270 Spring 2025: Problem Set 2 Agustín Esteva
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In this assignment, you may assume that holomorphic functions are infinitely differentiable. Moreover, if $O \subset \mathbb{C}$ is open, then for any $z \in \overline{D_r(z_0)} \subseteq O$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n}(z - z_0)^n,$$

where the series absolutely converges inside $D_r(z_0)$.

Problem 1

Suppose $O \subset \mathbb{C}$ is open, $f : O \to \mathbb{C}$ and $f'(z_0)$ exists for some $z_0 \in O$. If $z_0 = x_0 + iy_0$ and $u(x, y), v(x, y)$ are defined as the real and imaginary components of $f(z)$ respectively, then

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$$

Solution: Since f is differentiable at z_0, then it's partial's exist. By the hint,

$$\begin{aligned} f'(z_0) &= f'(x_0, y_0) \\ &= \lim_{h \to 0} \frac{[u(x_0 + h, y_0) + iv(x_0 + h, y_0)] - [u(x_0, y_0) - iv(x_0, y_0)]}{h} \\ &= \lim_{h \to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i \lim_{h \to 0} \frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \end{aligned}$$

and

$$\begin{aligned} f'(z_0) &= f'(x_0, y_0) \\ &= \lim_{h \to 0} \frac{[u(x_0, y_0 + h) + iv(x_0, y_0 + h)] - [u(x_0, y_0) - iv(x_0, y_0)]}{ih} \\ &= \frac{\partial v}{\partial y}(x_0, y_0) + \frac{1}{i}\frac{\partial u}{\partial y}(x_0, y_0) \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0) \end{aligned}$$

Thus, we have that

$$\mathrm{Re}\{f'(z_0)\} = \frac{\partial u}{\partial x}(z_0), \quad \mathrm{Re}\{f'(z_0)\} = \frac{\partial v}{\partial y}(z_0) \implies \frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0).$$

Similarly,

$$\frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$$

■

Problem 2

Suppose $f \in H(O)$, where $O \subset \mathbb{C}$ is open. If $f : O \to \mathbb{C}$. If $f(z) \in \mathbb{R}$ for all $z \in O$, then f is constant.

SOLUTION: Consider that for any $z \in O$, $f(z) = u(z)$. Thus, $f = u$ and $v = 0$. By problem 3 on the previous PSET, it suffices to see that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

Note that since $f \in H(O)$, f is differentiable for all of O. By Problem 1 above, we have that for any $z \in O$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. But $v = 0$, and thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ Similarly, $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$. Thus, we are done, since all the partials are zero. ■

Problem 3

Suppose $f \in H(O)$. Prove that if $z \in O$, then

$$\nabla u(z) = \nabla v(z) = 0.$$

SOLUTION: We compute using Problem 1. Let $z \in O$ such that $z = x + iy$. Then

$$\begin{aligned}
\nabla u(z) &= \nabla u(x, y) \\
&= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&= \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) \\
&= \frac{\partial}{\partial x}(\frac{\partial v}{\partial y}) + \frac{\partial}{\partial y}(-\frac{\partial v}{\partial x}) \\
&= \frac{\partial}{\partial x}(\frac{\partial v}{\partial y}) - \frac{\partial}{\partial y}(\frac{\partial v}{\partial x}) \\
&= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} \\
\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} & \\
&= 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nabla v(z) &= \nabla v(x, y) \\
&= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \\
&= \frac{\partial}{\partial x}(\frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(\frac{\partial v}{\partial y}) \\
&= \frac{\partial}{\partial x}(-\frac{\partial u}{\partial y}) + \frac{\partial}{\partial y}(\frac{\partial u}{\partial x}) \\
&= -\frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) + \frac{\partial}{\partial y}(\frac{\partial u}{\partial x}) \\
&= -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} \\
&= -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} \\
&= 0
\end{aligned}$$

Here, we use a few facts from multi-variable calculus. In particular, we use the fact that derivatives are linear and the Hessian matrix is symmetric. ■

Problem 4

Suppose $f \in H(O)$ where $O \subset \mathbb{C}$ is a connected open set. Suppose that for some $z_0 \in O$, f has a zero of infinite order. That is, $f^{(n)}(z_0) = 0$ for any $n \geq 0$. Show that $f(z) = 0$ for any $z \in O$.

Solution: Let

$$A := \{z \in O \mid f(z) = 0\}.$$

We claim that $A \neq \emptyset$, and that A is clopen.

Note that $A \neq \emptyset$ since $z_0 \in A$ since $f(z_0) = f^{(0)}(z_0) = 0$.

Let $z' \in A$. Since O is open, there exists some $r > 0$ such that $D_r(z') \subseteq O$. Thus, $\overline{D_{\frac{r}{2}}(z')} \subset O$. Let $z \in \overline{D_{\frac{r}{2}}(z')}$, then since $f \in H(O)$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z')}{n!}(z-z')^n = f(z') + f'(z')(z-z') + \frac{1}{2}f''(z')(z-z')^2 + \cdots + \frac{1}{n!}f^{(n)}(z')(z-z')^n.$$

We know that since $z' \in A$, then $f(z') = f'(z') = \cdots = f^{(n)}(z') = 0$, and thus $f(z) = 0$. Then we have that $z \in A$ and so A is open.

Let $(z_n) \in A$ such that $z_n \to z$. Since $z_n \in A$, then $f(z_n) = 0$ for each n. Since f is differentiable, then it absolutely must be continuous, and so $f(z_n) \to f(z)$, and thus $f(z) = 0$. We have showed that A is closed.

Since A is nonempty, open, and closed, and $O \supset A$ is connected, then $A = O$.[a] ■

[a] To see a proof of this, see previous PSET

Problem 5

Suppose that we define

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \sin z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}(-1)^n, \quad \cos z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}(-1)^n, \quad \forall\, z \in \mathbb{C}.$$

Prove that

$$e^{iz} = \cos z + i \sin z.$$

Solution: By definition, we compute:

$$\begin{aligned} e^{iz} &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\ &= 1 + iz - \frac{1}{2}z^2 - \frac{1}{3!}iz^3 + \frac{1}{4!}z^4 + \frac{1}{5!}iz^5 + \cdots \\ &= (1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \cdots) + (iz - \frac{1}{3!}iz^3 + \frac{1}{5!}iz^5 + \cdots) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\ &= \cos z + i \sin z \end{aligned}$$

$\blacksquare$

Problem 6

Suppose $O_1 \subseteq O_2$, where O_1 is open in $\mathbb{C}$ and O_2 is open and connected in $\mathbb{C}$. Suppose $f \in H(O)$. We say that $F \in H(O_2)$ is an **analytic continuation** of f on O_2 if $F(z) = f(z)$ for any $z \in O_1$. Prove that analytic continuations are unique.

SOLUTION: Let F_1, F_2 be analytic continuations of f on O_2. We want to show that $F_1 - F_2 = 0$. Since $F_1, F_2 \in H(O)$, then we can take infinite derivatives of $F_1 - F_2$. Consider that since $O_2 \subseteq \mathbb{C}$ is a connected open set, and if $z_0 \in O_1$, then

$$F_1^{(n)}(z_0) - F_2^{(n)}(z_0) = f^{(n)}(z_0) - f^{(n)}(z_0) = 0, \quad \forall n \geq 0.$$

Thus, by problem 4, we have that

$$(F_1 - F_2)(z) = 0, \quad \forall z \in O_2$$

That is, $F_1(z) = F_2(z)$ for any $z \in O_2$. ■

Problem 7

Assume that $f : O \to \mathbb{C}$ is continuous on an open connected set $O \subset \mathbb{C}$ such that

$$\int_\gamma f(z)dz = 0$$

for any closed path γ on O. Prove that f is holomorphic on O.

SOLUTION:

Lemma 1. Let $h \in \mathbb{C}$. For any $z \in \mathbb{C}$, we have that if γ is the straight line from z to $z + h$, then

$$\int_\gamma d\zeta = h$$

Proof. Clearly, w is a primitive to 1 since $w' = 1$. Thus, we have that by the fundamental theorem of path integrals,

$$\int_\gamma 1 d\zeta = \gamma(1) - \gamma(0) = z + h - z = h$$

$\square$

Let $z_0 \in O$. By the openness of O, there exists some $r > 0$ such that $\overline{D_r(z_0)} \subseteq O$. We define $F : O \to \mathbb{C}$ as

$$F(z) = \int_{\gamma(z)} f(\zeta)d\zeta,$$

where γ is a path from from z_0 to z. Indeed, such a path must exist since O is connected and thus polygonally connected.

To see that F is well defined, let γ and β be two paths that start at z_0 and end at z. Then $\gamma \circ \beta$ is a closed path starting from z_0 and ending z_0. Thus, we have by assumption that

$$\int_{\gamma \circ \beta} f(z)dz = 0$$

but

$$\int_{\gamma \circ \beta} f(z)dz = \int_\gamma f(z)dz - \int_\beta f(z)dz = 0 \implies \int_\gamma f(z)dz = \int_\beta f(z)dz.$$

Thus, F is indeed well defined.

Moreover, we claim that if $[z, z + h]$ is the straight path from z to $z + h$, then for small enough h such that $z + h \in O$, we claim that

$$F(z + h) - F(z) = \int_{[z,z+h]} f(\zeta)d\zeta.$$

To see this, we can, by the path independence shown above, take $\gamma(z), \gamma(z + h)$ to be the polygonal paths. For $h < r$, where $r > 0$ such that the convex set $D_r(z) \subseteq O$ by openness, we can take $[z, z+h]$ to be the straight line from z to $z+h$. Thus, since $\gamma(z)\circ[z, z+h]\circ(-\gamma(z+h))$ is the closed path starting and ending at z_0, we know that

$$0 = \int_{\gamma(z)\circ[z,z+h]\circ(-\gamma(z+h))} f(\zeta)d\zeta = F(z) + \int_{[z,z+h]} f(\zeta)d\zeta - F(z+h).$$

Thus,

$$F(z+h) - F(z) = \int_{[z,z+h]} f(\zeta)d\zeta$$

We claim that on O, we have that $F'(z) = f(z)$. Let $\epsilon > 0$. Then since f is continuous at z, then there exists some $\delta > 0$ such that if $|z-\zeta| < \delta$, then $|f(z)-f(\zeta)| < \epsilon$. Take $h < \min\{\delta, r\}$, where $r > 0$ such that $D_r(z) \subseteq O$ by the openness of O. Then $\max_{\zeta\in[z,z+h]} |f(\zeta) - f(z)| < \epsilon$. Thus, since $\text{length}|\eta| = |h|$, we have that

$$\begin{aligned}
\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| &= \left|\frac{\int_{\gamma(z+h)} f(\zeta)d\zeta - \int_{\gamma(z)} f(\zeta)d\zeta}{h} - f(z)\right| \\
&= \left|\frac{1}{h}\left[\int_{[z,z+h]} f(\zeta)d\zeta - f(z)h\right]\right| \\
&= \left|\frac{1}{h}\left[\int_{[z,z+h]} f(\zeta)d\zeta - f(z)\int_{[z,z+h]} d\zeta\right]\right| \\
&= \left|\frac{1}{h}\int_{[z,z+h]} f(\zeta) - f(z)d\zeta\right| \\
&\leq \max_{\zeta\in[z,z+h]} |f(\zeta) - f(z)||h| \\
&< \frac{1}{|h|}\epsilon|h| \\
&= \epsilon
\end{aligned}$$

Since F is differentiable in the disk, then it is infinitely differentiable. Since f is one of those derivatives, then it also is infinitely differentiable. Thus, f is holomorphic on this disk. ■