Nash equilibria in quantum games with generalized two-parameter strategies

Adrian P. Flitney [a,*], Lloyd C.L. Hollenberg [b]

[a] *School of Physics, University of Melbourne, Parkville VIC 3010, Australia*
[b] *Centre for Quantum Computer Technology and School of Physics, University of Melbourne, Australia*

Abstract

In the Eisert protocol for 2×2 quantum games [J. Eisert, et al., Phys. Rev. Lett. 83 (1999) 3077], a number of authors have investigated the features arising from making the strategic space a two-parameter subset of single qubit unitary operators. We argue that the new Nash equilibria and the classical–quantum transitions that occur are simply an artifact of the particular strategy space chosen. By choosing a different, but equally plausible, two-parameter strategic space we show that different Nash equilibria with different classical–quantum transitions can arise. We generalize the two-parameter strategies and also consider these strategies in a multiplayer setting.
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1. Introduction

The intersection of the mathematical theory of strategic conflict situations, known as game theory, and the tools of quantum mechanics was first made by Eisert and co-workers [1] and by Meyer [2]. Since then a body of theory on quantum games has built up. The original protocol for two-player, two-strategy (2×2) quantum games, introduced by Eisert et al. [1,3], has remained the main tool for exploring the properties of quantum games. In this, the players moves are quantum operators acting on qubits, with non-classical effects being introduced through entanglement.

In the seminal paper [1] and a number of subsequent publications [4–10] a particular two-parameter subset of SU(2) is chosen as the strategic space for the players. This has the advantage of mathematical simplicity and, in addition, new Nash equilibria appear that are able to solve some of the dilemmas of classical game theory. For example, in the well-known game of

Prisoners' Dilemma [11] there is a conflict between the Nash equilibrium (NE), that determines the strategies that two self-interested rational players would choose to maximize their payoffs, and the Pareto efficient outcome, that is the best overall for the players. With the commonly used two-parameter strategy space in the Eisert protocol this dilemma is resolved with a new NE appearing that coincides with the Pareto optimal (PO) outcome [1]. Subsequent investigations revealed three regions in the space of the entangling parameter: when the entanglement is greatest there is a quantum region with the new NE, at minimal entanglements the game behaves classically, and between these there is a classical–quantum region with intermediate behavior [12,13].

It was quickly observed by Benjamin and Hayden [14] that the two-parameter strategies were not the most general, and suffer from being incomplete under composition. Eisert et al. in their reply [15] emphasized that their protocol with the two-parameter strategy set is one possible quantum extension of Prisoners' Dilemma for which the dilemma disappears, that is, where there exists a Pareto optimal NE, without claiming universality for this equilibrium. Three parameters are necessary to describe (up to an arbitrary phase) a general SU(2) operator, and in the full three-parameter strategy space there is no NE in

* Corresponding author.
E-mail address: aflitney@physics.unimelb.edu.au (A.P. Flitney).
URL: http://aslitney.customer.netspace.net.au/default.html.

deterministic quantum strategies [14]. Nash equilibria in mixed quantum strategies exist, that is, in linear combinations of unitaries [3,16]. However, among these a different dilemma arises: they form a continuous set and the arrival at a unique NE may be problematic. In addition, in Prisoners' Dilemma, the payoff from the mixed quantum NE, though superior to that from the classical NE, is still worse than the PO outcome.

This Letter demonstrates that a different, but equally justifiable, two-parameter strategic space can give rise to different NE with different classical–quantum transitions. This result supports the position that the new NE and the classical–quantum boundaries to which they give rise are nothing but artifacts of the particular choice of sub-space. Indeed, though it is easy to see why allowable choices of the players may be limited by unitarity, since this gives the set of reversible quantum operators, rather than the space of all possible quantum operations, it is difficult to see a physical justification for the use of two-parameter strategies, in particular the preference of one two-parameter set over another.

The Letter is organized as follows. Section 2 summarizes the results for the existing two-parameter strategies for the game of Prisoners' Dilemma as well as presenting new results for the games of Chicken and Battle of the Sexes. In Section 3 we present an alternative two-parameter strategy space and consider the three games again, showing quite different behavior. In Section 4 we generalize the results for Prisoners' Dilemma to N players and in Section 5 we consider two-player games with generalized two-parameter strategy spaces.

2. Summary of the Eisert protocol and results for the existing two-parameter strategies

There have been many papers summarizing the Eisert protocol for 2×2 quantum games so it shall be described here as briefly as possible. For a more detailed description see, for example, Ref. [16].

The players' choice is encoded in a qubit, initially prepared in the $|0\rangle$ state. An operator $\hat{J}$, dependent on a parameter $\gamma \in [0, \pi/2]$,

$$\hat{J} = \cos\frac{\gamma}{2}\hat{I} \otimes \hat{I} + i\sin\frac{\gamma}{2}\hat{\sigma}_x \otimes \hat{\sigma}_x, \tag{1}$$

entangles the players' qubits to produce the state $(|00\rangle + i|11\rangle)/\sqrt{2}$. The players' strategies are the choice of local operator that they now make to act on their qubit. The space of allowable operators forms the strategic space of the players. Because of the entanglement, the actions of the players are not independent. The two classical pure strategies are represented by $\hat{I}$ and $i\hat{\sigma}_x$. After the players' moves $\hat{J}^{\dagger}$ is applied to the pair of qubits with the consequence that if both players choose classical strategies the result is entirely equivalent to the classical game. Finally, a measurement is made in the computational basis and payoffs are awarded using the classical payoff matrix. The state of the players' qubits prior to the final measurement can be computed by

$$|\psi_f\rangle = \hat{J}^{\dagger}(\hat{A} \otimes \hat{B})\hat{J}|\psi_i\rangle, \tag{2}$$

where $\hat{A}$ and $\hat{B}$ represent Alice's and Bob's moves, respectively. We are interested in the expectation value $\langle \$ \rangle$ of the players' payoffs. This can be calculated by

$$\langle \$ \rangle = \$_{00}\big|\langle\psi_f|00\rangle\big|^2 + \$_{01}\big|\langle\psi_f|01\rangle\big|^2 + \$_{10}\big|\langle\psi_f|10\rangle\big|^2 + \$_{11}\big|\langle\psi_f|11\rangle\big|^2, \tag{3}$$

where $\$_{ij}$ is the payoff to the player associated with the game outcome ij, $i, j \in \{0, 1\}$.

The two parameter quantum strategies of Eisert et al. [1] are drawn from the set[1]

$$S^{(1)} = \big\{\hat{M}^{(1)}(\theta, \phi): \theta \in [0, \pi], \phi \in [0, \pi/2]\big\},$$

$$\hat{M}^{(1)}(\theta, \phi) = \begin{pmatrix} e^{i\phi}\cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & e^{-i\phi}\cos(\theta/2) \end{pmatrix}. \tag{4}$$

A possible parameterization of the full space of SU(2) operators is

$$\hat{M}(\theta, \alpha, \beta) = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & ie^{i\beta}\sin(\theta/2) \\ ie^{-i\beta}\sin(\theta/2) & e^{-i\alpha}\cos(\theta/2) \end{pmatrix}, \tag{5}$$

where $\theta \in [0, \pi]$ and $\alpha, \beta \in [-\pi, \pi]$. If we use the latter set for the strategic space any operation carried out by one player can be reversed by the other. Consequently any strategy has a counter-strategy [3,14,16]. Pure strategy equilibria in the quantum game will only arise when the classical game, from which it is derived, has a purely co-operative solution. Such games are of little interest to game theory and, alas, generally do not model any real world situations either!

The two-parameter strategic space of Eq. (4) can give rise to interesting properties. As an example consider the famous Prisoners' Dilemma [11]. Here the players' moves are known as cooperation (C) or defection (D). The payoff matrix can be written as

	Bob: C	Bob: D
Alice: C	$(3, 3)$	$(0, 5)$
Alice: D	$(5, 0)$	$(1, 1)$

(6)

where the numbers in parentheses represent payoffs to Alice and Bob, respectively. The game is symmetric and there is a dominant strategy, that of always defecting, since it gives a better payoff regardless of the actions of the other player. The Nash equilibrium is $D \otimes D$ since from this outcome neither player can improve their payoff by a unilateral change in strategy. However, $D \otimes D$ is not such a good result for the players since had they both cooperated they would have both received a payoff of three rather than one. The outcome $C \otimes C$ is the PO result, the one from which neither player can improve their payoff without the other being worse off. It is this conflict between the NE and PO outcomes that forms the dilemma of the game's name. In social and political life this dilemma between individual and group rationality is responsible for much of the conflict throughout the world. In the quantum version of this game the strategy "always cooperate" is represented by $\hat{I}$, while

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[1] There are some notational differences with Ref. [1] but these are of no significance.
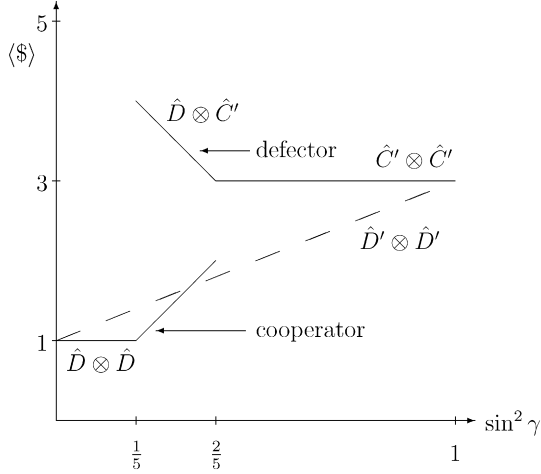
Fig. 1. The Nash equilibrium payoffs in a quantum version of Prisoners' Dilemma as a function of the degree of entanglement, as measured by $\sin^2 \gamma$, for the strategy space $S^{(1)}$ (solid lines), and for $S^{(2)}$ (dashed line). For $S^{(1)}$ there are three regions with different the Nash equilibria: $\hat{D} \otimes \hat{D}$, $\hat{C}' \otimes \hat{D}$ and $\hat{D} \otimes \hat{C}'$, and $\hat{C}' \otimes \hat{C}'$. The strategy $\hat{C}'$ is a cooperative strategy, becoming equivalent to $\hat{C}$ as $\gamma \to 0$.
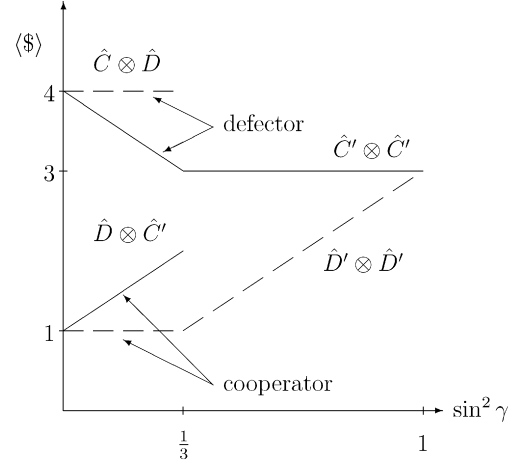

Fig. 2. The Nash equilibrium payoffs in a quantum version of the game of Chicken as a function of the degree of entanglement, as measured by $\sin^2 \gamma$, for the strategy space $S^{(1)}$ (solid lines), and for $S^{(2)}$ (dashed line). In the region $\sin^2 \gamma < \frac{1}{3}$, the Nash equilibria in the first case are $\hat{C}' \otimes \hat{D}$ and $\hat{D} \otimes \hat{C}'$, while for $S^{(2)}$ the Nash equilibria are the classical ones of $\hat{C} \otimes \hat{D}$ and $\hat{D} \otimes \hat{C}$. The strategy $\hat{C}'$ is a cooperative strategy, becoming equivalent to $\hat{C}$ as $\gamma \to 0$.

"always defect" is represented by $i\hat{\sigma}_x$. With maximal entanglement, $\gamma = \pi/2$, a new strategy

$$\hat{C}' = \hat{M}(0, \pi/2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{7}$$

emerges as the preferred strategy for both players,[2] with the NE being $\hat{C}' \otimes \hat{C}'$. The new NE gives an expected payoff of three to both players there by breaking the dilemma. The situation for non-maximal entanglement is described by Du et al. [12]. Once $\sin^2 \gamma$ slips below $\frac{2}{5}$, $\hat{C}' \otimes \hat{C}'$ is no longer a NE. Instead, two new asymmetric NE emerge, $\hat{C}' \otimes \hat{D}$ and $\hat{D} \otimes \hat{C}'$, with a payoff of $5 \sin^2 \gamma$ to the player choosing $\hat{C}'$ and $5 \cos^2 \gamma$ to the defecting player. Finally, for $\sin^2 \gamma < \frac{1}{5}$ the game behaves classically with a NE of $\hat{D} \otimes \hat{D}$. These regimes and their corresponding payoffs are indicated in Fig. 1. In the intermediate region, the existence of a pair of asymmetric NE each favoring one player over the other presents a dilemma equivalent to that found in the classical game of Chicken: How do the players arrive at a unique solution?

Chicken is similar to Prisoners' Dilemma except that mutual defection is the worst outcome:

	Bob: C	Bob: D	
Alice: C	(3, 3)	(1, 4)	(8)
Alice: D	(4, 1)	(0, 0)	

The classical game has two NE in pure strategies, $C \otimes D$ and $D \otimes C$, both of which are unsatisfactory for the cooperating player in comparison with the PO outcome of $C \otimes C$. A quantum version of this game have been considered in the Eisert protocol [3] and some of its features have been examined [18]. With the two parameter strategy set, the effect on the NE of varying the entanglement parameter has not previously been

published, though it follows the pattern of Prisoners' Dilemma. With $\sin^2 \gamma \geqslant \frac{1}{3}$ the strategy profile $\hat{C}' \otimes \hat{C}'$ is again a NE with payoffs of three to both players. For $0 < \sin^2 \gamma \leqslant \frac{1}{3}$, $\hat{C}' \otimes \hat{D}$ and $\hat{D} \otimes \hat{C}'$ are NE with payoffs of $4 - 3 \sin^2 \gamma$ to the $\hat{C}'$ player and $1 + 3 \sin^2 \gamma$ to the defector. As $\gamma \to 0$ these strategy profiles become equivalent to the two classical NE, as indicated in Fig. 2.

Another favorite dilemma of game theorists, and one studied in a number of quantum game papers [18–21] is the Battle of the Sexes, where a couple each have a preferred activity but are trying to coordinate their actions in the absence of communication. Alice wants to go to the opera (O) while Bob prefers to watch television (T). The payoff matrix is

	Bob: O	Bob: T	
Alice: O	(2, 1)	(0, 0)	(9)
Alice: T	(0, 0)	(1, 2)	

The classical game has two NE solutions, $O \otimes O$ and $T \otimes T$. The dilemma is similar to that in Chicken since there is no way to coordinate the decisions of the players and arrive at a unique NE. If the players choose differently the worst possible result for both players is obtained. The effect of using the two-parameter strategy set $S^{(1)}$ in this game has not previously been considered. If Alice plays $\hat{M}^{(1)}(\theta_A, \phi_A)$ while Bob plays $\hat{M}^{(1)}(\theta_B, \phi_B)$, the probability of the outcome OO is

$$p_{OO} = \left[\cos^2 \gamma \sin^2(\phi_A + \phi_B) + \cos^2(\phi_A + \phi_B)\right] c_A^2 c_B^2, \tag{10}$$

while the probability of TT is

$$p_{TT} = \sin^2 \gamma \sin^2(\phi_A + \phi_B) c_A^2 c_B^2 + s_A^2 s_B^2$$
$$- \frac{1}{2} \sin \gamma \sin \theta_A \sin \theta_B \sin(\phi_A + \phi_B), \tag{11}$$

where $s_k \equiv \sin(\theta_k/2)$, $c_k \equiv \cos(\theta_k/2)$, $k\{\in A, B\}$. If $\theta_A = 0$, that is, Alice tries to choose her favorite activity of Opera, then

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[2] In Ref. [1] the strategy $\hat{C}'$ is called $\hat{Q}$.

Bob's expected payoff is

$$\langle \$_B \rangle = \cos^2 \theta_B \big[1 + \sin^2 \gamma \sin^2(\phi_A + \phi_B) \big]. \tag{12}$$

Hence, as with the classical scenario, Bob will want to co-ordinate his selection of θ with Alice's by setting $\theta_B = 0$. However, for any ϕ_A chosen by Alice, Bob can always select $\phi_B = \pi/2 - \phi_A$ to give payoffs of

$$\langle \$_A \rangle = 2 - \sin^2 \gamma,$$
$$\langle \$_B \rangle = 1 + \sin^2 \gamma. \tag{13}$$

For $\gamma > \pi/4$ such a result favors Bob over Alice. Conversely, if Bob plays $\hat{M}^{(1)}(0, \phi_B)$ Alice's expected payoff is

$$\langle \$_A \rangle = \cos^2 \theta_A \big[2 - \sin^2 \gamma \sin^2(\phi_A + \phi_B) \big]. \tag{14}$$

Hence Alice will counter with $\hat{M}^{(1)}(0, \pi - \phi_B)$ to ensure a payoff of two compared with one for Bob. With the restriction in Eq. (4) of $0 \leqslant \phi \leqslant \pi/2$ the best she can be guaranteed of achieving is $\sin^2(\phi_A + \phi_B) \leqslant \frac{1}{2}$, resulting in

$$\langle \$_A \rangle \geqslant 2 - \frac{1}{2} \sin^2 \gamma,$$
$$\langle \$_B \rangle \leqslant 1 + \frac{1}{2} \sin^2 \gamma. \tag{15}$$

There is no equilibrium with $\theta_A = \theta_B = 0$ since every ϕ chosen by one party has an optimal counter by the other.

A cursory examination of Eqs. (10) and (11) reveal that for $\theta_B = \pi$ the phases ϕ_A and ϕ_B are not relevant to the payoffs of Alice or Bob. The result is a purely classical NE of $\hat{T} \otimes \hat{T}$, where $\hat{T} = \hat{M}^{(1)}(\pi, \phi)$ for arbitrary ϕ. That is, we have started with a symmetric game and by our choice of strategic space we have a unique NE with asymmetric payoffs! This is evidence that the strategy set $S^{(1)}$ has an inherent bias, as first noted in Ref. [14]. For the Battle of the Sexes this bias favors one player, Bob, over the other. Further indications of this bias are presented in the next section.

3. Alternate two-parameter strategy sets

To the present authors there seems no reason not to put the two-parameter strategy set

$$S^{(2)} = \big\{ M(\theta, \phi) : \theta \in [0, \pi], \phi \in [0, \pi/2] \big\},$$
$$\hat{M}^{(2)}(\theta, \phi) = \begin{pmatrix} \cos(\theta/2) & i e^{i\phi} \sin(\theta/2) \\ i e^{-i\phi} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{16}$$

on an equal footing to $S^{(1)}$. This strategic space has already been used to explore the NE payoff versus entanglement relationship in two and three player Prisoners' Dilemma [17] but the authors did not comment on the difference between their results and earlier ones for two player Prisoners' Dilemma [12, 13] found using the strategy space of $S^{(1)}$. We now explore the consequences of this choice of strategic space on the equilibria of the Prisoners' Dilemma, Chicken and the Battle of the Sexes. The results for the Prisoners' Dilemma are the equivalent, up to a minor notational change in Eq. (16), to those given in Ref. [17].

If Alice plays $\hat{M}^{(2)}(\theta_A, \phi_A)$ and Bob $\hat{M}^{(2)}(\theta_B, \phi_B)$, the probability of the four possible outcomes of a 2×2 quantum game in the Eisert protocol are

$$p_{CC} = \sin^2 \gamma \sin^2(\phi_A + \phi_B) s_A^2 s_B^2 + c_A^2 c_B^2$$
$$\quad + \frac{1}{2} \sin \gamma \sin \theta_A \sin \theta_B \sin(\phi_A + \phi_B),$$
$$p_{CD} = \big[\cos^2 \gamma \sin^2 \phi_B + \cos^2 \phi_B \big] c_A^2 s_B^2 + \sin^2 \gamma \sin^2 \phi_A s_A^2 c_B^2$$
$$\quad - \frac{1}{2} \sin \gamma \sin \theta_A \sin \theta_B \sin \phi_A \cos \phi_B,$$
$$p_{DC} = \big[\cos^2 \gamma \sin^2 \phi_A + \cos^2 \phi_A \big] s_A^2 c_B^2 + \sin^2 \gamma \sin^2 \phi_B c_A^2 s_B^2$$
$$\quad - \frac{1}{2} \sin \gamma \sin \theta_A \sin \theta_B \cos \phi_A \sin \phi_B,$$
$$p_{DD} = \big[\cos^2 \gamma \sin^2(\phi_A + \phi_B) + \cos^2(\phi_A + \phi_B) \big] s_A^2 s_B^2. \tag{17}$$

The equivalent to $\hat{C}'$ in the new strategy space is

$$\hat{D}' \equiv \hat{M}^{(2)}(\pi, \pi/4) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i - 1 \\ i + 1 & 0 \end{pmatrix}. \tag{18}$$

Consider the expected payoffs when Bob plays $M^{(1)}(\theta, \phi)$ while Alice plays $\hat{D}'$:

$$\langle \$ \rangle = \$_{CC} \sin^2 \frac{\theta}{2} \sin^2 \gamma (1 + \sin 2\phi)/2$$
$$\quad + \$_{CD} \cos^2 \frac{\theta}{2} \big(\sin^2 \gamma \big)/2$$
$$\quad + \$_{DC} \cos^2 \frac{\theta}{2} \left[1 - \frac{1}{2} \sin^2 \gamma \right]$$
$$\quad + \$_{DD} \sin^2 \frac{\theta}{2} \left[1 - \frac{1}{2} \sin^2 \gamma (1 + \sin 2\phi) \right]. \tag{19}$$

For the Prisoners' Dilemma payoffs of Eq. (6), Bob maximizes his results for all γ by setting $\theta = \pi$ and $\phi = \pi/4$. Thus $\hat{D}' \otimes \hat{D}'$ is a symmetric NE for all γ with

$$\langle \$_A \rangle = \langle \$_B \rangle = 1 + 2 \sin^2 \gamma. \tag{20}$$

The payoff is superior to that of the classical NE of $\hat{D} \otimes \hat{D}$ provided $\gamma > 0$, as indicated in Fig. 1. The NE is not strict since there is some flexibility in the allowable values of ϕ. The permitted values of ϕ_A, ϕ_B are those for which $\phi_A + \phi_B = \pi/2$ and both ϕ satisfy

$$\frac{3 \sin^2 \gamma - 1}{5 \sin^2 \gamma} < \sin^2 \phi < \frac{2 \sin^2 \gamma + 1}{5 \sin^2 \gamma}. \tag{21}$$

However, $\phi_A = \phi_B = \pi/4$ is a focal point [22] that will attract the players for psychological reasons.

In the game of Chicken the strategy profile $\hat{D}' \otimes \hat{D}'$ results in mutual payoffs of $3 \sin^2 \gamma$. This is a NE provided $\sin^2 \gamma \geqslant \frac{1}{3}$. Again there is some ambiguity in the values of ϕ that can give rise to a NE, but this is unimportant for our purposes.[3] Below this level, the classical strategy profiles $\hat{C} \otimes \hat{D}$ and $\hat{D} \otimes \hat{C}$ are NE, as indicated in Fig. 2.

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[3] For completeness, we require $\phi_A + \phi_B = \pi/2$ with both ϕ constrained by $\frac{1}{3 \sin^2 \gamma + 1} < \sin^2 \phi < \frac{3 \sin^2 \gamma}{3 \sin^2 \gamma + 1}$.

The reason for the behavior of the new quantum equilibrium in both Prisoners' Dilemma and Chicken results from the fact that $\hat{D}'$ becomes equivalent to $\hat{D}$ as $\gamma \to 0$. In the former case this means that $\hat{D}' \otimes \hat{D}'$ remains a NE for all γ since it smoothly asymptotes to the classical NE of $\hat{D} \otimes \hat{D}$ as $\gamma \to 0$, while for Chicken there is a switch to one of the classical NE $\hat{C} \otimes \hat{D}$ or $\hat{D} \otimes \hat{C}$ for small enough γ, since $\hat{D} \otimes \hat{D}$ is an undesirable result for both players. By comparison, in the strategic space $S^{(1)}$, $\hat{C}'$ becomes equivalent to $\hat{C}$ as $\gamma \to 0$ giving rise to different classical–quantum thresholds since at some point defection will be favored by one or both players.

In the Battle of the Sexes, using the strategic space $S^{(2)}$ reverses the results obtained for this game presented in the previous section. Now $\hat{O} \otimes \hat{O}$ is the unique NE, where $\hat{O} = \hat{M}^{(2)}(0, \phi)$ for arbitrary ϕ. The payoffs are

$$\langle \$_A \rangle = 1 + \sin^2 \gamma,$$
$$\langle \$_B \rangle = 2 - \sin^2 \gamma, \tag{22}$$

the reverse of those in Eq. (13). In the case of both strategic spaces, the addition of phase factors to only one diagonal of the matrix for the player operators $M^{(1)}$ or $M^{(2)}$ favors one of the players by giving them the means to respond to their opponent's preferred choice while there is no such response to their own desired selection. Consequently, one of the two classical NE is eliminated.

4. Nash equilibria in N-player quantum Prisoners' Dilemma

There is no standard accepted payoff matrix for N-player Prisoners' Dilemma, and for $N > 3$ it is little studied. However, we consider it of interest in the quantum case since it gives us some information about N-partite entanglement. Here an N-player mutual Prisoners' Dilemma is considered rather than the more frequently studied case of a series of two player interactions among multiple players (e.g., see Ref. [23]). The requirements on the payoffs are that

- Defection is always the dominant strategy, that is, a player always receives more for defecting than for cooperating regardless of the actions of the other players.
- A player is better off when more of the other players cooperate.
- When $N - 2$ players' moves are fixed, the remaining 2×2 game is a two-player Prisoners' Dilemma.

These constraints are satisfied if cooperators and defectors receive, respectively,

$$\$_C = \begin{cases} 0, & m = 1, \\ 3 + 4(m - 2), & m > 1, \end{cases}$$
$$\$_D = 5 + 4(m - 1), \tag{23}$$

where m is the number of players cooperating.

With this payoff structure and the strategy space $S^{(1)}$ we find the generalization of the $\hat{C}' \otimes \hat{C}'$ equilibrium in the fully entan-

gled game is $\hat{C}_N^{\otimes N}$, where we have introduced the notation

$$\hat{C}_n \equiv \hat{M}^{(1)}(0, \pi/n) = \begin{pmatrix} e^{i\pi/n} & 0 \\ 0 & e^{-i\pi/n} \end{pmatrix}. \tag{24}$$

The strategy $\hat{C}_N$ is cooperation ($\theta = 0$) but with the addition of a phase factor π/N. For the strategy set $S^{(1)}$, all $\hat{D}_n \equiv \hat{M}^{(1)}(\pi, \pi/n)$ are equivalent to $\hat{D}$. To demonstrate that $\hat{C}_N$ is a symmetric NE strategy, consider the payoff to the last player, Larry, when he plays $\hat{M}^{(1)}(\theta, \phi)$ while all the other players continue with $\hat{C}_N$:

$$\langle \$_L \rangle = \$_{C \ldots C} \cos^2 \frac{\theta}{2} \left[1 - \frac{1}{2} \sin^2 \gamma \left(1 - \cos \left(2\phi - \frac{2\pi}{N} \right) \right) \right]$$
$$+ \$_{C \ldots CD} \sin^2 \frac{\theta}{2} \left[1 - \frac{1}{2} \sin^2 \gamma \left(1 - \cos \frac{2\pi}{N} \right) \right]$$
$$+ \$_{D \ldots DC} \sin^2 \frac{\theta}{2} \sin^2 \gamma \left(1 - \cos \frac{2\pi}{N} \right) \Big/ 2$$
$$+ \$_{D \ldots D} \cos^2 \frac{\theta}{2} \sin^2 \gamma \left(1 - \cos \left(2\phi - \frac{2\pi}{N} \right) \right) \Big/ 2. \tag{25}$$

Since $\$_{C \ldots C} > \$_{D \ldots D}$, Larry chooses $\phi = \pi/N$ to maximize $\cos(2\phi - \frac{2\pi}{N})$. Given that the result $D \ldots DC$ yields Larry nothing,

$$\langle \$_L \rangle = \$_{C \ldots C} \cos^2 \frac{\theta}{2}$$
$$+ \$_{C \ldots CD} \sin^2 \frac{\theta}{2} \left[1 - \frac{1}{2} \sin^2 \gamma \left(1 - \cos \frac{2\pi}{N} \right) \right]. \tag{26}$$

Hence, Larry prefers $\theta = 0$ provided

$$\$_{C \ldots C} \geqslant \$_{C \ldots CD} \left[1 - \frac{1}{2} \sin^2 \gamma \left(1 - \cos \frac{2\pi}{N} \right) \right]. \tag{27}$$

Using Eq. (23) this reduces to

$$\sin^2 \gamma \geqslant \frac{4}{(4N - 3)(1 - \cos \frac{2\pi}{N})}. \tag{28}$$

Thus for sufficient high entanglement, $\hat{M}^{(1)}(0, \pi/N) \equiv \hat{C}_N$ is the best strategy for Larry when the strategies of the other players are fixed. By symmetry this demonstrates that $\hat{C}_N^{\otimes N}$ is a NE. When the entanglement parameter γ drops below the level given by Eq. (28) this strategy profile is no longer a NE. There is a new NE where one player selects the strategy $\hat{C}_2$ while all the others defect. Again we demonstrate that this is a NE for a certain range of entanglement by considering the payoff to a player that changes from their equilibrium strategy. The payoff to Larry when he plays $\hat{M}^{(1)}(\theta, \phi)$ while Alice plays $\hat{C}_2$ and the remaining players defect is

$$\langle \$_L \rangle = \$_{CD \ldots DC} \cos^2 \frac{\theta}{2} \left[1 - \frac{1}{2} \sin^2 \gamma \left(1 + \cos(2\phi) \right) \right]$$
$$+ \$_{CD \ldots D} \sin^2 \frac{\theta}{2} \cos^2 \gamma + \$_{DC \ldots C} \sin^2 \frac{\theta}{2} \sin^2 \gamma$$
$$+ \$_{DC \ldots CD} \cos^2 \frac{\theta}{2} \sin^2 \gamma \left(1 + \cos(2\phi) \right)/2. \tag{29}$$
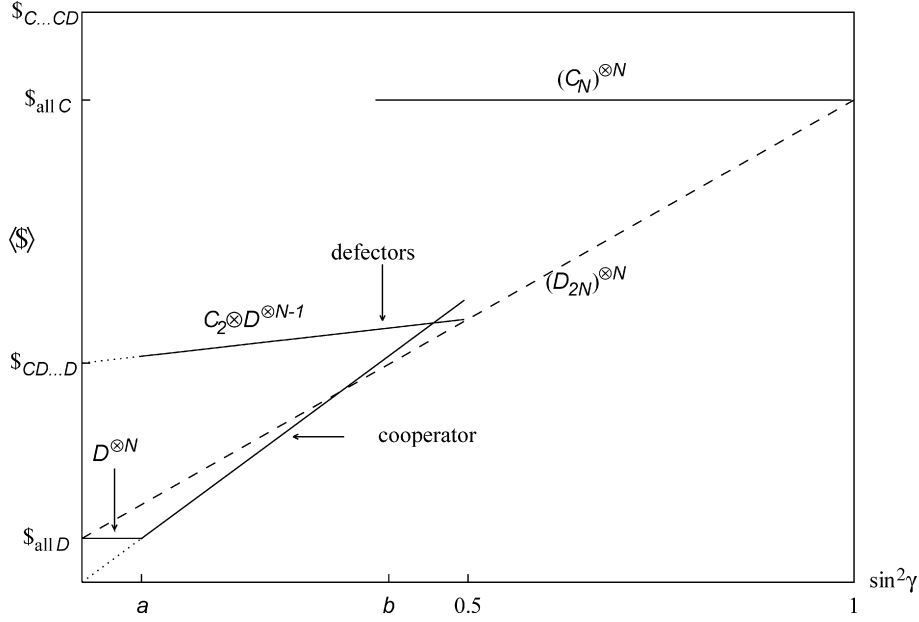
Fig. 3. The Nash equilibrium payoffs in a quantum version of a four-player Prisoners' Dilemma as a function of the degree of entanglement, as measured by $\sin^2 \gamma$, for the strategic space $S^{(1)}$ (solid lines), and $S^{(2)}$ (dashed line). In the case of $S^{(1)}$ there are three regions with different Nash equilibria. The same structure holds for arbitrary N, the main difference being that the gradient of the line for the cooperator's payoff in the intermediate region increases with N. The thresholds are, in general, $a = 1/(4N - 3)$ and $b = 4/((4N - 3)(1 - \cos \frac{2\pi}{N}))$.

For Larry, $\$_{DC...CD} > \$_{CD...DC}$ so he will choose $\phi = 0$. Then $\theta = \pi$ is his preferred choice provided

$$\sin^2 \gamma \leqslant \frac{\$_{CD...D} - \$_{CD...DC}}{\$_{DC...CD} + \$_{CD...D} - \$_{CD...DC} - \$_{DC...C}} = \frac{1}{2} \quad \forall N, \tag{30}$$

where the equality is calculated using the payoffs in Eq. (23). So for $\gamma \leqslant \pi/4$ Larry does best by sticking to $\hat{D}$. Similarly, if Alice plays $\hat{M}^{(1)}(\theta, \phi)$ while all the other players defect, her expected payoff is

$$\langle \$_A \rangle = \$_{CD...D} \left[1 - \frac{1}{2} \cos^2 \frac{\theta}{2} \sin^2 \gamma (1 - \cos(2\phi)) \right]$$

$$+ \$_{DC...C} \cos^2 \frac{\theta}{2} \sin^2 \gamma (1 - \cos 2\phi)/2$$

$$+ \$_{D...D} \sin^2 \frac{\theta}{2}. \tag{31}$$

Since, for Alice, $\$_{DC...C} > \$_{CD...D}$, $\phi = \pi/2$ is Alice's preferred value in the range $[0, \pi/2]$. Then she prefers $\theta = 0$ provided

$$\sin^2 \gamma \geqslant \frac{\$_{D...D} - \$_{CD...D}}{\$_{DC...C} - \$_{CD...D}} = \frac{1}{4N - 3}. \tag{32}$$

Hence for $1/(4N - 3) \leqslant \sin^2 \gamma \leqslant 1/2$ the strategy profile $\hat{C}_2 \otimes \hat{D}^{\otimes N-1}$ is a NE. There are N equilibria of this form depending on who plays $\hat{C}_2$. Of course, the existence of an equilibrium that "rational" players should select avoids the question of how to arrive at such an equilibrium. The asymmetric nature of this equilibrium would make reaching it extremely problematic in practice. Where both the asymmetric equilibria and the mutual cooperation equilibrium coexist, that is, for

$$\frac{4}{(4N - 3)(1 - \cos \frac{2\pi}{N})} \leqslant \sin^2 \gamma \leqslant \frac{1}{2}, \tag{33}$$

mutual cooperation is better for all the players and, being symmetric, there is no difficulty in reaching it. For $\gamma < 1/(4N - 3)$ the only NE is the classical one of mutual defection. Fig. 3 shows the NE payoffs for a four-player quantum Prisoners' Dilemma. The same structure applies for arbitrary N, the main difference being that the line representing the cooperator's payoff in the intermediate region becomes steeper for increasing N.

Straightforward calculations similar to the above demonstrate that there are no other NEs of the form $\hat{C}_m^{\otimes m} \otimes \hat{D}^{\otimes N-m}$, $m = 0, 1, 2, \ldots$, though we have not ruled out the existence of more complicated equilibria.

When the strategic space is chosen to be $S^{(2)}$ there is a symmetric NE analogous to that present in the two-player game, $\hat{D}_{2N}^{\otimes N}$, where

$$\hat{D}_n \equiv \hat{M}^{(2)}(\pi, \pi/n) = \begin{pmatrix} 0 & ie^{i\pi/n} \\ ie^{-i\pi/n} & 0 \end{pmatrix}. \tag{34}$$

This is demonstrated to be a NE by showing that any unilateral variation in strategy produces an inferior outcome for the varying player. Consider Larry's payoff when he plays $\hat{M}^{(2)}(\theta, \phi)$ while the others continue with $\hat{D}_{2N}$:

$$\langle \$_L \rangle = \$_{C...C} \sin^2 \frac{\theta}{2} \sin^2 \gamma \left[1 + \cos \left(2\phi - \frac{\pi}{N} \right) \right]/2$$

$$+ \$_{C...CD} \cos^2 \frac{\theta}{2} \sin^2 \gamma \left(1 + \cos \frac{\pi}{N} \right)/2$$

$$+ \$_{D...DC} \cos^2 \frac{\theta}{2} \left[1 - \frac{1}{2} \sin^2 \gamma \left(1 + \cos \frac{\pi}{N} \right) \right]$$

$$+ \$_{D...D} \sin^2 \frac{\theta}{2} \left[1 - \frac{1}{2} \sin^2 \gamma \left(1 + \cos \left(2\phi - \frac{\pi}{N} \right) \right) \right]. \tag{35}$$

Since $\$_{C...C} > \$_{D...D}$, Larry prefers $\phi = \pi/(2N)$. Larry will then choose $\theta = \pi$ provided

$$\$_{C...C}\sin^2\gamma + \$_{D...D} \geqslant \$_{C...CD}\left(1 + \cos\frac{\pi}{N}\right). \tag{36}$$

With the payoffs given by Eq. (23), Larry will select $\theta = \pi$ for all γ, provided $N \leqslant 8$. For $N \geqslant 9$ this NE begins to break down at large entanglements, however, mutual Prisoners' Dilemmas with such large numbers of players are of little or no practical interest.

Although the payoff structure versus entanglement presents an interesting picture with its different regions and bifurcations we would like to emphasize again that the existence of these NE and the associated entanglement thresholds are a function of the particular strategic space to which the players are restricted.

5. Two-player quantum games with generalized two-parameter strategies

The strategy spaces $S^{(1)}$ and $S^{(2)}$ can be generalized to

$$S_k^{(1)} = \begin{pmatrix} e^{i\phi}\cos\frac{\theta}{2} & ie^{ik\phi}\sin\frac{\theta}{2} \\ ie^{-ik\phi}\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix},$$

$$S_k^{(2)} = \begin{pmatrix} e^{ik\phi}\cos\frac{\theta}{2} & ie^{i\phi}\sin\frac{\theta}{2} \\ ie^{-i\phi}\sin\frac{\theta}{2} & e^{-ik\phi}\cos\frac{\theta}{2} \end{pmatrix}, \tag{37}$$

where $k \in [0, 1]$ is a fixed parameter. The two strategic spaces are distinct except when $k = 1$. Setting $k = 0$ reduces $S_k^{(j)} \rightarrow S^{(j)}$ for $j = 1, 2$.

Now consider a two-player quantum Prisoners' Dilemma using each of the two strategic spaces in Eq. (37). The strategic space $S_k^{(2)}$ is uninteresting since it yields the same payoff versus entanglement structure as $S^{(2)}$: $\hat{C}' \otimes \hat{C}'$ is not a NE for any γ, k while $\hat{D}' \otimes \hat{D}'$ is a NE for all γ, k with a payoff to each player of $1 + 2\sin^2\gamma$.

For $S_k^{(1)}$, $\hat{C}_2 \equiv \hat{C}'$ is a symmetric NE strategy with a payoff of $\$_{CC}$ for a range of γ and k. Consider Bob's payoff when he counters $\hat{C}_2$ with $\hat{M}_k^{(1)}(\pi, \phi)$ for some ϕ to be determined:

$$\langle\$_B\rangle = \$_{CD}(1 - \sin^2\gamma\cos^2 k\phi) + \$_{DC}\sin^2\gamma\cos^2 k\phi. \tag{38}$$

Bob maximizes this by selecting $\phi = \pi/2$. He prefers the resulting payoff over $\$_{CC}$ when

$$\sin^2\gamma\cos\frac{k\pi}{2} < \frac{\$_{CD} - \$_{CC}}{\$_{CD} - \$_{DC}}. \tag{39}$$

All $\$_{ij}$ are considered from Bob's perspective, so using the payoffs of Eq. (6), the right-hand side of the above equation evaluates to $\frac{2}{5}$. The region satisfying Eq. (39) is shown in Fig. 4.

The strategy $\hat{D}_{4k} \equiv \hat{M}_k^{(1)}(\pi, \pi/(4k))$, when played by both players, yields a payoff

$$\langle\$\rangle = \$_{CC}\sin^2\gamma + \$_{DD}\cos^2\gamma, \tag{40}$$

that, for $\gamma = \pi/2$, is the same as the cooperative payoff. If we maintain the restriction that $\phi \leqslant \pi/2$ this strategy can only be chosen when $k \geqslant \frac{1}{2}$. The region for which this is a NE can be deduced by considering the payoff to Bob if he instead switches
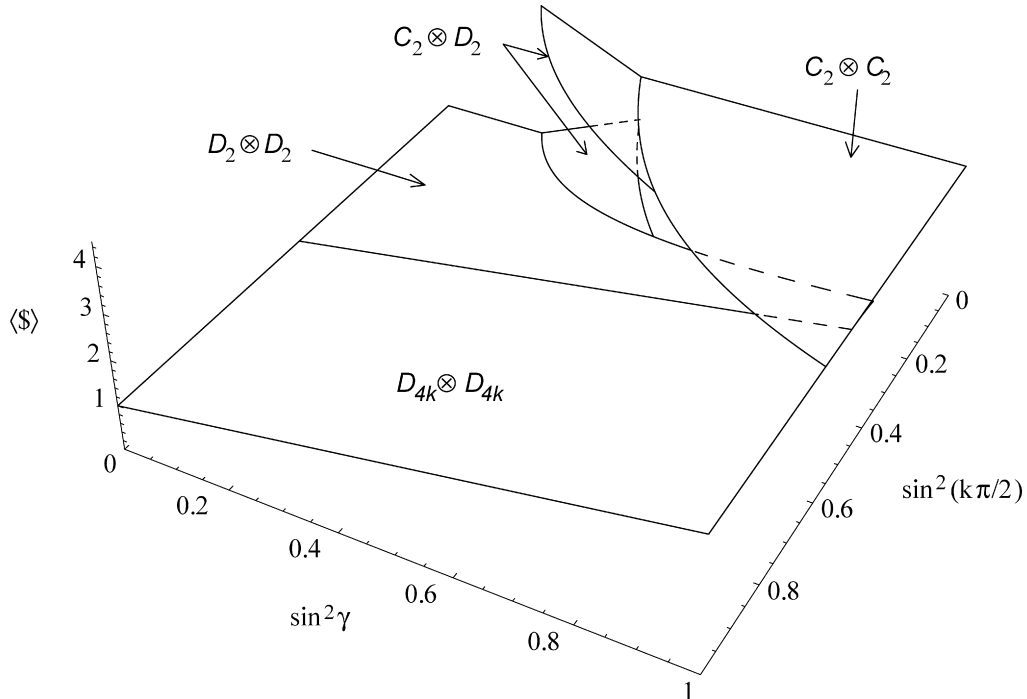

Fig. 4. The surfaces show the payoff to both players at the Nash equilibria for a two-player quantum Prisoners' Dilemma using the strategy space $S_k^{(1)}$, as a function of the entanglement and the strategy space parameter k. The strategy profile $\hat{D}_{4k} \otimes \hat{D}_{4k}$ is a Nash equilibrium for $k \geqslant \frac{1}{2}$, with the surface folding at $k = \frac{1}{2}$, below which the equilibrium is $\hat{D}_2 \otimes \hat{D}_2$. The strategy profile $\hat{C}_2 \otimes \hat{C}_2$ has a payoff of three but is only a Nash equilibrium for $\sin^2\gamma\cos^2\frac{k\pi}{2} \geqslant \frac{2}{5}$. There is a small region where there is an asymmetric Nash equilibrium of $\hat{C}_2 \otimes \hat{D}_2$ or $\hat{D}_2 \otimes \hat{C}_2$. Here the defector gets a payoff of between 3 and 4, while the cooperator must be satisfied with between 1 and 2.

to $\hat{M}_k^{(1)}(0, \phi)$:

$$\langle \$_B \rangle = \$_{CD}(1 - \sin 2k\phi)/2$$
$$+ \$_{DC}\big[1 - \sin^2 \gamma (1 - \sin 2k\phi)/2\big]. \tag{41}$$

For $k \geqslant \frac{1}{2}$ Bob maximizes this by setting $\phi = \pi/(4k)$. Bob prefers the resulting payoff over that of Eq. (40) when

$$\sin^2 \gamma > \frac{\$_{DD} - \$_{DC}}{\$_{DD} - \$_{CC} + \frac{1}{2}\$_{CD} - \frac{1}{2}\$_{DC} - \frac{1}{2}(\$_{CD} - \$_{DC})}, \tag{42}$$

which for the standard payoffs is never the case. That is, Bob cannot unilaterally improve upon Eq. (40), and by symmetry nor can Alice. Thus the strategy profile $\hat{D}_{4k} \otimes \hat{D}_{4k}$ is a NE for $k \geqslant \frac{1}{2}$. Following the same arguments, $\hat{D}_2 \otimes \hat{D}_2$ is a NE with

$$\langle \$ \rangle = \$_{CC} \sin^2 \gamma (1 - \cos 2k\pi)/2$$
$$+ \$_{DD}\big[1 - \sin^2 \gamma (1 - \cos 2k\pi)/2\big], \tag{43}$$

when $k < \frac{1}{2}$. As $k \to 0$ or $\gamma \to 0$, $\hat{D}_2$ becomes equivalent to $\hat{D}$. The region in (γ, k) space, and the associated payoff, for which $\hat{D}_2 \otimes \hat{D}_2$ is a NE is indicated in Fig. 4. Similarly, the regions for which $\hat{C}_2 \otimes \hat{D}_2$ and $\hat{D}_2 \otimes \hat{C}_2$ are NE can be determined. The payoffs and the region of applicability of these asymmetric NE are also indicated in Fig. 4. The defector gets the greater payoff.

We observe that by specifying the strategic space through the parameter k, the presence of Nash equilibria and, in general, the associated payoffs can be altered.

6. Conclusion

We have extended the examination of two-parameter quantum strategies in the Eisert protocol of quantum games. In the original form presented by Eisert et al. and subsequently taken up by a number of other authors we have presented results for the Nash equilibrium as a function of the entanglement parameter for the games of Chicken and the Battle of the Sexes, analogous to the results previously published for Prisoners' Dilemma. The asymmetry of the (unique) Nash equilibrium payoffs in Battle of the Sexes already raises alarm bells about the suitability of selecting this strategic space. By instead choosing an equivalent two-parameter strategic space where the phase factors are arranged differently, we show that different Nash equilibria can arise, with different payoffs and different classical–quantum thresholds.

We generalize these results in two ways: by examining an N-player quantum Prisoners' Dilemma and by considering an extension of the two-parameter strategies using an additional fixed parameter. The N-player Prisoners' Dilemma shows a similar, though more complex, structure in the Nash equilibrium payoffs to the two-player case. For the generalized two-parameter

strategies, the position of any entanglement thresholds between regions of differing Nash equilibria and the associated payoffs depend in most cases upon the (arbitrary) parameter used in generalizing the strategy space.

These results suggest that the new equilibria are artifacts of the choice of strategic space, that is, the particular slice of the space of unitary operators to which the players are restricted, rather than properties of the underlying game. In the absence of a physical justification for such a restriction, we feel the existing quantum game literature has attached too much importance to these equilibria.

Acknowledgements

References

[1] J. Eisert, M. Wilkins, M. Lewenstein, Phys. Rev. Lett. 83 (1999) 3077.
[2] D.A. Meyer, Phys. Rev. Lett. 82 (1999) 2543.
[3] J. Eisert, M. Wilkins, J. Mod. Opt. 47 (2000) 2543.
[4] A. Iqbal, A.H. Toor, Phys. Lett. A 280 (2001) 249.
[5] J. Du, H. Li, X. Xu, M. Shi, J. Wu, R. Han, Phys. Rev. Lett. 88 (2002) 137902.
[6] J. Du, X. Xu, H. Li, X. Zhou, R. Han, Fluct. Noise Lett. 2 (2002) R189.
[7] L.K. Chen, H. Ang, D. Kiang, L.C. Kwek, C.F. Lo, Phys. Lett. A 316 (2003) 317.
[8] J. Shimamura, S.K. Özdemir, F. Morikoshi, N. Imoto, Int. J. Quantum Inform. 2 (2004) 79.
[9] S.K. Özdemir, J. Shimamura, N. Imoto, Phys. Lett. A 325 (2004) 104.
[10] S.K. Özdemir, J. Shimamura, F. Morikoshi, N. Imoto, Phys. Lett. A 333 (2004) 218.
[11] A. Rapoport, A. Chammah, Prisoner's Dilemma: A Study in Conflict and Cooperation, University of Michigan Press, Ann Arbor, 1965.
[12] J. Du, X. Xu, H. Li, X. Zhou, R. Han, Phys. Lett. A 289 (2001) 9.
[13] J. Du, H. Li, X. Xu, X. Zhou, R. Han, J. Phys. A 36 (2003) 6551.
[14] S.C. Benjamin, P.M. Hayden, Phys. Rev. Lett. 87 (2001) 069801.
[15] J. Eisert, M. Wilkins, M. Lewenstein, Phys. Rev. Lett. 87 (2001) 069802.
[16] A.P. Flitney, D. Abbott, Fluct. Noise Lett. 2 (2002) R175.
[17] J. Du, H. Li, X. Xu, M. Shi, X. Zhou, Phys. Lett. A 302 (2002) 229.
[18] A.P. Flitney, D. Abbott, Proc. R. Soc. London A 459 (2003) 2463.
[19] L. Marinatto, T. Weber, Phys. Lett. A 272 (2000) 291.
[20] S.C. Benjamin, Phys. Lett. A 277 (2000) 180.
[21] A. Nawaz, A.H. Toor, J. Phys. A 37 (2004) 4437.
[22] S.C. Schelling, The Strategy of Conflict, Harvard Univ. Press, Cambridge, 1960.
[23] R. Axelrod, W. Hamilton, The Evolution of Cooperation, Basic Books, New York, 1984.