Advanced Macroeconomics II

Handout 4 - Optimization

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Short recap

Prototypical DP problem:

$$V(k,z) = \max_{\{c,k'\}} u(c) + \beta E\left[V\left(k^{'},z^{'}\right)|z\right]$$
$$\text{s.t.}c + k^{'} = f(k,z)$$
$$z^{'} = h(z,\eta)\,;\eta \text{ stochastic}$$

- We are looking for functions $\mathbf{V}, \mathbf{g^c}, \mathbf{g^k}$: We cannot solve this

We need to solve an approximate problem:

1. Discretize state space (functions are now vectors)
2. Approximate continuous function: **Interpolation**
   - Requires "exact" solution of maximization problem: **Optimization**

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Root finding:

- We ca solve the problem by looking at the FOC (Euler equation)
- We are looking for values that make the FOC be zero (hence the root)

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2. Guess a solution (say a value for $\{c, k', \ell, \ldots\}$)
3. Evaluate the objective function in that solution
4. Update your guess if needed

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⋆ We do have to help our optimizer with an initial guess of the solution

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⋆ Evaluating the function is the key step

- Evaluating might require solving intermediate problems (c vs ℓ)
- Requires evaluation in points off the grid (interpolation)
- Requires taking expectations (we are not there yet)

Optimization - Packages

- A good overview in quantecon (click here)
- Native Julia optimization module (Optim.jl)
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Ultimate source of all knowledge:

- Numerical Recipes: The Art of Scientific Computing by Press, Teukolki, Vetterling and Flannery

Local Optimizers

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- Local optimizers are generally faster than global methods
  - Drawback: We need to be "close to the solution"
- To overcome drawback we invest in **bracketing the solution**
  - Bracketing with theory: steady state convergence, minimum consumption, time constraints
  - Numerical procedures (see section 10.1 of Numerical Recipes)

Warning

We want to maximize... but computer scientists always want to minimize

- Make sure to operate on the negative of your value function
- Be careful! Most early bugs in your code are a misplaced minus sign

Bracketing a minimum (in one dimension)

- A minimum is bracketed by three points: a, b, and c
  - Without loss we will have: $a < b < c$
- A minimum is bracketed if: $f(a), f(c) > f(b)$

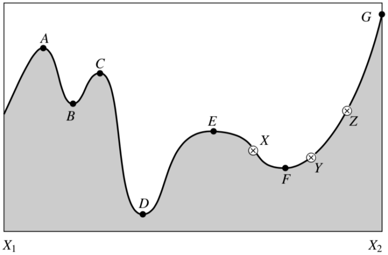

Figure 10.0.1. Extrema of a function in an interval. Points A, C, and E are local, but not global maxima. Points B and F are local, but not global minima. The global maximum occurs at G, which is on the boundary of the interval so that the derivative of the function need not vanish there. The global minimum is at D. At point E, derivatives higher than the first vanish, a situation which can cause difficulty for some algorithms. The points X, Y, and Z are said to "bracket" the minimum F, since Y is less than both X and Z.

Bracketing a minimum

Theory:

- We know that $c^{\star} \in [\epsilon, f(k,z) - \underline{k}]$ for some $\epsilon, \underline{k} > 0$
- We know some problems are globally convergent so:
  - If $k \leq k_{ss}$ then $k'^{\star} \in [k, k_{ss}]$, otherwise $k'^{\star} \in [k_{ss}, k]$

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Numerically:

- Start with some "arbitrary" interval $[a, b]$
- Get c so as to get $b \in [a, c]$ (be smart in choosing c, read NR)
- Check if $f(a), f(c) > f(b)$
- Rinse and repeat

One-dimensional optimizers

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How do optimizers work?

- Bisection (Golden section)
- Parabolic interpolation (Brent)
  - Parabolic interpolation with derivative (dBrent)
- Higher polynomials (Do not use!)

Golden section

- Start with $a < b < c$ so that $f(a), f(c) > f(b)$
- Choose a point in $x \in [a, b]$ or $x \in [b, c]$, say you picked $x \in [b, c]$
- Keep x if $f(c), f(b) > f(x)$, x is the new candidate for a min

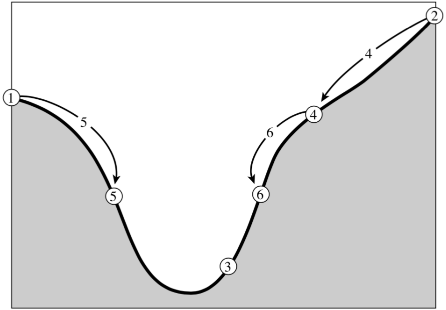

Figure 10.1.1. Successive bracketing of a minimum. The minimum is originally bracketed by points 1,3,2. The function is evaluated at 4, which replaces 2; then at 5, which replaces 1; then at 6, which replaces 4. The rule at each stage is to keep a center point that is lower than the two outside points. After the steps shown, the minimum is bracketed by points 5,3,6.

Golden section - How to pick intermediate point x?

- We can write b as a convex combination of a and b

$$b = (1-\gamma)\,a + \gamma c$$

Note: ${}^{b-a}/_{c-a} = \gamma$ and ${}^{c-b}/_{c-a} = 1-\gamma$

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- Two options for new segment: $[a, x]$ or $[b, c]$
  - First segment will have length $(\gamma+\eta)\,|c-a|$, second $(1-\gamma)\,|c-a|$
  - If we want to min worst case (having too large of an interval)

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- Magic! x is symmetric to b! Note: $|b-a| = |c-x|$
  - For $\eta > 0$ we need to place x in the longest of $[a, b]$ or $[b, c]$

Golden section - How to pick ratio γ?

- If we use the same γ for all iterations this imposes *scale similarity:*

$$\frac{x-b}{c-b} = \frac{b-a}{c-a} \longrightarrow \frac{{}^{x-b}\!/_{c-a}}{{}^{c-b}\!/_{c-a}} = \frac{b-a}{c-a} \longrightarrow \frac{\eta}{1-\gamma} = \gamma$$

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- Solution gives the golden ratio:

$$\gamma = \frac{3-\sqrt{5}}{2} \qquad 1-\gamma = \frac{1+\sqrt{5}}{2} - 1 = \frac{1}{\text{Golden Ratio}} \approx 0.618$$

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- Golden section guarantees that each new function evaluation will bracket min to an interval $\frac{1}{\varphi}$ the size of the preceding interval.

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Question is not why, but when!

- When facing complicated problems is good to start with robust methods
- Start with golden section and then move to more complex method
- Stop early when $|a - c| < tol$ for some "large" tol.

(Inverse) Parabolic interpolation - Brent

- Basic idea is that a function can be locally parabolic (quadratic)
- We can use the formula for the abscissa x that is the minimum of a parabola to guess our new point (if function is actually quadratic we are done in one step!)

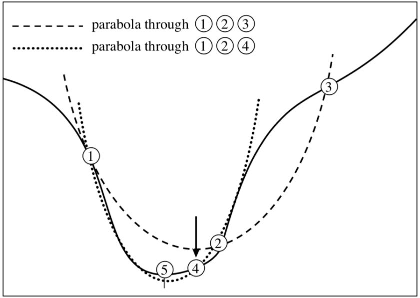

Figure 10.2.1. Convergence to a minimum by inverse parabolic interpolation. A parabola (dashed line) is drawn through the three original points 1,2,3 on the given function (solid line). The function is evaluated at the parabola's minimum, 4, which replaces point 3. A new parabola (dotted line) is drawn through points 1,4,2. The minimum of this parabola is at 5, which is close to the minimum of the function.

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Brent can be improved with derivatives:

- The sign of $f^{'}(b)$ indicates only whether the next test point should be taken in the interval (a, b) or in the interval (b, c).
- Avoids using derivatives in problematic ways

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- Also general problems with derivatives in the computer:

  "too many functions whose computed "derivatives" don't integrate up to the function value and don't accurately point the way to the minimum, usually because of roundoff errors, sometimes because of truncation error in the method of derivative evaluation"

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- Sequential solution
  - You might have to solve an auxiliary problem for each guess

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3. **Derivative-Free Nonlinear-Least-Squares (DFNLS):** Potentially very good
   - I don't know much about these
   - Use BOVYQA algorithm - Powell-like algorithm without derivatives

Global Optimizers

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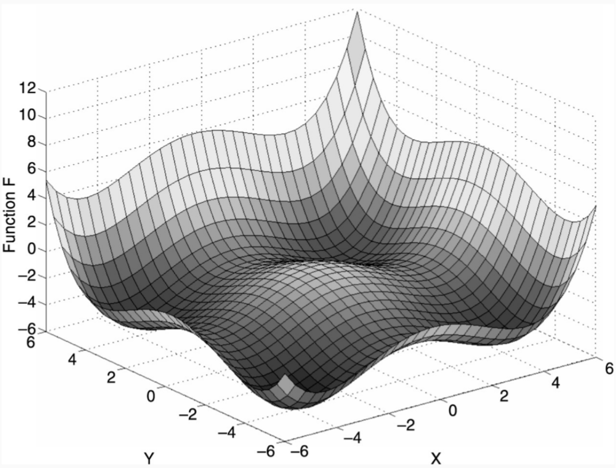
  - Couple with **Simulated Annealing** to search for global solution
- Nice alternative: TikTak algorithm, see Arnaud, Guvenen & Kleineberg (2020)

General problem: Functions are ugly



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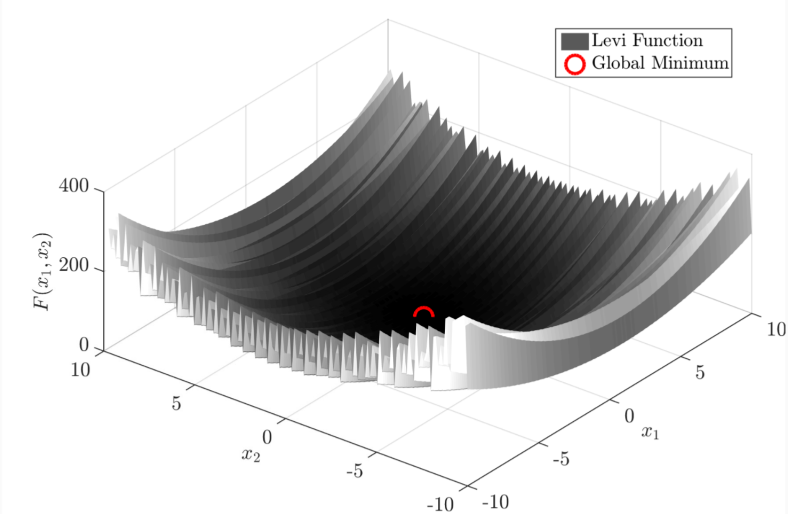

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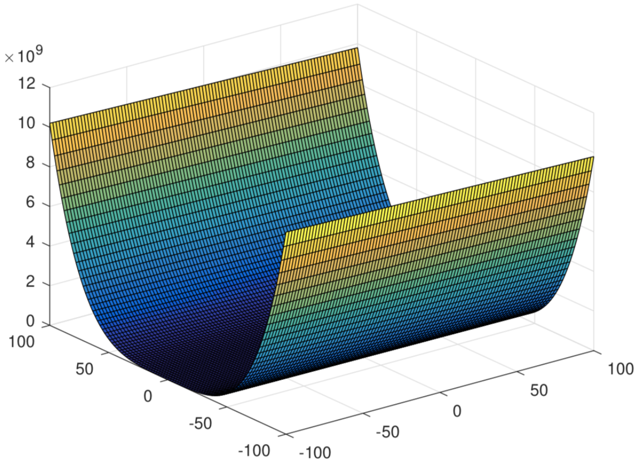

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- Use various starting points for your optimization routine (costly)

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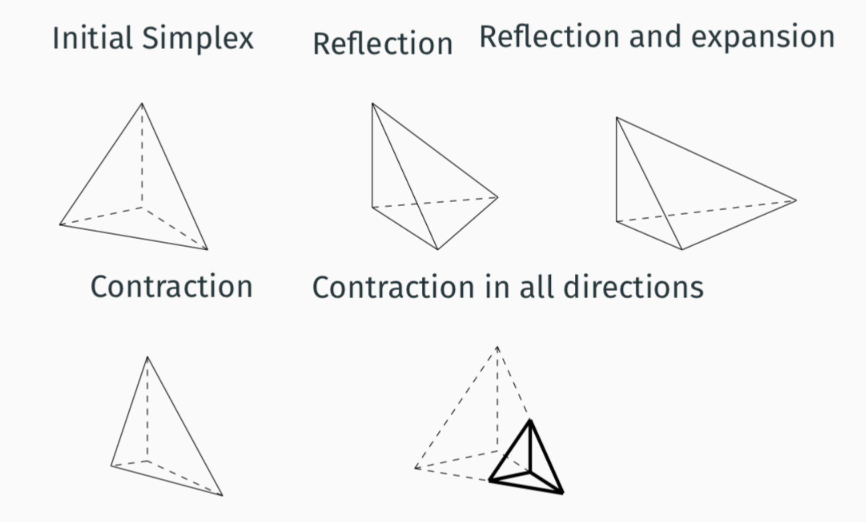

Initial Simplex
Reflection
Reflection and expansion
Contraction
Contraction in all directions

TikTak

Algorithm 1: TikTak Global Optimizer

input : Number of seeds (N_0) and number of candidates ($N^\star$)
output: Global optimum of function F

1. Generate a sequence of N_0 quasi-random **Sobol numbers** ;
2. Evaluate the function in N_0 points. Keep the best $N^\star$

   $x_1, x_2, \ldots x_{N^\star}$ s.t. $F(x_1) \leq F(x_2) \leq \ldots \leq F(x_{N^\star})$;
3. Set $x^\star = x_1$ and $y^\star = F(x_1)$;

for *i=1:*$N^\star$ **do**

- 4.1. Let $\tilde{x}_0 = (1-\theta_i)x_i + \theta_i x^\star$ with $\theta_i \in [0, \bar{\theta}]$ is increasing in i, $\bar{\theta} < 1$;
- 4.2. Get local optimum: $\tilde{x} = \text{Optim}(F, \tilde{x}_0)$;
- 4.3. Update: $y^\star = \max\{y^\star, F(\tilde{x})\}$ and $x^\star$;

5. Return best result $(x^\star, F(x^\star))$

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3. Quasi-random numbers: Deterministic sequences of numbers
   - Designed to spread maximally on a space
   - Build iteratively (next point in sequence fills out a portion of the space with less point density)

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Note: same idea as in Gauss-Kronrod quadrature integration

Other uses: (Quasi-)Monte Carlo integration

- Computing integrals by the Monte-Carlo method
  - Evaluate the function at many points and taking “expectations”

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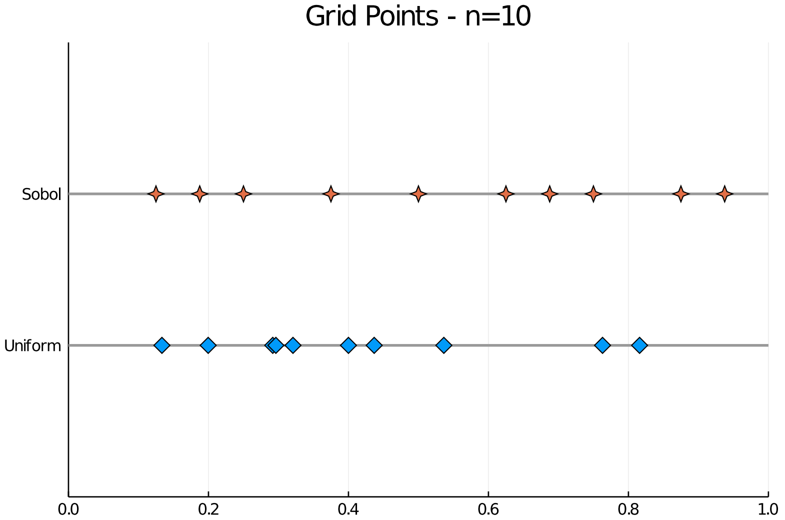
  - Monte Carlo is very robust but very demanding
  - Needs lots of points to get good evaluation
- Quasi-Monte Carlo integration uses quasi-random numbers
  - Increases convergence of integral
- Sobol numbers are particularly good

Sobol vs Uniform (1D)



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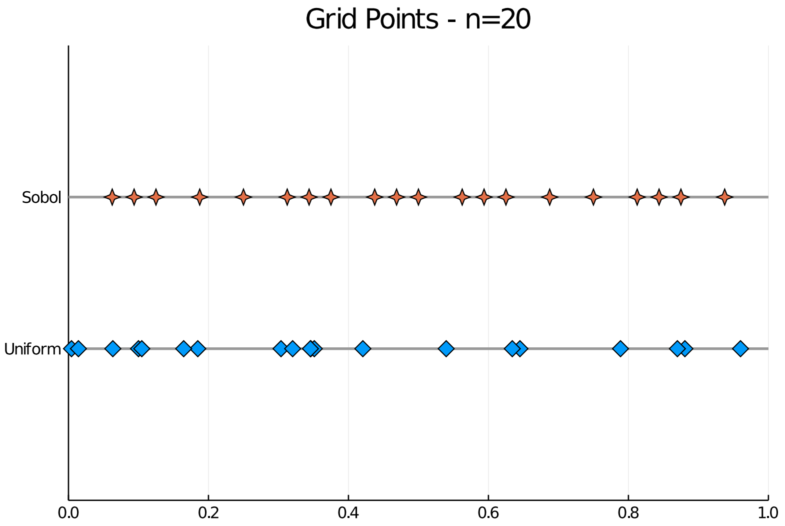

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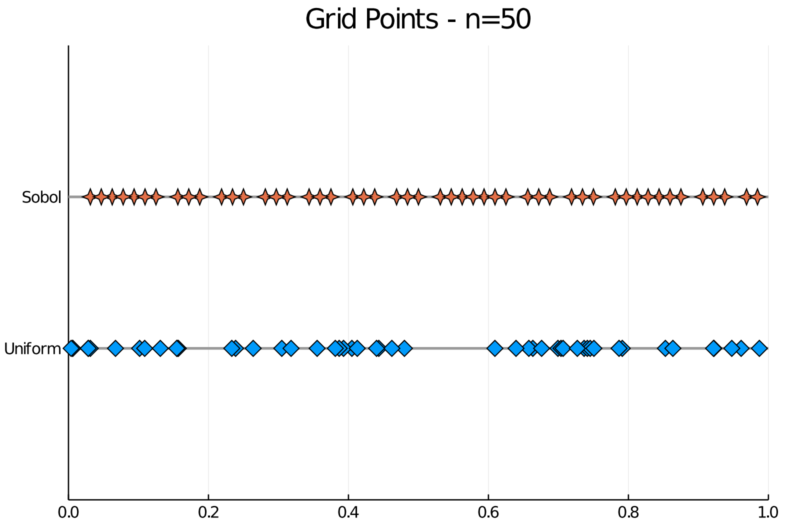
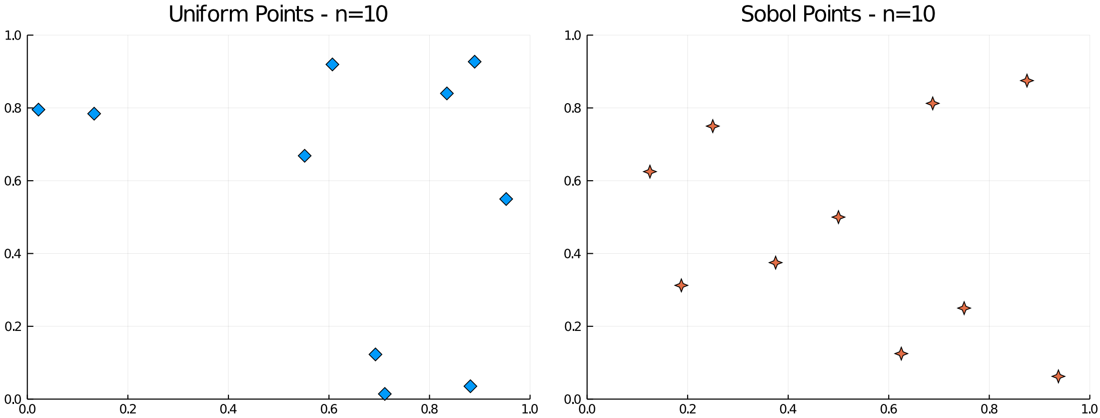


Sobol vs Uniform (2D) - Covering all the space



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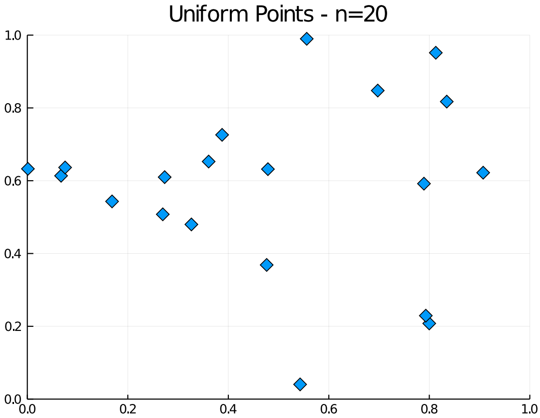

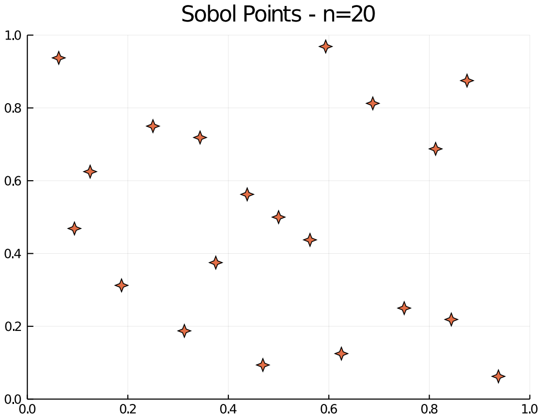

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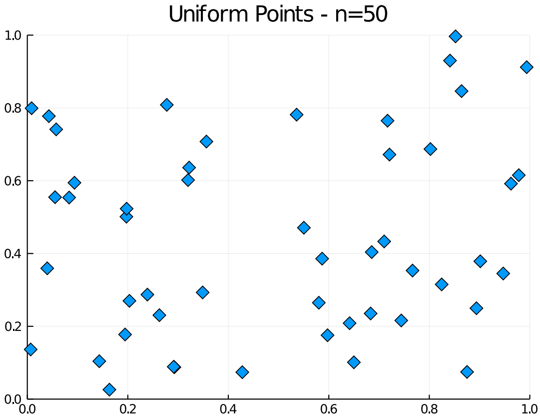

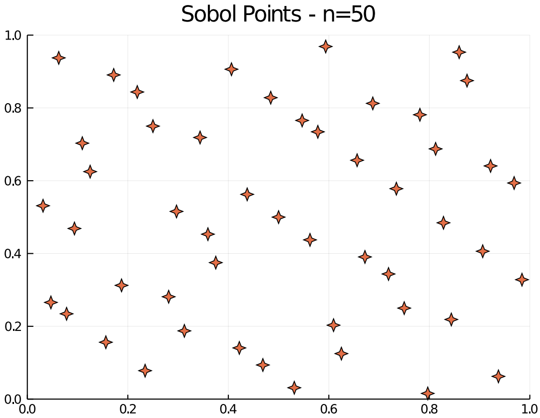

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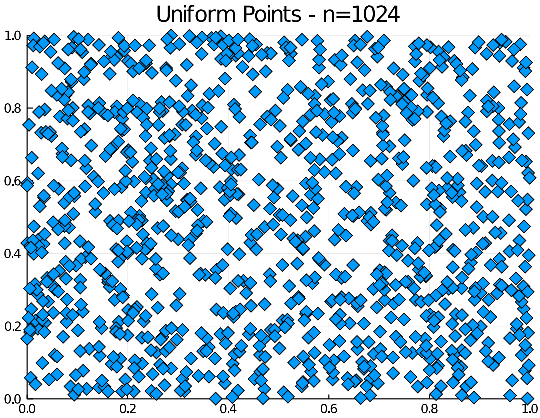
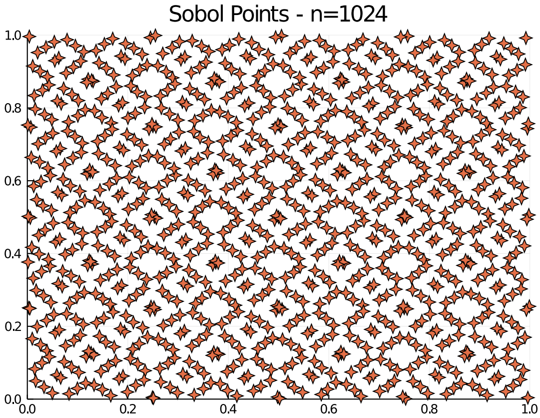




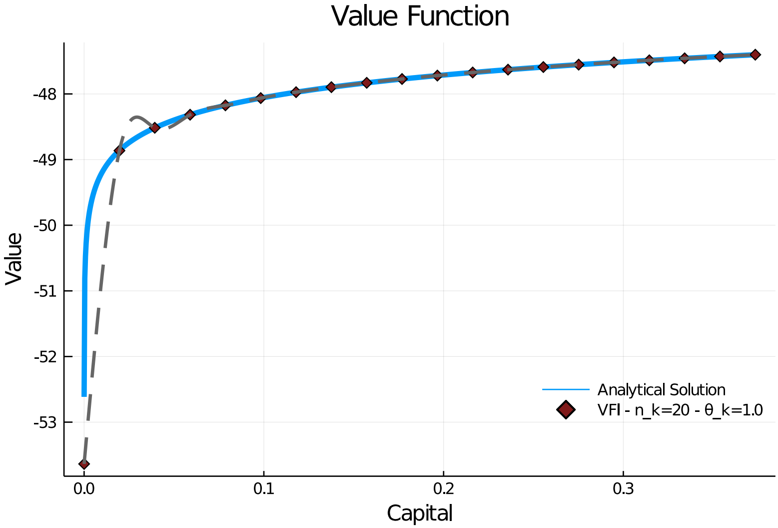
VFI with Continuous Optimization

VFI - Algorithm

Algorithm 2: Bellman Operator: Continuous choice

```
Function T(V_old,k_grid,n_k,parameters):
    for i = 1:n_k do
        # Define objective function
        F(kp) = -U(k_grid[i],kp) - β*Vp(kp)
              Vp(kp) = Interpolation(k_grid,V_old,kp)
        # Find feasible range of kp
        k_min= 0;     k_max = z*k_grid[i]^α - c_min
        #Check for corner solutions with derivative at bounds
        kp = k_min if -dF(k_min)<0;     kp = k_max if -dF(k_max)>0;
        # Solve min (Optional: Further bracket min before minimizing)
        G_kp[i] = Optimize(F,k_min,k_max);    V_new[i] = -F(kp)
    return V_new, G_kp
```

Value functions ($n_k = 20, 50, 100$)



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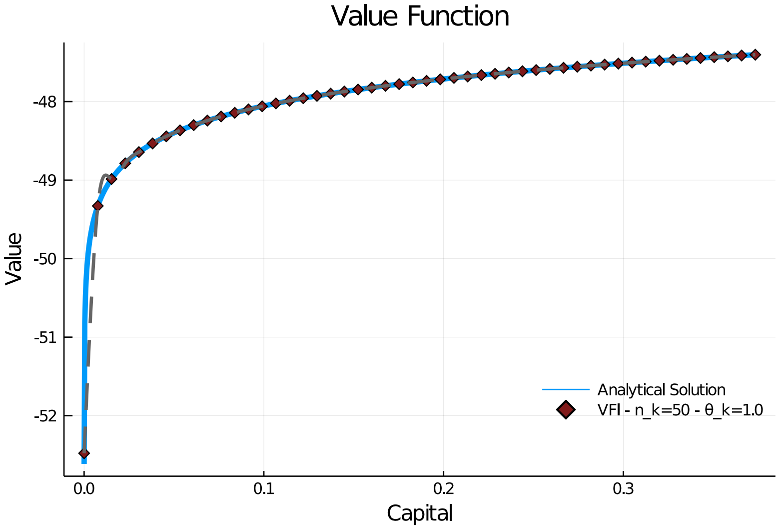

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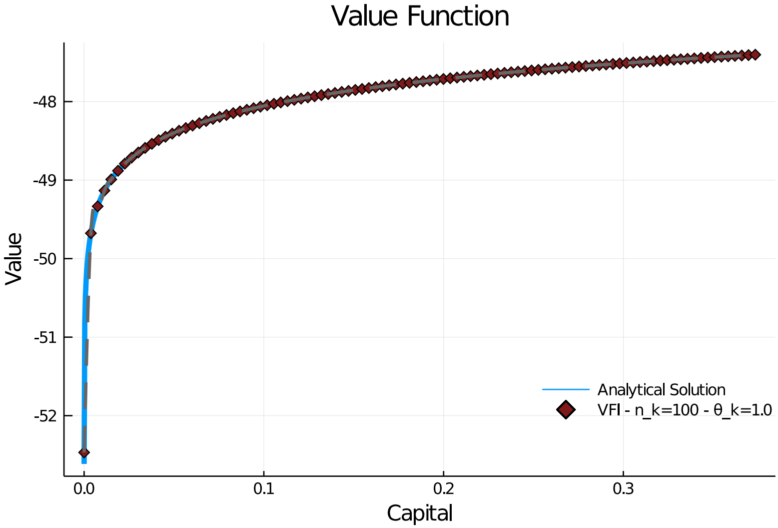

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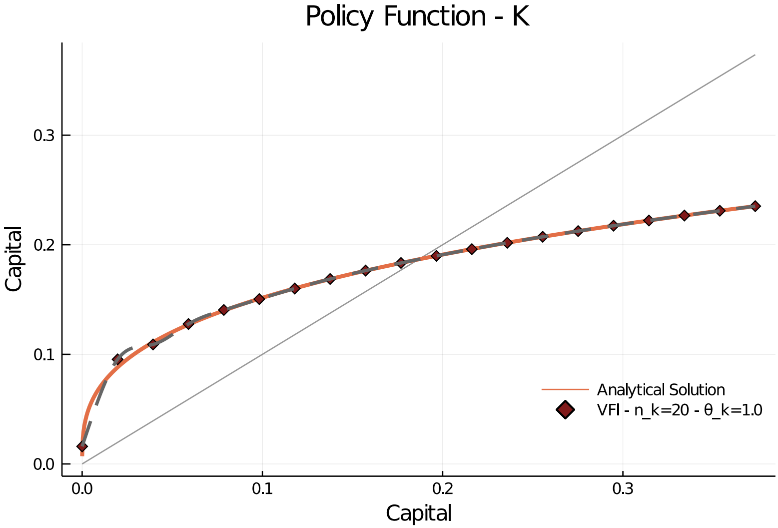

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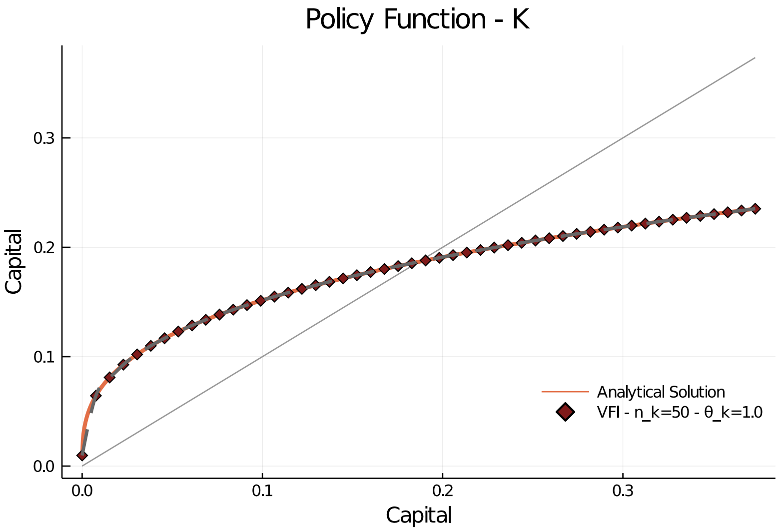

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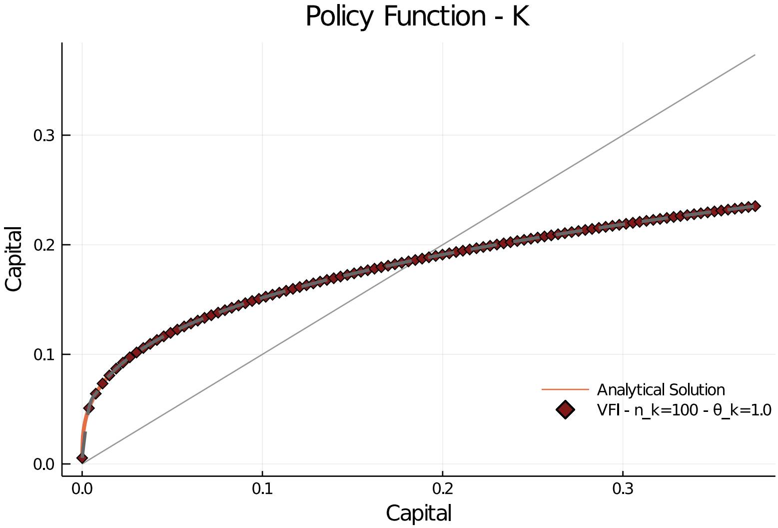
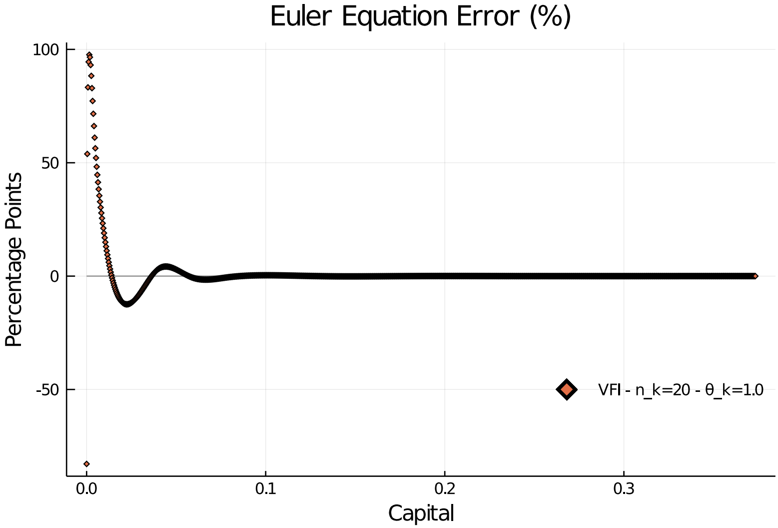


Euler Equation - Problems at the bottom



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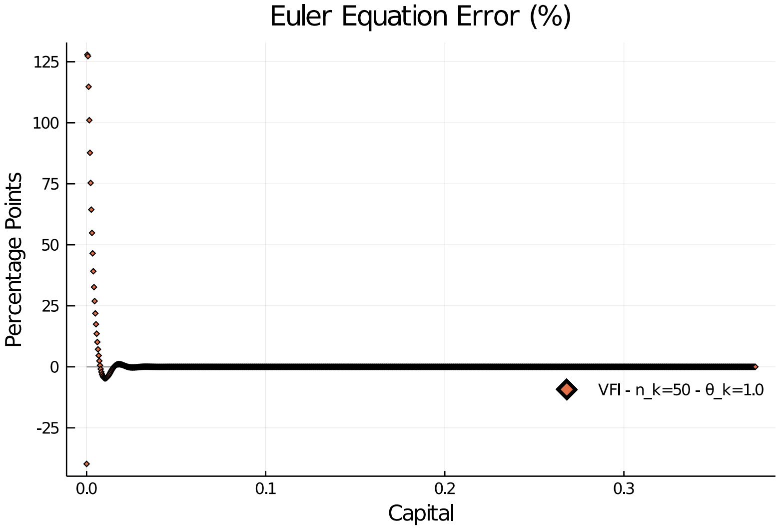

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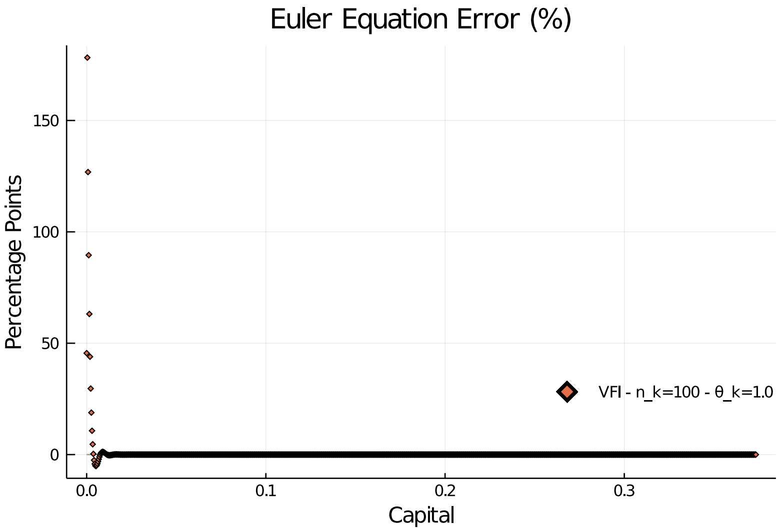

Revisiting Grid Spacing

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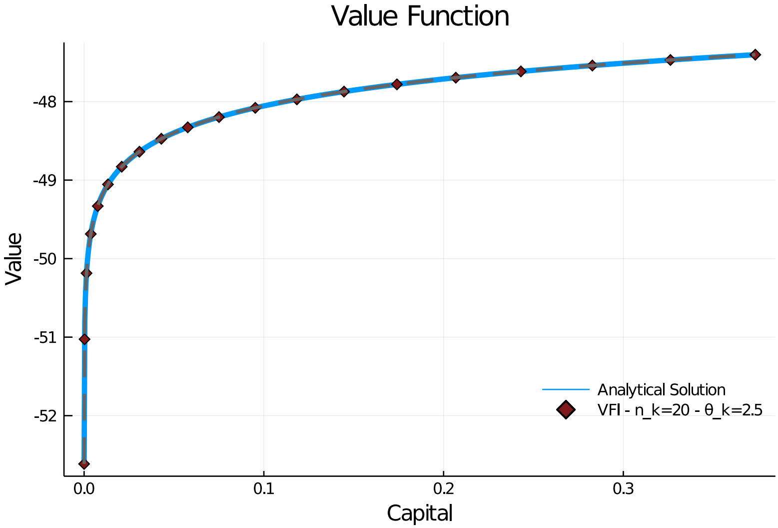

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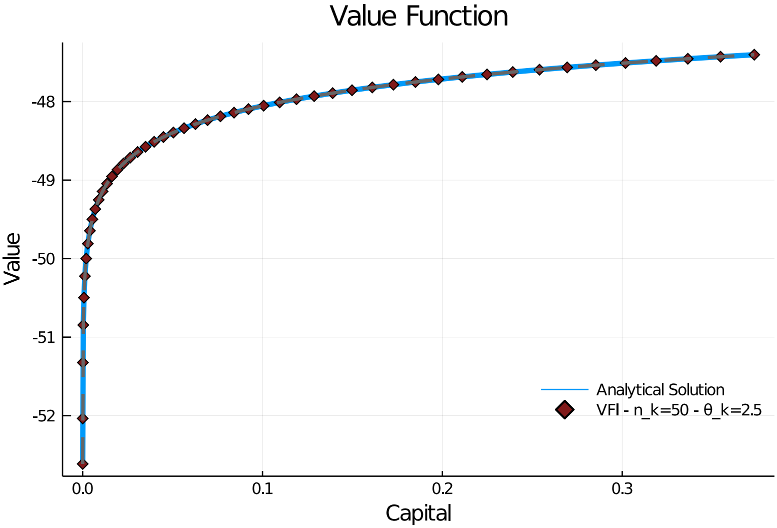

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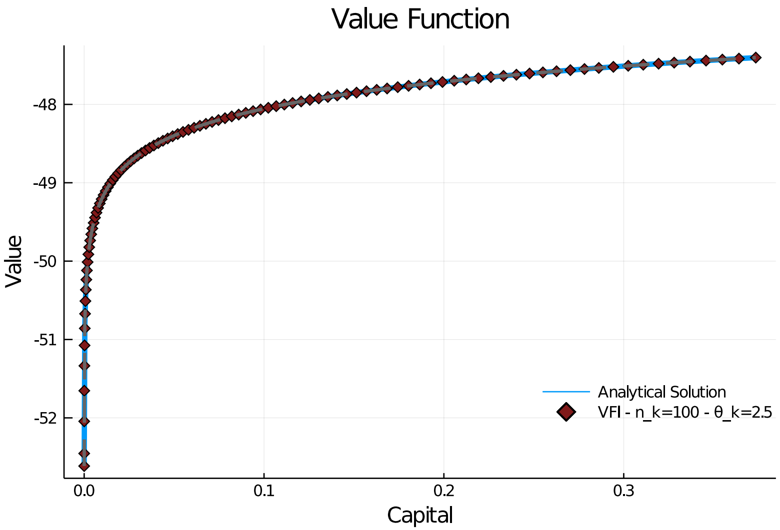

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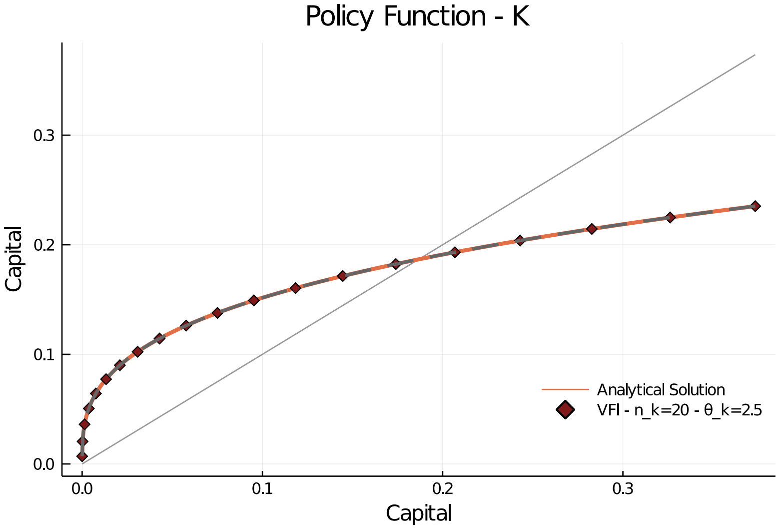

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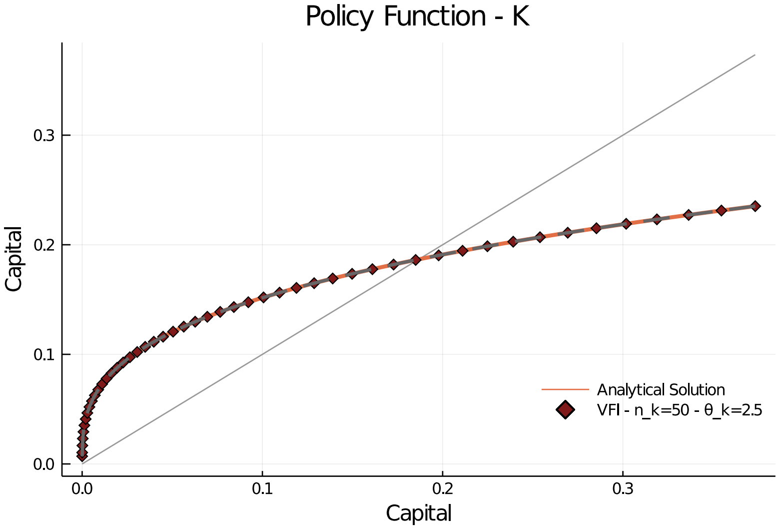

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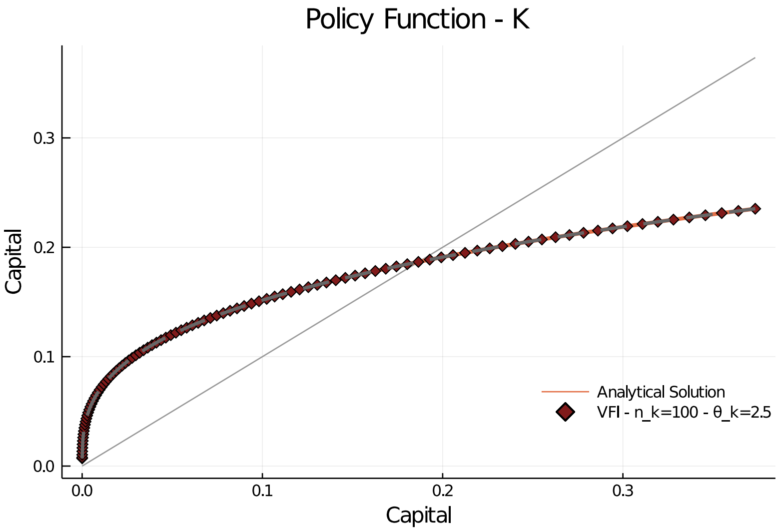
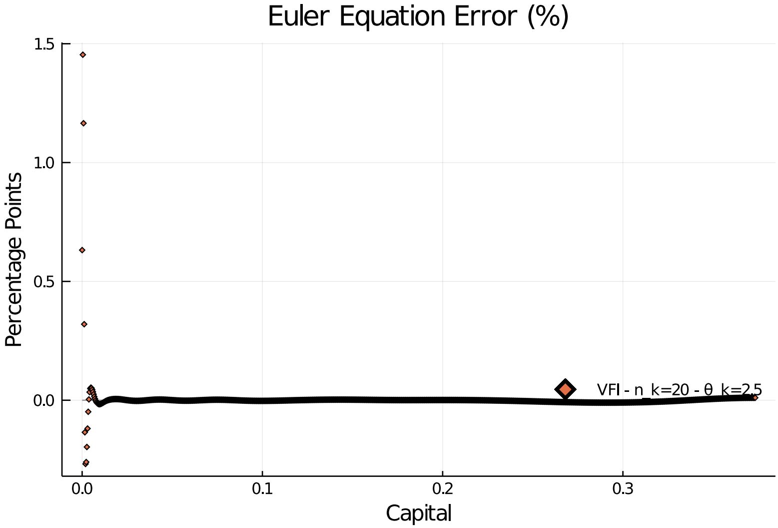


Euler Equation - No issues at the bottom



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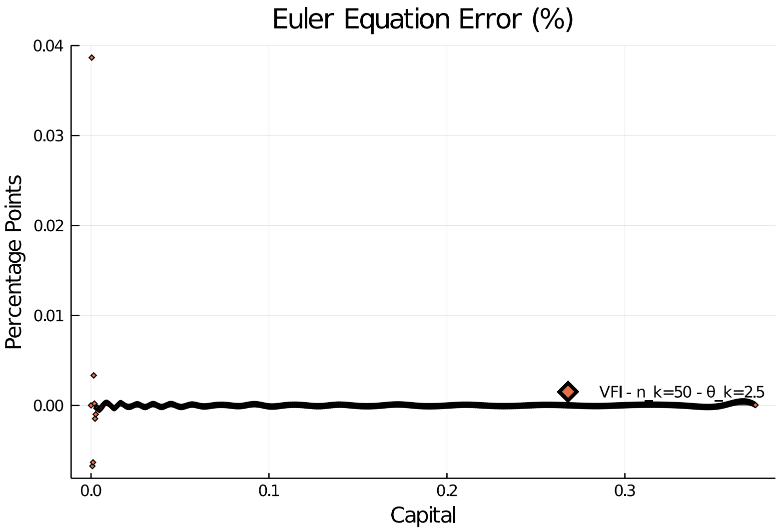

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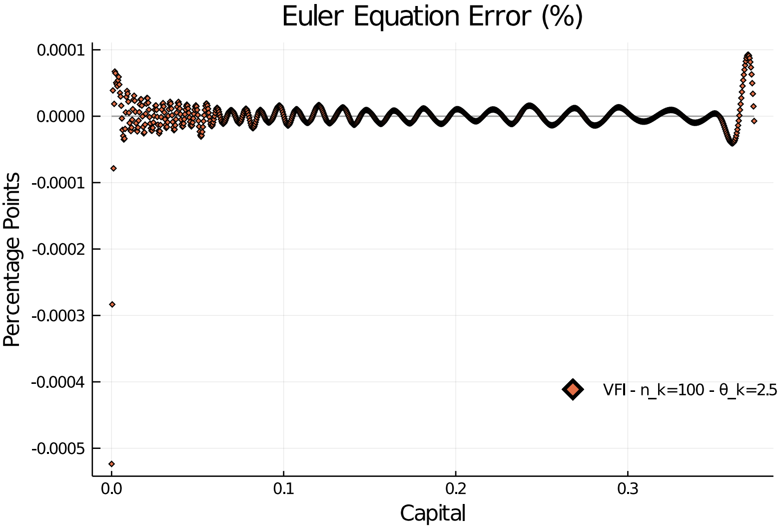

Root Finding

Overview

- Everything is very similar to what we have discussed
- Instead of maximizing the objective function we solve FOC
  - Find root of Euler equation

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- Instead of maximizing the objective function we solve FOC
  - Find root of Euler equation
- In fact is common to use minimization methods for FOC
  - Minimize square of residual
- Root finding is particularly useful to find equilibrium prices
  - Clear markets

Algorithms

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- Bisection (use Golden Section):
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- Brent (variants of the secant method):
  - Brent is the best (unless your problem is easy)
- There are many other methods... depends on what you are doing

VFI - Algorithm

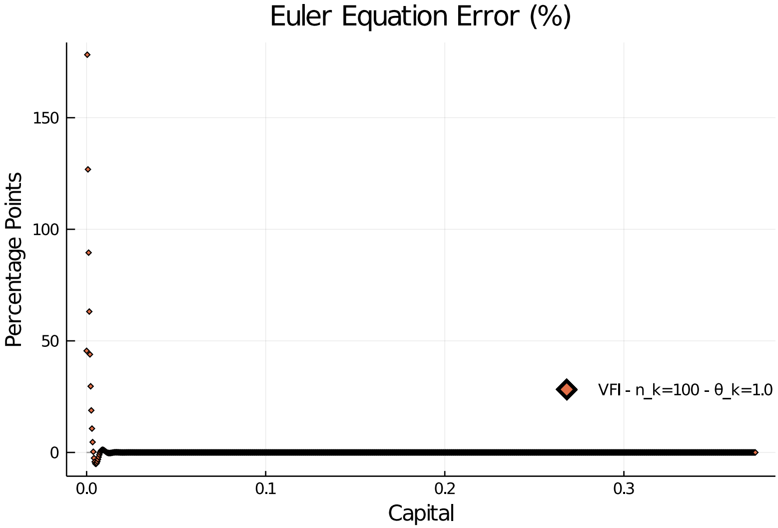
Algorithm 3: Bellman Operator: Continuous choice - Root finding

```
Function T(V_old,k_grid,n_k,parameters):
    for i = 1:n_k do
        # Define objective function
        F(kp) = -U(k_grid[i],kp) - β*Vp(kp)
        dF(kp) = dU(k_grid[i],kp) + β*dVp(kp)
            Vp(kp) = Interpolation(k_grid,V_old,kp)
        # Find feasible range of kp
        k_min= 0;    k_max = z*k_grid[i]^α - c_min
        #Check for corner solutions with derivative at bounds
        kp = k_min if -dF(k_min)<0;    kp = k_max if -dF(k_max)>0;
        # Solve min (Optional: Further bracket zero before minimizing)
        G_kp[i] = Roots(F,k_min,k_max);   V_new[i] = -F(kp)
```

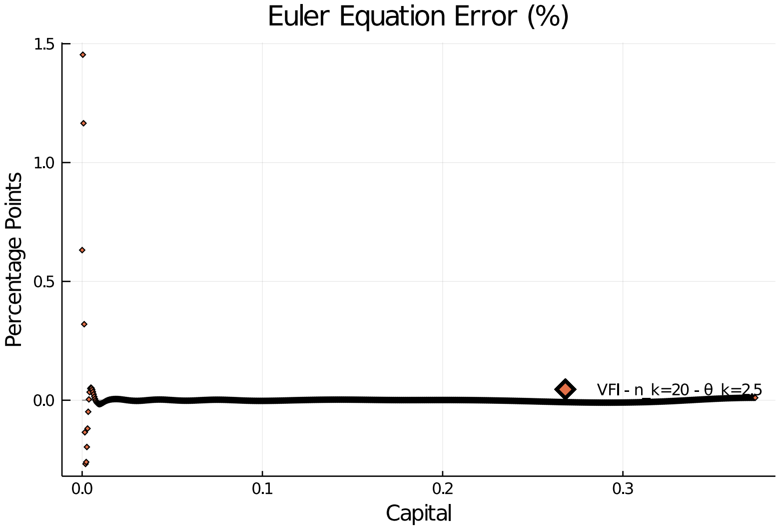
Euler Error ($n_k = 20, 50, 100$)

No Convergence for $n_k = 20, 50$ due to bad approximation to first derivative

Euler Error ($n_k = 20, 50, 100$)



Euler Equation - Curved Grid



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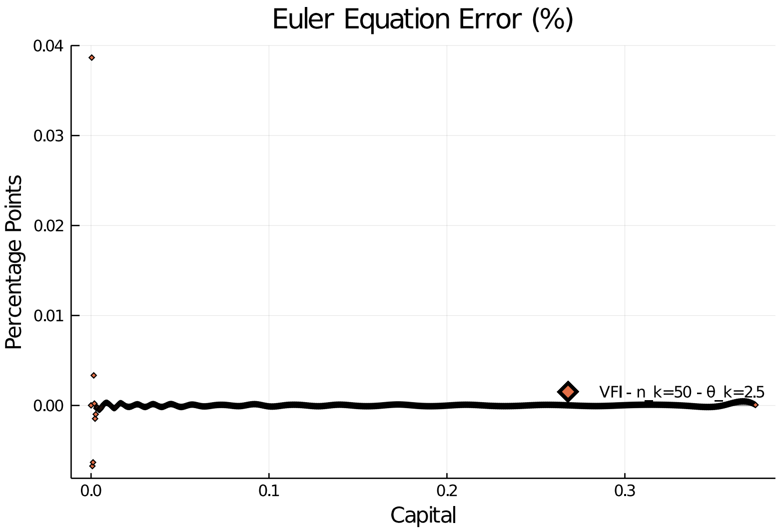

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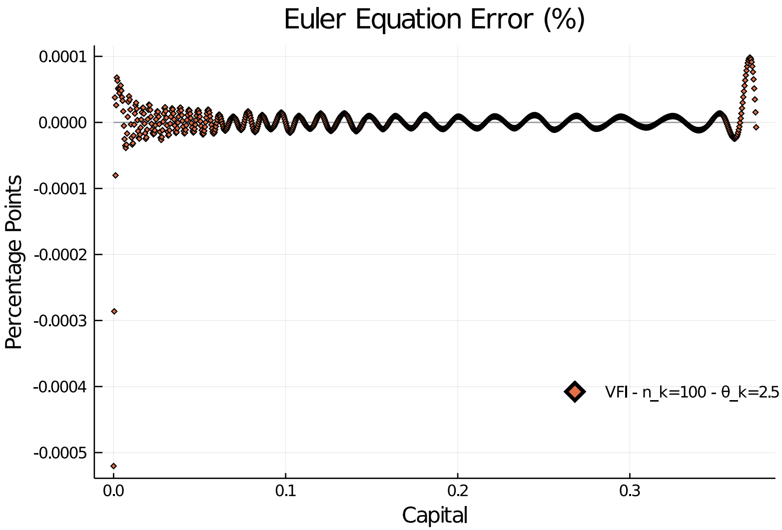

Two Choice Variables: Dealing with Labor Supply

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There are three main options:

1. Multi-dimensional maximization
2. Multi-dimensional root-finding
3. Nested problems

The problem

$$
\begin{aligned}
V(k) = \max_{\{c,k',\ell\}} \; & u(c,\ell) + \beta V\left(k'\right) \\
\text{s.t.} \quad & c + k' = f(k,\ell) \\
& \ell \in [0,1]
\end{aligned}
$$

First order conditions:

$$
\begin{aligned}
u_c(c,\ell) &= \beta V_k\left(k'\right) \\
-u_\ell(c,\ell) &= u_c(c,\ell)\, f_\ell(k,\ell) \\
c + k' &= f(k,\ell)
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c + k' &= f(k,\ell)
\end{aligned}
$$

Obviously eliminate consumption: $c = f(k,\ell) - k'$

Multi-dimensional maximization

Algorithm 4: Objective Function for Multi-Dimensinal Maximization

Function `Bellman_Objective`($k^{'}, \ell; k, V\_old, parameters$):

Compute implied consumption

$c = f(k, \ell) - k^{'}$

Check that consumption is allowable. If c is too low, use penalty

if $c < c\_min$ **then**

return F = penalty(c)

Evaluate Bellman objective

return $F = u(c, \ell) + \beta V\_old(k^{'})$

Note: You can pass V_old as a function (interpolation) rather than as a vector of values

Penalty Functions

- If the penalty is **too big**, it can completely throw off derivative-based methods and will at least confuse other methods.
  - Do not change the value of the objective function by orders of magnitude at the bound
- Always check for corner solutions before passing to a solver
  - Is $c = \underline{c}$ optimal if working at boundary?
- Avoid flat penalties, when function is equal to a "bad" value if violating constraint
  - Gives no information to solver... creates valleys

Penalty Functions

Better option: **Differentiable penalty functions**

1. Imagine the objective function is still defined at boundary, then:

$$P(c) = F(\underline{c}) + a(c - \underline{c})^2$$

   - Quadratic penalties are often enough, you can use something stronger
   - You can also use this if function is undefined at boundary by setting a base value close to the boundary

2. Alternative: Logarithmic barrier functions ()
   - See Boyd and Vandenberghe, 2013, Ch 11, Interior point methods
   - Approximate constrained optimization by adding a penalty
   - Instead of maximizing $F(x)$, maximize:

$$F(x) + \sum_{i=1}^{N} \frac{1}{t} \log(x_i)$$

   - Approximation improves with larger t

Multi-dimensional root-finding

Algorithm 5: Objective Function for Multi-Dimensinal Root-Finding

Function `FOC`($k', \ell; k, V\_old, parameters$)**:**

Compute implied consumption

$c = f(k, \ell) - k'$

Evaluate first order conditions

$F[1] = -u_c(c, \ell) + \beta V\_old_k(k')$

$F[2] = -u_\ell(c, \ell) + u_c(c, \ell) f_\ell(k, \ell)$

Note: You can pass V_old_k as a function

Return objective

return F (or return F.$\hat{2}$ for minimzation)

Penalties in root-finding

- It is harder to determine penalties in this case
- I am not aware of a standard practice
- You can always use the derivative from your differentiable penalty
  - Careful! Objective is differentiable but derivative is no longer continuous
- Best practice is still to check boundaries by hand

Nested problem - Outer function

Algorithm 6: Objective Function for Nested Optimization Problem

Function `Bellman_Objective`($k^{'}$;k,V_old,$parameters$):

  # Check consumption limits and apply penalty

  $c = f(k, 1) - k^{'}$

  if c<c_min **then**

    return F = penalty(c)

  # Solve for labor (inside bounds)

  $\ell^{\star} = Root(FOC_{\ell}; \underline{\ell}, 1, k^{'}, k)$ where $\underline{\ell} = max(\,0\,,\ f^{-1}(c\_min + k^{'}; k)\,)$

  # Compute implied consumption

  $c = f(k, \ell^{\star}) - k^{'}$

  # Evaluate Bellman objective

  return $F = u(c, \ell^{\star})) + \beta V\_old(k^{'})$

Nested problem - Inner function

Algorithm 7: Objective Function for Labor FOC

Function $FOC_\ell(\ell; k', k, V\_old, parameters)$**:**

Compute implied consumption
$c = f(k, \ell) - k'$

Evaluate first order condition
$F = -u_\ell(c, \ell) + u_c(c, \ell) f_\ell(k, \ell)$

Return objective
return F (or return F.2̂ for minimzation)

Why nest labor inside savings choice?

- Nesting makes the problem easier to solve
- But it is expensive because we solve too many inner problems
- Savings problem is harder to solve
  - Requires interpolation and potentially expectations
- Better to have the simpler problem be the inner problem