UChicago Honors Analysis Notes: 20800

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Contents

1 Lectures

1.1 Monday, Jan 6: The Stieltjes Integral

We begin by constructing the integral. Assume $f : [a, b] \to \mathbb{R}$ is continuous and $\alpha : [a, b] \to \mathbb{R}$ is monotonic.

Remark 1. The monotonicity assumption is made in order to control the variation of α, as monotonic functions can only have jump discontinuities (and countably many, prove this if you want, I don't care). Moreover, α is usually taken to be increasing to be consistent with all the upper sum being greater than the lower sum deal.

Suppose P partitions $[a, b]$ into $\{x_0, x_1, \ldots, x_n\}$, then we say that the upper sum is

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1})),$$

where M_i is defined as

$$M_i = \sup_{x \in (x_{i-1}, x_i]} f(x).$$

The lower sum $L(P, f, \alpha)$ is similarly defined. We say the upper integral of f with respect to α is

$$\overline{\int_a^b} f(x) d\alpha(x) = \inf_P U(P, f, \alpha).$$

We similarly define the lower integral. We say $f \in \mathscr{R}(\alpha)$, or f is *Riemann-Stieltjes integrable with respect to* α if

$$\underline{\int_a^b} f(x) d\alpha(x) = \overline{\int_a^b} f(x) d\alpha(x).$$

Definition 1. We denote the *Stietljes integral* with

$$\int_a^b f(x) d(\alpha(x)) \text{``} = \text{''} \int_a^b f d\alpha.$$

Definition 2. We say that P^* is a *refinement* of P if $P \subset P^*$.

Proposition 1. Suppose P^* is a refinement of P, then

$$U(P^*, f, \alpha) \le U(P, f, \alpha), \qquad L(P^*, f, \alpha) \ge L(P, f, \alpha).$$

We did not prove this. If I have time after I will go crazy. It is an induction proof.

Theorem 1. Suppose f is bounded and satisfies the usual conditions w α, then $f \in \mathscr{R}(\alpha)$ if and only if for all $\epsilon > 0$, there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

The proof for one direction is trivial, the other way is an $\frac{\epsilon}{2}$ proof.

Remark 2. We can approximate the integral by

$$\int_a^b f d\alpha = \sum_{i=1}^{n} f(t_i) \Delta\alpha_i, \qquad t_i \in [x_{i-1}, x_i].$$

Proposition 2. The Stieltjes integral satisfies linearity in both f and α.

Proofs follow immediately from the construction.

Theorem 2. If $f \in C([a, b])$, then $f \in \mathscr{R}(\alpha)$.

Proof. Since f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$, and so there exists a $\delta > 0$ such that if $|s - t| < \delta$ with $s, t \in [a, b]$, then $|f(s) - f(t)| < \frac{\epsilon}{\alpha(b)-\alpha(a)}$. Let P be a partition of $[a, b]$ with $||P|| < \delta$. Then we have that telescoping the sum,

$$U(P, f\alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i < \epsilon.$$

$\square$

Theorem 3. If f is monotone and α is continuous, then $f \in \mathscr{R}(\alpha)$.

Proof. By assumptions, we can choose a partition such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n},$$

where n is really large. Thus, we have that by the monotonicity of f,

$$U(P, f\alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))\Delta\alpha_i < f(b) - f(a)\frac{\alpha(b) - \alpha(a)}{n} < \epsilon.$$

$\square$

Theorem 4. If f is bounded with finitely many discontinuity points and α is continuous at the points of discontinuity of f, then $f \in \mathscr{R}(\alpha)$.

Proof. Let D denote the set of discontinuity points of α. Partition P such that each interval contains one discontinuity each. Then either $x \in D$ for all $x \in [x_{i-1}, x_i]$ or $x \notin D$ for some $x \in [x_{i-1}, x_i]$. If the former, then we can bound the sum by $\frac{\epsilon}{2}$ using Theorem 2. If the latter, then since we know f is bounded, we know that for that interval, $M_i - m_i \leq 2K$, where K is the max of the magnitude of the upper and lower bounds of f. Since α is continuous at the point of discontinuity, then we can bound $\alpha(x_i) - \alpha(x_{i-1})$ by whatever we want.

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i = \sum_{i\in D}(M_i - m_i)\Delta\alpha_i + \sum_{i\notin D}(M_i - m_i)\Delta\alpha_i.$$

$\square$

Theorem 5. Let $f \in \mathscr{R}(\alpha)$ and $m \leq f \leq M$. Suppose $\phi \in C[m, M]$ then $h(x) = \phi(f(x)) \in \mathscr{R}(\alpha)$.

Proof. Since ϕ is uniformly continuous, we know that there exists some $\delta > 0$ such that if $|s - t| < \delta$, then $|\phi(s) - \phi(t)| < \epsilon$. Since f is Stieltjes integrable, we know that there exists some partition P such that

$$\sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i = U(P, f, \alpha) - L(P, f, \alpha) < \delta^2.$$

Thus, we can choose a partition P^* of $[m, M]$ such that $||P^*|| <$, then

$$U(P^*, h, \alpha) - L(P^*, h, \alpha) = \sum_{i=1}^{n}(M_i^* - m_i^*)\Delta\alpha_i > \delta\sum_{i=1}^{n}\Delta\alpha_i$$

If $M_i - m_i < \delta$, then we are done by our choice of ϵ for those i. If, on the other hand, $M_i - m_i \geq \delta$, then we still know we can bound $M_i^* - m_i^*$ by EVT, and we can bound it by $2K$, where K is defined similarly in the above proof. Thus, we have that $\delta \sum_{i=1}^n \Delta\alpha_i \leq \sum_{i=1}^n (M_i - m_i)\Delta\alpha_i < \delta^2$, and thus we have bounded both cases and we are done. $\square$

1.2 Wednesday, Jan 8: Stieltjes Change of Variable and Fourier Series Introduction

We build up the tools to get to a change of variable formula.
"A *distribution mass function* maps functions with compact support to the real numbers."

Definition 3. A *heavy mass function* H_{x_0} is defined as

$$H_{x_0}(x) = \begin{cases} 0, & x \leq x_0 \\ 1, & x > x_0 \end{cases}.$$

Theorem 6. Suppose $I(x) = H_0$ and $f : [a, b] \to \mathbb{R}$ is bounded and continuous at $s \in (a, b)$, then if $\alpha(x) = I(x - s)$, we have that $f \in \mathscr{R}(\alpha)$ and $\int_a^b f(x) d(\alpha(x)) = f(s)$.

Proof. Consider a partition P on $[a, b]$ such that

$$P = \{x_0, x_1, x_2, x_3\}, \qquad \text{such that} \quad a = x_0 < x_1 < s < x_2 < x_3 = b.$$

Thus, we have that since $\Delta I(x - s) \neq 0$ only on the interval $[x_1, x_2]$, then

$$U(P, f, \alpha) - L(P, f, \alpha) = (M_2 - m_2) \to 0$$

since by continuity:

$$M_2 \to f(s), \qquad m_1 \to f(s).$$

To make this more precise, simply make the distance of x_1 and x_2 small enough. □

Theorem 7. Suppose $c_n \geq 0$ for all n and $\sum_{n=1}^{\infty} c_n$ converges, (s_n) is sequence of distinct points in (a, b), and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n) \tag{1}$$

If f is continuous on $[a, b]$, then

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n). \tag{2}$$

Proof. We know that by comparing (1) with the c_n series, (2) converges for every x. Moreover, since $c_n \geq 0$, we are adding positive numbers, and so $\alpha(x)$ is monotonic, and $\alpha(a) = 0$ and $\alpha(b) = \sum c_n$. Let $\epsilon > 0$, then we know that there exists an N such that

$$\sum_{N+1}^{\infty} c_n < \epsilon.$$

We can separate (22) into

$$\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n), \qquad \alpha_2(x) = \sum_{N+1}^{\infty} c_n I(x - s_n).$$

By the linearity of the integral and by Theorem 6, we have that

$$\int_a^b d\alpha_1 = \sum_{i=1}^{N} c_n f(s_n).$$

Since $\alpha_2(b) - \alpha_2(a) < \epsilon$, we get that
$$\left| \int_a^b f d\alpha_2 \right| \leq M\epsilon,$$
where $M = \sup |f(x)|$. Since $\alpha = \alpha_1 + \alpha_2$, we get that
$$\left| \int_a^b f d\alpha - \sum_{i=1}^{N} c_n f(s_n) \right| \leq M\epsilon,$$
letting $N \to \infty$ we obtain our result. □

Theorem 8. Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. In that case,
$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx.$$

Proof. Let $\epsilon > 0$. Since $\alpha' \in \mathscr{R}$, there exists a partition $P = \{x_0, \dots, x_n\}$ such that
$$U(P, f, \alpha') - L(P, f, \alpha') < \epsilon. \tag{3}$$
By the mean value theorem:
$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)\Delta x_i,$$
for all i and so if $s_i \in [x_{i-1}, x_i]$, then
$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i.$$
Moreover, we know that from (3),
$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i < \epsilon,$$
and so if M is defined as usual sup of $|f(x)|$, then we get that
$$\left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right| \leq M\epsilon.$$
Thus, we get that
$$|U(P, f, \alpha) - U(P, f\alpha')| \leq M\epsilon, \implies \left| \overline{\int_a^b} f d\alpha - \overline{\int_a^b} f(x)\alpha'(x)dx \right| < M\epsilon$$
for any bounded f. The same equality holds for lower integrals. □

We provide the following theorem without proof (since none was given and it looks lame).

Theorem 9. (Change of Variable) Suppose φ is a strictly increasing continuous function that maps and interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$, and finally that $f \in \mathscr{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by
$$\beta(y) = \alpha(\varphi(y)), \qquad g(y) = f(\varphi(y)).$$
Then $g \in \mathscr{R}(\beta)$ and
$$\int_A^B g d\beta = \int_a^b f d\alpha.$$

Remark 3. Suppose $\alpha(x) = x$. Then $\beta = \varphi$ and so by Theorem 8 and 9, we get that

$$\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\varphi'(x)dx = \int_{\varphi(a)}^{\varphi(b)} g d\varphi = \int_a^b f dx$$

We begin our discussion on Fourier series by introducing the heat equation, $\mu(x,t) : (0,1) \times (0,T) \to \mathbb{R}$ such that

$$\frac{\partial \mu}{\partial t} = \frac{\partial^2 \mu}{\partial x^2}, \qquad \mu(0,t) = \mu(1,t) = 0, \qquad \mu(x,0) = \mu_0(x) = \sin(n\pi x).$$

Fourier discovered the solution to be

$$\mu(x,t) = e^{-n^2\pi^2 t}\sin(n\pi x) \tag{4}$$

which you can check works. This implies that any linear combination of (4) solves the heat equation. Fourier then asks the obvious most logical question after this: *can we approximate any function f by trigonometric polynomials?*

Definition 4. We say f is a *trigonometric polynomial* if it is of the form

$$f(x) = a_0 + \sum_{i=1}^{n} (a_i \cos(nx) + b_i \sin(nx)) = \sum_{-N}^{N} c_n e^{-inx} \tag{5}$$

Remark 4. The second equality in (5) comes from the identity/definition that

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

This directly implies that

$$e^{ix} = \cos(x) + i\sin(x)$$

Definition 5. Suppose f is an integrable function on $[-\pi, \pi]$, then we define the *Fourier coefficients* of f to be

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx. \tag{6}$$

Definition 6. We define the *Fourier series* of f to be the sum

$$\sum_{-\infty}^{\infty} c_n e^{inx}. \tag{7}$$

Note that a partial sum of (7) yields a trigonometric polynomial.

Definition 7. Let (ϕ_n) be a sequence of complex functions on $[a,b]$ such that

$$\int_a^b \phi_n(x)\overline{\phi_m(x)}dx = 0, \quad (n \neq m),$$

then (ϕ_n) is an *orthonormal system of functions* and if

$$\int_a^b |\phi_n(x)|^2 dx = 1,$$

for all n, then (ϕ_n) is *orthonormal*

Remark 5. Essentially, we are equipping (ϕ_n) with a special inner product, that being

$$\langle \phi_n, \phi_m \rangle = \int_a^b \phi_n(x)\overline{\phi_m(x)}dx.$$

Sometimes to make calculations easier (as below), we have a factor of $\frac{1}{\pi}$ before the integral.

Example 1.1. An example of an orthonormal system on $[-\pi, \pi]$ are the functions

$$\frac{e^{inx}}{\sqrt{2\pi}}.$$

Another example is the basis for the real trigonometric polynomials equipped with a different dot product. As an exercise, show that if

$$\mathscr{B} = \{\frac{1}{\sqrt{2}}, \sin(x), \sin(2x), \ldots, \cos(x), \cos(2x) \ldots\},$$

and

$$\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg dx,$$

then $\mathscr{B}$ is an orthonormal basis.

Remark 6. While we do the more general case of complex functions in class with the Fourier coefficients given by (6) above, in the real case, we use the example above to solve for them:

$$a_k = \langle \cos(kx), f \rangle, \qquad b_k = \langle \sin(mx), f \rangle, \qquad a_0 = \langle \frac{1}{\sqrt{2}}, f \rangle$$

where the dot product is the one defined above. We also remark that while $f \in L^2_{2\pi-\text{periodic}}(\mathbb{R}, \mathbb{C})$, we can sometimes talk about it in L^1 space.

1.3 Friday, Jan 10: Bessel's Inequality and the Dirichlet Kernel

Theorem 10. Let (ϕ_n) be orthonormal on $[a, b]$. Let

$$s_n(x) = \sum_{m=1}^{n} c_m \phi_m(x)$$

be the partial sum of the Fourier series of f, that is

$$c_n = \int_a^b f(t)\overline{\phi_n(t)}dt, \qquad n = 1, 2, 3, \dots.$$

Suppose as well that

$$t_n(x) = \sum_{m=1}^{n} \gamma_m \phi_m(x).$$

Then

$$\int_a^b |f - s_n|^2 dx \leq \int_a^b |f - t_n|^2 dx,$$

where equality holds if and only if

$$\gamma_m = c_m, \qquad m = 1, 2, \dots, n$$

Proof. Consider that

$$\int f\overline{t_n} = \int f \sum \overline{\gamma_m}\overline{\phi_m} = \sum c_m \overline{\gamma_m},$$

and by the orthonormality of

$$\int |t_n|^2 = \int t_n \overline{t_n} = \int \sum \gamma_m \phi_m \sum \overline{\gamma_k}\overline{\phi_k} = \sum |\gamma_m|^2.$$

Thus,

$$\begin{aligned}
\int |f - t_n|^2 &= \int |f^2| - \int f\overline{t_n} - \int \overline{f}t_n + \int |t_n|^2 \\
&= \int |f^2| - \sum c_m \overline{\gamma_m} - \sum \overline{c_m}\gamma_m + \sum \gamma_m \overline{\gamma_m} \\
&= \int |f|^2 - \sum |c_m|^2 + \sum |y_m - c_m|^2,
\end{aligned}$$

which is minimized if and only if $y_m = c_m$. Subsituting y_m for c_m in both equations above we find that since $|f - t_n|^2 \geq 0$, then

$$\int |s_n|^2 = \sum |c_m|^2 \leq \int |f|^2.$$

□

Corollary 1. (Bessel Inequality) If $\{\phi_n\}$ is orthornormal on $[a, b]$ and if

$$f(x) \sim \sum c_n \phi_n(x),$$

then

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \int_a^b |f(x)|^2 dx$$

and thus

$$\lim_{n \to \infty} c_n = 0.$$

Remark 7. Applying Bessel's Inequality to the trig series, we find that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|s_n|^2 = \sum_{-N}^{N}|c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2$$

Definition 8. The *Dirichlet kernel* is defined to be

$$D_N(x) = \sum_{-N}^{N} e^{inx} = \frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}.$$

Remark 8.

$$\begin{aligned}
s_N(f,x) &= \sum_{-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} fe^{-int}dte^{inx} \\
&= \frac{1}{2\pi0}f(t)\sum_{-N}^{N}e^{in(x-t)}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt \\
&= \frac{1}{2\pi}(f*D_N)(t)
\end{aligned}$$

1.4 Monday, Jan 13: Parseval's Identity

Proposition 3. Let $n \to \infty$, then
$$\int_{-\pi}^{\pi} |D_N(t)|dt \to \infty.$$

Proof.
$$\begin{aligned}
\int_{-\pi}^{\pi} |D_N(t)|dt &= \int_{-\pi}^{\pi} \frac{\sin((N+1)x)}{\sin\left(\frac{x}{2}\right)}dx \\
&= \int_{-\pi}^{\pi} \frac{|\sin((2n+1)x)|}{\pi x}dx \\
&\geq \sum_{k=0}^{2n} \int_{k\pi}^{(k+1)\pi} \frac{|\sin(x)|}{\pi x}dx \\
&\geq \sum_{k=0}^{2n} \frac{|\sin(x)|}{(k+1)\pi}dx \\
&= \frac{1}{\pi}\int_{0}^{2\pi} |\sin(x)|dx \sum_{0}^{2n} \frac{1}{k+1} \to \infty.
\end{aligned}$$

□

Theorem 11. Suppose that for $t \in (-\delta, \delta)$, there exists a $C \in \mathbb{R}$ such that $|f(x-t) - f(x)| \leq C(t)$, (locally Lipshitz), then $S_n(x) \to f(x)$.

Proof. Define
$$g(t) = \frac{f(x-t) - f(x)}{\sin\left(\frac{t}{2}\right)},$$
then we have that by Remark 8:
$$\begin{aligned}
|S_n(x) - f(x)| &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x-t) - f(x))D_N(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\sin\left(Nt + \frac{t}{2}\right)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\cos\left(\frac{t}{2}\right)\sin(Nt)dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\sin\left(\frac{t}{2}\right)\cos(Nt)dt \\
&\to 0.
\end{aligned}$$

The last equality holds because $g(t)$ is bounded and because $|c_n| \to 0$, and thus both the real and imaginary components of c_n go to 0. □

Theorem 12. (Parseval) Suppose $f, g \in \mathscr{R}$ with period 2π defined on $[-\pi, \pi]$ with
$$f(x) \sim \sum_{-n}^{n} c_n e^{inx}, \qquad g(x) \sim \sum_{-n}^{n} \gamma_n\, e^{inx},$$

then

(a)
$$\lim_{n\to\infty} \frac{1}{2\pi}\int |f(x) - s_n(f,x)|^2 = 0$$

(b)

$$\frac{1}{2\pi}\int_{-n}^{n} f(x)\overline{g(x)} = \sum_{-\infty}^{\infty} c_n\overline{\gamma_n}$$

(c)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 = \sum_{-\infty}^{\infty} |c_n|^2.$$

Proof. For notation, we let

$$||h||_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |h(x)|^2 dx\right)^{\frac{1}{2}}.$$

Let $\epsilon > 0$. By an exercise (prove if time), there exists a continuous function h such that $||f - h||_2 < \epsilon$. By Stone-Weirstrass, there exists a trigonometric polynomial P such that $|h - P| < \epsilon$ and thus $||h - P||_2 < \epsilon$. By the best approximation theorem (Theorem 10), we know that

$$||h - s_n(h, x)||_2 \leq ||h - P||_2 < \epsilon.$$

Using Bessel's inequality and the fact that $\int |s_n(x)|^2 = \sum |c_n|^2$, we find that by putting $f - h$ into this as our function:

$$||s_n(f) - s_n(h)||_2 = ||s_n(f - h)||_2 \leq ||f - h||_2.$$

Note that this utilizes the fact that the s_n are linear (simplie exercise). Thus, we get that

$$||f - s_n(f, x)||_2 \leq ||f - h||_2 + ||h - s_n(h, x)||_2 + ||s_n(h, x) - s_n(g, x)|| < 3\epsilon.$$

To prove the second identity, it is easy to show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} s_n\overline{g} = \sum_{-N}^{N} c_n\overline{\gamma_n},$$

and then use C-S inequality to show that

$$|\int f\overline{g} - \int s_n\overline{g}| \leq \int |f - s_n||\overline{g}| \leq \sqrt{\int |f - s_n|^2 \int |\overline{g}|^2} \to 0.$$

Put these two together and you get the second result. Put $g = f$ to obtain final result. $\square$

1.5 Wednesday, Jan 15: Introduction to Fun Anal

We begin with a few (somewhat obvious) remarks:

Remark 9. (a) The canonical vector spaces are $\mathbb{R}^d$ and $C([0,1],\mathbb{R})$.

(b) We say V is finite dimensional if the cardinality of its basis is finite.

(c) $C([0,1],\mathbb{R})$ is infinite dimensional since $\{x^n\}$ are linearly independent.

(d) Suppose V is a normed vector space, then it is equipped with $\|\cdot\|$ such that if $v, w \in V$ and $\lambda \in$, then $\|v\| > 0$ and $\|0\| = 0$ if and only if $v = 0$; $\|v + w\| \leq \|v\| + \|w\|$; and $\|\lambda v\| = |\lambda|\|v\|$.

(e) Having 2/3 (not the positive definitiveness) makes a semi norm.

(f) A metric is induced by a norm with $d(v, w) = \|v - w\|$.

(g) Equivalent norms are those that have the same topology on V. An example is

$$\|v_n\|_p = \left(\sum_{k=1}^{\infty} (v_k)^p\right)^{\frac{1}{p}}, \qquad \|v_n\|_{\sup} = \sup_{n\in\mathbb{N}}(v_n).$$

Definition 9. We say that a sequence $(u_n) \in V$ **convergence in norm**, denoted by $\|u_n - u\| \to 0$ as $n \to \infty$ if and only if $u_n \to u$

Definition 10. We say V is a **Banach Space** if it is a complete normed space (with respect to convergence in its norm)

Example 1.2. $C_b(X)$, the space of bounded continuous functions, is a Banach Space with respect to the sup metric.

Proof. Let (f_n) be Cauchy in $C_b(X)$, then there exists some $N \in \mathbb{N}$ such that if $n, m \geq N$, we have that

$$\|f_n - f_m\|_{\sup} < \frac{\epsilon}{2}.$$

Thus, for each x, we have that $f_n(x)$ form a Cauchy of real numbers, and thus converge to some limit $f(x)$. Thus, for all $n \geq m(x)$, where $m(x) \geq N$, we have that

$$|f_n(x) - f(x)| \leq |f_n(x) - f_{m(x)}| + |f_{m(x)}(x) - f(x)| < \epsilon.$$

$\square$

Definition 11. Let V be a normed vector space. Let $(v_n) \in V$. We say that $\sum v_n$ is **summable** if the series

$$\sum_{n=1}^{\infty} v_n < \infty.$$

We say that $\sum v_n$ is **absolutely summable** if

$$\sum_{n=1}^{\infty} \|v_n\| < \infty$$

Proposition 4. If $\sum v_n$ is absolutely summable, then $\sum_{n=1}^{N} v_n$ is Cauchy.

Proof. Since the series is absolutely summable, then it is Cauchy, and let $n \geq m \geq N$ where

$$\sum_{N}^{\infty} \left\| \sum_{k=1}^{n} v_n - \sum_{k=1}^{m} v_n \right\| = \left\| \sum_{m+1}^{n} v_n \right\| \leq \sum_{m+1}^{n} \|v_n\| < \epsilon.$$

$\square$

Theorem 13. A normed vector space V is Banach if and only if every absolutely summable series is summable.

Proof. The forward direction is immediate by the above proposition. Let (v_n) be Cauchy, then for large $n, m \geq N_K$, we have

$$\|v_n - v_m\| < 2^{-k},$$

and define $n_K = \sum^K N_k$. Then we have that $n_K \geq N_k$ and thus

$$\left\|v_{n_{K+1}} - v_{n_K}\right\| < 2^{-K}.$$

We have that

$$\sum^{K} \left\|v_{n_{k+1}} - v_{n_k}\right\| < \sum^{K} \frac{2}{k},$$

and thus the series is absolutely summable, and thus by assumption

$$\sum v_{n_{k+1}} - v_{n_k} = v_{n_K} - v_{n_1}$$

converges, and thus v_{n_K} is a convergent subsequence of a Cauchy sequence, and thus v_n converges. $\square$

1.6 Friday, Jan 17: Linear Operators

Definition 12. Let V and W be two vector spaces. We say that $T : V \to W$ is a **linear operator** if for any constants λ_1, λ_2 and vectors $v_1, v_2 \in V$, we have that

$$T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T(v_1) + \lambda_2 T(v_2).$$

Remark 10. In finite dimensional vector spaces, all linear operators are continuous.

Theorem 14. Let $T : V \to W$ be a linear operator. Then T is continuous if and only if it is a bounder linear operator, that is, there exists some $C > 0$ such that for any $v \in V$,

$$\|T(v)\|_W \leq C\|v\|_V$$

Proof. For the backwards direction, let $\epsilon > 0$. If $\|v - w\|_V < \frac{\epsilon}{C}$, where $v, w \in V$, then

$$\|T(v) - T(w)\|_W = \|T(v - w)\|_W \leq C\|v - w\|_V < \epsilon.$$

For the forwards direction, suppose T is continuous. Consider $B_W(0, 1)$ to be the open ball in W around the origin of radius 1. By continuity, we have that $T^{-1}(B_W(0, 1)) = A$ is open in V, and thus for all $x_0 \in A$, there exists a $r > 0$ such that $B_V(x_0, r) \subset A$. In particular, since T is a linear operator, we must have that $0 \in A$, and thus

$$B_V(0, r) \subseteq A \implies T(B_V(0, r)) \subseteq T(A) = B_W(0, 1).$$

Thus, for all $v \in B_V(0, r)$, we have that $\|T(v)\|_W \leq 1$.

Let $v \in V$. Then consider that $r\frac{v}{\|v\|} \in B_v(0, r)$, and thus

$$\left\|T(r\frac{v}{\|v\|})\right\|_W \leq 1 \implies \|T(v)\|_W \leq r\|v\|_V$$

$\square$

Definition 13. Suppose V and W are two normed spaces. We denote the set of bounded linear operators $T : V \to W$ by $\mathscr{L}(V, W)$. The **operator norm** of $\mathscr{L}(V, W)$ is defined as

$$\|T\|_{\mathscr{L}(V,W)} = \sup_{v \in V,\ \|v\|=1} \|T(v)\|.$$

Proving that the norm is well defined is not hard.

Theorem 15. Suppose V and W are Banach Spaces, then $\mathscr{L}(V, W)$ is a Banach Space

Proof. Let $(T_n) \in \mathscr{L}(V, W)$ such that $\sum T_n$ is absolutely summable with

$$\sum^{\infty} \|T_n\| = C.$$

We let $v \in V$ and $m \in N$, then we have that

$$\sum_{n=1}^{m} \|T_n(v)\| \leq \|v\| \sum_{n=1}^{m} \|T\| = C\|v\|$$

Thus we have a bounded increasing sequence in W, which is Banach, and thus we have that the absolutely summable series is summable and thus

$$\sum^{\infty} T_n(v) = Tv.$$

Tv is a linear operator because of the linearity of limits. Since the norm is continuous, then we have that

$$\|Tv\| = \left\| \lim_{n\to\infty} \sum_{k=1}^{n} T_k v \right\| = \lim_{n\to\infty} \left\| \sum_{k=1}^{n} T_k(v) \right\| \leq \lim_{n\to\infty} \sum_{k=1}^{n} \|T_k v\| \leq C\|v\|.$$

Thus, T is a bounded linear operator.

We wish to show now that $T_n \to T$. Let $v \in V$ with $\|v\| = 1$. Then we get that

$$\begin{aligned}
\left\| Tv - \sum_{n=1}^{m} T_n v \right\| &= \left\| \lim_{m'\to\infty} \sum_{n=1}^{m'} T_n(v) - \sum_{n=1}^{m} T_n(v) \right\| \\
&= \left\| \lim_{m'\to\infty} \sum_{m+1}^{m'} T_n(v) \right\| \\
&= \lim_{m'\to\infty} \left\| \sum_{m+1}^{m'} T_n(v) \right\| \\
&\leq \lim_{m'\to\infty} \sum_{m+1}^{m'} \|T_n v\| \\
&\leq \lim_{m'\to\infty} \sum_{m+1}^{m'} \|T_n\|\|v\| \\
&= \lim_{m'\to\infty} \sum_{m+1}^{m'} \|T_n\| \\
&\to 0.
\end{aligned}$$

$\square$

1.7 Wednesday, Jan 21: The Hahn-Banach Theorem

Let E be a vector space over $\mathbb{R}$.

Definition 14. A **functional** is a function $f : E \to \mathbb{R}$.

We consider linear functionals.

Theorem 16. (Hahn-Banach) Let $p : E \to \mathbb{R}$ be a **Minkowski functional**, that is, it satisfies

$$p(\lambda x) = \lambda p(x), \qquad \forall x \in E, \lambda > 0$$

$$p(x + y) \leq p(x) + p(y), \qquad \forall x, y \in E.$$

Let $G \subset E$ be a linear subspace and let $g : G \to \mathbb{R}$ be a linear functional such that

$$g(x) \leq p(x) \quad \forall x \in G.$$

Then there exists a linear functional f defined on all of E that extends g ($g(x) = f(x)$ if $x \in G$) and such that

$$f(x) \leq p(x) \quad \forall x \in E.$$

To prove this we need a few definitions and a lemma:

Definition 15. Let P be a set with a partial order relation $\leq$. We say that a subset $Q \subset P$ is **totally ordered** if for any pair (a, b) in Q either $a \leq b$ or ba.

We say that $m \in P$ is a **maximal** element of P if there is no element $x \in P$ such that $m \leq x$ except for $x = m$.

We say that $c \in P$ is an **upper bound** for Q if $a \leq c$ for every $a \in Q$.

We say that P is **inductive** if every totally ordered subset Q in P has an upper bound

Remark 11. A maximal element may not be an upper bound, since it might not even be comparable.

Lemma 1. (Zorn's Lemma) Every nonempty ordered set that is inductive has a maximal element.

We now turn to the proof of the Hahn-Banach:

Proof. Suppose $D(h)$

$$P := \{h : D(h) \subset E \to\},$$

given that $D(h)$ is a linear subspace of E, h is linear, $G \subset D(h)$, and h extends g and $h(x) \leq p(x)$ for all $x \in D(h)$.

Define the order relation on P :

$$(h_1 \leq h_2) \iff (D(h_1) \subset D(h_2) \text{ and } h_2 \text{ extends } h_1).$$

Since $g \in P$, then $P \neq \emptyset$. Let QP be a totally ordered set, and write $Q := (h_i)_{i \in I}$, then set

$$D(h) = \bigcup_{i \in I} D(h_i), \qquad h(x) = h_i(x) \quad \text{if } x \in D(h_i) \text{ for some } i.$$

Evidently, we have that $h \in P$ is an upper bound for Q. By Zorn, there exists some maximal element $f \in P$. We claim that $D(f) = E$, which would complete the proof.

Suppose not. Let $x_0 \notin D(f)$, where $x_0 \in E$. Let $D(h) = D(f) + \mathbb{R}x_0$ such that

$$h(x + tx_0) = f(x) + t\alpha,$$

where $t \in \mathbb{R}$ and we must choose $\alpha \in \mathbb{R}$ such that $h \in P$. We must ensure that

$$f(x) + t\alpha \leq p(x + tx_0) \qquad \forall x \in D(f), \forall t \in \mathbb{R} \iff \begin{cases} f(x) + \alpha \leq p(x + x_0) & \forall x \in D(f) \\ f(x) + \alpha \leq p(x - x_0) & \forall x \in D(f) \end{cases}$$

Thus, it suffices to find an α such that

$$\sup_{y \in D(f)} \{f(y) - p(y - x_0)\} \leq \alpha \leq \inf_{x \in D(f)} \{p(x + x_0) - f(x)\}.$$

Since

$$f(y) - p(y - x_0) \leq p(x + x_0) - f(x) \qquad \forall x, y \in D(f),$$

then we have that α exists since

$$f(x) + f(y) \leq p(x + y) \leq p(x + x_0) + p(y - x_0)$$

Thus, we have that $f \leq h$, but f is maximal and $h \neq f$, a contradiction. $\square$

1.8 Friday, Jan 24 and Monday, Jan 27: Geometric Consequences of Hahn-Banach

I don't usually do a two in one, but there are two good reasons for this:

(a) I was too lazy and had to see about a girl this weekend to write up the notes.

(b) We used some lemmas on Friday which we then proved on Monday.

Remark 12. We call the **dual space** of E the space of continuous linear functionals on E, and denote it by E^*. However, we have already proved in Theorem 14 that this is equivalent to $\mathscr{L}(E, \mathbb{R})$, and so we will deviate from the (admittedly plagiarized lectures) and use this notation in order to remain consistent throughout these notes. Moreover, we say that $\langle , \rangle$ is the scalar product for the duality, and just write $\langle f, x\rangle$ instead of $f(x)$ for some reason.

Definition 16. An affine **hyperplane** is a subset $H \subset E$ such that

$$H = \{x \in E \mid \langle f, x\rangle = \alpha\} \text{``} = \text{''} [F = \alpha],$$

where f is a linear functional and $\alpha \in \mathbb{R}$.

Proposition 5. The hyperplane $H = [f = \alpha]$ is closed if and only if $f \in \mathscr{L} E, \mathbb{R}$.

Proof. The backwards direction is clear: $x_n \to x$ with $x_n \in H$, then by the continuity of f, we have that $\langle f, x_n\rangle = \alpha$ for all n and thus $\langle f, x_n\rangle \to \langle f, x\rangle = \alpha$.

For the backwards direction, we let H be closed, and so H^c is open and nonempty. We let $x_0 \in H^c$, and assume without loss of generality that $f(x_0) < \alpha$. By opennens, there exists an $r > 0$ such that $B_r(x_0) \subset H^c$, and thus for all $x \in B_r(x_0)$, we claim that $\langle f, x\rangle < \alpha$. If not, then there exists some y in this ball such that $\langle f, y\rangle > \alpha$. Since the ball is convex, and thus for any $x \in [0, 1]$, we have that

$$x(t) = (1 - t)x_0 + tx_1 \in B_r(x_0).$$

But we know that since $x_t \in B_r(x_0)$, then $\langle f, x_t\rangle < \alpha$, but then by then for $t = \frac{f(x_1)-\alpha}{f(x_1)-f(x_0)}$, we have that $f(x_t) = \alpha$ (use the linearity of f.) From this claim, we now have that for any $z \in B_1(0)$, $\langle f, x_0 + rz\rangle < \alpha$, and thus f is continuous since $\|f\| \leq \frac{1}{r}(\alpha - f(x_0))$. ☐

Definition 17. Let A and B be two subsets of E. We say that $H = [f = \alpha]$ **separates** A and B if for all $x \in A$, $y \in B$, we have that

$$f(x) \leq \alpha \leq f(y).$$

H **strictly separates** A and B if there exists some $\epsilon > 0$ such that for all $x \in A$ and $y \in B$, we have that

$$f(x) \leq \alpha - \epsilon < \alpha + \epsilon \leq f(y)$$

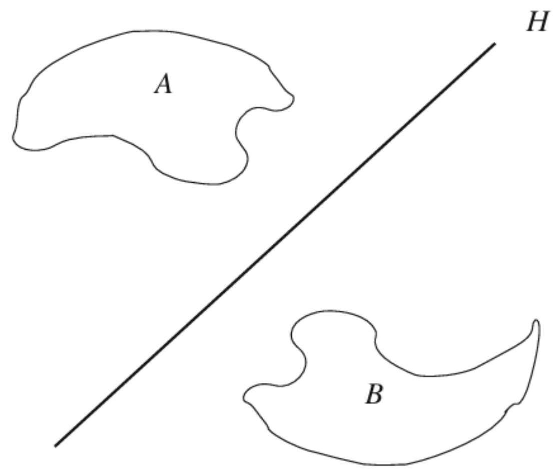

Figure 1: Seperation by Hyperplanes, from Brezis

Lemma 2. Let $C \subset E$ be an open convex set with $0 \in C$, then define $p : E \to \mathbb{R}$ by

$$p(x) = \{\alpha > 0 \mid \frac{1}{\alpha}x \in C\}.$$

Then p is a seminorm such that there is some $M \in \mathbb{R}$ with $0 \leq p(x) \leq M\|x\|$ for any $x \in E$, and

$$C = \{x \in E \mid p(x) < 1\}.$$

In a way, we can think of α as the smallest number required to pull an element into C when dividing it by α.

Proof. We have positive definitiveness by definition. By the openness of $C \ni 0$, we get that there exists some $r > 0$ such that $B_r(0) \subset C$. Informally, we can think of it taking more of an effort to pull in an x into $B_r(0)$ than it is to pull it into C (since C is 'larger'). Let $\alpha = \frac{\|x\|}{r}$, then we get that if $x \in E$,

$$\frac{1}{\alpha}x = \frac{x}{\|x\|}r \in B_r(0) \implies \frac{1}{\alpha} \in C,$$

and thus $p(x) \leq \frac{\|x\|}{r}$ by definition.

Let $x \in C$, then for small $\epsilon > 0$, we have by the openness of C (and convexity) that $(1+\epsilon)x \in C$, and thus

$$p(x) \leq \frac{1}{1+\epsilon} < 1.$$

Thus, $C \subset \{x \in E \mid p(x) < 1\}$. Let $x \in E$ with $p(x) < 1$, then there is an α such that

$$\frac{1}{\alpha}x \in C \implies \alpha(\frac{1}{\alpha}x) + (1-\alpha)(0) = x \in C.$$

To prove the triangle inequality, we note that if $x, y \in E$, $p(x), p(y) > 1$, and thus there exists an $\epsilon > 0$ such that

$$\frac{x}{p(x)+\epsilon} \in C, \qquad \frac{y}{p(y)+\epsilon} \in C \implies \frac{x}{p(x)+\epsilon} < 1 \implies x < p(x) + \epsilon.$$

Letting $0 \le t = \frac{p(x)+\epsilon}{p(x)+p(y)+2\epsilon} \le 1$, we see that using the convexity of C that

$$\frac{x+y}{p(x)+p(y)+2\epsilon} \in C$$

and thus we have that if we set $\alpha = p(x)+p(y)+2\epsilon$, then we see that

$$p(x+y) \le \alpha = p(x)+p(y)+2\epsilon.$$

□

Lemma 3. Let $C \subset E$ be a nonempty open convex set and suppose $x_0 \in EC$. Then there exists an $f \in \mathscr{L}(E,\mathbb{R})$ such that $f(x) < f(x_0)$ for all $x \in C$.

Proof. After translating (linear functions don't give a shit about translating) such that $0 \in C$, we let $G = \mathbb{R}x_0$ be the linear subspace of the line of x_0, and we define the functional

$$g : G \to \mathbb{R} \quad g(tx_0) = t.$$

Consider the **gauge** defined in Lemma 2, then we claim that $g(x) \le p(x)$: Obviously, $g(x_0) = 1$, and thus

$$g(x_0) \le p(x_0) \implies g(tx_0) \le p(tx_0) \implies g(x) \le p(x)$$

From the Hahn-Banach Theorem (Thm 16), we get that there exists an extension of $f : E \to \mathbb{R}$ such that $f(x) \le p(x) \le M\|x\|, \qquad \forall x \in E$, where the M comes from Lemma 2 above. Thus, we get that $f \in \mathscr{L}(E,\mathbb{R})$ and

$$f(x_0) = 1 \implies f(x) \le p(x) < 1 = f(x_0)\ \forall x \in C.$$

□

Theorem 17. (Geometric Hahn-Banach, first form) Suppose $A, B \subset E$ be nonempty, convex, and disjoint. Suppose that A is open. Then there exists a closed hyperplane that separates A and B.

Proof. Let $C = A - B$. Then C is convex. Since $C = \bigcup_{y\in B}(A-y)$, then C is open, and $0 \notin C$. By Lemma 3 there is some $f \in \mathscr{L}(E,\mathbb{R})$ such that $f(z) < f(0) = 0$ for all $z \in C$. Thus, we have that if $x \in A$ and $y \in B$, then

$$f(z) = f(x-y) = f(x) - f(y) < 0 \implies f(x) - f(y) \implies \sup_{x\in A} f(x) \le \alpha \le \sup_{y\in B} f(y).$$

□

Theorem 18. (Geometric Hahn-Banach, second form) Suppose $A, B \subset E$ be nonempty, convex, and disjoint. Suppose that A is closed and B is compact, then there exists a closed hyperplane that strictly separates A and B.

Proof. Again, we make $C = A - B$, then C is convex, $0 \notin C$, and C is closed (easy to see with the compactness of B). By the the Geometric first form of the Hahn Banach Theorem (Thm 17), we get that a closed hyperplane $H = [f = \alpha]$ that separates C and $B_r(0)$. Thus, we get that for all $x \in A$, $y \in B$, and $z \in B_1(0)$, we get

$$f(x-y) \le f(rz) \le \pm r\|f\|.$$

Let $\epsilon = \frac{r}{2}\|f\| > 0$, we see that

$$f(x) + \epsilon \le f(y) - \epsilon \implies \sup_{x\in A} f(x) + \epsilon \le \alpha \le \inf_{y\in B} f(y) - \epsilon.$$

□

We end with the most famous corollary of the Hahn Banach Theorem, an easy way for checking for density.

Corollary 2. Let $F \subset E$ be a linear subspace such that $\overline{F} \neq E$, then there exists some $f \in \mathscr{E}, \mathbb{R}$ with $f \not\equiv 0$ such that for all $x \in F$, we have that

$$\langle f, x \rangle = 0.$$

Proof. Let $x_0 \in E$ with $x_0 \in \overline{F}$. We can apply the second form of the geometric Hahn-Banach Theorem (Thm 18) since $\{x_0\}$ is compact that $\overline{F}$ is closed. Thus, there exists a closed hyperplane $H = [f = \alpha]$ that strictly seperates the two, that is, for any $x \in \overline{F}$,

$$\langle f, x \rangle < \alpha < \langle f, x_0 \rangle \implies \langle f, x \rangle = \equiv 0$$

since $\lambda \langle f, x \rangle < \alpha$ for every $\lambda \in \mathbb{R}$. $\square$

1.9 Bonus Problem Session: Riesz Representation Theorem

Definition 18. The **variation** of a function $\alpha : [a,b] \to \mathbb{R}$ is defined as

$$V_a^x(\alpha) = \sup_P \sum |\Delta\alpha_i|,$$

where P is a partition on $[a,x]$.

Definition 19. The **total variation** of α $V_a^b(\alpha)$.

Definition 20. We say α has bounded variation if

$$V_a^b(\alpha) < \infty.$$

We denote the space of all functions with bounded variation as BV.

Theorem 19. (Jordan Decomposition Theorem) If $\alpha \in BV([a,b])$, then there exist α^+ and α^- such that

$$\alpha = \alpha^+ - \alpha^-$$

and

$$V_a^b(\alpha) = V_a^b(\alpha^+) + V_a^b(\alpha^-).$$

For the proof of this, define

$$\alpha^+(0) := \alpha(a), \qquad \alpha^+(x) = \sup_P \sum_P \max\{0, \Delta\alpha_i\} + \alpha(a)$$

$$\alpha^-(0) := \alpha(a), \qquad \alpha^-(x) = \sup_P \sum_P \max\{0, -\Delta\alpha_i\}$$

(A version of the Riesz Representation Theorem)

Proposition 6. For all $F \in (C([a,b]))^*$, there exists an $\alpha \in BV([a,b])$ such that for all $f \in C([a,b])$,

$$F(f) = \int_a^b f d\alpha.$$

Obviously, $F(f)$ is a linear functional. To show that it is bounded, consider that for any $f \in C([a,b])$ with $\|f\| \leq 1$, we have that

$$\|F(f)\| \leq \int_a^b f d\alpha = \int_a^b f d\alpha^+ - \int_a^b f d\alpha^- \leq \alpha^+(b) - \alpha^+(a) - \alpha^-(b) + \alpha^-(a) < \infty.$$

We let $F \in C([a,b])^*$. Since $C([a,b]) \subset C^b([a,b])$, then by the Hahn Banach Theorem (Thm 16), there exists some linear functional $G \in (C^b([a,b]))^*$ such that $G|_{C([a,b])} = F$ and $\|G\| = \|F\|$. Define

$$\alpha(a) := 0, \qquad \alpha(x) := G(\chi_{[a,x]}).$$

Then note that

$$\alpha(y) - \alpha(x) = G(\chi_{[a,y]}) - G(\chi_{[a,x]}) = G(\chi_{[a,y]} - G(\chi_{[a,x]})) = G(\chi_{[x,y]}).$$

To see that α has bounded variation, let P be a partition of $[a,b]$. Then we have that

$$\sum_P |\Delta\alpha_i| = \sum_P \operatorname{sign}(\Delta\alpha_i)\Delta\alpha_i = \sum_P \operatorname{sign}(\Delta\alpha_i) G(\chi_{[x_{i-1},x_i]}) = G(\sum_P \operatorname{sign}(\Delta\alpha_i)\chi([_{x_{i-1},x_i]})) \leq \|G\| < \infty.$$

Let $f \in C([a,b])$ and choose $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2\|F\|}$ from uniform continuity. Choose P fine such that

$$\left| \sum_P f(x_i)\Delta\alpha_i - \int_a^b f d\alpha \right| < \frac{\epsilon}{2}.$$

Define

$$g := \sum_P f(x_i)\chi_{[x_{i-1},x_i]}.$$

Then

$$\begin{aligned}
\left| F(f) - \int_a^b f d\alpha \right| &\leq |F(f) - G(g)| + \left| G(g) + \int_a^b f d\alpha \right| \\
&< |F(f) - F(g)| + \frac{\epsilon}{2} \\
&= |F(f-g)| + \frac{\epsilon}{2} \\
&\leq \|F\|\|f-g\|\frac{\epsilon}{2\|F\|} + \frac{\epsilon}{2} \\
&< \|F\|\frac{\epsilon}{2\|F\|} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

1.10 Wednesday, Jan 29: Midterm

We had a midterm! I'll post the questions and my answers later. The first question was so ass though. It was not a hard midterm, but it felt like Soug was testing how fast we could write. Here are the questions:

(a) Define three functions, $\beta_1, \beta_2, \beta_3$ as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, and $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.

(i) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then $\int f d\beta_1 = f(0)$.

(ii) State and prove a similar result for β_2.

(iii) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.

(iv) If f is continuous at 0, prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0).$$

(b) Determine the Fourier coefficients:

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx,$$

where

$$f(x) = \begin{cases} -1, & x \in [-\pi, 0], \\ 1, & x \in (0, \pi] \end{cases}$$

and then use them to compute

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

(c) A map $\phi : V \to V$ where V is a vector space over $\mathbb{R}$ is **affine** if there exists some $x_0 \in V$ and a linear map $T : V \to V$ such that for all $x \in V$ we have $\phi(x) = x_0 + T(x)$. Prove that $\phi : V \to V$ is affine if and only if $\phi(\sum \lambda_i x_i) = \sum \lambda_i \phi(x_i)$ for all λ_i such that $\sum_{i=1}^{n} \lambda_i = 1$.

(d) Show that $C^n([0, 1])$ is a Banach space with the norm

$$\|f\| = \sup_{n} \sup_{x \in [0,1]} |f^n(x)|$$

(e) Show that the normed space $\mathscr{L}$ of all real value Lipshitz function on $\mathbb{R}$ that are equal to 0 at the origin under the norm

$$\|f\| = \sup_{x,y \in \mathbb{R}} \frac{|f(x) - f(y)|}{|x - y|}$$

is Banach.

1.11 Friday, Jan 31: The Baire, Uniform Boundedness, and Open Mapping Theorems

Big day today.

Theorem 20. (Baire's Category Theorem) Suppose X is a complete metric space and we can write it as

$$X = \bigcup_{i=1}^{\infty} F_i,$$

where each F_i is closed. Then there exists some i such that $\text{int}(F_i) \neq \emptyset$.

Proof. We will first prove that if G_n is a sequence such that each G_n is open and dense, then $\bigcap G_n$ is dense in X. To do this, we let $x \in X$ and $\epsilon > 0$. Consider that since G_1 is dense in X, then there exists some $x_1 \in G_1 \cap B_\epsilon(x)$. Since $G_1 \cap B_\epsilon(x)$ is open, then there exists some $r > 0$ such that

$$\overline{B_{\frac{r_1}{2}}(x_1)} \subset G_1 \cap B_\epsilon(x)$$

Since G_2 is dense in X, there exists some

$$x_2 \in G_2 \cap B_{\frac{r_1}{2}}(x_1) \subset G_1 \cap B_\epsilon(x),$$

and so there exists some $r_2 > 0$ such that

$$\overline{B_{\frac{r_2}{2}}}(x_2) \subset G_2 \cap B_{\frac{r_1}{2}}(x_1)$$

Thus, we create the sequence x_n such that

$$x_n \in G_n \cap B_{\frac{r_{n-1}}{2}}(x_{n-1}).$$

Evidently, (x_n) is Cauchy since $r_n \to 0$ and thus by the completeness of X, $x_n \to x_\infty \in G_n$ for all G_n. Thus,

$$x_\infty \in \bigcap G_n \cap B_\epsilon(x),$$

and so $\bigcap G_n$ is dense in X.

With this now proven, we note that if F is closed with an empty interior, then F^c is open and dense. Thus, suppose the interior of all F_n is empty, then

$$\emptyset = X^c = \left(\bigcup F_n\right)^c = \bigcap F_n^c,$$

and thus the emptyset is dense in X, which is obviously false (unless X is the emptyset). $\square$

Now for the Uniform Boundedness, which states that under certain conditions, pointwise convergence implies uniform convergence.

Theorem 21. (Banach-Steinhaus) Uniform Boundedness Theorem. Let E, F be Banach, and suppose $(T_\alpha)_{\alpha\in\mathscr{A}} \in \mathscr{L}(E, F)$. Suppose

$$\sup_{\alpha\in\mathscr{A}} \|T_\alpha x\| < \infty \qquad \forall\, x \in E,$$

then there exists some c such that

$$\sup_{\alpha\in\mathscr{A}} \|T_\alpha x\| < c\|x\| \qquad \forall\, x \in E. \tag{8}$$

That is,

$$\sup_{\alpha\in\mathscr{A}} \|T_\alpha\|_{\mathscr{L}(E,F)} < \infty$$

Proof. Define

$$C_n := \{x \in E \; ; \; \|T_\alpha x\| \leq n\}.$$

We know that $C_n \subset C_{n+1}$ and by assumption that $\bigcup C_n = E$. Closedeness is obvious. From Theorem 19 (Baire's), we have that there exists some C_n with nonempty interior. That is, there is some $x_0 \in C_{n_0}$ with $r > 0$ such that $B_r(x_0) \subset C_{n_0}$. Thus, for all $x \in B_r(x_0)$,

$$\|T_{n_0}(x)\| \leq n_0 \implies \|T_{n_0}(x_0 + rz)\| \leq n_0, \qquad \|z\| \leq 1,$$

thus, by linearity of the T, we find that

$$T_{n_0}(z) \leq \frac{n_0 + \|T_{n_0} x_0\|}{r},$$

implying uniform boundedness. □

We provide a couple of corollaries:

Corollary 3. Suppose E, F are Banach and $(T_n) \in \mathscr{L}(E, F)$ such that $\lim_{n\to\infty} T_n(x) = T(x)$, then

(a)
$$\sup_n \|T_n\| < \infty$$

(b)
$$T \in \mathscr{L}(E, F)$$

(c)
$$\|T\| \leq \liminf_{n\to\infty} \|T_n\|$$

Proof. By the existence of the limit, we have that the (T_n) are pointwise bounded, and thus (1) follows from Theorem 20. By taking the limit, we see that if

$$\|T_n x\| \leq c\|x\| \implies \|Tx\| \leq c\|x\|,$$

and so $T \in \mathscr{L}(E, F)$. If we let $x \in E$, then we have that

$$\|T_n x|\| \leq \|T_n\|\|x\|,$$

and so (3) is proved. □

Corollary 4. Suppose E is Banach and $A \subset E$. If $f(A)$ is bounded for any $f \in E^*$, then A is bounded.

Proof. From Theorem 20, we let E" = "E^*, F" = "$\mathbb{R}$, and $\mathscr{A}$" = "A. Then for all $a \in A$,

$$T_a(f) := f(a), \qquad f \in E^*.$$

By the boundedness of $f(A)$, we have that

$$\sup_{a\in A} \|T_a f\| < \infty \qquad \forall\, a \in A,$$

and so by Theorem 20:

$$|f(b)| \leq c\|f\| \qquad \forall\, b \in B, \quad f \in E^*,$$

and so

$$|b| \leq c \qquad \forall\, b \in B$$

□

I don't really fuck with Theorem 20. I do fuck with the open mapping theorem, although it's proof is rough.

Theorem 22. (Open Mapping Theorem) Suppose E, F are Banach and $T \in \mathscr{E}, \mathscr{F}$ such that T is surjective. Then there exists some $c > 0$ such that

$$B_c^F(0) \subset T(B_1^E(0)) \tag{9}$$

Proof. We claim that (9) implies that T is open. Indeed, let U be open in E, then if $y_0 \in T(U)$ (and let's say $y_0 = Tx_0$, where $x_0 \in U$), then by the openness of U, there exists some $r > 0$ such that $B_r^E(x_0) \subset U$. Thus,

$$B_r^E(x_0) = x_0 + B_r^E(0) \subset U,$$

and so

$$y_0 + T(B_r^E(0)) \subset T(U). \tag{10}$$

By (9), we have that there exists some $c > 0$ such that

$$B_c^F(0) \subset T(B_1^E(0)) \implies B_{rc}^F(0) \subset T(B_r^E(0))$$

So by (10), we know that

$$T(B_{rc}(y_0)) \subset T(U),$$

and so $T(U)$ is open.

(a) Our first step is to show that if T is a linear surjective map from E to F, then there exists some $c > 0$ such that

$$B_{2c}^E(0) \subset \overline{B_1^F(0)}.$$

To show this, we let

$$X_n = n\overline{T(B_1^F(0))}.$$

Obviously, X_n is closed, and since T is bounded, we have that $T(E) = F$ is bounded, and thus

$$F = \bigcup X_n.$$

Since F is complete, we use Baire's Category Theorem (Theorem 19) to say that there exists some i such that $\mathrm{int} X_n \neq \emptyset$. That is, there is some $y_0 \in X_n$ and $c > 0$ such that

$$B_{4c}(y_0) \subset \overline{T(B_1^F(0))}$$

and by symmetry

$$B_{4c}(-y_0) \subset \overline{T(B_1^F(0))}.$$

Thus, we get that by adding them up:

$$B_{4c}^E(0) \subset 2\overline{T(B_1^F(0))} \implies B_{2c}^E(0) \subset \overline{T(B_1^F(0))}.$$

(b) The second step is that if T is a linear continuous from E to F, then step (a) implies that

$$B_c^F(0) \subset T(B_1^E(0)).$$

To show this, let $y_0 \in B_c^F(0)$, that is $\|y_0\| < c$. We need to find an $x_0 \in E$ with $\|x_0\| < 1$ such that $Tx_0 = y_0$. Let $\epsilon > 0$. By (a), we know that there exists some $z \in E$ with $\|z_1\| < \frac{1}{2}$ such that

$$\|y - Tz_1\| < \epsilon.$$

Fixing $\epsilon = \frac{c}{2}$, we have that

$$\|y - Tz_1\| < \frac{c}{2}.$$

Thus, there exists some $\|z_2\| < \frac{1}{4}$ such that

$$\|(y - Tz_1) - Tz_2\| < \frac{c}{4},$$

and so on. Thus, we have that for $\|z_n\| < \frac{1}{2^n}$,

$$\|y - T(z_1 + z_2 + \cdots + z_n)\| < \frac{e}{2^n},$$

and so the sequence

$$x_n = \sum_{i=1}^{n} z_i$$

is Cauchy, and thus converges to some x_0 by the completeness of E. We have $\|x_0\| < 1$ and $y = Tx$.

□

Corollary 5. Suppose E, F are Banach and $T \in \mathscr{E}, \mathscr{F}$ such that T is bijective. Then $T^{-1} \in \mathscr{L}(F, E)$.

Corollary 6. Let E be a Banach space for two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If there exists a constant $C > 0$ such that

$$\|x\|_2 \leq C\|x\|_1, \qquad \forall\, x \in E,$$

then there exists a constant $c > 0$ such that

$$\|x\|_1 \leq c\|x\|_2, \qquad \forall\, x \in E.$$

Proof. To show the norms are equivalent, consider

$$I : (E, \|\cdot\|_1) \to (E, \|\cdot\|_2).$$

We know that $I \in \mathscr{L}(E)$ and that I is bijective. By the previous corollary, $I^{-1} \in \mathscr{E}$, and so we are done. □

Theorem 23. (Close Graph Theorem) Suppose E, F are Banach and $T : E \to F$ is linear. Then the graph of T, $(G(T)) = \{(x, Tx)\ x \in E\}$, is closed in $E \times F$ if and only if T is continuous.

Proof. The backwards direction is obvious. Consider the graph norm

$$\|\cdot\|_G = \|\cdot\|_E + \|T\cdot\|_F,$$

then one can probably see that E is Banach under both norms and obviously,

$$\|\cdot\|_E \leq \|\cdot\|_G,$$

and so by the previous corollary, the norms are equivalent and so there exists some $c > 0$ such that

$$\|Tx\|_F \leq c\|x\|_E$$

□

1.12 Monday, Feb 3: The Weakest Topology

We spent about 15 minutes today discussing how badly we did on the midterm. If only question 1 wasn't god awful huh. Anyways he talked about how his grading scheme is:

$$10\%\ \text{HW} \qquad 40\%\ \text{Midterm} \qquad 50\%\ \text{Final}$$

Discuss among yourselves how stupid 10% homework is when we had a 27 problem PSET (mostly Brezis problems) due on Week 6. But he assured us if we got a 30 on the midterm and did well on the final we would get an A. So I am not worried.

Then we proved the Open Mapping Theorem (which is proven above, where I stated it) and introduced weak convergence.

Remark 13. We construct the weak topology from X, a set, to $(Y_i)_{i\in I}$, a collection of topological spaces, such that it makes every map continuous. Let

$$\varphi_i : X \to Y_i$$

be continuous. Let $\omega_i \subset Y_i$ be open, then $(U_\lambda)_{\lambda\in|\Lambda} = \varphi^{-1}(\omega_i)$ is a collection of open sets in X. Consider $\bigcap_{\lambda\in\Gamma} U_\lambda$, where $\Gamma \subset \Lambda$ is finite. Then the finite intersection is open, and call this collection of intersections Φ. Call $\mathscr{F}$ the family of arbitrary unions of Φ. We say X is equipped with $\mathscr{F}$, which is the weakest topology associated with the φ_i.

Proposition 7. Suppose $(x_n) \in X$ with $x_n \to x$. Then for all $i \in I$, we have that $\varphi_i(x_n) \to \varphi(x)$.

Proposition 8. Suppose Z is a topological space and $\psi : Z \to X$. Then ψ is continuous if and only if

$$\varphi_i \circ \psi : Z \to X$$

is continuous for all $i \in I$.

1.13 Wednesday, Feb 5: Weak Convergence

From now on, whenever you see a $\varphi_f(x)$, think of it as $\langle f, x\rangle$

Definition 21. The **weak topology** of E, $\sigma(E, E^*)$ is the coarsest topology of E associated with the collection $(\varphi_f)_{f\in E^*}$

Proposition 9. Suppose $x_0 \in E$, then given $\epsilon > 0$ and a finite set $\{f_i\}_{i\in[k]}$, define

$$V := \{x \in E \,;\, |\langle f_i, x - x_0\rangle| < \epsilon, \quad \forall\, i \in [k]\}.$$

By varying the variables involved, we obtain a basis of the neighborhoods of x_0.

Proof. We have that

$$V = \bigcap_{i\in[k]} \varphi_{f_i}^{-1}(\langle f, x_0\rangle - \epsilon, \langle f, x_0\rangle + \epsilon),$$

and thus V is open in $\sigma(E, E*)$. To see that an open neighborhood contains V, see the construction of the topology. $\square$

Theorem 24. Suppose $(x_n) \in E$, then:

(a) $x_n \rightharpoonup x$ if and only if $\langle f, x_n\rangle \to \langle f, x\rangle$ for all $f \in E^*$.

(b) If $x_n \to x$, then $x_n \rightharpoonup x$.

(c) If $x_n \rightharpoonup x$, then $\|x_n\|$ is bounded and $\|x\| \le \liminf \|x_n\|$.

(d) If $x_n \rightharpoonup x$ and $f_n \to f$ in E^*, then $\langle f_n, x_n\rangle \to \langle f, x\rangle$.

Proof. We prove these one by one:

(a) Suppose $x_n \rightharpoonup x$, then since $f \in E^*$ then f is continuous and thus $f(x_n) \to f(x)$. See proposition 7.

(b) By (a), we have that it suffices to show $\langle f, x_n\rangle \to \langle f, x\rangle$. Let $f \in E^*$, then

$$\|\langle f, x_n\rangle - \langle f, x\rangle\| \le \|f\|\|x - x_n\| < \epsilon.$$

(c) By the uniform boundedness principle, we have that

$$\|\langle f, x_n\rangle\| \le \|f\|\|x_n\| \implies |\langle f, x\rangle\| \le \liminf \|f\|\|x_n\| \implies \|x\| \le \liminf \|x_n\|.$$

(d)
$$\|\langle f_n, x_n\rangle - \langle f, x\rangle\| \le \|\langle f_n - f, x_n\rangle\| + \|\langle f, x_n - x\rangle\|,$$
which is enough by (a) and (c).

$\square$

Proposition 10. If $\dim E < \infty$, then $x_n \rightharpoonup x$ if and only if $x_n \to x$.

Proof. One direction is clear. Let U be strongly open around some $x_0 \in E$. We need to find some $V \subset U$ such that $x_0 \in V$ and V is open in $\sigma(E, E^*)$. Thus, it suffices to find $\{f_1, f_2, \dots\} \subset E^*$ such that for all $\epsilon > 0$,

$$V = \{x \in E \,;\, |\langle f_i, x - x_0\rangle| < \epsilon\}.$$

Let $r > 0$ such that $B_r(x_0) \subset U$. Since E is finite, pick a basis $e_1, e_2, \dots, e_k$ such that $x = \sum_{n=1}^{k} e_n x_n$. Let $\pi_i : x \to x_i$ be the projection mapping, which is continuous, and thus

$$\|x - x_0\| \leq \sum \|\langle \pi_i, x - x_0\rangle\| < \epsilon k$$

for all $x \in V$. Choose $\epsilon = \frac{r}{k}$ and we are done. $\square$

We provide now an example of why the above proposition does not hold for infinite dimensions, that is, there exists open and closed sets in the strong topology that are *never* open or closed in the weak topology.

Example 1.3. We will show that the unit sphere, S, which is strongly closed and $U = \{x \in E \,;\, \|x\| < 1\}$, which is strongly open, are not weakly closed and not weakly open (respectively). We will see that

$$\overline{S}^{\sigma(E,E^*)} = B_E,$$

which is the closed unit ball in E.

Let $x_0 \in E$ with $\|x_0\| \leq 1$. Let $x_0 \in V$ be a neighborhood, it suffices to see that $V \cap S \neq \emptyset$ for every V in order for $x_0 \in \overline{S}^{\sigma(E,E^*)}$. We know by proposition 9 that there exists some $\{f_1, f_2, \dots, f_k\} \subset E^*$ such that

$$V = \{x \in E \,;\, |\langle f, x - x_0\rangle| < \epsilon\}.$$

There exists some $y_0 \neq 0$ such that for every $i \in [k]$, we have that $\langle f_i, y_0\rangle = 0$. This y_0 exists, since otherwise $\dim E < \infty$[1] Define $g(t) = \|x_0 + ty_0\|$, then $g(0) < 1$ and $g(t) \to \infty$ as $t \to \infty$, so there must exist some t_0 such that $g(t_0) = 1$. Thus, we have that $g(t_0) = x_0 + t_0 y_0 \in V \cap S$.

For the other inclusion, we have that $S \subset B_E$, and so $\overline{S}^{\sigma(E,E^*)} \subset \overline{B_E}^{\sigma(E,E^*)}$, so it suffices to show that B_E is weakly closed, but this comes from the fact that

$$B_E = \bigcap_{\substack{f \in E^* \\ \|f\| \leq 1}} \{x \in E \,;\, |\langle f, x\rangle| \leq 1\}$$

Thus, we see that S is not weakly closed. To see that U is not weakly open, suppose it is, then U^c is weakly closed, but then

$$S = B_E \cap U^c$$

is weakly closed, which is a contradiction.

[1]Prove this more later.

1.14 Friday, Feb 7: Weak Convexity

We mostly talked about example 3 above, but also stated a few results. Recall that in infinite dimensions, strongly closed sets are not necessarily weakly closed.

Theorem 25. Suppose $C \subset E$ is convex. Then C is strongly closed if and only if C is weakly closed.

Proof. If C is weakly closed, then of course it is strongly closed.

Suppose C is strongly closed, then let $x_0 \in E \setminus C$. $\{x_0\}$ is closed (strongly), convex, and nonempty. C is nonempty, closed, and convex, and thus we apply Hahn-Banach to find a continuous functional $f : E \to \mathbb{R}$. such that

$$f(x_0) < f(\alpha) < f(C).$$

Consider that

$$V := \{x \in E \ : \ \langle f, x \rangle < \alpha\}$$

is an open neighborhood containing x_0. Thus, $x_0 \in V \subset C^c$ and so C^c is weakly open and thus C is weakly closed. □

Definition 22. We say that f is **left semi continuous** atx_0 if

$$f(x_0) \leq \liminf x \to x_0 f(x)$$

Corollary 7. Suppose $\varphi : (-\infty, \infty]$ is convex and l.s.c. in the strong topoology, then φ is l.s.c in the weak topology.

Corollary 8. (Mazur) Suppose $x_n \rightharpoonup x$. Then there exists $(y_n) \to x$ such that $y_n \in \mathrm{Conv}[(x_n)]$.

1.15 Monday, Feb 10: Adjoint Operators

For this entire lecture, we suppose E, F are Banach, and $A : D(A) \subset E \to F$ is an unbounded linear operator.

Definition 23. We say that A is **densely defined** if $\overline{D(A)} = E$.

Definition 24. We say that A is bounded if $D(A) = E$ and there exists some $c > 0$ such that for all $u \in E$, we have that $\|A(u)\|_F \leq c\|u\|_E$.

Definition 25. Suppose $A : D(A) \subset E \to F$ is a densely defined unbounded linear operator. We define the **adjoint** $A^* : D(A^*) \subset F^* \to E^*$ such that

$$D(A^*) := \{v \in F^* \ : \exists c > 0 \text{ st } \langle v, Au\rangle \leq c\|u\|_E\} \ \forall u \in D(A)\}$$
$$\langle v, Au\rangle = \langle A^* v, u\rangle \qquad \forall\, v \in D(A^*),\ \forall\, u \in D(A).$$

We remark that $D(A^*)$ is a linear subspace and live the proof for you.

Remark 14. To prove the existence of such an operator, we use H-B. Define $g : D(A) \to \mathbb{R}$ such that $g_v(u) = \langle v, Au\rangle$, then extend g to E using Hahn-Banach to some $f : E \to \mathbb{R}$ such that $|f(u)| \leq c\|u\|_E \implies f \in E^*$. Define $A^* v = f$. A^* is unique because A is densely defined..

Proposition 11. If A is bounded, then A^* is bounded. Moreover,

$$\|A\|_{\mathscr{L}(E,F)} = \|A^*\|_{\mathscr{L}(F^*,E^*)}$$

Proof. We have that

$$|\langle A^* v, u\rangle| = |\langle v, Au\rangle| \leq \|v\|\|A\|\|u\|$$

and thus

$$\|A^* v\| \leq \|v\|\|A\| \implies \|A^*\| = \sup_{\|v\|\leq 1} \|A^* v\| \leq \|A\|.$$

For the other side, we have that

$$|\langle v, Au\rangle| \leq \|v\|\|A^*\|\|u\| \implies \|Au\| = \sup_{\|v\|\leq 1} |\langle v, Au| \leq \|A^*\|\|u\|.$$

Taking supremums, we see that

$$\|A\| \leq \|A^*\|$$

□

Proposition 12. If A is densely defined unbounded linear operator, then A^* is closed.

Proof. Suppose $(v_n) \in D(A^*)$ such that $v_n \to v$ in F^* and $A^* v_n \to f$ in E^*, then

$$\langle f, u\rangle \leftarrow \langle A^* v_n, u\rangle = \langle v_n, Au\rangle \to \langle v, Au\rangle.$$

Thus, $v \in D(A^*)$ since $|\langle v, Au\rangle| \leq \|f\|\|u\|$ for any $u \in D(A)$ and $A^* v = f$. □

Proposition 13. The following statements hold:

$$N(A) = R(A^*)^\perp$$
$$N(A^*) = R(A)^\perp$$
$$N(A)^\perp \supset \overline{R(A^*)}$$
$$N(A^*)^\perp = \overline{R(A)}$$

We proved this in our homework.

Remark 15. The inclusion in (3) is an inclusion in reflexive spaces since in the weak topology, $N(A)^\perp = \overline{R(A^*)}^{\sigma(E,E^*)}$

1.16 Friday, Feb 14: The Weak $\star$ Topology

Definition 26. The weak * topology $\sigma(E^*, E)$ is the coarsest topology E^* such that for every $x \in E$, the linear function $\varphi_x : E^* \to \mathbb{R}$ defined by $\varphi_x(f) = \langle f, x\rangle$ is continuous.

Remark 16. Since $E \subset E^{**}$, then $\sigma(E^*, E)$ is coarser than $\sigma(E^*, E^{**})$.

Proposition 14. Let $f_0 \in E^*$, then given $x_1, \ldots, x_k \in E$ and $\epsilon > 0$, a neighborhood of f_0 is

$$V := \{f \in E^* \ : \ |\langle f - f_0, x_i\rangle| < \epsilon, \ \ \forall\, i \in [k]\}$$

Theorem 26.
(a) $f_n \overset{*}{\rightharpoonup} f$ if and only if $\langle f_n, x\rangle \to \langle f, x\rangle$ for all $x \in E$.

(b) If $f_n \to f$, then $f_n \rightharpoonup f$ then $f_n \overset{*}{\rightharpoonup} f$ If $f_n \overset{*}{\rightharpoonup} f$, then $\|f_n\|$ is bounded and $\|f\| \le \liminf \|f_n\|$

(c) If $f_n \overset{*}{\rightharpoonup} f$ and $x_n \to x$, then $\langle f_n, x_n\rangle \to \langle f, x\rangle$

Theorem 27. The closed unit ball

$$B_{E^*} = \{f \in E^* \ : \ \|f\| \le 1\}$$

is compact in $\sigma(E^*, E)$.

That's pretty neat! I'm not going to prove it though!

1.17 Monday, Feb 17: Reflexive Spaces and Hilbert Spaces

Today we had double class for some fucking reason. Strap in. Bro didn't prove a thing about reflexive spaces, he just stated the important results.

Definition 27. Let E be a Banach space and suppose $J : E \to E^{**}$ is the canonical injection. The space E is **reflexive** if J is surjective.

Theorem 28. Let E be a Banach space. Then E is reflexive is and only if B_E is weakly compact in $\sigma(E, E^*)$

Theorem 29. Let E is be a Banach space and (x_n) is a bounded sequence in E. Then E is reflexive if and only if there exists a subsequence (x_{n_k}) that weakly converges in $\sigma(E, E^*)$.

Proposition 15. Suppose $M \subset E$ is a closed linear subspace of E reflexive Banach. Then M is a reflexive Banach space.

Corollary 9. A Banach Space E is reflexive if and only if E^* is reflexive.

Definition 28. A Banach space is **uniformly convex** if for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$x, y \in E, \|x\|, \|y\| \leq 1 \text{ and } \|x - y\| > \epsilon \implies \|\frac{x+y}{2}\| < 1 - \delta$$

Theorem 30. Every uniformly convex Banach space is reflexive.

Definition 29. Suppose H is a vector space. A **scalar product** is a bilinear map $(\cdot, \cdot) : H \times H \to \mathbb{R}$ such that if $u, v \in H$, then:

(a) $(u, v) = (v, u)$

(b) $(u, u) \geq 0$

(c) $(u, u) \neq 0$ if and only if $u \neq 0$.

Remark 17. Any vector space H with a scalar product induces a norm of the form

$$|u| = (u, u)^{\frac{1}{2}}.$$

Recall the Cauchy-Shwarz inequality:

$$(u, v) \leq |u||v|$$

and the Parallelogram Law:

$$|\frac{u+v}{2}|^2 + |\frac{u-v}{2}|^2 = \frac{1}{2}(|u|^2 + |v|^2) \tag{11}$$

Definition 30. A **Hilbert Space** is a vector space H equipped with a scalar product such that H is complete for the induced norm.

Proposition 16. H is uniformly convex.

Proof. Let $\epsilon > 0$ and $u, v \in H$ with $|u| \leq 1$, $|v| \leq 1$, $|u - v| > \epsilon$, then using (11) we see that

$$|\frac{u+v}{2}|^2 < 1 - \frac{\epsilon^2}{4},$$

and so $|\frac{u+v}{2}| < 1 - \delta$, where δ is chosen smartly from above. $\square$

Theorem 31. Let $K \subset H$ be a nonempty closed convex set. For every $f \in H$, there exists a unique element $u \in K$ such that

$$|f - u| = \text{dist}(f, K). \tag{12}$$

This is equivalent to

$$(f - u, u - v) \leq 0, \qquad \forall v \in K. \tag{13}$$

Remark 18. The element u is called the **projection** of f unto K and denoted by

$$u = P_K f$$

Proof. Let $(v_n) \in K$ such that

$$d_n = |f - v_n| \to d = \text{dist}(f, K).$$

Let $a = f - v_n$ and $b - f - v_m$. The parallelogram law (11) says that

$$|f - \frac{v_n + v_m}{2}|^2 + |\frac{v_n - v_m}{2}|^2 = \frac{1}{2}(d_n^2 + d_m^2)$$

and so

$$|\frac{v_n - v_m}{2}|^2 \leq \frac{1}{2}(d_n^2 + d_m^2) - d^2 \implies |v_n - v_m| \to 0$$

□

Proving the equivalence and the uniqueness is just annoying.

Proposition 17. Let $K \subset H$ be nonempty closed convex set, then P_k is a contraction.

Corollary 10. Suppose $M \subset H$ is a closed linear subspace. Then if $f \in H$, $P_M f$ is the orthagonal projection and P_M is linear. Moreover,

$$(f - P_M f, v) = 0, \qquad \forall u \in M.$$

Definition 31. By definition,

$$(f - P_M f, v - P_M f) \leq 0 \qquad \forall v \in M$$

and so for all $t \in \mathbb{R}$,

$$(f - P_M f, tv - P_M f) \leq 0$$

This next theorem is actually astounding. Let $f \in H$, then the map $u \mapsto (f, u)$ is in H^*! That's amazing!

Theorem 32. (Riesz) Let $\varphi \in H^*$. There exists a unique $f \in H$ such that

$$\langle \varphi, u \rangle = (f, u) \qquad \forall u \in H.$$

Moreover,

$$\|f\|_H = \|\varphi\|_{H^*}$$

Proof. Let $M = \ker \varphi$, which is a closed subspace of H. If $M = H$, then $\varphi \equiv 0$ and so we take $f = 0$ and conclude the proof. Take $M \neq H$. We claim there exists some $g \in H \setminus M$ such that

$$|g| = 1, (g, v) = 0 \qquad \forall v \in M.$$

Let $g_0 \in H \setminus M$. Let $g_1 = P_M g_0$ and

$$g = \frac{(g_0 - g_1)}{|g_0 - g_1|} \implies |g| = 1.$$

Let $u \in H$ and (since $g \notin \ker \varphi$) define

$$v := u - \frac{\langle \varphi, u \rangle}{\langle \varphi, g \rangle} g \implies (g, v) = (g, u - \frac{\langle \varphi, u \rangle}{\langle \varphi, g \rangle} g)$$

Notice that $v \in M$, and by Corollary 10 we have that $(g, v) = 0$ and so

$$(g, u)\langle \varphi, g \rangle = \langle \varphi, u \rangle g$$

$$\langle \varphi, u \rangle = \langle \varphi, g \rangle (g, u) \implies f = \langle \varphi, g \rangle g$$

$\square$

1.18 Friday, Feb 21: The Lax-Milgram Theorem and Orthonormal Bases

Definition 32. A bilinear form $a : H \times H \to \mathbb{R}$ is said to be:

(a) **continuous** if there exists some $C > 0$ such that

$$|a(u, v)| \leq C\|u\|\|v\|, \qquad \forall u, v \in H$$

(b) **coercive** if there is a constant $\alpha > 0$ such that

$$a(v, v) \geq \alpha\|v\|^2, \qquad \forall v \in H.$$

Theorem 33. (Lax-Milgram) Assume that $a(u, v)$ is a continuous coercive bilinear form on H. Then, given any $\varphi \in H^*$, there exists a unique element $u \in H$ such that

$$a(u, v) = \langle \varphi, v \rangle, \qquad \forall v \in H.$$

Proof. By the Riesz Representation theorem, since $v \mapsto a(u, v)$ is a continuous linear function on H, then there exists some $Au \in H$ such that $a(u, v) = (Au, v)$, where $Au : H \to H$ satisfies the same properties as a, namely

$$|Au| \leq C\|u\|, \qquad (Au, u) \geq \alpha|u|^2.$$

By the Riesz representation theorem, proving the Lax-Milgram theorem is equivalent to showing that for any $f \in H$, there exists a unique $u \in H$ such that $Au = f$. To do this, we need to show that A is injective and surjective.

We have injective by the fact that $|Au| \leq C\|u\|$.

We have surjective because $R(A)$ is closed and $R(A)$ is dense. It is closed by coerciveness. $\square$

Definition 33. Let (E_n) be a sequence of closed subspaces of H, then H is the **Hilbert Sum** of the $E_n's$ $H = \oplus_n E_n$ if

(a) E_n are mutually orthagonal.

(b) $H = \overline{\operatorname{span} \bigcup_{n=1}^{\infty} E_n}$

Recall Theorem 11 and 12.

Lemma 4. Suppose that (v_n) is a sequence in H such that

$$(v_m, v_n) = 0, \qquad m \neq n \tag{14}$$

$$\sum_{k=1}^{\infty} |v_k|^2 < \infty. \tag{15}$$

If $S_n = \sum_{k=1}^{n} v_n$, then $S = \lim_{n\to\infty} S_n$ exists and

$$|S|^2 = \sum_{k=1}^{\infty} |v_k|^2$$

Proof. Let $m > n$, then by the orthogonality of the $(v_n)'$s:

$$|S_m - S_n|^2 = |\sum_{k=n+1}^{m} v_k|^2 = \sum_{k=n+1}^{m} |v_k|^2.$$

Since the RHS is the tail of a series that converges, then $|S_m - S_n|^2$ is Cauchy, and since H is complete in its norm, $S = \lim_{n\to\infty} S_n$ exists. It is clear that as $n \to \infty$,

$$|S|^2 = \sum_{k=1}^{\infty} |v_k|^2$$

□

Theorem 34. Suppose H is the Hilbert sum of the E_n's. Let $u \in H$ and suppose

$$u_n = P_{E_n} u, \qquad S_n = \sum_{k=1}^{n} u_k$$

Then

$$\lim_{n\to\infty} S_n = u$$

and

$$\sum_{k=1}^{\infty} |u_k|^2 = |u|^2 \tag{16}$$

Proof. Let $m \neq n$, then

$$(u_n, u_m) = (P_{E_n} u, P_{E_m} u) = 0$$

since the E_n are orthogonal. Consider that by Corollary 10, we have that since $E_n \subset H$ is a linear subspace, then

$$(u - u_n, v) = 0 \quad \forall v \in H \implies (u - u_n, u_n) = 0 \implies (u, u_n) = |u_n|^2.$$

Thus,

$$(u, S_n) = (u, \sum_{k=1}^{n} u_n) = \sum_{k=1}^{\infty} |u_n|^2 = |S_n|^2,$$

where the last equality comes orthagonility. By C-S, we have that

$$|S_n|^2 = (u, S_n) \leq |u||S_n| \implies |S_n| \leq |u| \implies \sum_{k=1}^{n} |u_k|^2 \leq |u| < \infty.$$

So we apply the previous lemma, and thus $S_n \to S$ exists and

$$|S|^2 = \sum_{k=1}^{\infty} |u_k|^2.$$

We claim that

$$S = P_{\overline{F}} u,$$

where F is the space spanned by the E_n's. This is left as an exercise. □

Definition 34. A sequence $(e_n) \in H$ is an **orthonormal basis** of H (or a Hilbert basis) if it satisfies:

(a) $|e_n| = 1$, $(e_n, e_m) = 0$, if $m \neq n$.

(b) $\overline{\bigcup e_n} = H$.

We come back to Parseval's Indentity and Fourier Series

Corollary 11. Let (e_n) be an orthonormal basis. Then for every $u \in H$,

$$u = \sum_{k=1}^{\infty} (u, e_k) e_k$$

and

$$|u|^2 = \sum_{k=1}^{\infty} |(u, e_k)|^2$$

Proof. We have that $H = \oplus_n \mathbb{R} e_n$ and $P_{E_n} u = (u, e_n) e_n$. $\square$

1.19 Monday, Feb 24: Compact Operators

Let E, F be Banach.

Definition 35. A bounded operator, $T \in \mathscr{L}(E,F)$ is a **compact operator** if $\overline{T(B_E)}$ is compact in F. If T is compact, we say that $T \in \mathscr{K}(E,F)$.

Theorem 35. $\mathscr{K}(E,F)$ is a closed linear subspace of $\mathscr{L}(E,F)$.

Proof. Let $S, T \in \mathscr{K}(E,F)$, then clearly $S+T \in \mathscr{K}(E,F)$ and $\lambda T \in \mathscr{K}$. Also, $0 \in \mathscr{K}\underline{(E,F)}$ since $\{0\}$ is compact. Let $(T_n) \in \mathscr{K}(E,F)$ such that $T_n \to T$ in the operator norm. Let $\epsilon > 0$. Cover $\overline{T(B_E)}$ in B_ϵ balls. For large n, we have that

$$\|T_n - T\| < \frac{\epsilon}{2},$$

and so T_n has a finite covering of $T_n(B_E) \subset \bigcup_{i=1}^n B_{\frac{\epsilon_i}{2}}(f_i)$, and so $T \subset \bigcup_{i=1}^n B_{\epsilon_i}(f_i)$, □

Definition 36. An operator $T \in \mathscr{L}(E,F)$ is of **finite rank** if $\dim R(T) < \infty$.

Thus, we have that $R(T) \simeq \mathbb{R}^n$ and since $\overline{T(B_E)}$ is closed and bounded, then it is compact since $\overline{T(B_E)} \subset mB_{R(T)}$ for some m. We will see that $B_{R(T)}$ is compact.

Corollary 12. Suppose (T_n) is a sequence of finite rank operators and $T \in \mathscr{L}(E,F)$ such that $\|T_n - T\| \to 0$. Then T is compact.

Remark 19. Suppose F is a Hilbert space. Then the converse it true! Let T be compact. Then

$$T(B_E) \subset \bigcup B_{\epsilon_i}(f_i).$$

Let $G = \operatorname{span} \bigcup f_i$, then let $T_\epsilon = P_G T$. then T_ϵ is of finite rank since G is finite, and $\|T_\epsilon - T\| < \epsilon$ since for any $x \in B_E$, we have that

$$\begin{aligned}
\|T_\epsilon x - Tx\| &\leq \|P_G Tx - Tx\| \\
&\leq \|P_G Tx - P_G f_{i_0}\| + \|f_{i_0} - Tx\| \\
&\leq 2\|Tx - f_{i_0}\| \\
&< \epsilon
\end{aligned}$$

Where $f_{i_0} \in \overline{T(B_E)}$ is such that $Tx \in B_{\frac{\epsilon_{i_0}}{2}}(f_{i_0})$, and thus $\|Tx - f_{i_0}\| < \frac{\epsilon}{2}$.

Proposition 18. The composition of compact operators is compact.

Proof. Suppose $T \in \mathscr{K}(E,F)$ and $S \in \mathscr{K}(F,G)$ then $S \circ T \in \mathscr{K}(E,G)$. □

Theorem 36. We have that $T \in \mathscr{K}(E,F)$, if and only if $T^* \in \mathscr{K}(F^*, E^*)$.

Remark 20. Let $T \in \mathscr{K}(E,F)$. Suppose $u_n \rightharpoonup u \in E$, then $Tu_n \to Tu \in F$. To see this, we know that Tu_n has a convergent subsequence. Since T maps strong convergence to strong convergence, then it also maps weak convergence to weak convergence, and thus $Tu_n \rightharpoonup Tu$. Thus, $Tu_n \to Tu$.

In Hilbert spaces, the converse is true.

Lemma 5. Let E be a normed vector space and $M \subset E$ is a closed subspace such that $M \neq E$. Then for all $\epsilon > 0$, there exists some $u \in E$ with $\|u\| = 1$ such that $\|u - m\| \geq 1 - \epsilon$ for all $m \in M$.

Proof. Let $v \in E \setminus M$. Then $d = d(v, M) > 0$ since M is closed. Let $m_0 \in M$ with

$$d \leq \|v - m_0\| \leq \frac{d}{1-\epsilon},$$

then define

$$u := \frac{v - m_0}{\|v - m_0\|}.$$

We have that

$$d(u, M) = \inf_{m \in M} \|u - m\| = \inf_{m \in M} \|\frac{v - m_0}{\|v - m_0\|} - m_0\| \geq \frac{d}{\|v - m_0\|} \geq 1 - \epsilon$$

$\square$

Theorem 37. (Riesz) Suppose E is a normed vector space with B_E compact. Then $\dim E < \infty$.

Proof. Suppose $\dim E = \infty$. Let (E_n) be finite dimensional closed subspaces such that $E_{n-1} \subsetneq E_n$. Let $u_n \in E_n$ such that $\|u_n\| = 1$ with $d(u_n, E_{n-1}) \geq \frac{1}{2}$. Then $u_n \in B_E$ for all n but u_n has no convergent subsequence since it is not Cauchy. ABSURD. $\square$

1.20 Wednesday, Feb 26: The Friedholm Alternative

Suppose $u - Tu = f$. Can we fine such a u? The Friedholm Alternative (Theorem 38 below) says that either for every $f \in E$ $u - Tu$ has a unique solution OR it $u - Tu = 0$ has n linearly independent solutions, and so $u - Tu = f$ is solvable if and only if $f \in N(I - T^*)^\perp$.

In finite dimensions, T is injective if and only if T is surjective. The same is true if T is compact! That's insane!

Theorem 38. Suppose E is a n.v.s and $T \in \mathscr{K}(E)$. Then

(a) $N(I - T)$ is finite dimensional.

(b) $R(I - T)$ is closed with $R(I - T) = N(I - T^*)^\perp$.

(c) $N(I - T) = \{0\}$ if and only if $R(I - T) = E$.

(d) $\dim N(I - T) = \dim N(I - T^*)$.

Proof. (a) Let $E_1 = N(I - T)$. Then $B_{E_1} \subset T(B_E)$, and so since B_{E_1} is closed and a subset of a compact set, then E_1 is finite dimensional.

(b) Let $f_n \in R(I - T)$ such that $f_n \to f$. Then $f_n = u_n - Tu_n$. Let $d_n = d(u_n, N(I - T))$. There exist $v_n \in N(I - T)$ such that $d_n = \|u_n - v_n\|$, and so $v_n - Tv_n = 0$. Thus,

$$f_n = u_n - v_n - T(u_n - v_n) \tag{17}$$

Suppose there exists some subsequence such that $d_{n_k} \to \infty$, then if

$$w_n = \frac{u_n - v_n}{\|u_n - v_n\|} \implies w_{n_k} - T(w_{n_k}) \to 0,$$

where the implication comes from dividing (17) by $\|u_n - v_n\|$. Since T is compact, we can assume that $T(w_{n_{k_j}}) \to z$, and so $w_{n_{k_j}} \to z$ with $z \in N(I - T)$. Thus, $d(w_{n_{k_j}}, N(I - T)) \to 0$. But

$$d(w_n, N(I - T)) = \frac{d(u_n, N(I - T))}{\|u_n - v_n\|} = 1,$$

which is a contradiction. Thus, d_n is bounded and so $\|u_n - v_n\|$ is bounded and so T compact implies that $T(u_{n_k} - v_{n_k}) \to \ell$. Thus, rearranging (17) yields that $u_{n_k} - v_{n_k} \to f + \ell = g$, and thus $g - Tg = f$ and $f \in R(I - T)$. The second statement follows from Proposition 13 and what we just showed:

$$N(I - T^*) = R(I - T)^\perp \implies N(I - T^*) = R(I - T).$$

(c) Suppose $N(I - T) = \{0\}$, and suppose $R(I - T) \subsetneq E$. Then let $E_1 = (I - T)E = R(I - T)$, $E_2 = (I - T)^2 E$. We claim that $E_2 \subsetneq E_1$. Suppose not, then $E_2 = E_1$, and so

$$(I - T)^2 E = (I - T)E \implies (I - T)E = E \implies R(T) = E.$$

Preposturous! Inductively create $E_n = (I - T)^n E$. Each E_n is a closed subspace since $R(I - T)$ is closed. Let $u_n \in E_n \setminus E_{n+1}$ such that $\|u_n\| = 1$ and $d(u_n, E_{n+1}) \geq \frac{1}{2}$. Thus, u_n has no convergent subsequence. Moreover,

$$Tu_n - Tu_m = -(u_n - Tu_n) + (u_m - Tu_m) + (u_n - u_m),$$

where the first term is in E_{n+1} and the second is in E_{m+1}. Suppose $n > m$. Since E_n is decreasing, we have that $-(u_n - Tu_n) + (u_m - Tu_m) + u_n \in E_{m+1}$, and so $\|Tu_n - Tu_m\| \geq \frac{1}{2}$, but then how can Tu_n have a converging subsequence, even though T is compact?? Thus, $R(I - T) = E$.

Suppose $R(I - T) = E$. Then $R(I - T)^{\perp} = 0$ and $R(I - T)^{\perp} = N(I - T^*) = \{0\}$. But since $T^* \in \mathscr{K}(E^*)$, then $N(I - T^*) = \{0\}$ implies by what we just showed that $R(I - T^*) = E^*$ but $R(I - T^*)^{\perp} = NI - T) = \{0\}$.

□

1.21 Friday, Feb 28: The Spectrum

Definition 37. Let $T \in \mathscr{L}(E)$. The **resolvent set**, is defined by

$$\rho(T) := \{\lambda \in \mathbb{R} \ : \ T - \lambda I \text{ is bijective from } E \text{ to } E\}.$$

The **spectrum** of T is defined as

$$\sigma(T) := \mathbb{R} \setminus \rho(T)$$

Definition 38. The **eigenvalues** of T, $EV(T)$, are the $\lambda \in \sigma(T)$ such that $N(T - \lambda I) \neq \{0\}$.

Theorem 39. The spectrum $\sigma(T)$ of a bounded operator T is compact and

$$\sigma(T) \subset [-\|T\|, \|T\|]$$

Proof. Let $\lambda \in \mathbb{R}$ with $|\lambda| > \|T\|$. Let $f \in E$, then $Tu - \lambda u = f$ has a unique solution since $u = \lambda^{-1}(Tu - f)$ and thus the contraction mapping theorem applies.

Let $\lambda_0 \in \rho(T)$, then we will see that $\rho(T)$ is open. That is, for $\lambda \in \mathbb{R}$ close (we will see how close in a second) to λ_0, $T - \lambda I$ is bijective. Let $f \in E$, then we wish to solve

$$Tu - \lambda u = f \iff Tu - \lambda_0 u = f + (\lambda - \lambda_0)u \iff u = (T - \lambda_0 I)^{-1}[f + (\lambda - \lambda_0)u].$$

For $|\lambda - \lambda_0| \|T - \lambda_0 I\|^{-1} < 1$, the contraction mapping theorem applies and so we have a unique solution, and thus $T - \lambda I$ is bijective. $\square$

Lemma 6. Suppose $T \in \mathscr{K}(E)$ and $(\lambda_n) \in \mathbb{R}$ distinct such that $\lambda_n \to \lambda$ and $\lambda_n \in \sigma(T) \setminus \{0\}$. Then $\lambda = 0$.

Proof. We claim that $0 \in \sigma(T)$ if $\dim E = \infty$. Suppose not, that $0 \in \rho(T)$. Then T is bijective and so T^-1 exists. Thus, $I = T \circ T^{-1}$ is compact, and so $\overline{I(B_E)} = B_E$ is compact, and so E is finite dimensional.

Now we claim that $\sigma(T) \setminus \{0\} = EV(T) \setminus \{0\}$.. Let $\lambda \in \sigma(T) \setminus \{0\}$ such that $\lambda \notin EV(T)$. Then $N(T - \lambda I) = \{0\}$ and so by the Friedholm Alternative $R(T - \lambda I) = E$, and so $\lambda \in \rho(T)$, which is a contradiction. Evidently, we have that $EV(T) \subset \sigma(T)$.

Since $\lambda_n \in \sigma \setminus \{0\}$, we have that $\lambda_n \in EV(T) \setminus \{0\}$, and so $N(T - \lambda_n I) \neq \{0\}$. Let $e_n \in E$ such that $(T - \lambda_n I)e_n = 0$, and $E_n = \text{span} \bigcup^n e_k$. Then $E_n \subsetneq E_{n+1}$ is an increasing sequence and $E_n \neq E_{n+1}$ since $e_1, \ldots, e_n$ are linearly independent. Suppose they are not, and let $e_{n+1} = \sum_{i=1}^m \alpha_i e_i$, then

$$Te_{n+1} = \sum_{i=1}^n \alpha_i T(e_i) = \sum_{i=1}^n \alpha_i \lambda_i e_i$$

But also,

$$Te_{n+1} = e_{n+1}\lambda_{n+1} = \sum_{i=1}^n \alpha_i e_i \lambda_{n+1}.$$

Subtracting both equatinons, we see that $\alpha_i(\lambda_i - \lambda_{n+1}) = 0$ for any i, and thus $\alpha_i = 0$, and so we have a contradiction.

Since the E_n are closed subspaces, we let $u_n \in E_n$ such that $\|u_n\| = 1$ and $d(u_n, E_{n-1}) \geq \frac{1}{2}$. In particular, $\|u_n, u_{n-1}\| \geq \frac{1}{2}$. Let $m < n$. Then

$$\|\frac{Tu_n}{\lambda_n} - \frac{Tu_m}{\lambda_m}\| = \|\frac{(Tu_n - \lambda_n u_n)}{\lambda_n} - \frac{T(u_m - \lambda u_m)}{\lambda_m} + \lambda_n - \lambda_m\| \geq d(u_n, E_{n-1}) \geq \frac{1}{2},$$

and so Tu_n has no convergent subsequence if $\lambda \neq 0$, a contradiction to the fact that T is compact! $\square$

Corollary 13. Suppose T is compact and $\dim E = \infty$, then exactly one of the following hold:

(a) $\sigma(T) = \{0\}$

(b) $\sigma(T) \setminus \{0\}$ is finite

(c) $\sigma(T) \setminus \{0\}$ is a sequence converging to 0.

Proof. Consider $\sigma(T) \cap \{\lambda \in \mathbb{R} \ : \ |\lambda| \geq \frac{1}{n}\}$. Then if this set were infinite, it would have a converging subsequence to $\lambda \neq 0$ (since $\sigma(T)$ is compact), a contradiction to the previous theorem. Thus, the set is either finite or empty. If it is finite, then we can order them as converging them to 0. $\square$

Definition 39. Let H be a Hilbert space. $T \in \mathscr{L}(H)$ is **self-adjoint** if $T^* = T$, that is, for all $u, v \in H$,

$$(Tv, u) = (v, Tu).$$

Proposition 19. Suppose $T \in \mathscr{L}(H)$ is self adjoint, then if

$$m = \inf_{\substack{u \in H \\ \|u\|=1}} (Tu, u), \qquad M = \sup_{\substack{u \in H \\ \|u\|=1}} (Tu, u),$$

then $\sigma(T) \subset [m, M]$ and $m, M \in \sigma(T)$ and $\|T\| = \max(|m|, |M|)$.

Corollary 14. Suppose T is self adjoint such that $\sigma(T) = \{0\}$, then $T \equiv 0$.

Proof. Since $\sigma(T) = \{0\}$, then since $m, M \in \sigma(T)$, then $M = 0 = m$ and thus $\|T\| = 0$. $\square$

Theorem 40. Let H be a separable Hilbert space and let T be a compact self adjoint operator. There exists a Hilbert basis composed of eigenvectors of T.

Proof. Let $(\lambda)_n \in EV(T) \setminus \{0\}$. Let

$$\lambda_0 = 0, \qquad E_0 = N(T), \qquad E_n = N(T - \lambda_n I).$$

We know that $0 \leq \dim E_0 \leq \infty$ and $\dim E_n < \infty$ by the Friedholm Alternative. We claim that that $H = \oplus_n E_n$.

Let $u_n \in E_n$ and $u_m \in E_m$, then

$$Tu_n = \lambda_n u_n, \qquad Tu_m = \lambda_m u_m,$$

and so by the self adjointness of the T :

$$(Tu_n, u_m) = \lambda_n(u_n, u_m) = (u_n, Tu_m) = \lambda_m(u_n, u_m),$$

and thus $(u_n, u_m) = 0$.

Let $F = \operatorname{span} \bigcup E_n$. Consider that $T(F) \subset F$ and so $T(F^\perp) \subset F^\perp$ since for any $u \in F^\perp$, we have that for any $v \in F$,

$$(Tu, v) = (u, Tv) = 0 \implies Tu \in F^\perp.$$

Define

$$T_0 := T|_{F^\perp},$$

then T_0 is compact and self-adjoint. We claim that $\sigma(T_0) = \{0\}$, since if not, there is some $\lambda \in EV(T_0) \setminus \{0\}$ and so there is some $u \in F^\perp$ such that $Tu = \lambda u$, and thus $u \in F^\perp \cap F$, and so $u = \{0\}$, which is a contradiction. Since $\sigma(T_0) = \{0\}$, then $T_0 \equiv 0$ by Corollary 14 above, and so $F^\perp \subset N(T)$, and since $N(T) \subset F$, we have that $F^\perp \subset F$ and thus $F^\perp = \{0\}$. Thus, F is dense in H. $\square$

1.22 Tuesday, March 11: Final Exam

The day after my Birthday!!

(a) Let X be a Banach space. Suppose that Y and Z are closed linear subspaces of X such that for every $x \in X$, there exists unique $y \in Y$ and $z \in Z$ so that $x = y + z$. Show that there exists a constant $C > 0$ so that for all $x \in X$, we have $\|y\| \leq C\|x\|$, where $x = y + z$ for $y \in Y$ and $z \in Z$.

(b) (i) Let $(x^n) = (x_1^n, x_2^n, \dots) \in \ell^\infty$. Show that the following are equivalent.

   (a) $x^n \overset{*}{\rightharpoonup} x$ for some $x \in \ell^\infty$.

   (b) (x^n) is bounded and converges pointwise.

   (ii) Prove that $S = \{x \in \ell^p \ : \ \|x\|_p = 1\}$ is a closed subset of ℓ^p, where $p \in [1, \infty]$.

   (iii) Prove that S is not compact in ℓ^p.

(c) Let H be a Hilbert space and $T \in \mathscr{L}(H)$.

   (i) Let $\lambda > 0$. Assume $(Tx, x) \geq \lambda |x|^2$ for al $x \in H$, prove that T is a bijection.

   (ii) Assume T is self-adjoint and that $(Tx, x) \geq 0$ for all $x \in H$. Prove that $\sigma(T) \subset [0, \infty)$

   (iii) Suppose that H is separable and $T \in \mathscr{K}(H)$. Assume that T is self-adjoint and $(Tx, x) \geq 0$ for all $x \in H$. Prove that there exists $S \in \mathscr{L}(H)$ such that $S \circ S = T$.

(d) (i) Let X be a Banach space. Take $T \in \mathscr{L}(X)$ and $A \in \mathscr{K}(X)$. Prove that $T \circ A \in \mathscr{K}(H)$.

   (ii) Let X be an infinite-dinmensional Banach space and take $A \in \mathscr{K}(H)$. Prove that A is not invertible.

   (iii) Let H be a Hilbert space and $A \in \mathscr{L}(H)$. Prove that A compact if and only if $A^* \circ A$ is compact.

(e) Let H be a Hilber space and $\{e_n\}$ be an orthonormal basis of H.

   (i) Prove that $e_n \rightharpoonup 0$ weakly.

   (ii) Let (a_n) be a bounded sequence in $\mathbb{R}$. Define $u_n = \frac{1}{n} \sum_{k=1}^{n} a_k e_k$. Prove that $|u_n| \to 0$.

   (iii) Prove that $\sqrt{n} u_n \rightharpoonup 0$.