UChicago Point Set Topology

Notes by Agustín Esteva, Lectures by Calegari, Books by ,

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Contents

1 Lectures

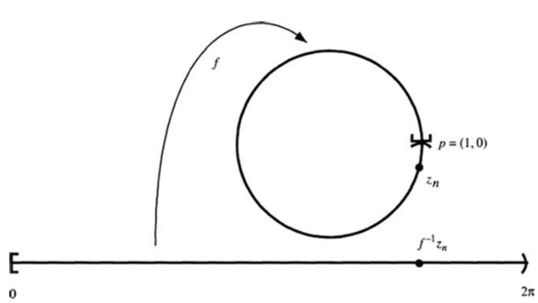
1.1 Tuesday, Jan 21: Continuous Functions and Homeomorphisms

Definition 1. We say X and Y are **homeomorphic** if there exits some $f : X \to Y$ and $g : Y \to X$ which are both continuous such that $f \circ g : Y \to Y$ is identity on Y and $g \circ f : X \to X$ is the identity on X.

We say that f and g are **homeomorphisms**.

Definition 2. Suppose $f : X \to Y$ is injective, then we say f is an **embedding** if f unto its image is a homeomorphism.

Remark 1. To give a non-example, we let $X = [0, 2\pi)$ and $Y = \mathbb{R}^2$, and we define $f; X \to Y$ by sending X to the unit circle:



Theorem 1. We assert that:

(a) Constant functions are continuous.

(b) If $A \subset X$ and i is the inclusion of A as a subset of X. That is, i is an injective map from $A \to X$, then i continuous.

(c) Suppose $f : X \to Y$ and $g : Y \to Z$, then if f and g are continuous, then $f \circ g$ is continuous.

(d) Suppose $A \subset X$ and $f : X \to Y$ is continuous, then $f|_A : A \to Y$ is continuous.

(e) Suppose $f : X \to Y$, where $Y \subset Z$, then if f is continuous, then $\hat{f} : X \to Z$ is continuous.

(f) Suppose $X = \bigcup U_\alpha$, where U_α is open for each α. If $f : X \to Y$ such that $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for each α, then f is continuous.

(g) Suppose $X = A \cup B$, where A, B are closed. Suppose we have $f : X \to Y$ such that $f|_A$ and $f|_B$ are both continuous, then f is continuous.

(h) We say $f : Y \to \prod X_\alpha$ is continuous if and only if the "coordinate functions," $f_\alpha = \prod_\alpha f$ is continuous for all α.

Proof. We give a proof for (f): Let U be an open set in Y. By continuity of each f restriction, we have that $f^{-1}|_{U_\alpha}(U)$ is open in U_α. Notice that $f^{-1}|_{U_\alpha}(U) = f^{-1}(U) \cap U_\alpha$, which is open in both U_α and in X (since the intersection of open is open). Moreover, we have that

$$f^{-1}(U) = \bigcup (f^{-1}(U) \cap U_\alpha),$$

which is open in X.

Proof of (g): Suppose $K \subset Y$ is closed, then $f^{-1}|_A(K)$ is closed in A and thus closed in X, similarly for B. Then

$$f^{-1}(K) = f^{-1}|_A(K) \cup f^{-1}|_B(K)$$

is closed. □

Remark 2. Note that (f) is not true if we replace U_α for closed sets. To see this, take $X = \bigcup K_\alpha$, where K_α is each point in X. Then there are a lot of examples.

Definition 3. Suppose X is a set and (Y, d) is a metric space. We say a sequence of functions $\{f_n : X \to Y\}$ converges uniformly to $f : X \to Y$ if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, we have that

$$d(f_n(x) - f(x)) < \epsilon, \quad \forall x \in X \iff ||f - f_n|| < \epsilon.$$

Theorem 2. Let $f_n : X \to Y$ be continuous, where X is a set and (Y, d) is a metric space. If $f_n \to f$ uniformly, then f is continuous.

Proof. Let $V \subset Y$ be open. Let $x \in f^{-1}(V)$. We want to find some open $U \subset f^{-1}(V)$ such that $x \in U$. That is, $f(U) \subset V$. Let $f(x) = y$. Since V is open, then there exists an $\epsilon > 0$ such that $B_\epsilon(y) \subset V$. Now we want to find an open neighborhood of x such that its image is contained in this ball. By uniform convergence, there exists an N such that if $n \geq N$, we have that $d(f_n(x), f(x)) < \frac{\epsilon}{3}$ for all $x \in X$. Since f_N is continuous, then there exists a $x \in U$ such that $f_N(U) \subset B_{\frac{\epsilon}{3}}(f_N(x))$. Thus, for all $y \in U$:

$$d(f(y), f_N(y)) < \frac{\epsilon}{3}, \quad d(f_N(y), f_N(x)) < \frac{\epsilon}{3}, \quad d(f_N(x), f(x)) < \frac{\epsilon}{3}.$$

□

1.2 Thursday, Jan 23: Connectedness

Definition 4. A **separation** of a topological space X is a decomposition

$$X = A \sqcup B$$

such that A, B are both open and nonempty.

Definition 5. A topological space X is **connected** if it does not admit a separation.

Lemma 1. X is connected if and only if, whenever we write $X = A \sqcup B$, where A and B are nonempty, then either $A \cap B' \neq \emptyset$ or $A' \cap B \neq \emptyset$.

Proof. Suppose X is connected, then without loss of generality, A is not closed. Thus,

$$A' \not\subset A \implies A' \cap B \neq \emptyset.$$

If, on the other hand, X is disconnected, then A and B are both closed, and thus

$$A' \cap B = A \cap B' = A \cap B = \emptyset.$$

□

Lemma 2. Suppose $X = C \sqcup D$, where C, D are both open. Suppose that $Y \subset X$ is connected in the subspace topoogy, then either $Y \subset C$ or $Y \subset D$.

Proof. Consider that

$$Y = Y \cap C \sqcup Y \cap D,$$

where both of the terms in the right are open in Y because both C and D are open. Thus, by connectedness of Y, at least one of these must be empty. □

Theorem 3. Suppose $X = \bigcup_\alpha X_\alpha$, where every X_α is connected and there exists some $p \in \bigcap X_\alpha$, then X is connected.

Proof. Suppose $X = A \sqcup B$, both open. By Lemma 2, we have that for all α, we have that either $X_\alpha \subset A$ or $X_\alpha \subset B$. Without loss of generality, $p \in A$, and thus $X_\alpha \subset A$ for all α, and thus B is empty. □

Theorem 4. Suppose $A \subset X$, where A is connected. If $A \subset B \subset \overline{A}$, then B is connected.

Proof. Suppose $B = C \sqcup D$, where C, D open and nonempty. Since A is connected, then by lemma 2, without loss of generality, we can say that $A \subset C$. Thus, $\overline{A} \subset \overline{C} = C$, which is disjoint from D, and thus

$$B \cap D = \emptyset.$$

□

Theorem 5. Suppose X is connected and $f : X \to Y$ is continuous. Then $f(X)$ is connected.

Proof. Let $f(X) = A \sqcup B$, A, B nonempty and open. Since f is continuous, then $X = f^{-1}(A) \sqcup f^{-1}(B)$, where they are both open by continuity and nonempty by surjectivity. □

Theorem 6. Suppose X and Y are connected, then $X \times Y$ is connected.

Proof. Let $x \in X$. We claim that $\{x\} \times Y$ is homeomorphic to Y. The homeomorphism is π, the projection map. Thus, $A_x = \{x\} \times Y$ is connected. Similarly, $B_x = X \times \{y\}$ is connected for every $y \in Y$. Thus, since $T_{x,y} = A_x \cap B_x = (x, y)$, then by Theorem 3, we have that

$$X = \bigcup_{y \in Y} T_{x,y}$$

is connected. $\square$

We can obviously extend this to a finite product of connected spaces. What about for infinite products?

Definition 6. Let X_α be a collection of spaces. We say that the **box topology** on $\prod X_\alpha$ is the topology separated by the basis

$$B = \{\prod U_\alpha, \,:\, U_\alpha \subset X_\alpha \text{ is open}\}.$$

.

Example 1.1. $\mathbb{R}^N$ is connected in the product topology but not connected in the box topology. To see this, think of $\mathbb{R}^= \{(x_1, x_2, \dots) \,:\, x_i \in \mathbb{R}\}$ and think of $\mathbb{R}^N =$ bounded sequences $\sqcup$ unbounded sequences. We claim that these are both open in the box topology: Let $a \in$ bounded sequence. Thus, there exists some C such that for all a_i, $a_i \leq C$. Consider

$$U_i = (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times \cdots,$$

which is an open set, and is a basis element in the box topology since every point in U_i is bounded by $C + 1$. An identical argument proves that $\{$unbounded sequences$\}$ are unbounded. Thus, $\mathbb{R}^N$ is not connected in the box topology.

Theorem 7. Suppose X_α is any collection of connected spaces, then $\prod X_\alpha$ is connected in the product topology.

Proof. It suffices to find a connected $K \subset X$ such that for every open $U \subset X$, we have that $K \cap U \neq \emptyset$. To find this, we will use Theorem 3. let $x \in X$, where $x = (x_\alpha)_\alpha$. Let I be a finite set of indices, and define

$$K_i := \{y \,:\, y_\alpha = x_\alpha, \quad \alpha \notin I\}.$$

We remark that K_I is homeomorphic to $\prod_{\alpha \in I} X_\alpha$ under the projection homeomorphism. $\prod_I X_\alpha$ is connected by Theorem 3, and thus K_I is connected and contains x.

$$K = \bigcup_{\text{all finite } I \text{ index sets}} K_I$$

is connected again by Theorem 3. For any nonempty open basis in the product topology $U \subset X$, we claim that $K \cap U \neq \emptyset$. To see this, let $U = \prod U$, where $U_\alpha = X_\alpha$ except for finitely many indices (I). Since each U_α is nonempty, then for $\alpha \in I$, we choose some $u_\alpha \in U_\alpha$. And for $\alpha \notin I$, then choose $u_\alpha = x_\alpha$. It is not hard to see that $u_\alpha \in K$ and that $u_\alpha \in U$. Thus, X is connected. $\square$

1.3 Tuesday, Jan 28: Compactness

Definition 7. Let X be a space, and let $x, y \in X$. A **path** in X is defined to be the continuous map $f : [0,1] \to X$ such that

$$f(0) = x, \qquad f(1) = y.$$

Definition 8. A space X is **path connected** if for any $x, y \in X$, there is a path in X from x to y.

Proposition 1. If X is path connected, then X is connected.

Proof. Consider that $f([0,1]) \subset X$ is a connected subspace of X by the continuity of f. Let $x \in X$. For any $y \in Y$, choose the path f_y such that $f_y(0) = x$ and $f_y(1) = y$. Let $P_y := f_y([0,1]) \subset X$. Since $y \in P_y$ for all y, then $X = \bigcup_y P_y$, and since $x \in P_y$ for all y, then X is connected. $\square$

Remark 3. The converse fails, see

$$X = \sin\left(\frac{1}{x}\right) \cup [0,1]$$

Theorem 8. (IVT) Let X be connected and let $f : X \to \mathbb{R}$ be continuous. If $a, b \in X$ and there exists some $c \in [f(a), f(b)]$, then there exists some $\gamma \in [a,b]$ such that $f(\gamma) = c$.

Proof. Suppose not, then $c \notin f(X)$, then $f(X) \subset [-\infty, c) \cup (c, \infty]$. Both of these sets are open, and thus the inverses are open and disjoint. Neither is empty and we get that their union is all of X. Thus, X is not connected, which is a contradiction. $\square$

Definition 9. We say that X is **compact** if any open cover has a finite open subcover.

Example 1.2. We give some examples.

(a) $\mathbb{R}$ is not compact. Let $\{(-n, n)\}_{n \in \mathbb{N}}$ be the open cover of $\mathbb{R}$. Obviously there is no finite subcover (say, of cardinality N), since then there would exist some $(-N, N)$ that does not contain $N \in \mathbb{R}$.

(b) $(0,1)$ is not compact since it is homeomorphic to $\mathbb{R}$.

(c) $[0,1]$ is compact. To see this, let $\{U_\alpha\}$ be a cover of X. Thus, $0 \in U_\alpha$ for some α, and thus there exists some $p > 0$ such that $B_p(0) = [0, p) \subset U_\alpha$ for some α. Since $p \in X$, then $p \in U_\beta$, and thus there exists some $q > p$ such that $[0, q] \subset U_\alpha \cup U_\beta$. Define

$$p = \sup\{q \;\; [0, q) \subset \text{finite subcover}\},$$

we claim that $p = 1$. Suppose that $p < 1$, then $p \in [0,1]$ and so $p \in U_\beta$ for some β. By definition, there must exist some q such that $[0, q] \subset \{U_i\}$. But then we see that since $p \in U_\beta$, and U_β is open, then $(p - \epsilon, p + \epsilon) \subset U_\beta$, but then $\{U_i\} \cup U_\beta \supset [0, p + \frac{\epsilon}{2}] \ni q$, which is a contradiction to the size of p.

Remark 4. To see that X is compact, it suffices to show that every open cover of X by basis elements has a finite subcover.

Theorem 9. If X is compact and $Y \subset X$ is closed, then Y is compact.

Proof. Let $\{U_\alpha\}$ be an open cover of Y. Then we have that for every α, there exist some open $V_\alpha \in X$ such that $V_\alpha \cap Y = U_\alpha$. Then we have that $Y \subset V_\alpha$. Moreover, since Y is closed, then XY is open in X, and so

$$\bigcup V_\alpha \cup (XY) \supset X.$$

By the compactness of X, we have a finite subset $\{V_{\alpha_i}\} \cup (X \setminus Y) \supset X$, and thus intersecting with Y gives an open finite subcover of X. $\square$

Theorem 10. Suppose f is Hausdorff. If $Y \subset X$ is compact, then Y is closed.

Proof. It suffices to show that for every $x \in X \setminus Y$, there exists some $r > 0$ such that $B_r(x) \cap Y = \emptyset$. Fix $x \in X \setminus Y$. Since X is Hausdorff, then for all $y \in Y$, there exists a set $V_y \ni y$ and $W_y \ni x$ such that $V_y \cap W_y = \emptyset$. Clearly, $\{V_y\}$ is an open cover, and thus let $\{V_{y_i}\}$ be the open finite subcover. Moreover, we have that

$$\bigcup V_{y_i} \cap \bigcap W_{y_i} = \emptyset.$$

Since $x \in W_{y_i}$ for all x, and each is open, then the finite intersection is open. Thus, we have that $Y \cap \bigcap W_{y_i} = \emptyset$ and $\bigcap W_{y_i} \ni x$. $\square$

Corollary 1. If X is compact and Hausdorff and $Y \subset X$ is closed, then Y is compact. Moreover, for any $x \in X \setminus Y$, there exist open $V \supset Y$ and $x \in U$ such that $V \cap U = \emptyset$.

This corollary separates a closed set from a point, and we say X is **regular**. We say X is **normal** if it separates from closed sets in $Y \setminus X$.

Theorem 11. Suppose $f : X \to Y$ be continuous with X compact. Then $f(X)$ is compact.

Proof. Let $\{U_\alpha\}$ be an open cover of $f(X)$. Then $\{f^{-1}(U_\alpha)\}$ is an open cover of X, and thus $\{f^{-1}(U_{\alpha_i})\}$ is a finite open cover with

$$X \subset \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i}) \implies f(X) \subset \bigcup_{i=1}^{n} U_{\alpha_i}.$$

$\square$

Theorem 12. Suppose $f : X \to Y$ is a continuous bijection with X compact and Y Hausdorff. Then f is a homeomorphism.

Proof. We have that f^{-1} is continuous if and only if $f(F)$ is closed (F closed). Let $K \subset X$ be closed, then K is compact, and thus $f(K)$ is compact, and thus since Y is closed, then $f(K)$ is closed. $\square$

1.4 Tuesday, Feb 4: Applications of Tychonoff's Theorem

Example 1.3. Suppose X is equipped with the discrete topology. Then if we say that

$$\beta X = \overline{F(X)} \subset \prod \text{Compact spaces with upper bound on card. and exist continuous function from X.}$$

Then

$$\beta X = \text{ultrafilters on X.}$$

For all $A \subset X, U_A \subset \beta X$, where

$$U_A := \{\mathscr{F} \ ; \ A \subset \mathscr{F}\}.$$

We claim that $\{U_A\}$ is a basis for a topology.

$$X \to \beta X$$

$$x \to \{\mathscr{F} \text{ principal ultrafilter gen by } x\}$$

Proposition 2. (a) This map is homeomorphic

(b) βX is compact and Hausdorff

Example 1.4. Supose $X = \mathbb{N}$, then an example from X to a compact Hausdorff space is $N \xrightarrow{a} [-C, C]$, where $|a_i| \leq C$.

Remark 5. (Universal Property) Any $a : \mathbb{N} \to [-C, C]$ admits a continuous extension from

$$\beta a : \beta\mathbb{N} \to [-C, C].$$

Let $\omega \in \beta\mathbb{N}$ be a non-principal ultrafilter. Then to find $\beta a(\omega)$, split $[-C, C] = [-C, 0] \cup (0, C]$, then if $C = 1$:

$$\mathbb{N} = a^{-1}[-1, 0] \sqcup a^{-1}(0, 1].$$

Suppose $a^{-1}(0, 1] \in \omega$, then split $(0, 1] = (0, \frac{1}{2}] \cup (\frac{1}{2}, 1]$, and suppose $a^{-1}(\frac{1}{2}, 1] \in \omega$. Keep going iteratively, and we find that $\beta a(\omega)$ is this limit.

Example 1.5. (Profinite Completions of groups) Let G be a group. Let $\phi_i : G \to F$ be all homeomorphisms, where F is a compact finite group. Then

$$G \xrightarrow{\Phi} \prod_{\phi_i, F} F,$$

Definition 10. A **continuum** is a nonempty compact connected metrizable space (the topology was induced by a metric)

Definition 11. A continuum K is **indecomposable** if whenever $K = A \cup B$, where A, B are continuum, then either $A = K$ or $B = K$ (or both.)

Example 1.6. (Indecomposable continuum with hmore than one point) The Knaster Continuum:

Figure 1: The Knaster Continuum

We claim that K is indecomposable.

Proposition 3. Suppose Q_n is a nested family of continua. Then $Q_\infty = \bigcap Q_n$ is a continua.

Proof. We know that Q_∞ is compact and metrizable and nonempty by properties of compactness. Suppose $Q_\infty = A \sqcup B$ where they're both nonempty and closed in $Q_\infty \subset Q_0$. Since Q_0 is metrizable, then A, B are compact and disjoint in Q_0, and so there exists an $\epsilon > 0$ such that $d(A, B) > \epsilon$. Thus there exist disjoint open in Q_0 $A \subset U$ and $B \subset V$ such that $U \cap V = \emptyset$. Define

$$F_n := Q_n - (U \cup V) = Q_n \cap (Q_0 - (U \cup V))$$

and so

$$\bigcap F_n = \emptyset$$

so some F_n is empty, and so $Q_n \subset U \sqcup V$, and so $(Q_n \cap U) \sqcup (Q_n \cap V)$ is separated and so Q_n is not connected. $\square$

Definition 12. We define the **tent map** $F : I \to I$ such that

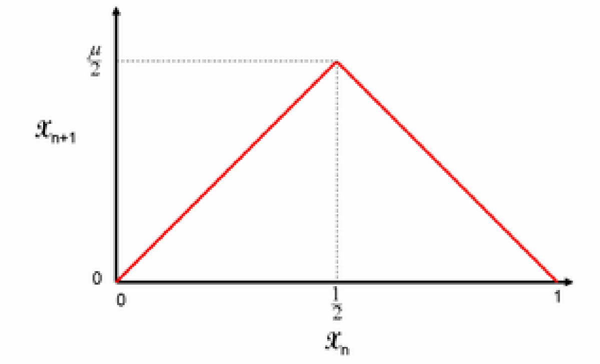


Figure 2: Tent Map

Definition 13. Let (X_i) be a sequence of continua such that for all $i \geq 1$,

$$f_{i+1} : X_{i+1} \to X_i$$

is a sequence of continuous surjective maps. Then we define

$$X_\infty := \lim_{\leftarrow}(X_i, f_i) = \{x_i \in \prod X_i \ ; \ x_i = f_{i+1} \ \forall i\}$$

as the **inverse limit**, as a subset of $\prod X_i$.

$$\cdots \to X_3 \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1.$$

Theorem 13. X_∞ is a continuum and if $A_\infty \subset X_\infty$ is a sub-continuum, then $A = \lim_{\leftarrow}(A_i, g_{i+1})$ where $A_i = \pi_i(A), g_{i+1} = f_{i+1}|_{A_{i+1}}$

Proof. Define

$$Q_{n,i} := \{(x_i) \in \prod X_i \ ; \ x_i = f_{i+1}(x_{i+1}), \ \forall i \in [n]\}.$$

Since $Q_n \approx \prod_{i \geq n} X_i$, then Q_n is nonempty, compact, Hausdorff, and connected. - $\square$

Tuesday, Feb 11: Regular and Normal results

Remark 6. (a) T0- Points are closed (a point)

(b) T1-Hausdorff

(c) T2-Regular

(d) T3-Normal

Lemma 3. Suppose points are closed in X. Then

(a) X is regular if and only if for all $u \in X$, for all open $U \ni u$, there exists a $V \ni u$ open such that $\overline{V} \subset U$

(b) X is normal if and only if for all $A \subset X$ closed, for all $U \supset A$ open, there exists a $V \supset A$ open such that $\overline{V} \subset U$.

Proof. (a) If X is regular, then there exists some open set U containing x. Thus, $X \setminus U$ is closed and disjoint from $\{x\}$. By regularity, there exists $V \ni u$ open such that $W \supset X \setminus U$ open and $V \cap W = \emptyset$. We have that $V \supset X \setminus W$, the latter of which is closed, and thus $\overline{V} \subset X \setminus W \subset U$

Let $x \in X$, and suppose $K \subset X$ is closed with $x \notin K$. $X \setminus K = U$ is open and contains x, and thus by assumption, there exists some $V \ni x$ such that $\overline{V} \subset U$, and thus $X \setminus \overline{V}$ is open and contains K and is disjoint from V.

(b) Replace x with A above. □

Theorem 14. The following hold:

(a) The subspace of a Hausdorff space is Hausdorff. Moreover, an arbitrary product of Hausdorff spaces is Hausdorff

(b) A subspace of a regular space is regular and an arbitrary product of regular spaces is regular.

Remark 7. The subspace of a normal space is not necessarily normal, and the product of normal spaces is not necessarily normal.

Proof. (b) Suppose X is our regular space, and $Y \subset X$ is a subspace. Suppose $y \in X$ and $A \subset Y$ is closed and $\{x\} \cap A = \emptyset$. Thus, $A = Y \cap K$ for some $K \subset X$ closed. So then $\{x\} \cap K = \emptyset$, and so there exists $U \ni x$ and $V \supset K$ both open and disjoint. Then $Y \cap U$ and $Y \cap V$ are both open and disjoint, and we are done.

Suppose $\{X_\alpha\}$ are all regular, then they are Hausdorff, and so $X = \prod X_\alpha$ are all Hausdorff, and so points are closed. Let $x = (x_\alpha) \in X$. Then if $U \ni x$ open (where U is a basis). Thus, for all α, $x_\alpha \in U_\alpha \subset X_\alpha$, and there exists $x_\alpha \in V_\alpha \subset U_\alpha$ where $\overline{V_\alpha} \subset U_\alpha$. Evidently, $V = \prod V_\alpha$, and $V \ni x$, and we claim that

$$\overline{V} = \overline{\prod V_\alpha} = \prod \overline{V_\alpha} \subset \prod U_\alpha = U$$

□

Theorem 15. If X is regular with a countable basis, then X is normal.

Proof. Let $\mathscr{B}$ be a countable basis. Let A, B be closed disjoint subsets of X. For all $x \in A$, $x \in X \setminus B$ open, and thus there exist $U_x \ni x$ such that $\overline{U_x} \subset X \setminus B$. Without loss of generality, we can assume U_x is a basis element. Since $\mathscr{B}$ is countable, we can find $W_1, W_2, \ldots$ basis elements such that $\overline{W_i} \cap B = \emptyset$ for all i, and $\bigcup W_i \supset A$. Similarly for B, there exists $V_1, V_2, \ldots$ such that $\overline{V_i} \cap A = \emptyset$ for all i and $\bigcup V_i \supset B$.

Let

$$W_1' := W_1 \setminus \overline{V_1} = W_1 \cap (\overline{V_1}^c), \quad V_1' = V_1 \cap (\overline{W_1}^c),$$

and note both are open and $A \cap W_1 = A \cap W_1'$ and similarly for V_1'. Define

$$W_2' = W_2 \cap \overline{V_1}^c \cap \overline{V_2}^c, \quad V_2' = V_2 \cap \overline{W_1}^c \cap \overline{W_2}^c$$

Build this recursively. The for any n, W_n' is disjoint from V_j' with $j \leq n$ and V_n' is disjoint from W_j' with $j \leq n$. Define

$$W := \bigcup W_i' \supset A, \qquad V := \bigcup V_i' \supset B$$

open and disjoint. □

Theorem 16. Every metric space is normal.

Proof. Suppose X is a metric space induced by the topology and let A, B be disjoint closed sets. For all $a \in A$, ther exists an $\epsilon_a > 0$ such that

$$B_{\epsilon_a}(a) \cap B = \emptyset, \qquad B_{\epsilon_b}(b) \cap A = \emptyset$$

Let

$$U := \bigcup_{a \in A} B_{\frac{\epsilon_a}{2}}(a), \quad V := \bigcup_{b \in B} B_{\frac{\epsilon_b}{2}}(b)$$

Both are open. Suppose $x \in U \cap V$, then $v \in B_{\frac{\epsilon_a}{2}}(a) \cap B_{\frac{\epsilon_b}{2}}(b)$, for some $a \in A$, $b \in B$, and thus

$$d(a,b) \leq d(a,x) + d(x,b) < \epsilon = \min\{\epsilon_a, \epsilon_b\} \implies b \in B_{\epsilon_a}(a).$$

□

Theorem 17. Compact Hausdorff spaces are normal.

Proof. Let A, B be closed disjoint sets. For all $a \in A$, there exists $U_a \ni a$ and $V_a \supset B$ open and disjoint.

$$A \subset \bigcup_{a \in A} U_a \implies A \subset \bigcup_{i=1}^{N} U_{a_i} =: U.$$

Moreover,

$$V := \bigcap_{i=1}^{N} V_{a_i} \supset B.$$

U and V are open disjoint. □

Remark 8. To recap: For compact X, the following are equivalent:

(a) X is regular

(b) X is Hausdorff

(c) X is normal

For second countable X:

(a) X is regular

(b) X is normal

(c) X is metrizable.

Thus, we have yet to prove the last equivalence.

Lemma 4. (Urysohn's Lemma) Let X be normal, $A, B \subset X$ be closed and disjoint. Then there exists a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.

Proof. Let $P = \mathbb{Q} \cap [0,1]$. For each $p \in P$, we want to find some U_p open such that $A \subset U_0$, $U_1 = X \setminus B$, and if $p < q$, then $\overline{U_p} \subset U_q$.

Let $U_1 = X \setminus B$ and since $A \subset U_1$, then by normality, there exists some $A \subset U_0$ such that $\overline{U_0} U_1$. By normality, there exists some open $\overline{U_0} \subset U_{\frac{1}{2}}$ such that $\overline{U_{\frac{1}{2}}} \subset U_1$. Keep going with the fairy rationals. Let $U_{\frac{p}{q}} = X$ if $\frac{p}{q} > 1$, and Define

$$f(x) := \inf\{\frac{p}{q} \text{ such that } x \in U_{\frac{p}{q}}\}.$$

We claim that f is our function. $\square$

1.5 Tuesday, Feb 18:

Theorem 18. Suppose X is regular with a countable basis. Then X is metrizable.

Proof. We use Urysohn's Lemma. For all $x \in X$, for all U open, there exists $f : X \to [0, 1]$ continuous such that $f(x) = 1$ and $f(X^c) = 0$.

We claim that if X is completely regular, then there exists an embedding from $X \mapsto [0, 1]^J$, for some J index set In fact, X completely regular and a countable basis implies there exist an embedding from $X \mapsto [0, 1]^N$. It suffices to show this, since X would be homeomorphic to a subset of a metric space.

(a) If X is compltely regular the for all $x \in X$, for all $U \ni x$ open, then we choose $f : X \to [0, 1]$ with $f(x) = 1$ and $f(X^c) = 0$. We take these f to be the coordinates of

$$F : X \to [0, 1]^J.$$

If $x \neq y$, then we can choose $x \in U$ and $y \notin U$ such that $f(x) = 1$ and $f(y) = 0$, and so the map is injective. F is continuous and injective, to show that F is a homeomorphism unto its image, we want to show that $F^{-1} : F(X) \to X$ is continuous. That is, for all $U \subset X$ is open, then $F(U) \subset F(X)$ is open. That is, we want to show that $F(U) = F(X) \cap V$, $V \subset [0, 1]^N$ is open. Let $F(x) \in F(U)$ for some $x \in U$. We want to find some open $W = F(X) \cap V$ such that $F(x) \in W$. Let

$$V := \pi_f^{-1}((0, 1]) \subset \prod_J [0, 1],$$

which is obviously open, then $F(x) \in V \cap F(X) \subset F(U)$

(b) If F has a countable basis, then we claim that we can find a countable set $f_n : X \to [0, 1]$ continuous such that for all $x \in X$, for all $U \ni x$, open, there exists some n such that $f_n(x) = 1$ and $f_n(X^c) = 0$. Let $\{U_i\}$ be a countable basis, and suppose $x \in U$ open. Then $x \in U_i \subset U$, then since X is regular, $x \in \overline{U_j} \subset U_i \subset U$. Define $f_{i,j} : X \to [0, 1]$ such that $f_{i,j}(\overline{U_j}) = 1$ and $f_{i,j}(U_i^c) = 0$.

(c) Take $f_{i,j}$ from above as the coordinates of F, and so $F : X \to [0, 1]^N$ is a homeomorphism unto its image. Thus, X is metrizable.

□

Proposition 4. Let $\overline{X} = \overline{F(X)} \subset [0, 1]^J$.

(a) $\overline{X}$ is compact and Hausdorff.

(b) If $X \subset \overline{X}$ is dense and X has subspace topology.

(c) For all $f : X \to [0, 1]$, there exists a unique $\overline{f} : \overline{X} \to [0, 1]$.

Lemma 5. Suppose $\overline{X}$ is compact and Hausdorff. Then $\overline{f} : \overline{X} \to [0, 1]$ and $\overline{f}|_X = f$, then $\overline{f}$ is defined by f. That is, the extension is unique.

Theorem 19. (Stone-Ĉech compactication) Let X be a completely regular space. There exists a compact Hausdorff space βX and a homeomorphism mapping X into a dense subset of βX such that if f is a bounded continuous function from X to $\mathbb{R}$, then f has a bounded continuous extension to βX.

Theorem 20. (Alexandroff- Hausdorff) Let X be Hausdorff, the following are equivalent:

(a) There is a continuous surjective map $f : C \to X$

(b) X is nonempty, compact, and metrizable.

(c) X is nonempty, compact, and has a countable basis.

Remark 9. For a compact Hausdorff X, metrizable is equivalent to X having a countable basis from before.

Lemma 6. Suppose $f : A \to B$ is continuous and surjective. If A is compact and metrizable, and B is Hausdorff, then B is compact and metrizable.

Proof. It suffices to show that B has a countable basis. Let $\mathscr{U}$ be a countable basis for A. Let $\mathscr{U}'$ be another countable basis for A such that each $U_i' = \bigcup_{j=1}^{N} U_j$. Define

$$V_i := B - f(A - U_i').$$

We claim that $\{V_i\}$ is a basis for B. V_i is obviously open. Let $b \in B$. Let $b \in V \subset B$ be open. It suffices to find some $V_i \in \{V_i\}$ such that $b \in V_i \subset V \subset B$. $f^{-1}(\{b\})$ is compact and contained in $f^{-1}(V)$ open. For all $a \in f^{-1}(\{b\})$, $a \in U_j$ for $U_j \subset f^{-1}(V)$. By compactness, there exist finitely many of these, U_i so take the union to make U_i' and thus

$$f^{-1}(\{b\}) \subset U_i' \subset f^{-1}(V) \implies A - f^{-1}(\{b\}) \supset A - U_i' \supset A - f^{-1}(V) \implies f(A - f^{-1}(\{b\})) \supset f(A - U_i') \supset f(A - f^{-1}(V))$$

and so

$$B - f(A - f^{-1}(\{b\})) \subset B - f(A - U_i') \subset B - f(A - f^{-1}(V))$$

and it can be shown that this is equivalent to

$$\{b\} \subset V_i \subset V$$

□

Proof. (Alexandroff-Hausdorff) Suppose $f : X \to X$ is continuous and surjective, then clearly, X is nonempty and X is compact. We claim that X has a countable basis, which is obvious from our lemma. □

Definition 14. Suppose X is Hausdorff. Let $x \in X$. The **connected component** of X containing x is the **maximal connected** subset of X containing x.

Remark 10. This is equivalent to saying that

$$\text{component} = \bigcup Q \quad \text{st } Q \subset X \text{ is connected}$$

Definition 15. X is totally disconnected if every connected component is a single point.

Definition 16. X is **perfect** if no point is open.

Remark 11. X is perfect if for all $x \in X$, for all $U \ni u$ open, there exists $y \in U - x$. That is, there are no isolated points.

Proposition 5. Suppose $X = \prod_{i=1}^{\infty} X_i$, where each X_i is finite, nonempty, and is equipped with the discrete topology. Then X is compact, nonempty, metrizable, and totally disconnected. Moreover, if infinitely many X_i have more than 1 point, then X is perfect

Proof. Compact comes from Tychonoff. X is metrizable from before. X is obviously nonempty. To show that X is totally disconnected

□

Feb 25: