1. (a) Give the definition of a normal form game. [2]

(b) Consider the normal form game with the following matrix representation:

$$A = \begin{pmatrix} 30 & 6 \\ 12 & 30 \end{pmatrix} \qquad B = \begin{pmatrix} 12 & 30 \\ 42 & 12 \end{pmatrix}$$

Consider the mixed strategies for the row player $\sigma_r = (x, 1-x)$ and the column player $\sigma_c = (y, 1-y)$. Sketch a plot of:

- the row player's utilities: $u_r((1,0),\sigma_c)$ and $u_r((0,1),\sigma_c)$. [1]
- the column player's utilities: $u_c(\sigma_r,(1,0))$ and $u_c(\sigma_r,(0,1))$. [1]

Using the plot, obtain the best responses of both players. [1]

(c) Consider a modification of the above game where a third strategy is given to the column player:

$$A = \begin{pmatrix} 30 & 6 & 36 \\ 12 & 30 & 18 \end{pmatrix} \qquad B = \begin{pmatrix} 12 & 30 & 22 \\ 42 & 12 & 6 \end{pmatrix}$$

Sketch a plot of $u_c(\sigma_r,(0,0,1))$. [2]

Using this plot and the plots of part (b) above, obtain the best responses for both players for this modified game. [4]

(d) Give the definition of the row/column best response polytopes for a 2 player game $(A,B) \in \mathbb{R}^{m\times n^2}$. [2]

(e) State the Lemke-Howson algorithm. [4]

(f) Using Tableaux, carry out the Lemke-Howson algorithm on the modified game. Describe how this confirms your finding of part (c) [8]

2. (a) Give the definition of a repeated game. [2]

(b) Give the definition of strategy in a repeated game. [2]

(c) Consider the following stage game:

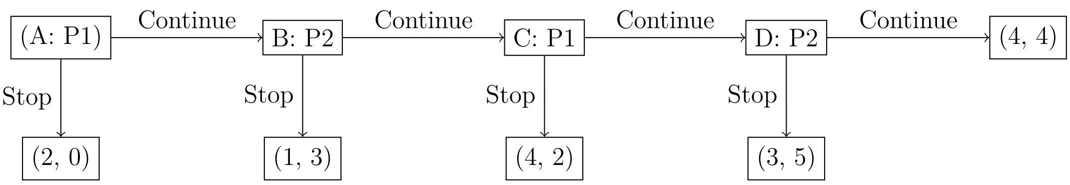
$$A = \begin{pmatrix} 1 & 4 & -1 \\ -1 & 0 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 5 & 5 \\ 18 & -1 & -2 \end{pmatrix}$$

Obtain all possible histories for the corresponding 2 stage repeated game. [3]

(d) Obtain all Nash equilibria for the 2 stage repeated game of part (c) that are sequences of stage Nash equilibria. [4]

(e) Obtain a Nash equilibrium that is not a sequence of stage Nash equilibria for the 2 stage repeated game of part (c). Justify this. [5]

(f) Consider the Centipede game which can be represented by the following diagram:



Players take turns:

- At step (A), the first player can choose to stop in which case they get a utility of 2 and the second player a utility of 0.
- If the first player chooses to continue, then at step (B), the second player can choose to stop in which case they get a utility of 3 and the first player a utility of 1.
- If the second player chooses to continue, then at step (C), the first player can choose to stop in which case they get a utility of 4 and the second player a utility of 2.
- If the first player choose to continue, then at step (D), the second player can choose to stop in which case they get a utility of 5 and the first player a utility of 3.
- If the second player chooses to continue then both players get a utility of 4.

Show how this game can also be represented as a Normal form game with the following matrices:

$$A = \begin{pmatrix} 4 & 3 & 1 & 1 \\ 4 & 4 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 5 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[6]

(g) Obtain the Nash equilibria in pure strategies for the Centipede game described in part (f). [3]

3. (a) Give the general definition of the Prisoner's Dilemma. [2]

(b) What values of S, T, if any, give valid Prisoner's Dilemma games for:

(i) $A = \begin{pmatrix} 3 & S \\ 6 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & T \\ -1 & 1 \end{pmatrix}$ [4]

(ii) $A = \begin{pmatrix} 2 & S \\ -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -2 \\ S & 1 \end{pmatrix}$ [4]

(c) For the remainder of this question, consider the following game:

$$A = \begin{pmatrix} b-c & c \\ b & 1 \end{pmatrix} \qquad B = \begin{pmatrix} b-c & b \\ c & 1 \end{pmatrix}$$

What conditions need to hold on b, c to ensure that the game is a Prisoners Dilemma? [5]

(d) Consider two reactive players $p = (p_1, p_2)$ and $q = (q_1, q_2)$. Stating any results you use show that the expected utility for player p is given by:

$$s_1 s_2 \times (b-c) + s1(1-s_2) \times c + (1-s_1)s_2 \times b + (1-s_1)(1-s_2)$$

where:

$$s_1 = \frac{q_2 r_1 + p_2}{1 - r_1 r_2} \qquad s_2 = \frac{p_2 r_2 + q_2}{1 - r_1 r_2}$$

for:

$$r_1 = p_1 - p_2 \qquad r_2 = q_1 - q_2$$

[3]

(e) Using this, for $(b, c) = (2, 1/2)$, show that the utility of the first player is given by: [3]

$$s_2 - s_1/2 + 1$$

(f) What is the optimal behaviour against a player q for which $q_1 = q_2$? [4]

4. (a) For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, obtain the following equation describing the corresponding evolutionary game:

$$\frac{dx}{dt} = x(f - \phi)$$

where $f = Ax$ and $\phi = fx$. [2]

(b) Define a mutated population. [2]

(c) Define an evolutionary stable strategy. [2]

(d) State and prove a theorem giving a general condition for an evolutionary stable strategy. [4]

(e) Consider the following game $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Obtain all evolutionary stable strategies. [5]

(f) Consider the accompanying 2008 paper entitled "Studying the emergence of invasiveness in tumours using game theory" by Basanta et al.

(i) Give a general summary of the paper. [3]

(ii) There is a minor error in this paper in the game matrix. Describe and suggest the fix. [2]

(iii) How does the theorem in part (d) of this question relate to the findings of the paper? [2]

(iv) Suggest an alternative area of game theory that could also be used. [3]