Solutions to some Project Euler Problems

Problem 26

$$\frac{1}{x} = 0.a_0 a_1 \dots (a_j \dots a_n)$$

Where the () indicate the period.
Then we can define:

$$\alpha_0 = \frac{10}{x}, \quad \alpha_n = \lfloor \alpha_n \rfloor + \frac{\alpha_{n+1}}{10}$$

$$\alpha_n = \frac{10 b_{n-1}}{x}$$

$$b_{n+1} = (10 * b_n) \mod x, \quad b_{-1} = 1$$

And the a_n's are given by $a_n = \lfloor \alpha_n \rfloor$

Example: Take $x = 7$. Then:

$$\alpha_0 = \frac{10}{7}, \quad a_0 = 1, \quad b_0 = 3$$

$$\alpha_1 = \frac{30}{7}, \quad a_1 = 4, \quad b_1 = 2$$

$$\alpha_2 = \frac{20}{7}, \quad a_2 = 2, \quad b_2 = 6$$

$$\alpha_3 = \frac{60}{7}, \quad a_3 = 8, \quad b_3 = 4$$

$$\alpha_4 = \frac{40}{7}, \quad a_4 = 5, \quad b_4 = 5$$

$$\alpha_5 = \frac{50}{7}, \quad a_5 = 7, \quad b_5 = 1$$

$$\alpha_6 = \frac{10}{7} = \alpha_0, \quad a_6 = 1 = a_0, \quad b_6 = 3 = b_0$$

So that $\frac{1}{7} = 0.(142857)$.

We also have

$$\frac{1}{10}\alpha_0 = \frac{1}{x} = \frac{1}{10}\left(a_0 + \frac{\alpha_1}{10}\right) = 0.a_0 + \frac{1}{10^2}\left(a_1 + \frac{\alpha_2}{10}\right)$$

$$= 0.a_0 a_1 \ldots a_{n-1} + \frac{\alpha_n}{10^{n+1}}$$

When $\alpha_n = \alpha_0 = \frac{10}{x}$

$$\frac{1}{x} = 0.a_0 a_1 \ldots a_{n-1} + \frac{1}{10^n x} \to \frac{1}{x} = \frac{0.a_0 a_1 \ldots a_{n-1}}{1 - 1/10^n}$$

Since $\frac{1}{1-1/10^n} = \sum_{k\geq 0} \frac{1}{10^{kn}}$. Then

$$\frac{1}{x} = \frac{0.a_0a_1\ldots a_{n-1}}{1-1/10^n}$$

$$= 0.a_0a_1\ldots a_{n-1} + (0.a_0a_1\ldots a_{n-1}) \times 10^{-n} + \ldots$$

$$= 0.(0.a_0a_1\ldots a_{n-1})$$

We repeat the code until $b_n = b_j$ for some $j < n$; then the period of $1/x$ is $n-j$.

$$\alpha_{n+1} = \frac{10b_n}{x} = \frac{10b_j}{x} = \alpha_{j+1} \to a_{n+1} = a_{j+1}$$

$$\to \frac{1}{x} = 0.a_0a_1\ldots a_ja_{j+1}\ldots a_{n+1}a_{n+2}\ldots$$

$$= 0.a_0\ldots a_j(a_{j+1}\ldots a_n)$$

Problem 38

We have $x*1, x*2, \ldots x*n$ for some natural n. We then form the number $x2x3x\ldots nx$ such that the total number of digits in this last number is 9.

0.1 Code

Let Ω be the ordered set of all permutations to $1, 2, \ldots 9$. Starting from the last (biggest) element and moving to the first we find numbers in the form $abcdefghi$, we firstly grab a, and check whether $2\times a$ appears in the beginning of the string bcdefghi; otherwise take ab and repeat, until we reach the number abcd. For a if 2a is in the list, eliminate it and check whether 3a is in the list if len(3a) > len(abcdefghi)-len(a)-len(2a) Same with ab, abc, abcd.

1 Problem 39

$$(a,b,c)/a+b+c=p, a^2+b^2=c^2 \to (a,b,c)=(3,4,5) \to p \geq 12$$

Then

$$a^2+b^2=(p-a-b)^2 \to 0=p^2-2pa-2pb+2ab$$

$$b=\frac{p^2-2pa}{2p-2a}=\frac{p}{2}\frac{p-2a}{p-a}$$

We check if $b \in \mathbb{N}$.

$$b>0 \to a<\frac{p}{2} \to 1 \leq a < \frac{p}{2}$$

Also assume a ¡ b ¡ c:

$$a<b=\frac{p}{2}\frac{p-2a}{p-a} \implies 2ap-2a^2<p^2-2ap$$

$$0<(a-p)^2-\frac{p^2}{2}$$

$$|a-p|>\frac{p}{\sqrt{2}}$$

$$p-a>\frac{p}{\sqrt{2}}$$

$$a<p\left(1-\frac{1}{\sqrt{2}}\right)<\frac{p}{2}$$

Hence $1 \leq a < p\left(1-\frac{1}{\sqrt{2}}\right)$

2 Problem 65

$e = [2; 1, 2, 1, \dots 1, 2k, 1, \dots] \to c_n = \dfrac{p_n}{q_n}; \quad a_1 = 2, \quad a_{3k} = 2k, \quad a_{3k\pm1} = 1, \quad k \geq 1$

$c_1 = \dfrac{2}{1}, \quad c_0 = \dfrac{1}{0}, \quad p_n = a_n p_{n-1} + p_{n-1}, \quad p_0 = 1, \quad p_1 = 2$

Note:

$$\begin{aligned} p_{3k} &= 2kp_{3k-1} + p_{3k-2} \\ p_{3k\pm1} &= p_{3k\pm1-1} + p_{3k\pm1-2} \end{aligned}$$

Problem 64

$$\sqrt{m} = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\dots\frac{1}{a_n+}\frac{1}{a_1+}\frac{1}{a_2+}\dots$$

$$= [a_0; (a_1, \dots a_n)]$$

Now:

$$\begin{aligned} b_0 &= \sqrt{m}; \quad b_{n+1} = \frac{1}{\{b_n\}}; \quad b_1 = \frac{1}{m - b_0} = \frac{\sqrt{m} + \lfloor\sqrt{m}\rfloor}{m - \lfloor\sqrt{m}\rfloor^2} \\ b_0 &= \frac{\sqrt{m}\cdot 1 + 0}{1}. \end{aligned}$$

Where $\{x\} = x - \lfloor x \rfloor$. And $a_n = \lfloor b_n \rfloor$

If $b_n = \frac{\sqrt{m}+d_n}{c_n} \to b_{n+1} = \frac{\sqrt{m}+d_{n+1}}{c_{n+1}}; \quad n \geq 0$ Now

$$\begin{aligned} b_{n+1} &= \frac{1}{\frac{\sqrt{m}+d_n}{c_n} - \lfloor b_n \rfloor} \\ &= \frac{c_n(\sqrt{m} + c\lfloor b_n \rfloor - d_n)}{m - (c * \lfloor b_n \rfloor - d_n)^2} \\ &\to d_{n+1} = c_n\lfloor b_n \rfloor - d_n, \quad d_0 = 0, \qquad c_{n+1} = \frac{m - d_{n+1}^2}{c_n}, \quad c_0 = 1 \end{aligned}$$

Problem 58

Going along the spiral, defining $a_0 = 1$ and a_n to be the next number that is on a diagonal we have:

$$(a_n)_n = (1, \quad 3, 5, 7, 9, \quad 13, 17, 21, 25, \quad 31, 37, 43, 49, \ldots)$$

Note that numbers 3 to 9 have a common difference of 2, 13 to 25 a difference of 4, 31 to 49 a difference of 6, and so on. Every 1+4*k term the common difference increments by 2.
Notice that

$$a_n = a_{n-1} + l_{n-1} \rightarrow a_n = a_0 + \sum_{j=1}^{n}(l_j - 1) = 1 + \sum_{j=1}^{n}(l_j - 1)$$

For $k \leq n \leq k+4; \quad k = 1 \mod 4$ we have $l_n = \frac{k-1}{4} * 2 + 3 = \frac{k+5}{2}$.
And

$$k = n-(n-1) \mod 4 = 1+n-1-(n-1) \mod 4 = 1+4*\left\lfloor \frac{n-1}{4} \right\rfloor = 4\left\lfloor \frac{n+3}{4} \right\rfloor - 3$$

Hence

$$\begin{aligned} a_n &= 1 + \sum_{j=1}^{n}(l_j - 1) = 1 + \sum_{j=1}^{n} \frac{k+3}{2} \\ &= 1 + 2\sum_{j=1}^{n} \left\lfloor \frac{j+3}{4} \right\rfloor \end{aligned}$$

For a square of size $l \times l$, there are

$$1 + 4\Big(\frac{l-1}{2}\Big) = 2l - 1$$

diagonals.
And there are

$$\sum_{a_n prime} 1; \quad \text{for } n \leq 4 * \left(\frac{l-1}{2}\right) = 2(l-1)$$

prime numbers. Note that if $4|n$

$$s_n = \sum_{j=1}^{n} \left\lfloor \frac{j+3}{4} \right\rfloor = \left\lfloor \frac{4}{4} \right\rfloor + \left\lfloor \frac{5}{4} \right\rfloor + \left\lfloor \frac{6}{4} \right\rfloor + \left\lfloor \frac{7}{4} \right\rfloor + \left\lfloor \frac{8}{4} \right\rfloor + \dots \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n+3}{4} \right\rfloor$$

$$= 1 + 1 + 1 + 1 + 2 + \dots + \frac{n}{4} + \frac{n}{4} + \frac{n}{4} + \frac{n}{4}$$

$$= 4\left(1 + 2 + \dots + \frac{n}{4}\right) = 4 * \frac{(n/4)(n/4+1)}{2}$$

If $4|n-1$

$$s_n = 1 + 1 + 1 + 1 + \dots + \left\lfloor \frac{n-1}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n+3}{4} \right\rfloor$$

$$= 4\left(1 + \dots \frac{n-1}{4}\right) + \frac{n+3}{4}$$

$$= 4 * \frac{((n-1)/4)((n-1)/4+1)}{2} + \frac{n+3}{4}$$

Similarly if $4|n-2$

$$s_n = 4\frac{((n-2)/4)((n-2)/4+1)}{2} + 2\frac{n+2}{4}$$

if $4|n-3$

$$s_n = 4\frac{((n-3)/4)((n-3)/4+1)}{2} + 2\frac{n+1}{4}$$

So in general:

$$s_n = 4\left(1 + \dots \left\lfloor \frac{n}{4} \right\rfloor\right) + (n \mod 4)\left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right)$$

$$= \frac{4}{2}\left\lfloor \frac{n}{4} \right\rfloor \left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right) + (n \mod 4)\left\lfloor \frac{n+4}{4} \right\rfloor$$

$$= 2\left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n+4}{4} \right\rfloor + (n \mod 4)\left\lfloor \frac{n+4}{4} \right\rfloor$$

$$= \left\lfloor \frac{n+4}{4} \right\rfloor \left(2 * \left\lfloor \frac{n}{4} \right\rfloor + (n \mod 4)\right)$$

$$= \left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right)\left(2 * \left\lfloor \frac{n}{4} \right\rfloor + (n \mod 4)\right)$$

Hence

$$a_n = 1 + 2s_n = 1 + 2\left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right)\left(2 * \left\lfloor \frac{n}{4} \right\rfloor + (n \mod 4)\right)$$

The percentage of primes for a $l \times l$ square is :

$$\frac{1}{2l-1} \sum_{a_n prime, 0 \le n \le 2(l-1)}$$

Problem 57

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+\dots}} \to d_n = 1 + b_n$$

Let the n-th (approx-1) be $b_n = a_n/c_n$, then:

$$b_{n+1} = \frac{1}{2 + b_n} = \frac{c_n}{a_n + 2c_n} = \frac{a_{n+1}}{c_{n+1}}$$
$$\to a_{n+1} = c_n; \quad c_{n+1} = a_n + 2c_n \quad (\text{Note: } c_{n+1} - 2c_n - c_{n-1} = 0) \quad c_0 = 1, \quad c_1 = 2$$

Now $b_n = \frac{a_n}{c_n} = \frac{c_{n-1}}{c_n} \to d_n = 1 + b_n = \frac{c_n + c_{n-1}}{c_n}$
We look for n's such that $digits(c_n + c_{n-1}) > digits(c_n)$.

3 Problem 243

$$R(d) = \frac{\phi(d)}{d-1} = \frac{d}{d-1} \prod_{p|d} \left(1 - \frac{1}{p}\right) < \frac{a}{b} < 1$$
$$\to bd \prod (p-1) < a(d-1) \prod p = ad \prod p - a \prod p$$
$$\to a \prod p < d\left(a \prod p - b \prod (p-1)\right)$$
$$\to d > \frac{a \prod p}{a \prod p - b \prod (p-1)}$$

Also since $1 < \frac{d}{d-1} \to \prod_{p|d}(1-\frac{1}{p}) < \frac{a}{b}$. Or

$$\frac{a}{b} - \prod\left(\frac{p-1}{p}\right) > 0 \to a\prod p > b\prod(p-1) \leftarrow \text{(for the code)}$$

$$\frac{a}{b} - \prod_{p\le p_n}\left(1-\frac{1}{p}\right) > 0 \leftarrow \text{((1) in practice)}$$

The least d is the number bigger than $\eta = \left\lceil \frac{a\prod p}{a\prod p - b\prod(p-1)} \right\rceil$ which is divisible by $p_1, \dots, p_n$ which make (1) true.

Example

For $\frac{a}{b} = \frac{4}{10}$ we have $\frac{4}{10} - (1-\frac{1}{2})(1-\frac{1}{3}) > 0$. And $\eta = 6 = 2*3$, then $d = 2*2*3 = 12$ (next divisor of 2*3).

For $\frac{a}{b} = 0.32 \to \frac{a}{b} - \prod_{p\le 5}(1-\frac{1}{p}) > 0$, and $\eta = \left\lceil \frac{a/b}{a/b - \prod_{p\le 5}(1-\frac{1}{p})} \right\rceil = 6 = 2*3 \to$ $d = 2*3*5 = 30$ (first divisor of $\prod_{p\le 5} p$).

For $\frac{a}{b} = \frac{15499}{94744} \to \frac{a}{b} - \prod_{p\le 23}(1-\frac{1}{p}) > 0$, and $\eta = 805,994,497$. The least divisor of $\prod_{p\le 23}(1-\frac{1}{p})$ is $223,092,870$, its multiples are:

$$\prod p = 223,092,870 < \eta$$
$$2\prod p = 446,185,740 < \eta$$
$$3\prod p = 669,278,610 < \eta$$
$$4\prod p = 892,371,480 >= \eta$$

So the answer is $4\prod p = 892,371,480$.

For the code:

We look for the prime p_m s.t.

$$\frac{a}{b} - \prod_{p\le p_m}\left(1-\frac{1}{p}\right) > 0 \equiv a\prod p - b\prod(p-1) > 0$$

Having found p_m, we calculate $\eta = \left\lceil \frac{a\prod p}{a\prod p - b\prod(p-1)} \right\rceil$ and letting $d_1 = \prod_{p\le p_m} p,\ d_n = nd_1$, we find the least d_n for which $d_n > \eta \to n > \frac{\eta}{d_1} \to n = \lceil \frac{\eta}{d_1} \rceil$. The answer therefore is: $d_n = d = \lceil \frac{\eta}{d_1} \rceil d_1,\ d_1 = \prod_{p \le p_m} p$.

$$\rightarrow d = \left\lceil \frac{a}{a \prod p - b \prod (p-1)} \right\rceil \prod p$$

Problem 120

$$\begin{aligned} r_n(a) &= ((a-1)^n + (a+1)^n) \mod a^2 \\ \rightarrow S_n &= (a-1)^n + (a+1)^n = \sum_k \binom{n}{k}(1+(-1)^k)a^{n-k} \\ &= 2\sum_k \binom{n}{2k} a^{n-2k} = 2(\binom{n}{0}a^n + \binom{n}{2}a^{n-2} + \ldots) \end{aligned}$$

Hence

$$S_{2n} = 2(\binom{2n}{0}a^{2n} + \binom{2n}{2}a^{2(n-1)} + \ldots + \binom{2n}{2n}) \equiv 2 \mod a^2$$
$$S_{2n+1} = 2(\binom{2n+1}{0}a^{2n+1} + \binom{2n+1}{2}a^{2n-1} + \ldots + \binom{2n+1}{2n}a) \equiv 2\binom{2n+1}{2n} \mod a^2$$

So that

$$\begin{aligned} r_{2n} &= 2 \\ r_{2n+1} &= 2a(2n+1) \mod a^2 \qquad (1) \end{aligned}$$

We search for the maximum n such that (1) is maximum. Now we have $a|r_n$ and then $\frac{r_{2n+1}}{a} = 2(2n+1) \mod a$. So that

$$r_{max}(a) = a * max(2(2n+1) \mod a : n \in \mathbb{N})$$

For some $k \in \mathbb{N}$, choose $2(2n+1) = ka - d, d \ge 1$ So that $d + 2 = ka - 4n$. (0)

For there to be some solution (k, n) we must have $(a, 4)|d + 2$, since $\eta = (a, 4) \in \{1, 2, 4\}$. For $\eta = 1$ we choose $d = 1$, otherwise $d = 2$.(2)

Due to the euclidean algorithm, under conditions (2) we can always find a solution to (0), hence $2(2n+1) \equiv a - d \mod a$, and $r_{max}(a) = a(a -$

$2), 2|a, \quad r_{max}(a) = a(a-1), otw.$

So for example if $a = 7$, (0) becomes $3 = 7k - 4n$ with solution $k = n = 1 \rightarrow r_{max} = 7 * 6 = 42$
For $a = 504; (504, 4) = 4 \rightarrow 4 = 504k - 4n \rightarrow 1 = 126k - n \rightarrow k = 1, n = 125 \rightarrow 2(2n+1) = 502 \equiv 502 \mod a$. Therefore $r_{max} = 504*502 = 253,008$.
We also have

$$\sum_{a=3}^{N} r_{max} = \frac{1}{12}(4N^3 - 3N^2 - 10N) = \frac{N}{12}(4N^2 - 3N - 10) = \frac{N}{12}(N-2)(4N+5)$$

Problem 40

If one takes the numbers 1 to x and concatenates them like so:

1234567891011...

The length of this integer is

$$S = \sum_{j=1}^{n} 9j \cdot 10^{j-1} + (n+1)(x - 10^n + 1)$$

Where $n = \lfloor \log_{10} x \rfloor$ (i.e. $10^n \leq x < 10^{n+1}$) . We require this sum to be at least 10^6. So for $n = 5$, and taking $S \geq 10^6$. We get $x \geq 185186$, with a length of $S \geq 1000005$.

Problem 69

Since

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Then

$$\frac{n}{\phi(n)} = \prod_{p|n} \left(1 - \frac{1}{p}\right)^{-1}.$$

This ratio is maximum when $\prod_{p|n} 1$ is maximum. For this we choose the first prime numbers whose product does not exceed n.

Problem 101

Using Lagrange's interpolation formula we obtain that given the first k terms of a sequence $U_1, U_2, ..., U_k$ the value of $OP(k,x)$ is given as:

$$OP(k,x) = \sum_{j=1}^{k} \frac{U_j(-1)^{k+j}}{(j-1)!(k-j)!} \cdot \prod_{0 \le h \le k, h \ne j} (x-h)$$

The sum of all the FIT's is given by:

$$S = \sum_{t=1}^{k} OP(k, k+1)$$