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SSI: Formal Theory
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Problem 1

Let A, B, C be sets. Suppose $A \subseteq B, B \subseteq C, B \subseteq A, C \subseteq A$. Show that $A = B = C$.

SOLUTION: Let $b \in B$. Since $B \subseteq C$, we have that $b \in C$. Since $C \subseteq A$, we have that $b \in A$. Thus, $B \subseteq A$ and since by assumption $A \subseteq B$, we have by double inclusion that $A = B$. Let $c \in C$. Then since $C \subseteq A$, we have that $c \in A$ and thus $c \in B$ and therefore $C \subseteq B$. Since $B \subseteq C$ by assumption, we have that $B = C$ and thus $A = B = C$. ■

Problem 2

Let $X = (0,5)$, $Y = (2,4)$, $Z = (1,3)$, and $W = (3,5)$ be intervals in $\mathbb{R}$. Find the following sets:

(a) $Y \cup Z$

Solution: $(1,4)$ ■

(b) $Z \cap W$

Solution: $\emptyset$ ■

(c) $Y \setminus W$

Solution: $(2,3]$ ■

(d) $(W \cap Y) \cup Z$

Solution: $((3,5) \cap (2,4)) \cup (1,3) = (1,4)$ ■

(e) $X \setminus (Z \cup W)$

Solution: $(0,5) \setminus ((1,3) \cup (3,5)) = (0,1] \cup \{3\}$ ■

Problem 3

Let R be a complete and reflexive binary relation. Use the definition of quasitransitivity and acyclicity to show that if R is quasitransitive, then R is acyclic.

SOLUTION: Suppose $x_1 P x_2, x_2 P x_3, \ldots, x_{n-1} P x_n$. Because R is quasitransitive, then we must necessarily have that $x_1 P x_n$. Thus, by definition of P, we have $x_1 R x_n$ and $\neg(x_n R x_1)$. Thus, $x_1 P x_2, x_2 P x_3, \ldots, x_{n-1} P x_n \implies x_1 R x_n$, showing acyclicity. ■

Problem 4

Provide a counterexample when:

(a) Binary relation is acyclic but not quasi-transitive

> SOLUTION: Let $X = \{x, y, z\}$ and impose the binary relation R such that xPy, yPz, and xIz, then R on X is acyclic since $xIz \implies xRz$ but it is obviously not quasitransitive since $xIz \implies \neg(xPz)$. ∎

(b) Binary relation is quasi-transitive but not transitive.

> SOLUTION: Let $X = \{x, y, z\}$ and impose the binary relation R such that xPy, yIz, and xIz. Then R is quasitransitive since we have that $\neg(xPz), \neg(zPy)$ and $\neg(yPx)$. R is not transitive since xRz, zRy but $\neg(yRx)$ ∎

Problem 5

Suppose that R is a complete and transitive preference relation on some finite set of alternatives X. Show that

(a) The corresponding strict preference relation P is transitive ($xPy, yPz \implies xPz$).

SOLUTION: Suppose xPy and yPz, then by definition of P and transitivity,

$$xRy, yRz \implies xRz.$$

Assume, for the sake of contradiction, that zRx, then since xRy, we have by transitivity that zRy, and thus $\neg yPz$, a contradiction! Thus, we have that xRz and $\neg(zRx)$. ■

(b) The corresponding indifference relation I is transitive ($xIy, yIz \implies xIz$).

SOLUTION: Easy! Suppose xIy and yIz. By definition of I, we have that:

$$xIy \implies xRy, yRx;$$

$$yIz \implies yRz, zRy.$$

By transitivity:

$$xRy, yRz \implies xRz;$$

$$zRy, yRx \implies zRx,$$

and thus xIz. ■

Problem 6

Let us call a relation R intransitive on the set S if and only if for any elements $x, y, z \in S$, if xRy and yRz, it's definitely not true that xRz (i.e, xRy and yRz but $\neg(xRz)$) According to this definition, which of the following relations are intransitive? Explain.

(a) If S is a finite set of line segments, "being longer in length"

> SOLUTION: Let x, y, z be line segments such that x is longer than y, y is longer than z, then it is definitely the case by the transitive property of Euclidean distance that x is longer than z. Thus, R is transitive and thus **not intransitive**. ■

(b) If S is a finite set of people, "being the mother of"

> SOLUTION: Let $x, y, z \in S$ and let x be the mother of y, y be the mother of z, then it better not be the case that x is the mother of z, since x is the grandmother of z. Thus, R is **intransitive.** ■

(c) If S is a finite set of people, "being a sister of"

> SOLUTION: Let x, y, z be in S. Suppose x is a sister of y and y is a sister of z, then evidently, x is a sister of z. Thus, R is transitive and thus **not intransitive** ■

(d) If S is a finite set of straight lines on a plane, "being perpendicular to"

> SOLUTION: Let $x, y, z \in S$. Thus, x, y, z are straight lines lying in the same plane. Suppose x is perpendicular to y and y is perpendicular to z, then x is parallel to z, and thus not perpendicular to each other. Thus, R is **intransitive.** ■

(e) If S is a finite set of members of the U.S. House of Representatives, "vote for in the Speaker election"

> SOLUTION: Let $x, y, z \in S$. If x votes for y in the election, y votes for z in the election, then z can vote for whomever he wants in the election. However, since the definition of intransitive is that it is definitively not true that z votes for x, but that is a succinct possibility in this case, R is **not intransitive.** ■

Is every relation either transitive or intransitive?

> SOLUTION: Consider xPy, yIz, and xIz. It has been shown that R is not transitive. Note that xRy and yRz and xRz, not thus R is not intransitive! ■

Problem 7

Suppose that R is a complete and transitive preference relation on some finite set of alternatives X. Define a new binary relation PP ("*way better than*") as $xPPy$ if there exists an element $z \in X$ such that xPz and zPy. Further define a corresponding weak preference relation $xRRy$ if y is not way better than x. Is the binary relation RR complete? Is it transitive? Explain.

SOLUTION: Let $x, y \in X$. Since R is complete, we have that (without loss of generality), xRy. Thus, y is not way better than x, and so $xRRy$. Thus, RR is complete.

Suppose $X = \{1, 2, 3\}$ and the preference is the usual ordering on the naturals ($>$). Thus, we have that $3RR2$ and $2RR1$, but $3PP1$, since there exists 2 such that $3P2$ and $2P1$. Thus, RR is not transitive, since we do not have that $3RR1$. Thus, RR is not necessarily transitive.

■

Problem 8

Prove that if WARP is satisfied, then if $A \cap C(B) \neq \emptyset$, then $C(A) \cap B \subset C(B)$.

Solution: Let $x \in C(A) \cap B$. Suppose $x \notin C(B)$. Since $x \in B$, then $x \in B \setminus C(B)$. Let $y \in C(B)$. By WARP, since $y \in C(B)$, $x \in B \setminus C(B)$, and $x \in C(A)$, we have that $y \notin A$. Thus $A \cap C(B) = \emptyset$, which is a contradiction. ∎