270 Spring 2025: Problem Set 3 Agustín Esteva
Complex Variables aesteva@uchicago.edu
Instructor(s): Robert Fefferman Due Date: 4-24-2025

Problem 1

Let $O \subseteq \mathbb{C}$ be open and connected and let $f \in H(O)$. If $\{z \in O \mid f(z) = 0\}$ has a limit point, then $f(z) = 0$ for all $z \in O$.

SOLUTION: Note that since $f \in H(O)$, f is infinitely differentiable. Suppose, for the sake of contradiction, that $f(z') > 0$ for some $z' \in O$. Then by a problem on a previous PSET, we have that for any $z \in O$, there exists some $n_z \in \mathbb{N}$ such that $f^{(n_z)}(z) \neq 0$. Let $z_0 \in O$ such that z_0 is a limit point of the vanishing set. Then by Cauchy's theorem, there is some $r > 0$ such that if $z \in D_r(z_0)$, then

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n = \sum_{n=1}^{\infty} a_n (z - z_0)^n.$$

Let k be the smallest derivative such that $f^k(z_0) = a_k \neq 0$. Then we have that

$$f(z) = a_k(z - z_0)^k + \sum_{k+1}^{\infty} a_n (z - z_0)^n = a_k(z - z_0)^k + O((z - z_0)^{k+1}).$$

Note that the second term goes to 0 as $z \to z_0$. Let $(z_n) \in \{z \in O \mid f(z) = 0\}$ such that $z_n \to z_0$. Without loss of generality, we can take $z_n \neq z_0$, and thus since $|z_0 - z_n| < r$ for large n, we have that

$$|f(z_n)| = |a_k||(z_n - z_0)|^k \neq 0,$$

which is a contradiction to the fact that $f(z_n) = 0$. Thus, we must have that $f(z) = 0$ for all $z \in O$. ∎

Problem 2

Let $(a_k) \in \mathbb{C}$ and $(b_j) \in \mathbb{C}$ and define

$$c_n := \sum_{k+j=n} a_k b_j, \quad n \geq 2.$$

If $\sum_{k=1}^{\infty} |a_k| < \infty$ and $\sum_{j=1}^{\infty} |b_j| < \infty$, then $\sum_{n=2}^{\infty} |c_n| < \infty$ and

$$\sum_{n=2}^{\infty} c_n = \sum_{k=1}^{\infty} a_k \sum_{j=1}^{\infty} b_j.$$

Solution: Let

$$A = \sum_{k=1}^{\infty} |a_k| < \infty, \quad B = \sum_{j=1}^{\infty} |b_j| < \infty.$$

Then

$$\sum_{n=2}^{N} |c_n| = \sum_{k=1}^{N-1} \sum_{j=1}^{N-k} |a_k||b_j| = \sum_{k=1}^{N} |a_k| \sum_{j=1}^{N-k} |b_j| \to \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{\infty} |b_k| = AB < \infty.$$

Thus, $\sum |c_n|$ converges. Moreover, we have that

$$\begin{aligned}
\left| \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} a_k \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} b_j - \sum_{n=2}^{N} c_n \right| &= \left| \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} a_k \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} b_j - \sum_{n=2}^{N} \sum_{j+k=n} a_k b_j \right| \\
&= \left| \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} a_k b_j - \sum_{j=1}^{N-1} a_1 b_j + \sum_{j=1}^{N-2} a_2 b_j + \cdots + a_{N-1} b_1 \right| \\
&= \left| \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} a_k b_j - \sum_{k=1}^{N-1} \sum_{j=1}^{N-k} a_k b_j \right| \\
&= \left| \sum_{j=\lfloor \frac{N}{2}+1 \rfloor}^{N-1} b_j \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} a_k - \sum_{k=\lfloor \frac{N}{2}+1 \rfloor}^{N-1} a_k \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} b_j \right| \\
&\leq \sum_{j=\lfloor \frac{N}{2}+1 \rfloor}^{N-1} |b_j| \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} |a_k| + \sum_{k=\lfloor \frac{N}{2}+1 \rfloor}^{N-1} |a_k| \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} |b_j| \\
&\leq \sum_{j=\lfloor \frac{N}{2}+1 \rfloor}^{\infty} |b_j| \sum_{k=1}^{\infty} |a_k| + \sum_{k=\lfloor \frac{N}{2}+1 \rfloor}^{\infty} |a_k| \sum_{j=1}^{\infty} |b_j|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=\lfloor \frac{N}{2}+1 \rfloor}^{\infty} |b_j| A + \sum_{k=\lfloor \frac{N}{2}+1 \rfloor}^{\infty} |a_k| B \\
&= A \sum_{j=\lfloor \frac{N}{2}+1 \rfloor}^{\infty} |b_j| + B \sum_{k=\lfloor \frac{N}{2}+1 \rfloor}^{\infty} |a_k| \\
&< A \frac{\epsilon}{2A} + B \frac{\epsilon}{2B} \\
&= \epsilon
\end{aligned}
$$

where the second to last inequality holds because both series absolutely converge and thus their tail ends can be arbitrarily small. Thus, for N large,

$$
\sum_{n=2}^{N} c_n = \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} a_k \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} b_k \to \sum_{n=2}^{\infty} c_n = \sum_{k=1}^{\infty} a_k \sum_{j=1}^{\infty} b_k
$$

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Problem 3

Using the previous problem, prove that

$$e^{w+z} = e^w e^z.$$

SOLUTION: For $n \geq 1$, define

$$c_n := \sum_{k+j=n} \frac{1}{(n)!}\binom{n}{k} w^k z^j = \sum_{k=0}^{n} \frac{1}{(n)!}\binom{n}{k} w^k z^{n-j} = \frac{1}{(n)!}(w+z)^n.$$

Then note that

$$\frac{1}{n!}\binom{n}{k} = \frac{1}{n!}\frac{n!}{(n-k)!(k)!} = \frac{1}{(j)!(k)!}.$$

Thus, define

$$a_k := \frac{1}{k!}w^k, \quad b_k := \frac{1}{j!}z^j,$$

then

$$c_n = \sum_{k+j=n} a_k b_j.$$

By the previous problem, we have that

$$e^{z+w} = \sum_{n=0}^{\infty} \frac{1}{n!}(w+z)^n = \sum_{n=0}^{\infty} c_n = \sum_{k=0}^{\infty} a_n \sum_{j=0}^{\infty} b_n = \sum_{k=0}^{\infty} \frac{w^k}{k!} \sum_{j=0}^{\infty} \frac{z^j}{j!} = e^w e^z$$

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Problem 4

Let $O \subseteq \mathbb{C}$ be open. Let $(f_n) \in H(O)$ such that $f_n \rightrightarrows f$ on every compact subset. Then $f \in H(O)$.

Solution: Let $z \in O$. Since O is open, there exists some closed disk $\overline{D_r(z)} \subseteq \mathbb{C}$. Note that this disk is compact and connected. Thus, $f_n \rightrightarrows f$ on $\overline{D_r(z)}$. Since each f_n is continuous, then f is continuous on $\overline{D_r(z)}$. Define

$$D_z := D_{\frac{r}{2}}(z) \subset \overline{D_r(z)}$$

as an open disk. D_z is open and connected and f is continuous on D_z. Let γ be a closed path on D_z. Then γ is compact since it is closed and bounded, and thus $f_n \rightrightarrows f$ on γ, and so

$$\int_\gamma f_n(\zeta)d\zeta \to \int_\gamma f(\zeta)d\zeta.$$

By Cauchy's theorem, since γ is also a closed path on O and each $f_n \in H(O)$, then for every n,

$$\int_\gamma f_n(\zeta)d\zeta = 0 \implies \int_\gamma f(\zeta)d\zeta = 0.$$

Thus, by Problem 7 on the previous PSET, we have that $f \in H(D_z)$. Because this is true for every $z \in O$, then $f \in H(O)$. ■

Problem 5

(a) Suppose γ is a closed path in $\mathbb{C}$. Let O be the open, unbounded complement of γ. Prove the winding number is 0 for any $z \in O$.

Solution: Without loss of generality up to a translation, we can assume that $z = 0$. Also assume $\gamma(t) : [a, b] \to O$. Then because we are in $\mathbb{C}$, we can write

$$\gamma(t) = \rho(t)e^{i\phi(t)},$$

where $\rho(t) = |\gamma(t) - 0|$ and $\phi(t)$ is the angle between 0 and $\varphi(t)$. Note that since $\gamma(t)$ is piecewise differentiable by assumption, then so are $\rho(t)$ and $\phi(t)$. Thus,

$$\begin{aligned}\int_\gamma \frac{d\zeta}{\zeta - 0} d\zeta &= \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt \\ &= \int_a^b \frac{\rho'(t)e^{i\phi(t)} + i\rho(t)\phi'(t)e^{i\phi(t)}}{\rho(t)e^{i\phi(t)}} dt \\ &= \int_a^b \frac{\rho'(t)}{\rho(t)} dt + i\int_a^b \phi'(t) dt\end{aligned}$$

Since both have primitives, we can evaluate them using FTC. Moreover, we know that since γ is closed, we must necessarily have $\rho(a) = \rho(b)$ and $\phi(a) = \phi(b) + 2\pi n$ for some $n \in \mathbb{Z}$ Thus,

$$\int_\gamma \frac{d\zeta}{\zeta - 0} d\zeta = \log(\rho(t))\Big|_a^b + i\phi(t)\Big|_a^b = i2\pi n.$$

Thus,

$$\mathrm{Ind}_\gamma(0) = n.$$

Because it is clearly invariant over a translation, we have that

$$\mathrm{Ind}_\gamma(z) = n.$$

By problem 2 on PSET 2, since $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$ for all $z \in O$, we have that Ind_γ is constant over O. It suffices to show that $n = 0$. To see this, we note that as O is unbounded, we can see that

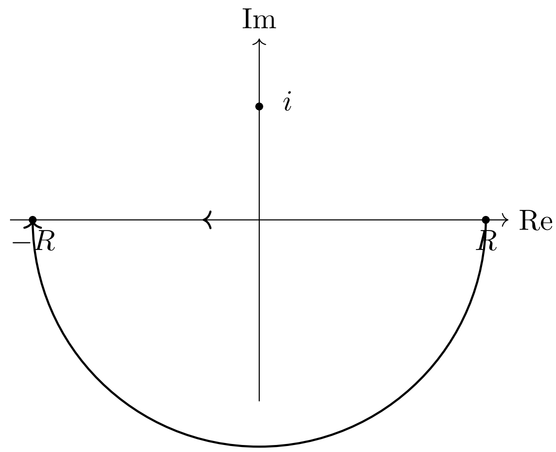
$$\lim_{z\to\infty} \mathrm{Ind}_\gamma(z) = \lim_{z\to\infty} \int_\gamma \frac{d\zeta}{\zeta - z} d\zeta = \lim_{z\to\infty} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z} dt = \int_a^b \lim_{z\to\infty} \frac{\gamma'(t)}{\gamma(t) - z} = 0.$$

Note that we can exchange limits because of the uniform convergence to 0 of the function within the integrand. Thus, Ind_γ is constant over O, we have that

$$\mathrm{Ind}_\gamma(z) = 0, \quad \forall\, z \in O.$$

$\blacksquare$

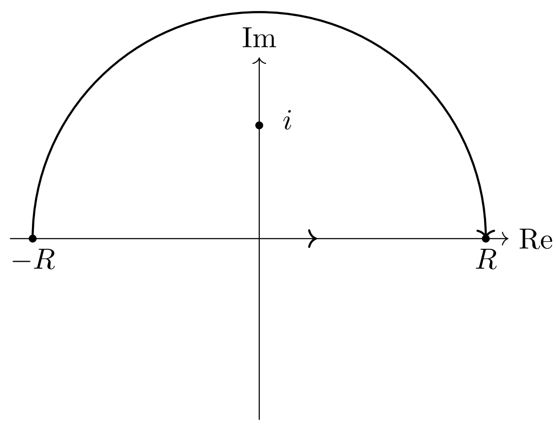
(b) Let γ_d be the path as pictured below.



What is $\text{Ind}_\gamma(i)$.

SOLUTION: Clearly, $i \in O$, where O is the unbounded connected complement of γ. By part (a), we know that $\text{Ind}_{\gamma_d}(i) = 0$. ■

(c) Let $R > 1$ and define γ_u as pictured below.



What is $\gamma_u(i)$?

SOLUTION: Consider that since the orientations cancel out the middle interval, we have that $\gamma_d + \gamma_u$ is the closed circle around 0 of radius R. We have seen in class that $\text{Ind}_{C_R}(0) = 1$, where $C_R = \gamma_d + \gamma_u$ is the circle of radius R around 0. By a corollary of what we proved in the proof of part (a), we know that the winding number is also constant in the bounded complement of C_R, and since i is in this connected open bounded complement, then

$$1 = \text{Ind}_{C_r}(i) = \text{Ind}_{\gamma_u}(i) + \text{Ind}_{\gamma_d}(i) = \text{Ind}_{\gamma_u}(i)$$

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Problem 6

(a) Prove that

$$\lim_{z\to i}\left(\frac{1}{z^2+1}-\frac{1}{2i(z-i)}\right)$$

exists.

SOLUTION: Since $z^2+1=(z-i)(z+i)$, we can write

$$\lim_{z\to i}\left(\frac{1}{z^2+1}-\frac{1}{2i(z-i)}\right)=\lim_{z\to i}\left[\frac{1}{z-i}\left(\frac{1}{z+i}-\frac{1}{2i}\right)\right]=\lim_{z\to i}\frac{\frac{1}{z+i}-\frac{1}{2i}}{z-i}\to\frac{0}{0}.$$

We apply L'Hopital's rule:

$$\begin{aligned}\lim_{z\to i}\left(\frac{1}{z^2+1}-\frac{1}{2i(z-i)}\right)&=\lim_{z\to i}\frac{\left(\frac{1}{z+i}-\frac{1}{2i}\right)'}{(z-i)'}\\&=\lim_{z\to i}\frac{-\frac{1}{(z+i)^2}}{1}\\&=\frac{1}{4}\end{aligned}$$

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(b) Call this limit L. Then, define

$$f(z)=\begin{cases}\frac{1}{z^2+1}-\frac{1}{2i(z-i)}, & z\neq i\\ L, & z=i\end{cases}$$

What is

$$\int_{\gamma_{\text{up}}} f(z)\,dz?$$

SOLUTION: Note that γ_u is a closed path in $O:=D_{R+1}(0)\subseteq\mathbb{C}$, which is open and convex in $\mathbb{C}$. Also observe that $f\in H(O)$ except for one point, $z=i$. Note that by part (a), $f(z)$ is continuous at $z=i$.

We use Goursat's and Cauchy's theorem, which forgive functions which are holmorphic at all but one point, at which they are continuous, that states that since γ_u is closed in an open and convex set, then

$$\int_{\gamma_u} f(\zeta)d\zeta=0$$

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(c) Taking the limit of the integral in part (b) as $R \to \infty$, evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

Solution: We have that

$$\int_{-R}^{R} \frac{1}{1+x^2} dx = \int_{-R}^{R} f(x) + \frac{1}{2i(x-i)} dx = \int_{-R}^{R} f(x)dx + \int_{-R}^{R} \frac{1}{2i(x-i)} dx$$

Denote the top arc of the half circle by γ_{arc}. Then we have by part (a) that

$$0 = \int_{\gamma_u} f(\zeta)d\zeta = \int_{[-R,R]} f(\zeta)d\zeta + \int_{\gamma_{\text{arc}}} f(\zeta)d\zeta = \int_{-R}^{R} f(x)dx \implies \int_{-R}^{R} f(x)dx = -\int_{\gamma_{\text{arc}}} f(\zeta)d\zeta$$

But we know that

$$\begin{aligned} \left| \int_{\gamma_{\text{arc}}} f(\zeta)d\zeta \right| &\leq \text{arclength}(\gamma_{\text{arc}}) \max_{z \in \gamma_{arc}} \left(\left|\frac{1}{z^2+1} - \frac{1}{2i(z-i)}\right|\right) \\ &\leq \pi R\left(\frac{1}{R^2+1} + \frac{1}{|2(R-1)|}\right) \\ &\to 0 \end{aligned}$$

as $R \to \infty$. Thus, it suffices to compute

$$\frac{1}{2i} \int_{-R}^{R} \frac{1}{(x-i)} dx.$$

To do this, we will use Cauchy's residue formula, which states that

$$\int_{-R}^{R} \frac{1}{(x-i)} dx = 2\pi i \operatorname{res}_i(f) = 2\pi i \lim_{z \to i} \left[(x-i)\frac{1}{x-i}\right] = 2\pi i.$$

Thus,

$$\int_{-R}^{R} \frac{1}{1+x^2} dx = \int_{-R}^{R} \frac{1}{2i(x-i)} dx = 2\pi i \frac{1}{2i} = \pi.$$

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