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Problem 1

Consider a variant of the anarchy game in Ellingsen discussed in class. There are two players (i = 1, 2) who choose actions $y_i \in [0, 1]$ representing weapons, with the remaining portion $x_i = 1 - y_i$ representing food. The players' preferences are over their consumption c_i, which is given by:

$$c_1 = \begin{cases} x_1 + x_2 & \text{if } y_1 + \alpha \geq y_2 \\ 0 & \text{otherwise} \end{cases}$$

$$c_2 = \begin{cases} x_1 + x_2 & \text{if } y_2 > y_1 + \alpha \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha \in (0, 1)$ gives player 1 an advantage in combat.

(a) Show that there is a Nash equilibrium of this game in which $(y_1, y_2) = (1 - \alpha, 1)$.

SOLUTION: Let $(y_1^*, y_2^*) = (1 - \alpha, 1)$ represent our candidate Nash Equilibrium. Thus, we have that since $y_1^* + \alpha = 1 = y_2^*$, then

$$u_1(y_1^*, y_2^*) = c_1(y_1^*, y_2^*) = x_1 + x_2 = \alpha + 0$$

Consider some other action profile, (y_1, y_2^*). We have two options, suppose $y_1 > 1 - \alpha$, in which case we have that

$$u_1(y_1, 1) = 1 - y_1 < 1 - (1 - \alpha) = \alpha.$$

Our only alternative profile for player 1 is where $y_1 < 1 - \alpha$. Thus, consider that

$$u_1(y_1 = 1, y_2^*) = c_1(y_1, y_2^*) = 0, \qquad (y_1 + \alpha < 1)$$

Thus, player 1 deviating from this (y_1^*) is not profitable.

Consider now (y_1^*, y_2), where $y_2 = c$ and $c < 1$ (the only alternative profile to $y_2 = 1$,) then

$$u_2(y_1^*, y_2) = 0 \qquad (c \not> 1),$$

and so deviating is not more profitable that y_2^*.

Thus, we have that for $i = 1, 2$

$$u_i(y^*) \geq u_i(y_i, y_{-i}^*),$$

and thus $(1 - \alpha, 1)$ is Nash equilibrium. ■

(b) Is this the unique Nash equilibrium? If so, explain why. If not, provide at least one example of another equilibrium.

SOLUTION: Yes, it is unique. Suppose not, then there exists some $(y_1, y_2) \neq (1-\alpha, 1)$. If $y_1 < y_2 - \alpha$, then $c_1 = 0$. However, player 1 can just deviate such that $y_1 = y_2 - \alpha$ and get $c_1 > 0$. Similarly, if $y_1 > y_2 - \alpha$, then y_1 can deviate to some $y_1 - \epsilon > y_2 - \alpha$. Thus, we must have that $y_1 = y_2 - \alpha$. and so $c_1 > 0$ and $c_2 = 0$. However, then player 2 stands to profit by increasing production of weapons such that $y_1 < y_2 - \alpha$, which we have already seen is not an equilibrium. ■

(c) What is the weakest punishment that a (nonstrategic) state could impose on any actor i who chooses $y_i > 0$ that would ensure that $(y_1, y_2) = (0, 0)$ is a Nash equilibrium?

SOLUTION: Suppose the state imposes a punishment λ_i on y_i such that

$$u_i = c_i - \lambda_i y_i.$$

Note that we have that

$$u_1(0,0) = 2, \qquad u_2(0,0) = 0.$$

For any action profile (y_1, y_2), we have that if $y_1 \geq y_2$, then $y_1 + \alpha \geq y_2$ and so

$$c_1 = x_1 + x_2 = (1 - y_1) + (1 - y_2) = 2 - (y_1 + y_2)$$

and so

$$u_1(y_1, y_2 = 0) = 2 - (y_1 + y_2) - \lambda y_1 = 2 - (y_1 + y_2) - \lambda y_1 y_1 = 2 - y_1(1 + \lambda y_1) < 2,$$

and so $u_1(y_1, 0) \leq u_1(0,0)$ Thus, player 1 does not stand to profit from deviating from $(y_1, y_2) = (0, 0)$ for any λy_1.

Now consider player 2, and suppose he deviates to some $y_2 > 0$. If $y_2 \leq \alpha$, then his utility is obviously still 0, but if $y_2 > \alpha$, then

$$u_2(0, y_2) = x_1 + x_2 - \lambda_2 y_2 = 1 + (1 - y_2) - \lambda_2 y_2 = 2 - y_2 - \lambda_2 y_2 = 2 - y_2(1 + \lambda_2) < 0$$

for $\lambda_2 \leq \frac{2}{y_2} - 1$, thus, $\lambda_i = \frac{2}{\alpha} - 1$ is the weakest punishment such that $(0, 0)$ is the Nash equilibrium. Note that this punishment depends on how much weapons player 2 produces, which is different from what we did in class. In class, we did it using

$$u_i - c_i - \lambda \mathbf{1}_{y_i > 0},$$

which would result in the same thing for player 1 (and by that I mean us not caring about them), and so for player 2, if he/she/they[a] deviates to $y_2 = \alpha + \epsilon$, then we have that

$$u_2(0, y_2) = 1 + (1 - y_2) - \lambda = 1 + (1 - (\alpha + \epsilon)) - \lambda = 2 - \alpha - \epsilon - \lambda < 0$$

when $2 - \alpha - \epsilon < \lambda$, and so $\lambda = 2 - \alpha$ is smallest punishment available. ■

[a] I am not conforming to the follow executive order https://www.nytimes.com/2025/01/31/us/politics/trump-pronouns.html, sorry!

Problem 2

Each player extracts c_i $i = 1, 2$ from the first period. The amount not extract, $y - c_1 - c_2$, renews into $\sqrt{y - c_1 - c_2}$ for the second period. In the second period, the total is divided evenly between both players.

(a) Write down the best response problem for player 1.

SOLUTION: We have the utility function of player 1 is given by

$$u_1(c_1, c_2) = \log(c_1) + \log\left(\frac{\sqrt{y - c_1 - c_2}}{2}\right),$$

thus, the best response problem is to maximize this utility with respect to the player's own consumption, that is, to solve for

$$\arg\max_{c_1} \log(c_1) + \log\left(\frac{\sqrt{y - c_1 - c_2}}{2}\right) = \arg\max_{c_2} \log(c_1) + \log\left(\sqrt{y - c_1 - c_2}\right) - \log(2)$$

■

(b) Show that the best response function is given by

$$R_1(c_2) = \frac{2(y - c_2)}{3}$$

SOLUTION: Solving the above problem requires us to find the critical points and setting equal to 0 :

$$\frac{\partial}{\partial c_1} \log(c_1) + \log\left(\sqrt{y - c_1 - c_2}\right) - \log(2) = \frac{1}{c_1} + \frac{1}{\sqrt{y - c_1 - c_2}} \frac{-1}{2\sqrt{y - c_1 - c_2}} = \frac{1}{c_1} - \frac{1}{2(y - c_1 - c_2)}$$

Setting equal to 0 :

$$\frac{1}{c_1} - \frac{1}{2(y - c_1 - c_2)} = 0 \iff c_1 = 2y - 2c_1 - 2c_2 \iff 3c_1 = 2(y - c_2) \iff c_1 = \frac{2}{3}(y - c_1),$$

thus,

$$R_1(c_2) = \frac{2(y - c_2)}{3}$$

■

(c) Compute the Nash Equilibrium.

Solution: By symmetry, we have that

$$R_2(c_1) = \frac{2(y - c_1)}{3},$$

and thus solving for when $R_2(c_1) = R_1(c_1)$, that is solving the system

$$c_1 = \frac{2}{3}(y - c_2), \qquad c_2 = \frac{2}{3}(y - c_1),$$

by plugging in the first into the second:

$$c_2 = \frac{2}{3}(y - \frac{2}{3}(y - c_2)) = \frac{2}{9}y + \frac{4}{9}c_2 \iff \frac{5}{9}c_2 = \frac{2}{9}y \iff c_2 = \frac{2}{5}y.$$

Again, by symmetry, we must have that $c_1 = \frac{2}{5}y$. Thus, our Nash equilibrium is when $(c_1, c_2) = (\frac{2}{5}y, \frac{2}{5}y)$. ■

Reflections: Comparing our results, we see that renewing resources over time increases the amount people are going to take in the first period, but it it still not socially optimal.