D_N is closed. Thus, D_N are nowhere dense. Hence, $D = \bigcup_{n=1}^\infty (D_n \cap \mathscr{D})$ is a countable union of nowhere dense sets, and thus it is of first category. Hence, D^c is residual.

$\square$

1.26 Friday, May 23: Corollaries of Baire Category

Theorem 34. Suppose that $\{f_n\}$ is a sequence of continuous real valued functions on $\mathbb{R}$. Suppose that $f_n \to f$ pointwise. The set of points at which f is continuous is generic.

Theorem 35. Let B be the set of complex valued functions on $[-\pi, \pi]$. The set of $f \in B$ whose Fourier series diverges on a generic set of $[-\pi, \pi]$ is itself generic.

1.27 Thursday, May 29: Final

The first question was an easy ahh integration over a one-form. The second one was an easy ahh pullback of a two-form.

Proposition 6. Suppose $\alpha \in \Lambda^k(\mathbb{R}^n)$ and $\beta \in \Lambda^\ell(\mathbb{R}^n)$. Show that if

(a) α is closed and β is closed, then $\alpha \wedge \beta$ is closed.

(b) α is closed and β is exact, then $\alpha \wedge \beta$ is exact.

Proof. (a) Use the wedge product rule to conclude. (b) There is some λ such that $d(\lambda) = \beta$. Use the wedge product rule on $d(\lambda \wedge \alpha)$ to conclude. $\square$

Proposition 7. Suppose f is continuous and $2\pi-$periodic and Riemann integrable. Show that $\|S_N(f) - f\|_{L_1} \to 0$.

Proof. Using C-S inequality (up to a constand of 2π)

$$\|S_N(f) - f\|_{L_1} = (S_N(f) - f, 1) \leq \|S_N - f\|_{L_2} \cdot \|1\|_{L_2} \to 0$$

$\square$

Proposition 8. Suppose $f_n \to f$ pointwise. Define

$$F_n = \int_{(0,x)} f_n \qquad F = \int_{(0,x)} f.$$

Then

$$\int_{(0,1)} f + F \leq \liminf_{n\to\infty} \left(\int_{(0,1)} f_n + F_n \right)$$

Proof. Using Fatou's lemma, linearity of the integral, and the fact that

$$\liminf(a_n) + \liminf(b_n) \leq \liminf(a_n + b_n),$$

then it suffices to show that

$$\int_{(0,1)} F \leq \liminf \int_{(0,1)} F_n.$$

Use Fatou's lemma again to show this. $\square$