ROYAL HOLLOWAY UNIVERSITY OF LONDON

Department of Physics

# PH3010 MSci Skills Project
# Observation and analysis of solar limb darkening

## Important safety notice

**The Physics Department's Rules on Observatory Safety must be observed at all times. Never point the telescope at the sun until the solar filters have been checked and securely attached. Never remove the solar filters until there is no chance of having sunlight enter the telescope, e.g., until the dome is closed or the telescope has been steered well away from the sun.**

## 1 Introduction

In this project you will measure the variation in intensity of the sun's radiation at different positions over its surface. What you will observe is that the sun is brighter in the centre of the disc than at the edge (or limb); this is called *limb darkening*. These observations can be used to infer the temperature of the sun as a function of depth. The project provides the opportunity to become familiar with the operation of the LX-200 Schmidt-Cassegrain/Skywatcher Equinox telescopes and one of the CCD cameras. Analysis of the data will involve using the method of least squares to fit the parameters of a function describing the centre-to-limb variation of intensity.

## 2 The assignment

The main tasks in this project are listed below.

1. Familiarize yourself with the safe operation of the telescope and CCD camera (must be done under the supervision of the member of staff responsible).
2. As a warm-up exercise, take some photographs of the sun's surface, including sunspots if any are present, and describe the features that you observe.
3. Measure the intensity of the sun's radiation at several different wavelengths as a function of the position over the surface of the solar disc.
4. Use the data to determine the source function $S$ and the temperature $T$ as a function of optical depth.
5. Briefly discuss the meaning of your results in the context of solar models.

## 3 The physics of limb darkening

In this section we recall the ingredients necessary to relate the observed intensity of the sun to quantities that can be predicted by solar models, such as the temperature as a function of depth.

### 3.1 Qualitative picture

The key to understanding limb darkening is that not all of the photons that escape the sun are emitted at the same radius. Rather, they are emitted from a layer called the *photosphere*, which is several hundred km thick (small compared to the 700,000 km radius of the sun). The temperature increases as one goes deeper into the photosphere, and according to laws of blackbody radiation, the light from the hotter regions is more intense.

The light from the interior of the photosphere must, however, pass through the outer layers in order to escape. In doing so there is given probability that it will be absorbed or scattered. The distance scale for this to happen is characterized by a certain mean free path, which in general depends on the photon's wavelength. Photons that traverse one mean free path coming from the centre of the disc are originating at a vertical depth equal to this distance. Photons coming from closer to the limb that travel the same distance through the photosphere, however, originate closer to the surface of the sun, where it is cooler. Therefore, the intensity of the centre of the sun's disc is brighter than at the edge. This is illustrated schematically in Fig. 1.

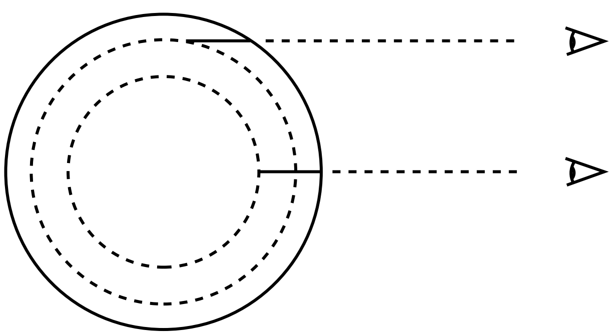

Figure 1: A schematic illustration of limb darkening (see text).

### 3.2 Intensity

To describe limb darkening quantitatively, we need to define the *intensity* of the sun's radiation. Consider an area element $dA$ with a normal unit vector $\mathbf{n}$, as shown in Fig. 2. This area element could be anywhere in the radiation field, e.g., inside the sun, although we will finally be interested in the case where $dA$ is on the sun's surface. Suppose radiation with energy $dE_\lambda$ passes through $dA$ into a cone of solid angle $d\omega$, which has an angle $\theta$ from the vector $\mathbf{n}$, in a wavelength interval from $\lambda$ to $\lambda + d\lambda$ in a time $dt$. The cone should be regarded as originating anywhere on $dA$. The intensity $I_\lambda$ is defined by

$$dE_\lambda = I_\lambda \cos\theta \, d\lambda \, dA \, d\omega \, dt \; . \tag{1}$$

The factor of $\cos\theta$ is included in the definition because the area element $dA$ has an projected area of $dA\cos\theta$ when viewed along the direction of the cone $d\omega$.

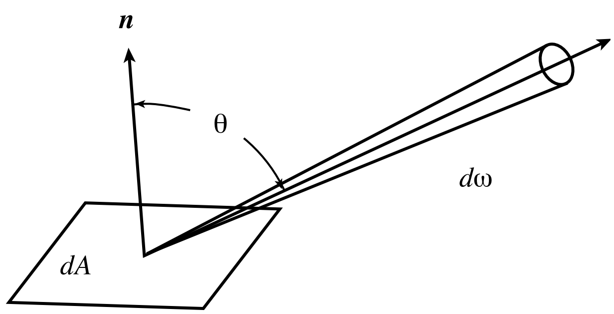


Figure 2: Illustration of the definition of intensity (see text).

The intensity $I_\lambda$ has units of W m$^{-3}$ sr$^{-1}$. Note that one of the powers of length in the m$^{-3}$ is related to the wavelength $d\lambda$ and the other two are from the area $dA$. Equivalently, the intensity $I_\nu$ can be defined using the energy $dE_\nu$ in a frequency interval from $\nu$ to $\nu + d\nu$, which has units of W m$^{-2}$ sr$^{-1}$ s.

The intensity is what we can measure when we observe the surface of a resolved object such as the sun, i.e., when we measure the amount of energy coming from a given area of its surface projected along our line of sight. It is related to but should not be confused with the flux or the luminosity. If we consider the energy passing per unit time through the area element $dA$ and integrate over all solid angle, then we get the *radiative flux*,

$$F_\lambda = \int I_\lambda \cos\theta \, d\omega \; . \tag{2}$$

If we integrate the flux over the surface of a sphere of radius $r$ centred about a star, we obtain the *monochromatic luminosity*,

$$L_\lambda = \int_S F_\lambda \, dA \; = \; 4\pi r^2 F_\lambda \; , \tag{3}$$

where the second equality holds if the flux is independent of direction. The total luminosity is obtained by integrating $L_\lambda$ over all wavelengths; this is what determines the absolute magnitude of a star.

### 3.3 The transfer equation

We will now derive the *transfer equation*, which describes how the intensity of a radiation field changes as it traverses matter. Consider an element of matter of thickness $ds$ which could be, for example, somewhere in the middle of the photosphere. As radiation with a certain incident intensity $I_\lambda$ traverses $ds$ it can be scattered or absorbed. The resulting reduction in intensity is proportional to $ds$, to the density of matter $\rho$ and to the incident $I_\lambda$:

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda \, ds \; . \tag{4}$$

The constant of proportionality, $\kappa_\lambda$, is called the *absorption coefficient*, which depends in general on the wavelength $\lambda$.

The same element of matter can emit photons which will increase the intensity. The change in $I_\lambda$ due to emission is proportional to $\rho\, ds$ but is assumed independent of the incident intensity, i.e.,

$$dI_\lambda = j_\lambda \rho\, ds\ . \tag{5}$$

The constant of proportionality, $j_\lambda$, is called the *emission coefficient*, and it also depends in general on the wavelength $\lambda$.

Putting together equations (4 ) and (5 ) gives the rate of change of the intensity,

$$\frac{1}{\rho}\frac{dI_\lambda}{ds} = j_\lambda - \kappa_\lambda I_\lambda\ . \tag{6}$$

We now define the *source function* $S_\lambda$ as the ratio

$$S_\lambda = \frac{j_\lambda}{\kappa_\lambda}\ , \tag{7}$$

which has the same units as the intensity. Equation (6) then becomes

$$\frac{1}{\kappa_\lambda \rho}\frac{dI_\lambda}{ds} = S_\lambda - I_\lambda\ , \tag{8}$$

which is called the equation of radiative transfer or simply the *transfer equation.*

It is convenient to introduce the *optical depth* $\tau_{\lambda s}$ of a region in the photosphere by defining

$$d\tau_{\lambda s} = -\kappa_\lambda \rho\, ds\ . \tag{9}$$

Notice the minus sign since here optical depth is defined to increase moving away from the observer, whereas $s$ increases as one moves towards the observer. Both $\kappa_\lambda$ and $\rho$ can in general vary as a function of position.

The optical depth $\tau_{\lambda s}$ describes the thickness of matter that a ray of light passes through from somewhere inside the sun to the observer. Except at the centre of the sun's disc, this thickness does not correspond directly to the depth below the surface. It is therefore useful to define the *vertical* optical depth $\tau_\lambda$ as the optical depth of a given region inside the sun, measured vertically downward from the sun's surface. If the normal to the sun's surface directly above the region in question makes an angle $\theta$ relative to our line of sight, then $d\tau_\lambda$ is given by

$$d\tau_\lambda = \cos\theta\, d\tau_{\lambda s} = -\rho\kappa_\lambda \cos\theta\, ds\ . \tag{10}$$

This relation holds as long as one can regard the photosphere as a slab between parallel planes. Since the photosphere is only several hundred km thick, which is small compared to the sun's 700,000 km radius, this is a good approximation except very close to the edge of the disc. In terms of the vertical optical depth, the transfer equation becomes

$$\mu \frac{dI_\lambda(\tau_\lambda, \mu)}{d\tau_\lambda} = I_\lambda(\tau_\lambda, \mu) - S_\lambda(\tau_\lambda) \ , \tag{11}$$

where we will use the abbreviation $\mu = \cos\theta$. Keep in mind that both $I_\lambda$ and $S_\lambda$ depend on position, and in addition, $I_\lambda$ depends on the direction $\theta$, or equivalently on $\mu$.

### 3.4 Solving the transfer equation

What we can measure is the intensity of the sun's radiation as it leaves the photosphere, i.e., at $\tau_\lambda = 0$, and we would like to be able to solve the transfer equation in order to relate $I_\lambda(0, \mu)$ to the source function $S_\lambda$. We start by writing down a formal solution which in effect only converts the differential equation (11) into an integral equation. To simplify the notation we will temporarily drop the subscript $\lambda$ from $I$, $S$, and $\tau$, but keep in mind that these quantities all depend in general on the wavelength. Multiplying both sides of (11) by $e^{-\tau/\mu}$ and rearranging terms gives

$$\mu \frac{dI}{d\tau} e^{-\tau/\mu} - I e^{-\tau/\mu} = -S e^{-\tau/\mu} \ , \tag{12}$$

which can be written as

$$\frac{d}{d\tau}\left(I \mu e^{-\tau/\mu}\right) = -S e^{-\tau/\mu} \ . \tag{13}$$

Integrating both sides from a depth 0 (the surface) down to $\tau$ and solving for the surface intensity $I(0, \mu)$ gives

$$I(0, \mu) = I(\tau, \mu) e^{-\tau/\mu} + \frac{1}{\mu} \int_0^\tau S(\tau') e^{-\tau'/\mu} \, d\tau' \ . \tag{14}$$

If we now consider that the sun extends many optical depths below the surface, we can let $\tau$ go to infinity. Reinserting the subscript $\lambda$, we obtain

$$I_\lambda(0, \mu) = \int_0^\infty S_\lambda(\tau_\lambda) e^{-\tau_\lambda/\mu} \, \frac{d\tau_\lambda}{\mu} \ . \tag{15}$$

Equation (15) with $\mu = \cos\theta$ gives the intensity we see from the sun where the normal to surface makes an angle $\theta$ relative to our line of sight. This angle is directly related to the position that we view on the sun's disc, as illustrated in Fig. 3. If the sun's radius is $R$ and we look at a radius $r$ on the disc, we have

$$\mu = \cos\theta = \sqrt{1 - \left(\frac{r}{R}\right)^2} \ . \tag{16}$$

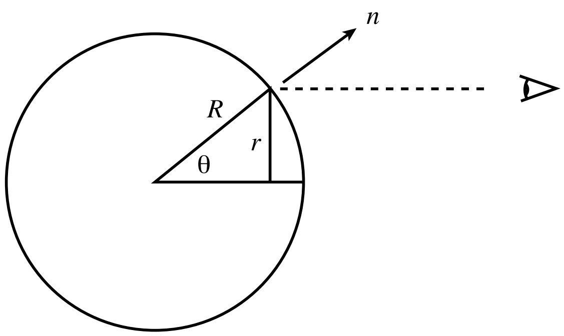


Figure 3: Illustration of the relation between $\cos\theta$, the radius on the disc $r$, and the sun's radius $R$ (see text).

### 3.5 Measuring the source function

We would like to solve equation (15) for the source function $S_\lambda(\tau_\lambda)$ as a function of the vertical optical depth. If we can then measure $I_\lambda(0,\mu)$, the intensity emerging from the surface at different angles (i.e., at different positions on the disc), we can use this to determine $S_\lambda(\tau_\lambda)$.

In practice, we do not solve (15) for $S_\lambda(\tau_\lambda)$ explicitly. We can, however, suppose a reasonably general functional form for the source function which depends on certain adjustable parameters, and we can use the measured intensity to determine the values of the parameters. For example, suppose that the source function is a power series in $\tau_\lambda$

$$S_\lambda(\tau_\lambda) = \sum_{n=0}^{m} a_{\lambda n} \tau_\lambda^n \; . \tag{17}$$

Substituting this into (15 ) and using

$$\int_0^\infty x^n e^{-x}\, dx = n! \; , \tag{18}$$

we find

$$I_\lambda(0,\mu) = \sum_{n=0}^{m} a_{\lambda n} n! \mu^n \; . \tag{19}$$

It turns out that a good description of the intensity can often be obtained by using only three terms ($m = 2$) in the power series.

We cannot measure the scale of intensity $I_\lambda(0,\mu)$ using the RHUL telescope. In practice we take an image of the sun and measure the ADU count $C_\lambda(\mu)$ on a CCD pixel corresponding to a point on the sun's disc defined by $\mu$. We expect that $I_\lambda(0,\mu) = AC_\lambda(\mu)$ for some constant of proportionality $A$. This implies that we can write

$$C_\lambda(\mu) = \frac{1}{A}\sum_{n=0}^{2} a_{\lambda n} n! \mu^n = \frac{1}{A}(a_{\lambda 0} + a_{\lambda 1}\mu + 2a_{\lambda 2}\mu^2) = a'_{\lambda 0} + a'_{\lambda 1}\mu + 2a'_{\lambda 2}\mu^2 \tag{20}$$

where we have defined $a'_{\lambda n} = a_{\lambda n}/A$. In order to correctly calibrate the model we can use the values for $I_\lambda(0,1)$ shown in Table 1 (from [Bö89] p. 233) and the fact that at the centre of the sun's disk

$$I_\lambda(0,1) = a_{\lambda 0} + a_{\lambda 1} + 2a_{\lambda 2} = A(a'_{\lambda 0} + a'_{\lambda 1} + 2a'_{\lambda 2}). \tag{21}$$

The fitting procedure therefore involves using equation (20) to determine the coefficients $a'_{\lambda 0}$, $a'_{\lambda 1}$ and $a'_{\lambda 2}$ from the measured values of $C_\lambda$ at different values of $\mu$. This can be done using the *method of least squares*, which is described in Section 8.1. Then we use equation (21) to fit the parameter $A$ using the data from Table 1. Finally we can use the relation $a_{\lambda n} = Aa'_{\lambda n}$ to determine the coefficients $a_{\lambda 0}$, $a_{\lambda 1}$ and $a_{\lambda 2}$ from which we can find $S_\lambda(\tau_\lambda)$ and $I_\lambda(0,\mu)$ from equations (17) and (19).

Table 1: Values of the sun's surface intensity at the centre of the disc $I_\lambda(0,1)$ for different wavelengths $\lambda$ (from [Bö89] p. 233).

| $\lambda$ (nm) | $I_\lambda(0,1)$ (W m$^{-3}$ sr$^{-1}$) |
|---|---|
| 373.7 | $4.20 \times 10^{13}$ |
| 426.0 | $4.49 \times 10^{13}$ |
| 501.0 | $4.03 \times 10^{13}$ |
| 699.0 | $2.50 \times 10^{13}$ |

### 3.6 Determining the temperature as a function of depth

Once we have found the source function $S_\lambda(\tau_\lambda)$, we can use this to determine the temperature of the sun as a function of optical depth. To do this we need to introduce the concept of *local thermal equilibrium* (LTE).

Consider first a situation where a radiation field is in complete thermal equilibrium with a blackbody of temperature $T$. The intensity is then given by the Planck function, $B_\lambda(T)$,

$$I_\lambda = B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1} . \tag{22}$$

Furthermore, if we consider an element of matter in thermal equilibrium, then the intensity of the radiation passing through it does not change, since if it did, the element would either heat up or cool down. This means that in thermal equilibrium, $dI_\lambda/ds = 0$, and therefore from the transfer equation (8 ) we have

$$I_\lambda = S_\lambda . \tag{23}$$

Both of these quantities are then equal to the Planck function $B_\lambda(T)$ given by equation (22).

The photosphere is not, however, in thermal equilibrium, since there is a temperature gradient of several thousand degrees over a distance of several hundred km and a net flow of radiation outwards. However we will assume that these changes are sufficiently gradual that there is a well-defined temperature at any given position. The assumption of *local thermal equilibrium* means that we can find a temperature at each depth such that

$$S_\lambda(\tau_\lambda) = B_\lambda(T) \tag{24}$$

holds. Further discussion on the validity of LTE can be found in the book by Böhm-Vitense [Bö89].

Assuming LTE, i.e., setting $S_\lambda(\tau_\lambda) = B_\lambda(T)$, we can solve for the temperature to obtain

$$T(\tau_\lambda) = \frac{hc/k\lambda}{\ln\left(1 + \frac{2hc^2}{\lambda^5 S_\lambda(\tau_\lambda)}\right)} \ . \tag{25}$$

By substituting the source function derived from the measured intensity, we can determine the temperature of the sun as a function of the vertical optical depth.

## 4 The telescope

You will be observing the sun using the Department's Skywatcher Equinox 80mm telescope mounted onto a Meade LX-200 10-inch (25 cm) Schmidt-Cassegrain telescope. More information on telescopes and astronomical observations in general can be found in the book by Kitchin [Ki98].

The telescope is equatorially mounted. The axis of the fork points north and its angle with respect to the horizontal is equal to the local latitude of 51 degrees, i.e., the fork is parallel to the earth's axis. This happens to point about 0.8° from Polaris. As the earth turns to the east, the telescope turns about the same axis at the same rate but to the west, so that the telescope stays pointing at a fixed direction in space.

The Skywatcher Equinox is a refracting telescope with an achromatic lens to limit the effects of spherical aberration. It has a diameter of 80mm and a focal length of 500mm, so the focal ratio is $f/6.25$. The field of view of this telescope is almost large enough to view the entire sun.

The diffraction limit to the angular resolution is about 1.6 arc seconds. In fact the limit from atmospheric turbulence is more like 1 to 2 arc seconds at night and perhaps 3 arc seconds during the day. Thus the diffraction limited resolution of the telescope is more than necessary for observing the sun. Since the sun is extremely bright, there is no advantage to using a larger diameter telescope for reasons of light collection, and in fact the solar filter that we use will drastically reduce the sun's intensity.

## 5 CCD cameras

The CCD camera that you will use for this project is a ZWO ASI462MM 12 bit camera. This camera is designed to be used for planetary, lunar, and solar imaging. It has a resolution of $1936 \times 1096$ and a pixel size of $2.9\mu$m. The CCD camera is operated using the SharpCap software installed on the pc in the dome.

If you remove the lens cover you can peer in and see the chip. Please don't do this more than for quick looks as we would like to keep the CCD as dust free as possible.

When photons hit the pixels of the CCD, electrons are liberated by the photo-electric effect. The photo-electrons can be stored in the pixels with very little loss of charge. At the end of an exposure, the contents of each pixel are read out row by row into the computer. The number of photo-electrons in each pixel gives a direct measure of the number of photons collected.

The quantum efficiency of the device is 89% at 500nm (dropping to 20% at 1000nm; 86% at 400nm). For purposes of this project it is sufficient to assume that it has a reasonably uniform response to different colours across the visible part of the spectrum. As an extension to this project you could devise (and if time permits, implement) a scheme for calibrating the CCD as a function of wavelength. More information on the use of CCDs in astronomy can be found in the book by Howell [Ho00].

# 6 Electronic filter wheel

The ZWO EFWmini electronic filter wheel installed between the telescope and the CCD camera houses 5 different filters which can be selected via the ASIcap application on the pc. The different filters enable us to limit the spectral response of the CCD camera to within a fairly narrow band of wavelengths. The wavelengths listed in Table 1 correspond to the mean wavelength of filters U, B, V, and R respectively.

# 7 Experimental procedure

## 7.1 Setting up the telescope and CCD

Please consult with your project supervisor if you have any questions on safe and proper use of the Meade LX-200/Skywatcher Equinox telescopes. Copies of operating manuals can be found in the observatory and they are also available online.

The electronic filter wheel mounts onto the rear cell of the telescope and the CCD mounts onto the filter wheel. Fit the CCD so that the cable points down towards the telescope's fork to ensure that the image is correctly oriented when viewed on the computer screen. Do not over-tighten any of the screws (firmly finger-tight is sufficient). Connect the cable to the PC and start the SharpCap program. From the Cameras dropdown menu select ZWO ASI462MM and click on Live View. You may not see anything if the telescope is not focussed.

Before trying to observe the sun, it is best to have the telescope reasonably well focussed. This can be done by focussing on one of the towers of Founders Building. As has no doubt been pointed out to you numerous times, **keep the telescope pointed well away from the sun during this part of the operation. Never point the telescope at the sun until the proper filters are in place**.

For solar observation you will use a neutral density filter made with Baader AstroSolar safety film. It is made out of a very strong but thin (12 $\mu$m) film with a reflective coating on both sides. It transmits only $10^{-5}$ of the incident light, which for solar observation is plenty. Note that not any 'ordinary' coated film will do. Some films which may appear to be as opaque as proper solar filters could transmit infrared radiation, which can severely damage your eyes. This can occur without any immediate sensation of pain, which makes it all the more dangerous. The filters used in this experiment are especially designed to absorb over a broad range of wavelengths and are CE tested and certified.

Before using the filters, they must be inspected and approved by the member of staff responsible. Make sure the telescope is pointed well away from the sun before removing the lens cover. Mount the filter carefully on the Skywatcher Equinox telescope, taking care not to damage the frames. Leave lens covers on all other telescopes.

Finding the sun with the telescope is sometimes more difficult than one might think. For a start, slew the telescope so that its shadow on the wall is made as small as possible. The telescope mount includes two sets of pins which hold the telescope in place. When their shadows are aligned the telescope should be pointing towards the sun. Additional focussing can be performed at this stage using the image on the computer screen.

From the Tools dropdown menu select Histogram. This enables you to see the numbers of pixels on the chip at each excitation level. When the sun is in view there should be a large peak of high excitation associated with those pixels where the sun's light is focussed and another large peak at low excitation arising from the background. Adjust the exposure and gain settings on the right hand side to ensure that ADU counts are high (in order to obtain accurate readings) but pixels are not saturated. Make sure that any automatic control settings are turned off to ensure consistent readings. Be sure to record all of the relevant settings in your lab book and include these in your report.

Once you have achieved a good focus, take some pictures of the sun, with sunspots if any are visible, and include this in your report. You should be able to see the granularity of the sun's surface, which is caused by the turbulence at the top of the convection zone, just below the photosphere.

## 7.2 Measuring the centre-to-limb intensity variation

The sun subtends an angle of about $0.5°$. The field of view of the CCD chip is slightly greater than this in the horizontal direction and slightly less in the vertical. In a single image the brightness is significantly non-uniform, with decreased response near the edges. This is an effect called vignetting, and it would interfere with our attempt to measure the actual variation in the sun's intensity.

Rather than having to correct for vignetting, it is easier to only use a single pixel from the central region of the CCD's image. A measurement of the centre-to-limb variation can be made by using the *drift scan* method. First, adjust the declination of the telescope to be equal to that of the sun. This can be done by centring the image of the sun in the vertical direction on the screen.

Set the filter wheel by opening the ASIcap application. Press the settings cog at the top right and then press EFW to go to the filter wheel settings. The different filters are listed and the desired filter can be chosen from the dropdown list under Goto. This automatically sets the filter to the appropriate choice.

Then, turn off the telescope's tracking motors. To do this, bring the arrow on the hand paddle's display to 'Setup' using the up or down arrow keys in the lower right and press 'Enter'.

Then move the arrow to 'Targets' and press 'Enter'. There should be a check by 'Astronomical', meaning that the tracking is on. Move the arrow down to 'Terrestrial', and press enter; this turns the tracking off. Then press 'Mode' twice to return to the main menu. You will now see the sun drift in the field of view to the west as the earth rotates to the east. You will have to keep moving the right ascension to the west with the hand paddle to keep the image centred.

Next, without changing the telescope's declination, move the telescope past the western limb of the sun and simply let it drift all the way through the field of view. Take pictures at regular time intervals, e.g., every 1 second. Choose the output format to be .fits file on the right hand side. Click on Start Capture; set a suitable target name, Number of Frames, Sequence Length = 1 and Interval between Captures = 00:00:01. Make sure that the Perform a Sequence of Captures box is checked. When ready press Start. The images will be saved in the SharpCap folder on the desktop.

Then repeat the procedure using different colour filters. If time and the clouds permit, repeat the procedure several times in order to investigate the reproducibility of your measurements. You should also investigate different camera settings, recording each time the relevant numbers in your lab book.

You should also try to estimate the accuracy of your intensity measurements. One method for doing this is to turn the telescope's tracking back on (go to 'Setup', then 'Targets', and select 'Astronomical'), and bring the telescope to the centre of the sun's disc. Then take a series of pictures at the same intervals as used for the drift scan. There will be in general some variation from picture to picture due to changes in the atmospheric conditions or drift in the camera's response. By analysing these variations you can obtain an estimate of your measurement accuracy for the drift scans.

# 8 Data analysis

Transfer your data from the dome pc to another computer for analysis using a USB stick.

There are several steps necessary to go from the raw images to the desired measurements, e.g., the temperature as a function of optical depth. They include:

1. Extract the measured intensities from the central regions of your images and estimate the measurement accuracy. [In python this can be done with the following code fragment:
   ```
   from astropy.io import fits
   f = fits.open('fileNameAndPath.fits')
   d = f[0].data
   ```
   This creates a 2D array `d` of ADU values associated to each pixel.

2. Determine for each measurement the position on the sun's disc, or equivalently, $\mu = \cos\theta$.

3. Choose a functional form $f(\mu; \mathbf{a}')$, which will depend on some adjustable parameters $\mathbf{a}' = (a'_0, a'_1, \ldots)$ to describe the count $C_\lambda(\mu)$ as a function of $\mu$. Using your measurements and the method of least squares, estimate the values of the parameters (must be done separately for each value of $\lambda$).

4. Use the values of the parameters to determine the intensity function, the source function and then use this to find the temperature as a function of vertical optical depth (also done separately for each value of $\lambda$).

You can work out the value of $\mu$ for each measured point by using the known value at the edge of the sun, $\theta = 90°$ or $\mu = 0$.

To estimate the uncertainty in your measurements, you can find the counts $C_\lambda$ from the series of $n$ repeated images taken pointing at the centre of the sun's disc. To obtain a measure of their spread, you can compute the standard deviation, $\sigma$, using

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(C_i - \overline{C})^2} = \sqrt{\frac{n}{n-1}(\overline{C^2} - \overline{C}^2)} \;, \tag{26}$$

where $\overline{C}$ and $\overline{C^2}$ are the averages of the $n$ values of $C_i$ and $C_i^2$, respectively. If you have time you could try estimating the measurement accuracy in this way for different positions over the sun's surface. Of course, the standard deviation obtained from (26) may not reflect the entire uncertainty in your measurement. You should try to think of other potential sources of experimental error and propose how their influence on the measurement could be quantified. Use error propagation to determine the errors for $I_\lambda$ and $S_\lambda$.

## 8.1 The method of least squares

In this section we will use a general statistical technique, the *method of least squares*, to fit a curve with a specified functional form through a set of data points. This is a very powerful data analysis tool that is described in more detail in Refs. [Br97] and [Co98].

At this point you should have a series of measured counts,

$$y_i = C_\lambda(\mu_i) \;, \quad i = 1, \ldots, n \;, \tag{27}$$

at $n$ values of $\mu$. In this section we will drop the subscript $\lambda$, but it is understood that the analysis must be repeated separately for each wavelength. Each value of $y_i$ is characterized by a certain standard deviation $\sigma_i$, which you can take to be the value of $\sigma$ from (26) for all $n$ measurements, or if you have time you can try to estimate individual errors for each data point.

Furthermore, we have from equation (20) a hypothesis for a function $f(\mu)$ that should describe our measurements of $y$ vs. $\mu$. If we take only three terms in the power series for the source function, we have

$$f(\mu; \mathbf{a}') = a_0' + a_1'\mu + 2a_2'\mu^2 \;, \tag{28}$$

where $\mathbf{a}' = (a_0', a_1', a_2')$ are the parameters that determine the shape of the function. We would like to find the values of $a_0'$, $a_1'$ and $a_2'$ so that $f(\mu; \mathbf{a}')$ best describes the measured values $y_i$. We do this by finding those parameter values which minimize the following quantity (called chi-squared),

$$\chi^2(\mathbf{a}') = \sum_{i=1}^{n} \frac{(y_i - f(\mu_i; \mathbf{a}'))^2}{\sigma_i^2} \ . \tag{29}$$

The resulting values are called the *least-squares estimators*, $\hat{a}_0'$, $\hat{a}_1'$, and $\hat{a}_2'$, for the parameters. These are written with hats to distinguish them from the true parameter values, which may forever remain unknown.

Roughly speaking, the idea of minimizing $\chi^2(\mathbf{a}')$ is that this minimizes the sum of squares of the distances between each data point and the fitted curve, where for each point, the 'yardstick' of distance is taken to be the standard deviation $\sigma_i$. In this way, data points with large uncertainties are allowed to deviate more from the curve than those which are well measured.

Finding the estimators $\hat{a}_0'$, $\hat{a}_1'$, and $\hat{a}_2'$ can be performed using the Scipy function scipy.optimize.curve_fit or some other fitting function if you prefer.

Once you have estimated the parameters, you can plot the measured intensities and compare them with the fitted curve. For a more visually appealing plot use $r$ instead of $\mu$ for the $x$-axis, i.e.,

$$f(r; \mathbf{a}') = a_0' + a_1'[1 - (r/R)^2]^{1/2} + 2a_2'[1 - (r/R)^2] \ . \tag{30}$$

Hopefully the curve will pass more or less through the points, and the typical difference between $f(r_i; \hat{\mathbf{a}}')$ and $y_i$ should be similar to the size of the error bar $\sigma_i$. In this case, each point will contribute about one unit to the $\chi^2$, and the minimum value $\chi^2_{\min}$ will be equal to the number of data points. If the functional form of the hypothesis is correct and under certain other reasonably general conditions, one can show that the expected value for $\chi^2_{\min}$ is equal to the number of data points minus the number of fitted parameters, which is called the *number of degrees of freedom* of the fit.

If the value of $\chi^2_{\min}$ is much larger than the number of degrees of freedom, then one would reject the hypothesized functional form as incorrect. You should try the fit with different numbers of terms in the power series (19), each time comparing the value of $\chi^2_{\min}$ to the number of degrees of freedom.

## 9 Interpretation of the results

Having used your measurements to determine the temperature of the sun as a function of optical depth at several wavelengths, you should discuss briefly what your results imply for the composition and structure of the photosphere. As a start you can consult, for example, the books by Böhm-Vitense [Bö89], Carroll and Ostlie [Ca96], and Gibson [Gi73].

## 10 Further work

There are many tasks concerning both the telescope observations and the analysis and interpretation of the data which extend beyond the scope of this project. You may wish to consider some of them if you have time. For the observations you could try, for example, to determine the response of the camera to different wavelengths using light sources of known frequencies available in the teaching lab. In a similar way we could attempt to measure the transmission of the filters as a function of wavelength.

A more complete analysis of the data would include a study of the statistical errors in the fitted parameters and more quantitative tests of the hypothesized functions used to describe the intensity. For more information see Brandt [Br97] or Cowan [Co98].

The book by Carroll and Ostlie [Ca96] includes a computer program called `STATSTAR` for a simple solar model. The program is available from the book's web site (see below) and it can be used to predict the temperature of the sun as a function of optical depth. You can use the program to investigate how your measurements can constrain the model's parameters.

## References

This script and links to other resources can be found on the project web page: `www.pp.rhul.ac.uk/~cowan/msci_skills.html` .

The physics of radiative transfer and solar limb darkening is described in:

Bö89 Erika Böhm-Vitense, *Introduction to stellar astrophysics, volume 2: stellar atmospheres*, Cambridge University Press, 1989.

Ca96 Bradley Carroll and Dale Ostlie, *An Introduction to Modern Astrophysics*, Addison-Wesley, 1996. The book's website, `astrophysics.weber.edu`, contains the program `STATSTAR`.

Gi73 Edward Gibson, *The Quiet Sun*, NASA SP-303, Washington, DC, 1973.

Information on the Schmidt-Cassegrain telescope and CCD camera can be found in:

Ki98 C.R. Kitchin, *Astrophysical Techniques*, 3rd edition, IoP Publishing, Bristol, 1998.

Ho00 Steve B. Howell, *Handbook of CCD Astronomy*, Cambridge University Press, 2000.

The method of least squares and other data analysis techniques are described in:

Br97 S. Brandt, *Statistical and Computational Methods in Data Analysis*, Springer, New York, 1997.

Co98 G. Cowan, *Statistical Data Analysis*, Clarendon Press, Oxford, 1998.

G. Cowan, S Boogert, D. Bedingham, October, 2024