208 Spring 2025: Problem Set 3
Accelerated Analysis in $\mathbb{R}^n$ III
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Problem 1

Suppose $f \in \mathcal{L}(m)$ on E and g is measurable and bounded. Then $fg \in \mathcal{L}(m)$.

SOLUTION: Since f, g are measurable then fg is measurable. Since $f \in \mathcal{L}(m)$, then $|f| \in \mathcal{L}(m)$. Since g is bounded, then $|g(E)| \leq C$ for some $C > 0$. Decompose fg into

$$fg = (fg)^+ - (fg)^-.$$

It suffices to show that both terms are integrable. For all $x \in E$, we have that

$$|(fg)(E)| = |f(E)||g(E)| \leq C|f(E)|.$$

Thus, since $(fg)^+, (fg)^- \leq |fg|$, then by a remark proved in the previous homework, since $C|f| \in \mathcal{L}(m)$

$$(fg)^+ \leq |fg| \leq C|f| \implies \int_E (fg)^+ \leq C \int |f| < \infty.$$

Similarly,

$$(fg)^- \leq |fg| \leq C|f| \implies \int_E (fg)^- \leq C \int |f| < \infty.$$

Thus, $fg \in \mathcal{L}(m)$. ■

Problem 2

(Egorov) Let $E \subseteq \mathbb{R}$ be measurable with $m(E) < \infty$. Let (f_n) be measurable such that $f_n : E \to \mathbb{R}$ for all n and $f_n \to f$ pointwise to some function $f : E \to \mathbb{R}$. Then, for all $\epsilon > 0$, there exists a closed $F \subset E$ such that

$$f_n(x) \rightrightarrows f(x) \quad \forall x \in F, \quad m(E \setminus F) < \epsilon$$

(a) Show that, under these assumptions, for every $\eta > 0$ and $\delta > 0$, there is a measurable subset $A \subset E$ and $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \eta \quad \text{for all } x \in A \text{ and } n \geq N \text{ and } \quad m(E \setminus A) < \delta.$$

Solution: Let $\delta > 0$, let $\eta = \frac{1}{k}$. For any $k \in \mathbb{N}$, define for each $N \in \mathbb{N}$:

$$A_N^{(k)} := \{x \in E \mid |f_n(x) - f(x)| < \frac{1}{k}, \quad \forall n \geq N\}.$$

Since $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in E$, then f is measurable, and thus $A_N^{(k)}$ is measurable for each N and each k.

We claim that

$$\lim_{N\to\infty} m(A_N^{(k)}) = m(E) \tag{1}$$

To see this, it suffices to show that

(i) A_N^k is ascending with respect to N;

(ii) $\bigcup_{N=1}^{\infty} E_N^{(k)} = E$.

To see (i), let $x \in A_N^{(k)}$. By definition, since $N + 1 \geq N$, then $|f_{N+1}(x) - f(x)| < \frac{1}{k}$. Thus, $x \in A_{N+1}^{(k)}$. One inclusion of (ii) is obvious. To see the other, let $x \in E$. Since $f_n(x) \to f(x)$, then there exists some $N_x \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \frac{1}{k}$. Thus, $x \in E_{N_x}^k$ (and in fact, $x \in A_n^{(k)}$) for each $n \geq N_x$) and so $E \subseteq \bigcup_{N=1}^{\infty} A_N^{(k)}$.

By (i) and (ii), and the fact that each $A_N^{(k)}$ is measurable, we have (1) by a theorem in class; so for each $k \in \mathbb{N}$, there is some $N_k \in \mathbb{N}$ such that

$$m(E \setminus A_{N_k}^{(k)}) < \frac{1}{2^k}.$$

Define

$$A := \bigcap_{k \geq K} A_{N_k}^{(k)},$$

where $K \in \mathbb{N}$ is chosen such that

$$\sum_{i=K}^{\infty} \frac{1}{2^i} < \frac{\delta}{2}.$$

Since each $A_{N_k}^{(k)}$ has already been shown to be measurable and this is a countable intersection, A is measurable. Let $\eta > 0$ and $x \in A$. There is some $k > 0$ such that $\frac{1}{k} < \eta$. Thus, since $x \in A$, then by definition, $x \in A_{N_k}^{(k)}$, and thus if $n \geq N_k$, we have that

$$|f_n(x) - f(x)| < \frac{1}{k} < \eta.$$

It suffices to show that $m(E \setminus A) < \delta$.

$$\begin{aligned}
m(E \setminus A) &= m\left(E \setminus \bigcap_{k \geq K} A_{N_k}^{(k)}\right) \\
&= m\left(E \cap \left(\bigcap_{k \geq K} A_{N_k}^{(k)}\right)^c\right) \\
&= m\left(E \cap \bigcup_{k \geq K} (A_{N_k}^{(k)})^c\right) \\
&= m\left(\bigcup_{k \geq K} E \cap (A_{N_k}^{(k)})^c\right) \\
&= m\left(\bigcup_{k \geq K} E \setminus A_{N_k}^{(k)}\right) \\
&\leq \sum_{k \geq K} m(A_{N_k}^{(k)}) \\
&= \sum_{k \geq K} \frac{1}{2^k} \\
&< \delta
\end{aligned}$$

■

(b) Prove Egorov's theorem.

Solution: Let $\epsilon > 0$. By part (a), there exists some $A \subset E$ such that $f_n \rightrightarrows f$ on A and $m(E \setminus A) < \frac{\epsilon}{2}$. Since A is measurable, we have proved that there is a closed

$F \subset A \subset E$ such that $m(A \setminus F) < \frac{\epsilon}{2}$. Thus,

$$m(E \setminus F) = m\big((E \setminus A) \cup (A \setminus F)\big) \leq m(E \setminus A) + m(A \setminus F) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $F \subset A$, and the convergence on A is uniform, then the convergence on F must be uniform, since we can just use the $N \in \mathbb{N}$ from A for any $x \in F$. $\blacksquare$

Problem 3

(Luzin) Let $f : E \to \mathbb{R}$ be a measurable function with $E \in \mathcal{M}$ such that $m(E) < \infty$. For every $\epsilon > 0$, there exists a closed set $F \subseteq E$ with $m(E \setminus F) < \epsilon$ such that $f|_F$ is continuous.

(a) Prove this when f is a simple function on E.

Solution:

Lemma 1. Let E be measurable with $m(E) < \infty$. For all $\epsilon > 0$, there exists some $K \subseteq E$ compact such that $m(E \setminus K) < \epsilon$.

Proof. Let $\epsilon > 0$. By the regularity of m, there exists a closed set $F \subseteq E$ such that $m(E \setminus F) < \frac{\epsilon}{2}$. Let $K_n = F \cap \overline{B_n(0)}$. Then since K_n is closed and bounded in $\mathbb{R}^n$, K_n is compact. Clearly

$$E \setminus K_n \downarrow E \setminus F \implies m(E \setminus F) = \lim_{n\to\infty} m(E \setminus K_n).$$

Thus, there exists some N large such that $m(E \setminus K_N) < \epsilon$. $\square$

Let $\epsilon > 0$. Let φ be a simple function on E, we claim there exists some $F \subseteq E$ closed with $m(E \setminus F) < \epsilon$ such that $\varphi|_F$ is continuous. Note that since E is measurable, then φ is a measurable simple function.

There exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\epsilon}{2}$. Since φ is simple on E, then we can take it, without loss of generality, to be defined as

$$\varphi = \sum_{k=1}^{N} c_k \chi_{E_k}, \quad \bigsqcup_{k=1}^{N} E_k. = E,$$

where each E_k is measurable. By the lemma above, for each E_k, there exists a compact set $X_k \subseteq E_k$ such that

$$m(E_k \setminus X_k) < \frac{1}{Nn}$$

Each X_k is measurable and since it is a subset of disjoint sets, then the X_k are pairwise disjoint. Define

$$X = \bigsqcup_{k=1}^{N} X_k.$$

Moreover, note that X is measurable since it is the countable union of measurable sets. We claim that

(i) φ is continuous on X

(ii) $m(E \setminus X) \leq \frac{1}{n}$.

To see (i), let $x \in X$ let $\eta > 0$. Then $x \in X_k$ for some k and $\varphi(x) = c_k$. Since the X_k are compact and disjoint,

$$d(X_k, X_{k+1}) =: r_k > 0$$

is achieved and is positive for each k. Thus, we take $\delta < \min\{r_k\}_k$ If $y \in (x-\delta, x+\delta)$, then, $y \in X_k$, and so $\varphi(y) = c_k$. Thus,

$$|\varphi(y) - \varphi(x)| = 0 < \eta.$$

Thus, $\varphi|_X$ is continuous. To see (ii), consider that

$$\begin{aligned} m(E \setminus X) &= m(E \setminus \bigsqcup_{k=1}^{N} X_k) \\ &= m(\bigsqcup_{k=1}^{N} E_k \setminus \bigsqcup_{k=1}^{N} X_k) \\ &= m(\bigsqcup_{k=1}^{N} E_k \setminus X_k) \\ &= \sum_{k=1}^{N} m(E_k \setminus X_k) \\ &< \sum_{k=1}^{N} \frac{1}{Nn} \\ &= \frac{1}{n} \\ &< \epsilon \end{aligned}$$

Since X is the finite union of closed sets, then X is closed. We use the lemma provided in the PSET to conclude that since φ is continuous on the closed X, then φ has a continuous extension g on $\mathbb{R}$ such that $\varphi = g$ on X. ■

(b) Prove Luzin's Theorem.

Solution: Let f be a measurable function and let $\epsilon > 0$. By a theorem proved in the previous problem set, there exists a sequence $(\varphi_n) : E \to \mathbb{R}$ of measurable simple functions such that $\varphi_n \to f$ pointwise. By Egorov's theorem in the previous problem, there exists a closed $A \subseteq E$ such that $\varphi_n \rightrightarrows f$ on A and $m(E \setminus A) < \frac{\epsilon}{3}$. By the above problem for each n, there exists some closed $F_n \subset E$ such that $\varphi_n|_{F_n}$ is continuous and $m(E \setminus F_n) < \frac{1}{2^n}$. There exists some $N > 0$ such that

$$\sum_{n \geq N} \frac{1}{2^n} < \frac{\epsilon}{3}.$$

Define

$$F' := A \setminus \left(\bigcup_{n \geq N} (F_n^c \cap E) \right)$$

Since $F' \subset A$, then $\varphi_n \rightrightarrows f$ on F'. We claim that φ_n is continuous on A for any $n \geq N$. To see this, suppose not, then φ_n is discontinuous for some $x \in A$. Thus, $x \in F_n^c$ due to how the F_ns were defined in the above problem which is a contradiction. Then we see that since continuous functions uniformly converge to continuous functions, f is continuous on F'. Moreover, note that

$$\begin{aligned}
m(E \setminus F') &= m\left(E \setminus \left(A \setminus \left(\bigcup_{n \geq N} (F_n^c \cap E) \right) \right) \right) \\
&= m\left(E \setminus \left(A \cap \left(\bigcup_{n \geq N} (F_n^c \cap E) \right)^c \right) \right) \\
&= m\left(E \setminus \left(A \cap \bigcap_{n \geq N} F_n \right) \right) \\
&= m\left(E \cap \left(A \cap \bigcap_{n \geq N} F_n \right)^c \right) \\
&= m\left(E \cap \left(A^c \cup \bigcup_{n \geq N} F_n^c \right) \right) \\
&= m\left((E \setminus A) \cup (E \cap \bigcup_{n \geq N} F_n^c) \right) \\
&= m\left((E \setminus A) \cup \bigcup_{n \geq N} (F_n^c \cap E) \right) \\
&\leq m(E \setminus A) + m(\bigcup_{n \geq N} F_n^c \cap E) \\
&< \frac{\epsilon}{3} + \sum_{n=N}^{\infty} \frac{1}{2^n} \\
&< \frac{2\epsilon}{3},
\end{aligned}$$

where the second to last inequality is from work on the previous part. Thus, we have found a set F' such that $m(E \setminus F') < \frac{2\epsilon}{3}$ and f is continuous on F'. There exists some closed set $F \subseteq F' \subseteq E$ such that $m(F' \setminus F) < \frac{\epsilon}{3}$. Then since $F \subseteq F'$ and f is continuous on F', then f is continuous on F. We conclude by noting that

$$m(E \setminus F) = m\big((E \setminus F') \sqcup (F' \setminus F)\big) \leq m(E \setminus F') + m(F' \setminus F) < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Conclude by using the lemma provided to get the continuous extension on $\mathbb{R}$, g such that $f = g$ on F. ■

Problem 4

Use Fatou's Lemma to prove the Monotone Convergence Theorem.

SOLUTION: Let $E \in \mathcal{M}$, let $(f_n) : E \to \mathbb{R}$ be a sequence of non-negative measurable functions such that $f_n \uparrow f$ pointwise. We claim that

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Since each f_n is measurable, then $\lim_{n\to\infty} f_n = f$ is measurable. Moreover, since $\liminf_{n\to\infty} f_n = f$, and each f_n is non-negative, then Fatou's lemma states that

$$\int_E f dm \leq \liminf_{n\to\infty} \int_E f_n dm \tag{2}$$

Define

$$g_n(x) := f(x) - f_n(x), \quad x \in E$$

Then $g_n \geq 0$ for each n and $g_n \to 0$ pointwise, and thus

$$\liminf_{n\to\infty} g_n = 0$$

and then it follows that g_n is measurable. By Fatou's lemma, we have that

$$\int_E 0 dm \leq \liminf_{n\to\infty} \int_E g_n = \liminf_{n\to\infty} \int_E f - f_n dm \implies \limsup_{n\to\infty} \int_E f_n dm \leq \int_E f dm \tag{3}$$

(2) and (3) prove the MCT. ■

Problem 5

Suppose that $E \subseteq \mathbb{R}^n$ is measurable. Let (f_n) be a sequence of non-negative functions such that $0 \leq f_1(x)$ for almost every $x \in E$, and for $n \geq 1$, $f_n(x) \leq f_{n+1}(x)$ for almost every $x \in E$. Prove that

$$\lim_{n\to\infty} \int_E f_n dm = \int_E f dm,$$

where, for every x, we have that

$$f(x) := \begin{cases} \lim_{n\to\infty} f_n(x), & \text{if the limit exists} \\ 0, & \text{else} \end{cases}$$

Solution: For every n, define

$$X_1 := \{x \in E \mid f_1(x) < 0\}, \quad X_n := \{x \in E \mid f_n(x) < f_{n+1}(x)\} \quad n > 1.$$

Let $X = \bigcup_{\mathbb{N}} X_n$, then by the assumption of the problem and sub-additivity (and the fact that each X_n is measurable since f_n is measurable),

$$m(X) \leq \sum_{n=1}^{\infty} m(X_n) = 0.$$

Thus, $m(X) = 0$. Take

$$g_n := f_n \chi_{E\setminus X}.$$

Let $x \in E$. Either $x \in X \cap E$ or $x \in X^c \cap E$. If $x \in X \cap E$, then $g_n(x) = 0$ for any n, and thus

$$0 \leq g_1(x) \leq g_2(x) \cdots .$$

If $x \in X^c \cap E$, then $g_n = f_n$ and thus

$$0 \leq g_1(x) \leq g_2(x) \leq \cdots .$$

Either way since we are in the extended reals and the sequence is monotonic, the limit function $g(x) = \lim_{n\to\infty} g_n(x)$ exists for every $x \in E$. By the normal monotone convergence theorem, we have that

$$\lim_{n\to\infty} \int_E g_n dm = \int_E g dm.$$

Which by definition implies that

$$\lim_{n\to\infty} \int_E f_n \chi_{E\setminus X} dm = \int_E f_n \chi_{E\setminus X} dm = \int_{E\setminus X},$$

which in turn directly implies that

$$\lim_{n\to\infty} \int_{E\setminus X} f_n dm = \int_{E\setminus X} f_n dm.$$

Since $E = (E \setminus X) \sqcup X$ and $m(X) = 0$, we have that since the integrals are countably additive, then

$$\begin{aligned}
\lim_{n\to\infty} \int_E f_n dm &= \lim_{n\to\infty} \left(\int_{E\setminus X} f_n dm + \int_X f dm \right) \\
&= \lim_{n\to\infty} \int_{E\setminus X} f_n dm \\
&= \int_{E\setminus X} f dm \\
&= \int_{E\setminus X} f dm + \int_X f dm \\
&= \int_E f dm
\end{aligned}$$

■