RANDOM MODELS FOR THE JOINT TREATMENT OF RISK AND TIME PREFERENCES

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Abstract. The aim of this paper is to develop a simple, tractable and theoretically sound stochastic framework to deal with heterogeneous risk and time preferences. This we do in three steps: (i) study the comparative statics of the main deterministic model of risk and time preferences, the discounted expected utility, (ii) embed the model and its comparative statics within the random utility framework, and (iii) illustrate the empirical implementation of the model using several experimental datasets. The solidity of the proposed framework and its effectiveness in delivering novel methodological and empirical results of interest for the understanding of risk and time preferences are demonstrated throughout.

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1. Introduction

Economic situations simultaneously involving risk and time pervade most spheres of everyday live, and the presence of heterogeneous attitudes in these situations is the rule. Unfortunately, the empirical strategies currently being applied in discrete choice frameworks either fail to capture heterogeneity or present serious theoretical shortcomings. The main aim of this paper is to develop a simple, tractable and theoretically sound stochastic framework for the treatment of heterogeneity in discrete choice problems involving risk and time attitudes. Essentially, we build upon intuitive comparative

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statics of risk and time preferences to provide a sound framework for econometric purposes. This is a foundational issue in Economics, and to the best of our knowledge, this is the first paper to address it.

We study two settings with widespread impact, which appear in by far the greater part of the experimental literature on risk and time preferences. In the first, the individual expresses preferences over *dated lotteries* offering a contingent prize at a given time period.[1] This setting includes the particular case in which risk and time are treated separately, in the sense that the only choice options are immediate payoffs or, alternatively, delayed but certain payoffs.[2] The second setting involves *convex budgets*, where the individual decides how to distribute an endowment between an earlier stock and a later stock. It leads to a situation that can ultimately be defined as a pair of lotteries that will be paid out at two different times.[3]

We start by studying the comparative statics of the most standard model of risk and time preferences: the so-called discounted expected utility (DEU) model.[4] This model directly combines the expected utility treatment of risk with the exponentially discounted utility treatment of time. The comparative statics of DEU can be described as follows: (i) more risk aversion, i.e. a greater preference for present degenerate lotteries over other present lotteries, is captured by the curvature of the monetary utility function, exactly as in expected utility, and (ii) more delay aversion, i.e. a greater preference for present degenerate lotteries over future degenerate lotteries, is captured by the curvature of a normalized monetary utility function, exactly as in exponentially discounted utility. We show that these simple classic comparisons extend to more general pairs of alternatives simultaneously involving risk and time considerations, when properly controlling for time and risk attitudes. These comparative statics constitute a solid basis for the treatment of heterogeneity.

In Section 4 we embed DEU into the stochastic framework of random utility models. Crucially, we show that all the comparative statics of DEU are immediately extensible

[1] For papers exploiting the generality of this setting, see Ahlbrecht and Weber (1997), Coble and Lusk (2010), Baucells and Heukamp (2012) and Cheung (2015).

[2] See Andersen et al. (2008), Burks et al. (2009), Dohmen et al. (2010), Tanaka et al. (2010), Abdellaoui et al. (2013), Benjamin et al. (2013), Falk et al. (2018) or Jagelka (2019).

[3] This setting was proposed by Andreoni and Sprenger (2012). See also Cheung (2015), Miao and Zhong (2015), Epper and Fehr-Duda (2015), and Kim et al. (2018).

[4] In the Supplementary Material we show that our treatment of heterogeneity can be immediately extended to generalizations of DEU.

to the random utility model built upon it, thereby guaranteeing a proper understanding of risk and time attitudes with stochastic data. To place this result in perspective, let us recall that Apesteguia and Ballester (2018) show that the standard approach in the study of a single trait, either risk or time, involves using additive iid random utility models, which have counterintuitive properties.[5] Apesteguia and Ballester (2018) also show the good stochastic properties of parametric random utility models built upon a one-parameter family of utilities. Fundamentally, this paper shows, for the first time, that multidimensional models can be successfully embedded into (not necessarily parametric) random utility models.

Section 4 continues with some relevant results regarding the estimation of DEU random utility models. In actual practice, the estimation of stochastic models is typically facilitated by the use of particular probability distributions over the relevant parameters. We show that, unless carefully designed, these simplifications can lead to undesirable outcomes. We illustrate with the case of a homogeneous Bernoulli function, under the assumption of independent probability distributions for the discount parameter and the curvature of the Bernoulli function. In this case, we show that, for sufficiently risk-averse individuals, any earlier lottery is almost sure to be chosen over any later one. This, of course, is nonsense, since the earlier lottery may involve very low payoffs, while the later one may involve arbitrarily large payoffs. We show that if the interest lies in imposing independence, a flawless practical approach requires us to assume independence not on the parameters, but on the normalized parameters suggested by the comparative statics of the model, as shown in Section 3.

In the third part of the paper, we illustrate our approach by structurally estimating risk and time preferences using the DEU random utility model studied in Section 4, under the assumption of homogeneous monetary functions. We use three different, well-known experimental datasets, which represent the diversity of experimental elicitation methods in common use. Although the literature has used different models and estimation strategies with the various experimental elicitation methods, we show that our approach offers a unified framework, and reconciles the conflicting results reported in the literature.

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[5] Additive iid random utility models add an error term to the utility valuation and generically predict higher choice probabilities for a risky lottery over a degenerate one in individuals displaying more risk aversion in the baseline, deterministic, utility.

We start with the dataset of Andersen et al. (2008), which exemplifies the oft-used approach in which risk and time attitudes are elicited separately by way of multiple price lists. An immediate implication of our study of the comparative statics of risk and time in Section 3 is that, in this type of elicitation procedures, the separate estimation of risk and time preferences is but equivalent to their joint estimation. To stress the relevance of this point, let us note that the literature has settled around the idea that time preferences cannot be estimated without estimating risk preferences. That is, actual experimental practice uses either double multiple price lists, as in Andersen et al. (2008), or the convex budget setting of Andreoni and Sprenger (2012). When using the right notion of delay aversion established in Section 3, the estimation of risk aversion is unnecessary in multiple price lists. That is, delay aversion can be estimated using a single multiple price list, and the results are valid estimates that can be used to compare individual time preferences.[6] This is potentially important for the design of field experiments where the consideration of time and budget constraints is often crucial.

We then use the dataset of Coble and Lusk (2010) to study the estimation of data in a general dated lottery setting. This enables empirical testing of the external validity of experimental designs, which, like the previous one, use separate elicitations of risk and time preferences, as opposed to those incorporating choice problems in which both dimensions are simultaneously relevant. Our results suggest that the separate elicitation of risk and time preferences is a reasonable option, since the overall estimates are remarkably similar in both scenarios.

Finally, we draw on Andreoni and Sprenger (2012) to illustrate the applicability of our stochastic framework to settings involving convex budgets. We obtain that DEU random utility models are remarkably effective in capturing idiosyncratic choice patterns. This accounts for the large fraction of corner choices, while at the same time rationalizing the inverse-U shape of interior choices. This is a noteworthy result from our stochastic framework, especially in light of the discussion in the literature on the nature of the data generated by convex budgets, and the explanatory challenges they involve. Using non-linear least squares, Andreoni and Sprenger (2012) obtain estimates of risk aversion close to 0 (linear utility). Their model is good at predicting the mean choices, but fails to explain corner choices, which constitute almost half the

[6] Risk attitudes would be needed to estimate the discount parameter, but we show that this is just an auxiliary parameter with no valid comparative statistics and should not be the object of estimation.

observed choices in their sample. Harrison et al. (2013), on the other hand, use an additive iid random utility model, which is effective in explaining the observed corner choices but only at the cost of poorly fitting the observed interior choices. Furthermore, their estimates result in negative risk-aversion levels (convex utility function). The implausibility of this result leads them to conclude that the convex budget design is flawed and subjects probably have a hard time understanding the questions. Our methodology reconciles these two results by applying an estimation procedure which results in an estimated median risk aversion close to zero, and is able to predict both the corner choices and the interior choices observed in the data. Our results show that the use of a valid stochastic methodology enables us to smoothly account for the heterogeneity of observed behavior in convex budget settings.

To conclude, this paper proposes a stochastic framework for the estimation of heterogeneous risk and time preferences, and shows this framework to be well-founded in the sense that it respects the comparative statics of the deterministic model. Hence it contributes to the latest active methodological literature on preference estimations (see, e.g., Della Vigna, 2018; Barseghyan et al. 2018; Dardanoni et al. 2020). The implementation of the framework in practice is illustrated on a diversity of existing experimental datasets. Our methodology is shown to yield reasonable estimates, while also providing a unifying and theoretically solid framework for understanding estimates in both multiple price lists and convex budget designs, and with the ability to account for highly heterogeneous datasets. The quick and easy implementation of our stochastic framework is also illustrated.[7]

2. Framework

The set of monetary payouts is $X = \mathbb{R}_+$. A lottery is a finite collection of payouts and the probabilities with which they are awarded, i.e., a vector $l = [p_1, \ldots, p_N; x_1, \ldots, x_N]$ with $p_n \geq 0$, $\sum_n p_n = 1$ and $x_n \in X$. A degenerate lottery is a lottery composed of one sure payoff, i.e., a lottery of the form $[1; x]$. A basic lottery is a lottery containing at most one strictly positive payoff, i.e., a lottery of the form $[p, 1-p; x, 0]$ with $x > 0$. We denote by $\mathcal{L}, \mathcal{D}$ and $\mathcal{B}$ the space of all lotteries, all degenerate lotteries and all basic lotteries, respectively. Time can take any positive real value, i.e. $T = \mathbb{R}_+$.

[7] All our estimation programs are available for use at https://github.com/agutieda/Estimating-Risk-and-Time-Preferences.

The literature has primarily used two different settings in the study of risk and time preferences.

2.1. **Dated Lotteries.** A setting that has been intensively studied in the joint treatment of risk and time preferences involves individuals facing menus made up of alternatives $(l,t) \in \mathcal{L} \times T$, that represent the situation in which lottery $l \in \mathcal{L}$ is awarded at time $t \in T$.[8] We call these alternatives dated lotteries. The general case is analyzed in Coble and Lusk (2010) and Cheung (2015). Andersen et al. (2008), Burks et al. (2009), Dohmen et al. (2010), Tanaka et al. (2010), Benjamin et al. (2013), Falk et al. (2018) or Jagelka (2019) elicit risk and time attitudes separately, such that subjects face menus made up exclusively either of present lotteries, i.e., elements in $\mathcal{L} \times \{0\}$, or, alternatively, dated degenerate lotteries, i.e., elements in $\mathcal{D} \times T$. Ahlbrecht and Weber (1997) and Baucells and Heukamp (2012) study the case of dated basic lotteries, i.e., elements in $\mathcal{B} \times T$.

2.2. **Convex Budgets.** In an alternative setting, in increasing use since it was pioneered by Andreoni and Sprenger (2012), subjects are faced with convex budget menus. Here, two independent, dated, basic lotteries $([p, 1-p; x, 0], t)$ and $([q, 1-q; y, 0], s)$, with $t < s$, are presented to the individual, who chooses a budget share $\alpha \in [0,1]$ to be invested in the first dated lottery, leaving $1-\alpha$ to be invested in the second. Thus, if α is chosen, the individual receives the sequence of dated basic lotteries, $([p, 1-p; \alpha x, 0], t)$ and $([q, 1-q; (1-\alpha)y, 0], s)$.[9]

3. Discounted Expected Utility

The most commonly-used representation for the joint analysis of risk and time preferences is the discounted expected utility (DEU) model.[10] Formally, denote by $\mathcal{U}$ the set of all continuous and strictly increasing functions $u : \mathbb{R}_+ \to \mathbb{R}_+$ such that $u(0) = 0$. DEU requires us to consider a monetary utility function $u \in \mathcal{U}$ and a discount factor

[8] These menus are typically binary, as in multiple price lists settings, thereby allowing preferences and choices to be treated as equivalents.

[9] It is typically understood that, if both prizes are awarded, individuals perceive them as being consumed on reception.

[10] See Phelps (1962) for an early application of the model, and Fishburn (1970) for an axiomatic treatment of DEU in the context of lotteries over sequences of monetary payouts. In the Supplementary Material we provide an axiomatization of DEU and other models, in the context of dated lotteries, as a way of establishing the formal relationships between the models.

$\delta \in (0,1)$, to arrive at the following evaluation of the dated lottery (l,t) or the share α:

$$DEU_{\delta,u}(l,t) = \delta^t \sum_{n=1}^{N} p_n u(x_n)$$
$$DEU_{\delta,u}(\alpha) = \delta^t pu(\alpha x) + \delta^s qu((1-\alpha)y).$$

In many applications, monetary utility functions are parameterized. The most common family of monetary utility functions assumes homogeneity, adopting the well-known homogeneous functional form $u_h(x) = \frac{x^{1-h}}{1-h}$, with $h < 1$.[11] When homogeneous monetary utility functions are being considered, we refer to DEU as DEU-H. This particular model will be used extensively throughout the paper.

3.1. **More Risk Aversion and More Delay Aversion.** It is well-known that, in expected utility, individual 1 is said to be more risk averse than individual 2 if, whenever individual 2 prefers a degenerate lottery to a non-degenerate lottery, so does individual 1; which is simply equivalent to saying that the utility function of individual 1 is more concave than that of individual 2. The first part of Proposition 1 below formalizes this notion in the context of DEU by closing down the time component, comparing a riskless present lottery with any other potentially risky present lottery. Importantly, as in the standard treatment of lotteries in expected utility, this notion allows for comparative static exercises far beyond the simple comparisons used in the definition.[12] Similarly, it is well-know that, in exponentially discounted utility, individual 1 is said to be more delay averse than individual 2 if, whenever individual 2 prefers a present payoff over a delayed one, so does individual 1; which is equivalent to saying that a specific normalized utility function of individual 1 is more concave than the corresponding one for individual 2.[13] The second part of Proposition 1 formalizes the notion of more delay aversion for DEU by closing down the risk component, and comparing a present riskless payoff with another potentially delayed riskless payoff. Again, the notion of

[11] In the context of risk preferences, this family is typically called CRRA. Notice how the assumption $h < 1$ is fundamental to guarantee that $u_h \in \mathcal{U}$.

[12] For example, lotteries related by mean preserving spreads, or Holt and Laury (2002) pairs of lotteries $[p, 1-p; x_1, x_4]$ and $[p, 1-p; x_2, x_3]$ such that $x_1 < x_2 < x_3 < x_4$.

[13] It is important to stress that, on its own, the discount parameter is uninformative about delay aversion; a fact that is often overlooked in empirical applications. To illustrate, consider two exponentially discounted utilities, $.97\sqrt{x}$ and $.95x$. Although the discount parameters suggest that the second individual is more delay averse than the first, since the corresponding present values of \$1 paid at $t = 1$ are .94 and .95, it is the first individual who is the more delay averse.

more delay aversion in DEU is informative about other comparisons beyond the simple one used in the definition.[14] For subsequent analysis, let us briefly stress the nature of this normalization, which was already introduced in Fishburn and Rubinstein (1982). When DEU evaluates a unique payoff at a given moment in time, it can be shown that $DEU_{\delta,u}$ is equivalent to $DEU_{\theta,\bar{u}}$, where $\theta \in (0,1)$ can be freely chosen, and $\bar{u} = u^{\frac{\log\theta}{\log\delta}}$. Hence, a proper comparison of delay aversion simply requires us to set a common discount factor across individuals.

Based on the above discussion, we now formalize the basic notions of more risk aversion and more delay aversion in the context of DEU.[15]

Proposition 1.

(1) *More risk aversion: u_1 is a concave transformation of u_2 if, and only if, for every $l \in \mathcal{L}$ and every $x \in X$, $DEU_{\delta_2,u_2}([1;x],0) \geq DEU_{\delta_2,u_2}(l,0)$ implies that $DEU_{\delta_1,u_1}([1;x],0) \geq DEU_{\delta_1,u_1}(l,0)$.*

(2) *More delay aversion: Fix $\theta \in (0,1)$. $u_1^{\frac{\log\theta}{\log\delta_1}}$ is a concave transformation of $u_2^{\frac{\log\theta}{\log\delta_2}}$ if, and only if, for every $x, y \in X$ and every $s \in T$, $DEU_{\delta_2,u_2}([1;x],0) \geq DEU_{\delta_2,u_2}([1;y],s)$ implies that $DEU_{\delta_1,u_1}([1;x],0) \geq DEU_{\delta_1,u_1}([1;y],s)$.*

The analysis of the parametric family DEU-H is more direct. More risk aversion simply requires us to compare the curvature of power functions $\frac{x^{1-h_1}}{1-h_1}$ and $\frac{x^{1-h_2}}{1-h_2}$, which reduces to the comparison of the parameters h_1 and h_2. Hence, individual 1 is more risk averse than individual 2 if, and only if, $1-h_1 \leq 1-h_2$, i.e., if, and only if, $h_1 \geq h_2$. More delay aversion requires us to compare the curvature of the normalized (power) functions $(\frac{x^{1-h_1}}{1-h_1})^{\frac{\log\theta}{\log\delta_1}}$ and $(\frac{x^{1-h_2}}{1-h_2})^{\frac{\log\theta}{\log\delta_2}}$. That is, we need to evaluate whether $(1-h_1)\frac{\log\theta}{\log\delta_1} \leq (1-h_2)\frac{\log\theta}{\log\delta_2}$, which holds if, and only if, $\hat{\delta}_1 \equiv \delta_1^{\frac{1}{1-h_1}} \leq \delta_2^{\frac{1}{1-h_2}} \equiv \hat{\delta}_2$. The following argument may help in the interpretation of this comparison. Since every monetary utility in the homogeneous family is a power transformation of the linear utility function, we can represent the choice behavior of individual i over dated degenerate lotteries by using the alternative DEU-H composed of the corrected discount factor $\hat{\delta}_i$ and the linear monetary utility function. It then becomes evident that individual 1 is more delay averse than individual 2 if, and only if, $\hat{\delta}_1 \leq \hat{\delta}_2$.

[14] For example, settings where payoffs or streams of payoffs can be clearly ordered in terms of delay. See Benoît and Ok (2007) for a general treatment of the notion of more delay aversion.

[15] The proofs are contained in Appendix A.

3.2. **Comparative Statics with Dated Lotteries.** When both risk and time features are present, it is only in very restrictive scenarios that the analyst is able to interpret choices exclusively through the notion of more risk aversion or that of more delay aversion. For example, whenever the dated lotteries are awarded at the same time period, the choice is uniquely governed by risk aversion. Similarly, whenever dated basic lotteries with the same probability of winning are being considered, choice is uniquely governed by delay aversion. The correct approach to obtain rich comparative statics requires us to control for one of these two notions, and then establish comparative statics for the other. The following result on dated lotteries illustrates this.

Proposition 2.

(1) *Consider two DEU individuals such that both are equally delay averse but the first is more risk averse than the second. Then, for every $l \in \mathcal{L}$, every $x \in X$ and every pair $t_1, t_2 \in T$, if the second individual prefers $([1; x], t_1)$ to (l, t_2), so does the first.*
(2) *Consider two DEU individuals such that both are equally risk averse but the first is more delay averse than the second. Then, for every $l, l' \in \mathcal{L}$, and every $t, s \in T$ with $t < s$, if the second individual prefers (l, t) to (l', s), so does the first.*

The first part of Proposition 2 analyzes the case in which two DEU individuals display the same level of delay aversion. Under this condition, more risk aversion unequivocally generates a higher preference for riskless lotteries, irrespective of the moment of payoff. The second part of Proposition 2 analyzes the case in which two DEU individuals share the same level of risk aversion. Under this restriction, more delay aversion unequivocally generates a higher preference for earlier lotteries, irrespective of the risk involved.

Controlling for delay aversion requires us to set $(\delta_2, u_2) = (\delta_1^k, u_1^k)$ for some $k > 0$. Under these restrictions, if individual 1 is more risk averse, it must be that $k \geq 1$, and hence u_2 must be a convex transformation of u_1 and $\delta_2 \leq \delta_1$. Similarly, controlling for risk aversion requires monetary utilities to be equal, in which case more delay aversion comes with a smaller discount factor. For the case of DEU-H, the relations between (δ_1, h_1) and (δ_2, h_2) that must be considered are straightforward. Part 1 requires us to set $\hat{\delta}_1 = \delta_1^{\frac{1}{1-h_1}} = \delta_2^{\frac{1}{1-h_2}} = \hat{\delta}_2$ and $h_1 \geq h_2$, while part 2 requires us to set $h_1 = h_2$ and $\delta_1 \leq \delta_2$.

3.3. **Comparative Statics in Convex Budgets.** We now discuss the convex budget problem.[16] Denote by $\overline{\alpha}_i$ the share that equates the discounted expected utilities of individual i in both periods, i.e., $\delta_i^t p u_i(\overline{\alpha}_i x) = \delta_i^s q u_i((1-\overline{\alpha}_i)y)$, and by α_i^* the share that maximizes her discounted expected utility.

Proposition 3.

(1) *If individual i is a risk lover, then $\alpha_i^* \in \{0,1\}$. Moreover, if both individuals are risk lovers and individual 1 is more delay averse than individual 2 then $\alpha_1^* \geq \alpha_2^*$.*

(2) *If individual i is (strictly) risk averse, then $\alpha_i^* \in (0,1)$. Moreover:*

  (a) *If both individuals have the same level of risk aversion, and individual 1 is more delay averse than individual 2 then $\alpha_1^* \geq \alpha_2^*$.*

  (b) *If both individuals have the same level of delay aversion, and individual 1 is more risk averse than individual 2 then $\alpha_1^* \leq \alpha_2^*$ (resp. $\alpha_1^* \geq \alpha_2^*$) whenever $\alpha_1^* \geq \overline{\alpha}_1$ (resp. $\alpha_1^* \leq \overline{\alpha}_1$).*

The first part of Proposition 3 shows that risk lovers, i.e. those with convex monetary utility functions, allocate their whole share either to the earlier lottery or to the later one.[17] For illustrative purposes, consider the case of a risk-neutral individual. One unit of money invested in the earlier period brings a constant marginal utility return of $\delta^t p$ units, while one invested in the later period brings a constant marginal utility return of $\delta^s q$ units. Ultimately, the individual places all her money in only one of the two lotteries. Any risk-lover faces a similar dilemma and hence $\alpha_i^* \in \{0,1\}$. In this situation, more delay aversion implies a higher preference for the present.

With risk-averse individuals, the solution will be interior and the interplay between risk and time becomes relevant.[18] Part (2)(a) of Proposition 3 analyzes the case in which we control for risk aversion, where a higher level of delay aversion unequivocally generates the choice to invest a higher share in the earlier lottery. Part (2)(b) analyzes the case with equal delay aversion, where a higher level of risk aversion results in less

[16] To simplify the exposition, we assume that u is differentiable and that $\lim_{x\to 0} u'(x) = +\infty$, as is typically the case in standard parameterizations.

[17] Convexity is only required in the relevant range $[0, y]$.

[18] The first-order condition of the optimization problem is $\frac{u'(\alpha x)}{u'((1-\alpha)y)} = \frac{\delta^s q y}{\delta^t p x}$ and hence the solution depends on the ratio $\frac{p}{q}$, an observation made and tested in Andreoni and Sprenger (2012).

extreme solutions involving a compound lottery across time that is less risky. If the solution of an individual is on the right-hand (respectively, left-hand) side of $\overline{\alpha}$, that of a more risk-averse individual will be smaller (respectively, larger) than that of the first individual. That is, the same level of delay aversion together with a higher level of risk aversion unequivocally leads to a more balanced allocation of money across time.

4. Random utility models

In this section, we discuss the structure of random utility models and their implementation for the treatment of risk and time preferences under DEU.[19] Formally, we define DEU-RUM as the simplex over the set of DEU representations, with an instance of DEU-RUM corresponding to a particular probability distribution f, which captures the prevalence of each DEU. At the choice stage, one DEU is realized according to f, and maximized, thereby generating random choices.[20] We now discuss two fundamental properties of DEU-RUM: its stochastic comparative statics and the potential implications of restricting the set of allowable probability distributions.

4.1. **Stochastic Comparative Statics.** A crucial virtue of random utility models is that the comparative statics of a deterministic model extend immediately to the random utility model built upon it. This is because results based on degenerate distributions over the set of utilities (the deterministic model) naturally extend to probability distributions over the set of utilities (the random utility model built upon it). We illustrate this feature by establishing the stochastic counterparts of the deterministic notions of more risk aversion and more delay aversion for DEU.

In the deterministic world, more risk aversion corresponds to the equivalence of: (i) a greater preference for or, equivalently, a higher inclination to choose the degenerate lottery, and (ii) higher concavity of the monetary utility. The stochastic version of part (i) can be written simply as a larger mass assigned to preferences for which the degenerate lottery is better or, equivalently, as a larger probability of choice for the degenerate lottery. The stochastic implementation of part (ii) requires us to consider the following equivalent formulation of more risk aversion: for every utility function

[19] Again, in the Supplementary Material we argue that the use of random utility models coupled with more general deterministic models of risk and time preferences follows immediately from the analysis of DEU-RUM.

[20] In the results that follow, we assume that f is measurable in the respective sets. This assumption is easily met in the parametric versions used in our data analysis, as discussed later.

$u \in \mathcal{U}$, if u_2 is more concave than u, then u_1 is also more concave than u. The stochastic version of this is now direct: $\overline{MCT}_{f_1}(u) \geq \overline{MCT}_{f_2}(u)$, where $\overline{MCT}_{f_i}(u)$ denotes the mass, according to f_i, of DEU utilities with a monetary utility that is more concave than u. The same logic applies to more delay aversion and thus we denote by $\widetilde{MCT}_f(u)$ the mass, according to f, of DEU utilities with a normalized utility that is more concave than u. The next result follows immediately.

Proposition 4. *Consider two instances, f_1 and f_2, of DEU-RUM.*

(1) *Stochastic more risk aversion: $\overline{MCT}_{f_1}(u) \geq \overline{MCT}_{f_2}(u)$ for every $u \in \mathcal{U}$ if, and only if, $\rho_{f_1}(([1;x],0),(l,0)) \geq \rho_{f_2}(([1;x],0),(l,0))$ for every $l \in \mathcal{L}$ and every $x \in X$.*
(2) *Stochastic more delay aversion: Fix $\theta \in (0,1)$. $\widetilde{MCT}_{f_1}(u) \geq \widetilde{MCT}_{f_2}(u)$ for every $u \in \mathcal{U}$ if, and only if, $\rho_{f_1}(([1;x],0),([1;y],s)) \geq \rho_{f_2}(([1;x],0),([1;y],s))$ for every $x, y \in X$ and $0 < s \in T$.*

In words, the probability of choosing a present payoff over a present lottery is larger for the distribution which has stochastically more concave monetary utilities. Similarly, the probability of choosing a present payoff over a future payoff is larger for the distribution that has stochastically more concave normalized utilities. Accordingly, we call these notions stochastic more risk-aversion and stochastic more delay-aversion.

Parametric models, such as DEU-H, simplify further the analysis. Notice that $\overline{MCT}_f(u)$ simply corresponds to one minus the cumulative mass, according to f, of the set of utilities with curvatures below that of u. Hence, f_1 is stochastically more risk averse than f_2 if, and only if, the CDF over the monetary utility curvatures of f_1 first-order stochastically dominates that of f_2. Similarly, f_1 is stochastically more delay averse than f_2 if, and only if, the CDF over the normalized curvatures of f_1 first-order stochastically dominates that of f_2, which is equivalent to say that the CDF over the corrected discount factors of f_1 is first-order stochastically dominated by that of f_2. That is, stochastically more risk-averse individuals' will have distributions over h that are biased towards higher values of risk aversion and stochastically more delay-averse individuals will have distributions over normalized curvatures that are biased towards higher values or, equivalently, distributions over corrected discount factors that are biased towards lower values. All these intuitive results are fundamental in that they enable reliable estimations and interpretations of the preference parameters of interest.

4.2. **Distributional Assumptions.** In applications, the analyst typically simplifies the treatment of random models by restricting the set of probability distributions governing the preference parameters. For instance, the distribution may be assumed to belong to a well-known family or have marginals over the parameters that are independent. Clearly, this restriction has no relevant implications for stochastic comparative statics, as it simply reduces the set of admissible instances of the model and, consequently, the set of behaviors it is able to explain. However, as we are about to discuss, some distributional assumptions may have important undesirable consequences.

We illustrate using DEU-H-RUM, showing that the assumption of independence of the parameters h and δ leads to problematic conclusions. We show that, whenever risk aversion is sufficiently high, an earlier dated lottery is almost surely preferred to a later dated lottery, *however low the earlier payoffs and however high the later payoffs.* Similarly, in a convex budget setting, the shares chosen will in no way depend on the magnitude of payoffs x and y. That is, whenever risk aversion is sufficiently high, when it comes to the choice of endowment share, it is irrelevant whether the later payoff y is similar or markedly higher than the earlier payoff x. Formally, denoting by $\rho_f(\alpha)$ the distribution over shares induced by f, we have:

Proposition 5. *Consider an instance f of the DEU-H-RUM satisfying distributional independence for h and δ. Then:*

(1) *For any $(l,t),(l',s)$ with $l \neq [1;0]$ and $t<s$, $\lim_{h\to 1}\rho_f((l,t),(l',s)) = 1$.*
(2) *$\lim_{h\to 1}\rho_f(\alpha)$ is independent of x and y.*

The intuition of Proposition 5 is as follows. In a dated lottery setting, the earlier lottery is chosen if, and only if, the ratio of expected utilities between the later and the earlier lotteries is not high enough to compensate the discounting δ^{s-t}. When h approaches 1, the ratio of expected utilities converges to 1. Under distributional independence, the discounting δ^{s-t} is independent of h and hence the mass of utilities for which the earlier lottery is chosen converges to 1. Similarly, in a convex budget setting, the choice of endowment share depends on the term $(\frac{y}{x})^{\frac{1-h}{h}}(\delta^{s-t}\frac{q}{p})^{\frac{1}{h}}$. When h approaches 1, this converges to a constant depending neither on x nor on y.

These features are of course nonsensical and may severely affect the estimation exercise. For illustrative purposes, imagine that an individual is highly risk averse but only moderately delay averse. An estimation exercise using DEU-H-RUM with distributional independence cannot correctly estimate both traits because, if, say, a correct estimation of risk aversion is attempted, high levels of risk aversion must come with an extreme preference for the present, thus contradicting the behavior of the individual. If simplification of the analysis through the assumption of independence is desired, our discussion on comparative statics in Section 3 shows that independence can be safely imposed on the (normalized) parameters governing the notions of more risk aversion and more delay aversion. We illustrate this methodology in our next section.

5. Estimation of Risk and Time Preferences

In this section we empirically implement the framework developed in the previous sections, using the homogeneous monetary functions DEU-H, on the experimental datasets of Andersen et al. (2008), Coble and Lusk (2010) and Andreoni and Sprenger (2012).[21]

5.1. Dated Lotteries: Risk and Time Preferences Independently Elicited.

The literature has often designed experiments in which individuals must choose, separately, from menus involving only present lotteries and from menus involving only dated payoffs. This separate elicitation enables us to illustrate an important advantage of DEU-RUM; namely, the possibility of the joint estimation of risk and delay aversion using the entire dataset or, alternatively, the separate estimation of risk and time attitudes using the relevant sub-samples of the dataset.

To illustrate, we use the influential dataset of Andersen et al. (2008), who designed 100 menus, which we index by m, each involving either a pair of present lotteries or a pair of degenerate dated lotteries. A group of 253 individuals, which we index by i, made choices from these menus. This provided a collection of 23,108 observations, i.e. pairs of menus and corresponding choices, which we denote by $\mathcal{O}$.[22]

Our first implementation of DEU-H-RUM adopts a representative agent approach, in which the stochasticity of every individual in the population is assumed to be governed by the same distribution over DEU-H, denoted by f. In accordance with the discussion in Section 4.2, we assume distributional independence between risk aversion and delay aversion, i.e., f will be the product of two independent probability distributions, $\bar{f}$

[21] Other influential datasets are Tanaka et al. (2010), Dohmen et al. (2010), Cheung (2015) and Miao and Zhong (2015). The corresponding estimation results are available upon request.

[22] Not every individual faced all menus, but each was required to make between 84 and 100 choices. The Supplementary Material contains further details of all three experimental datasets used in this section.

and $\hat{f}$, defined, respectively, on the risk aversion parameter and the corrected discount factor.[23] For the risk aversion parameter, we assume that $\bar{f}$ is a truncated normal distribution in the interval $(-\infty, 1)$ with parameters μ_h and σ_h^2.[24] For the delay aversion parameter, which we measure in months, we assume that $\hat{f}$ is a beta distribution with parameters $a_{\hat{\delta}}$ and $b_{\hat{\delta}}$.

Now, let an individual i confront menu $m = \{1, 2, \ldots, \mathcal{T}_m\}$. The probability of choosing alternative τ, denoted by $\rho_{im\tau}(f)$, corresponds to the measure of all parameters for which the associated DEU-H utility ranks τ as the best alternative within menu m. Denoting by $\mathbf{1}$ the usual indicator function and by j a generic alternative in the menu, this is

$$\rho_{im\tau}(f) = \int_h \int_{\hat{\delta}} \mathbf{1}\left(\tau = \max_{j\in\{1,2,\ldots,\mathcal{T}_m\}} DEU_{\delta,h}(j)\right) \bar{f}(h)\hat{f}(\hat{\delta})\mathrm{d}h\mathrm{d}\hat{\delta}.$$

Denote by $y_{im\tau}$ the indicator variable, which takes the value 1 when individual i chooses alternative τ from menu m. The log-likelihood function is

$$\log \mathcal{L}\left(f|\mathcal{O}\right) = \frac{1}{|\mathcal{O}|}\sum_{i=1}^{I}\sum_{m=1}^{M}\sum_{\tau=1}^{\mathcal{T}_m} y_{im\tau}\log\left(\rho_{im\tau}\left(f\right)\right).$$ [25]

Consistent estimation of $(\mu_h, \sigma_h^2, a_{\hat{\delta}}, b_{\hat{\delta}})$ can be achieved via maximization of the log-likelihood, and this estimator summarizes all the information about the estimated distributions of both risk and time attitudes. Robust standard errors for these estimates are computed using the delta method and clustered at the individual level. The computation of integrals is facilitated by means of a Quasi-Monte Carlo method, which can be easily implemented in most statistical packages, and which delivers log-likelihood

[23] Notice that, for any realization of the risk-aversion coefficient h and the corrected discount factor $\hat{\delta}$, the implied discount factor can be backed out as $\delta = \hat{\delta}^{1-h}$, and used for the relevant computations.

[24] In practice, we simplify the computational analysis by considering the subinterval $[-\underline{h}, 1)$ instead of $(-\infty, 1)$, where $\underline{h}$ is chosen small enough as not to bound the estimation. See Appendix B for details.

[25] In order to allow for positive choice probabilities of dominated lotteries, we introduce a fixed small tremble, such that, with very large probability $1-\nu$, the individual chooses according to $\rho_{im\tau}(f)$ and with very small probability ν, the individual uniformly randomizes. In the Supplementary Material, we report the results of a version of the baseline estimation of each model, using all three datasets studied in this section, where ν is estimated as an additional parameter. In general, we find that the estimation of the tremble probability improves the fit of the models by allowing them to explain the observed positive probability of making dominated choices. However, the estimated distributions of risk and time preferences do not change substantially from that obtained by fixing ν, as we do here.

functions with smooth parameters that can be quickly maximized using gradient-based methods.[26]

Table 1 shows the estimated risk and time preferences, including medians, standard deviations and the corresponding standard errors. Columns 2 and 3 show the results when risk and delay aversion are estimated separately, while column 4 shows the results from the joint estimation of their distributions. As expected, the results are identical in both cases. Figure 1 shows the estimated PDFs of the risk and delay aversion parameters, $\bar{f}$ and $\hat{f}$, and the implied estimate of δ.[27] We observe high levels of risk aversion, with the mode of the distribution at the upper bound of 1, suggesting that a sizable portion of the generated data may show risk-aversion levels above 1.[28]

Given that the dataset is rich enough to perform individual estimations, we now assume that the governing distributions, denoted by f_i, are individual-specific and use the sub-sample of the corresponding individual observations.[29] Column 5 of Table 1 reports the median value of the median individual estimations of the parameters. Figure 2 shows a scatter-plot with the estimated individual median risk and corrected discount factor for each of the 253 subjects in the sample, and Figure 3 plots the ordered individual estimates against the CDFs of the pooled estimations. This exercise illustrates a series of points. In the first place, notice the substantial heterogeneity of preference across the population.[30] The distributions of median-traits in the population are far from degenerate and, indeed, closely reproduce those of the representative agent. In particular, and in consonance with our previous observation, it is apparent that there is a number of individuals for whom the bound of 1 on the risk aversion level

[26] In Appendix B, we discuss the numerical evaluation of the log-likelihood function. The Matlab code for implementing these methods is available at https://github.com/agutieda/Estimating-Risk-and-Time-Preferences.

[27] The figure plots the normal kernel estimates of the PDF of $\delta = \hat{\delta}^{1-h}$ using the draws from the distributions of $\hat{\delta}$ and h. The results are consistent with the theoretical discussion, in that the high levels of risk aversion result in substantially higher discount factors.

[28] In the Supplementary Material we show how to extend the model to allow for higher levels of risk aversion, and report the resulting estimates under this extension. We obtain a similar median risk aversion, but capture the upper part of the distribution more closely.

[29] In this case, the asymptotic properties of the maximum likelihood estimator hold as the number of menus grows large.

[30] The variability across estimates may also reflect sampling/estimation variability. Accordingly, the observed heterogeneity can be interpreted as an upper bound of the underlying preference heterogeneity.

is binding. Next, notice that not all heterogeneity is due to the existence of different individual preferences, since, as reported in column 5 of Table 1, the median of the estimated individual standard deviations is clearly non-null. Thirdly, the correlation between the risk-aversion coefficients and the corrected discount factors is slightly positive (0.050), i.e. there is slightly negative correlation between risk and delay aversion, but it is not significant at conventional levels (p-value = 0.425).[31]

5.2. **Dated Lotteries: General Case.** A different strand of the literature elicits risk and time preferences over general menus of dated lotteries. Using the techniques discussed in the previous section, we illustrate using the dataset of Coble and Lusk (2010), which reports on an experiment involving 47 subjects each choosing from 94 menus involving either: (i) pairs of same-dated lotteries, (ii) pairs of dated degenerate lotteries, or (iii) pairs of non-degenerate lotteries awarded at different time periods.

For the sake of comparison, we first run an estimation exercise equivalent to that presented in the previous section. That is, we use only the subset of the data involving, basically, only risk or only time considerations, parts (i) and (ii) above. Columns 2 and 3 of Table 2 report that, in this substantially different population of subjects, we find a relatively small decrease in the median levels of risk aversion and of the corrected discount factor. We can now use part (iii) of the dataset to evaluate whether behavior is substantially affected when both risk and time considerations are active. This joint estimation of risk and time preferences is reported in column 4. Interestingly, the conclusions reached using (i) and (ii) vary little with respect to those obtained using (iii). This supports the view that the large body of literature using independent elicitations of risk and time preferences is obtaining a picture that is close to the one that emerges with dated lotteries involving both risk and time dimensions. Column 5 reports the estimation results with the pooled data, and Figure 4 plots the PDFs of the parameters using this pooled dataset.

We next perform individual estimations. Column 6 of Table 2 reports the median and standard deviation of the individual estimates; Figure 5 shows a scatter-plot with

[31] This dataset also contains information on individual characteristics. In the Supplementary Material, we show how to incorporate this sort of information into the estimations by modeling the parameters of the distribution as a linear function of the observable characteristics. We can assume, for example, that $\mu_h = \gamma_0 + \gamma_l x_l$, where x_l is either a dummy or a real variable and then estimate parameters γ_0 and γ_l. In a recent paper, Jagelka (2019), using a version of DEU-H-RUM, implements a similar methodology to study the influence of personality traits on risk and time preferences.

the estimated individual medians of risk and corrected discount factors for each of the 47 sample subjects, and Figure 3 plots the ordered individual estimates against the CDFs of the pooled estimations. Again, this analysis suggests great interpersonal heterogeneity, with a slightly negative correlation between individual risk parameters and corrected discount factors, and the presence of some intrapersonal variability of preferences.

Interestingly, the richness of this dataset enables further exploration of the idea of correlation between risk and time, since we can now capture both intrapersonal and interpersonal correlation. Since risk and time parameters are jointly responsible for choices in part (iii), we can now run a version of the pooled estimation without the independence assumption. Essentially, we now express the joint distribution of h and $\hat{\delta}$ in terms of their marginal distributions (which follow the same parametric forms used in the independent case) and a Gaussian copula allowing for correlation between the two.[32] Column 7 in Table 2 shows the results of this exercise. We observe that the estimated correlation coefficient is negative albeit not statistically different from zero, due to high variation. Furthermore, the estimated moments of the marginal distributions are close to those obtained assuming independence, thereby showing that the estimates in this dataset vary very little even when allowing for the correlation of risk and time preferences.

5.3. **Convex Budgets.** We now analyze the setting of convex budgets. We use the original dataset of Andreoni and Sprenger (2012), which involves 80 subjects, each making 84 decisions from convex budget menus, for a total of 6,720 observations. The experimental implementation uses discretized versions of the continuous share problem, with $\alpha \in \{\frac{0}{100}, \frac{1}{100}, \dots, \frac{100}{100}\}$. Moreover, since the vast majority of participants tend to choose multiples of 10, for practical reasons we discretize the choice of α to 11 equidistant possible shares. The DEU-H-RUM specification of $\rho_{im\tau}(f)$ described in Section 5.1 and its associated log-likelihood immediately extend to this setting.

Column 2 of Table 3 reports the baseline estimation, column 3 the results when allowing for correlation between the distributions of the parameters using a Gaussian copula, and column 4 the estimations at the individual level, providing the median and standard deviation of the medians estimated for each individual. Figure 7 plots the estimated PDFs of the preference parameters, Figure 8 the observed and predicted choice

[32] See Fan and Patton (2014) for an introduction to the use of Copulas in econometrics.

probabilities across the different experimental parameters, and Figure 9 the scatter-plots with the estimated individual median parameters for each of the 80 subjects. All the figures use the baseline estimated parameters reported in Table 3.

Here, we would like to stress the following findings. The simple model DEU-H-RUM appears to perform remarkably well. Figure 8 shows that the estimated DEU-H distribution is able to capture the two main empirical regularities in the dataset; namely, a large task-dependent fraction of corner choices, followed by a task-dependent distribution of interior choices. This is because DEU-H-RUM allows for preference heterogeneity, and, as we know from Proposition 3, interior choices are predicted for risk-averse attitudes while corner choices are predicted for risk-seeking attitudes. The estimation of a positive but close-to-zero median risk-aversion coefficient is the reason why approximately half of the predicted choices are made using negative risk-aversion coefficients leading to corner choices. This is a remarkable result, that previous empirical strategies fail to achieve.[33]

As already mentioned in Section 3, DEU-H imposes the same predictions across tasks with the same ratio of probabilities. However, Figure 8 clearly shows that this is not observed in the data. For instance, corner solutions appear to be chosen more often when both lotteries are degenerate than when both outcomes are realized with equally low probability. Given that DEU-H must predict the same choices in both scenarios, there is room for improvement when considering more general models.[34] It is also worth noting that the corrected discount factor estimates are rather stable across estimations, indicating a level of patience somewhere between those observed in the previous two datasets involving dated lotteries. Finally, Figure 9 shows scatter plots of the individually-estimated parameters. There seems to be a negative correlation between risk and the corrected discount factor across individuals. Interestingly, the correlation obtained in the copula estimate of column 3 is also negative. Notice, however, the small magnitude of the correlations (approximately -0.345 (p-value=0.002)).

[33] The reason being that previous analyses either sacrifice the large heterogeneity in the data, or use additive iid random utility models which, as commented in Section 1, not only lack sufficient theoretical grounding, but are also unable to reproduce the bimodality in the corners and the interior distributions.

[34] In the Supplementary Material we show that an extension of DEU-H, which we call CEPV-H, has no such restriction and fits the data better across these tasks.

6. Final Remarks

In this paper, we have developed a sound stochastic framework for the analysis of risk and time preferences. Using the discounted expected utility model as our base model, we have established its risk and time comparative statics, have incorporated them into the framework of random utility models, and have empirically illustrated their potential on several experimental datasets. Our framework offers a unique tractable tool for gaining a deeper understanding of risk and time preferences; one of the cornerstones of economics.

Appendix A. Proofs

Proof of Proposition 1: When $t = 0$, DEU reduces to expected utility and hence, the first part follows immediately from standard results. When lotteries are degenerate, DEU reduces to exponentially discounted utility and we can use the normalization of Fishburn and Rubinstein (1982) to prove the result. Since this normalization plays a key role in this paper, we now discuss its details. When the space $\mathcal{D} \times T$ of dated degenerate lotteries is considered, $DEU_{\delta,u}$ is equivalent to $DEU_{\theta,\bar{u}}$, where θ is any value in $(0, 1)$ and $\bar{u} = u^{\frac{\log\theta}{\log\delta}}$. To see this, notice that $DEU_{\delta,u}([1; x], t) \geq DEU_{\delta,u}([1; y], s)$ if and only if $\delta^t u(x) \geq \delta^s u(y)$. For any $\theta \in (0, 1)$, since $\frac{\log\theta}{\log\delta} > 0$, the above inequality is equivalent to $(\delta^t u(x))^{\frac{\log\theta}{\log\delta}} \geq (\delta^s u(y))^{\frac{\log\theta}{\log\delta}}$ or, alternatively, $\theta^t \bar{u}(x) \geq \theta^s \bar{u}(y)$. This shows that the normalized model represents the same preferences. Now consider the pair of degenerate lotteries $([1; x]; 0)$ and $([1; y], s)$ with $0 < s$. We start with the 'only if' part. Clearly, if $x \geq y$, both individuals prefer the present payoff, and the claim follows. Let, then, $x < y$, and suppose that the second individual expresses a preference for the present payoff, i.e., $\bar{u}_2(x) \geq \theta^s \bar{u}_2(y)$, or equivalently, $\frac{\bar{u}_2(x)}{\bar{u}_2(y)} \geq \theta^s$. Then, it must be that $0 < x$. Suppose that $\bar{u}_1$ is more concave than $\bar{u}_2$. Without loss of generality, we can re-scale one of the two normalized utility functions to set $\bar{u}_1(x) = \bar{u}_2(x)$ and then, more concavity of $\bar{u}_1$ implies $\bar{u}_1(y) \leq \bar{u}_2(y)$, or equivalently, $\frac{\bar{u}_1(x)}{\bar{u}_1(y)} \geq \frac{\bar{u}_2(x)}{\bar{u}_2(y)} \geq \theta^s$, leading the first individual also to prefer the present payoff, as desired. We prove the converse by way of contradiction. Assume the existence of two payoffs $x^* < y^*$ and $\gamma \in (0, 1)$ such that $\frac{\bar{u}_1(x^*)}{\bar{u}_1(y^*)} > \gamma > \frac{\bar{u}_2(x^*)}{\bar{u}_2(y^*)}$. Trivially, we can find $t^* \in T$ such that $\gamma = \theta^{t^*}$. Thus, selecting the dated lotteries $([1; x^*], 0)$ and $([1; y^*], t^*)$, we obtain a contradiction. ■

Proof of Proposition 2: Denote by $CE_u(l)$ the certainty equivalent of lottery l using expected utility with monetary utility u. Given any pair of dated lotteries

$(l_1 \equiv [p_1, \dots, p_N; x_1, \dots, x_N], t_1)$ and $(l_2 \equiv [q_1, \dots, q_M; y_1, \dots, y_M], t_2)$, it is evident that $\delta^{t_1} \sum_n p_n u(x_n) \geq \delta^{t_2} \sum_m q_m u(y_m)$ is equivalent to $\frac{\sum_n p_n u(x_n)}{\sum_m q_m u(y_m)} \geq \delta^{t_2 - t_1}$ and thus, equivalent to $\frac{u(CE_u(l_1))}{u(CE_u(l_2))} \geq \delta^{t_2 - t_1}$. We now prove the first part. Suppose that DEU_{δ_1, u_1} is equally delay averse and more risk averse than DEU_{δ_2, u_2}. From Proposition 1, this means that, for every x, $\bar{u}_1(x) = [u_1(x)]^{\frac{\log \theta}{\log \delta_1}} = [u_2(x)]^{\frac{\log \theta}{\log \delta_2}} = \bar{u}_2(x)$.[35] Taking logarithms, this is equivalent to $\frac{\log u_1(x)}{\log \delta_1} = \frac{\log u_2(x)}{\log \delta_2}$ for every x. That is, the ratio $\frac{\log u_2(x)}{\log u_1(x)}$ is equal to the constant $\frac{\log \delta_2}{\log \delta_1} = k > 0$, and hence: (i) $\delta_2 = \delta_1^k$ and (ii) $u_2 = u_1^k$. Since the first individual is more risk averse than the second it must be that $k \geq 1$. We can then rewrite the preference of the second individual for (l_1, t_1) as $\frac{[u_1(CE_{u_2}(l_1))]^k}{[u_1(CE_{u_2}(l_2))]^k} \geq \delta_1^{k(t_2 - t_1)}$, which is equivalent to $\frac{u_1(CE_{u_2}(l_1))}{u_1(CE_{u_2}(l_2))} \geq \delta_1^{t_2 - t_1}$. Whenever $l_1 = [1; x]$, we have $CE_{u_2}(l_1) = CE_{u_1}(l_1)$. As u_1 is more concave than u_2, we also have $CE_{u_2}(l_2) \geq CE_{u_1}(l_2)$. Hence, $\frac{u_1(CE_{u_1}(l_1))}{u_1(CE_{u_1}(l_2))} \geq \delta_1^{t_2 - t_1}$ and the first individual also prefers the (degenerate) dated lottery (l_1, t_1).

For the second part, notice that, if the two individuals are equally risk averse, their monetary utility functions must have the same curvature. If the first individual is more delay averse than the second, the normalized utility function of the first individual must have greater curvature. With fixed risk aversion, it is immediate to see that the first individual must have a lower discount factor. Hence, if $t_1 < t_2$ and the second individual prefers (l_1, t_1) over (l_2, t_2), we have $\frac{\sum_n p_n u_2(x_n)}{\sum_m q_m u_2(y_m)} = \frac{\sum_n p_n u_1(x_n)}{\sum_m q_m u_1(y_m)} \geq \delta_2^{t_2 - t_1} \geq \delta_1^{t_2 - t_1}$, and the first individual also prefers the earlier lottery, as desired. ■

Proof of Proposition 3: For the first part, let u_i be convex. Then, $\delta_i^t p u_i(\alpha x) + \delta_i^s q u_i((1-\alpha) y) \leq \alpha \delta_i^t p u_i(x) + (1-\alpha) \delta_i^t p u_i(0) + (1-\alpha) \delta_i^s q u_i(y) + \alpha \delta_i^s q u_i(0) = \alpha \delta_i^t p u_i(x) + (1-\alpha) \delta_i^s q u_i(y) \leq \max\{\delta_i^t p u_i(x), \delta_i^s q u_i(y)\}$ and hence, the solution must be corner. As the problem has been reduced to the DEU comparison of the two basic dated lotteries $([p, 1-p; x, 0], t)$ and $([q, 1-q; y, 0], s)$, the result follows immediately.

For the second part, let u_i be strictly concave. Given the differentiability assumption, the first-order condition of the optimization problem is $\frac{u_i'(\alpha x)}{u_i'((1-\alpha) y)} = \frac{\delta_i^s q y}{\delta_i^t p x}$, with the left hand side strictly decreasing in α. The strict concavity of u_i guarantees that the solution is interior. We now start by fixing risk aversion, i.e. $u_1 = u_2$. In this case, we know that individual 1 is more delay averse than individual 2 if and only if

[35] More formally, $\bar{u}_1(x) = a \bar{u}_2(x)$ with $a > 0$. The constant a is inessential to our arguments, and hence we normalize to $a = 1$ without loss of generality. Similar conventions will be adopted in the following proofs.

$\delta_1 \leq \delta_2$. This implies that $\frac{\delta_1^s qy}{\delta_1^t px} \leq \frac{\delta_2^s qy}{\delta_2^t px}$, and hence, given the strict concavity of u_i and $\frac{u_1'(\alpha x)}{u_1'((1-\alpha)y)} = \frac{u_2'(\alpha x)}{u_2'((1-\alpha)y)}$, it must be that $\alpha_1^* \geq \alpha_2^*$.

We now fix delay aversion. As discussed in Proposition 2, this implies that $\delta_2 = \delta_1^k$ and $u_2 = u_1^k$. Next, suppose that the first individual is more risk averse than the second, i.e., $k \geq 1$. The first order condition of the second individual can be written as $\left(\frac{u_1(\alpha x)}{u_1((1-\alpha)y)}\right)^{k-1} \frac{u_1'(\alpha x)}{u_1'((1-\alpha)y)} = \frac{\delta_1^{ks} qy}{\delta_1^{kt} px} = \frac{\delta_1^s qy}{\delta_1^t px} \frac{\delta_1^{(k-1)s}}{\delta_1^{(k-1)t}}$, or equivalently $\left(\frac{\delta_1^t u_1(\alpha x)}{\delta_1^s u_1((1-\alpha)y)}\right)^{k-1} \frac{u_1'(\alpha x)}{u_1'((1-\alpha)y)} = \frac{\delta_1^s qy}{\delta_1^t px}$. Clearly, considering the constant $\overline{\alpha}_1$ defined in the text, $g(\alpha) = \left(\frac{\delta_1^t u_1(\alpha x)}{\delta_1^s u_1((1-\alpha)y)}\right)^{k-1} \leq 1$ if and only if $\alpha \leq \overline{\alpha}_1$. Let $f(\alpha) = \frac{u_1'(\alpha x)}{u_1'((1-\alpha)y)}$, which we know is strictly decreasing in α. Then, for values of α below (respectively, above) $\overline{\alpha}_1$ the function $h(\alpha) = f(\alpha)g(\alpha)$ on the left hand side of the first-order condition falls below (respectively, is above) $f(\alpha)$. Since the right hand side is a constant, the result follows immediately. ■

Proof of Proposition 5: For the first part, consider an instance f of the random utility model built upon DEU-H, where h and δ are independently distributed, and a pair of dated lotteries $(l \equiv [p_1, \ldots, p_N; x_1, \ldots, x_N], t)$ and $(l' \equiv [q_1, \ldots, q_M; y_1, \ldots, y_M], s)$, with $l \neq [1; 0]$ and $t < s$. We know that, for DEU-H, (l, t) is preferred to (l', s) if and only if $\delta^t \sum_n p_n \frac{x_n^{1-h}}{1-h} > \delta^s \sum_m q_m \frac{y_m^{1-h}}{1-h}$. Since $l \neq [1; 0]$, this is equivalent to $1 > \delta^{s-t} \frac{\sum_m q_m y_m^{1-h}}{\sum_n p_n x_n^{1-h}}$. Since the result is trivial for $l' = [1; 0]$, assume that $l' \neq [1; 0]$. Having fixed δ, the right hand side converges to δ^{s-t} whenever h converges to 1. Hence, the independence assumption guarantees that the proportion of choices for which the inequality holds must converge to 1 whenever h approaches 1, thus proving the result.

For the second part, notice that the first order condition of DEU-H can be rewritten as $\frac{(1-\alpha)}{\alpha} = \left(\frac{\delta^s q}{\delta^t p}\right)^{\frac{1}{h}} \left(\frac{y}{x}\right)^{\frac{1-h}{h}}$. Having fixed δ, the right hand side converges to $\delta^{s-t} \frac{q}{p}$ whenever h converges to 1. $\alpha^* = \frac{\delta^t p}{\delta^t p + \delta^s q}$ solves the first-order condition corresponding to the limit value $\delta^{s-t} \frac{q}{p}$. The independence assumption then guarantees that as h converges to 1, the mass of choices belonging to a neighborhood of α^* approaches 1. Since this result does not depend on x and y, the result has been proved. ■

Appendix B. Computational Considerations

In practice, the implementation of the maximum likelihood estimator may be complicated by the computation of probabilities $\rho_{im\tau}$, since these are given by multiple integrals with no closed-form solution. Numerical evaluation of these integrals using quadrature methods can be very slow and fall prey to the curse of dimensionality.

Monte-Carlo integration, by directly drawing from the distributions of the parameters, avoids the curse of dimensionality but can still be slow for the problem at hand. Ultimately, this method may lead to log-likelihood functions that are not smooth in the estimated parameters, preventing the use of traditional gradient-based methods to maximize the log-likelihood and compute standard errors. As an alternative, we use Quasi Monte-Carlo methods to evaluate $\rho_{im\tau}$.[36]

Formally, we generate K Halton draws $\{h_k,\, \hat{\delta}_k\}_{k=1}^{K}$ on the domain of the parameters h and $\hat{\delta}$. Notice that these are not draws from the distributions of h and $\hat{\delta}$, but quasi-random low-discrepancy sequences dependent upon these distributions. To simplify computation, we assume sufficiently large compact supports, characterized by the intervals $[\underline{h}, \overline{h}]$ and $[\underline{\hat{\delta}}, \overline{\hat{\delta}}]$, and formally work with the associated truncated distributions over these intervals. Using these draws, we can approximate $\rho_{im\tau}$ as follows

$$\rho_{im\tau}(f) \approx \frac{V}{K} \sum_{k=1}^{K} \sum_{i=1}^{I} \sum_{m=1}^{M} \mathbf{1}\left(\tau = \max_{j \in \{1,2,\ldots,\mathcal{T}_m\}} DEU_{\delta_k, h_k}(j)\right) \bar{f}(h_k) \hat{f}(\hat{\delta}_k),$$

where h_k and $\hat{\delta}_k$ are the k-th draw of the parameters, δ_k is derived from the former as usual, and $V = \int_h \int_{\hat{\delta}} \mathrm{d}h \mathrm{d}\hat{\delta} = (\overline{h} - \underline{h})(\overline{\hat{\delta}} - \underline{\hat{\delta}})$ is a normalization constant.

The advantages of this approach are based on the fact that, once the domain of the parameters is specified and the points on the domain of the parameters are drawn, the indicator function is independent of $\bar{f}$ and $\hat{f}$. The indicator function can thus be computed at first, stored, and then used in every step of the maximization of the log-likelihood function, thereby dramatically speeding up the estimation. This is especially useful when additional parameters are included, as in the models PVCE-H and CEPV-H, studied in the Supplementary Material.

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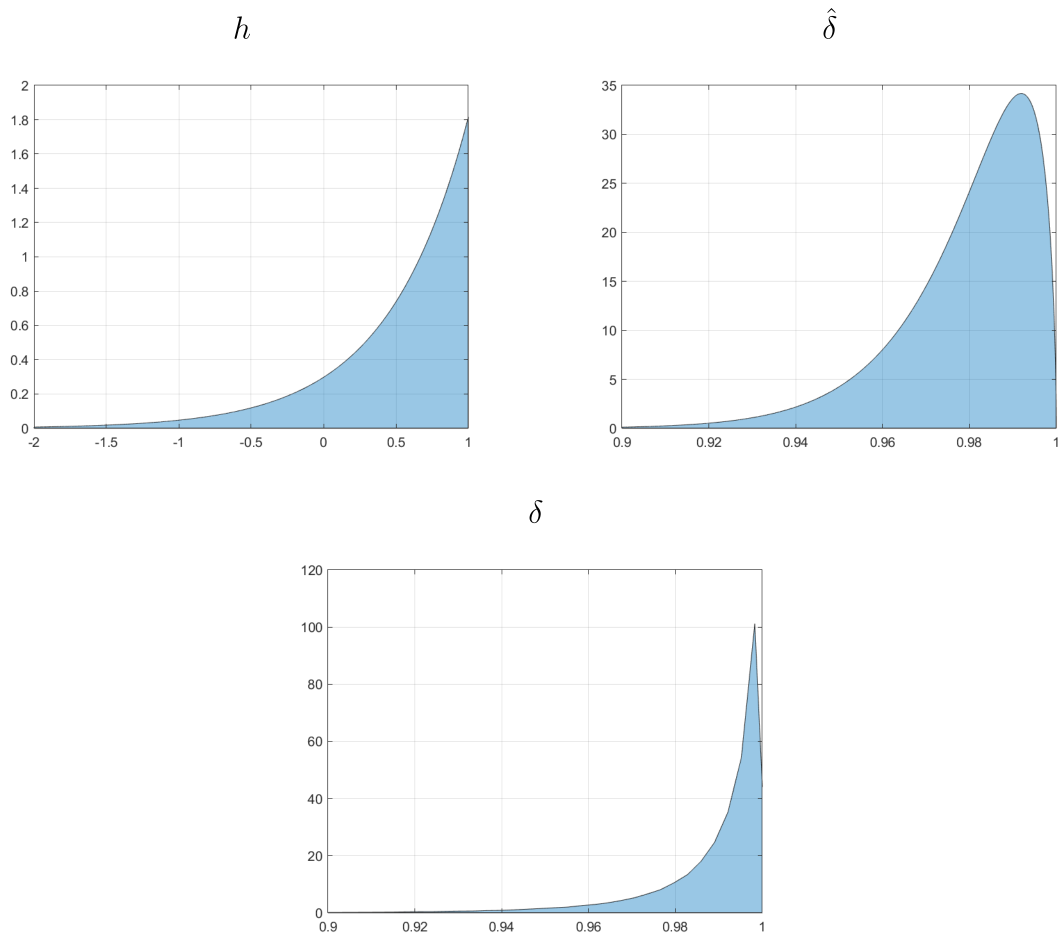
TABLE 1. Estimated Risk and Time Preferences: Andersen et al. (2008)

Dataset	**Risk Only**	**Time Only**	**Joint**	**by Individual**
Median r	0.620		0.620	0.718
	[0.023]		[0.023]	(0.517)
Std. Dev. r	0.512		0.512	0.377
	[0.023]		[0.023]	(0.284)
Median $\hat{\delta}$		0.983	0.983	0.980
		[0.001]	[0.001]	(0.085)
Std. Dev. $\hat{\delta}$		0.016	0.016	0.007
		[0.001]	[0.001]	(0.051)
# Obs.	7928	15180	23108	253
Log-Likelihood	-2.128	-0.543	-1.087	-0.991

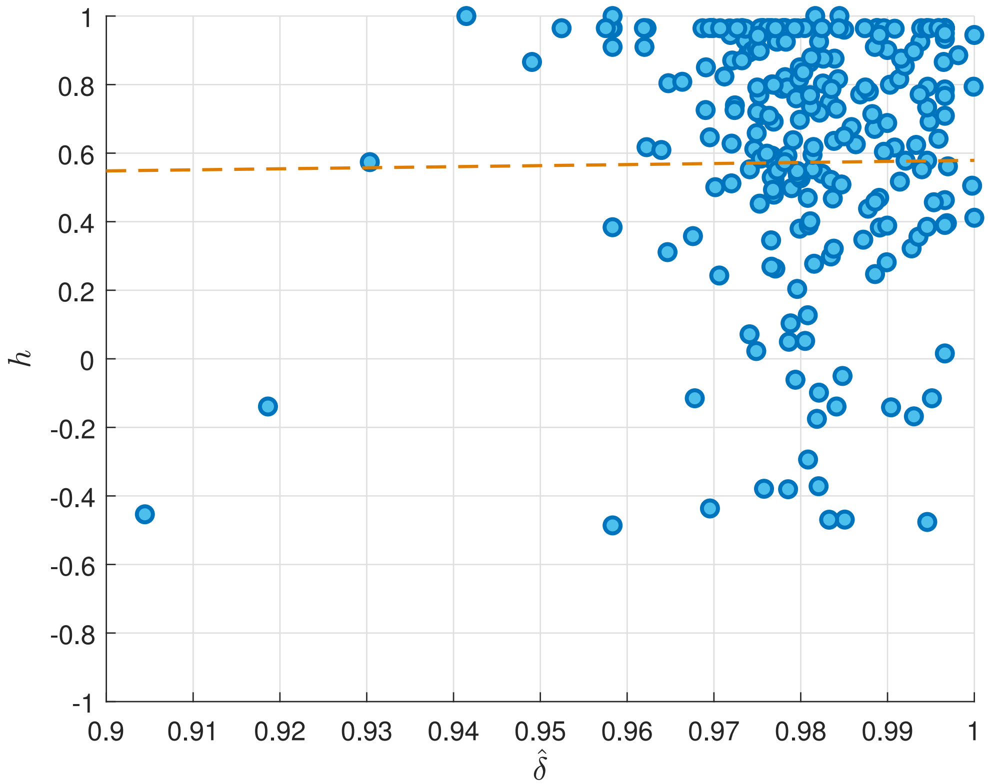
NOTES.- The above table reports the maximum-likelihood estimates of the median and the standard deviation of the distributions of risk and time preferences under the DEU-H representation, using data from Andersen et al. (2008). The second column shows the results obtained using the subsample of menus eliciting risk aversion only. The third column shows the results obtained using the subsample of menus eliciting delay aversion only. The fourth column shows the results of the joint estimation of risk aversion and delay aversion using the pooled menu sample. Standard errors, shown in brackets, are computed using the delta method and clustered at the individual level. The last column shows the median and standard deviation (in parentheses) of the distribution of individual estimates of the respective parameter. In all cases, the coefficient of risk aversion h is assumed to follow a normal distribution truncated at 1, while the corrected discount factor $\hat{\delta}$ follows a beta distribution.

Figure 1. PDFs of Estimated Risk and Time Preferences: Andersen et al. (2008)



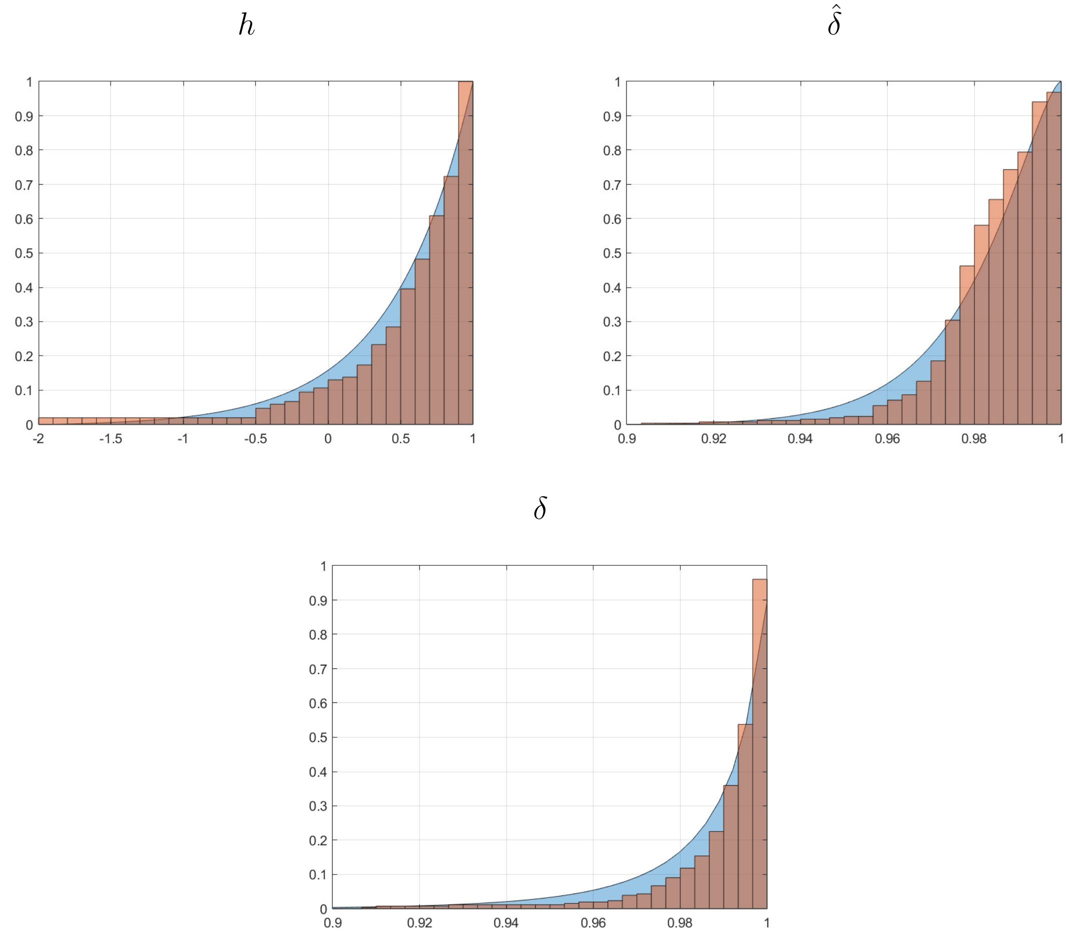
NOTES.- PDFs of the estimated distributions reported in Table 1. The PDF of the discount factor $\delta = \hat{\delta}^{1-h}$ is estimated from the distributions of risk and delay aversion using a normal kernel.

FIGURE 2. Individual Estimates: Andersen et al. (2008)



NOTES.- Each point represents the median of the estimated distributions of the coefficient of risk aversion h and the corrected discount factor $\hat{\delta}$ for the subsample of choice data for a particular individual, following the estimation procedure reported in Table 1.

Figure 3. CDFs of Estimated Risk and Time Preferences and Histograms of Individual Estimates: Andersen et al. (2008)



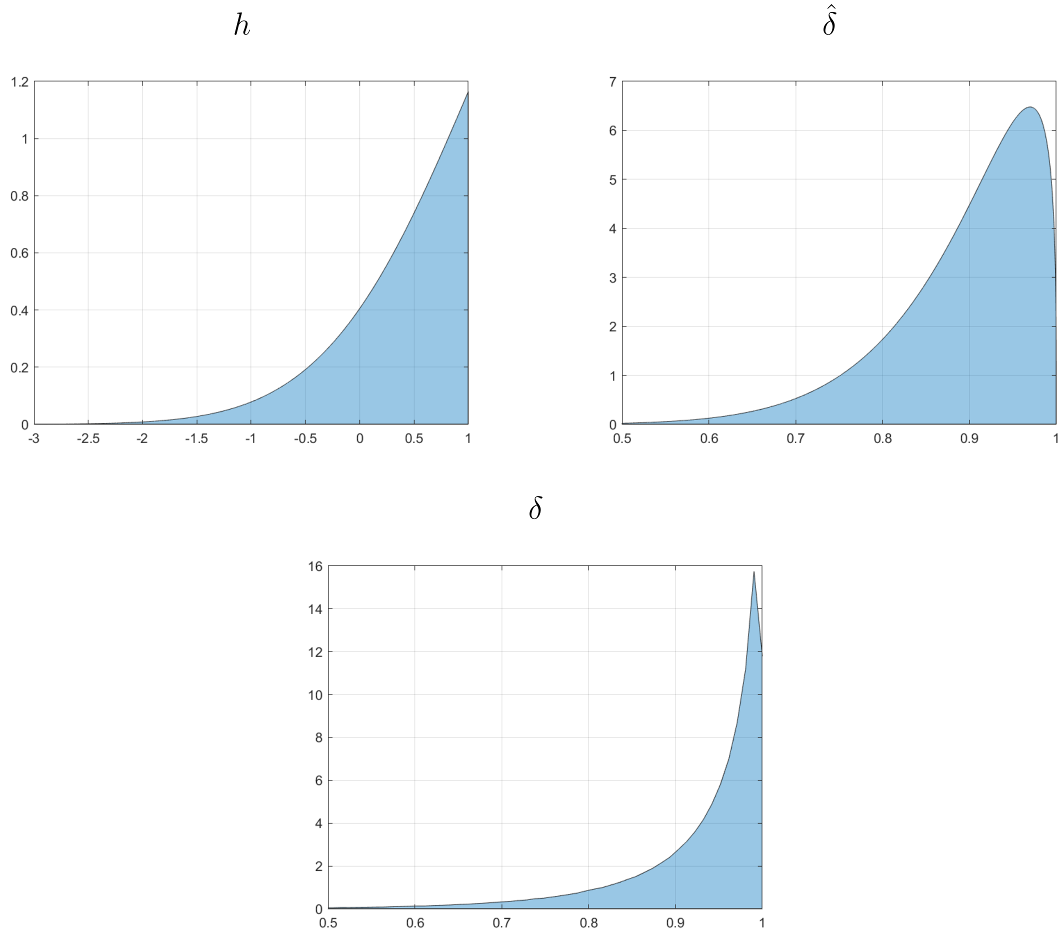
NOTES.- CDFs of the pooled estimation and histograms of the empirical distributions of the individual estimates.

Table 2. Estimated Risk and Time Preferences: Coble and Lusk (2010)

Dataset	**Using Risk Tasks Only**	**Using Discount Tasks Only**	**Using Joint Tasks Only**	**All Tasks**	**All Tasks - Correlated Preferences**	**Pooled Individual Estimates**
Median r	0.503 [0.072]	–	0.485 [0.109]	0.464 [0.076]	0.490 [0.074]	0.238 (0.530)
Std. Dev. r	0.569 [0.084]	–	0.413 [0.061]	0.582 [0.085]	0.554 [0.079]	0.012 (0.278)
Median $\hat{\delta}$	–	0.903 [0.012]	0.939 [0.017]	0.915 [0.010]	0.913 [0.010]	0.918 (0.236)
Std. Dev. $\hat{\delta}$	–	0.089 [0.017]	0.130 [0.034]	0.085 [0.013]	0.082 [0.013]	0.054 (0.031)
Corr(r,$\hat{\delta}$)	–	–	–	–	−0.453 [0.254]	0.023
# Obs.	1880	1128	1410	4418	4418	47
Log-Likelihood	−0.878	−0.436	−0.362	−0.606	−0.604	−0.406

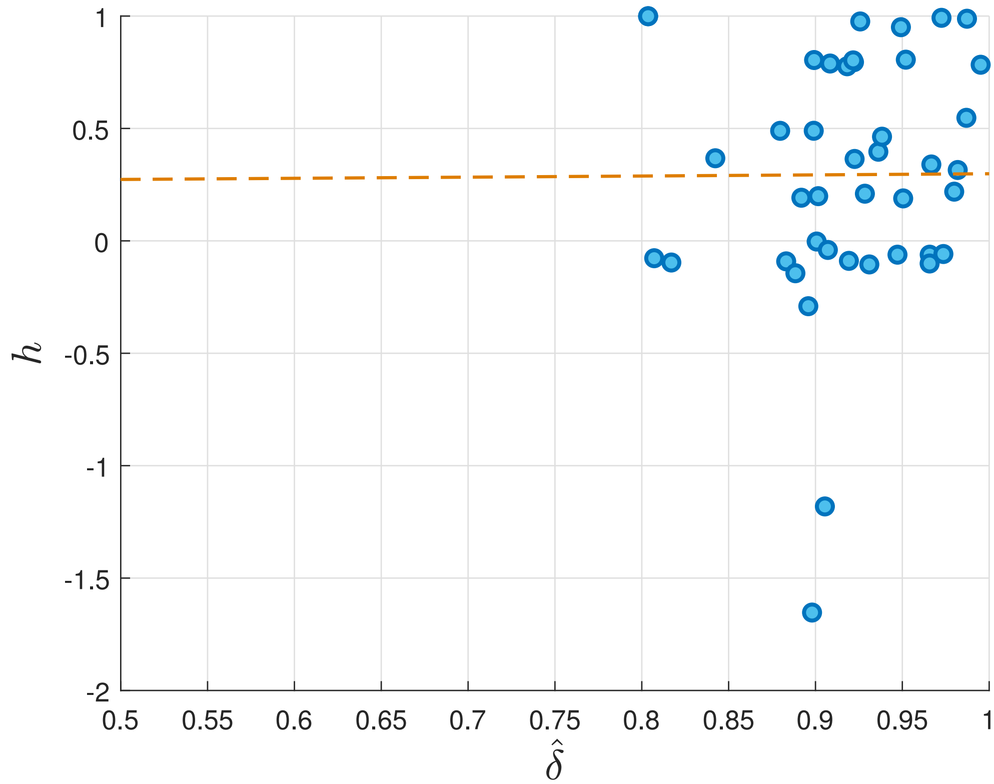
NOTES.- The above table reports the maximum-likelihood estimates of the median and the standard deviation of the distributions of risk and time preferences under the DEU-H representation, using data from Coble and Lusk (2010). The second column shows the results obtained using the subsample of menus eliciting risk aversion only. The third column shows the results obtained using the subsample of menus eliciting delay aversion only. The fourth column shows the results using menus with pairs of non-degenerate lotteries awarded at different time periods. The fifth column shows the results of the joint estimation of risk aversion and delay aversion, using the pooled menu sample. The sixth column shows the estimates obtained when allowing correlation between parameters using a Gaussian copula. Standard errors, shown in brackets, are computed using the delta method and clustered at the individual level. The last column shows the median and standard deviation (in parentheses) of the distribution of individual estimates of the respective parameter. In all cases, the coefficient of risk aversion h is assumed to follow a normal distribution truncated at 1, while the corrected discount factor $\hat{\delta}$ follows a beta distribution.

Figure 4. PDFs of Estimated Risk and Time Preferences: Coble and Lusk (2010)



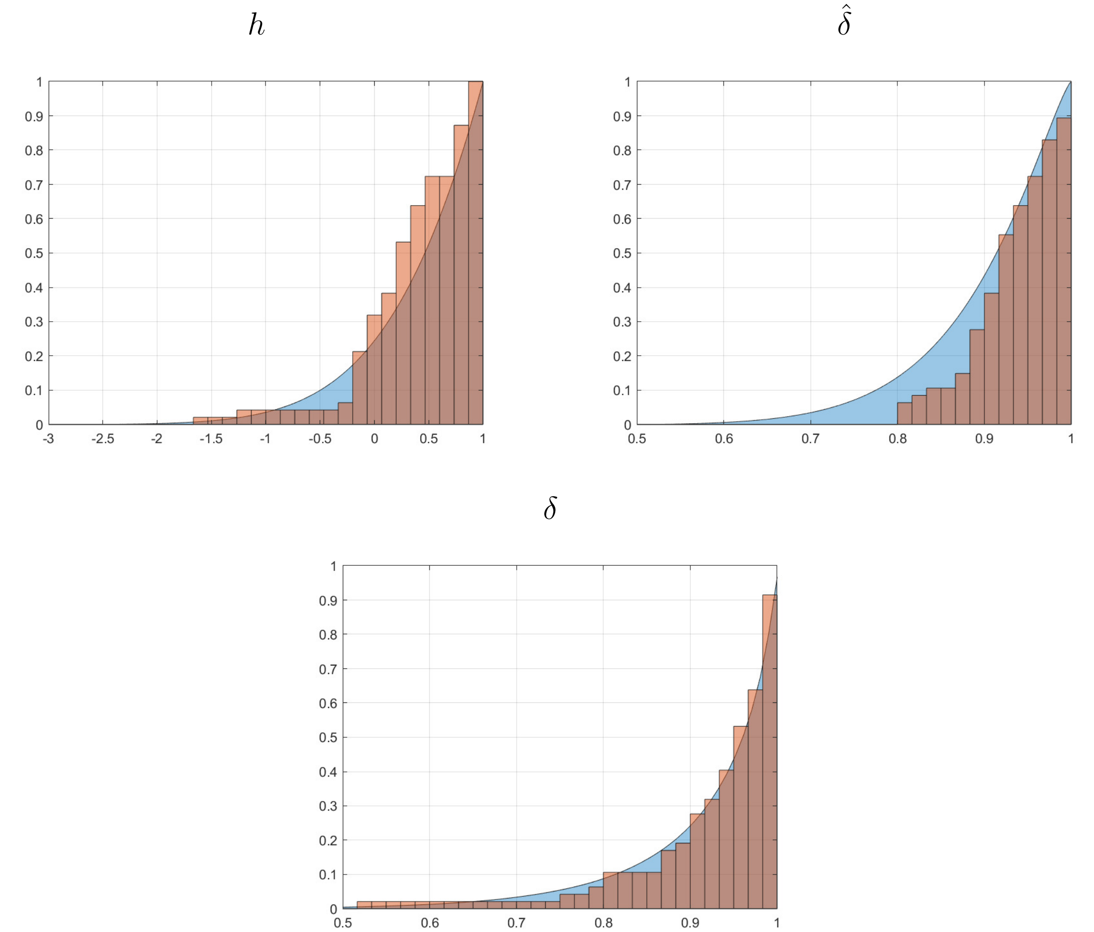
NOTES.- PDFs of the estimated distributions reported in Table 2. The PDF of the discount factor $\delta = \hat{\delta}^{1-h}$ is estimated from the distributions of risk and delay aversion using a normal kernel.

FIGURE 5. Individual Estimates: Coble and Lusk (2010)



NOTES.- Each point represents the median of the estimated distributions of the coefficient of risk aversion h and the corrected discount factor $\hat{\delta}$ for the subsample of choice data for a particular individual, following the estimation procedure reported in Table 2.

Figure 6. CDFs of Estimated Risk and Time Preferences and Histograms of Individual Estimates: Coble and Lusk (2010)



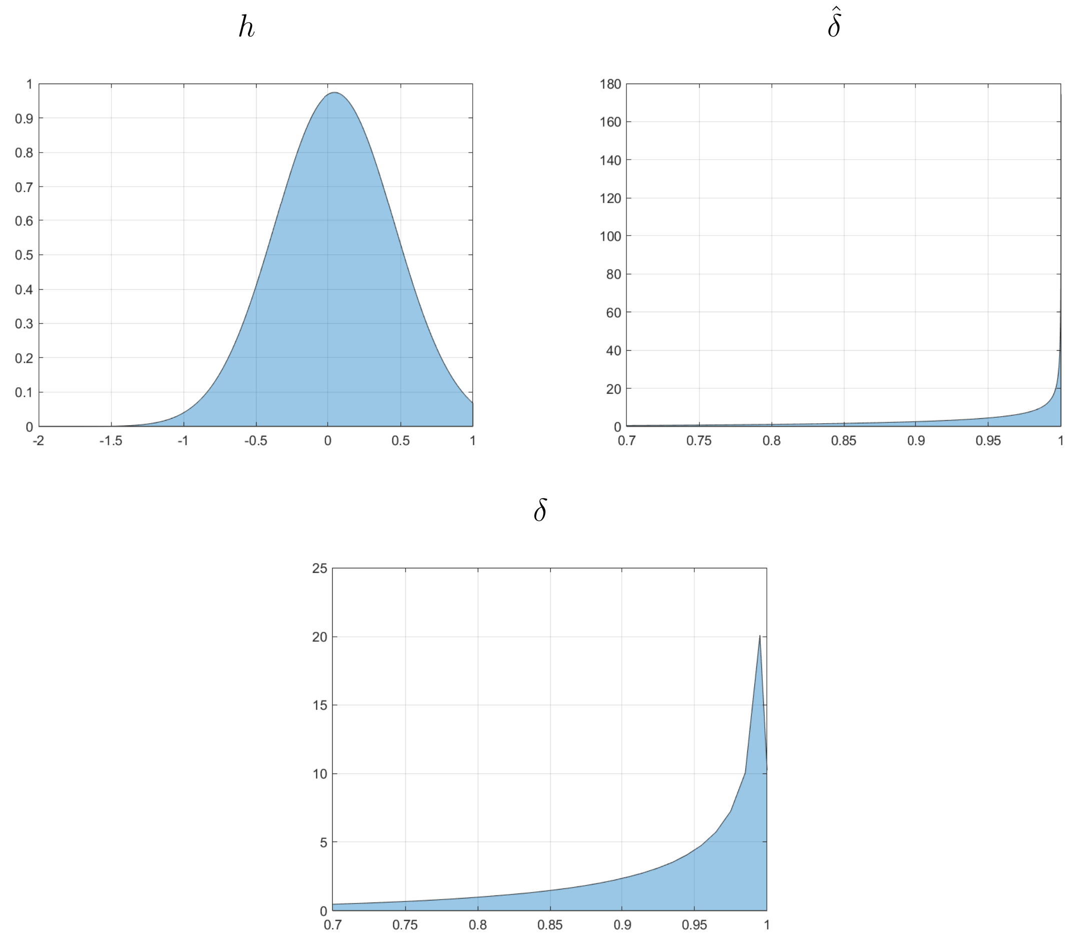
NOTES.- CDFs of the pooled estimation and histograms of the empirical distributions of the individual estimates.

TABLE 3. Estimated Risk and Time Preferences: Andreoni & Sprenger (2012)

Dataset	**All Tasks**	**All Tasks - Correlated Preferences**	**Pooled Individual Estimates**
Median h	0.039 [0.035]	0.048 [0.035]	0.100 (0.443)
Std. Dev. h	0.400 [0.032]	0.420 [0.020]	0.238 (0.215)
Median $\hat{\delta}$	0.953 [0.005]	0.958 [0.005]	0.957 (0.055)
Std. Dev. $\hat{\delta}$	0.113 [0.008]	0.135 [0.012]	0.079 (0.060)
Corr(h,$\hat{\delta}$)	–	-0.601 [0.046]	-0.345
Log-Likelihood	-3.650	-3.602	-3.284

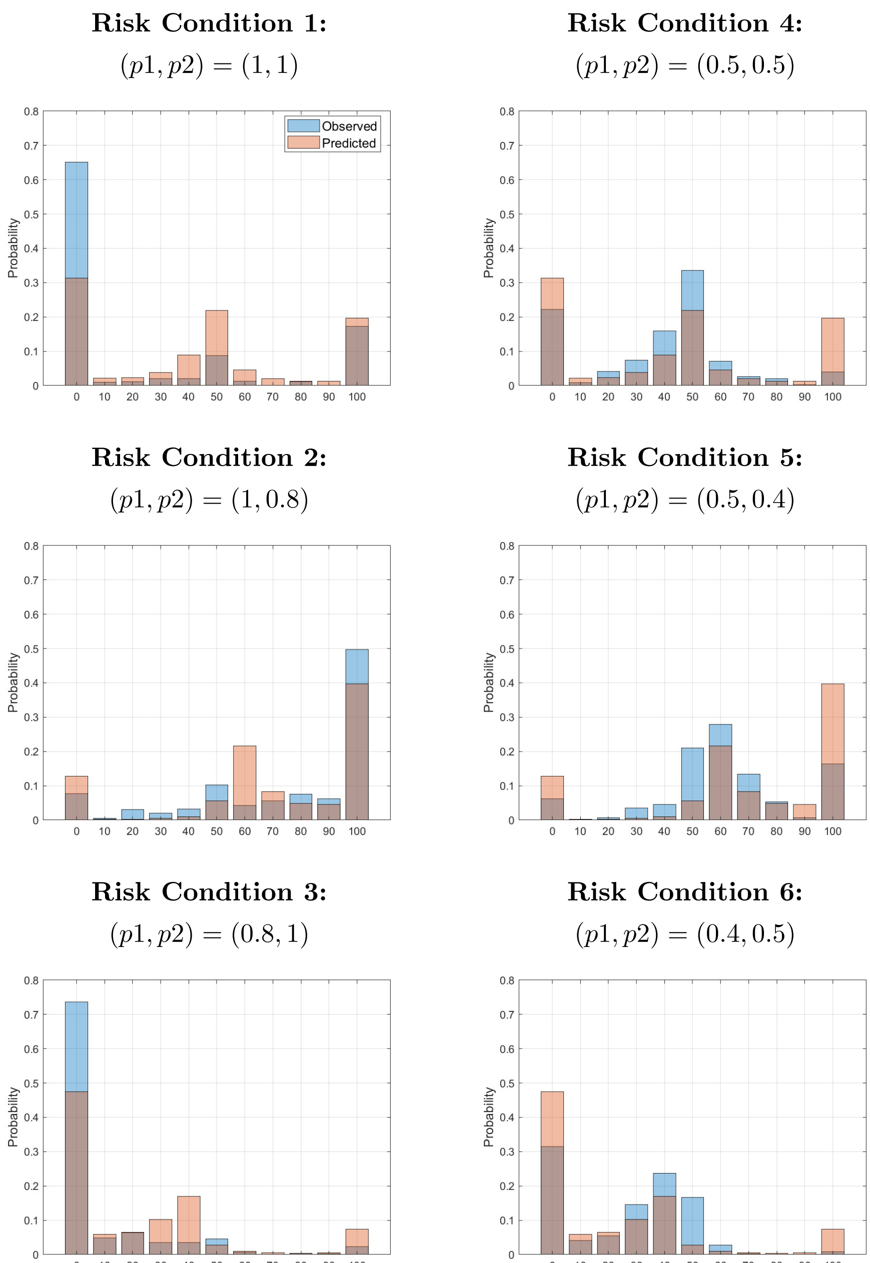
NOTES - The table reports the maximum-likelihood estimates of the median and the standard deviation of the distributions of risk and delay aversion under the DEU-H representation, using data from Andreoni and Sprenger (2012). The second column shows the results from estimating the model pooling all observations in the sample, assuming independent preferences. The third column shows the resulting estimates when we allow preferences to be correlated using a Gaussian copula. The last column shows the median and standard deviation (in parenthesis) of the distribution of individual estimates of the corresponding parameter. In all cases, it is assumed that the coefficient of risk aversion h follows a Normal distribution trunctated at 1 and the corrected discount factor $\hat{\delta}$ follows a Beta distribution. Standard errors, shown in brackets, are computed using the delta method and are clustered at the individual level.

FIGURE 7. PDFs of Estimated Risk and Time Preferences: Andreoni and Sprenger (2012)



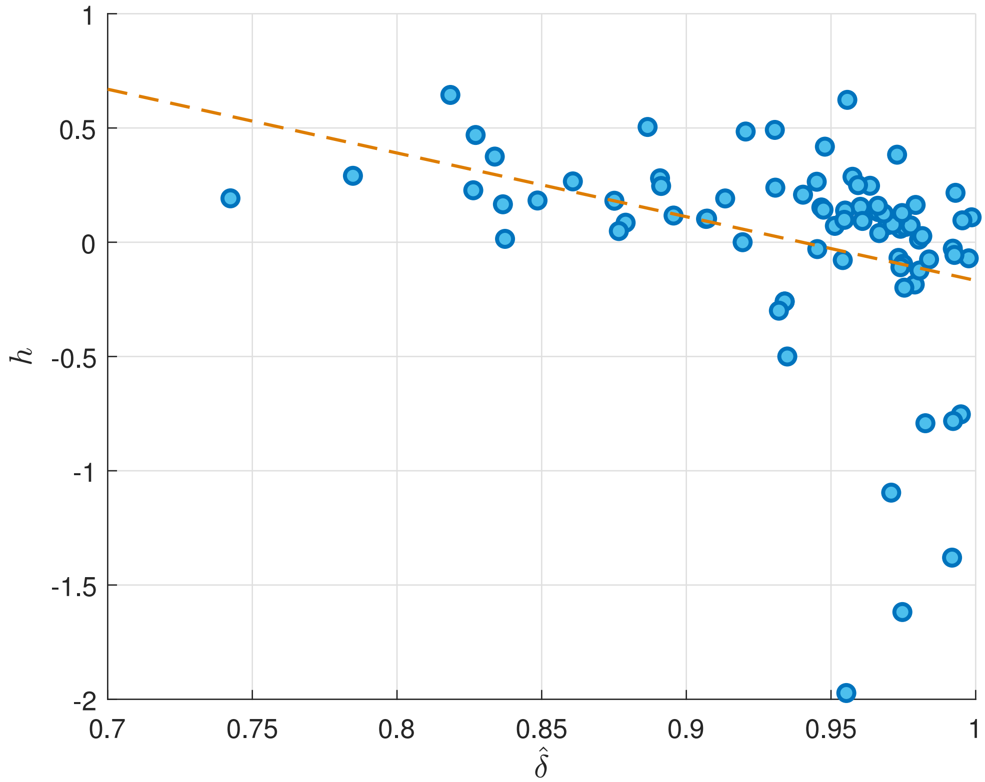
NOTES.- PDFs of the estimated distributions reported in Table 3. The PDF of the discount factor $\delta = \hat{\delta}^{(1-h)}$ is estimated non-parametrically from the distributions of risk aversion and the corrected discount factor using a normal kernel.

FIGURE 8. Observed and Predicted Distributions of Choices in Andreoni and Sprenger (2012)



NOTES.- Observed frequencies and predicted probabilities of choosing share α_τ ($\times 100$) in each risk condition considered in Andreoni and Sprenger (2012). The observed distributions show the relative frequency of each allocation in the data, grouped to the closest multiple of 10. The predicted distributions are computed based on the estimated parameters of the DEU-H representation, as shown in, as shown in Table 3.

Figure 9. Risk and Delay Aversion by Individual: Andreoni & Sprenger (2012)



NOTES.- Each point represents a combination of the medians of the estimated distributions of the coefficients of risk aversion h, and the corrected discount factor $\hat{\delta}$, for the subsample of choice data for a particular individual, following the estimation procedure reported in Table 3.