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Problem 1

Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and $f(x + 2\pi) = f(x)$ for all x.

(a) Compute the Fourier coefficients of f.

SOLUTION:

$$f(x) = \begin{cases} 1, & |x| \leq \delta \\ 0, & \delta < |x| \leq \pi \end{cases}$$

We split the integral into the natural segments and use the definition of f to compute a_n and b_n and a_0 (the complex coefficients of f, wouldn't make sense since it is a real valued function).[a] We use the fact that $\sin(-kx) = -\sin(kx)$

$$\begin{aligned} a_k &= \langle f, \cos(kx) \rangle \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \\ &= \frac{1}{\pi} \int_{-\delta}^{\delta} f(x) \cos(kx) dx + 2 \int_{\delta}^{\pi} f(x) \cos(kx) dx \\ &= \frac{1}{\pi} \int_{-\delta}^{\delta} \frac{\cos(kx)}{k} dx \\ &= \frac{2\sin(\delta k)}{\pi k} \end{aligned}$$

and for b_k :

$$\begin{aligned} b_m &= \langle f, \sin(mx) \rangle \\ &= \frac{1}{\pi} \int_{-\delta}^{\delta} f(x) \sin(mx) dx + 2 \int_{\delta}^{\pi} f(x) \sin(mx) dx \\ &= \frac{1}{\pi} \int_{-\delta}^{\delta} \sin(mx) dx \\ &= 0 \end{aligned}$$

and finally:

$$a_0 = \frac{1}{\pi} \int_{-\delta}^{\delta} f(x) dx = \frac{2\delta}{\pi}$$

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[a] Or at least I could not figure out how to make them work lol

(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}, \quad (0 < \delta < \pi).$$

SOLUTION: We apply the localization theorem for $x = 0$, since we know that

$$|f(x+t) - f(x)| = 1$$

for $0 < |t| < \delta$, then f is locally Lipshitz. Thus, we have that $s_n(f, 0) = f(0)$ in the limit, and so by the previous part:

$$s_n(f, 0) = a_0 + \sum_{n=1}^{N} a_n \cos(n(0)) = \frac{2\delta}{\pi} + \sum_{n=1}^{N} \frac{2\sin(\delta n)}{\pi n} \to f(x)$$

Thus,

$$\frac{2\delta}{\pi} + \sum_{n=1}^{\infty} \frac{2\sin(\delta n)}{\pi n} = 1 \iff \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}$$

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(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}.$$

SOLUTION: From Parseval's theorem, we get that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right).$$

Thus, plugging in all the stuff from before, we get that after some algebraic manipulation (multiplying by $\frac{\pi^2}{4\delta}$ and moving stuff around):

$$\frac{2\delta}{\pi} = \frac{1}{2}(\frac{2\delta}{\pi})^2 + \sum_{n=1}^{\infty} \frac{4\sin^2(\delta n)}{\pi^2 n^2} \iff \frac{\pi - \delta}{2} = \sum_{n=1}^{\infty} \frac{\sin^2(\delta n)}{n^2\delta}$$

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(d) Let $\delta \to 0$ and prove that

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

SOLUTION: The right hand side is fine by letting $\delta \to 0$ in the above equation, so we need to show that for all $\epsilon > 0$, as $\delta \to 0$

$$\left| \int_0^\infty \left(\frac{\sin x}{x} \right)^2 dx - \sum_{n=1}^\infty \frac{\sin^2(n\delta)}{n^2 \delta} \right| < \epsilon.$$

To provide intuition, we need to show that

$$\lim_{\delta \to 0} \left(\lim_{N \to \infty} \sum_{i=1}^N \frac{\sin^2(n\delta)}{n^2\delta} \right) = \int_0^\infty \left(\frac{\sin x}{x} \right)^2.$$

Let b be large, and partition $[0, b]$ by $P = \{0, \delta, 2\delta, \dots, N\delta = b\}$. By letting $x_n = n\delta$ and $\Delta x_n = \delta$, then

$$\lim_{\delta \to 0} \sum_{n=1}^N \frac{\sin^2(n\delta)}{n^2\delta} = \lim_{\Delta x_n \to 0} \sum_{n=1}^N \frac{\sin^2(x_n)}{x_n^2} \delta = \int_0^b \frac{\sin^2(x)}{x^2} dx \tag{1}$$

Let $\epsilon > 0$. Let b large such that

$$\left| \int_0^\infty \frac{\sin^2(x)}{x^2} dx - \int_0^b \frac{\sin^2(x)}{x^2} dx \right| < \frac{\epsilon}{2}.$$

Since $\delta \to 0$, we find δ small enough such that by (1):

$$\left| \int_0^b \frac{\sin^2(x)}{x^2} dx - \sum_{n=1}^N \frac{\sin^2(n\delta)}{n^2\delta} \right| < \frac{\epsilon}{2}$$

Combining the inequalities:

$$\begin{aligned} \left| \int_0^\infty \left(\frac{\sin x}{x} \right)^2 dx - \sum_{n=1}^\infty \frac{\sin^2(n\delta)}{n^2\delta} \right| &\le \left| \int_0^\infty \frac{\sin^2(x)}{x^2} dx - \int_0^b \frac{\sin^2(x)}{x^2} dx \right| \\ &\quad + \left| \int_0^b \frac{\sin^2(x)}{x^2} dx - \sum_{n=1}^N \frac{\sin^2(n\delta)}{n^2\delta} \right| \\ &< \epsilon. \end{aligned}$$

Thus, by part (c), we are done (one could have done an $\frac{\epsilon}{3}$ argument to not be so wishy washy about connecting the integral to part (c), but the author thinks that is obvious enough).

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(e) Put $\delta = \pi/2$ in (c). What do you get?

Solution: Define the Esteva function as

$$\chi_{\mod 4}(n) = \begin{cases} 1, & n \mod 4 = 1 \\ 0, & n \mod 4 = 0 \\ -1, & n \mod 4 = 3 \end{cases}$$

Then we can make the result prettier:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2\sin^2(n\frac{\pi}{2})}{n^2\pi} = \frac{\pi}{4} \iff \sum_{n=1}^{\infty} \frac{\sin^2(\frac{n\pi}{2})}{n^2} = \frac{\pi^2}{8} \iff \sum_{n=1}^{\infty} \frac{\chi_{\mod 4}(n)}{n^2} \\ = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \cdots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \\ = \frac{\pi^2}{8} \end{aligned}$$

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Problem 2

Put $f(x) = x$ for $x \in [0, 2\pi)$, and apply Parseval's Theorem to conclude that

$$\sum_{n=1}^{\infty} = \frac{\pi^2}{6}$$

SOLUTION: We get that since x is odd, we do not worry about the even cosine, and thus:

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} x dx = 2\pi,$$

and using integration by parts:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{-2}{n}.$$

Thus:

$$f(x) = \pi + \sum_{n=1}^{\infty} \frac{-2}{n} \sin(nx).$$

Parseval gives us then that

$$\frac{1}{\pi} \int_0^{2\pi} x^2 dx = \frac{8\pi^2}{3} = 2\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} \iff \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

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Problem 3

If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

SOLUTION: We calculate the Fourier series, by first noting that $f(x)$ is even and thus

$$\int_{-\pi}^{\pi} (\pi - |x|)^2 dx = 2 \int_0^{\pi} (\pi - x)^2 dx.$$

Since f is even, we need not even both calculating the coefficients of the odd $\sin(x)$ function, since we know them to be 0. Thus

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 = \frac{2}{\pi} \int_0^{\pi} (\pi - x)^2 dx = -\frac{1}{3\pi} \left[(\pi - x)^3 \right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\pi - x)^2 \cos(nx) dx = \frac{4}{n^2}$$

To deduce the two relations, we first let $x = 0$ in the equation derived above and notice that

$$f(0) = \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \iff \frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2} \iff \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

For the second relation, we use Parseval's identity:

$$\begin{aligned}
\frac{1}{\pi} \int_{-\pi}^{\pi} |f^2| dx &= \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - |x|)^4 dx \\
&= \frac{2}{\pi} \int_0^{\pi} (\pi - x)^4 dx \\
&= \frac{2\pi^4}{5} \\
&= \frac{1}{2} (\frac{2\pi^2}{3})^2 + \sum_{n=1}^{\infty} (\frac{4}{n^2})^2 \\
&= \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}
\end{aligned}$$

Thus, we get that

$$\frac{36\pi^4}{90} = \frac{20\pi^4}{90} + \sum_{n=1}^{\infty} \frac{16}{n^4} \iff \frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{16}{n^4}.$$

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Problem 4

With D_n as defined in (77), put

$$K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x}$$

SOLUTION: By work done in class, we know that

$$D_n(x) = \frac{\sin\left(nx + \frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}.$$

Thus, it suffices to show that

$$\sum_{n=0}^{N} \frac{\sin\left(nx + \frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} = \frac{1-\cos(Nx+x)}{1-\cos(x)}$$

Thus, consider that

$$(1-\cos(x))K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} \frac{\sin\left(nx + \frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}(1-\cos(x)) = \frac{1}{N+1}\sum_{n=0}^{N} 2\sin\left(\frac{x}{2}\right)\sin\left(nx + \frac{x}{2}\right)$$

We use the multiplication of sin identity

$$(1-\cos(x))K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} \cos(-nx) - \cos(x(n+1)) = \frac{1}{N+1}\sum_{n=0}^{N} \cos(nx) - \cos(nx+x)$$

we then telescope the above sum:

$$1-\cos(x)K_N(x) = 1-\cos((N+1)x) \iff K_N(x) = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x}$$

■

and that

(a) $K_N \geq 0$,

SOLUTION: This is fairly obvious, as $\cos(x) \leq 1$, then the denominator $1-\cos(x) \geq 0$. Similarly, $\cos((N+1)x) \leq 1$ for all N. ■

(b) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,dx = 1$,

SOLUTION: Recall that

$$\int_{-\pi}^{\pi} D_N(x)dx = 2\pi,$$

thus we have that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{N+1}\sum_{n=0}^{N} D_n(x) = \frac{1}{2\pi}\frac{1}{N+1}\sum_{n=0}^{N}\int_{-\pi}^{\pi} D_n(x) = \frac{1}{2\pi}\frac{1}{N+1}(2\pi(N+1)) = 1$$

∎

(c) $K_N(x) \le \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}$ if $0 < \delta \le |x| \le \pi$.

SOLUTION: Since $\cos(x)$ is decreasing on $[0,\pi]$, then $1-\cos(x)$ is increasing on the interval, and thus since $\delta \le |x|$, then $1-\cos(\delta) \le 1-\cos(|x|) = 1-\cos(x)$, which implies that $\frac{1}{1-\cos(\delta)} \ge \frac{1}{1-\cos(x)}$. We also know that $\cos((N+1)x) \le 2$ for all N. Thus,

$$K_N(x) = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x} \le \frac{2}{1-\cos(\delta)}.$$

∎

If $s_N = s_N(f;x)$ is the Nth partial sum of the Fourier series of f, consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$\sigma_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\,dt,$$

SOLUTION: This follows directly from the definition of $K_N(t)$:

$$\begin{aligned}\sigma_N &= \frac{\sum_{n=0}^{N} s_n(f,x)}{N+1}\\ &= \frac{\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)}{N+1}dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\frac{1}{N+1}\sum_{n=0}^{N} D_n(t)dt\end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt.$$

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and hence prove Fejér's theorem: *If f is continuous, with period 2π, then $\sigma_N(f;x) \to f(x)$ uniformly on $[-\pi,\pi]$.*

Solution: Let $\epsilon > 0$ and $x \in [-\pi,\pi]$. Since f is continuous, then f is uniformly continuous on $[-\pi,\pi]$. Thus, there exists some $\delta > 0$ such that if $|t| < \delta$, then $|f(x-t) - f(x)| < \epsilon$ for any $x \in [-\pi,\pi]$.

$$\begin{aligned}
|\sigma_N(f,x) - f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt - f(x)\right| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt - f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)dt\right| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x-t) - f(x)\right)K_N(t)dt\right| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(x-t) - f(x)\right| K_N(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{-\delta} |f(x-t) - f(x)|K_N(t)dt \\
&\quad + \frac{1}{2\pi}\int_{-\delta}^{\delta} |f(x-t) - f(x)|K_N(t)dt \\
&\quad + \frac{1}{2\pi}\int_{\delta}^{\pi} |f(x-t) - f(x)|K_N(t)dt
\end{aligned}$$

We know f to be uniformly continuous, and thus f is bounded. Let $M = \sup_{x\in[-\pi,\pi]} |f(x)|$. Thus, for $x \in [\pm\pi, \pm\delta]$, we get that $|f(x-t) - f(x)| \leq 2M$. Thus, we can pick some large N, not dependent on x, and then

$$\begin{aligned}
|\sigma_N(f,x) - f(x)| &\leq \frac{2}{\pi}\int_{\delta}^{\pi} K_N(t)dt + \frac{1}{2\pi}\int_{-\delta}^{\delta} |f(x-t) - f(x)|K_N(t)dt \\
&< \frac{4}{\pi(N+1)}\int_{\delta}^{\pi} \frac{1}{1-\cos\delta}dt + \epsilon\frac{1}{2\pi}\int_{-\delta}^{\delta} K_N(t)dt \\
&\to 0
\end{aligned}$$

as $N \to \infty$ and $\epsilon \to 0$. ■

REFLECTIONS: I've realized that I have stopped caring so much about making my $\epsilon's$ prettier as time has gone along. Nowadays, I see an ϵ and I let it go to 0 without arguing very hard.

Problem 5

If $f \in \mathcal{R}$ and $f(x+)$ and $f(x-)$, exist for some x, then

$$\lim_{n\to\infty} \sigma_n(f,x) = \frac{1}{2}[f(x+) + f(x-)].$$

Solution: Let $\epsilon > 0$. By the existence of the left hand limit, we know that there exists some δ_L such that if $t \in (\delta_L, 0)$, then $|f(x-t) - f(x-)| < \epsilon$.

Similarly for the existence of such a δ_R. Thus, we get that

$$\begin{aligned}
\left|\sigma_n(f,x) - \frac{1}{2}[f(x+) + f(x-)]\right| &= \left|\frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-t)K_n(t)dt - \frac{1}{2}[f(x+) + f(x-)]\right| \\
&= \left|\frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-t)K_n(t)dt - \frac{1}{2\pi}\int_{[x-\pi,x+\pi]\setminus\{x\}} \frac{1}{2}[f(x+) + f(x-)]K_n dt\right| \\
&= \left|\frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-t)K_n(t)dt - \frac{1}{2\pi}\int_{x-\pi}^{x} f(x-)K_n(t)dt - \frac{1}{2\pi}\int_{x}^{x+\pi} f(x+)K_n dt\right| \\
&\leq \frac{1}{2\pi}\int_{x-\pi}^{x} |f(x-t) - f(x-)|K_n(t)dt + \int_{x}^{x+\pi} |f(x-t) - f(x+)|K_n(t)dt
\end{aligned}$$

First, in order to make my life simpler, we know by Rudin that the interval over which we integrate does not matter as long as its total length is 2π. Thus, we translate everything by $-x$, and we now split up the terms and argue symmetrically,

$$\frac{1}{2\pi}\int_{-\pi}^{0} |f(-t) - f(0-)|K_n(t)dt = \frac{1}{2\pi}\int_{-\pi}^{-\delta_L} |f(-t) - f(0-)|K_n(t)dt + \frac{1}{2\pi}\int_{-\delta_L}^{0} |f(-t) - f(0-)|K_n(t)dt$$

Since $f \in \mathcal{R}$, then f is bounded, so we can bound $|f(x-t) - \frac{1}{2}f(x)| \leq 2M$ for $x \in [-\pi, \delta_L]$. Then we do an identical argument to the previous problem that the difference goes to 0. Thus,

$$\left|\sigma_n(f,x) - \frac{1}{2}[f(x+) + f(x-)]\right| \to 0$$

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Assume f is bounded and monotonic on $[-\pi, \pi)$, with Fourier coefficients c_n, as given by (62).

(a) Use Exercise 17 of Chap. 6 to prove that $\{nc_n\}$ is a bounded sequence.

Solution: Since f is monotonic on $[-\pi, \pi)$, e^{-inx} is continuous, $g(x) = \frac{1}{-in}e^{-inx} = \frac{i}{n}e^{-inx}$ for $-\pi \leq x \leq \pi$, then exercise 17 of chapter 6 tells us that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{i}{n}e^{-in\pi}f(\pi) - \frac{i}{n}e^{in\pi}f(-\pi) - \frac{i}{2\pi n}\int_{-\pi}^{\pi} e^{-inx}df(x)$$

By definition, the left hand side is equal to c_n. The first two terms in the right hand side go away since f is $2-\pi$ periodic and thus $f(\pi) = f(-\pi)$ and thus using trig properties:

$$\begin{aligned}
\frac{i}{n}e^{-in\pi}f(\pi) - \frac{i}{n}e^{in\pi}f(-\pi) &= \frac{i}{n}(e^{-in\pi}f(\pi) - e^{in\pi}f(-\pi)) \\
&= \frac{i}{n}(\cos(-n\pi)f(\pi) + i\sin(-n\pi)f(\pi) - \cos(n\pi)f(-\pi) - i\sin(n\pi)f(-\pi)) \\
&= \frac{i}{n}(\cos(-n\pi)f(\pi) - \cos(n\pi)f(-\pi)) \\
&= \frac{i\cos(n\pi)}{n}(f(\pi) - f(-\pi)) \\
&= \frac{i}{n}(f(\pi) - f(-\pi))
\end{aligned}$$

Thus,

$$\begin{aligned}
c_n &= \frac{i}{n}(f(\pi) - f(-\pi)) - \frac{i}{2\pi n}\int_{-\pi}^{\pi} e^{-inx}df(x) \\
\implies |c_n| &= \frac{1}{n}f(\pi) - f(-\pi)) + \frac{1}{2\pi n}\left|\int_{-\pi}^{\pi} e^{-inx}df(x)\right| \\
&\leq \frac{1}{n}f(\pi) - f(-\pi)) + \frac{1}{2\pi n}\int_{-\pi}^{\pi} |e^{-inx}|df(x)
\end{aligned}$$

Consider that

$$|e^{inx}| = 1$$

Thus, since $|e^{-inx}|$ is bounded, then we have that if $M = \sup_{x\in[-\pi,\pi]} e^{-inx} = 1$, then by Theorem 6.12 on Rudin, we get that

$$|c_n| \leq \frac{1}{n}f(\pi) - f(-\pi)) + \frac{1}{2\pi n}\int_{-\pi}^{\pi} df(x) \leq \frac{1}{n}f(\pi) - f(-\pi)) + \frac{1}{2\pi n}[f(\pi) - f(-\pi)]$$

and by multiplying by n on both sides we find that

$$|nc_n| \leq \frac{1}{2\pi n}[f(\pi) - f(-\pi)]$$

and is thus bounded.

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(b) Combine (a) with Exercise 16 and with Exercise 14(e) of Chap. 3, to conclude that

$$\lim_{N\to\infty} s_N(f;x) = \frac{1}{2}[f(x+) + f(x-)]$$

for every x.

SOLUTION: By exercise 14 in chapter 3, we know that since $|nc_n| \leq M$ for some M, and since $f \in \mathcal{R}[(-\pi,\pi)]$[a], then by the previous problem,

$$\lim_{n\to\infty} \sigma_n(f,x) = \frac{1}{2}[f(x-) + f(x+)],$$

and thus

$$\lim_{n\to\infty} s_n(f,x) = \frac{1}{2}[f(x-) + f(x+)].$$

The only assumption left to check[b] is that

$$s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} k a_k,$$

where $|ka_k| \leq M$ This is just definitional however, since

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{s_0 + s_1 + \cdots + s_n}{n+1} \\ &= \frac{(n+1)s_n}{n+1} - \frac{s_0 + s_1 + \cdots + s_n}{n+1} \\ &= \frac{1}{n+1}\sum_{k=1}^{n} k c_k e^{inx}, \end{aligned}$$

and thus $a_k = c_k e^{inx}$ which has a norm of just c_k.

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[a] We know this because f is monotonic and bounded, and thus has at most countably many discontinuities, and is thus integrable by the Riemann-Lebesgue Theorem

[b] We know $f(x-), f(x+)$ exist on every point because f is mononic

(c) Assume only that $f \in \mathcal{R}$ on $[-\pi,\pi]$ and that f is monotonic in some segment $(\alpha,\beta) \subset [-\pi,\pi]$. Prove that the conclusion of (b) holds for every $x \in (\alpha,\beta)$. (This is an application of the localization theorem.)

SOLUTION: Let

$$\varphi(x) = \begin{cases} f(\alpha), & x \in (-\pi,\alpha] \\ f(x), & x \in (\alpha,\beta) \\ f(\beta), & x \in (\beta,\pi) \end{cases} .$$

Since φ is bounded and monotonic on $[\alpha,\beta)$, and we know that $\varphi(x-), \varphi(x+)$ exist on the interval, then we must have that

$$\lim_{N\to\infty} s_N(\varphi, x) = \frac{1}{2}[\varphi(x+) + \varphi(x-)] = .$$

By the localization theorem, we know moreover that since $\varphi(x) = f(x)$ for $x \in (\alpha, \beta)$, then:

$$\lim_{N\to\infty} s_N(\varphi - f, x) = \lim_{N\to\infty} s_N(\varphi, x) - s_N(f, x) = 0,$$

and so

$$\lim_{N\to\infty} s_N(f, x) = \frac{1}{2}[\varphi(x+) + \varphi(x-)] = \frac{1}{2}[f(x+) - f(x-)].$$

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