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Problem 1

Let $\mathcal{R}$ be the ring of all elementary subsets of $(0, 1]$. If $0 < a \leq b \leq 1$, define

$$\phi([a, b]) = \phi([a, b)) = \phi((a, b]) = \phi((a, b)) = b - a,$$

but define

$$\phi((0, b)) = \phi((0, b]) = 1 + b$$

if $0 < b \leq 1$. Show that this gives an additive set function ϕ on $\mathcal{R}$, which is not regular and which cannot be extended to a countably additive set function on a σ-ring.

SOLUTION: (Additive) Let $A, B \in \mathcal{R}$ disjoint. Suppose neither A nor B contain intervals of the form $(0, b]$ or $(0, b)$. Since both A and B are elementary, then we can write A and B as a disjoint union of finite intervals. That is

$$A = \bigcup_{i=1}^{\infty}(a_i, b_i), \qquad B = \bigcup_{i=1}^{\infty}(c_i, d_i), \qquad \text{The parenthesis may be replaced by } [\cdot,\cdot], [\cdot,\cdot), \text{ or } (\cdot,\cdot]$$

Since $A \cap B = \emptyset$, $(a_i, b_i) \cap (c_j, d_j) = \emptyset$ for all i, j. Thus we can write

$$A \cup B = \bigcup_{i=1}^{\infty}(e_i, d_i)$$

where each interval is disjoint and moreover,

$$\phi(A \cup B) = \phi\left(\bigcup_{i=1}^{\infty}(e_i, d_i)\right) = \sum_{n=1}^{\infty}\phi((e_i, d_i)) = \sum_{n=1}^{\infty}\phi(a_i, b_i) + \sum_{n=1}^{\infty}\phi(c_i, d_i) = \phi(A) + \phi(B)$$

Suppose A contains an interval (a_k, b_k) such that $a_k = 0$. Then since A and B are disjoint, it is clear that B cannot also contain some interval $(c_{k'}, d_{k'})$ such that $c_{k'} = 0$, as then the sets would not be disjoint. But then the above formula still holds, noting that we can separate the infinite sum by singling out the (a_k, b_k) interval and adding 1 on both sides.

(Not Regular) Suppose ϕ were regular. Let $A = (0, 1]$. Then $A \in \mathcal{R}$ and $\phi(A) = 1 + 1 = 2$. Let $\epsilon = \frac{1}{2}$ Since ϕ is regular, there exists some closed $F \in \mathcal{R}$ such that $F \subset A$ and

$$\phi(A) \leq \phi(F) + \frac{1}{2}.$$

However, since F is closed and $F \subset A$, then F cannot contain an interval of the form $(0, b)$ or $(0, b]$, as otherwise, it would not be closed. So then $\phi(F) \leq \phi((\frac{1}{n}, 1]) = 1 - \frac{1}{n}$ for any n, and we arrive at a contradiction, since

$$\phi(F) + \frac{1}{2} \leq 1.5 < 2 = \phi(A).$$

(Not countably additive) Suppose that ϕ is countably additive, then consider that

$$(0,1] = \bigcup_{n=1}^{\infty} (\frac{1}{2^n}, \frac{1}{2^{n-1}}].$$

We know that $\phi(0,1] = 2$, but

$$\phi\left(\bigcup_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} \phi\left((\frac{1}{2^n}, \frac{1}{2^{n-1}}]\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Since $1 \neq 2$, we arrive at a contradiction and so ϕ is not countably additive on $\mathcal{R}$. ■

Problem 2

(a) If A is open, then $A \in \mathcal{M}(m)$. If B is closed, then $B \in \mathcal{M}(m)$.

SOLUTION:

Lemma 1. Suppose $U \subset \mathbb{R}^d$ is open. Then it can be uniquely expressed as a countable union of open intervals, where the endpoints of the intervals don't belong to U.

Proof. We will first show the result for $\mathbb{R}$. Let $x \in U$. Define then

$$a_x = \inf\{a \mid (a, x) \subseteq U\}, \qquad b_x = \sup\{b \mid (x, b) \subseteq U\}, \qquad I_x = (a_x, b_x).$$

Suppose, for the sake of contradiction, that $b_x \in U$. There exists some open $J_x \subseteq U$ such that $J \ni b_x$ (by the above construction), so then $A_x = J_x \cup I_x$ is an open interval containing b_x and x. Since A_x is open, then $b_x \in B_r(b_x) \subseteq A_x$, and thus there exists some $b' > b_x$ such that $b' \in A_x \subset U$, and so $b_x \neq \sup\{b \mid (x, b) \subseteq U\}$. Thus, neither b_x nor a_x are in U. Let $y \in U$. Either I_x, I_y are disjoint, or by the above argument, since $I_x \cup I_y$ is an open interval containing x, and y, $I_x = I_y$. Let $q_x \in I_x$ for each disjoint interval in U, where $q_x \in \mathbb{Q}$, then we can count the I_x by the rationals within them.

As a side note, it is valid for $a_x = -\infty$ and $b_x = \infty$.

Consider that since A is open, then for any $x \in A$, there exists some open cube containing x of side length $\frac{1}{2^N}$ where N is large such that the cube is entirely contained in A.

For a general $\mathbb{R}^n$ with $n \geq 2$, consider the grid of cubes formed by the integer lattice $\mathbb{Z}^n$. We know that $A \subset \bigcup_{i=1}^{k} Q_k$, where Q_k are open cubes of side length 1 and volume 1. Refine $\{Q_k\}$ to be just $\{Q_k^{\text{good},1}\} \cup \{Q_k^1\}$. where $Q_k^{\text{good, }1} \subset A$ and $Q_k^1 \cap A \neq \emptyset$ and $Q_k^1 \cap A^c \neq \emptyset$. Throw out all other Q_k. Refine $\{Q_k^1\}$ dividing each Q_k^1 into open cubes of side length $\frac{1}{2}$. Repeat the process of accepting these new cubes into $\{Q_k^{\text{good },2}\}$ and $\{Q_k^2\}$, and divide the Q_k^2 and repeat indefinitely. Because every $x \in A$ is contained in one of these $Q_k^{\text{good},N}$, then

$$A \subset \bigcup_{i,k=1}^{\infty} Q_k^{\text{good},i}$$

□

By the above lemma, $A = \bigcup_{k=1}^{\infty} I_k$, where each I_k is a disjoint open interval. We have shown in class that if $I_k \in \mathcal{E}$, then $I_k \in \mathcal{M}(m)$. Since $\mathcal{M}(m)$ is a $\sigma-$algebra, then it is closed under countable unions, and thus $U \in \mathcal{M}(m)$.

We know that since B is closed, then B^c is open, and thus $B^c \in \mathcal{M}(m)$. Since it is a $\sigma-$algebra closed under complements, then $(B^c)^c = B \in \mathcal{M}(m)$. ■

(b) Let $\epsilon > 0$ and suppose $E \in \mathcal{M}(m)$. Then there exist closed F and open G such that $F \subset E \subset$

G and

$$m(G \setminus E) < \epsilon, \qquad m(E \setminus A) < \epsilon.$$

SOLUTION:

Lemma 2. Suppose A, B are measurable, then $A \setminus B$ is measurable.

Proof. It is easy to show that

$$A \setminus B = A \cap B^c.$$

Thus, since B is measurable, then B^c is measurable. Since $\mathcal{M}(m)$ is closed under finite (indeed countable) intersections, and A is measurable, then $A \cap B^c \in \mathcal{M}(m)$. □

Lemma 3. Suppose G and A are measurable, then

$$m(G) - m(A) = m(G \setminus A).$$

Proof. We have that $G \setminus A$ is measurable by Lemma 1. We can write

$$G = (G \setminus A) \cup A.$$

Since the decomposition is disjoint and m is countably additive, then

$$m(G) = m(G \setminus A) + m(A),$$

and we are done. □

Suppose $E \in \mathcal{M}_F(m)$, then $m(E) = m^*(E) < \infty$. There exist $\{E_n\}$ countable open cover of elementary sets such that[a]

$$E \subset \bigcup_{n=1}^{\infty} E_n, \qquad \sum_{n=1}^{\infty} m^*(E_n) \leq m^*(E) + \epsilon.$$

Define $G = \bigcup E_n$, then since $m^* = m$ on elementary sets and G is an open set and thus measurable by part (a), we have that

$$m(G) = m(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} m^*(E_n) \leq m^*(E) + \epsilon = m(E) + \epsilon.$$

By the previous lemma, we have that subtracting the $m(E)$ from the right hand side,

$$m(G \setminus A) = m(G) - m(A) < \epsilon.$$

For the general case, let $E \in \mathcal{M}(m)$. Then there exist $(E_n) \in \mathcal{M}_F(m)$ such that

$$E = \bigcup_{n=1}^{\infty} E_n.$$

For each E_n, we have showed that there exists some open G_n such that

$$m(G_n \setminus E_n) < \frac{\epsilon}{2^n}.$$

Take $G = \bigcup_{n=1}^{\infty} G_n$, which is open. Then $E \subset G$ and

$$m(G \setminus E) = m(\bigcup_{n=1}^{\infty} G_n \setminus E_n) \leq \sum_{n=1}^{\infty} m(G_n \setminus E_n) < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

Since $E \in \mathcal{M}(m)$, then $E^c \in \mathcal{M}(m)$, and thus by the above, there exists some open $G \supset E^c$ such that $m(G \setminus E^c) < \epsilon$. Because G^c is closed and clearly $G^c \subset E$, it suffices to show that $m(E \setminus G^c) < \epsilon$. Clearly, $E \setminus G^c = G \setminus E^c$ and so $m(E \setminus G^c) = m(G \setminus E^c) < \epsilon$. Call $F = G^c$ and you are done. ■

[a] Here we recall that if $E \in \mathcal{E}$, then $m^*(E) = \mathrm{Vol}(E)$

(c) $\mathcal{B}$ are the Borel sets, where $\mathcal{B} = \sigma(\mathcal{G})$, where $\mathcal{G}$ is the collection of open sets in $\mathbb{R}$. Show that if $A \in \mathcal{B}$, then $A \in \mathcal{M}(m)$.

Solution: Suppose $A \in \mathcal{B}$, then A is made up of countable unions, intersection, and/or complements of open sets. But since open sets are measurable by part (a) and $\mathcal{M}(m)$ is closed under all of these operations, then $A \in \mathcal{M}(m)$. ■

(d) Suppose $E \in \mathcal{M}(m)$, then there exist $F, G \in \mathcal{B}$ such that $F \subset E \subset G$.

$$m(G \setminus B) = m(B \setminus F) = 0$$

Solution: Let $E \in \mathcal{M}(m)$, let $n \in \mathbb{N}$. By part (b), there exists an open $G_n \supset E$ such that

$$m(G_n \setminus E) < \frac{1}{n}.$$

Since G_n is open, then $G_n \in \mathcal{B}$. Define $G := \bigcap_{n=1}^{\infty} G_n$. Since $\mathcal{B}$ is a $\sigma-$algebra, then $G \in \mathcal{B}$. Moreover, we have that for any n, since $G \in \mathcal{B} \subset \mathcal{M}(m)$

$$G \setminus E \subseteq G_n \setminus E \implies m(G \setminus E) \leq m(G_n \setminus E) < \frac{1}{n},$$

and so $m(G \setminus E) = 0$.

We have that $E^c \in \mathcal{M}(m)$, and so there exists some $G \in \mathcal{B}$ containing E^c such that $m(G \setminus E^c) = 0$. But then $G \setminus E^c = E \setminus G^c$. $G^c \in \mathcal{B}$, so call $F = G^c$, and we get that $m(E \setminus F) = 0$. ■

(e) Let $\mathcal{N}(m) = \{A \in \mathcal{M}(m) \mid m^*(A) = 0\}$. Then $\mathcal{N}(m)$ is a $\sigma-$algebra.

SOLUTION: Since $m^*(\emptyset) = 0$, then $\emptyset \in \mathcal{N}(m)$.

Suppose $A_1, \cdots \in \mathcal{N}(m)$. Then since m^* is countably subadditive,

$$m(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m(A_n) = 0,$$

and so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{N}(m)$.

Suppose $A, B \in \mathcal{N}(m)$, then $A \setminus B \subset A$ and thus by monotonicity of m^*, we have that

$$m^*(A \setminus B) \leq m^*(A) = 0,$$

and so $m^*(A \setminus B) = 0$ and thus $A \setminus B \in \mathcal{N}(m)$ ■

Problem 3

Let $x \in \mathbb{R}^n$, and $A \subseteq \mathbb{R}^n$. Prove that

$$\mathbf{m}^*(x + A) = \mathbf{m}^*(A),$$

where $x + A = \{x + y \mid y \in A\}$. Prove that, if $A \in \mathcal{M}(\mathbf{m})$, then $x + A \in \mathcal{M}(\mathbf{m})$.

SOLUTION:

Lemma 4. Let X be a metric space and $A \subset X$ be open. Let $x \in X$. Then $x + A$ is open.

Proof. Let $z \in x + A$. Then $z = x + a$ for some $a \in A$. Since $a \in A$ and A is open, there exists some $r > 0$ such that $B_r(a) \subset A$. We claim that $x + B_r(a) \subset x + A$. Indeed, it is not hard to see that $x + B_r(a) = B_r(a + x) = B_r(z)$, so then it suffices to show that $x + B_r(a) \subset x + A$. Let $z' \in x + B_r(a)$, then $z' = x + a'$ for some $a' \in B_r(a) \subset A$, and so $z' \in x + A$. Thus, we have found some $r > 0$ such that $B_r(z) \subset x + A$, and thus $x + A$ is open. $\square$

Lemma 5. Let $\{A_n\}_{n=1}^{\infty}$ be a countable open cover of $A \subset X$, where X is a topological space. If $x \in X$, then $\{x + A_n\}$ is a countable open cover of $x + A$.

Proof. Let $z \in x + A$. Then $z = x + a$ for some $a \in A$ and thus there exists some $k \in \mathbb{N}$ such that $A_k \in \{A_n\}$ and $a \in A_k$. Thus, $z \in x + A_k$. By Lemma 4, we know that $x + A_k$ is open, and thus $\{x + A_n\}_{n=1}^{\infty}$ is a countable open cover of $x + A$. $\square$

Note that the converse holds as well by identical logic.

Lemma 6. Suppose $E \in \mathcal{E}$, and $E \subset \mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$, $x + E \in \mathcal{E}$.

Proof. It suffices to show that $x + E$ is the union of finite intervals. Since $E \in \mathcal{E}$, then $E = \bigcup_{k=1}^{n} I_k$. Thus, we claim that

$$x + E = x + \bigcup_{k=1}^{N} I_k = \bigcup_{k=1}^{N} (x + I_k).$$

The only equality that needs explaining is the second one. To see it, let $z \in x + \bigcup_{k=1}^{N} I_k$, then there exists some $j \in [N]$ and some $x_j \in I_j$ such that $z = x + x_j$. Thus, $z \in x + I_j$, and so $x \in \bigcup_{k=1}^{N} (x + I_k)$. For the other inclusion, let $z \in \bigcup_{k=1}^{N} (x + I_k)$, then there exists some $j \in [N]$ such that $z \in x + I_j$. Thus, $z = x + x_j$ for some $x_j \in I_j$. Since $x_j \in \bigcup I_k$, then $z \in x + \bigcup I_k$.

We claim that $x + I_k$ is an interval for any $k \in [n]$. It should be obvious that if I_k is the interval made of points $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $a_i \leq x_i \leq b_i$ (where the $\leq$ can be replaced with $<$), then $x + I_k$ is the set of points $\mathbf{x}' = (x_1', \ldots, x_n')$ such that $a_i + x \leq x_i' \leq b_i + x$, and thus $x + I_k$ is an interval. $\square$

Lemma 7. Suppose $E \in \mathcal{E}$, and $E \subset \mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$, we have that $\mathrm{Vol}(E) = \mathrm{Vol}(x + E)$.

Proof. First, we know that $\text{Vol}(x+E)$ is well defined since by Lemma 6, $x+E \in \mathcal{E}$. By work in class, we know that we can decompose E into $E = \bigsqcup_{k=1}^{N} I_k$, where each the I_k are disjoint intervals. By work done in the Lemma 6, we know that $x+E = \bigsqcup_{k=1}^{N}(x+I_k)$, where again, $(x+I_k)$ are disjoint nonempty intervals made of points such that $\mathbf{x} \in (x+I_k)$ if and only if

$$a_i + x \le x_i \le b_i + x,$$

where (a_i, b_i) are the endpoints of the edges of each I_i. Thus, we have that

$$\begin{aligned}
\text{Vol}(E) &= \text{Vol}(\bigsqcup_{k=1}^{N} I_k) \\
&= \sum_{k=1}^{N} \text{Vol}(I_k) \\
&= \sum_{k=1}^{N} \prod_{i=1}^{n} (b_i^k - a_i^k) \\
&= \sum_{k=1}^{N} \prod_{i=1}^{n} (b_i^k - a_i^k + x - x) \\
&= \sum_{k=1}^{N} \prod_{i=1}^{n} ((b_i^k + x) - (a_i^k + x)) \\
&= \sum_{k=1}^{N} (x + I_k) \\
&= \text{Vol}(x+E)
\end{aligned}$$

□

Finally, we are ready to prove the statement. Let $\epsilon > 0$. Let $\{A_n\}_{n=1}^{\infty}$ be a countable open cover of A such that each $A_n \in \mathcal{E}$ and

$$\sum_{n=1}^{\infty} \text{Vol}(A_n) \le m^*(A) + \epsilon.$$

By Lemma 7, we have that

$$\sum_{n=1}^{\infty} \text{Vol}(A_n) = \sum_{n=1}^{\infty} \text{Vol}(x + A_n).$$

By Lemma 5, we know that $\{x + A_n\}_{n=1}^{\infty}$ is a countably open cover of $x + A$. By Lemma 6, $x + A_n \in \mathcal{E}$ for all n. Thus,

$$m^*(x+A) \le \sum_{n=1}^{\infty} \text{Vol}(x + A_n) = \sum_{n=1}^{\infty} \text{Vol}(A_n) \le m^*(A) + \epsilon,$$

and so because this holds for any $\epsilon > 0$, we have that.

$$m^*(x+A) \le m^*(A) \tag{1}$$

Let $\{x+A_n\}_{n=1}^{\infty}$ be a countable open cover of $x+A$ such that each $x+A_n \in \mathcal{E}$ and

$$\sum_{n=1}^{\infty} \text{Vol}(x+A_n) \le m^*(x+A) + \epsilon$$

We can apply all our lemmas to A_n since $A_n = (-x)+x+A_n$. Thus, $\{A_n\}_{n=1}^{\infty}$ is a countable cover of A with $A_n \in \mathcal{E}$ for all n. Thus,

$$m^*(A) \le \sum_{n=1}^{\infty} \text{Vol}(A_n) \le \sum_{n=1}^{\infty} \text{Vol}(x+A_n) \le m^*(x+A) + \epsilon.$$

Because this holds for all $\epsilon > 0$, we have that

$$m^*(A) \le m^*(x+a) \tag{2}$$

Putting together (1) and (2), we see that $m^*(A) = m^*(x+A)$.

Lemma 8. Suppose $A, B \subset \mathbb{R}^d$. Then for any $x \in \mathbb{R}^d$,

$$m^*(A \triangle B) = m^*\big[(x+A)\triangle(x+B)\big]$$

Proof. It should be easy to see that under some set manipulation, we get that

$$(x+A)\triangle(x+B) = x + A\triangle B.$$

So then by the translation invariance we proved above,

$$m^*\big[(x+A)\triangle(x+B)\big] = m^*(x+A\triangle B) = m^*(A\triangle B).$$

□

Suppose $A \in \mathcal{M}(m)$, then there exists some $(A_n) \in \mathcal{M}_F(m)$ such that $A = \bigcup_{n=1}^{\infty} A_n$. We claim that (1) $x + A = \bigcup_{n=1}^{\infty}(x+A_n)$ where (2) $x+A_n \in \mathcal{M}_F(m)$. To see the second claim, consider that since $A_n \in \mathcal{M}_F(m)$, then for each n, there exists a sequence $(E_k^{(n)}) \in \mathcal{E}$ such that $E_k^{(n)} \to A_n$ (in exterior measure) as $k \to \infty$. Thus, for every n, we have by Lemma 8 that for large k,

$$m^*(E_k^{(n)}\triangle A_n) = m^*\big[(x+E_k^{(n)})\triangle(x+A_n)\big] < \epsilon$$

By lemma 6, we know that $x + E_k^{(n)} \in \mathcal{E}$, so then $x + A_n \in \mathcal{M}_F(m)$ for every n. It remains to show that

$$x + A = \bigcup_{n=1}^{\infty}(x+A_n),$$

but the logic of this proof is identical to that of Lemma 5's. ■

Problem 4

Consider the real line in $\mathbb{R}^2$,

$$X = \{(x, 0) \mid x \in \mathbb{R}\}.$$

What is $m^*(X)$ (where m^* is the Lebesgue outer measure defined on $\mathbb{R}^2$)? Show that $X \in \mathcal{M}(m)$.

Solution: Define

$$X_n := \{(x, 0) \mid x \in (-n, n)\}$$

Evidently, $X = \bigcup_{n=1}^{\infty} X_n$. Thus, we know by Theorem 11.8 in Rudin that

$$m^*(X) \leq \sum_{n=1}^{\infty} m^*(X_n) \tag{3}$$

Let $\epsilon > 0$. For each n, define

$$E_\epsilon^n = \{(x, y) \in \mathbb{R}^2 \mid x \in (-n, n), y \in (-\frac{\epsilon}{2^n \cdot 4n}, \frac{\epsilon}{2^n \cdot 4n})\}.$$

Clearly, we know that

(a) $X_n \subset E_\epsilon^n$ for any n, for any $\epsilon > 0$. Thus, E_ϵ^n is an open cover of X_n, and so since E_ϵ^n is an interval,

$$m^*(X_n) \leq \mathrm{Vol}(E_\epsilon^n), \qquad \forall\, n \in \mathbb{N} \tag{4}$$

(b) E_ϵ^n is an interval, and thus

$$\mathrm{Vol}(E_\epsilon^n) = (2n)(\frac{\epsilon}{2^n \cdot 2n}) = \frac{\epsilon}{2^n} \tag{5}$$

By (3), (4), and (5):

$$m^*(X) \leq \sum_{n=1}^{\infty} m^*(X_n) \leq \sum_{n=1}^{\infty} \mathrm{Vol}(E_\epsilon^n) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Because this holds for all $\epsilon > 0$, $m^*(X) = 0$.

Since $X_n \in \mathcal{E}$ for all n because they are all intervals, then $X_n \in \mathcal{M}(m)$ for all n. Thus, since $\mathcal{M}(m)$ is closed under countable unions and $X = \bigcup_{n=1}^{\infty} X_n$, then $X \in \mathcal{M}(m)$. ■

Problem 5

Let $A, B \subseteq \mathbb{R}^n$, and suppose that

$$d := \inf_{x \in A, y \in B} |x - y| > 0.$$

Prove that

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

SOLUTION: From Theorem 11.8 in Rudin, we have that

$$m^*(A \cup B) \leq m^*(A) + m^*(B),$$

so it suffices to show the other inequality.

Let $\epsilon > 0$. There exists $\{X_n\}_{n=1}^{\infty}$ countable open cover of $A \cup B$ such that

$$\sum_{n=1}^{\infty} \mathrm{Vol}(X_n) \leq m^*(A \cup B) + \epsilon. \tag{6}$$

Each $X_n \in \mathcal{E}$, so we can subdivide the X_n into $\tilde{X}_n$ so that each $_n$ has diameter less than $\frac{d}{2}$. Then each $\tilde{X}_n$ intersects at least one A or B (we can throw out the ones that don't without any consequence). Thus, there exists some index set I such that if $i \in I$, then $\tilde{X}_i \cap B = \emptyset$. Since $\mathrm{diam}(\tilde{X}_n) < \frac{d}{2}$, then there exists if $j \in I^c$, then $\tilde{X}_j \cap A = \emptyset$. Thus, $\{\tilde{X}_i\}_{i \in I}$ is a countable open cover of A disjoint completely from $\{\tilde{X}_j\}_{j \in I^c}$, which is a countable open cover of B (for a more precise proof, see my proof for problem 2, (a)). Thus by (6),

$$m^*(A) + m^*(B) \leq \sum_{i \in I} \mathrm{Vol}(\tilde{X}_i) + \sum_{j \in I^c} \mathrm{Vol}(\tilde{X}_i) \leq \sum_{n=1}^{\infty} \mathrm{Vol}(X_n) \leq m^*(A \cup B) + \epsilon.$$

Because this holds for any $\epsilon > 0$, we are done. ■

Problem 6

Let $f : \mathbb{R} \to \mathbb{R}$. We define the **graph** of f to be the set

$$G(f) = \{(x, y) \in \mathbb{R}^2 \mid f(x) = y\}$$

Show that if f is continuous, then $m(G(f)) = 0$.

Solution: First, we will show that $G(f)$ is measurable. Let $\epsilon > 0$. If we define

$$A_n = \{(x, f(x)) \in \mathbb{R}^2 \mid x \in [-n, n]\},$$

then

$$G(f) = \bigcup_{n=1}^{\infty} A_n.$$

We want to show that $A_n \in \mathcal{M}_F(m)$ for each n. Thus, it suffices to find a sequence $E_k^{(n)} \in \mathcal{E}$ such that $E_k^{(n)} \to A_n$ (in outer measure) as $k \to \infty$. Consider that since f is continuous and $[-n, n] \subset \mathbb{R}$ is compact, then f is uniformly continuous on $[-n, n]$. There exists some $\delta_n > 0$ such that if $x, y \in [-n, n]$ and $|x - y| < \delta_\epsilon$, then $|f(x) - f(y)| < \frac{\epsilon}{2^{n+1}}$.

Partition $[-n, n]$ into a partition P, $-n = t_0 < \cdots < t_K = n$ such that $\|P\| < \min\{1, \delta_n\}$. Thus if $x, y \in [t_i, t_{i+1}]$, then $|f(x) - f(y)| < \frac{\epsilon}{2^{n+1}}$. Call $E_i^{(n)}$ the boxes of height $2 \cdot \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2^n}$ for each subinterval $[t_i, t_{i+1}]$ of P. Notice that each $E_i^{(n)}$ is an interval and thus

$$m^*(E_i^{(n)}) = \text{Vol}(E_i^{(n)}) = (t_i - t_{i-1}) \cdot \frac{\epsilon}{2^{n+1}}$$

Call $E_K^{(n)} = \bigcup_{i \in [K]} E_i^{(n)}$ the union of all such boxes. Note that again, $E_K^{(n)} \in \mathcal{E}$ since it is the finite union of intervals.

Since $A^n \subset E_K^{(n)}$ by the construction of $E_K^{(n)}$, we have that $E_K^{(n)} \triangle A_n = E_K^{(n)} \setminus A_n$. Therefore, using the finite additivity of m^*, we see that by telescoping the sum:

$$\begin{aligned}
m^*(E_K^{(n)} \triangle A_n) &\leq m^*(E_K^{(n)}) \\
&= m^*\left(\bigcup_{i \in [K]} E_i^{(n)}\right) \\
&= \sum_{i=1}^{K} m^*(E_i^{(n)}) \\
&= \sum_{i=1}^{K} (t_i - t_{i-1}) \cdot \frac{\epsilon}{2^{n+1}} \\
&= (2n) \cdot \frac{\epsilon}{2^{n+1}} \\
&= \frac{\epsilon}{2^n} \\
&< \epsilon
\end{aligned}$$

Thus, we see that $A_n \in \mathcal{M}_F(m)$, and since $G(f) = \bigcup_{n=1}^{\infty} A_n$, then $G(f)$ is measurable. Using the countable additivity of the Lebesgue measure, and the above calculation:

$$m(G(f)) = \sum_{n=1}^{\infty} m(A_n) < \sum_{n=1}^{\infty} m(E_K^n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Because this holds for any $\epsilon > 0$, we see that $m(G(f)) = 0$. ■