A quantum *battle of the sexes* cellular automaton

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The dynamics of a spatial quantum formulation of the iterated *battle of the sexes* game is studied in this work. The game is played in the cellular automata manner, i.e. with local and synchronous interaction. The effect of spatial structure is assessed in the quantum versus quantum players contest as well as in the unfair quantum versus classical players contest. The case of partial entangling is also scrutinized.

1. Introduction: the non-quantum context

The so-called *battle of the sexes* (BOS) is a simple example of a two-person asymmetric (or bi-matrix) game, i.e. a game whose payoff matrices are not coincident after transposition (Maynard Smith 1982; Hofbauer & Sigmund 2003). In this game, the preferences of a conventional couple are assumed to fit the traditional stereotypes: the male prefers to attend a Football match, whereas the female prefers to attend a Ballet performance. Both players (which are treated symmetrically) decide in the hope of getting together, so that their payoff matrices are given in table 1a, with rewards $R > r > 0$. There are both coordination and conflict elements in the BOS game. While both players want to go out together, the *conflict* element is present because their preferred activities differ, and the *coordination* element is present because they may end up going to different events. Thus, in the absence of preplay communication, it is natural to expect that coordination failure (of ending up in one of the inefficient outcomes) will occur frequently.
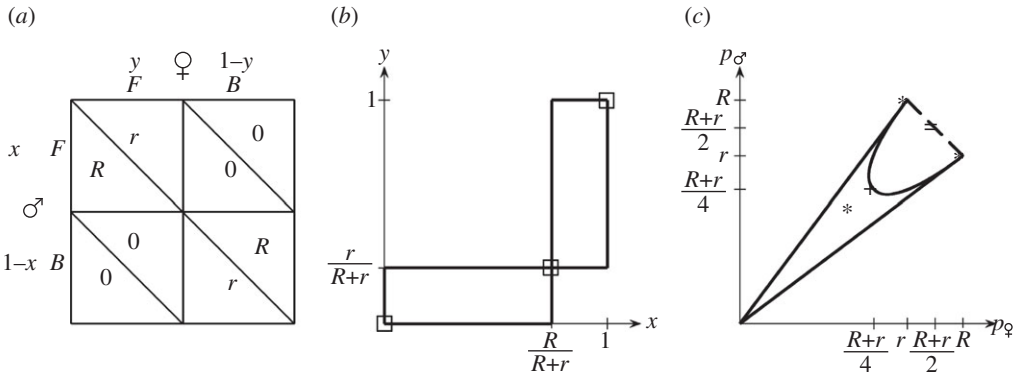
(a) Uncorrelated strategies

The expected payoffs (p) in the BOS game, using uncorrelated mixed probabilistic strategies $(x, 1-x)$ and $(y, 1-y)$, are as follows:

$$p_{\male}(x;y) = (x, 1-x) \begin{pmatrix} R & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = ((R+r)y - r)x + r(1-y) \quad (1.1a)$$

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Table 1. (*a*) The payoff matrices, (*b*) reaction correspondences and (*c*) payoff region in the *battle of the sexes* game.

and

$$p_{♀}(y; x) = (x, 1-x) \begin{pmatrix} r & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = ((R+r)x - R)y + R(1-x). \quad (1.1b)$$

The pair of strategies $((x, 1-x), (y, 1-y))$ are in Nash equilibrium if x is a best response to y, and y is a best response to x. Formally, $p_A(x, y) \geq p_A(z, y)$ and $p_B(x, y) \geq p_B(x, z)$, $\forall z$. Nash equilibria are shown with points where the correspondences of the two players agree, i.e. cross, in the reaction correspondence graphs. Thus, according to the reaction correspondences given in table 1, the pairs of strategies in Nash equilibrium are the pure, $(0,0)$ and $(1,1)$, and the mixed, $(x^* = R/(R+r), y^* = r/(R+r))$. In the latter, $p_{♂}(x^*, y^*) = p_{♀}(x^*, y^*) = rR/(R+r) < r < R$.

Both players get the same payoff if $y = 1 - x$, in which case, $p = (R+r)(1-x)x$. This egalitarian payoff is maximum when $x = y = 1/2$, with $p^+ = (R+r)/4$, the point marked with a plus symbol ('+') in panel (*c*) of table 1. Thus, the set of payoffs that can be obtained by both players (or payoff region) is closed by the parabola passing by (R, r), (r, R) and (p^+, p^+), as shown in panel (*c*) of table 1.

(*b*) *Joint probability distribution*

In a broader game scenario, a probability distribution $\mathbb{A} = (a_{ij})$ assigns probability to every combination of player choices; so $\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in 2×2 games (Owen 1995). Thus, the expected payoffs in the BOS are

$$p_{♂} = a_{11}R + a_{22}r$$

and

$$p_{♀} = a_{11}r + a_{22}R.$$

The probability distribution $\mathbb{A}$ is in *correlated equilibrium* (Aumann 1974, 1987) if the players cannot gain by disobeying the signals given by this randomization

device. Thus, in the BOS game, $\mathbb{A} = \left(\begin{smallmatrix} a & 0 \\ 0 & 1-a \end{smallmatrix} \right)$ is in correlated equilibrium, giving

$$p_{\mathcal{O}^{\!\top}} = aR + (1-a)r$$

and

$$p_{\mathcal{Q}} = ar + (1-a)R,$$

so that the payoff region limited by the parabola and the segment that joins (R, r) and (r, R) becomes accessible. In this scenario, both players reach a maximum egalitarian payoff $p^{=} = (R + r)/2$ (the point marked '=' in the payoff region of table 1), with $a = 1/2$, i.e. fully discarding the mutually inconvenient FB and BF combinations and adopting FF and BB with equal probability.

The paper is structured as follows. In §2, a brief introduction to quantum games is given. The spatialized, cellular-automata (CA)-like BOS is studied in §3: first in its classical form and then in its quantum formulation. The unfair situation in which one type of players is restricted to classical strategies, whereas the other one is allowed to use quantum ones is considered in §4. The case of partial entangling is analysed in §5. Conclusions and potential areas of further study are discussed in §6.

2. Quantum games

In the quantization scheme introduced by Eisert *et al.* (1999), the classical strategies F and B are assigned two basic vectors $|0\rangle$ and $|1\rangle$, respectively, in a Hilbert space of a two-level system. The state of the game at any instant is a vector in the tensor product space spanned by the basis vectors $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, where the first entry in the key refers to the male player (termed A in the general scheme) and the second entry refers to the female player (termed B in the general scheme).

The game protocol starts with an initial entangled state $|\psi_i\rangle = \hat{J}|00\rangle$, where $\hat{J}$ is a unitary operator that *entangles* the players qubits and which is known to both players. The operator $\hat{J}$ is symmetric for fair games. The players perform their quantum strategies as local unitary operators ($\hat{U}$) belonging to a particular two-parameter subset of SU(2),

$$\hat{U}(\theta, \alpha) = \begin{pmatrix} e^{i\alpha} \cos\left(\dfrac{\theta}{2}\right) & \sin\left(\dfrac{\theta}{2}\right) \\ -\sin\left(\dfrac{\theta}{2}\right) & e^{-i\alpha} \cos\left(\dfrac{\theta}{2}\right) \end{pmatrix}, \quad \theta \in [0, \pi], \; \alpha \in \left[0, \dfrac{\pi}{2}\right].$$

With classical strategies, $\hat{D} = \hat{U}(\pi, 0) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$, $\hat{I} = \hat{U}(0, 0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$.

To ensure that the classical game is a subset of its quantum version, it is necessary that $\hat{J} = \exp(i(\gamma/2)\hat{D}^{\otimes 2})$, where $\gamma \in [0, \pi/2]$ (table 2).

The strategic moves of both players are the unitary operators $\hat{U}_A(\theta_A, \alpha_A)$ and $\hat{U}_B(\theta_B, \alpha_B)$. After the application of these strategies, which the players chose independently, the state of the game evolves to $|\psi_{f_o}\rangle = (\hat{U}_A \otimes \hat{U}_B)\hat{J}|00\rangle$. Prior to measurement for finding the payoffs of the players, a reversible two-bit gate $\hat{J}^{\dagger}$ is

Table 2. The Eisert protocol. Adapted from Eisert *et al.* (1999).
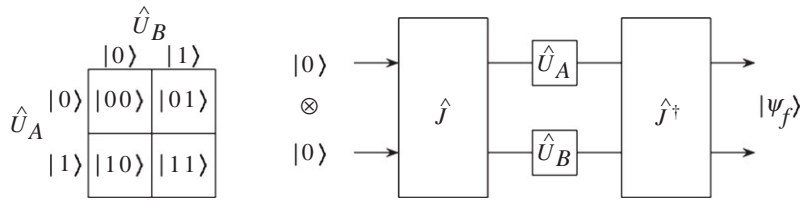
applied and the state of the game becomes

$$|\psi_f\rangle = \hat{J}^{\dagger}(\hat{U}_A \otimes \hat{U}_B)\hat{J}|00\rangle. \tag{2.1}$$

This follows a pair of Stern–Gerlach-type detectors for measurement. As a result, using the notation $|\psi_f\rangle = (\psi_1\psi_2\psi_3\psi_4)'$, $\mathbb{A} = \begin{pmatrix} |\psi_1|^2 & |\psi_2|^2 \\ |\psi_3|^2 & |\psi_4|^2 \end{pmatrix}$. Consequently, the expected payoffs become

$$p_{\left\{{A \atop B}\right\}} = \left\{{R \atop r}\right\} |\psi_1|^2 + \left\{{r \atop R}\right\} |\psi_4|^2. \tag{2.2}$$

With maximal entangling, $\gamma = \pi/2$, so that $\hat{J} = \frac{1}{\sqrt{2}}(\hat{I}^{\otimes 2} + i\hat{D}^{\otimes 2})$. Thus,

$$\hat{J}|00\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle)$$

and

$$\hat{J}^{\dagger} = \frac{1}{\sqrt{2}}(\hat{I}^{\otimes 2} - i\hat{D}^{\otimes 2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}.$$

If $\alpha_A = \alpha_B = 0$, the one-parameter operators $\hat{U}(\theta, 0) \equiv \tilde{U}(\theta)$ are *classical* mixtures between $\hat{I}$ and $\hat{D}$. In a $\tilde{U}(\theta_A)$ versus $\tilde{U}(\theta_B)$ contest, noting $\omega \equiv \theta/2$,

$$|\psi_f\rangle = \begin{pmatrix} \cos\omega_A \cos\omega_B \\ -\cos\omega_A \sin\omega_B \\ -\sin\omega_A \cos\omega_B \\ \sin\omega_A \sin\omega_B \end{pmatrix},$$

so that the probability distribution matrix is

$$\mathbb{A} = \begin{pmatrix} \cos^2\omega_A \cos^2\omega_B & \cos^2\omega_A \sin^2\omega_B \\ \sin^2\omega_A \cos^2\omega_B & \sin^2\omega_A \sin^2\omega_B \end{pmatrix} = \begin{pmatrix} \cos^2\omega_A \\ \sin^2\omega_A \end{pmatrix} \begin{pmatrix} \cos^2\omega_B & \sin^2\omega_B \end{pmatrix}.$$

Thus, the joint probabilities factorize as in the classical game by employing independent strategies.

In contrast, if $\theta_A = \theta_B = 0$,

$$|\psi_f\rangle = \begin{pmatrix} \cos(\alpha_A + \alpha_B) \\ 0 \\ 0 \\ \sin(\alpha_A + \alpha_B) \end{pmatrix} \Rightarrow \mathbb{A} = \begin{pmatrix} \cos^2(\alpha_A + \alpha_B) & 0 \\ 0 & \sin^2(\alpha_A + \alpha_B) \end{pmatrix};$$

in this case, the joint probability distribution is not factorizable. In the particular case $\alpha_A + \alpha_B = \pi/4$, $a_{11} = a_{22} = 1/2$.

3. The spatialized *battle of the sexes*

In the spatial version of the BOS that we deal with, each player occupies a site (i,j) in a two-dimensional $N \times N$ lattice. We will consider that *males* and *females* alternate in the site occupation, so that in the chessboard form shown in table 3a, every player is surrounded by four partners (♀–♂, ♂–♀) and four mates (♀–♀, ♂–♂).

In a CA[1]-like implementation, in each generation (T), every player plays with his four adjacent partners, so that the payoff $p_{i,j}^{(T)}$ of a given individual is the sum over these four interactions. In the next generation, every player will adopt the parameter choice $(\alpha_{i,j}^{(T)}, \theta_{i,j}^{(T)})$ of his nearest-neighbour mate (including himself) that received the highest payoff. In the case of a tie, the player maintains his choice.

All the simulations in this work are run in a 100×100 lattice with periodic boundary conditions. In order to ensure that the players in the borders are in the regular conditions regarding mate and partner neighbourhood, an even side size lattice ($N = 100$) was stated.

(a) Classical strategies

As stated before, if $\alpha = 0$, the joint probabilities factorize so that the quantum component of the game vanishes and we are referred to the classical game employing independent strategies.

This is the case of the simple ($R = 5$, $r = 1$) example of table 3. In the initial scenario of table 3, every player chooses his preferred choice ($\theta_{\male} = 0 \equiv x = 1$, $\theta_{\female} = \pi \equiv y = 0$), except a male in the central part of the lattice that chooses θ at the $\pi/2$ level. As a result, at time step $T = 1$, the general income is nil, with the only exception arising from the $\theta_{\male} = \pi/2$ choice. This reports two units ($4 \times \frac{1}{2}$) to the initial $\pi/2$ male and 2.5 units ($1 \times \frac{5}{2}$) to its female neighbours. The change in payoffs from zero to two units fires the change to $\theta_{\male} = \pi/2$ of the four males connected with the initial $\theta_{\male} = \pi/2$, as indicated under $T = 2$ in table 3c. The change $\theta_{\male} = 0$ into $\theta_{\male} = \pi/2$ advances in this way at every time step, so that in this simple example, every male player will choose $\theta = \pi/2$ in the long term.

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[1]CA are spatially extended dynamical systems that are discrete in all their constitutional components: space, time and state variable. Uniform, local and synchronous interactions, as assumed here, are landmark features of CA. A compact account on CA can be found in the book by Schiff (2008).

Figure 1. A simulation of the classical (5,1)-BOS cellular automaton. (a) The mean θ parameter values and mean payoffs per encounter (p). (b,c) The patterns at $T = 200$ of θ and p. Increasing grey levels indicate increasing parameter values. (Online version in colour.)

Table 3. A classical BOS cellular automaton. $R = 5$, $r = 1$.

(a)						(b)		$\theta_{\male},\theta_{\female}$		$T=1$				$p_{\male},p_{\female}$				(c)		$\theta_{\male},\theta_{\female}$		$T=2$				$p_{\male},p_{\female}$		
♂	♀	♂	♀	♂	♀	0	π	0	π	0	π	0	0	0	0	0	0	0	π	0	π	0	π	0	2.5	0	2.5	0	0
♀	♂	♀	♂	♀	♂	π	0	π	0	π	0	0	0	2.5	0	0	0	π	π/2	π	π/2	π	0	2.5	2.0	7.5	2.0	2.5	0
♂	♀	♂	♀	♂	♀	0	π	π/2	π	0	π	0	2.5	2.0	2.5	0	0	0	π	π/2	π	0	π	0	7.5	2.0	7.5	0	0
♀	♂	♀	♂	♀	♂	π	0	π	0	π	0	0	0	2.5	0	0	0	π	π/2	π	π/2	π	0	2.5	2.0	7.5	2.0	2.5	0
♂	♀	♂	♀	♂	♀	0	π	0	π	0	π	0	0	0	0	0	0	0	π	0	π	0	π	0	2.5	0	2.5	0	0
♀	♂	♀	♂	♀	♂	π	0	π	0	π	0	0	0	0	0	0	0	π	0	π	0	π	0	0	0	0	0	0	0

There is no way of altering the universal $\theta_{\female} = 0$ choice in this simple example. Consequently, in the long term, the male players will get two units and the female players will get 10 units ($4 \times \frac{5}{2}$) at every time step.

Figure 1 deals with a simulation in the classical scenario, thus $\alpha = 0$, starting at random with respect to the θ parameter values in a (5,1)-BOS cellular automaton. The figure shows, up to $T = 200$, the evolution of the mean values across a 100×100 lattice of $\theta_{\male}$ and $\theta_{\female}$, as well as the evolution of the payoffs. As a result of the random assignment of θ values in the $[0, \pi]$ interval, initially, $\bar{\theta}_{\male} \simeq \bar{\theta}_{\male} = \pi/2 \simeq 1.57$, and the mean payoffs commence at the arithmetic mean of the payoff values, i.e. $p^+ = (R + r)/4 = 1.5$.

After the first round, both types of players drift to their preferred θ levels, i.e. nil for the male player and π for the female player, and as a consequence, both payoffs plummet at $T = 2$. Thus, $\bar{\theta}_{\male}^{(2)} = 0.73$, $\bar{\theta}_{\female}^{(2)} = 2.45$, $\bar{p}_{\male}^{(2)} = 0.78$, $\bar{\theta}_{\female}^{(2)} = 0.84$ in figure 1. But immediately the θ drift becomes moderated, and both p recover. After a fairly short transition period, the θ values, and consequently the payoffs, stabilize. In the simulation of figure 1, the stable values of the mean

payoffs are $\bar{p}_{\female} = 2.96$, $\bar{p}_{\male} = 1.97$, a pair of payoff values that are not accessible in the uncorrelated formulation of the game: the equation of the parabola closing the payoffs region in such a scenario is $3(p_{\male} - p_{\female})^2 - 16(p_{\male} + p_{\female}) + 48 = 0$, so that the maximum feasible female payoff with fixed $p_{\male} = 1.97$ is the value on the parabola, $p_{\female} = 1.15$, which 2.96 notably exceeds.

The curves labelled p^* in figure 1*a* show the *theoretical* payoffs of both players in a two-person game with independent strategies, using as probabilities those induced by the evolving mean values of θ_A and θ_B, namely

$$p_{\male}^* = R\cos^2\left(\frac{\bar{\theta}_{\male}}{2}\right)\cos^2\left(\frac{\bar{\theta}_{\female}}{2}\right) + r\sin^2\left(\frac{\bar{\theta}_{\male}}{2}\right)\sin^2\left(\frac{\bar{\theta}_{\female}}{2}\right) \qquad (3.1a)$$

and

$$p_{\female}^* = r\cos^2\left(\frac{\bar{\theta}_{\male}}{2}\right)\cos^2\left(\frac{\bar{\theta}_{\female}}{2}\right) + R\sin^2\left(\frac{\bar{\theta}_{\male}}{2}\right)\sin^2\left(\frac{\bar{\theta}_{\female}}{2}\right), \qquad (3.1b)$$

which is a reformulation of equations (1.1*a*) and (1.1*b*) with $x = \cos^2(\bar{\theta}_{\male}/2)$, $y = \cos^2(\bar{\theta}_{\female}/2)$.

The actual mean payoffs of both kinds of players shown in figure 1 are over these *expected* values due to the spatial structure, which allows for the emergence of *agreement* clusters, shown in figure 1*b* as black (high θ, low x and y, so *BB* drift) and white (*FF*) regions, with interfaces of disagreement among the clusters also shown in the payoffs snapshot in figure 1*c*.

The study of spatial games was pioneered by Nowak & May (1992,1993) with regard to the prisoner's dilemma (PD) (Axelrod 2008). They concluded in their original work that spatial structure (or territoriality) can facilitate the survival of cooperators. Thus, the spatialized PD has proved to be a promising tool to explain how cooperation can hold out against the ever-present threat of exploitation.

The notable case of self-organization in the BOS just presented appears as a novel example of the boosting effect induced by the spatial ordered structure, which allows the access to payoffs that are feasible only with correlated strategies in the two-person game. The initial configuration ((x, y) or (θ_A, θ_B) assignments) also plays a role in the evolving dynamics, in such a way that, although the main features of the dynamics are common to any simulation, the details vary from one simulation to another. In particular, which kind of player predominates in terms of mean payoffs of both kinds of players. The same consideration applies to the classical spatialized binary ($x \in \{0, 1\}$, $y \in \{0, 1\}$) BOS studied in previous articles, both with deterministic updating of strategies (Alonso-Sanz 2011*a*), and with a probabilistic mechanism of election of the next choice (Alonso-Sanz 2011*b*).

(b) Quantum strategies

The final state (2.1) becomes

$$|\psi_f\rangle = \begin{pmatrix} \cos\omega_A\cos\omega_B\cos(\alpha_A + \alpha_B) \\ -\cos\omega_A\sin\omega_B\cos\alpha_A + \sin\omega_A\cos\omega_B\sin\alpha_B \\ \cos\omega_A\sin\omega_B\sin\alpha_A - \sin\omega_A\cos\omega_B\cos\alpha_B \\ \sin\omega_A\sin\omega_B + \cos\omega_A\cos\omega_B\sin(\alpha_A + \alpha_B) \end{pmatrix}.$$

Thus, the payoffs (2.2) result,

$$
p_{\left\{ {A \atop B} \right\}} = \left\{ {R \atop r} \right\} (\cos\omega_A \cos\omega_B \cos(\alpha_A + \alpha_B))^2
$$
$$
+ \left\{ {r \atop R} \right\} (\sin\omega_A \sin\omega_B + \cos\omega_A \cos\omega_B \sin(\alpha_A + \alpha_B))^2. \qquad (3.2)
$$

Starting from $|\psi_i\rangle = |00\rangle$, the quantum protocol is not fair as it favours the B (or ♀) player in the following way: let us suppose an equal middle-level election of the α and θ parameters of both players, i.e. $\theta_A = \theta_B = \pi/2$, $\alpha_A = \alpha_B = \pi/4$; in such a scenario, $a_{11} = a_{12} = a_{21} = 0$, $a_{22} = 1$. As a result, $p_{\circlearrowleft} = r < p_{\circ\!\!\!+} = R$. The same holds when $\omega_A = \omega_B$, $\alpha_A + \alpha_B = \pi/2$. In the same vein, starting from $|11\rangle$, or adopting the space of strategies ([Benjamin & Hayden 2001](#), [Flitney & Hollengerg 2007](#)),

$$
\hat{U}(\theta, \alpha) = \begin{pmatrix} \cos\left(\dfrac{\theta}{2}\right) & e^{i\alpha} \sin\left(\dfrac{\theta}{2}\right) \\[3mm] -e^{-i\alpha} \sin\left(\dfrac{\theta}{2}\right) & \cos\left(\dfrac{\theta}{2}\right) \end{pmatrix},
$$

would favour the A (or ♂) player.[2]

Nash equilibrium is achieved in two ways ([Flitney & Hollengerg 2007](#)),

$$
\begin{cases} \theta_A = \theta_B, & \alpha_A + \alpha_B = \dfrac{\pi}{2}, \\[3mm] \theta_A = \theta_B = \pi, & \forall \alpha_A, \alpha_B. \end{cases}
$$

All these Nash equilibria produce the same payoffs: $\bar{p}_A = r < \bar{p}_B = R$, which again makes apparent the bias inherent to the quantization scheme introduced by [Eisert *et al.* (1999)](#) when dealing with an asymmetric game such as the BOS.

In the spatial formulation we follow here (play with partner neighbours, imitate the most rewarded mate neighbourhood), every player will adopt the (α, θ) parameter values of the mate neighbourhood with the highest payoff. In the case of a tie, i.e. several mate neighbours with the same maximum payoff, the average of the θ and α parameter values corresponding to the best mate neighbours will be adopted.

[Figure 2](#) deals with a simulation starting at random with respect to the θ and α parameter values in a quantum (5,1)-BOS cellular automaton. [Figure 2*a*](#) shows up to $T = 100$ the evolution of the mean values across a 100×100 lattice of θ and α, and the actual mean payoffs. The figure also shows the *theoretical* payoffs, i.e. the payoffs achieved for both kinds of players in a single hypothetical two-person game with players adopting the mean parameters appearing in the spatial simulation. Thus, these theoretical payoffs calculated from equation (3.2) can be

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[2]In the scheme introduced by [Marinatto & Weber (2000)](#), the initial state of the game $|\psi_i\rangle$ is a linear combination of the vectors of the base $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$. Operationally, the scheme differs from the earlier proposed of [Eisert *et al.* (1999)](#) by the absence of the reverse gate $\hat{J}^\dagger$, i.e. $|\psi_f\rangle = (\hat{U}_A \otimes \hat{U}_B)\hat{J}|\psi_i\rangle$. This quantization scheme is often adopted when dealing with the BOS, e.g. [Frcąckiewicz (2009)](#), but will not be considered in this study.
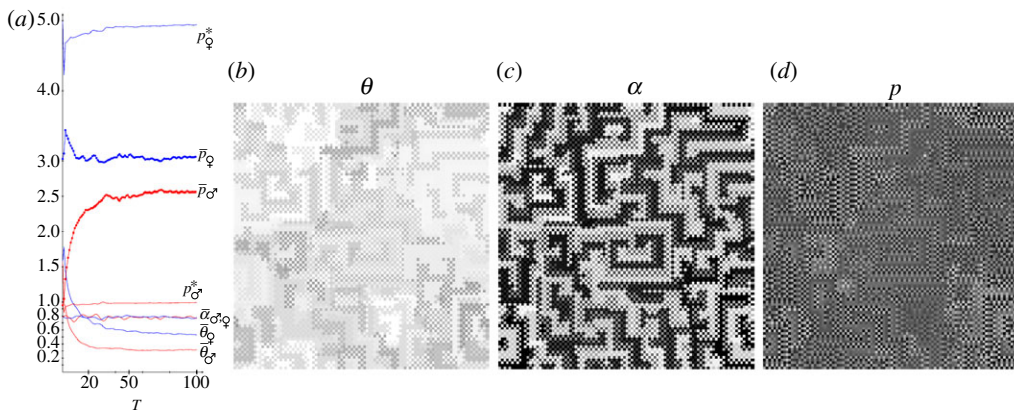
Figure 2. A simulation of the quantum (5,1)-BOS cellular automaton. (a) The mean θ and α parameter values, and the actual mean ($\bar{p}$) and theoretical (p^*) payoffs per encounter. (b–d) The parameter and payoff patterns at $T = 100$. (Online version in colour.)

rewritten as

$$p^*_{\left\{\begin{array}{c}\sigma\\\female\end{array}\right\}} = \left\{\begin{array}{c}R\\r\end{array}\right\} (\cos \bar{\omega}_{\sigma} \cos \bar{\omega}_{\female} \cos(\bar{\alpha}_{\sigma} + \bar{\alpha}_{\female}))^2$$

$$+ \left\{\begin{array}{c}r\\R\end{array}\right\} (\sin \bar{\omega}_{\sigma} \sin \bar{\omega}_{\female} + \cos \bar{\omega}_{\sigma} \cos \bar{\omega}_{\female} \sin(\bar{\alpha}_{\sigma} + \bar{\alpha}_{\female}))^2. \qquad (3.3)$$

As a result of the random assignment of the parameter values, initially, $\bar{\theta}_{\sigma} \simeq \bar{\theta}_{\female} \simeq \pi/2 = 1.57$ and $\bar{\alpha}_{\sigma} \simeq \bar{\alpha}_{\female} \simeq \pi/4 = 0.78$. Consequently, the theoretical mean payoffs are initially $p^*_{\sigma} \simeq 1.0$, $p^*_{\female} \simeq 5.0$. The actual mean payoffs are initially biased as the theoretical payoffs are, but not in such a marked extent: $p^{(1)}_{\sigma} = 0.95$, $p^{(1)}_{\female} = 3.03$. After the first round, both types of players tend to *moderate* this initial trend, particularly, in the case of the male payoff, which quickly grows, stabilizing around $p_{\sigma} \simeq 2.56$, whereas the female payoff descends only to $p_{\female} \simeq 3.06$. In any case, the female player outperforms the male one. In contrast to what happens in the classical simulation in figure 1, in the quantum simulation in figure 2, the θ parameters of both players stabilize in low values, which constitutes an indicator of correlation, as stated in §2. Last but not least, the mean α parameter values of both kinds of players remain almost unaltered during the whole simulation, so that they remain notably coincident oscillating around the initial $\alpha \simeq \pi/4 = 0.78$.

The spatial parameter and payoff patterns in the quantum simulation shown in figure 2 differ very much from those in the classical context in figure 1. Thus, instead of consistent coordination clusters being generated when starting at random, maze-like structures emerge for both parameter and payoff patterns in the quantum simulation.

Again, as in the classical context, the initial configurations do matter, and the details of the dynamics vary from one simulation to another. Albeit now the ordering $\bar{p}_{\female} > \bar{p}_{\sigma}$ remains, regardless of the initial configuration, owing to the bias inherent to the quantization scheme considered here.

4. Unfair contest

Let us assume the unfair situation: B is restricted to classical strategies $\tilde{U}(\theta_B, 0)$, whereas A may use quantum $\hat{U}(\theta_A, \alpha_A)$ ones. In this context,

$$p_{\left\{{A \atop B}\right\}} = \left\{{R \atop r}\right\} (\cos \omega_A \cos \omega_B \cos \alpha_A)^2$$
$$+ \left\{{r \atop R}\right\} (\sin \omega_A \sin \omega_B + \cos \omega_A \cos \omega_B \sin \alpha_A)^2. \qquad (4.1)$$

As a result, player A would chose $\alpha_A = \pi/2$ in order to give preference for getting the R payoff. Without knowing the player B move, player A may assume that B has played $\theta_B = \pi/2$ (the average move); in this case,

$$p_{\left\{{A \atop B}\right\}} = \frac{1}{2} \left\{{r \atop R}\right\} (\sin \omega_A + \cos \omega_A)^2 = \frac{1}{2} \left\{{r \atop R}\right\} 2 \sin \theta_A.$$

Consequently, player A would chose $\theta_A = \pi/2$ in order to get the R payoff. In summary, A would play the so-called *miracle* move (Flitney & Abbott 2003),

$$\hat{M} \equiv \hat{U}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}.$$

Figure 3 deals with a simulation of this kind of unfair quantum (5,1)-BOS CA, where the female players are restricted to classical strategies. As a result of the unfair scenario, figure 3a shows how male players rapidly get mean payoffs near the maximum $R = 5$, whereas the female players are induced to get mean values very close to $r = 1$. Only a number of small and sparse clusters deviate from this general trend. These clusters become apparent in the patterns at $T = 100$ in figure 3: those of θ and α, which are mainly blank, correspond to very low values of both kinds of parameters ($\bar{\theta}_B = 0.181$, $\bar{\theta}_A = 0.092$, $\bar{\alpha}_A = 0.034$), and in the alternating black (payoff near 5 achieved by male players)—soft grey (payoff near 1 achieved by female players) p-pattern shown in figure 3c.

5. Partial entangling

Adopting an entanglement factor γ,

$$\hat{J} = \exp\left(i\frac{\gamma}{2}\hat{D}^{\otimes 2}\right) = \cos\left(\frac{\gamma}{2}\right)\hat{I}^{\otimes 2} + i\sin\left(\frac{\gamma}{2}\right)\hat{D}^{\otimes 2}, \quad \gamma \in \left[0, \frac{\pi}{2}\right].$$

Thus,

$$\hat{J}|00\rangle = \begin{pmatrix} \cos\left(\frac{\gamma}{2}\right) & 0 & 0 & i\sin\left(\frac{\gamma}{2}\right) \\ 0 & \cos\left(\frac{\gamma}{2}\right) & -i\sin\left(\frac{\gamma}{2}\right) & 0 \\ 0 & -i\sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) & 0 \\ i\sin\left(\frac{\gamma}{2}\right) & 0 & 0 & \cos\left(\frac{\gamma}{2}\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\gamma}{2}\right) \\ 0 \\ 0 \\ i\sin\left(\frac{\gamma}{2}\right) \end{pmatrix}$$
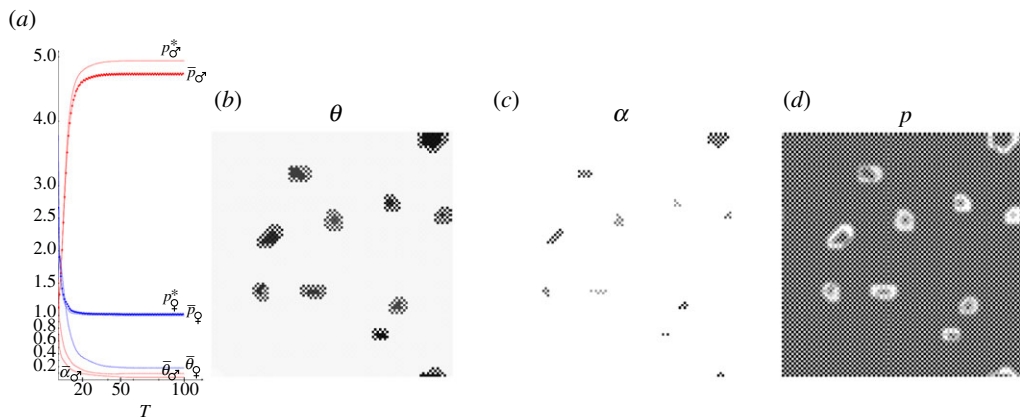
(a)

Figure 3. A simulation of a quantum (5,1)-BOS cellular automaton, where the female players are restricted to classical strategies. (a) Evolving mean parameters and payoffs. (b–d) Parameter and payoff patterns at $T = 100$. (Online version in colour.)

and

$$\hat{J}^{\dagger} = \exp\left(-\mathrm{i}\left(\frac{\gamma}{2}\right)\hat{D}^{\otimes 2}\right).$$

Maximal entangling is achieved with $\gamma = \pi/2$, whereas $\gamma = 0$ would refer to the classical uncorrelated context. The diagonal elements of $\mathbb{A}$ are

$$a_{11} = \cos^2 \omega_A \cos^2 \omega_B (\cos^2 \gamma \sin^2(\alpha_A + \alpha_B) + \cos^2(\alpha_A + \alpha_B))$$

and

$$a_{22} = \sin^2 \gamma \cos^2 \omega_A \cos^2 \omega_B \sin^2(\alpha_A + \alpha_B) + \sin^2 \omega_A \sin^2 \omega_B$$
$$+ \tfrac{1}{2}\sin \gamma \sin \theta_A \sin \theta_B \sin(\alpha_A + \alpha_B).$$

If $\gamma > \pi/4$, the B player will out score the A player (Du *et al.* 2001, 2003, Flitney & Abbott 2002, 2003).[3]

A simulation of the quantum (5,1)-BOS cellular automaton with $\gamma = \pi/4$ is shown in figure 4. Opposite to what happens in the fully entangled simulation of figure 2, now it is the male type of players who outperforms the female type. The theoretical payoffs (p^*) are much closer in figure 4 compared with those in figure 2, and still preserve the $p^*_{\female} > p^*_{\male}$ relation. The mean α parameter values of both kinds of players are not dramatically altered during the simulation, so that they remain fairly close, oscillating around the initial $\bar{\alpha} \simeq \pi/4 = 0.78$. Also much as in the fully entangled case of figure 2, maze-like structures emerge for both parameter and payoff patterns in the $\gamma = \pi/4$ simulation in figure 4.

[3] Such as in the particular case: $\begin{array}{l}\theta_A = \theta_B = 0 \\ \alpha_A + \alpha_B = \pi\end{array} \Rightarrow \begin{array}{l}p_A = R\cos^2\gamma + r\sin^2\gamma = R + (r-R)\sin^2\gamma \\ p_B = r\cos^2\gamma + R\sin^2\gamma = r + (R-r)\sin^2\gamma\end{array}$ $p_A = p_B \rightarrow \sin^2\gamma = \frac{1}{2} \Rightarrow \gamma > \pi/4 \rightarrow p_B > p_A$.
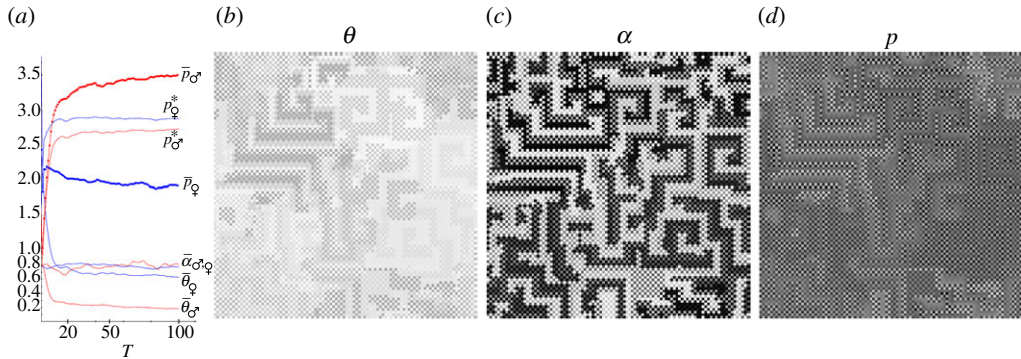
Figure 4. A simulation of the quantum (5,1)-BOS cellular automaton with $\gamma = \pi/4$. (*a*) The mean θ and α parameter values and mean payoff per encounter (p). (*b–d*) Parameter and payoff patterns at $T = 100$. (Online version in colour.)

6. Conclusions and future work

The quantum formulation of the spatial BOS evolves in a manner notably distinct from that of its classical counterpart. Thus, instead of the emergence of coordination clusters achieved in the classical context, a more homogeneous distribution of payoffs across the lattice is achieved in the quantum formulation. The parameter strategies show a clear maze-like aspect, also appreciable in the payoff snapshots, though in much more fuzzy manner. In the long term, the mean payoffs per encounter are notably high, often accessible only in the correlated two-person game scenario. But this property is also found in the classical formulation, which already exhibits a powerful self-organization capacity.

Within the seminal paper of Eisert *et al.* (1999), a specific parametrization was additionally used in order to reduce the two sets of two parameters (θ_A, α_A), (θ_B, α_B) into two parameters (t_A, t_B), with $t \in [-1, 1]$. It would be interesting to see the results of the iterated quantum BOS CA adopting this t-parametrization, which makes the numerical results more comparable to their classical counterpart, as only one parameter is to be arranged by the players (an example of the use of the t-parametrization is given in Hanauske *et al.* (2010).

Other quantization schemes (Nawaz & Toor 2004) deserve particular study in the spatial context. Our priority will be given to the consideration of the full space of SU(2) strategies, i.e. the three-parameter strategies (Benjamin & Hayden 2001),

$$\hat{U}(\theta, \alpha, \beta) = \begin{pmatrix} e^{i\alpha} \cos\left(\dfrac{\theta}{2}\right) & e^{i\beta} \sin\left(\dfrac{\theta}{2}\right) \\ -e^{-i\beta} \sin\left(\dfrac{\theta}{2}\right) & e^{-i\alpha} \cos\left(\dfrac{\theta}{2}\right) \end{pmatrix}, \quad \theta \in [0, \pi], \ \alpha, \beta \in \left[0, \dfrac{\pi}{2}\right],$$

free of the bias assumed in the parametrization scheme chosen here.

Turning the deterministic mechanism of updating the configurations into a probabilistic one will allow one to relate somehow the results presented here with other *ordering* processes found in spatialized social relations.

In such a probabilistic updating mechanism, the individuals will adopt the mate parameters with a probability proportional to its payoff among their mate neighbours.

As long as only the results from the last round are taken into account and the outcomes of previous rounds are neglected, the model considered here may be termed *ahistoric*, although it is not fully memoryless as there is chain (or Markovian) mechanism inherent in it, so that previous results affect further outcomes. We plan to deal with models with (proper) memory in the near future. We will adopt a simple approach to memory implementation: the general scheme of the transition rule will remain unaltered, but it will operate on the trait payoffs and parameters of every player, previously constructed from the payoffs and parameter values from the previous rounds. We have studied the effect of probabilistic updating and of embedded memory in the classical spatial BOS in previous articles (Alonso-Sanz 2011*a*,*b*). Technically, its extension to the quantum context will demand only the additional effort of dealing with two parameters (θ, α) instead of one (the F-probability).

In real-life situations, the preferential choice and refusal of partners play an important role in the emergence of cooperation (Szabo & Fáth 2007). We have studied the effect of memory on a simple, deterministic, structurally dynamic PD game, in which state and link configurations are *both* dynamic and continually interacting (Alonso-Sanz 2009*c*). Further study is due on the structurally dynamic BOS. The study of the effect of memory in other structurally spatialized games, as well as in games on networks (Alonso-Sanz 2009*a*) is also planned in the future.

Asynchronous updating and the effect of increasing degrees of spatial dismantling (via rewiring, for example, as carried out in Alonso-Sanz (2009*b*) in the PD context), which would lead to more realistic models, are tasks left to be scrutinized in further studies.

Last but not least, other games, particularly the PD, will be taken into account in the near future.

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