Sigma-Point Kalman Filtering for tightly-coupled GPS/INS

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***Abstract*—This paper proposes the fusion of GPS measurements and inertial sensor data from gyroscopes and accelerometers in tightly-coupled GPS/INS navigation systems. Usually, an extended Kalman fiter (EKF) is applied for this task. However, as system dynamic model as well as the pseudorange and pseudorange rate measurement models are nonlinear, the EKF is sub-optimal choice from theoretical point of view, as it approximates the propagation of mean an covariance of Gaussian random vectors through these nonlinear models by a linear transformation, which is accurate to first-order only. The sigma-point Kalman filter (SPKF) family of algorithms use a carefully selected set of sample points to more accurately map the probability distribution than linearization of the standard EKF, leading to faster convergence from inaccurate initial conditions in position and attitude estimation problems, which achieves an accurate approximation to at least second-order. Therefore, the performance of EKF and SPKF applied to tightly-coupled GPS/INS integration is compared in numerical simulations. It is found that the SPKF approach offers better performances over standard EKF**

Keywords-GPS/INS SPKF EKF tightly-coupled pseudorange pseudorange rate

I. INTRODUCTION

The integration of Global Positioning System (GPS) signals with Inertial Navigation System (INS) has become a standard approach for position and attitude determination of a moving vehicle. There are many approaches to mechanize an integrated GPS/INS. One aspect involves how GPS observations are used in the filter design. The term "tightly-coupled" configuration utilizes the GPS pseudo ranges directly. The main advantage of a tightly-coupled system is that state quantity estimates can still be provided even when the minimum number of four GPS satellites is not available, where the GPS receiver cannot provide position and velocity information anymore. The widely used EKF [1] has one well-known drawback. If the errors are not within the "linear region", the filter divergence may occur. Sigma-point filters essentially provide derivative-free higher-order approximations by approximating a Gaussian distribution rather than approximating an arbitrary nonlinear function as the EKF does. They can provide more accurate results than an EKF, especially when accurate initial condition states are not known.

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The organization of this paper proceeds as follows. Firstly, the concept of tightly-coupled GPS/INS integration and the nonlinear system dynamic and measurement models are introduced. Then, an SPKF-based tightly-coupled GPS/INS integration is proposed. After that, EKF and SPKF are compared by means of numerical simulations for different inertial sensors. Finally, conclusions are drawn.

II. TIGHTLY-COUPLED GPS/INS

In a tightly-coupled GPS/INS system, GPS pseudo range and pseudo range rate measurements are processed in the data fusion algorithm, with the aim to calibrate the INS and to prevent the growth of navigation error with time exhibited by an unaided INS. The processing of the raw GPS measurements automatically takes into account the GPS constellation. In most cases, the errors of pseudo range and pseudo range rate measurements to different satellites can be treated as uncorrelated with each other. Despite of that, the errors of the position and velocity measurements constructed from this raw data show cross-correlations depending on the satellite constellation. [2]

The main drawback of a tightly-coupled system is the increased complexity that is connected with the need to handle raw GPS data. Next, the system dynamic model and the pseudo range and pseudo range rate measurement models are discussed, which are the key to the development of GPS/INS data fusion algorithms.

A. System Dynamic model

The core of the system dynamic model is given by the differential equation (1)-(2) of the INS mechanization. In this paper, navigation frame mechanization was chosen.

$$
\begin{aligned}
&\dot{\varphi} = \frac{V^n_{en,n}}{R_m - h} \\
&\dot{\lambda} = \frac{V^n_{en,e}}{(R_n - h)\cos\varphi} \\
&\dot{h} = V^n_{en,d} \\
&\dot{\bar{V}}^n_{en} = C^b_b \bar{f}^b_{ib} - (2\bar{\omega}^n_{ie} + \bar{\omega}^n_{en}) \times \bar{V}^n_{en} + \bar{g}^n
\end{aligned}
\tag{1}
$$

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_{nbx}^b & -\omega_{nby}^b & -\omega_{nbz}^b \\ \omega_{nbx}^b & 0 & \omega_{nbz}^b & -\omega_{nby}^b \\ \omega_{nby}^b & -\omega_{nbz}^b & 0 & \omega_{nbx}^b \\ \omega_{nbz}^b & \omega_{nby}^b & -\omega_{nbx}^b & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{2}$$

The bias estimates provided by filters based on this system dynamic model correspond to the net effect of bias, bias drift, scale factor errors and sensor misalignment. The choice of an appropriate model for the inertial sensor biases is dependent on the INS that is used, for a higher-grade FOG-INS these bias states could be modeled as constant, whereas for MEMS-INS a random walk process seems to be the more approximate choice.

B. GPS Measurement model

Here we concentrate on the mathematical models of the pseudo range and delta range measurement. Pseudo range measurement is obtained by measuring the time-of-flight of the satellite signal. The time-of-flight is converted into a distance by multiplication with the satellite signal group velocity. [3]

The pseudo range measurement to ith satellite can be modeled as follows [4]:

$$\rho_i = |r_i - r_u| + cb_u + \xi_{\rho i} \tag{3}$$

where r_i is satellite position at transmit time; r_u is the receiver position at receive time; b_u is the bias in receiver clock; and $\xi_{\rho i}$ is the composite of errors produced by atmospheric delay, satellite ephemeris mismodeling, receiver noise, etc. Of course, the pseudo range measurement is related to vehicle position and attitude in a nonlinear way, but this nonlinearity is very weak.

We can predict that the pseudo range measurement should be as follow:

$$\hat{\rho}_i = |r_i - \hat{r}_u| + c\hat{b}_u + \hat{\xi}_{\rho i} \tag{4}$$

The measurement residual $\Delta\rho$, which is the difference between the predicted and actual measurement, can be modeled as linearly related to the error, by performing a Taylor expansion about the current state estimate. The linearized result is given by the following:

$$\Delta\rho_i = \hat{\rho}_i - \rho_i = \begin{bmatrix} -\hat{1}_i^T & 1 \end{bmatrix} \begin{bmatrix} \Delta r \\ c\Delta b \end{bmatrix} + \Delta\xi_{\rho i} \tag{5}$$

Where, $\hat{1}_i^T \equiv \dfrac{r_i - \hat{r}_u}{|r_i - \hat{r}_u|}$, $\Delta b = \hat{b}_u - b_u$, $\Delta\xi_{\rho i} = \hat{\xi}_{\rho i} - \xi_{\rho i}$

$\hat{1}_i^T$ is the estimated line of sight unit vector from the user to the satellite; and $\Delta\xi_i$ is the residual error after the known biases have been removed. This linearized model is the fundamental GPS pseudo range measurement equation.

The pseudo range rate measurement $\dot{\rho}_i$ is obtained by measuring the Doppler shift of the satellite carrier signal. A mathematical formulation of the measurement model is given by

$$\dot{\rho}_i = \frac{r_i - r_u}{|r_i - r_u|}(v_i - v_u) + f + \xi_{\dot{\rho} i} \tag{6}$$

where f is the receiver clock drift in m/s; and $\xi_{\dot{\rho} i}$ is the error in the observation in m/s.

We can predict that the pseudo range rate measurement should be as follow:

$$\dot{\rho}_i = (v_i - v_u)\hat{1}_i + \hat{f} + \hat{\xi}_{\dot{\rho}} \tag{7}$$

The linearized pseudo range rate measurement equation is then as follow

$$\Delta\dot{\rho}_i = \hat{\dot{\rho}}_i - \dot{\rho}_i = \begin{bmatrix} -\hat{1}_i^T & 1 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta f \end{bmatrix} + \Delta\xi_{\dot{\rho} i} \tag{8}$$

C. Sigma-point Kalman filter

Instead of linearization required by EKF, the SPKF does not require to approximate nonlinear system dynamic and measurement models using the Jacobian in order to calculate the covariance of random vector propagated through the nonlinear models. Instead, a set of deterministically selected sigma-points is chosen, which have the same mean and covariance as the original random vector. Then, these sigma-points are propagated through the nonlinear models, and the mean and the covariance accurately to the second order for arbitrary nonlinear functions, while the EKF achieves first order accuracy only. The different types of sigma-point filters, such unscented Kalman filter or central difference Kalman filter, are distinguished by the weights and the scaling parameter associated with the sigma-points. Throughout this paper, the unscented Kalman filer is used.

The sigma-point Kalman filter is implemented as follow:

First, an augmented state vector is constructed, which includes the process noise and measurement noise.

$$\hat{x}^\alpha = E[x^\alpha] = [x^T w^T V^T] \tag{9}$$

The covariance of the augmented state vector estimation error is given by

$$P^\alpha = E[(\hat{x}^\alpha - x^\alpha)(\hat{x}^\alpha - x^\alpha)^T] = \begin{pmatrix} P_x & 0 & 0 \\ 0 & Q_w & 0 \\ 0 & 0 & R_v \end{pmatrix} \tag{10}$$

Next, a set of 2L+1 sigma-points is chosen, where L is the dimension of the augmented state state vector:

$$\begin{aligned} &x_0 = \bar{x} \\ &x_i = \bar{x} + (\xi\sqrt{P_x})_i, i = 1,\ldots,L \\ &x_i = \bar{x} - (\xi\sqrt{P_x})_i, i = L+1,\ldots,2L \end{aligned} \tag{11}$$

Each sigma-points is a $L\times 1$ vector, ξ is a scaling parameter denoting how far the sigma-points spread from the mean. ξ is often set to a small positive value.

In the filter prediction step, the system dynamics model is reformulated as a nonlinear function of the augmented state vector. Then, the sigma-points are propagates through this function:

$$x_{k/k-1} = f(x_{k-1}, u_{k-1}) \tag{12}$$

From the transformed set of sigma-points, the propagated mean and covariance of the augmented state vector is calculated:

$$\begin{aligned} &\bar{x}_k^- = \sum_{i=0}^{2L} \omega_i^m x_{i,k/k-1}^i \\ &P_k^- = \sum_{i=0}^{2L} \omega_i^c [x_{i,k/k-1}^i - \bar{x}_k^-][x_{i,k/k-1}^i - \bar{x}_k^-]^T + Q_k \end{aligned} \tag{13}$$

Again, ω_i^m and ω_i^c are weights specific to the type of SPKF. Of course, these weights and the scaling parameter ξ have to satisfy certain conditions in order to obtain an unbiased estimator.

In the filter measurement step, a set of predicted measurements is calculated from the sigma-points by propagating them through the nonlinear measurement model:

$$y_i = g(x_i), i = 0,\ldots,2L \tag{14}$$

With this set of predicted measurements, mean, covariance and correlation can be calculated using:

$$\begin{aligned} &\bar{y} \approx \sum_{i=0}^{2L} \omega_i^m y_i, P_y \approx \sum_{i=0}^{2L} \omega_i^c y_i y_i^T, \bar{y}_{k/k-1} = h(\chi_{k/k-1}), \bar{y}_k^- = \sum_{i=0}^{2L} \omega_i^m y_{i,k/k-1} \\ &P_{y_k} = \sum_{i=0}^{2L} \omega_i^c [y_{i,k/k-1} - \bar{y}_k^-][y_{i,k/k-1} - \bar{y}_k^-]^T + R_k \\ &P_{x_k y_k} = \sum_{i=0}^{2L} \omega_i^c [x_{i,k/k-1} - \bar{x}_k^-][y_{i,k/k-1} - \bar{y}_k^-]^T \end{aligned} \tag{15}$$

The knowledge of these quantities allows calculating the posteriori mean and covariance of the augmented state vector. The following completes the measurement step.

$$k = P_{x_k y_k} P_{y_k}^{-1}, \bar{x}_k = \bar{x}_k^- + k(y_k - \bar{y}_k^{-1}), P_k = P_k^- - kP_{y_k}k^T \tag{16}$$

III. COMPUTER SIMULATION

We now describe the application of the SPKF to the problem of tightly-coupled GPS/INS integration for navigation of Autonomous Underwater vehicles (AUV). In order to investigate the performance of the SPKF in the area of tightly-coupled GPS/INS integration, an 18-state total space SPKF was implemented. The performance of EKF-based and SPKF-based tightly coupled GPS/INS navigation system was compared in numerical simulations.

The generation of the raw GPS data, such as the error characteristics could be referred to [5].

Firstly, four different types of IMUs were investigated. A low-cost MEMS IMU ($10°/h$), a medium grade MEMS IMU ($1°/h$), a tactical grade FOG IMU ($0.1°/h$) and a high grade RLG IMU ($0.01°/h$). With these IMUs, a wide range of applications is addressed, from low-cost MEMS IMU navigation system to mission application. All measurements are assumed to be sampled every 1 second. The accelerometer parameters are given by 10-4g. The vehicle motion is comprised of several segments: level acceleration; pitch up transition; velocity; roll back to straight; level off. The initial quaternion is given so that the vehicle body frame is aligned with the local geographical frame. The tightly-coupled GPS/INS integration performances are shown in Fig.1 and Fig.2. From the plots, IMU RMS errors decrease different extents under the aid of GPS respectively. Sensor bias, scale factor and misalignment will be included in the models. IMU sensor calibration with GPS data is performed during the whole navigation process. SPKF application in tightly-coupled GPS/INS integration produces relatively good results.

For the further investigation of SPKF algorithm, we study the attitude error of INS in order to assess the performance of EKF and SPKF in situations with tightly-coupled GPS/INS. Numerical simulations were performed as follows.

The sway motion model is represented as below

$$\left.\begin{aligned} &\theta = \theta_m \sin(\omega_\theta t + \theta_0) \\ &r = r_m \sin(\omega_r t + r_0) \\ &\varphi = \varphi_m \sin(\omega_\varphi t + \varphi_0) \\ &V_x = V_{x0}, \quad V_y = V_{y0} \end{aligned}\right\} \tag{17}$$

where θ, r, φ are pitch, roll and yaw respectively; the sway magnitudes θ_m, r_m, φ_m are 1°, 1°, and 10° respectively. The sway frequents ω_θ, ω_r, ω_φ are 0.1 Hz, 0.1Hz, and 0.05 Hz respectively. The initial phase angles θ_0, r_0 and φ_0 are 0°, 0°, and 45° respectively. This error is not unrealistic for an actual application. $V_x = V_y = 5m/s$. All gyro constant drift are 0.01°/h and all the accelerometer bias are 10^{-4}g. The initial quaternion equals [1, 0, 0, 0]. The initial gyros and accelerometer biases and scale factors are set to zero. The random noise of initial covariance for the gyros and accelerometers are supposed as Gaussian white noise N (0, $(0.001\text{deg/h})^2$) and N (0,$(10^{-6}\text{g})^2$) respectively. The parameters used in SPKF are $\alpha = 0.005$, $\beta = 2$, $\kappa = 3 - n$, $n = 18$. Other parameters are also initialized in the above sections.

TABLE I. THE MEAN AND VARIANCE OF THE MSE

Algorithm	Mean	Var
EKF	0.3852	0.0138
SPKF	0.2715	0.0088

The estimated state mean and covariance of the mean-square-error are obtained in Table I, the SPKF make the MSE much better than EKF in larger initial yaw condition.

To compare the estimation performance of the two methods, the total time of simulation is 500 seconds. The pitch, roll and yaw estimation of simulation results are shown in Fig.3-5.

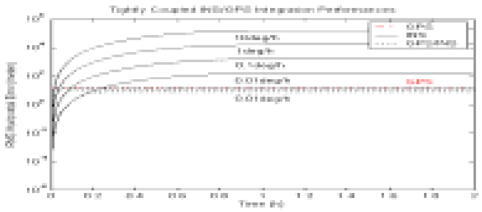

Figure 1. Horizontal position error of tightly-coupled GPS/INS

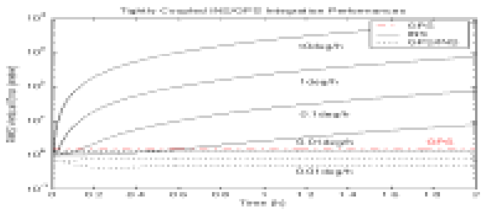

Figure 2. Vertical position error of tightly-coupled GPS/INS

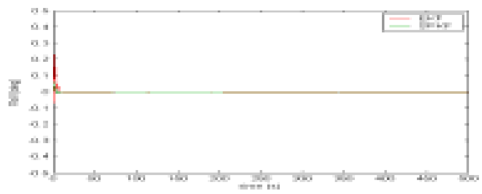

Figure 3. Roll estimation using SPKF and EKF

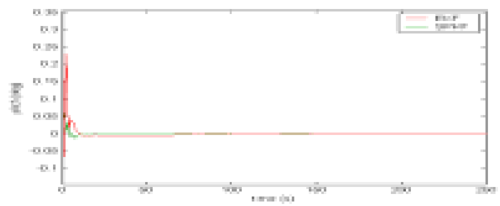

Figure 4. Pitch estimation using SPKF and EKF

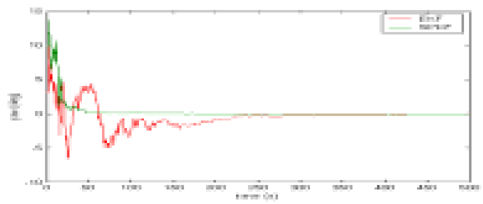

Figure 5. Yaw estimation using SPKF and EKF

Fig.3 and Fig.4 reveal the estimation errors of the horizontal attitude angle (so-called roll and pitch), respectively. Fig.5 shows that estimation error of azimuth attitude angle (so-called yaw). From the previous two figures, the estimation errors of horizontal angle quickly converge in less than 10 seconds, and SPKF has the slightly better results for horizontal attitude angle than EKF. Moreover, the estimated error is less biased for SPKF than that of EKF. Both of the horizontal estimation can converge almost at the same time, while the azimuth estimation of SPKF shown in Fig.5 is obviously different from that of EKF. The estimation error of azimuth angle converges in less than 100 seconds using SPKF. In contrast, the estimation error of azimuth angle by EKF can not converge within the 250 seconds. This is due to the large initial errors that are not handled well in the linearization of the dynamic model in the EKF. So in the sense, the SPKF is able to provide more robust characteristics than EKF for GPS/INS application.

Of course, the computational cost of SPKF is significantly greater than that of EKF; the execution time difference is related to the number of times GPS/INS integration equations are evaluated for each fusion algorithm. With the SPKF, these equations are evaluated many times, once for sigma point, while with EKF, the Taylor series expansion of these equations are only evaluated once at each iteration. However, with the ability of computer powerful operation, the computational cost of SPKF has been cut short enormously.

IV. CONCLUSION

As tightly-coupled GPS/INS integration poses a nonlinear estimation problem, this paper investigated the performance of SPKF-based tightly-coupled GPS/INS system by means of numerical simulations.

It was found that for different inertial sensor grades SPKF-based tightly-coupled GPS/INS system shows good performance concerning the accuracy of the navigation solution and the ability to calibrate the INS. The significant advantage of the SPKF was found for extremely large initial attitude errors, the attitude precision of SPKF is much higher than that of EKF especially for azimuth angle. So this clearly indicates that the SPKF may be a better option considering the design of the GPS/INS system in order to improve its performance.

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