1. (a) Give the definition of Normal Form Game. [2]

(b) For the rest of this question, consider the Normal Form Game with the following matrix representation:

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Give the utilities to both players using the following strategy pairs:

(i) $\sigma_r = (1, 0) \qquad \sigma_c = (0, 1)$ [1]

(ii) $\sigma_r = (1/2, 1/2) \qquad \sigma_c = (1/3, 2/3)$ [1]

(iii) $\sigma_r = (1/4, 3/4) \qquad \sigma_c = (0, 1)$ [1]

(c) Give the definition for the Lemke-Howson algorithm. [5]

(d) Show that the vertices and their labels for the best response polytopes are given by:

For the row player best response player $\mathcal{P}$:

- $(0, 0)$ with labels: $\{0, 1\}$
- $(1/3, 0)$ with labels: $\{1, 2\}$
- $(0, 1/4)$ with labels: $\{0, 3\}$
- $(3/10, 1/10)$ with labels: $\{2, 3\}$

For the column player best response player $\mathcal{Q}$:

- $(0, 0)$ with labels: $\{2, 3\}$
- $(1/4, 0)$ with labels: $\{0, 3\}$
- $(0, 1/3)$ with labels: $\{1, 2\}$
- $(1/10, 3/10)$ with labels: $\{0, 1\}$

[4]

(e) Draw the best response polytopes. [2]

(f) Use the plots to carry out the Lemke-Howson algorithm with all possible initial dropped labels. [4]

(g) Consider a modified Lemke-Howson algorithm that uses any pair of fully labeled vertices (and not necessarily (0, 0)). Use any pair of fully labeled vertices found in the previous question and find an initial dropped label that gives a different Nash equilibrium than the ones obtained in the previous question. [1]

(h) Give a sketch of a proof, including potential assumptions that in a non degenerate game the number of Nash equilibria is odd. [4]

2. (a) Give the general definition of the Prisoner's Dilemma. [2]

(b) What values of S, T give valid Prisoner's Dilemma games:

(i) $A = \begin{pmatrix} 3 & S \\ 5 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & T \\ -1 & 1 \end{pmatrix}$ [4]

(ii) $A = \begin{pmatrix} 2 & S \\ -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -2 \\ S & 1 \end{pmatrix}$ [4]

(c) Consider the following reactive players:

$$p = (3/5, 3/4) \qquad q = (1/2, 1/4)$$

Obtain the Markov chain representation of a match between these two players. [4]

(d) State a theoretic result giving the utility of two general reactive players in a Prisoner's dilemma match (as a function of (R, S, T, P)). [5]

(e) Consider a reactive player $p = (x, x/2)$ and an opponent $q = (1/2, 1/4)$. Show that, for $(R, S, T, P) = (3, 0, 4, 1)$ the utility to the player p is given by:

$$u(x) = \frac{3x - 14}{(x - 8)}$$

[5]

(f) Using the above, show that:

$$\frac{du}{dx} = -\frac{10}{(x - 8)^2}$$

and use this to identify the optimal value of x.

[4]

3. (a) For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, obtain the following equation describing the corresponding evolutionary game:

$$\frac{dx}{dt} = x(f - \phi)$$

where $f = Ax$ and $\phi = fx$. [2]

(b) Define a mutated population. [2]

(c) Define an evolutionary stable strategy. [2]

(d) State and prove a theorem giving a general condition for an Evolutionary stable strategy. [4]

(e) Consider the following game $A = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$, obtain all evolutionary stable strategies. [5]

(f) Consider the accompanying 2008 paper entitled “Studying the emergence of invasiveness in tumours using game theory” by Basanta et al.

(i) Give a general summary of the paper. [3]

(ii) There is a minor error in this paper in the game matrix, describe and suggest the fix. [2]

(iii) How does the theorem in part 4 of this question relate to the findings of the paper? [2]

(iv) Suggest an alternative area of game theory that could also be used. [3]

4. (a) Give the definition of a Moran process on a game. [4]

(b) State and prove a theorem giving the fixation probabilities for a general birth death process. [6]

(c) Consider the Moran process on the Prisoners Dilemma: $A = \begin{pmatrix}4 & 1\\ 6 & 2\end{pmatrix}$ Use the above theorem to obtain the fixation probabilities for each strategy for $N = 5$. [4]

(d) Consider the following two strategies for the Prisoners Dilemma:

- Tit For Tat: start by cooperating and then repeat the opponents previous message.
- Alternator: start by cooperating and then alternate between defecting and cooperating.

Assuming a match lasting 5 turns show that the utility matrix between these two strategies corresponds to:

$$\begin{pmatrix}20 & 18\\ 18 & 16\end{pmatrix}$$

[5]

(e) Obtain the fixation probabilities x_1 for each strategy for $N = 5$. [6]