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Due Date: 05/02/2025

Problem 1

Consider the following variant of the branching process. At each time n, each individual produces offspring independently with offspring distribution $\{p_j\}_{j\geq 0}$, then dies with probability $q \in (0,1)$. So, each individual reproduces k times where k is its lifetime (k is a random positive integer). Note that $q = 1$ for the version of the branching process discussed in class. Assuming that $X_0 = 1$, show that if $\phi(t) = \sum_{j=0}^{\infty} t^j p_j$ is the generating function, then the extinction probability is the smallest positive solution to

$$q\phi(t) + (1-q)t\phi(t) = t.$$

Solution: Conditioning,

$$\begin{aligned}
a &= \mathbb{P}\{\text{extinction} \mid X_0 = 1\} \\
&= \sum_{k=0}^{\infty} \mathbb{P}\{X_1 = k\}\mathbb{P}\{\text{extinction} \mid X_1 = k\} \\
&= \sum_{k=0}^{\infty} \mathbb{P}\{X_1 = k \mid \text{fucking dies}\}\mathbb{P}\{\text{fucking dies}\} + \mathbb{P}\{X_1 = k \mid \text{lives}\}\mathbb{P}\{\text{lives}\})a^k \\
&= qp_0 + \sum_{k=1}^{\infty}(p_k q + p_{k-1}(1-q))a^k \\
&= \sum_{k=0}^{\infty}(p_k q + p_{k-1}(1-q))a^k \\
&= \sum_{k=1}^{\infty} p_k q a^k + \sum_{k=1}^{\infty}(1-q)p_{k-1}a^k \\
&= \sum_{k=1}^{\infty} p_k q a^k + \sum_{k=0}^{\infty}(1-q)p_k a^{k+1} \\
&= q\phi(a) + (1-q)a\phi(a)
\end{aligned}$$

We claim that the extinction probability is the smallest possible solution to $\varphi(a) = a$. We note by the fourth line above that the generating function is

$$\varphi(a) = qp_0 + \sum_{k=1}^{\infty}(p_k q + p_{k-1}(1-q))a^k,$$

and so by a result in class, the extinction probability is given by the smallest positive solution to $a = \varphi(a) = \cdots = q\phi(a) + (1-q)a\phi(a)$ ■

Problem 2

Let $\{X_n\}$ be a branching process with offspring distribution $\{p_j\}_{j\geq 0}$ and let $\phi(a)$ be the generating function $\sum_{j=0}^{\infty} p_k a^k$. We let $\phi^{(n)} = \phi \circ \phi \circ \cdots \circ \phi$ composed n times. Show that for $n \geq 1$,

$$\mathbb{P}\{X_n = 0 \mid X_{n-1} \neq 0\} = \frac{\phi^{(n-1)}(p_0) - \phi^{(n-1)}(0)}{1 - \phi^{(n-1)}(0)}.$$

Solution: By the law of total probability, we have that

$$\mathbb{P}\{X_n = 0\} = \mathbb{P}\{X_n = 0, X_{n-1} = 0\} + \mathbb{P}\{X_n = 0, X_{n-1} \neq 0\}$$

So we use Bayes rule:

$$\begin{aligned}
\mathbb{P}\{X_n = 0 \mid X_{n-1} \neq 0\} &= \frac{\mathbb{P}\{X_n = 0, X_{n-1} \neq 0\}}{\mathbb{P}\{X_{n-1} \neq 0\}} \\
&= \frac{\mathbb{P}\{X_n = 0, X_{n-1} \neq 0\}}{1 - \mathbb{P}\{X_{n-1} = 0\}} \\
&= \frac{\mathbb{P}\{X_n = 0\} - \mathbb{P}\{X_n = 0, X_{n-1} = 0\}}{1 - \mathbb{P}\{X_{n-1} = 0\}} \\
&= \frac{\mathbb{P}\{X_n = 0\} - \mathbb{P}\{X_{n-1} = 0\}}{1 - \mathbb{P}\{X_{n-1} = 0\}} \\
&= \frac{\phi^{(n)}(0) - \phi^{(n-1)}(0)}{1 - \phi^{(n-1)}(0)},
\end{aligned}$$

and by a result in class

$$\phi^{(n)}(0) = \phi^{(n-1)}(\phi(0)) = \phi^{(n-1)}(p_0),$$

and so we are done.

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Problem 3

Let $\{X_t\}$ and $\{Y_t\}$ be independent Poisson processes with rates λ_1 and λ_2, respectively. We imagine that $\{X_t\}$ and $\{Y_t\}$ count the number of calls on two different phone lines, with the time t measured in hours.

(a) Find the probability that there were 5 calls on line 1 between times 0 and 1 and 5 calls on line 1 between times 1 and 2.

SOLUTION: We are asked to find $\mathbb{P}\{X_1 - X_0 = 5, X_2 - X_1 = 5\}$. By the memorylessness of the Poisson-Process, these events are independent and are both distributed Poisson with parameter λ_1. Thus

$$\mathbb{P}\{X_1 - X_0 = 5, X_2 - X_1 = 5\} = \mathbb{P}\{X_1 = 5\}\mathbb{P}\{X_2 - X_1 = 5\} = \boxed{(\frac{e^{\lambda_1}\lambda_1^5}{5!})^2}$$

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(b) Given that there were 10 total calls (on both lines) between time 1 and time 2, find the conditional probability that all 10 calls were on line 1.

SOLUTION: We use the formula derived in problem session to see that

$$\begin{aligned}\mathbb{P}\{X_2 - X_1 = 10 \mid (X_2 - X_1) + (Y_2 - Y_1) = 10\} &= \mathbb{P}\{X_1 = 10 \mid X_1 + Y_1 = 10\} \\ &= \binom{10}{10}(\frac{\lambda_1}{\lambda_1 + \lambda_2})^{10}(\frac{\lambda_2}{\lambda_1 + \lambda_2})^0 \\ &= \boxed{(\frac{\lambda_1}{\lambda_1 + \lambda_2})^{10}}\end{aligned}$$

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(c) Let T_1 (resp. T_2) be the time of the first call on line 1 (resp. line 2). Find $\mathbb{E}[\min\{T_1, T_2\}]$.

SOLUTION: From class, we know that if

$$\tau := \min\{T_1, T_2\},$$

then since $T_1 \sim \exp\{\lambda_1\}$ and $T_2 \sim \exp\{\lambda_2\}$, then $\tau \sim \exp\{\lambda_1 + \lambda_2\}$, and thus

$$\mathbb{E}[\tau] = \boxed{\frac{1}{\lambda_1 + \lambda_2}}$$

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(d) Find the distribution of the number of calls on line 1 before time T_2 (i.e., find $\mathbb{P}[X_{T_2} = k]$ for each k).

SOLUTION: We use the Gamma function,

$$\begin{aligned}
\mathbb{P}\{X_{T_2} = k\} &= \int_0^\infty \mathbb{P}\{X_t = k \mid T_2 = t\}\mathbb{P}\{T_2 = t\}\, dt \\
&= \int_0^\infty \mathbb{P}\{X_t = k \mid T_2 = t\}(\lambda_2 e^{-\lambda_2 t})\, dt \\
&= \int_0^\infty \frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}\lambda_2 e^{-\lambda_2 t}\, dt \\
&= \frac{\lambda_2}{k!}\int_0^\infty (\lambda_1 t)^k e^{-(\lambda_1+\lambda_2)t}\, dt \\
&= \frac{\lambda_2 \lambda_1^k}{k!}\int_0^\infty t^k e^{-(\lambda_1+\lambda_2)t}\, dt \\
&= \frac{\lambda_2 \lambda_1^k}{(\lambda_1+\lambda_2)k!}\int_0^\infty (\frac{u}{\lambda_1+\lambda_2})^k e^{-u}\, du \\
&= \frac{\lambda_2 \lambda_1^k}{k!(\lambda_1+\lambda_2)^{k+1}}\Gamma(k+1) \\
&= \frac{k!}{k!}\frac{\lambda_2}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k
\end{aligned}$$

Thus,

$$\boxed{X_{T_2} \sim \text{Geometric}(\frac{\lambda_2}{\lambda_1+\lambda_2})}$$

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Problem 4

Suppose that traffic on a road follows a Poisson process with rate $\lambda > 0$ cars per minute. A chicken needs a (time) gap of length at least c minutes in the traffic to cross the road. In this problem, we wish to compute the time the chicken will have to wait to cross the road. To that end, we let τ_j be the arrival time of the j-th car, and let

$$J := \min\{j : \tau_j - \tau_{j-1} > c\}.$$

Ideally, the chicken will start to cross the road at time τ_{J-1} and complete its journey at time $\tau_{J-1} + c$.

(a) Suppose T is exponentially distributed with rate $\lambda > 0$. Compute $\mathbb{E}[T \mid T < c]$.

Solution: First, note that

$$\mathbb{P}\{T < c\} = \int_0^c \mathbb{P}\{T = t\}\, dt$$

Using Bayes rule and the law of total expectation, we have that

$$\begin{aligned}
\mathbb{E}[T \mid T < c] &= \int_0^\infty t\mathbb{P}\{T = t \mid T < c\}\, dt \\
&= \int_0^\infty t\frac{\mathbb{P}\{T = t, T < c\}}{\mathbb{P}\{T < c\}}\, dt \\
&= \int_0^c t\frac{\mathbb{P}\{T = t\}}{\mathbb{P}\{T < c\}}\, dt \\
&= \int_0^c t\frac{f_T(t)}{F_T(t)}\, dt \\
&= \int_0^c t\frac{\lambda e^{-\lambda t}}{1 - e^{-\lambda c}}\, dt \\
&= \frac{1}{1 - e^{-\lambda c}} \int_0^c t\lambda e^{-\lambda t}\, dt \\
&= \frac{1}{1 - e^{-\lambda c}} \frac{1 - e^{-c\lambda}(c\lambda + 1)}{\lambda} \\
&= \boxed{\frac{1}{\lambda} - \frac{c}{1 - e^{-\lambda c}}}
\end{aligned}$$

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(b) Use part (a) to find $\mathbb{E}[\tau_{J-1} + c]$.

Solution: Consider that $J \sim$ Geometric since we need to 'fail' $J - 1$ times, that is, the first $J - 1$ cars must come within c of each other, and succeed once, the Jth car will be $> c$ time from the last. Using various properties of the Poisson process, such as memorylessness and independence of arrival times, then

$$\begin{aligned}
\mathbb{P}\{J = j\} &= \mathbb{P}\{\tau_{j-1} - \tau_{j-2} \leq c, \tau_{j-2} - \tau_{j-3} \leq c, \ldots, \tau_1 \leq c\}\mathbb{P}\{\tau_j - \tau_{j-1} > c\} \\
&= \mathbb{P}\{\tau_1 \leq c\}^{j-1}\mathbb{P}\{\tau_1 > c\} \\
&= (1 - e^{-\lambda c})^{j-1}(e^{-\lambda c})
\end{aligned}$$

and so

$$\mathbb{E}[J-1] = \frac{1 - e^{-\lambda c}}{e^{-\lambda c}}$$

Using the fact that the arrival times are independent, wenote that

$$\begin{aligned}
\mathbb{E}[\tau_{J-1} + c] &= \mathbb{E}[\tau_{J-1}] + c \\
&= \mathbb{E}[\mathbb{E}[\tau_{j-1} \mid J = j]] + c \\
&= c + \sum_{j=1}^{\infty} \mathbb{E}[\tau_{j-1} \mid J = j]\mathbb{P}\{J = j\} \\
&= c + \sum_{j=1}^{\infty} \mathbb{P}\{J = j\} \left(\mathbb{E}[\tau_{j-1} - \tau_{j-2} \mid J = j] + \mathbb{E}[\tau_{j-2} - \tau_{j-3} \mid J = j] + \cdots + \mathbb{E}[\tau_1 - \tau_0 \mid J = j]\right) \\
&= c + \sum_{j=1}^{\infty} \mathbb{P}\{J = j\} \left(\mathbb{E}[\tau_{j-1} - \tau_{j-2} \mid \tau_{j-1} - \tau_{j-2} < c] + \cdots + \mathbb{E}[\tau_1 - \tau_0 \mid \tau_1 - \tau_0 < c]\right) \\
&= c + \sum_{j=1}^{\infty} \mathbb{P}\{J = j\} \left((j-1)\mathbb{E}[\tau_1 \mid \tau_1 < c]\right) \\
&= c + \mathbb{E}[\tau_1 \mid \tau_1 < c] \sum_{j=1}^{\infty} \mathbb{P}\{J - 1 = j - 1\}(j-1) \\
&= c + \mathbb{E}[\tau_1 \mid \tau_1 < c]\mathbb{E}[J-1] \\
&= \boxed{c + \frac{1 - e^{-\lambda c}}{e^{-\lambda c}}\left(\frac{1}{\lambda} - \frac{c}{1 - e^{-\lambda c}}\right)}
\end{aligned}$$

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Problem 5

Consider the continuous-time Markov chain $\{X_t\}_{t\geq 0}$ with state space $\{1,2,3\}$ and infinitesimal generator matrix:

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 4 & -5 & 1 \\ 0 & 4 & -4 \end{pmatrix}$$

(a) If we start in state 1, what is the expected time that we move to a different state?

SOLUTION: The rate of leaving state 1 is given by $A_{1,2}+A_{1,3}=1$, and thus if T denotes the time when we leave, we have that $T \sim \exp\{\alpha(1)\}$ and thus

$$T \sim \exp\{1\} \implies \boxed{\mathbb{E}[T]=1}$$

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(b) If we start in state 2, what is the expected time that we move to a different state?

SOLUTION: Similarly to above, if we define T as leaving state two, then $T \sim \exp\{\alpha(2)\}$, and so

$$\boxed{\mathbb{E}[T] = \frac{1}{5}}$$

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(c) Explain why this Markov chain is irreducible and find the stationary distribution π.

SOLUTION:

$$\begin{aligned} 0 &= \det(A-\lambda I) \\ &= \det\left(\begin{pmatrix} -1-\lambda & 1 & 0 \\ 4 & -5-\lambda & 1 \\ 0 & 4 & -4-\lambda \end{pmatrix}\right) \\ &= -\lambda^3 - 10\lambda^2 - 21\lambda \\ &= \lambda(\lambda+7)(\lambda+3) \end{aligned}$$

Solving for the null spaces we find that

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \frac{1}{8} \\ \frac{-3}{4} \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ 1 \end{pmatrix},$$

and so

$P_t = e^{tA}$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \\
&= \begin{pmatrix} 1 & \frac{1}{8} & \frac{-1}{8} \\ 1 & \frac{-3}{4} & \frac{1}{4} \\ 1 & 1 & 1 \end{pmatrix} \left(\sum_{n=1}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -7t & 0 \\ 0 & 0 & -3t \end{pmatrix}^n \right) \begin{pmatrix} 1 & \frac{1}{8} & \frac{-1}{8} \\ 1 & \frac{-3}{4} & \frac{1}{4} \\ 1 & 1 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & \frac{1}{8} & \frac{-1}{8} \\ 1 & \frac{-3}{4} & \frac{1}{4} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-7t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{8} & \frac{-1}{8} \\ 1 & \frac{-3}{4} & \frac{1}{4} \\ 1 & 1 & 1 \end{pmatrix}^{-1} \\
&= \frac{e^{-7t}}{84} \begin{pmatrix} 2\left(7e^{4t} + 32e^{7t} + 3\right) & -7e^{4t} + 16e^{7t} - 9 & -7e^{4t} + 4e^{7t} + 3 \\ -4\left(7e^{4t} - 16e^{7t} + 9\right) & 2\left(7e^{4t} + 8e^{7t} + 27\right) & -2\left(-7e^{4t} - 2e^{7t} + 9\right) \\ 16\left(-7e^{4t} + 4e^{7t} + 3\right) & -8\left(-7e^{4t} - 2e^{7t} + 9\right) & 4\left(14e^{4t} + e^{7t} + 6\right) \end{pmatrix}
\end{aligned}
$$

Thus, the Markov chain is irreducible because $p_t(x, y) > 0$ for any $x, y \in S$. Finding the stationary distribution, we send $t \to \infty$ in P_t above, noting that any term with a power less than $7t$ is going to get obliterated:

$$
\begin{aligned}
\lim_{t\to\infty} P_t &= \lim_{t\to\infty} \frac{e^{-7t}}{84} \begin{pmatrix} 2\left(32e^{7t}\right) & 16e^{7t} & 4e^{7t} \\ -4\left(-16e^{7t}\right) & 2\left(8e^{7t}\right) & -2\left(-2e^{7t}\right) \\ 16\left(4e^{7t}\right) & -8\left(-2e^{7t}\right) & 4\left(e^{7t}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{64}{84} & \frac{16}{84} & \frac{4}{84} \\ \frac{64}{84} & \frac{16}{84} & \frac{4}{84} \\ \frac{64}{84} & \frac{16}{84} & \frac{4}{84} \end{pmatrix} \\
&\implies \boxed{\pi = \begin{pmatrix} \frac{16}{21} & \frac{4}{21} & \frac{1}{21} \end{pmatrix}}
\end{aligned}
$$

■

(d) Find $p_t(1, 3)$ for each $t > 0$.

Solution: From the above, one can see that

$$
\boxed{p_t(1,3) = \frac{e^{-7t}}{84}(-7e^{4t} + 4e^{7t} + 3)}
$$

■

(e) Let $\tau_1, \tau_2, \ldots$ be the times of the successive jumps of $\{X_t\}_{t\geq 0}$. Let $\tilde{X}_n = X_{\tau_n}$. Find the transition matrix for the discrete-time Markov chain $\{\tilde{X}_n\}$.

Solution: We have that the state space is given by $S = \{1, 2, 3\}$, but since $\{\tilde{X}_n\}$ is antsy, then $p(i, i) = 0$ for $i = 1, 2, 3$. From A, we note that $\alpha(1, 3) = \alpha(3, 1)$, and so

$p(3,1) = p(1,3) = 0$, since the rate at which X_n traverses both is 0. Thus,

$$P = \begin{pmatrix} 0 & \cdot & 0 \\ \cdot & 0 & \cdot \\ 0 & \cdot & 0 \end{pmatrix} \implies P = \begin{pmatrix} 0 & 1 & 0 \\ \cdot & 0 & \cdot \\ 0 & 1 & 0 \end{pmatrix}$$

Note that

$$\mathbb{P}\{\tilde{X}_1 = 1 \mid \tilde{X}_0 = 2\} = \mathbb{P}\{\tau(2,1) < \tau(2,3)\},$$

where

$$\tau(i,j) := \min\{t \geq 0 : X_\tau = j \mid X_0 = i\}.$$

That is, the probability our antsy Markov chain makes it to 1 is the probability that it switches to 1 before it switches to 3. Since $\tau(2,1) \sim \exp\{4\}$ and $\tau(2,3) \sim \exp\{1\}$. Thus, we see that since both of these r.v. are independent, then

$$\begin{aligned}
\mathbb{P}\{\tau(2,1) < \tau(2,3)\} &= \int_0^\infty \mathbb{P}\{\tau(2,1) = t, \tau(2,3) > t\}\, dt \\
&= \int_0^\infty \mathbb{P}\{\tau(2,1) = t\}\mathbb{P}\{\tau(2,3) > t\}\, dt \\
&= \int_0^\infty 4e^{-4t}e^{-t}\, dt \\
&= 4\int_0^\infty e^{-5t}\, dt \\
&= \frac{4}{5}
\end{aligned}$$

Thus,

$$\boxed{P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 \end{pmatrix}}$$

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Problem 6

Let G be a finite connected graph. The continuous random walk on G with state space $V(G)$ is given by rates $\alpha(x, y) = 1$ if x and y share an edge and zero else. Find the stationary distribution.

Solution: Suppose $V(G) = \{1, 2, \ldots, N\}$. Then

$$A = \begin{pmatrix} \alpha(1,1) & \alpha(1,2) & \cdots & \alpha(1,N) \\ \alpha(2,1) & \alpha(2,2) & \cdots & \alpha(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(N,1) & \alpha(N,2) & \cdots & \alpha(N,N) \end{pmatrix} = \begin{pmatrix} -\deg(1) & \alpha(1,2) & \cdots & \alpha(1,N) \\ \alpha(2,1) & -\deg(2) & \cdots & \alpha(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(N,1) & \alpha(N,2) & \cdots & -\deg(N) \end{pmatrix}.$$

But since sharing edges is an equivalence class, then $\alpha(i, j) = \alpha(j, i)$, and thus

$$A = \begin{pmatrix} -\deg(1) & \alpha(1,2) & \cdots & \alpha(1,N) \\ \alpha(1,2) & -\deg(2) & \cdots & \alpha(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(1,N) & \alpha(2,N) & \cdots & -\deg(N) \end{pmatrix}.$$

Since the graph is finite and connected, the Markov chain is clearly irreducible, and so the unique stationary distribution will satisfy $\pi A = 0$. That is,

$$\begin{aligned} -\deg(1)\pi_1 + \alpha(2,1)\pi_2 + \cdots + \alpha(N,1)\pi_N &= 0 \\ \alpha(1,2)\pi_1 - \deg(2)\pi_2 + \cdots + \alpha(N,2)\pi_N &= 0 \\ &\vdots \\ \alpha(1,N)\pi_1 + \alpha(2,N)\pi_2 + \cdots - \deg(N)\pi_N &= 0 \\ \pi_1 + \pi_2 + \cdots + \pi_N &= 1 \end{aligned}$$

Consider the candidate distribution

$$\pi_1 = \pi_2 = \cdots = \pi_N.$$

Then

$$\begin{aligned} -\deg(1)\pi_1 + \alpha(2,1)\pi_1 + \cdots + \alpha(N,1)\pi_N = -\deg(1)\pi_1 + \deg(1)\pi_1 &= 0 \\ &\vdots \\ \alpha(1,N)\pi_N + \alpha(2,N)\pi_N + \cdots - \deg(N)\pi_N = \deg(N)\pi_N - \deg(N)\pi_N &= 0 \end{aligned}$$

and thus plugging into the final constraint, we see that

$$\boxed{\pi_i = \frac{1}{N}}$$

for all $i = 1, 2, \ldots N$ ■