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Problem 1

Let $f \in \mathcal{R}$ be $2-\pi$ periodic.

(a) Show that the Fourier Series of f can be written as

$$\hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n) + \hat{f}(-n))\cos(nx) + i(\hat{f}(n) - \hat{f}(-n))\sin(nx)$$

SOLUTION: By definition, we have that for any $N \in \mathbb{N}$,

$$\begin{aligned}
S_N(f) &= \sum_{-N}^{N} (f, e_n)e_n \\
&= \sum_{-N}^{N} \hat{f}(n)e_n \\
&= \sum_{-N}^{N} \hat{f}(n)e^{inx} \\
&= \sum_{-N}^{N} \hat{f}(n)(\cos(nx) + i\sin(nx)) \\
&= \hat{f}(0) + \sum_{n=1}^{N} \hat{f}(n)\cos(nx) + i\hat{f}(n)\sin(nx) + \sum_{n=-N}^{-1} \hat{f}(n)\cos(nx) + i\hat{f}(n)\sin(nx) \\
&= \hat{f}(0) + \sum_{n=1}^{N} \hat{f}(n)\cos(nx) + i\hat{f}(n)\sin(nx) + \sum_{n=1}^{N} \hat{f}(-n)\cos(-nx) + i\hat{f}(-n)\sin(-nx) \\
&= \hat{f}(0) + \sum_{n=1}^{N} \hat{f}(n)\cos(nx) + i\hat{f}(n)\sin(nx) + \sum_{n=1}^{N} \hat{f}(-n)\cos(nx) - i\hat{f}(-n)\sin(nx) \\
&= \hat{f}(0) + \sum_{n=1}^{N} (\hat{f}(n) + \hat{f}(-n))\cos(nx) + i(\hat{f}(n) - \hat{f}(-n)\sin(nx)
\end{aligned}$$

■

(b) Prove that if f is even, then $\hat{f}(n) = \hat{f}(-n)$ and we get a cosine series.

SOLUTION: Let f be even so that $f(x) = f(-x)$. Then

$$\begin{aligned}
\hat{f}(-n) &= (f, e_{-n}) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-i(-n)x}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-x)e^{inx}\,dx \\
&= -\frac{1}{2\pi}\int_{\pi}^{-\pi} f(u)e^{-inu}\,du \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)e^{-inu}\,du \\
&= (f, e_n) \\
&= \hat{f}(n)
\end{aligned}$$

Moreover, we use the identity derived in part (a) to notice that

$$\begin{aligned}
S_N(f) &= \hat{f}(0) + \sum_{n=1}^{N}(\hat{f}(n) + \hat{f}(-n))\cos(nx) + i(\hat{f}(n) - \hat{f}(-n)\sin(nx) \\
&= \hat{f}(0) + \sum_{n=1}^{N}(\hat{f}(n) + \hat{f}(n))\cos(nx) + i(\hat{f}(n) - \hat{f}(n)\sin(nx) \\
&= \hat{f}(0) + 2\sum_{n=1}^{N}\hat{f}(n)\cos(nx)
\end{aligned}$$

as desired. ■

(c) Prove that if f is odd, then $\hat{f}(n) = -\hat{f}(-n)$ and we get a sine series.

SOLUTION: Let f be odd such that $f(x) = -f(-x)$. Then

$$\begin{aligned}
-\hat{f}(-n) &= -(f, e_{-n}) \\
&= -\frac{1}{2\pi}\int_{0}^{2\pi} f(x)e^{inx}\,dx \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} -f(x)e^{inx}dx \\
&= \frac{1}{2\pi}\int_{0}^{-2\pi} -(-f(-u)e^{-inu})\,du \\
&= -\frac{1}{2\pi}\int_{0}^{-2\pi} f(u)e^{-inu}\,du
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} f(u)e^{-inu}\,du \\
&= (f, e_n) \\
&= \hat{f}(n).
\end{aligned}
$$

Moreover, we use the identity derived in part (a) to show that

$$
\begin{aligned}
S_N(f) &= \hat{f}(0) + \sum_{n=1}^{N}(\hat{f}(n) + \hat{f}(-n))\cos(nx) + i(\hat{f}(n) - \hat{f}(-n)\sin(nx) \\
&= \hat{f}(0) + \sum_{n=1}^{N}(-\hat{f}(-n) + \hat{f}(-n))\cos(nx) + i(\hat{f}(n) + \hat{f}(n)\sin(nx) \\
&= \hat{f}(0) + \sum_{n=1}^{N} i(2\hat{f}(n))\sin(nx) \\
&= \hat{f}(0) + 2i\sum_{n=1}^{N}\hat{f}(n)\sin(nx)
\end{aligned}
$$

as desired ■

(d) Suppose that $f(x+\pi) = f(x)$ for all $x \in \mathbb{R}$. Show that $\hat{f}(n) = 0$ for all odd n.

Solution: Let n be odd, then

$$
\begin{aligned}
\hat{f}(n) &= (f, e_n) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)e^{-inx}\,dx + \int_{0}^{\pi} f(x)e^{-inx}\,dx\right] \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)e^{-inx}\,dx + \int_{-\pi}^{0} f(u+\pi)e^{-in(u+\pi)}\,dx\right] \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)e^{-inx}\,dx + \int_{-\pi}^{0} f(u)\frac{e^{-inu}}{e^{in\pi}}\,dx\right]
\end{aligned}
$$

Since we have that n is odd, then

$$e^{in\pi} = \cos(n\pi) + i\sin(n\pi) = -1 + 0 = -1,$$

then

$$\hat{f}(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)e^{-inx}\,dx - \int_{-\pi}^{0} f(u)e^{-inu}\,dx\right] = 0.$$

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(e) Show that f is real valued if, and only if, $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all $n \in \mathbb{N}$.

SOLUTION: ($\Longrightarrow$) Suppose f is real valued. Then $\overline{f(x)} = f(x)$ for all $x \in \mathbb{R}$. Thus, we note that since the conjugate of the integral is the integral of the conjugate and similarly for products, we can compute

$$\begin{aligned}
\overline{\hat{f}(n)} &= \overline{\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}\,\overline{e^{-inx}}\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{inx}\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-i(-n)x}\,dx \\
&= \hat{f}(-n)
\end{aligned}$$

($\Longleftarrow$) Suppose f is continuous and $\overline{\hat{f}(n)} = \hat{f}(-n)$. To see that f is real value, it suffices to show that $\overline{f(x)} = f(x)$ for any $x \in \mathbb{R}$. By a corollary in class, it suffices to show that the Fourier coefficients of $\overline{f}$ and $\hat{f}(n)$ are equal.

$$\begin{aligned}
(\overline{f}, e_n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)}e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)e^{inx}}\,dx \\
&= \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx} \\
&= \overline{(f, e_{-n})} \\
&= \overline{\hat{f}(-n)} \\
&= \overline{\overline{\hat{f}(n)}} \\
&= \hat{f}(n)
\end{aligned}$$

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Problem 2

Let $f(x) = |x|$ be defined on $[-\pi, \pi]$.

(a) Calculate $\hat{f}(0)$.

Solution:

We have that

$$\hat{f}(0) = (f, e_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}$$

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(b) Calculate $\hat{f}(n)$ when $n \neq 0$.

Solution: We integrate by parts

$$\begin{aligned}\hat{f}(n) &= (f, e_n)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|e^{-inx}\,dx\\ &= \frac{1}{2\pi}\left[\int_0^{\pi} xe^{-inx}\,dx - \int_{-\pi}^{0} xe^{-inx}\,dx\right].\end{aligned}$$

First term:

$$\begin{aligned}\int_0^{\pi} xe^{-inx}\,dx &= \frac{-1}{in}e^{-inx}x\Big|_0^{\pi} + \frac{1}{in}\int_0^{\pi} e^{-inx}\,dx\\ &= \frac{-1}{in}e^{-i\pi n}\pi + \frac{1}{in}\frac{-1}{in}\left[e^{-in\pi} - 1\right]\\ &= \frac{-e^{-i\pi n}\pi}{in} + \frac{e^{-in\pi} - 1}{n^2}\\ &= \frac{e^{-i\pi n}i\pi n}{n^2} + \frac{e^{-in\pi} - 1}{n^2}\\ &= \frac{-1 + e^{-i\pi n}(1 + i\pi n)}{n^2}\end{aligned}$$

With similar algebra, we see that

$$\int_{-\pi}^{0} xe^{-inx} = \frac{1 + e^{i\pi n}(-1 + i\pi n)}{n^2}$$

Thus,

$$\hat{f}(n) = \frac{1}{2\pi}\left[\int_0^{\pi} xe^{-inx}\,dx - \int_{-\pi}^{0} xe^{-inx}\,dx\right].$$

$$= \frac{1}{2\pi}\left[\frac{-1+e^{-i\pi n}(1+i\pi n)}{n^2} - \frac{1+e^{i\pi n}(-1+i\pi n)}{n^2}\,dx\right]$$
$$= \frac{1}{2\pi}\left[\frac{-2+e^{-i\pi n}(1+i\pi n)-e^{i\pi n}(-1+i\pi n)}{n^2}\right]$$
$$= \frac{1}{2\pi}\left[\frac{-2+(e^{-i\pi n}+e^{i\pi n})+i\pi n(e^{-i\pi n}-e^{i\pi n})}{n^2}\right]$$

It can be shown without too much work that this can further be simplifyed into

$$\hat{f}(n) = \frac{(-1)^n - 1}{\pi n^2}$$

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(c) Calculate the Fourier Series in terms of sines and cosines.

SOLUTION: By the first question, since $|x|$ is even, we will have a Fourier series of the form

$$S_N(f) = \hat{f}(0) + 2\sum_{n=1}^{N} \hat{f}(n)\cos(nx).$$

From part (b), we have that for $n \neq 0$,

$$\hat{f}(n) = \frac{1}{2\pi}\left[\frac{-2+(e^{-i\pi n}+e^{i\pi n})+i\pi n(e^{-i\pi n}-e^{i\pi n})}{n^2}\right]$$

We have that using properties of sin and cos,

$$e^{-i\pi n} + e^{i\pi n} = \cos(-\pi n) + i\sin(-\pi n) + \cos(\pi n) + i\sin(\pi n) = 2\cos(\pi n)$$

$$e^{-i\pi n} - e^{i\pi n} = \cos(-\pi n) + i\sin(-\pi n) - \cos(\pi n) - i\sin(\pi n) = -2i\sin(\pi n) = 0$$

Plugging back in:

$$\hat{f}(n) = \frac{1}{2\pi}\left[\frac{-2+(e^{-i\pi n}+e^{i\pi n})+i\pi n(e^{-i\pi n}-e^{i\pi n})}{n^2}\right]$$
$$= \frac{1}{2\pi}\left[\frac{-2+2\cos(\pi n)}{n^2}\right]$$
$$= \frac{-1+\cos(\pi n)}{\pi n^2}$$

Thus,

$$S_N(f) = \hat{f}(0) + 2\sum_{n=1}^{N} \hat{f}(n)\cos(nx)$$
$$= \frac{\pi}{2} + 2\sum_{n=1}^{N} \frac{-1+\cos(\pi n)}{\pi n^2}\cos(nx)$$

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(d) Deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

SOLUTION:

Lemma 1. Suppose that for $t \in (-\delta, \delta)$, there exists a $C \in \mathbb{R}$ such that $|f(x-t)-f(x)| \leq C(t)$, (locally Lipshitz), then $S_n(x) \to f(x)$.

Proof. Define

$$g(t) = \frac{f(x-t) - f(x)}{\sin\left(\frac{t}{2}\right)},$$

and define

$$D_N(x) = \sum_{-N}^{N} e^{inx} = \frac{\sin\left(N + \frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}.$$

Then

$$\begin{aligned}
S_N(f,x) &= \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f e^{-int} e^{inx}\, dt \\
&= \frac{1}{2\pi 0} f(t) \sum_{-N}^{N} e^{in(x-t)} dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(x-t) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt \\
&= \frac{1}{2\pi} (f * D_N)(t)
\end{aligned}$$

Thus,

$$\begin{aligned}
|S_n(x) - f(x)| &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x)) D_N(t) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \sin\left(Nt + \frac{t}{2}\right) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \cos\left(\frac{t}{2}\right) \sin(Nt) dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \sin\left(\frac{t}{2}\right) \cos(Nt) dt \\
&\to 0.
\end{aligned}$$

The last equality holds because $g(t)$ is bounded and because $|\hat{f}(n)| \to 0$ by Bessel's inequality and thus both the real and imaginary components of $\hat{f}(n)$ go to 0. $\square$

We first show that $|x|$ is locally Lipshitz around $x = 0$. Let $t \in (-\delta, \delta)$, then

$$|f(0-t) - f(0)| = |t| = C(t).$$

Thus, by our lemma, $S_N(f,0) \to f(0) = 0$, thus,

$$\begin{aligned} 0 &= \frac{\pi}{2} + 2\sum_{n=1}^{\infty} \frac{-1+\cos(\pi n)}{\pi n^2}\cos(n(0)) \\ &= \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{-1+(-1)^n}{n^2} \\ &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}}^{\infty} \frac{1}{n^2} \end{aligned}$$

Thus,

$$\sum_{n \text{ odd}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \frac{\pi^2}{8} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{\pi^2}{8},$$

and thus

$$\frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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Reflections: In this problem, we actually have stronger convergence. In fact, we have that $S_N(f) \rightrightarrows f$. To see this, it suffices to note that f is continous, $2\pi-$periodic on $[-\pi, \pi]$, and $S_N(f)$ converges absolutely since

$$\|S_N(f)\| \leq \frac{\pi}{2} + 2\sum_{n=1}^{N} \frac{2}{\pi n^2} < \infty$$

Problem 3

Show that in $\mathcal{R}$, the space of $2\pi-$integrable functions, the Pythagorean theorem, the C-S inequality, and the triangle inequality all hold.

SOLUTION: (Pythagorean Theorem) Let $f, g \in \mathcal{R}$ such that $f \perp g$. Then

$$\begin{aligned}\|f+g\|^2 &= (f+g, f+g) \\ &= (f, f+g) + (g, f+g) \\ &= (f,f) + (f,g) + (g,f) + (g,g) \\ &= \|f\|^2 + 0 + 0 + \|g\|^2 \\ &= \|f\|^2 + \|g\|^2\end{aligned}$$

(Hölder's Inequality). Let $p, q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Consider the degenerate case when

$$\int_{-\pi}^{\pi} |f|^p = \int_{-\pi}^{\pi} |g|^q = 1.$$

Note that for any $x \in [-\pi, \pi]$, we have that by properties of the logarithm and by its convexity,

$$\log(|f(x)||g(x)|) = \frac{1}{p}\log(f(x)^p) + \frac{1}{q}\log(g(x)^q) \leq \log\left(\frac{1}{p}f(x)^p + \frac{1}{q}g(x)^q\right).$$

Since the logarithm function is monotonically increasing, we have that since the integral is monotonic,

$$|f(x)||g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q \implies \int_{-\pi}^{\pi} |f||g| \leq \frac{1}{p}\int_{-\pi}^{\pi} f^p + \frac{1}{q}\int_{-\pi}^{\pi} g^q = \frac{1}{p} + \frac{1}{q} = 1.$$

Thus, in the degenerate case,

$$\int_{-\pi}^{\pi} |f||g| \leq 1. \tag{1}$$

Now for the general case. Define

$$f^* := \frac{f}{\left(\int_{-\pi}^{\pi}|f|^p\right)^{\frac{1}{p}}} \implies \int_{-\pi}^{\pi} |f^*|^p = \int_{-\pi}^{\pi} \frac{|f|^p}{\left(\left(\int_{-\pi}^{\pi}|f|^p\right)^{\frac{1}{p}}\right)^p} = 1$$

$$g^* := \frac{g}{\int_{-\pi}^{\pi}|g|^q} \implies \int_{-\pi}^{\pi} |g^*|^q = 1.$$

By our degenerate case, we know that

$$\begin{aligned}1 &\geq \int_{-\pi}^{\pi} |f^*||g^*| \\ &= \int_{-\pi}^{\pi} \frac{|f|\cdot|g|}{\left(\int_{-\pi}^{\pi}|f|^p\right)^{\frac{1}{p}} \cdot \left(\int_{-\pi}^{\pi}|g|^p\right)^{\frac{1}{q}}},\end{aligned}$$

and so

$$\int_{-\pi}^{\pi} |f| \cdot |g| \leq \left(\int_{-\pi}^{\pi} |f|^p \right)^{\frac{1}{p}} \left(\int_{-\pi}^{\pi} |g|^p \right)^{\frac{1}{q}}.$$

Thus, letting $p = q = 2$, we have that

$$\begin{aligned}
|(f,g)| &= \left| \int_{-\pi}^{\pi} f \cdot \overline{g} \right| \\
&\leq \int_{-\pi}^{\pi} |f| \cdot |\overline{g}| \\
&= \int_{-\pi}^{\pi} |f| \cdot |g| \\
&\leq \left(\int_{-\pi}^{\pi} |f|^2 \right)^{\frac{1}{2}} \left(\int_{-\pi}^{\pi} |g|^2 \right)^{\frac{1}{2}} \\
&= \|f|| \cdot \|g\|
\end{aligned}$$

(Triangle Inequality) We have that for any $f, g, h \in \mathcal{R}$,

$$\begin{aligned}
\|f - g\| &= \int_{-\pi}^{\pi} |f - g| \\
&= \int_{-\pi}^{\pi} |(f - h) + (h - g)| \\
&\leq \int_{-\pi}^{\pi} |f - h| + |h - g| \\
&= \int_{-\pi}^{\pi} |f - h| + \int_{-\pi}^{\pi} |h - g| \\
&= \|f - h\| + \|h - g\|
\end{aligned}$$

A more general proof:

$$\begin{aligned}
\|f + g\|^2 &= (f + g, f + g) \\
&= (f, f) + (f, g) + (g, f) + (g, g) \\
&\leq \|f\|^2 + \|g\|^2 + 2|(f, g)| \\
&\leq \|f\|^2 + \|g\|^2 + 2\|f\|\|g\| \\
&= (\|f\| + \|g\|)^2.
\end{aligned}$$

Take square roots of both sides and conclude.

$\blacksquare$

Problem 4

Find the values of

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^4}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4}$$

SOLUTION: Parseval's Theorem states that

$$\lim_{n\to\infty} (S_n(f))^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2.$$

Using Problem (1), we see that since

$$(S_n(f))^2 = \frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^4}$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |x|^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 = \frac{\pi^2}{3}$$

Thus, we have that

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^4} \iff \frac{\pi^2}{96} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^4}.$$

Moreover, we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n=1}^{\infty} \frac{1}{(2n)^4} + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^4} = \frac{1}{16} \sum_{n=1}^{\infty} + \frac{\pi^2}{96}$$

and so

$$\frac{15}{16} \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{96} \implies \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{90}$$

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