# Advanced Macroeconomics II

## Handout 3 - Interpolation

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Western University

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# Short recap

Prototypical DP problem:

$$V(k,z) = \max_{\{c,k'\}} u(c) + \beta E\left[V\left(k', z'\right)|z\right]$$
$$\text{s.t.}\, c + k' = f(k,z)$$
$$z' = h(z,\eta)\,;\eta \text{ stochastic}$$

- We are looking for functions $\mathbf{V}, \mathbf{g^c}, \mathbf{g^k}$: We cannot solve this.

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- We are looking for functions $\mathbf{V},\mathbf{g^c},\mathbf{g^k}$: We cannot solve this.

We need to solve an approximate problem:

1. Discretize state space (functions are now vectors)
2. Approximate continuous function: **Interpolation**
    - Requires "exact" solution of maximization problem: **Optimization**

# Interpolation: The problem

- We want to know function $V$...
  - But we only know $\{V(x_1), \ldots, V(x_N)\}$
- When working with $V$ we will often need $V(x)$ for $x \notin \{x_1, \ldots, x_N\}$

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- When working with $V$ we will often need $V(x)$ for $x \notin \{x_1, \ldots, x_N\}$
- We want a function $\tilde{V}$ that we can evaluate at any $x$
  - It must be that $\tilde{V}(x_i) = V(x_i)$ for all $x \in \{x_1, \ldots, x_N\}$
- The problem now is how to find this function $\tilde{V}$

# Interpolation: Two approaches

1. "Global" approximation
   - Approximate with a known function and evaluate that!
   - But functions are infinite dimensional...
   - Choose functions from some vector space! Basis is finite dimensional
   - Problem is to find coefficients for linear combination
   - Ex: Polynomial approximation

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2. Local approximation
   - Match the function locally (between two nodes)
   - The local function is called a **Spline**
   - Splines can be as flexible as you need them to be
   - Ex: Cubic splines, shape preserving splines

# Polynomial Approximation

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$$\tilde{V}(x) = \sum_{m=0}^{M} a_m \phi_m(x)$$

3. We are looking for $\{a_0, \ldots, a_M\}$ such that

$$y_i = V(x_i) = \tilde{V}(x_i) = \sum_{m=0}^{M} a_m \phi_m(x_i) \qquad x_i \in \{x_1, \ldots, x_N\}$$

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Then what we have is a linear problem:

$$y = Aa$$

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- We need to set $M = N - 1$ to fit the values
- We need to choose a basis for our polynomial
  - Monomial basis ($\phi_m(x) = x^m$)
  - Newton basis $\left(\phi_m(x) = \Pi_{j=0}^{m-1}(x - x_j)\right)$

# Weierstrass Approximation Theorem

## Theorem

*Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function.*
*For all* $\epsilon > 0$*, there exists a polynomial of order* $n$*,* $P_M(x)$*, such that for all* $x \in [a, b]$*, we have* $\|f(x) - P_M(x)\|_\infty < \epsilon$*.*
*Further,* $\lim_{M\to\infty} \|f(x) - P_M(x)\|_\infty = 0$*.*

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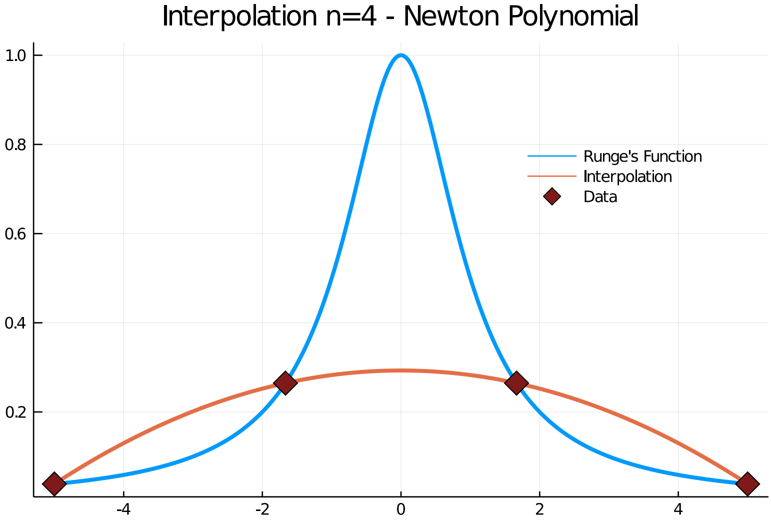
- *It looks like using polynomials is a great idea!*
- *With enough nodes $\{x_i\}$ we can approximate any continuous function*
- *Success comes at a cost: Higher order polynomials*
  - *Polynomials start to oscillate dramatically at higher orders*

# Runge example: $f(x) = 1/_{1+x^2}$



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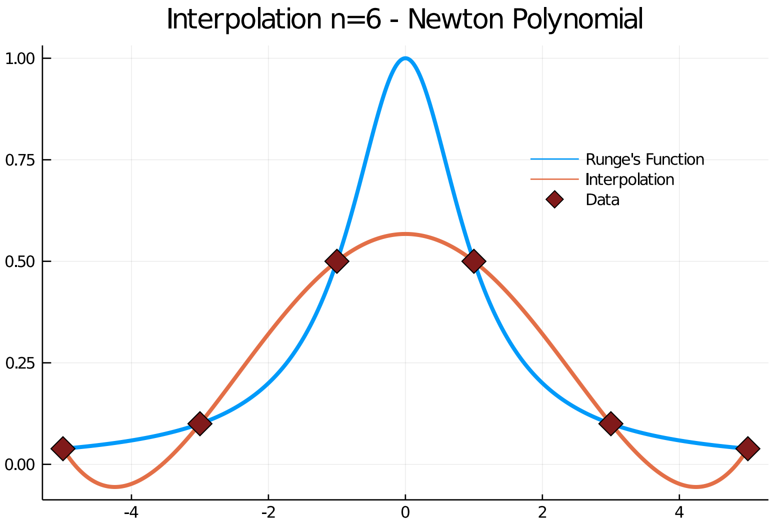

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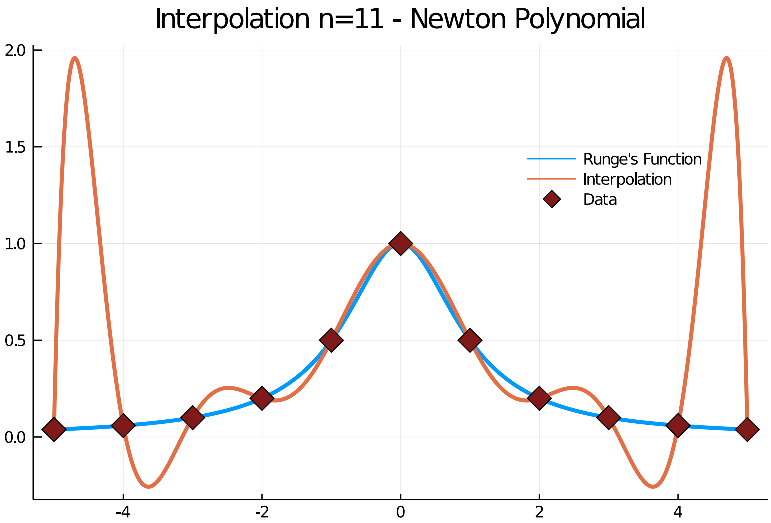

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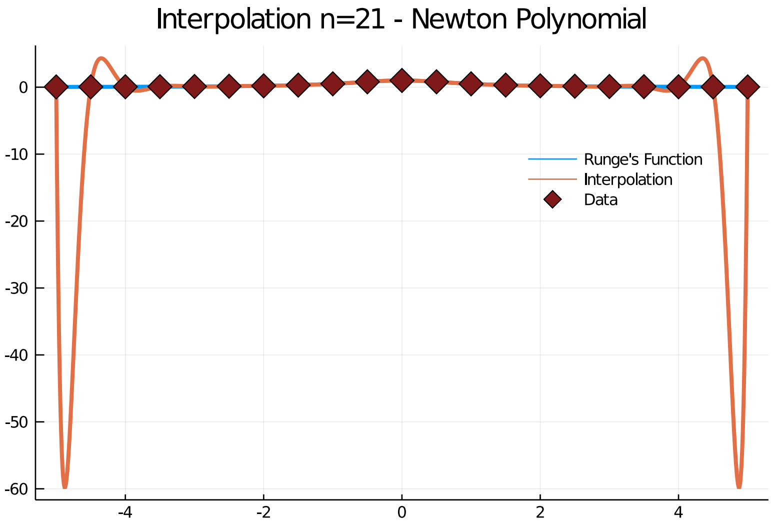

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   - Hard to know which node placement works for your particular problem
   - We will come back to this at the end
2. Avoid "global" approximation (one size does not fit all)
   - Lets talk about splines

# Splines

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$$\tilde{V}(x) = A(x) \cdot V(x_i) + B(x) \cdot V(x_{i+1})$$

where: $A(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i}$ $\qquad B(x) = 1 - A(x) = \frac{x - x_i}{x_{i+1} - x_i}$

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  - Resulting function is continuous but not smooth
    - Curse of dimensionality applies if looking for good approximation
  - First derivatives do not exist at nodes $\{x_1, \ldots, x_N\}$ (FOC)
  - However: Fast, robust method

# Splines - Linear splines

**Algorithm 1:** Linear Splines

**Result**: Interpolated value y_hat at point x

Define grids ;

x_grid = $(x_1, \ldots, x_N)$ ;

y_grid = $(y_1, \ldots, y_N)$ ;

Locate closest indeces to x on grid ;

```
ind = findmax(sign.(x_grid .- x))[2] - 1 ;
```

Compute interpolation ;

```
A_x = (x_grid[ind+1] - x)/(x_grid[ind+1]-x_grid[ind]) ;
y_hat = A_x*y_grid[ind] + (1-A_x)*y_grid[ind+1] ;
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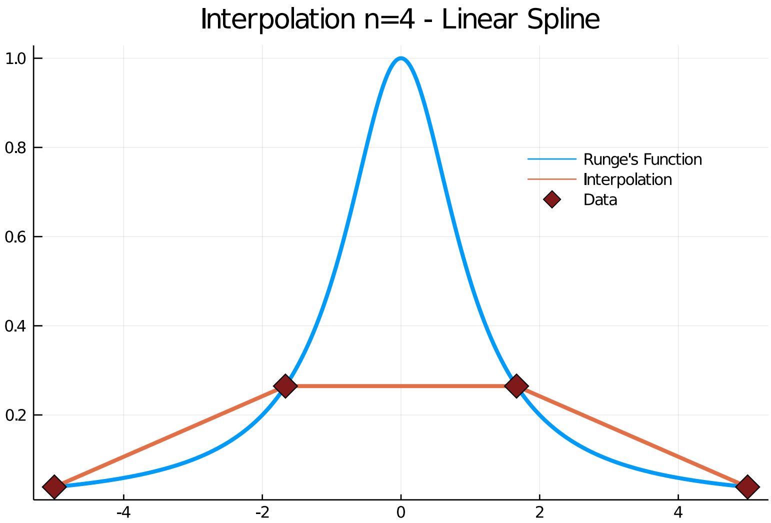

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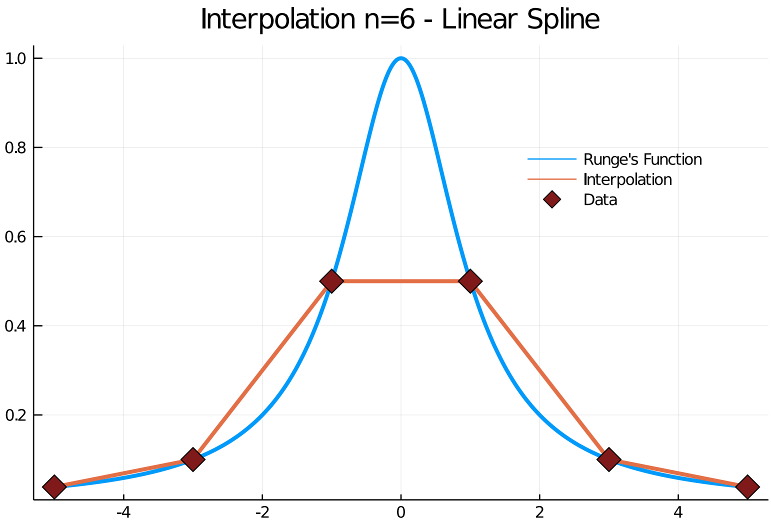

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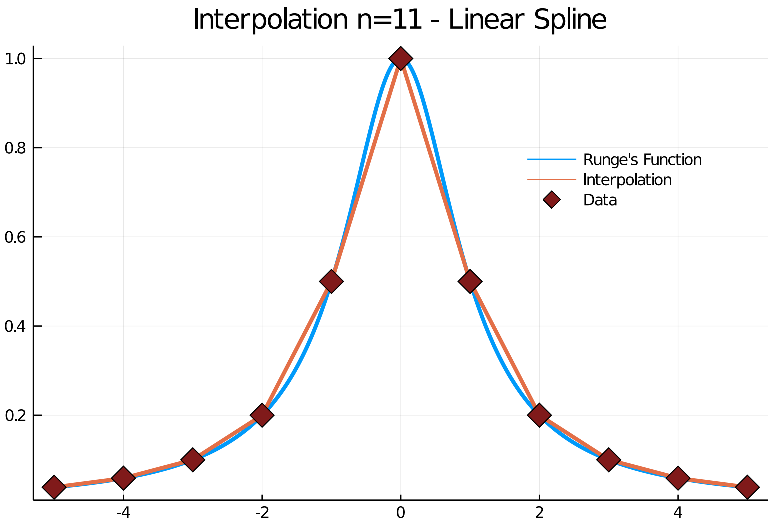

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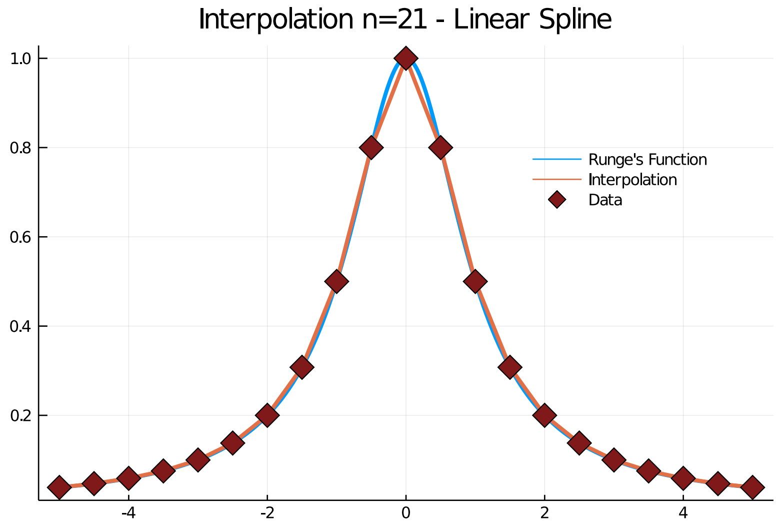

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2. First derivatives are continuously differentiable everywhere
3. Second derivatives are continuous everywhere

# Cubic splines - Importance of derivatives

This is a preferred method for economic applications:

- First derivatives obtained as a byproduct of interpolation
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Derivatives are key:

- Many optimization algorithms are gradient-based
- First order conditions (Euler equation) depend on $V'$
- EGM depends on $V'$ to avoid solving the Euler equation

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**Note:** For easy of exposition I will change from $V(x)$ to $y$ notation

# (How to) Cubic splines

Twice continuous differentiability implies that:

$$\tilde{y}(x) = A(x) \cdot y_i + B(x) \cdot y_{i+1} + C(x) \cdot y_i^{''} + D(x) \cdot y_{i+1}^{''}$$

where: $C(x) = \frac{1}{6}\left(A^3(x) - A(x)\right)(x_{i+1} - x_i)^2$ and
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# (How to) Cubic splines

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$$\tilde{y}(x) = A(x) \cdot y_i + B(x) \cdot y_{i+1} + C(x) \cdot y_i'' + D(x) \cdot y_{i+1}''$$

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- **Good news:** You only need to compute $A$ and $B$ to get all coefficients
- **Bad news:** You need to find out values for $\left\{y_i''\right\}$...

# (How to) Cubic splines

How to solve for the unknown second derivatives? With first derivatives!

- First derivative of $\tilde{V}$ satisfies:

$$\frac{\partial y}{\partial x}(x) = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{x_{i+1} - x_i}{6}\left[\left(3A(x)^2 - 1\right) y_i^{''} - \left(3B(x)^2 - 1\right) y_{i+1}^{''}\right]$$

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  Note that once we know $y^{''}$ we get first derivative for free!

- But we want these derivatives to be continuous (at the grid nodes):

$$\underbrace{\frac{y_i - y_{i-1}}{x_i - x_{i-1}} - \frac{x_i - x_{i-1}}{6}\left[-y_{i-1}^{''} - 2y_i^{''}\right]}_{\lim_{x \to x_i^-} \frac{\partial y}{\partial x}} = \underbrace{\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{x_{i+1} - x_i}{6}\left[2y_i^{''} + y_{i+1}^{''}\right]}_{\lim_{x \to x_i^+} \frac{\partial y}{\partial x}}$$

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  Note that once we know $y''$ we get first derivative for free!

- Rearrange:

$$\underbrace{\underbrace{\frac{x_i - x_{i-1}}{6}}_{c_{i-1}} y_{i-1}'' + \underbrace{\frac{x_{i+1} - x_{i-1}}{3}}_{d_i} y_i'' + \underbrace{\frac{x_{i+1} - x_i}{6}}_{c_i} y_{i+1}''}_{\text{Linear system on } y''} = \underbrace{\underbrace{\frac{y_{i+1} - y_i}{x_{i+1} - x_i}}_{s_i} - \underbrace{\frac{y_i - y_{i-1}}{x_i - x_{i-1}}}_{s_{i-1}}}_{\text{Diff. of Slopes (Known)}}$$

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  - **Natural Spline:** Spline is linear at boundaries $y_1'' = y_N'' = 0$
    - This is the normal assumption
    - Helps for extrapolation (more on this at the end)
  - **Flat Spline:** Spline is flat at boundaries $y_1' = y_N' = 0$

# (How to) Cubic splines

To find cubic splines solve this (tri-diagonal) linear system:

$$\begin{bmatrix} 2c_1 & -c_1 & & & & & & \\ c_1 & d_i & c_2 & & & & & \\ & & \ddots & & & & & \\ & & c_{i-1} & d_i & c_i & & & \\ & & & & \ddots & & & \\ & & & & c_{N-2} & d_{N-1} & c_{N-1} & \\ & & & & & & -c_N & 2c_N \end{bmatrix} \cdot \begin{bmatrix} y_1'' \\ y_2'' \\ \vdots \\ y_i'' \\ \vdots \\ y_{N-1}'' \\ y_N'' \end{bmatrix} = \begin{bmatrix} s_1 - b_1 \\ s_2 - s_1 \\ \vdots \\ s_i - s_{i-1} \\ \vdots \\ s_{N-1} - s_{N-2} \\ s_N - b_N \end{bmatrix}$$

$$Cy'' = S$$

# (How to) Cubic splines

**Algorithm 2:** Cubic Splines

**Result**: Interpolated value y_hat at point x

Define grids and boudnary conditions (either on y' or y'') ;

x_grid = $(x_1, \ldots, x_N)$ y_grid = $(y_1, \ldots, y_N)$ ;

Solve tri-diagonal system for vector of y'': y_pp = C\S ;

Locate closest indeces to x on grid ;

ind = findmax(sign.(x_grid .- x))[2] - 1 ;

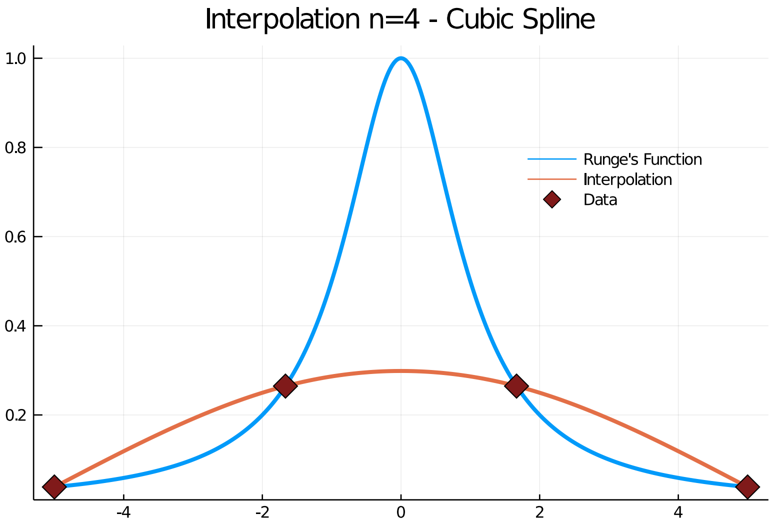
Compute interpolation ;

A_x = (x_grid[ind+1] - x)/(x_grid[ind+1]-x_grid[ind]) ;

Compute B_x, C_x, D_x accordingly ;

y_hat = A_x*y_grid[ind] + (1-A_x)*y_grid[ind+1] +
C_x*y_pp[ind] + D_x*y_pp[ind+1] ;

# Runge example: $f(x) = {}^{1}/_{1+x^2}$ - Cubic Splines



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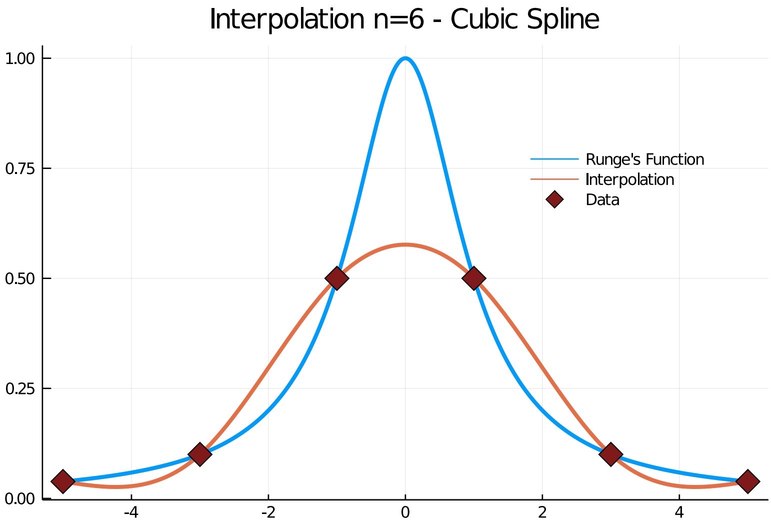

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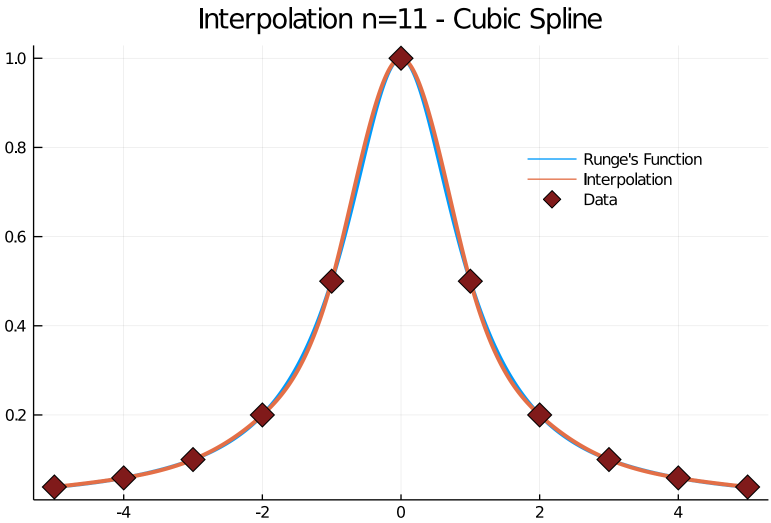

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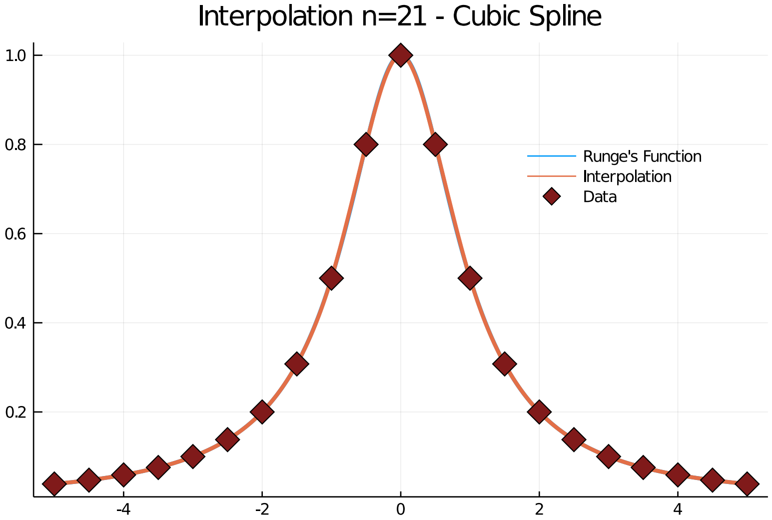
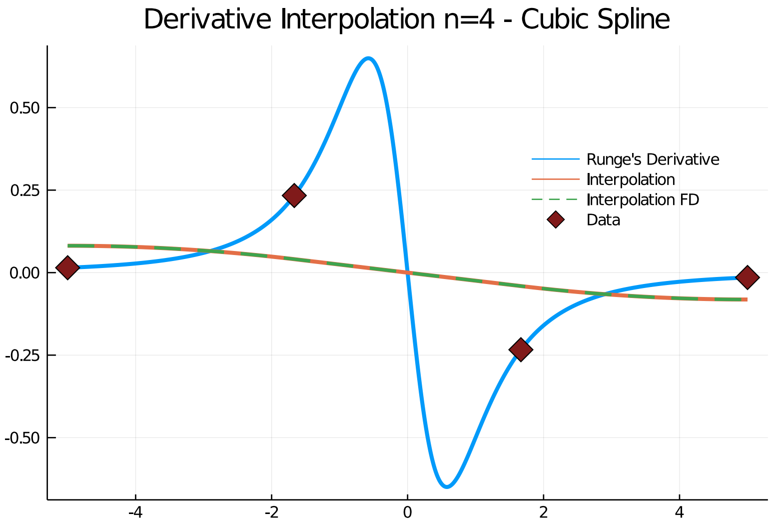


# Runge example: $f(x) = {}^{1}/_{1+x^2}$ - Derivative



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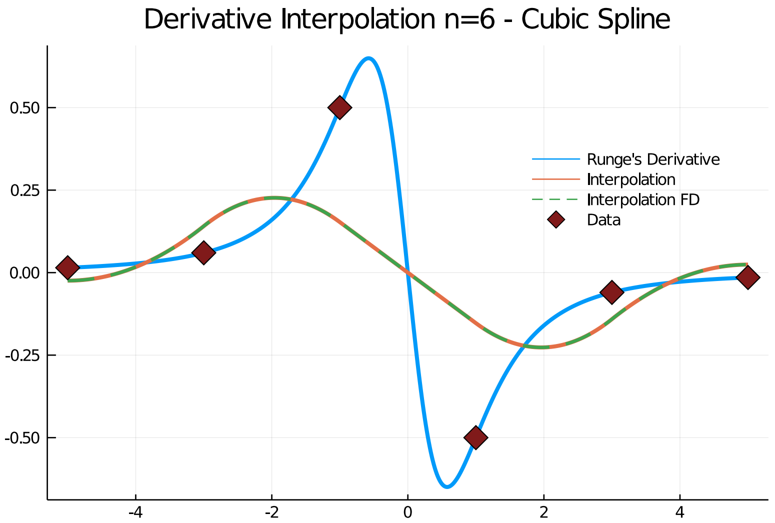

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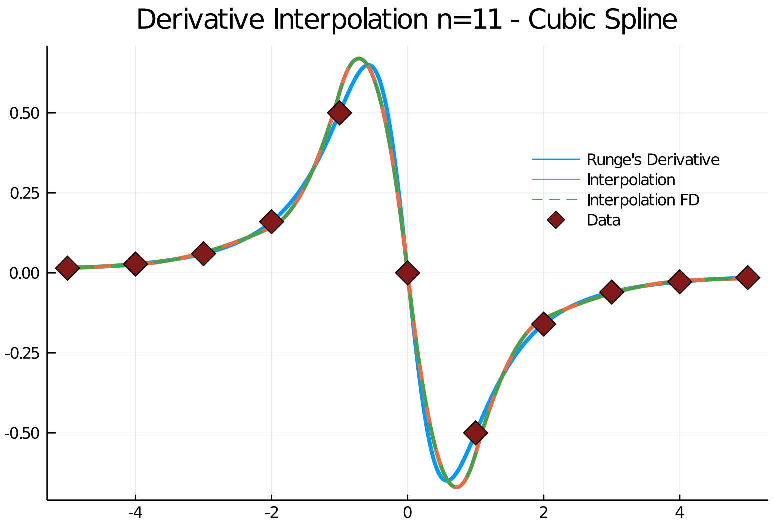

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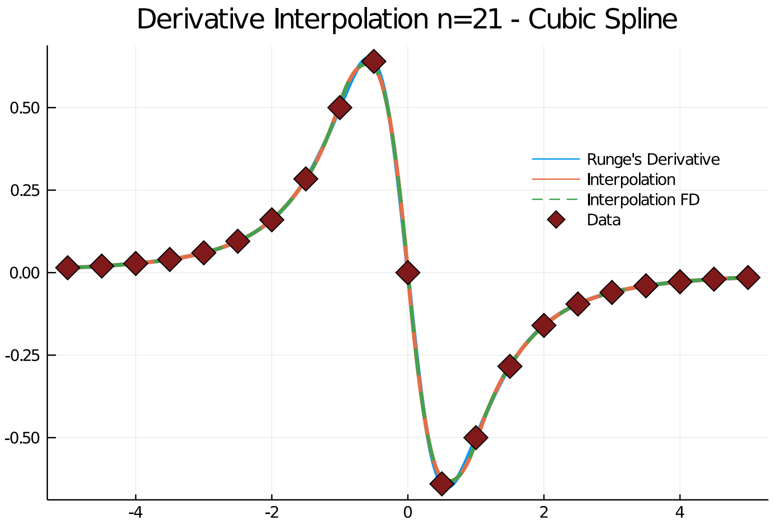

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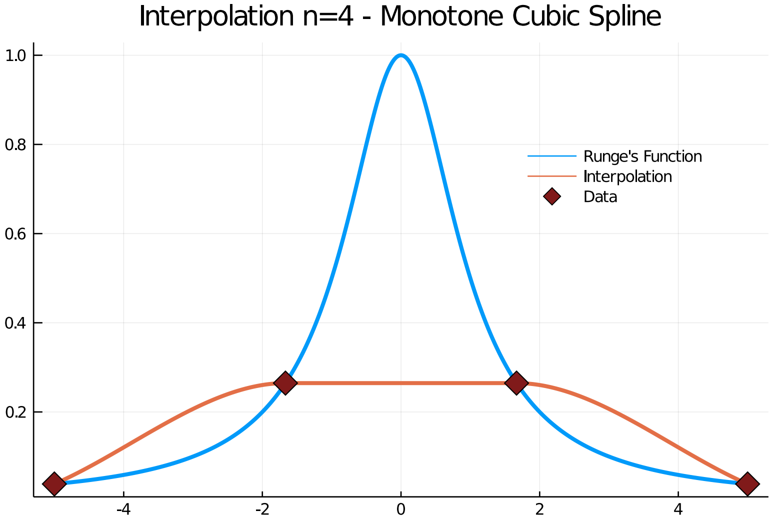
- **Schumaker Splines:**
  - Quadratic splines preserving monotonicity or concavity
  - Faster to compute, oscillates less, worth checking out
  - Shape restrictions already mess up second derivatives

# Runge example: $f(x) = 1/_{1+x^2}$ - Montone Splines



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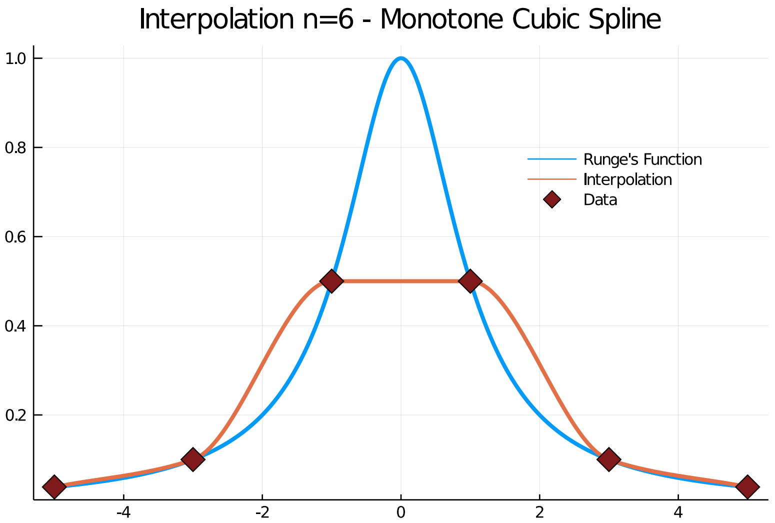

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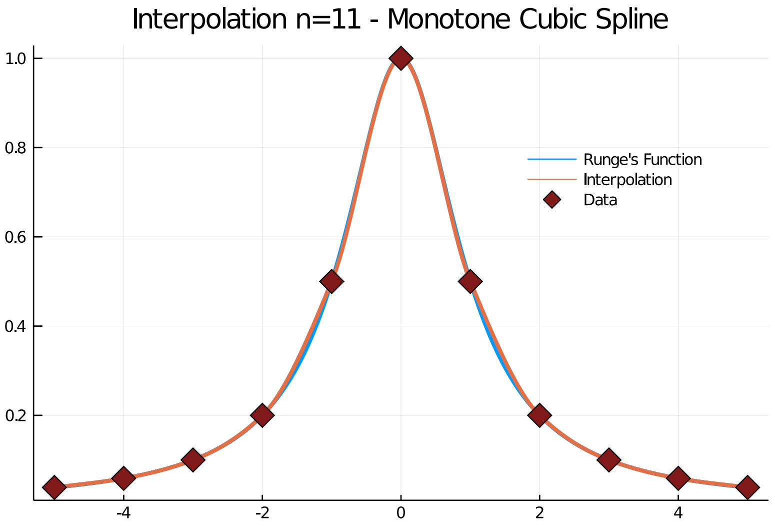

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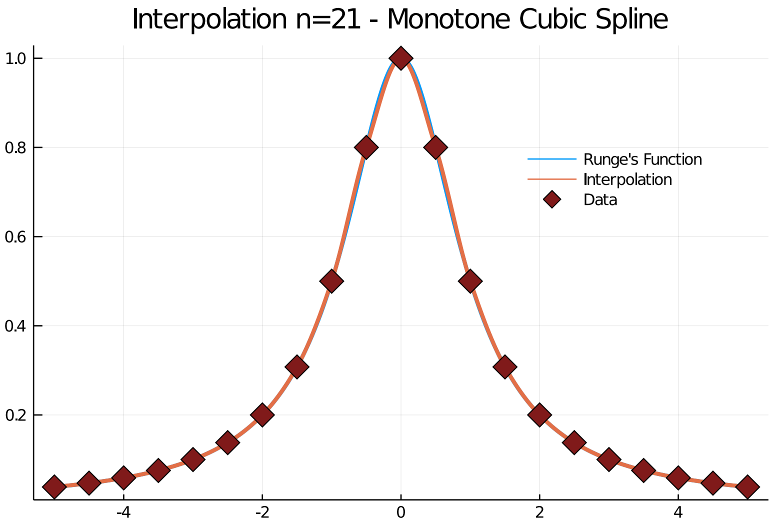

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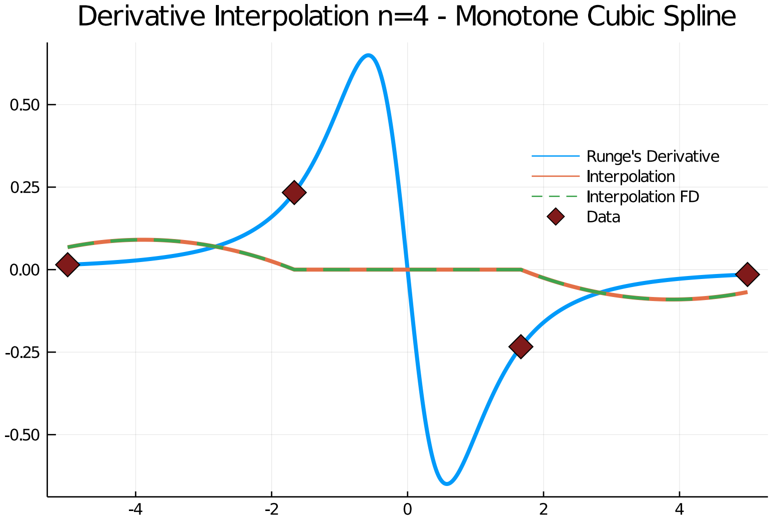

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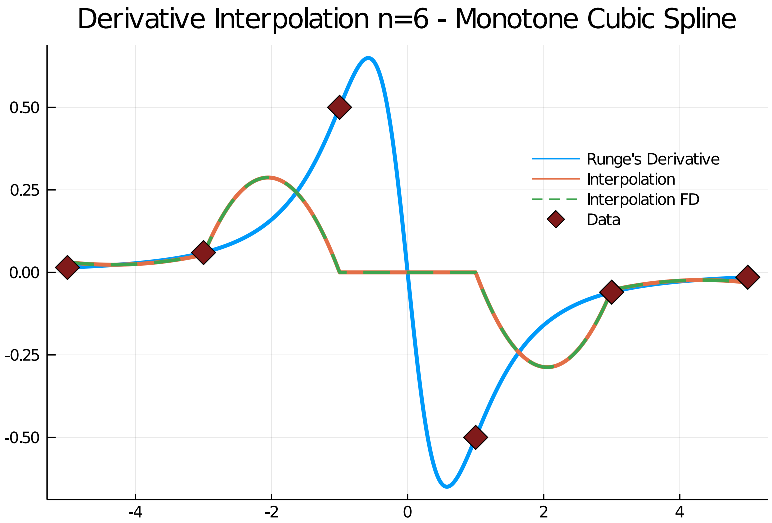

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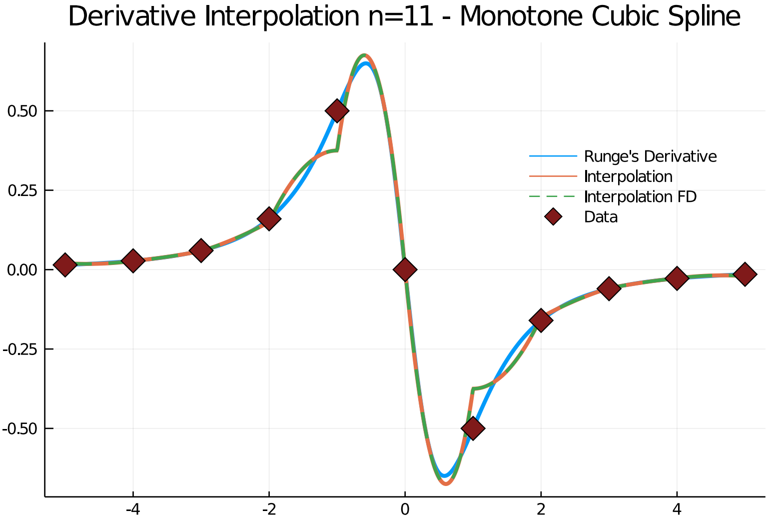

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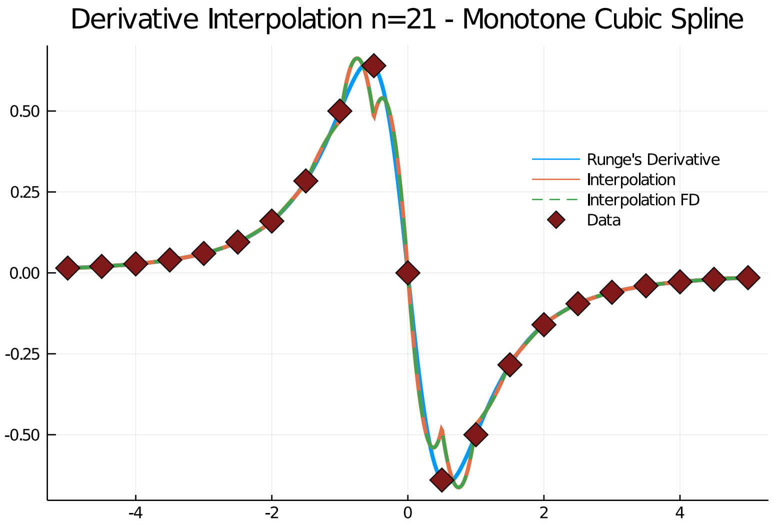

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- A good idea when cubic splines are too wavy or jumpy
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- You pay the price with potentially funky first derivatives
- Important to test your interpolation on the type of functions you use
  - Hard to know ex-ante what will work

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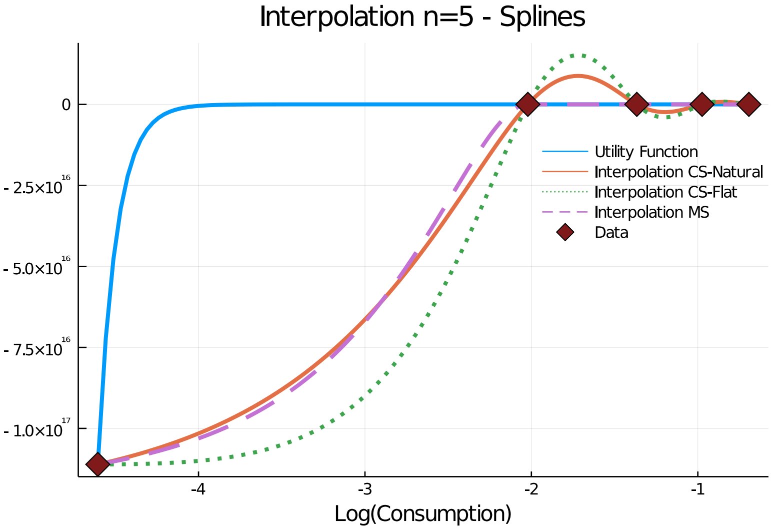

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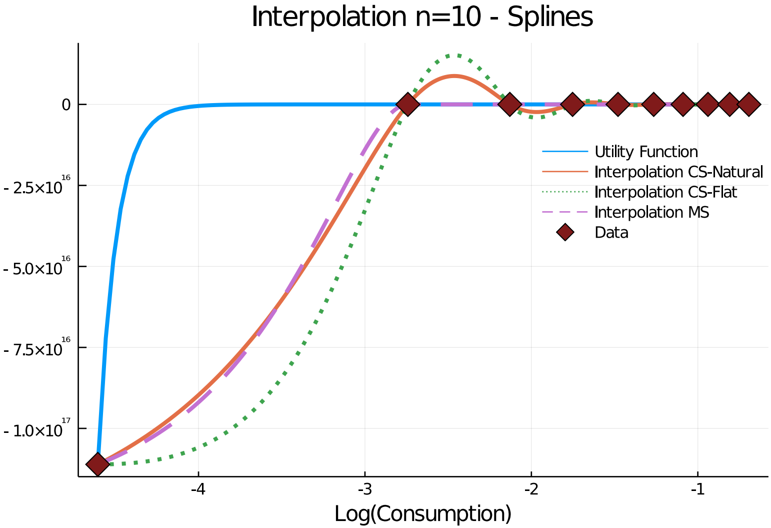

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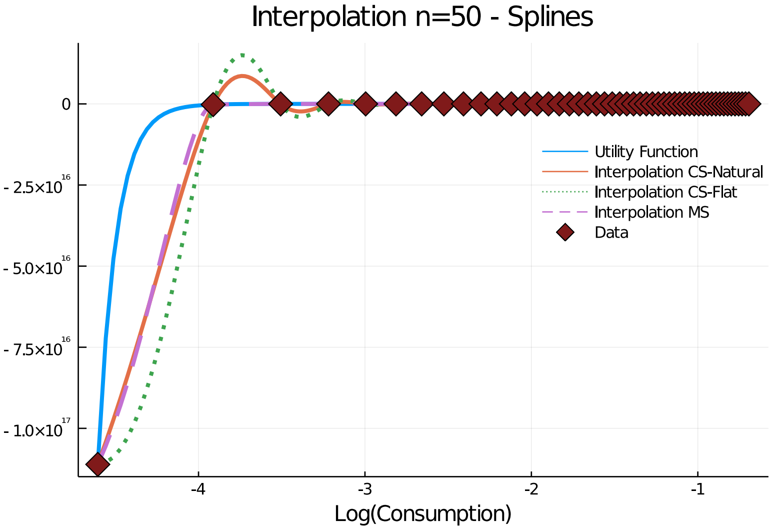

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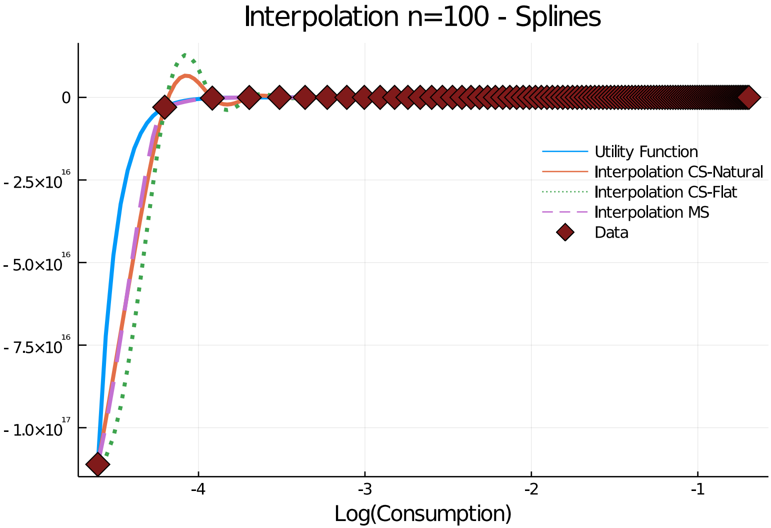

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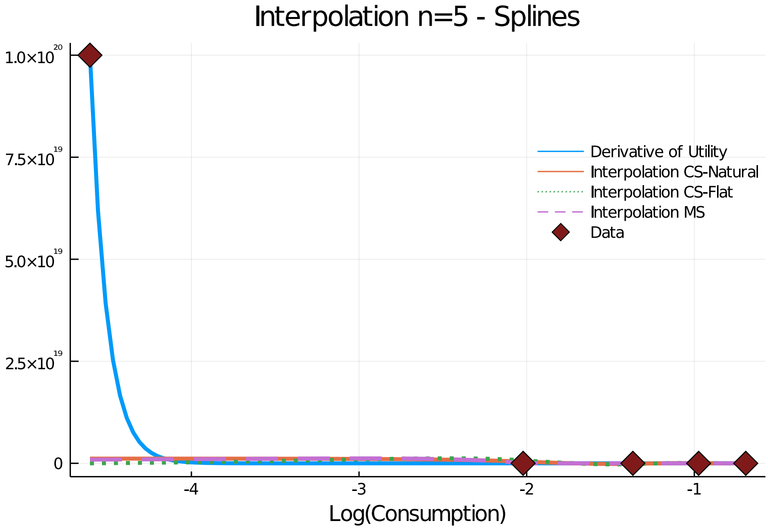

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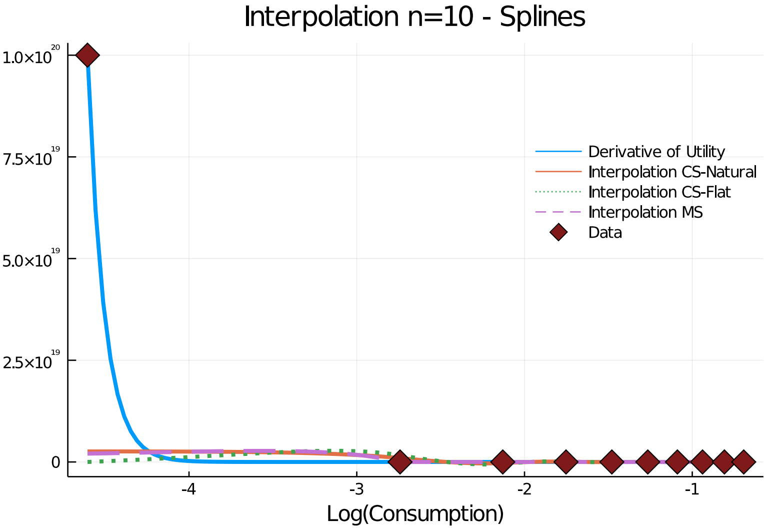

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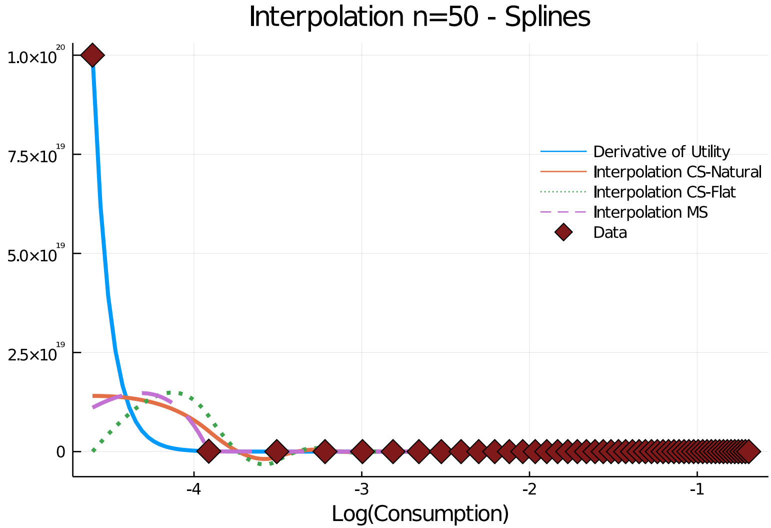

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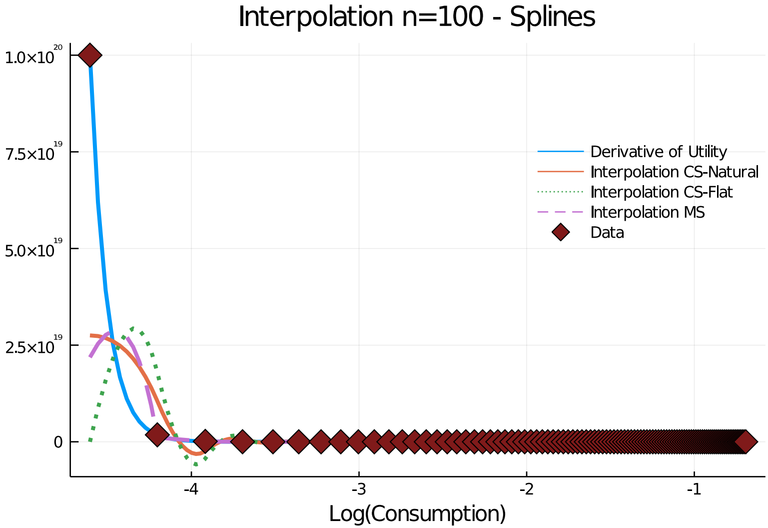

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**Solution:** Supply your own first order conditions

- You have to write your own function for this

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- This also affects kinks
  - Kinks (coming from a discrete choice) change curvature
  - Better to deal with them with linear interpolation
  - You need more points there

# Grid spacing - Algorithm

**Algorithm 3:** Curved Grid: Polynomial or Exponential Scaling

**Function** `Curved_Grid`($n$,$a$,$b$,$\theta$, $Type$):

    grid = range(0,1,length=n)

    **if** $Type$==$Polynomial$ **then**

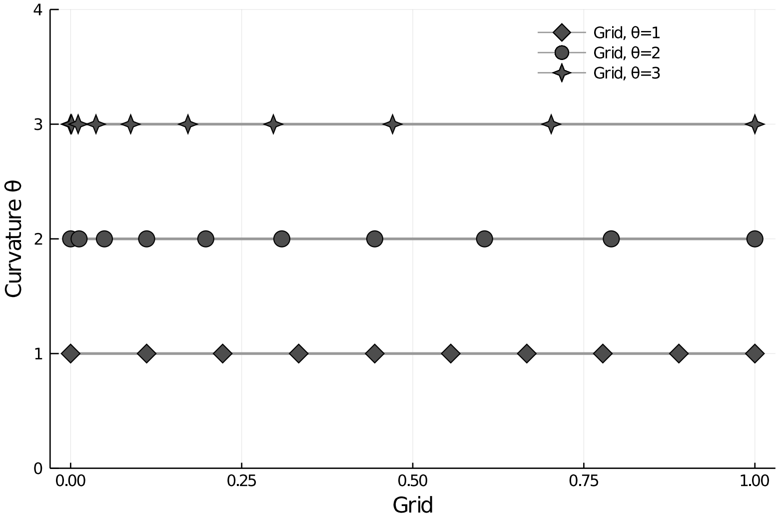
        grid = a + (b-a)*$grid^{\theta}$

    **if** $Type$==$Exponential$ **then**

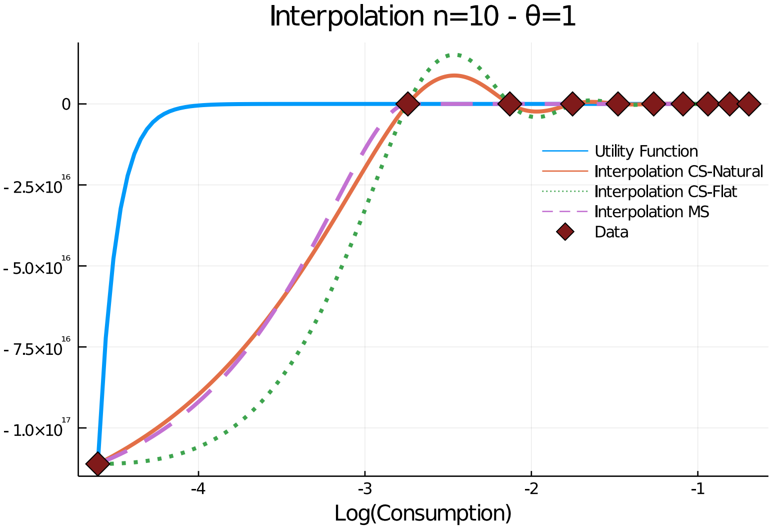
        grid = a + (b-a)*$\frac{exp(\theta * grid) - 1)}{exp(\theta) - 1}$

    return grid

# Grid spacing - Polynomial grid example



# Grid spacing - Back to CRRA



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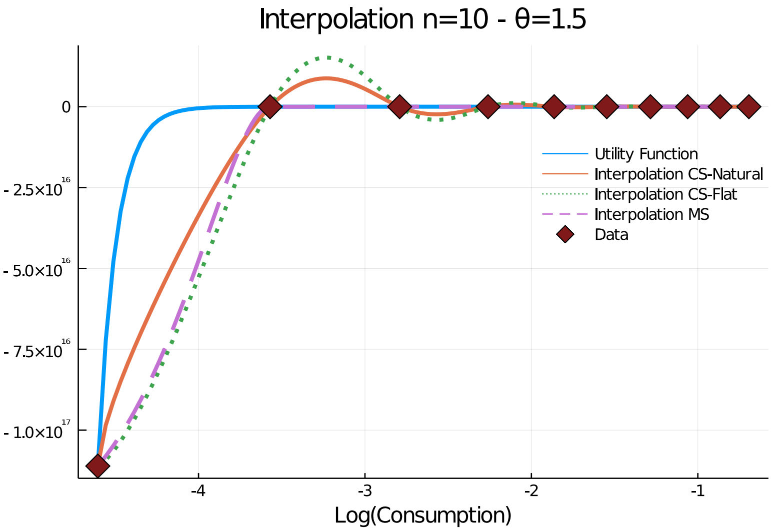

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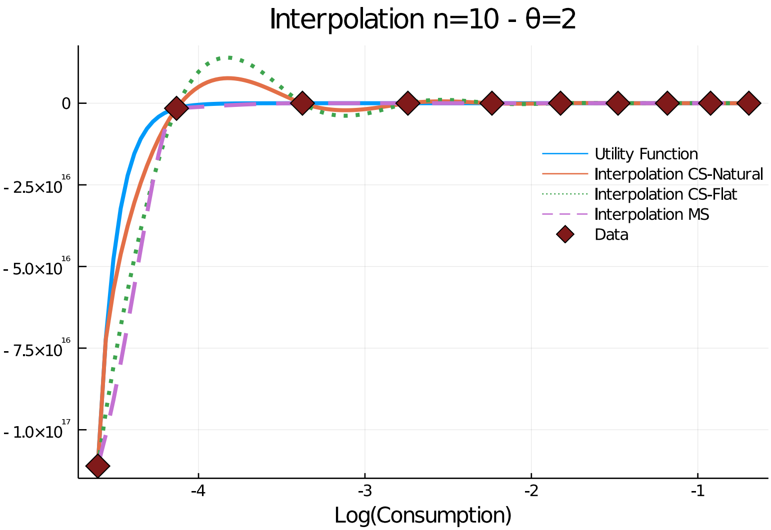

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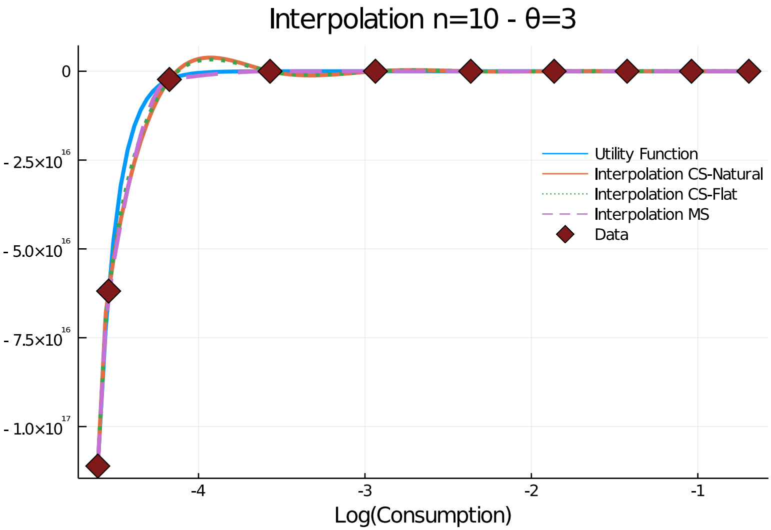

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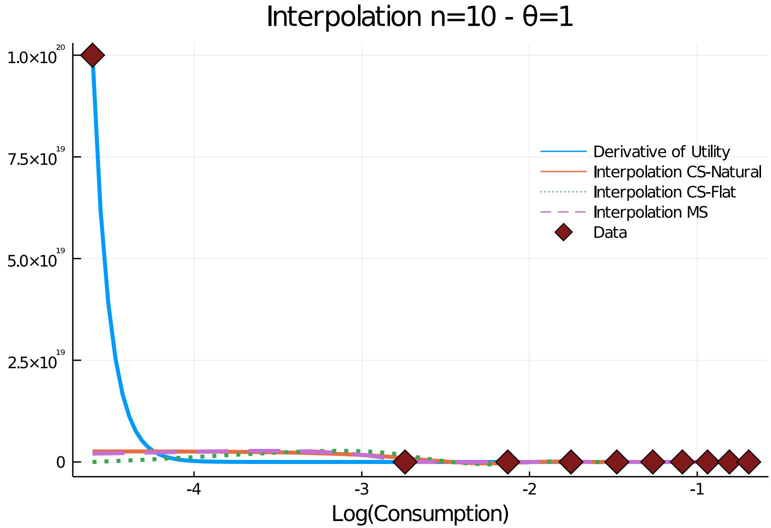

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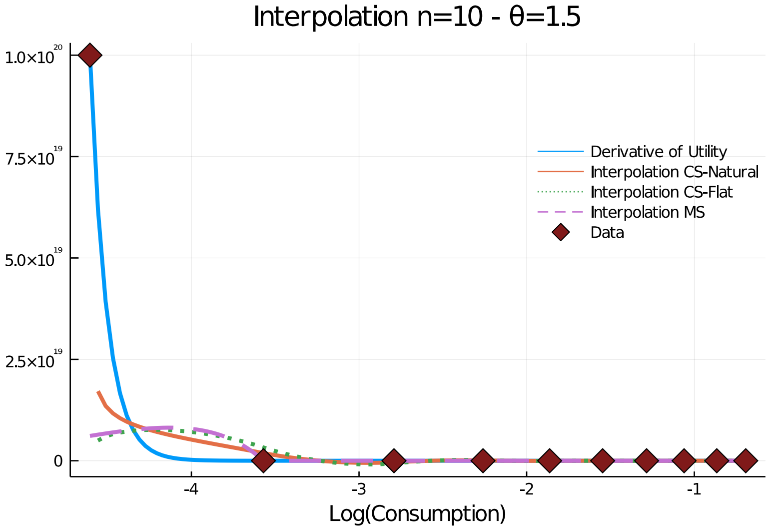

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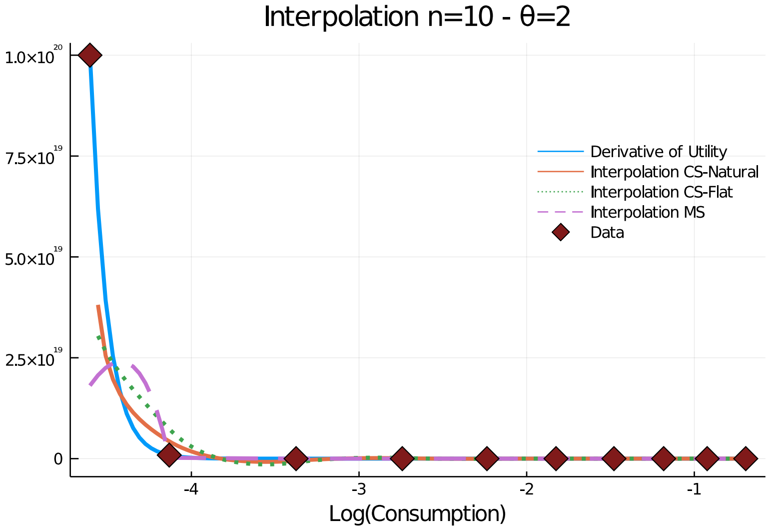

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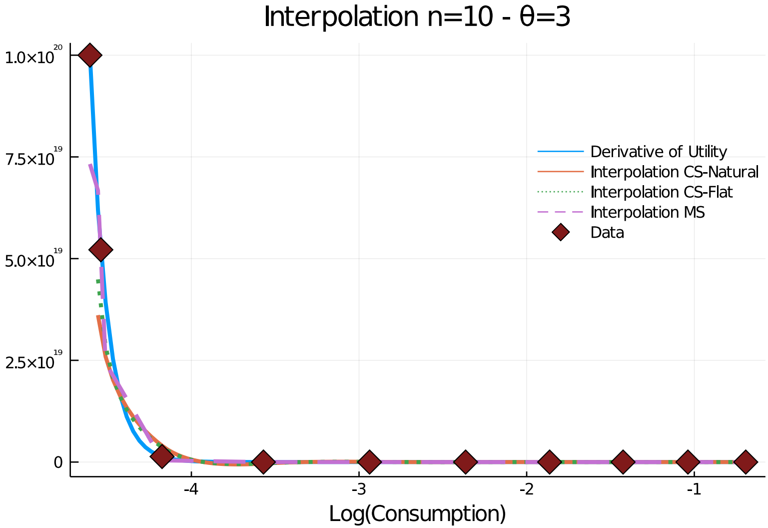

# Final Words

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    - Extrapolating is lethal if you use high degree polynomials
- Abstain at all costs from extrapolating
- If you must extrapolate use linear extrapolation
- Unless you have some theory on your side
    - Theory is great because it tells you what to do!
    - Ex: Pareto Extrapolation:
      An Analytical Framework for Studying Tail Inequality by Akira-Toda & Gouin-Bonenfant

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- All rules have exceptions... Sometimes you cannot make approximation errors, you will need specialized algorithms tailored to your problem