Advanced Macroeconomics II

Handout 2 - Dynamic Programming, VFI+

Sergio Ocampo

Western University

September 23, 2020

What does a typical problem look like?

1. A dynamic programming problem with:
   - At least two choice variables (c, ℓ)
   - Two to four continuous state variables $(a/k, h, \epsilon, z)$
   - At least two discrete state variables (age, occupation)
   - Non-concavities (fixed costs, adjustment costs, asymmetries)

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2. Part of a general equilibrium environment
    - At least two prices (r, w) solved as function of aggregate state
    - Keep track of distribution of agents
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   - No analytical solution for moments
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Dynamic programming

Prototypical DP problem:

$$V(k,z) = \max_{\{c,k'\}} u(c) + \beta E\left[V\left(k',z'\right)|z\right]$$
$$\text{s.t.} c + k' = f(k,z)$$
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- Useful in representative and heterogeneous agent problems
- What constitutes a solution?
  - Value function ($\mathbf{V}$) and policy functions $(\mathbf{g^c}, \mathbf{g^k})$

Dynamic programming PROBLEMS

1. We are looking for functions $\mathbf{V}$ and $\mathbf{g^c}, \mathbf{g^k}$

$$
\begin{aligned}
\mathbf{V}(k,z) = \max_{\{\mathbf{c},\mathbf{k}'\}} \; & u(c) + \beta E\left[V\left(k',z'\right)|z\right] \\
\text{s.t.}\, c + k' &= f(k,z) \\
z' &= h(z,\eta)\,;\eta \text{ stochastic}
\end{aligned}
$$

- Functions are infinite-dimensional objects... unclear how to find them

Dynamic programming PROBLEMS

2 The problem involves solving a maximization

$$V(k,z) = \max_{\{c,k'\}} u(c) + \beta E\left[\mathbf{V}\left(\mathbf{k}',\mathbf{z}'\right)|z\right]$$

$$\text{s.t.} c + k' = \mathbf{f}(\mathbf{k},\mathbf{z})$$

$$z' = h(z,\eta)\,;\eta \text{ stochastic}$$

- Maximization depends on the solution to the problem!
- Control variables can be continuous (hard... we need derivatives)
- Control variables can be discrete (also hard... no derivatives)
- Choice set can be non-convex

Dynamic programming PROBLEMS

3 The problem involves taking expectations

$$
\begin{aligned}
V(k,z) = \max_{\{c,k'\}} \; & u(c) + \beta \mathbf{E}\left[\mathbf{V}\left(\mathbf{k}',\mathbf{z}'\right)|\mathbf{z}\right] \\
\text{s.t.}\, c + k' &= f(k,z) \\
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\end{aligned}
$$

- Expectation is over the solution of the problem!
- Expectations are hard... they involve integrals... integrals are the worst

Importance of analytical results

- How do you know if there is a (unique) solution to your problem?
- What do you know about how your solution looks like?
  - Monotone? Increasing? Concave? Linear?

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  - Key for stability and speed of numerical methods
- Answers let you contrast numerical solution to predictions
  - How do you know if you found the right answer?

Contraction mappings - Quick review

Contraction Mapping: Let (S, d) be a metric space and $T : S \to S$ be a mapping of S into itself. T is a contraction with modulus β, if for some $\beta \in (0, 1)$ we have:

$$\forall_{v_1, v_2 \in S} \quad d\left(Tv_1, Tv_2\right) \leq \beta d\left(v_1, v_2\right)$$

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- Turns out the DP problem above defines a contraction on the space of functions (verify with Blackwell's sufficient conditions)

$$\begin{aligned} Tv\left(k,z\right) = \max_{\left\{c,k'\right\}} &\; u\left(c\right) + \beta \mathbf{E}\left[\mathbf{v}\left(\mathbf{k}^{'}, \mathbf{z}^{'}\right)|\mathbf{z}\right] \\ \text{s.t.} c + k^{'} &= f\left(k,z\right) \\ z^{'} &= h\left(z,\eta\right); \eta \text{ stochastic} \end{aligned}$$

- Solution to DP problem is a fixed point of the contraction: $V = \mathbf{T}V$

Contraction mapping theorem

Turns out all contractions have a unique fixed point!

Contraction Mapping Theorem: Let (S, d) be a **complete** metric space and $T : S \to S$ a contraction mapping on S. Then, T has a unique fixed point $v^\star \in S$ such that:

$$\forall_{v_0 \in S} \quad v^\star = Tv^\star = \lim_{n \to \infty} T^n v_0$$

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The CMT is the best result you can ever hope for

1. Gives you a solution
2. Gives you a unique solution
3. Gives you an algorithm that converges globally

But it gets better!

Contraction mapping corollary

Corollary - Contraction Mapping Theorem: Let (S, d) be a complete metric space, $T : S \to S$ a contraction mapping on S and $v^\star$ the fixed point of T on S.

- If $\overline{S}$ is a closed subset of S, and $T(\overline{S}) \subset \overline{S}$, then $v^\star \in \overline{S}$.
- If in addition there is a set $\tilde{S}$ such that $T(\overline{S}) \subset \tilde{S} \subset \overline{S}$, then $v^\star \in \tilde{S}$.

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The corollary lets us apply the CMT to non-complete spaces

- S can be the space of continuous, bounded functions
- $\overline{S}$ can add weak concavity
- $\tilde{S}$ can add strict concavity

Analytical solution

Some problems can be solved analytically

1. Guess and verify
2. Manual VFI or backwards induction (finite horizon)
3. Euler equations

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Very limited in practice

- Very few problems can be solved this way
  - Exceptions: Angeletos (2007), Moll (2014), Itskhoki & Moll (2019), Achoud, et al (2020), Benhabib, Bisin (2018), Akira Toda, et al (2019)
- Euler equations still useful - Reduce problem
- Problems provide good initial conditions

Analytical solution: Guess and verify

$$V(k) = \max_{\{c,k'\}} \log(c) + \beta V\left(k^{'}\right) \qquad \text{s.t. } c + k^{'} = zk^{\alpha}$$

Guess and verify (problem set): $V(k) = a_0 + a_1 \log k$

1. Get Euler equation given guess.
2. Solve for policy function given guess.
3. Replace back and solve for coefficients.

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1. Get Euler equation given guess.
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Result:

$$a_1 = \frac{\alpha}{1-\beta\alpha} \qquad k' = g^{k'}(k) = \beta\alpha z k^{\alpha} \qquad c = g^{c}(k) = (1-\beta\alpha) z k^{\alpha}$$

Analytical solution: VFI/Backward induction

$$V^{\mathbf{n+1}}(k) = \max_{\{c,k'\}} \log(c) + \beta V^{\mathbf{n}}\left(k'\right) \qquad \text{s.t. } c + k' = zk^{\alpha}$$

1. Start from initial value, say $V^0(k) = 0$
2. Iterate: $V^1(k) = \max_{k'} \log\left(zk^{\alpha} - k'\right) = \log z + \alpha \log k$
3. Iterate, again: $V^2 = \max_{k'} \log\left(zk^{\alpha} - k'\right) + \beta \log z + \beta\alpha \log k'$
   3.1 Euler: $\frac{1}{zk^{\alpha} - k'} = \frac{\beta\alpha}{k'} \longrightarrow k' = \frac{\beta\alpha}{1+\beta\alpha} zk^{\alpha}$
   3.2 Replace back: $V^2(k) = [\text{Constant}] + (1 + \beta\alpha)\,\alpha \log k^{\alpha}$
4. Keep going... you can see that $1 + \beta\alpha + (\beta\alpha)^2 + \ldots = \frac{1}{1-\beta\alpha}$

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Result:

$$a_1 = \frac{\alpha}{1-\beta\alpha} \qquad k' = g^{k'}(k) = \beta\alpha zk^{\alpha} \qquad c = g^{c}(k) = (1-\beta\alpha)\, zk^{\alpha}$$

Analytical solution: Euler equation

$$V(k) = \max_{\{c,k'\}} \log(c) + \beta V\left(k^{'}\right) \qquad \text{s.t. } c + k^{'} = zk^{\alpha}$$

Euler equation (obtained with envelope theorem):

$$\frac{1}{zk^{\alpha} - g(k)} = \frac{\beta\alpha z \left(g(k)\right)^{\alpha-1}}{z\left(g(k)\right)^{\alpha} - g\left(g(k)\right)}$$

Objective is to find the policy function **g** directly

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Euler equation (obtained with envelope theorem):

$$\frac{1}{zk^{\alpha} - g(k)} = \frac{\beta\alpha z\,(g(k))^{\alpha-1}}{z\,(g(k))^{\alpha} - g(g(k))}$$

Objective is to find the policy function **g** directly

- Guess and verify works here: $g(k) = szk^{\alpha} \longrightarrow s = \beta\alpha$
- More generally we might try to solve this problem numerically
- Fit a parametric function that approximates the solution
- Particularly useful for life cycle models - No need to solve V

Value Function Iteration

Value Function Iteration

Objective is to solve Bellman's equation:

$$
\begin{aligned}
V(k,z) = \max_{\left\{c,k'\right\}} u(c) + \beta E\left[V\left(k', z'\right)|z\right] \\
\text{s.t.} c + k' = f(k,z) \\
z' = h(z,\eta)\,; \eta \text{ stochastic}
\end{aligned}
$$

Value Function Iteration

Solution is fixed point of the mapping T:

$$V(k,z) = \mathbf{TV}(\mathbf{k},\mathbf{z}) = \max_{\{c,k'\}} u(c) + \beta E\left[V\left(k',z'\right)|z\right]$$

$$\text{s.t.}\, c + k' = f(k,z)$$

$$z' = h(z,\eta)\,;\eta \text{ stochastic}$$

Value Function Iteration

CMT gives us a solution by iterating over functions:

$$\mathbf{V^{n+1}}(k,z) = T\mathbf{V^{n}}(k,z) = \max_{\{c,k'\}} u(c) + \beta E\left[\mathbf{V^{n}}\left(k^{'},z^{'}\right)|z\right]$$
$$\text{s.t.} c + k^{'} = f(k,z)$$
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CMT lets us start from an arbitrary function

VFI - Algorithm

Algorithm 1: Value Function Iteration

Result: Fixed Point of Bellman Operator T

$n = 0$; $V^0 \in S$; $dist_V = 1$;
while $n \leq N$ & $dist_V > tol_V$ **do**
  $V^{n+1} = TV^n$;
  $dist_V = d\left(V^{n+1}, V^n\right)$;
end
if $dist_V \leq tol_V$ **then**
  Obtain g from TV^n;
else
  You are in trouble... something went wrong;
end

VFI - Algorithm implementation I

Algorithm 2: VFI: Discrete grid with loops

input : Grid size n_k, model par. z, α, β, code par. max_iter, tol_V
output: Value function V and policy functions G_kp, G_c

```
k_grid = range(1E-5,2*k_ss;length=n_k) ;
V_old = zeros(n_k) ; iter = 0 ; V_dist = 1 ;
while iter<=max_iter && dist_V>tol_V do
    V_new,G_kp,G_c = T(V_old,k_grid,z,α,β);
    dist_V = maximum(abs.(V_new./V_old.-1)) ;
    iter += 1;
if dist_V<=tol_V then
    return V_new,G_kp,G_c;
else
    error("You are in trouble... something went wrong");
```

VFI - Algorithm implementation II

Algorithm 3: VFI: Discrete grid with loops

input : Grid size n_k, model par. z, α, β, code par. max_iter, tol_V
output: Value function V and policy functions G_kp, G_c

```
k_grid = range(1E-5,2*k_ss;length=n_k) ;
V_old = zeros(n_k) ; iter = 0 ; V_dist = 1 ;

for iter = 1:max_iter do
    V_new,G_kp,G_c = T(V_old,k_grid,z,α,β);
    dist_V = maximum(abs.(V_new./V_old.-1)) ;
    if dist_V<=tol_V then
        return V_new,G_kp,G_c;

error("You are in trouble... max_iter reached");
```

VFI - What does it actually mean?

- It means solving a maximization problem many times
- Inside maximization problem you need expectations

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This is hard... and slow... convergence at rate β... but $\beta \approx 1$

- How to speed up?
  1. Speed up solution (EGM)
  2. Skip solution (Howard's PFI)
  3. Speed up update (MPB)

VFI - Grid Search

We will start with the simplest implementation of VFI

- No continuous choice
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- Robust to kinks, asymmetries, etc.
- Easy to implement

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Limitations

- It is an approximation... not very precise
- Low rate of convergence
- Curse of dimensionality - Pay for precision (and even then)

VFI - Grid Search

Original problem:

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$$V\left(k\right)=\max_{\left\{c,k'\right\}}\log\left(c\right)+\beta V\left(k^{'}\right)\qquad \text{s.t. } c+k^{'}=zk^{\alpha}$$

Approximation:

$$V\left(k_i\right)=\max_{k'\in\{k_1,\ldots,k_I\}}\log\left(zk_i^{\alpha}-k^{'}\right)+\beta V\left(k^{'}\right)$$

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Approximation:

$$V(k_i) = \max_{k' \in \{k_1,\ldots,k_I\}} \log\left(zk_i^{\alpha} - k^{'}\right) + \beta V\left(k^{'}\right)$$

Note: Everything is a vector or a matrix now

$$\vec{V} = [V_1, \ldots, V_I]^T \qquad \vec{k} = [k_1, \ldots, k_I]^T \qquad \vec{U} = \left[U_{ij} = u\left(zk_i^{\alpha} - k_j^{'}\right)\right]$$

VFI - Grid Search - Code I

Algorithm 4: Bellman Operator: Discrete grid with loops

```
Function T(V_old, k_grid, z, α, β):
    n_k = length(k_grid)
    V = zeros(n_k); G_kp = fill(0,n_k); G_c = zeros(n_k)
    for i = 1:n_k do
        V_aux = zeros(n_k)
        for j = 1:n_k do
            V_aux[j] = u(k_grid[i],k_grid[j],z, α, β) + β*V_old[j]
        V[i], G_kp[i] = findmax(V_aux)
        G_c[i] = z*k_grid[i]^α - k_grid[G_kp[i]]
    return V, G_kp, G_c
```

VFI - Grid Search - Code II

Algorithm 5: Bellman Operator: Discrete grid with matrices

Function T($V\_old, U\_mat, k\_grid, z, \alpha, \beta$):

```
n_k = length(V_old)
V, G_kp = findmax( U_mat .+ β*repeat(V_old',n_k,1) , dims=2)
G_kp = [G_kp[i][2] for i in 1:n_k]
G_c[i] = z*k_grid[i]^α - k_grid[G_kp[i]]
return V, G_kp, G_c
```

Where:

U_mat = [utility(k_grid[i],k_grid[j],z, α, β) for i in 1:n_k, j in 1:n_k]

How do we judge the solution?

- Plot as much as you can
- Summary statistics can hide large mistakes
- Report what is most relevant for what you are doing

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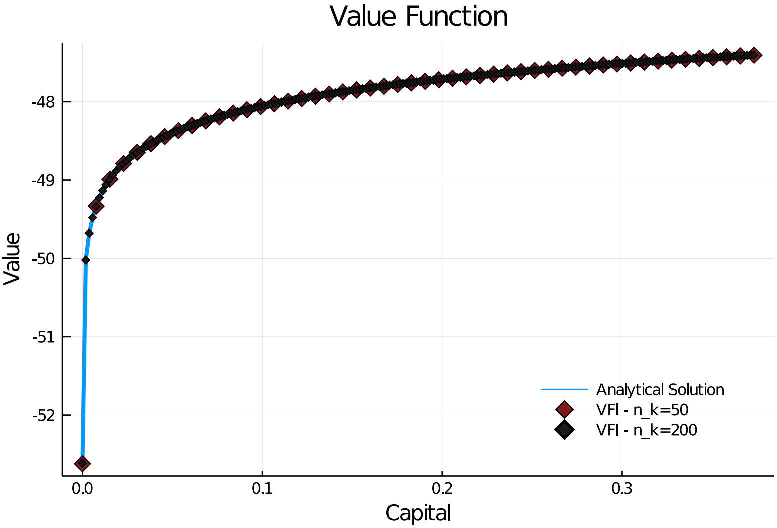
In this case we know the solution

1. Plot value function
2. Plot policy function

Value and policy functions



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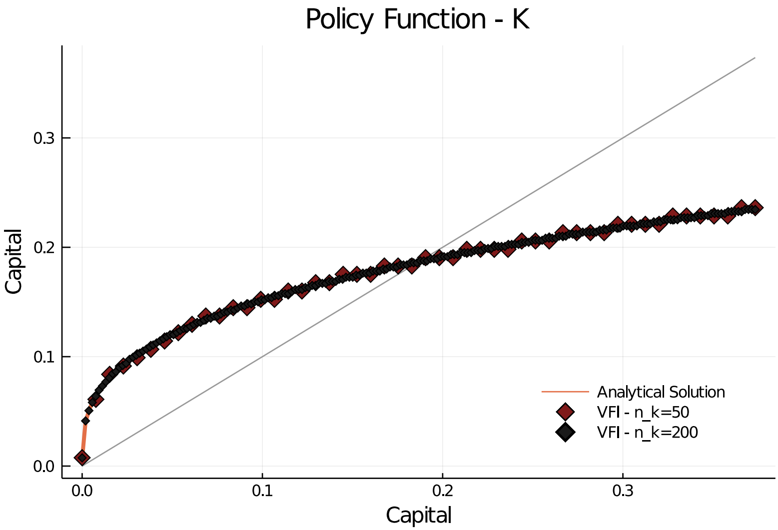

Judging the solution

- Graphs point at a great fit
  - Even with $n_k = 50$ the fit is really good
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- But these graphs can be misleading
  - They are approximations: Discrete problem vs continuous problem

Judging the solution

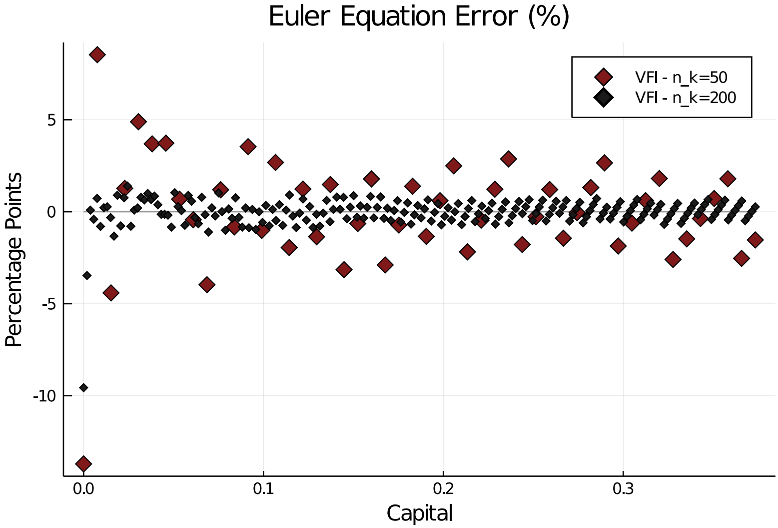
- Graphs point at a great fit
  - Even with $n_k = 50$ the fit is really good
  - $n_k = 200$ seems more than enough
- But these graphs can be misleading
  - They are approximations: Discrete problem vs continuous problem

Judge the solution with the optimization of the agent:

$$\frac{1}{zk^{\alpha} - g(k)} = \frac{\beta\alpha z (g(k))^{\alpha-1}}{z(g(k))^{\alpha} - g(g(k))}$$

$$0 = \underbrace{\frac{\beta\alpha z (g(k))^{\alpha-1} zk^{\alpha} - g(k)}{z(g(k))^{\alpha} - g(g(k))} - 1}_{\text{\% Error in Euler Equation}}$$

Euler Equation - Not a great fit



Howard's Policy Iteration

Howard's policy iteration: The idea

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  - Even for discrete grid

Using the policy function only once is such a waste...

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  Using the policy function only once is such a waste...

- Howard's policy iteration:

  Solve for the policy function once and use it to update many times!

$$V^{n+1}(k) = T^H V^n = u\left(\overline{\mathbf{c}}\left(\mathbf{k}\right)\right) + \beta V^n\left(\overline{\mathbf{k}}'\left(\mathbf{k}\right)\right)$$

where $\overline{c}$ and $\overline{k}'$ are fixed policy functions

Howard's policy iteration: The idea

Why would applying the same policy function many times work?

- Turns out the mapping T^H with given $\overline{c}$ and $\overline{k}'$ is also a contraction.
- So the iteration process will converge to a unique fixed point...
  just not to the solution to our original problem

Howard's policy iteration: The idea

Why would applying the same policy function many times work?

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  just not to the solution to our original problem

So, why do policy iteration?

- Algorithm does not necessarily take us where we want, but it (can) take us close and fast (mostly fast)

Howard's policy iteration

Algorithm 6: VFI with Howard's Policy Iteration

Result: Fixed Point of Bellman Operator T

$n = 0$; $V^0 \in S$; $dist_V = 1$;

while $n \leq N$ & $dist_V > tol_V$ **do**

% Compute current policy function ;

$G^n = \text{argmax}\{TV^n\}$;

% Obtain fixed point under G^n ;

$V^{n+1} = \lim_{m\to\infty} T^m_{G^n} V^n$;

$dist_V = d\left(V^{n+1}, V^n\right)$;

end

Howard's policy iteration: Properties

Results from Puterman & Brumelle (1979)

- Policy iteration is equivalent to the Newton-Kantorovich method in the context of dynamic programming
- HPI behaves like Newton's method:
  1. The method is guaranteed to converge if initial guess is in some neighborhood of the true solution ("Basin of Attraction").
  2. If $V_0 \in$"Basin of Attraction" the method converges at a quadratic rate in the iteration index n.

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  - Each iteration takes a long time because we want the fixed point of T_G

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- So the new algorithm is potentially very fast ...
  But it no longer has the global convergence properties of VFI
- Quadratic convergence is misleading because it operates over n
  - Each iteration takes a long time because we want the fixed point of T_G
- Overall it is not clear that it is faster...
  To make matters worse the "Basin of Attraction" can be small (and is definitely unknown)

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  But it no longer has the global convergence properties of VFI
- Quadratic convergence is misleading because it operates over n
  - Each iteration takes a long time because we want the fixed point of T_G
- Overall it is not clear that it is faster...
  To make matters worse the "Basin of Attraction" can be small (and is definitely unknown)

Solution: Use the policy iteration only for n_H steps

(Modified) Howard's policy iteration

Algorithm 7: VFI with Howard's Policy Iteration

Result: Fixed Point of Bellman Operator T

$n = 0$; $V^0 \in S$; $dist_V = 1$;

while $n \leq N$ & $dist_V > tol_V$ **do**

% Compute current policy function ;

$$G^n = \text{argmax}\{TV^n\} \;;$$

% Iterate n_H times under G^n ;

$$V^{n+1} = T_{G^n}^{n_H} V^n \;;$$

$dist_V = d\left(V^{n+1}, V^n\right)$;

end

HPI: Algorithm Implementation

Algorithm 8: Howard's Policy Iteration

Function T^{HPI}(*V_old,U_mat,k_grid,z*, α, β,***n_H***):

```
n_k = length(V_old)
V, G_kp = findmax( U_mat .+ β*repeat(V_old',n_k,1) , dims=2)
U_vec = U_mat[G_kp]
for i=1:n_H do
    V = U_vec .+ b*repeat(V_old',n_k,1)[G_kp]
    if maximum(abs.(V./V_old.-1))<=tol then
        break
    V_old = V
G_kp = [G_kp[i][2] for i in 1:n_k]
G_c[i] = z*k_grid[i]^α - k_grid[G_kp[i]]
return V, G_kp, G_c
```

MacQueen-Porteus Bounds

Convergence and Stopping Criteria

How do we know when we are close to the solution?

- The CMT gives us an answer for VFI:

$$d\left(V^{\star}, V^{n}\right) \leq \frac{1}{1-\beta} d\left(V^{n}, V^{n-1}\right)$$

- Stop if ϵ away from solution: $d\left(V^{n}, V^{n-1}\right) \leq \epsilon\left(1-\beta\right)$

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This bound on distance is not too informative:

- Bound is a worst case scenario (and covers all the function's domain)

MacQueen-Porteus Bounds

Can we get a better bound for how far we are from the solution?

- The MacQueen-Porteus Bounds (MPB) provide us with better bounds
  - New bounds close faster, they are more informative
  - But for a different specification of the DP problem

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Discrete-State Dynamic Programming:

$$V(x_i) = \max_{y \in \Gamma(x_i)} \left\{ U(x_i, y) + \beta \sum_{j=1}^{N_x} \pi_{ij}(y) V(x_j) \right\}$$

- State x is discrete but control y is continuous
- Transition matrix depends on control: $\Pi(y)$
- Very common in other fields
  - See Bertsekas & Shreve (1996) or Bertsekas & Ozdaglar (2003)

MacQueen-Porteus Bounds

Theorem

Consider the discrete-state dynamic programming problem

$$V^n(x_i) = TV^{n-1}(x_i) = \max_{y \in \Gamma(x_i)} \left\{ U(x_i, y) + \beta \sum_{j=1}^{N_x} \pi_{ij}(y) V^{n-1}(x_j) \right\}$$

Define $\underline{c}_n = \frac{\beta}{1-\beta} \min\{V_n - V_{n-1}\} \quad \wedge \quad \overline{c}^n = \frac{\beta}{1-\beta} \max\{V_n - V_{n-1}\}$

Then, for all $x \in X$ *and* V^0*, it holds that:*

$$T^n V^0(x) + \underline{c}_n \leq V^\star(x) \leq T^n V^0(x) + \overline{c}_n$$

Further, the two bounds approach the solution monotonically as n grows.

MacQueen-Porteus Bounds - Algorithm

Algorithm 9: VFI with MacQueen-Porteus Bounds

Result: Fixed Point of Bellman Operator T

$n = 1$; $V^0 \in S$; $dist_V = 1$;

while $n \leq N$ & $dist_V > tol_V$ **do**

$V^n = TV^n - 1$;

$\underline{c}_n = \frac{\beta}{1-\beta} \min \{V^{n+1} - V^n\}$; $\bar{c}_n = \frac{\beta}{1-\beta} \max \{V^{n+1} - V^n\}$;

$dist_V = \bar{c}_n - \underline{c}_n$;

end

$V = V^n + \frac{\bar{c}_n + \underline{c}_n}{2}$;

$G = \operatorname{argmax} TV$;

MacQueen-Porteus Bounds - Properties

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  - For an AR(1) process the subdominant eigenvalue is ρ (persistence)
  - If persistence is low convergence is very fast

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Results from Bertsekas (1987)

- The MPB converge monotonically to the true solution
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  - For an AR(1) process the subdominant eigenvalue is ρ (persistence)
  - If persistence is low convergence is very fast
- Compare with VFI:
  - Convergence proportional to dominant eigenvalue
  - Always 1 because Π is a stochastic matrix
  - Multiplied by β gives convergence rate... but we often have $\beta \approx 1$

Coda: Convergence in policy functions

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- Value functions critical for welfare comparisons