UChicago SSI: Formal Theory: 13210

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Contents

1 Lectures

1.1 Monday, Jan 6: Interactions

Solutions to making people behave themselves:

(a) **Sacrifice:** voluntary individual sacrifice for others.

- *Example of people not taking wallets with money (except in Mexico)*

(b) **Cooperation:** coordinating actions for mutual benefit.

- *Example of firms*

(c) **Coercion:** individuals forced to take actions which will benefit others.

- *Example of laws (driving)*

Differences in outcomes are driven by differences in institutions

Remark 1. We can think of them as constitutions and laws, or as conventions and customs. Acemoglu/Robinson argue that political institutions influence economic institutions which influence economic growth.

1.2 Wednesday, Jan 8

Definition 1. We say a person is *selfish* when she lacks kindness toward and responsibility for others.

Definition 2. We say a person is *rational* if she takes actions that are consistent with her goals.

Remark 2. We say a person is *irrational* if they are not rational. There are three main types:

(a) To forget or neglect goals (drunk before a test)

(b) To draw wrong conclusions from the information at hand (classic human errors)

(c) To hold incorrect beliefs for other reasons (Coronavirus beliefs)

Definition 3. *Expected utility theory* describes the decision-maker's preferences through a utility function and the beliefs through a probability function. The action with the highest expected utility is then chosen by the decision maker.

Definition 4. (Under certainty) Given a set of alternatives X and a preference relation R on X, we define the *utility function* $u(x)$ represents R if and only if, for all $x, y \in X$, $u(x) \geq u(y)$ if and only if xRy.

Remark 3. If $u(\cdot)$ is a utility representing R on X, then

$$M(R, X) = \arg\max_{x \in U} u(x)$$

Definition 5. We represent *utility functions* in generality with the following form

$$\mathbf{u}_i(\mathbf{s}) = U_i(c(s)) + \lambda_i v(s), \tag{1}$$

where $\mathbf{s} = (s_1, \ldots, s_n)$ denotes the actions that relevant people take, $c = (c_1, \ldots, c_n)$ denotes the consumption of the n people that u cares about, U_i denotes person $i's$ utility, λ_i denotes the extent to which i cares about social values, v denotes the social values.

Remark 4. *Economics is about individual's choices, sociology is about how individuals don't have any choices to make.* When $\lambda \to \infty$, the individual follows society, or a *logic of appropriateness.* When $\lambda = 0$, individuals follow a *logic of consequences.*

Remark 5. When a person is selfish, a persons utility and goals coincide with their own consumption, and we denoted this by saying that if i is selfish, then

$$u_i(s) = U_i(s_i(s))$$

Theorem 1. Under certain conditions, there exists a function u that assigns a number u_j for each outcome x_j such that the expected utility of a lottery $\mathbf{p}_j = (p_{i1}, \ldots, p_{in})$ induced by action i is given by

$$U(\mathbf{p}_i) = \sum_{j=1}^{n} p_{ij} uj$$

and $\mathbf{p}_i R \mathbf{p}_k$ if and only if $U(\mathbf{p}_i) \geq U(\mathbf{p}_k)$.

Definition 6. When a person's action has an impact on somebody else's utility, we say the action causes an *externality.*

We can impose a positive, negative, or neutral externality to other people.

Definition 7. An outcome (a strategy profile s or a consequence c) is said to be efficient if there does not exist another outcome that yields higher utility (that is better for some and worse for none).

1.3 Monday, Jan 13: Situations, Games, and Cooperation

Definition 8. A *social situation* (or a *game form*) comprises of three elements:

(a) The people involved in the situation (the *players*)

(b) The information and actions that are available to them (the *strategies*)

(c) The potential results of those actions (the *consequences*)

Definition 9. A *game* is a game form together with a utility function for each player.

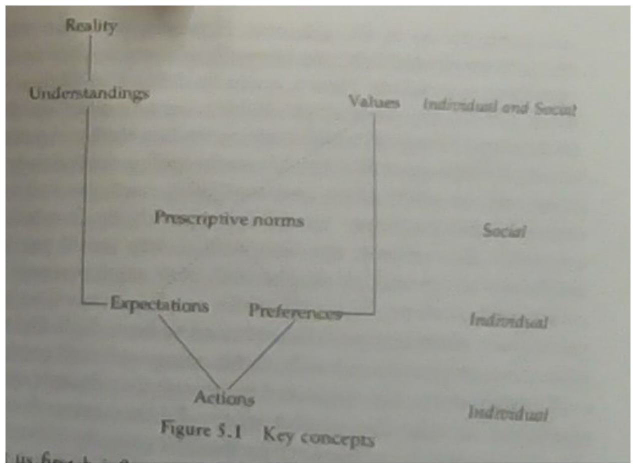

Figure 1: The Group and the Individual

1.4 Wednesday, Jan 15: Nash Equilibrium

Definition 10. A **strategic game** is defined by:

(a) Players: $i \in [n]$.

(b) Actions: An action is an element $a_i \in A_i$ where A_i is player i's **action space** and an **action profile** is defined to be $a = (a_1, \ldots, a_N) \in A$. Some more notation, a_{-i} denotes the action profile for all players other than player i. I.e, $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$.

(c) Preferences: preferences over action profiles are typically represented in utility form.

Remark 6. In Ellingsen, a game form is a social situation that includes the players, the strategies/actions, and the consequences. A game is a game form and the preferences. I.e, a game includes the utility functions necessary for the calculations later on. Consider the following Prisioner's dilemna, where person A can either sacrifice (-1 dollar to A, +2 dollars to B) or not (-0 dollars to A, +0 dollars to B), and B can do the same. This is a game form in Table 1.

	S	N
S	(2,2)	(0,3)
N	(3,0)	(1,1)

Table 1: Bilateral Sacrifice Situation (Consequences)

Now consider selfish players with utility function $U_i = c_i$, this is now a game!

	S	N
S	(2,2)	(0,3)
N	(3,0)	(1,1)

Table 2: Bilateral Sacrifice Game (Selfish Utility)

Consider now another game of altruistic players with α level of altruisim[1]:

	S	N
S	$(2 + 2\alpha, 2 + 2\alpha)$	$(3\alpha, 3)$
N	$(3\alpha, 3)$	$(1 + \alpha, 1 + \alpha)$

Table 3: Bilateral Sacrifice Game (Altruistic Utility)

Finally consider when both players are egalitarian[2]:

	S	N
S	(2,2)	(-3$\beta, 3 - 3\beta$)
N	(3 - 3$\beta, -3\beta$)	(1,1)

Table 4: Bilateral Sacrifice Game (Egalitarian Utility)

[1]Recall that the utility function of an altruistic player is $U_i = c_i + \alpha c_j$

[2]Recall that the utility function of an egalitarian player is $U_i = c_i - \beta|c_i - c_j|$

Definition 11. We say an action a_i'' **strictly dominates** a_i' if and only if for all a_{-i}, we have that

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i})$$

Remark 7. In Table 2, N strictly dominates S for every player.

Consider the battle of the sexes:

	Opera	Boxing
Opera	(2,1)	(0,0)
Boxing	(0,0)	(1,2)

Table 5: The Battle of the Sexes

In words, Nash equilibrium is everybody doing the best they can given what other people are doing:

Definition 12. An action profile $a^* \in A$ is a **Nash equilibrium** if and only if

$$u_i(a^*) \geq u_i(a_i, a_{-i}^*)$$

for all $a_i \in A_i$ for each player i.

Remark 8. Nash equilibrium corresponds to a steady state of the game, it embodies a stable social norm. No individual has any reason to deviate from it!

Thus, in the battle of the sexes, $a^* \in \{(\text{Opera-Opera}), (\text{Boxing-Boxing})\}$ is a Nash equilibrium.
Why is it reasonable? Why are the player's beliefs about other's actions reasonable?

(a) Repeated play

(b) Mediator

(c) Communication

(d) Focal point

Definition 13. We define a **best response** to be a set valued function such that

$$B_i(a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{i-1}) \geq u_i(a_i', a_{-i}) \quad \forall\, a_i' \in A_i\}.$$

Remark 9. In the battle of the sexes, we have that

$$B_1(\text{Box}) = \text{Box}, \quad B_1(\text{Opera}) = \text{Opera}$$

Proposition 1. An action profile $a^* \in A$ is a Nash equilibrium if and only if $a_i^* \in B_i(a_{i-1})$ for all $i \in [n]$.

Definition 14. An outcome is a **Pareto efficient** outcome if there is no other outcome such that every actor is at least as well of and at least one actor that is strictly better off. That is, if an outcome is PE, then moving to any other outcome results in players getting worse than before.

Remark 10. Nash equilibrium does not imply Pareto efficient (consider Table 2's equilibria of $(1, 1)$.)

Definition 15. A group engages in voluntary **cooperation** if each member of the group takes an action that benefits others (given what they do), even though that action is not certain to benefit oneself.

1.5 Wednesday, Jan 22: Models of Anarchy

Hobbes, ever the optimist, describes in *Leviathan* in 1651 his views on anarchy (Part I, Ch. 13, Para 9):

> ...wherein men live without other security, that what their own strength, and their own invention shall furnish them withall,...and the life of a man, solitary, poor, nasty, brutish, and short.

Thus, man cannot truly live without governance. Can we model this? Well I can't but Ellingsen can. Let there be two individuals, i and j, who can divide their labor-time (normalized to 1) between making food (x_i) and making weapons (y_i):

$$y_i = 1 - x_i.$$

The total food production is $x_1 + x_2 = x$ and the total weapons production is $y_1 + y_2 = y$. Both i and j are selfish[3] and $c_1 + c_2 = x$ since people cannot eat weapons[4] and all the food will be eaten. $x_{\max} = 2$ when $y = 0$ and vice versa:

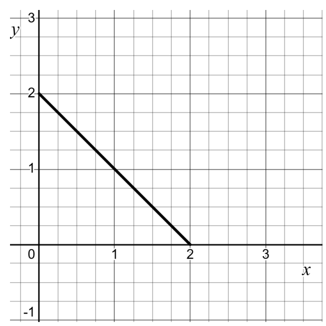

Figure 2: Production Possibilities

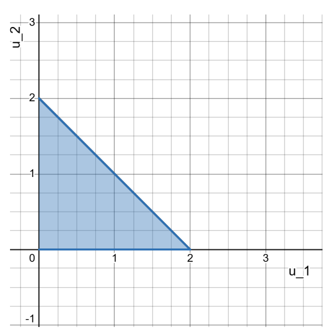

Figure 3: Consumption Possibilities

By Hobbes, the only thing that matters is strength, and thus the person with the most weapons gets all the food:

$$u_i(y_i, y_j) = \begin{cases} x, & y_i > y_j \\ \frac{x}{2}, & y_i = y_j \\ 0, & y_i < y_j \end{cases}$$

[3]Recall that the utility function of a selfish player is $U_i = c_1$

[4]Evidently Ellingsen is not American

Thus, the only Nash equilibrium is $y_1 = y_2 = 1$. Everybody spends all their time making weapons...

Remark 11. How do we fix this? Hobbes argued for a leviathan (a king/strong ruler with the power to control the people). Rousseau argues that humans are not governed by the selfish natured we imposed on them, and that we would need to establish a social contract in order for the few to not only benefit the few. Montesquieu argued that this government should be split in order to check and balance itself.[5]

Consider the following game:

	S	N
S	(2,2)	(0, 3)
N	(3,0)	(1,1)

Table 6: Pareto vs Nash

Here, we have that (N, N) is the Nash Equilibrium, but all the other outcomes are Pareto Efficient. Now lets consider this next game with rule compliance:

	S	N
S	(2,2)	(0, 3 - $\lambda\mu$)
N	(3 - $\lambda\mu, 0$)	(1 - $\lambda\mu, 1 - \lambda\mu$)

Table 7: Nash Equilibrium with Rule Compliance

Intuitively, when $\lambda\mu$ gets really large, the cost of not sacrificing grows immense. Thus, (S, S) is a Nash equilibrium when $\lambda\mu \geq 1$. If the inequality is an equality, then everything is an equilbria...

	S	N
S	$(2 + 2\alpha, 2 + 2\alpha)$	$(3\alpha, 3)$
N	(3, 3α)	$(1 + \alpha, 1 + \alpha)$

Table 8: Nash Equilibrium with Altruism

The Nash Equilibrium will change when $(2 + 2\alpha) \geq 3 \implies \alpha \geq \frac{1}{2}$, which is the tipping point. If more than the only (S, S). If equality, then all. If strictly less than, then only (N, N).

	S	N
S	(2,2)	(-3$\beta, 3 - 3\beta$)
N	(3 - 3$\beta, -3\beta$)	(1,1)

Table 9: Nash Equilibrium with Egalitarianism

Tipping points of $2 \geq 3 - 3\beta \implies 1 \leq 3\beta \implies \beta \geq \frac{1}{3}$ and $-3\beta \geq 1 \implies \beta \geq \frac{-1}{3}$.

Example 1.1. Consider the game where two players choose $x_1, x_2 \in [0, 100]$ and in which the player who chooses the closest to half of the mean wins. Players prefer winning to tying to losing.

(Existence) Consider the action profile $(0, 0)$. Then suppose WLOG that x_1 deviates to some $x > 0$. Then since $\frac{x}{4}$ is closer to 0 than to x, player 2 wins and thus both players choosing 0 is a Nash Equilibrium.

[5] Of course, both Rousseau and Montesquieu are French, so take them with a grain of salt. Hobbes is english, so ignore him. Listen to Tommy Jefferson.

(Uniqueness) Let (x_1, x_2) be the numbers chosen by the two players. Then the player closest to $\frac{x_1+x_2}{4}$ will win. If $x_2 < x_1$, then x_2 is closer to the required quantity.

Example 1.2. (Cournot Model)

- Two firms, $i = 1, 2$
- Each firm chooses a quantity $q_i \in \mathbb{R}^+$
- Firms prefer more profit to less
- Profit:
$$\pi_i = pq_i - cq_i = q_i(p - c),$$
where c is the marginal cost of production and price P is determined by the inverse demand function
$$p = \max[\alpha - q_1 - q_2, 0],$$
where $\alpha > c$.

Looking at best response function:

$$b_1 = \arg\max_{q_1} \pi_1 = \max_{q_1} q_1(\alpha - q_1 - q_2 - c).$$

is a convex function, and thus taking the derivative and setting equal to 0, we get that

$$q_1(q_2) = \frac{\alpha - c - q_2}{2}$$

is the best response for player 1. Symmetrically

$$q_2(q_1) = \frac{\alpha - c - q_1}{2}$$

is the best response for player 2. Best responses coincide at

$$q_1 = \frac{\alpha - c}{3}, \qquad q_2 = \frac{\alpha - c}{3}.$$

1.6 Monday, Jan 27: The Leviathan

Assume the presence of a (nonstrategic) state (the leviathan) that imposes a cost $\gamma > 1$ if any player i that chooses $y_i > 0$. Let players $1, 2$ be in the Ellingsen-Hobbes scenario, but now

$$u_i = c_i - \gamma \mathbf{1}_{y_i > 0},$$

then the Nash Equilibrium is $y_1 = y_2 = 1$.

1.7 Wednesday, Jan 29: The Tragedy of the Commons

Two players, $i = 1, 2$, and each player extracts $c_i \in [0, y]$ in period 1, where $y > 0$ us a stock of a common-property resource. There is a second period they also worry about:

$$u_i(c_1, c_2) = \ln(c_i) + \ln\left(\frac{y - c_1 - c_2}{2}\right).$$

Note that everything will be consumed by the end of period 2. Finding the Nash Equilibrium with the best response method, we must first find the best response of player 1 from the consumption of player 2 as c_2. Maximizizng the utility function u_1 :

$$0 = \frac{1}{c_1} - \frac{1}{y - c_1 - c_2} \implies b_1(c_2) = \frac{y - c_2}{2}$$

By symmetry,

$$b_2(c_1) = \frac{y - c_1}{2}$$

Thus, we get that setting them equal to each other, we find that the Nash Equilibrium is uniquely defined by:

$$(c_1^*, c_2^*) = (\frac{y}{3}, \frac{y}{3}).$$

Thus, the equilibrium utility is

$$u(c_1^*, c_2^*) = \ln\Big(\frac{y}{3}\Big) + \ln\Big(\frac{y}{6}\Big),$$

and thus the commons are overgrazed. Comparing this to the social optimum of

$$\max_{c_1, c_2}\left[\ln(c_1) + \ln(c_2) + 2\ln\left(\frac{y - c_1 - c_2}{2}\right)\right] \implies (c_1^W, c_2^W) = (\frac{y}{4}, \frac{y}{4}).$$

How do we solve this issue?

(a) privatization of the commons: Each player will receive $\frac{y}{2}$ of the commons:

$$u_1(c_1) = \ln(c_1) + \ln\left(\frac{y}{2} - c_1\right) \implies c_1^* = \frac{y}{4}$$

and equally,

$$c_2^* = \frac{y}{4}$$

If men were angels, no government would be necessary. If angels were to govern men, neither external nor internal controls on government would be necessary.

-Federalist 51

Monday, Feb 3: Coordination Games

Rousseau's Stag Hunt:

	S	H
S	(9,9)	(0,8)
H	(8,0)	(7,7)

Table 10: Stag Hunt

The weak link has a massive effect on performance. Solutions to coordination problems:

(a) Sending a costless message to the other players *might help*

(b) Have a boss (manager)

Definition 16. A message is *self committing* if the sender wants to do whatever the message entailed if the receiver believed the message.

Definition 17. A message is *self-signaling* if the sender intends to do what she says.

When is coordination not desirable? The game of chicken:

	H	D
H	(0,0)	(4,1)
D	(1,4)	(3,3)

Table 11: Chicken!

Common Knowledge Problem:

	Stay	Leave
Stay	(1,1)	(0,0)
Leave	(0,0)	(2,2)

Table 12: Should I Stay or Should I Go?

The above problem is solved with common knowledge. Creating Common Knowledge:

(a) Ads

(b) Meetings

(c) cc:

(d) Rituals

Wednesday, Feb 5: Midterm

(a) Consider an elaboration of a two players, two actions coordination game in which, prior to play of the game, one player can send a costless message stating which of the two actions she intends to play. Such an action is necessarily:

(i) Self-signaling but not self-committing.

(ii) Self-committing but not self-signaling.

(iii) Both self-signaling and self-committing.

(iv) Neither self-signaling nor self-committing.

Solution: b.

(b) The idea athat women with young children should stay at home is an exmaple of a One the one hand, two are stirctly dominant, but not pareto efficient

Monday, Feb 10: Mixed Strategy Games

Schelling: Coordination is more imagination than rationality.

Wednesday: Feb 13: Mixed-Strategy Games

Definition 18. A **strategic game with vNM preferences** is defined by

(a) Players

(b) Actions

(c) Preferences over lotteries over action profiles that may be represented by an expected payoff function.

Definition 19. A **mixed strategy** is a probability distribution over actions.

Definition 20. A **pure strategy** is a mixed strategy that assigns probability 1 to some action.

Definition 21. A mixed strategy profile, α^*, is a **mixed-strategy Nash equilibrium** if and only if

$$U_i(\alpha^*) \geq U_i(\alpha_i, \alpha^*_{-i}), \qquad \forall\, \alpha_i,\ \forall\, i \in [n].$$

	Left	Right
Left	(1,-1)	(-1,1)
Right	(-1,1)	(1,-1)

Table 13: Penalties with von Neumann

Suppose player 1 believes player 2 will play L with probability q and thus R with probability $(1-q)$. Thus:

$$u_1(L, q) = q(1) + (1-q)(-1) = 2q - 1$$

$$u_1(R, q) = q(-1) + (1-q)(1) = 1 - 2q$$

Plotting these gives:

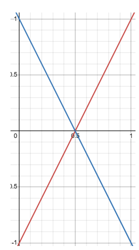

Figure 4: Red is $u_1(L)$, blue is $u_1(R)$

Thus, we see that $u_1(L) = u_1(R)$ when $q = \frac{1}{2}$. If $q < \frac{1}{2}$, then player 1's first response is to go right, and so $p = 1$ and thus player 1 has a pure strategy of R. If $q > \frac{1}{2}$, then player 1 wants to play L and thus $p = 1$. Thus, the best response function of player 1 is given by:

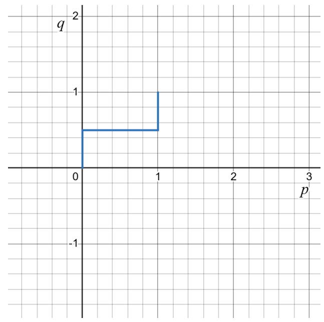

Figure 5: Best Response of player 1

Symmetrically, we find that the black graph is player 2s best response function:

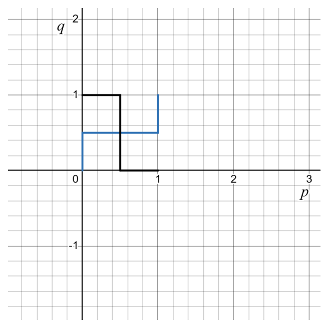

Figure 6: Best Response functions

Thus, we find that $(p^*, q^*) = (\frac{1}{2}, \frac{1}{2})$ or equivalently, $\left((\frac{1}{2}L, \frac{1}{2}R), (\frac{1}{2}L, \frac{1}{2}R)\right)$.

Theorem 2. Every strategic game with vNM preferences in which each player has finitely many actions has a mixed-strategy Nash equilibrium.

	O	B
O	(2,1)	(0,0)
B	(0,0)	(1,2)

Table 14: BoS with von Neumann

Example 1.3.

$$u_1(O) = 2q, \qquad u_1(B) = 1 - q \implies q' = \frac{1}{3}, u_1(O) = u_1(B)$$

$$u_2(O) = p, \qquad u_2(B) = 2(1-p) \implies p' = \frac{2p}{3}, u_2(O) = u_2(B)$$

Plotting the best response functions: And thus

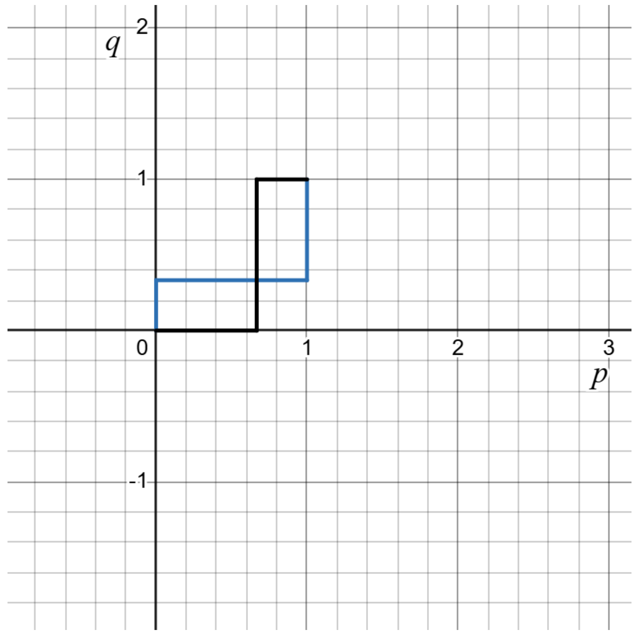

Figure 7: BoS Best Response Functions

$$(p^*, q^*) = \{(\frac{2}{3}, \frac{1}{3}), (0,0), (1,1)\}$$

1.8 Monday, Feb 17: Extensive Games

Definition 22. An **extensive game with perfect information** is defined by:

(a) Players

(b) Terminal Histories:

- *sub history:* For some sequence of actions $(a^1, a^2, \ldots, a^k)$, a sequence of actions, $(a^1, a^2, \ldots, a^m)$ with $m \in [k]$ or the empty history.
  - A *proper sub history* is a sub history such that $m < k$ or the empty history
- A *terminal history* is a sequence of actions that is not a proper sub history of any other sequence.
- A *history* is a sub history of some terminal history.

(c) Player function, which signs a player for every proper sub history of some terminal history

(d) Preferences over terminal histories.

Example 1.4. Prisoner's Dilemma: Players: 1,2 Terminal Histories: $(C,C), (C,D), (D,C), (D,D)$ Player function: $P(\emptyset) = 1, P(C) = 2, P(D) = 2$ Preferences: Obvious from picture

Definition 23. The **action space** for player assigned to history h is

$$A(h) := \{a \ : \ (h,a) \text{ is a history}\}$$

Example 1.5. In our example, $A(\emptyset) = \{C, D\}$.

Definition 24. A **strategy** s_i for player i assigns an action in $A(h)$ at each history h assigned to i (a complete contingent plan)

Example 1.6. $s_1(A(\emptyset)) = \{C, D\}$

$s_2(A(C)) = \{CC, CD\}$ or $s_2(A(D)) = \{DC, DD\}$ and so $s_2(A(h)) = \{CC, CD, DC, DD\}$

Definition 25. The **outcome** of s, $O(s)$, is the terminal history if each player i players s_i.

Example 1.7. Suppose $s = (C, CD)$, then $O(s) = (C, C)$

Suppose $S = (D, DC)$, then $O(s) = (D, C)$

Remark 12. In multistage games where a player plays more than once, think of the doofus friend when coming up with all the strategies, and so you need to have a strategy which has an action at *every* history where each player has a choice to make.

Definition 26. A strategy profile s^* is a Nash equilibrium if and only if

$$u_i(O(s^*)) \geq u_i(O(s_i, s^*_{-i})), \qquad \forall s_i, \forall i$$

Example 1.8. To find the Nash equilibrium of the multi-stage P.D. game above, we can write down the strategic form of the game:

	CC	CD	DC	DD
C	(2,2)	$(2^*, 2)$	$(0,3^*)$	$(0,3^*)$
D	$(3^*, 0)$	$(1,1^*)$	$(3^*, 0)$	$(1^*, 1^*)$

Table 15: Multi-Stage P.D. Strategic Form

For sub-game perfect Nash equilibrium, start at the end of the game, find optimal action, and prune the suboptimal actions and repeat.

1.9 Wednesday, Feb 19: Bargaining Games

Definition 27. A strategy profile s^* is a **subgame-perfect Nash equilibrium** if and only if in no subgame can any player i do better by choosing a strategy different from s_i^* given that every other player j adheres to s_j^*.

Remark 13. Note that SPNE are Nash Equilibrium.

hi

Example 1.9. (Ultimatum Bargaining Game)

- Player 1 makes an offerer of:

$$x = (x_1, x_2), \quad x_i = \text{share of pie for player } i$$

$$x_i, x_2 \in [0, 1]$$

$$x_1 + x_2 = 1$$

- Player 2 can accept or reject, if player 2 rejects, they both receive 0.
- Player's preferences represented by share of pie they receive.

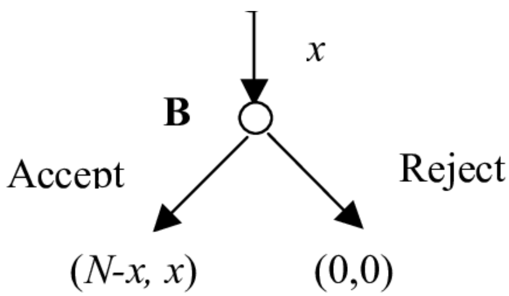

Figure 8: Ultimatum Bargaining Game in Extended Form, where $N = 1$ and x is x_1.

Obviously, it is a best response for player 2 to accept if $x_2 > 0$. If $x_2 = 0$, then they are both best responses. Thus, Nash equilibrium is $(1, 0)$. where $P2$ accepts if $x_2 > 0$ and accepts if $x_2 = 0$.

Monday, Feb 24: Repeated Game

Definition 28. A **stage game** G is a strategic game with actions $a_i \ni A_i$ and payoff function u_i for each player i.

Definition 29. A **repeated game** $G(T, \delta)$ is a stage game G for T periods where $T \in (0, \infty] \cap \mathbb{N}$. At each period t, each player chooses $a_i \in A_i$ having observed the full history. Preferences over terminal histories $(a^1, a^2, \dots)$ given by

$$\sum_{t=1}^{T} \delta^{t-1} u_i(a^t)$$

where a^t is the action profile played in period t and $\delta \in (0, 1)$.

Example 1.10. Play this game twice:

	C	D
C	(6,6)	(-1,8)
D	(8,-1)	(1,1)

Table 16: Finite Prisoner's Dilemma

There are 5 different subgames (one subgame per outcome of first game and the entire game)

To find SPNE, we solve by backwards induction. For each subgame in period 2, the Nash Equilibrium is (D, D). Thus, in the first period, we have that if δ is the discount factor, then

	C	D
C	$(6+\delta, 6+\delta)$	$(-1+\delta, 8+\delta)$
D	$(8+\delta, -1+\delta)$	$(1+\delta, 1+\delta)$

Table 17: Finite Prisoner's Dilemma, Backward's Induction

And so period 1 is (D, D). Thus,

$s_1* = (D, D|O(s^1) \in \{(C,C), (C,D), (D,C), (D,D)\})$, $\qquad s_2^* = (D, D)|O(s^1) \in \{(C,C), (C,D), (D,C), (D,D)\}$

	C	D	E
C	(6,6)	(-1,8)	(1,7)
D	(8,-1)	(**1**,**1**)	(4,0)
E	(7,1)	(0,4)	(**4**,**4**)

Table 18: Finitely Repeated Modified PD

Example 1.11. Suppose we play Table 18 twice. At time 2, NE are (D, D) and (E, E). Thus, SPNE:

$$s_i = \begin{cases} C, & t = 1 \\ E & t = 2|(C,C)@t = 1, \\ D, & \text{else} \end{cases}$$

since the 'temptation' is $7 - 6$ and the punishment is $(4 - 1)\delta$. Thus, in order for the players to coordinate, we need $3\delta \geq 1 \implies \delta \geq \frac{1}{3}$.

Suppose now Table 16 (classic PD) is repeated infinitely. Then if both players coordinate every period, we will have that

$$u_1 = 6 + 6\delta + 6\delta^2 + \cdots = 6\sum_{t=0}^{\infty} \delta^t = \frac{6}{1-\delta}$$

Definition 30. The **one deviation property** states that no player can increase her payoff by changing her action at the start of any subgame in which she is the first mover, given the other player's strategies and the rest of her own strategy.

Proposition 2. A strategy profile is a SPNE if and only if it satisfies the one-deviation property.

Wednesday, Feb 26: Infinitely Repeated Prisoner's Dilemma

Definition 31. In an infinitely repeating prisioner's dillema, a **grim trigger** strategy is for each player i,

$$s_i = \begin{cases} C, & \forall t \text{ if always } (C,C) \text{ previously} \\ D, & t > t_0 \text{ Where at } t_0 \text{ something other than } (C,C) \text{ was played} \end{cases}$$

Remark 14. In state-transition represenation, we have that

$$\boxed{\underline{C} : C} \xrightarrow{(D,C),(C,D),(D,D)} \boxed{\underline{D} : D},$$

where the right box is self absorbing (can't leave). We need to check SPNE in both states:

- ($\underline{D}$) Both players today recieve:

$$u_i = \sum_{i=1}^{\infty} \delta^i = \frac{1}{1-\delta}$$

For a player to not deviate, we need to have that the player should not benefit from switching to cooperating in this period, and thus

$$\frac{1}{1-\delta} \geq -1 + \delta + \delta^2 + \dots,$$

which is true. Thus, the one deviation property is met and so $\underline{D}$ is an SPNE.

- ($\underline{C}$) Both players recieve:

$$u_i = 6 + 6\delta + \dots = \frac{6}{1-\delta}$$

If a player deviates, then they recieve

$$u_i' = 8 + 1\delta + \delta^2 + \dots = 8 + \frac{\delta}{1-\delta}$$

Thus, we must have that

$$6 \geq 8(1-\delta) + \delta \iff \delta \geq \frac{2}{7}$$

Example 1.12. Consider the following infinitely repeated prisoner's dilemma:

	C	D
C	(6,6)	(-1,8)
D	(8,-1)	(1,1)

Table 19: Infinite Prisoner's Dilemma

To see that

$$\boxed{\underline{C} : C} \xrightarrow{(D,C),(C,D),(D,D)} \boxed{\underline{D} : D},$$

the state transitions are SPNE, we must have that

- ($\underline{D}$) : In the current state, we have that both players recieve

$$u_i = 0 + 0 + \dots = 0.$$

If they defect, then

$$u_i = -1 + 0 + \dots .$$

Thus, not defecting is optimal since $0 \geq -1$.

- ($\underline{C}$) In the current state, we have that both players recieve

$$u_i = 3 + 3\delta + \cdots = \frac{3}{1-\delta}$$

If they defect then

$$u_i' = 7 + 0 + \cdots = 7.$$

Thus, we must have that in order for $\underline{C}$ to be an SPNE, then

$$\frac{3}{1-\delta} \geq 7 \iff 3 \geq 7 - 7\delta \implies \delta \geq \frac{4}{7},$$

which is always true.

Definition 32. The **discounted average payoff** is

$$\overline{u}_\delta = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} w_t$$

Example 1.13. Suppose that in the prevoius example, the players oscillate bewteen periods by changing from (C, C) to (D, D), then

$$u_i = 3 + 0\delta + 3\delta^2 + \cdots = \frac{3}{1-\delta^2}.$$

Thus, we have that

$$\overline{u_\delta} = \frac{3}{1-\delta^2}(1-\delta) = \frac{3}{(1+\delta)(1-\delta)}(1-\delta) = \frac{3}{1+\delta}$$

Definition 33. The **feasible payoffs** are all the weighted averages payoffs profiles in a strategic game.

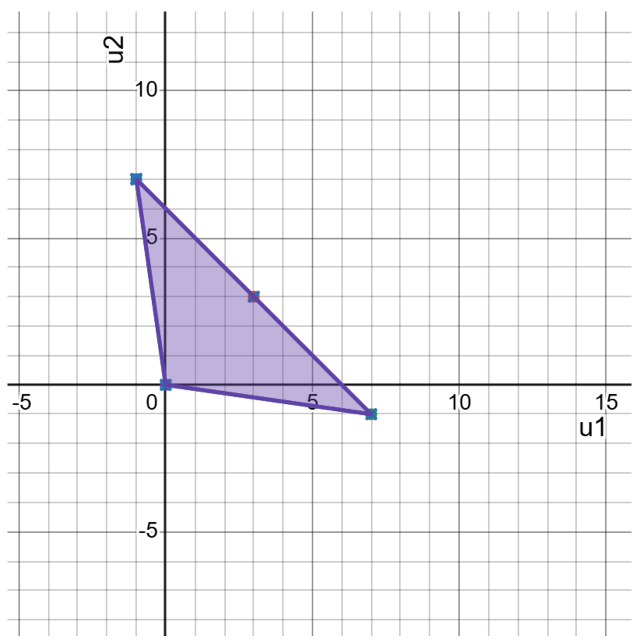

Figure 9: Feasible Payoffs

Example 1.14.

Theorem 3. (Folk Theorem) Let (x_1, x_2) be a pair of feasible payoffs in the stage game G for which x_i is greater than the worst payoff player i can receive in a Nash equilibrium of G. Then there exists $\overline{\delta} < 1$ such that if $\delta > \overline{\delta}$, the infinitely repeated game G has SPNE in which the discounted average payoff of each player i is x_i.

Proof. The proof is constructive. □

Ostrom's Design Principles for Self-Governance

(a) Clearly defined boundaries

(b) Congruence tween rules and local conditions

(c) Collective-choice arrangements

(d) Local monitoring

(e) Graduated sanctions

(f) Conflict-resolution dilemmas

(g) Minimal recognition of right to organize

(h) Nested enterprises

Definition 34. (a) **First-level dilemma** Incentives to underinvest, overfish , . . . , can be solved through self-governance. But!

(b) **second-level dilemma** institutions themselves are public goods, and thus have an incentive to underinvest.

1.10 Monday, March 3: Enforcers

11- Pick, 1- Keller

S	(0,3)
NS	(1,0)

Table 20: Rowena (First Stage)

	R	NR
	(2,0)	(0,2)

Table 21: Colin (Second Stage)

P	(-ρ, π)
NP	(0,0)

Table 22: Rowena (Third Stage)

Suppose the game is played only once, then using backwards induction to find SPNE:

(a) Stage 3: NP ($0 \geq -\rho$)

(b) Stage 2: NR ($2 \geq 0$)

(c) Stage 3: NS ($1 \geq 0$)

Strategy:

- Stage 1: Play S is previously (S, R) or (S, NR, P), otherwise NS.
- Stage 2: Play R is S this period and always either (S, R) or (S, NR, P), otherwise NR.
- Play P if (S, NR) this period and always previously (S, R) or (S, NR, P), otherwise (N, P)
- Stage 3:

SPNE:

- Stage 1: current strategy:

  $$u_R = 2 + 2\delta + \cdots$$

  Deviating:

  $$u'_R = 1 + 1\delta + 1\delta^2 + \cdots$$

  $u_R > w\prime_{R,}$ and so it is not profitable for Rowena to deviate.

- Stage 2:

$$u_C = 1 + \delta + \delta^2 + \cdots$$

Deviating:

$$u'_C = 2 - \pi + 1\delta + 1\delta^2 + \cdots$$

Which will hold for $\pi \geq 2$.

- Stage 3

$$u_R = -\rho + 2\delta + 2\delta^2 + \cdots = -\rho + \frac{2\delta}{1-\delta}$$

Deviating

$$u'_R = 1 + 1\delta + 1\delta^2 + \cdots = \frac{\delta}{1-\delta},$$

which will hold for $\rho \leq \frac{\delta}{1-\delta}$

F	(-f,0, f)
NF	(0,0, 0)

Table 23: Rowena (First Stage)

S	(0,3, 0)
NS	(1,0, 0)

Table 24: Rowena (Second Stage)

	R	NR
	(2,0, 0)	(0,2, 0)

Table 25: Colin (Third Stage)

P	(0,-π, $-\rho$)
NP	(0,0,0)

Table 26: Rowena (Third Stage)

Suppose Enfo has m clients, then consider:

- Stage 1: Play F if never NP after F and NR. Otherwise NF.
- Stage 2: Play S if F this period, else NS.
- Stage 3: Play R if (F, S) this period, else NR.
- Play P if (F, S, NR), else NP.

Solving:

- Stage 1:
$$u_R = \delta \frac{-f+2}{1-\delta}$$
$$u'_R = 1 + 1 - f - \delta \frac{-f+2}{1-\delta}$$
Thus, $f \leq 1$.
- Stage 2: $\pi \geq 2$.
- Stage 3: $p \leq mf \frac{\delta}{1-\delta}$.

1.11 Wednesday, March 5: Coase Theorem

	Contribute	Not
Contribute	2v-c, 2v-c	v-c + $\pi, v - \pi$
Not	v - $\pi, v - c + \pi$	0,0

Table 27: Illustrating Coase ($v < c < 2v$)

(C, C) is Nash when $\pi \geq c - v$. Consider π to be the contract signed by players punishing for not contributing and benefiting to contribute. Some actor must enforce contract (might need a helping hand).

With more than two actors, consider 3 actors with $v < c < 2v$. The timing is as follows:

(1) Negotiate/Not

(2) Negotiating actors write a contract (π)

(3) Contribute/Not

The result is that any actor is better off not negotiating. We can fix this with a state!

S	(0,1)
NS	(2,0)

Table 28: Rowena (First Stage)

	R	NP
	(0,0)	(-$\pi, -\rho$)

Table 29: Colin (Second Stage)

Suppose $\pi > 2$ and $\rho < \frac{\delta}{1-\delta}$. Consider the strategy:

(a) Rowena plays S if always S or always P after NS. Else NS.

(b) Colin plays P if NS this period and never NP after NS. Otherwise, NP.

To see if this is SPNE:

(a)
$$u_R = 0, \quad u_R' = 2 - \pi + 0$$
and thus Rowena sacrifices if $0 \geq 2 - \pi$, which is always true.

(b)
$$u_C = -\rho + \delta + \delta^2 = -\rho + \frac{\delta}{1-\delta}, \quad u_C' = 0 + 0 + \cdots$$
and thus Colin punishes if $\rho \leq \frac{\delta}{1-\delta}$.