Problem Set 3
207 Fall 2024:
Honors Analysis in $\mathbb{R}^n$
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Due Date: 2024-14-10

Problem 1

A map $f : M \to N$ is said to be *open* if for all open $U \subset M$, we have that $f(U)$ is open in N.

(a) If f is open, is it continuous?

> Solution: **Not necessarily.** Let $Id : (M, d) \to (N, d')$, where d is the euclidean metric and d' is the discrete metric. Let U be open in M, then $id(U) = U$ is open in N since we can write $U = \bigcup_{\alpha \in \mathscr{A}} u_\alpha$, where $\{u_\alpha\}$ is every point in U. Since $B_{\frac{1}{2}}(u_\alpha) = \{u_\alpha\}$, and the ball of radius $\frac{1}{2}$ is open in N, then $\{u_\alpha\}$ is open. Thus, since unions of open sets are open, U is open in N and thus id is open.
> Let $\{x\} \subset N$. Then $\{x\}$ is open in N. Since $id^{-1}(\{x\}) = \{x\}$ is closed in M, then id is not continuous. ■

(b) If f is a homeomorphism, is it open?

> Solution: **Yes.** Let $U \subset M$ be open. Since f is bijective, there exists $L \subset N$ such that $f^{-1}(L) = U$. Suppose L is closed, then by continuity, U is closed. Thus, L is open, and so $f(U) = f(f^{-1}(L)) = L$ is open. ■

(c) If f is an open continuous bijection, is a homeomorphism?

> Solution: **Yes.** Suppose $U \subset M$ is open. Since f is bijective, there exists some $L \subset N$ such that $f^{-1}(L) = U$. Since f is open, then $f(U) = f(f^{-1}(L)) = L$ is open. Thus, $f^{-1}(U)$ is continuous. ■

(d) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and surjective, is it open?

> Solution: **Not necessarily.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous surjection. We claim that we can map $(0, 1) \mapsto [0, 1]$. Let $a < b$ with $a, b \in (0, 1)$ such that $f(a) = 0$ and $f(b) = 1$, then by the IVT, $f([a, b]) = [0, 1]$, and so if we let $0 < x < a$ be mapped by $f(a) = 0$ and $b < x < 1$ be mapped by $f(x) = 1$, we are sending an open set to a closed set. ■

(e) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous, open, and a surjection, must it be a homeomorphism?

> Solution: **Yep.** By (c), it suffices to show that f is an injection. Suppose $f(x) = f(y)$ with $x < y$, then by the extreme value theorem, there exists m and M maximum and minimum achieved in the interval $f((x, y))$. Then by the IVT, $f((x, y)) = [m, M]$, and thus f is not open. Therefore, f must be an injection. ■

(f) What happens in (e) if $\mathbb{R}$ is replaced by the unit circle S^1.

> Solution: The same is not true. Consider $f : S^1 \to S^1$, where if $z \in S^1$, then

Problem 2

Consider a function $f : M \to \mathbb{R}$. Its graph is the set

$$G := \{(p, y) \in M \times \mathbb{R} | y = f(p)\}$$

(a) Prove that if f is continuous then its graph is closed (as a subset of $M \times \mathbb{R}$).

> SOLUTION: Let $s = (p, y)$ be a limit point of G. Thus, there exists a sequence $(s_n) \in G$ such that $s_n \to s$. Since $s_n \in G$ for all n, we have that $s_n = (p_n, f(p_n)) \to (p, y)$. We have $p_n \to p$, then because f is continuous, $f(p_n) \to f(p)$, and thus we use the fact that limits are unique to show that $y = f(p)$. Thus, $s = (p, f(p))$. Therefore, $s \in G$. ■

(b) Prove that if f is continuous and M is compact then its graph is compact.

> SOLUTION: Let $(s_n) \in G$. We want to find a convergent subsequence (s_{n_k}) such that it converges to a limit in G. Suppose $s_n = (p_n, f(p_n))$ for each n, then since M is compact and (p_n) is a sequence in M, we have (p_{n_k}) convergent sequence to some $p \in M$. Thus, $s_{n_k} = (p_{n_k}, f(p_{n_k})) \to (p, y) = s$. Since G is closed (part a), we have that $s \in G$. ■

(c) Prove that if the graph of f is compact then f is continuous.

> SOLUTION: Suppose $(p_n) \in M$ is convergent with $p_n \to p$. We claim $f(p_n)$ converges. Let $(s_n) = (p_n, f(p_n))$, then since G is compact, there exists a convergent subsequence $(s_{n_k}) = (p_{n_k}, f(p_{n_k})) \to (p', y')$. Note that $s = (p', y') \in G$ because G is closed). However, since (p_{n_k}) is a convergent subsequence of a convergent sequence, then we necessarily must have that $p' = p$, and thus $f(p) = y'$. Thus, $f(p_{n_k}) \to f(p)$. It suffices to show that $f(p_n) \to f(p)$. Suppose not, then there exists some $\epsilon > 0$ such that for N large enough, if $n \geq N$, we have that $d(f(p_n), f(p)) > \epsilon$. Consider $(s_{n_j}) = (p_{n_j}, f(p_{n_j}))$ as the subsequence of (s_n) such that $d(f(p_{n_j}), f(p)) > \epsilon$. By compactness, there exists some sub-subsequence that converges to some (p, γ), where evidently, $\gamma \neq f(p)$. Thus $(p, \gamma) \notin G$, which is a contradiction to the fact G is closed. ■

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(d) What if the graph is merely closed? Give an example of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed.

> SOLUTION: Consider the function $f(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 0, & = 0. \end{cases}$ Evidently, the function is discontinuous at $x = 0$. We need to show that there does not exist some $y \in \mathbb{R}$ such

that $(0, y)$ is a limit of $G = \{(p, y) \in \mathbb{R} \times \mathbb{R} | y = \frac{1}{p^2}\}$. Suppose not, that is, there exists some $y \in \mathbb{R}$ such that there is some sequence $s_n = (x_n, y_n)$ in G that converges to $(0, y)$. Thus, $s_n = (x_n, \frac{1}{x_n^2})$. Thus, there exists some $N \in \mathbb{N}$ such that for $n \geq N$, we have that

$$d(s_n, (0, y)) = \sqrt{(x_n - 0)^2 + (y_n - y)^2} < \epsilon.$$

However, if we take n large, then $x_n \to 0$, and thus $\frac{1}{x_n} \to \infty$. then $d(s_n, (0, y)) > \epsilon$ since

$$\begin{aligned} d(s_n, (0, y)) &= \sqrt{(x_n - 0)^2 + (y_n - y)^2} \\ &= \sqrt{(x_n)^2 + (\frac{1}{x_n^2} - y)^2} \end{aligned}$$

Thus, the graph G only has limit points where it can achieve them, and is thus closed.

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Problem 3

Suppose that (K_n) is a nested sequence of compact nonempty sets, $K_1 \supset K_2 \supset \dots$, and $K = \bigcap K_n$. If for some $\mu > 0$, $\text{diam} K_n \geq \mu$ for all n, is it true that $\text{diam} K \geq \mu$

Solution: **Yes.** Let $x_n, y_n \in K_n$ with $d(x_n, y_n) \geq \mu$. We know these exist by assumption. Evidently, $(x_n) \in K_1$, and since K_1 is compact, there exists some convergent subsequence $x_{n_k} \to x$. Since K_1 is closed, then $x \in K_1$. We must have that $x \in K_2$ since except for maybe a few terms of x_{n_k}, $(x_{n_k}) \in K_2$, for if it weren't, then $x_n \notin K_n$. Continuing with this logic, $x \in K_n$ for all n, and thus $x \in K$. Similarly, $y \in K$. Thus, for k large.

$$\begin{aligned}
\mu &\leq d(x_{n_k}, y_{n_k}) \\
&\leq d(x, x_{n_k}) + d(x, y) + d(y_{n_k}, y) \\
&< \frac{\epsilon}{2} + d(x, y) + \frac{\epsilon}{2} \\
&= d(x, y) + \epsilon.
\end{aligned}$$

Because this is true for all $\epsilon > 0$, we have that $\text{diam} K \geq \mu$. Note that here we use the fact that we can sample from the same subsequence by taking sub subsequences as shown in class.

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Problem 4

The distance from a point p in a metric space M to a nonempty subset $S \subset M$ is defined to be $\mathrm{dist}(p, S) = \inf\{d(p, s) : s \in S\}$.

(a) Show that p is a limit of S if and only if $\mathrm{dist}(p, S) = 0$.

SOLUTION:

- ($\Longrightarrow$) Suppose p is a limit of S, and suppose for the sake of contradiction that $\mathrm{dis}(p, S) = \mu$ for some $\mu > 0$. Thus, there does not exist any $s \in S$ such that $d(p, s) < \mu$. Since p is a limit of S, then there exists some sequence $s_n \in S$ such that $s_n \to p$, and thus there exists some n large enough such that

$$d(p, s_n) < \frac{\mu}{2}.$$

Oops!

- ($\Longleftarrow$) Suppose $\mathrm{dist}(p, S) = 0$. For all $n \in \mathbb{N}$, there exists some $s_n \in S$ such that

$$d(p, s_n) < \frac{1}{n},$$

as otherwise, $s \geq \frac{1}{n}$ for all $s \in S$, and so $\mathrm{dist}(p, S) \geq \frac{1}{n}$. Thus, we have built a sequence $s_n \to p$ with all $s_n \in S$, and so p is a limit of S.

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(b) Show that $p \to \mathrm{dist}(p, S)$ is a uniformly continuous function of $p \in M$.

SOLUTION: Suppose $p, q \in M$ and let $\epsilon > 0$. There exists $\delta = \frac{\epsilon}{2}$ such that if $p, q \in M$ with $d(p, q) < \delta$, we have the following. For all $s \in S$, we have that

$$d(p, s) \leq d(p, q) + d(q, s), \qquad d(q, s) \leq d(q, p) + d(p, s)$$

and thus

$$\mathrm{dist}(p, S) \leq d(p, q) + \mathrm{dist}(q, S), \qquad \mathrm{dist}(q, S) \leq d(p, q) + \mathrm{dist}(p, S)$$

Thus, combining the inequalities,

$$d(\mathrm{dist}(q, S), \mathrm{dist}(q, S)) \leq 2d(p, q) < 2\frac{\epsilon}{2}.$$

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Problem 5

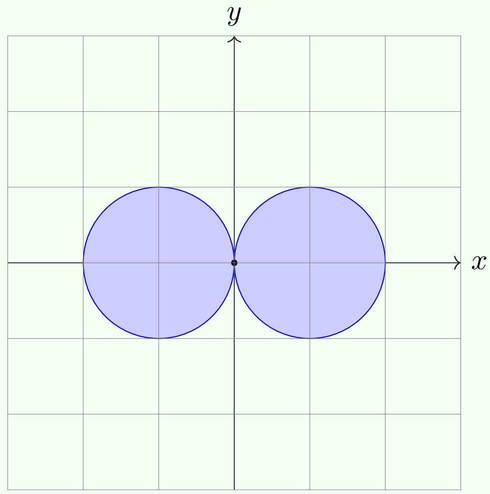
Prove that the 2-sphere is not homeomorphic to the plane.

Solution: S^2 is compact since it fits inside a box $([0,1]^3)$ in $\mathbb{R}^2$. The plane is not compact. Thus, they are not homeomorphic. ■

Problem 6

If S is connected, is the interior of S connected? Prove this or give a counterexample.

Solution:



Consider the union of two balls who intersect at the origin. Evidently, each ball is connected since they are path connected. The union of connected sets sharing a common point is connected (Theorem in book). Thus, the union is connected. Consider that the interior of the union does not contain the origin since any ball around the origin contains points in the plane not in either ball. Thus, we can express the interior of the union as two open balls which are disjoint and nonempty, and thus the interior of the union is disconnected. ■

Problem 7

A function $f : (a, b) \to \mathbb{R}$ satisfies a $\alpha-$Hölder condition of order α if $\alpha > 0$, H is a constant, and for all $u, x \in (a, b)$, we have

$$|f(u) - f(x)| \leq H|u - x|^{\alpha}$$

(a) Prove that an $\alpha-$Hölder function defined on (a, b) is uniformly continuous and infer that it extends uniquely to a continuous function defined on $[a, b]$. Is the extended function $\alpha-$Hölder?

Solution: Let $\epsilon > 0$. For all $u, x \in (a, b)$, there exists a $\delta = \min\{1, \frac{\epsilon}{|H|+1}\}$ such that if $|u - x| < \delta$, we have that by the $\alpha-$Hölder condition,

$$|f(u) - f(x)| \leq H|u - x|^{\alpha} < H|u - x| < |H|\frac{\epsilon}{|H| + 1} < \epsilon.$$

To extend this function to $[a, b]$, we need to define $f(a)$ and $f(b)$. To do this, let

$$f(a) = \lim_{x \to a^+} f(x), \qquad f(b) = \lim_{x \to b^-} f(x).$$

Since limits are unique, then this is a unique extension on $[a, b]$. Continuity comes directly from the construction (See PSET 2). We claim that the extended function is $\alpha-$Hölder

Let $x \in (a, b]$, we want to show that

$$|f(a) - f(x)| \leq H|a - x|^{\alpha}.$$

By continuity, for all $\epsilon > 0$, there exists some $\delta > 0$ and $u \in (a, x)$ such that if $x - a < \delta$, we have that $|f(a) - f(x)| < \epsilon$. Thus,

$$\begin{aligned} |f(a) - f(x)| &\leq |f(a) - f(u)| + |f(u) - f(x)| \\ &< \epsilon + H|u - x|^{\alpha} \\ &\leq \epsilon + H|a - x|^{\alpha} \end{aligned}$$

Where the last inequality is justified since $|u - x| \leq |a - x|$ because $u \in (a, x)$. Thus, as $\epsilon \to 0$, we have that f is $\alpha-$Hölder continiuos at a. A similar argument can be shown for b. ■

(b) What does $\alpha-$Hölder continuity mean when $\alpha = 1$?

Solution: If $\alpha = 1$, we have that for all $u, x \in (a, b)$,

$$|f(u) - f(x)| \leq H|u - x|,$$

and thus f satisfies a global Lipschitz condition. Specifically, if f is differentiable, then

$|f'(x)| \leq H$ for all $x \in (a,b)$. Thus, the derivative is bounded for all $x \in (a,b)$. ■

(c) Prove that $\alpha-$Hölder continuity when $\alpha > 1$ implies that f is constant.

SOLUTION: Let $x \in (a,b)$. Because f is $\alpha-$Hölder, then

$$|f(u) - f(x)| \leq H|u-x|^{\alpha}.$$

Thus,

$$0 \leq \left|\frac{f(u)-f(x)}{u-x}\right| \leq H|u-x|^{\alpha-1}$$

Because $\alpha > 1$, then $|u-x|^{\alpha-1}$ is a well defined denominator. Consider that by continuity:

$$\lim_{u\to x}(0) = 0, \qquad H\lim_{u\to x}|u-x|^{\alpha-1} = H(0) = 0,$$

where the second limit comes from the fact that for u close to x, because $\alpha - 1 > 0$, we have that $|u-x| > |u-x|^{\alpha-1}$. Thus, by the squeeze theorem, we have that

$$\lim_{u\to x}\left|\frac{f(u)-f(x)}{u-x}\right| = |f'(x)| = 0$$

It suffices to show that if $f'(x) = 0$ for any $x \in (a,b)$, we have that f is constant. Let $y < x \in (a,b)$ By mean value theorem, we have that $f(y) - f(x) = f'(\lambda)(x-a) = 0$, and thus $f(y) = f(x)$. ■

Problem 8

For each $r \geq 1$, find a function that is C^r but not C^{r+1}.

SOLUTION: Let $f_r : \mathbb{R} \to \mathbb{R}$ be defined by $f_r(x) = x^r|x|$. We claim that f_r is C^r but not C^{r+1}. We proceed by induction on $\mathbb{N}$ over the variable r.

(a) For $r = 1$, we have that $f_1(x) = x|x|$. We claim that $f'(x) = 2|x|$. To see this, let $x > 0$, then $f_1'(x) = 2x$. If $x < 0$, then $f_1'(x) = -2x$. If $x = 0$, then $f_1'(0) = 0$. Thus, $f_1'(x) = 2|x|$, which is continuous everywhere. Evidently, $f_1''(x)$ is not differentiable at $x = 0$.

(b) Assume $f_r(x) = x^r|x|$. is C^r. Note that by similar work done above, $f_r^{(r)}(x) = r!|x|$. Also assume that $f_r(x)$ is not C^{r+1}

(c) We claim that $f_{r+1}(x) = x^{(r+1)}|x|$ is C^{r+1}. We have that, $f_{r+1}^{(r+1)}(x) = (r+1)!|x| = r!|x| + (r+1)|x|$, which is continuous by hypothesis. Evidently, since $(r+1)|x|$ is not differentiable, we have that $f_{r+1}^{(r+2)}(x)$ is not continuous at $x = 0$, and thus $f_{r+1} \notin C^{r+1}$.

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