Topic: Mathematical pedagogy and the use of heuristics

Research Question: How pedagogically effective are heuristic proofs?

Word Count: 3266

Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding.

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1 Introduction

The definition of heuristics is as follows: "enabling someone to discover or learn something for themselves[1]". Teaching with this hands-on approach encourages autonomy in the learner, allowing ideas to be understood more effectively. Mathematics is a vast interlinking web of knowledge. These associations are rarely revealed within high school, in favor of modularisation. Deeper conceptual understanding of mathematics is born in the recognition of why disparate concepts reveal common mathematical attributes. We try to circumvent the sentiment, as Bryan observes, "I know the proofs, but I still don't believe it[2]". I argue that by deploying familiar tangible examples, some of these abstract algebraic ideas are contextualised. Further, the process of generalising from the specific - that which is the method of mathematics - is espoused. This method of teaching, in promoting self-discovery, better reflects the work of professional mathematicians, rather than silo mentality.

My initial discussion reviews the idea of geometrically solving quadratic equations by completing the square (the vertex form). This is then generalised for the less familiar case of the cubes, and how they facilitate a more conceptual understanding of the linkage between the Binomial Coefficients and Pascal's Triangle. Thenceforth, a more multifaceted example is explored - the counting of squares and rectangles in chessboards, followed by the same with cubes and cuboids in Rubik's Cubes. This heuristic inquiry draws to-

[1] From (University 2021)
[2] (Dawson 2016)

gether the concepts of Sequences and their summation, and Combinatorics, again showing their representations within Pascal's Triangle. Concrete examples are provided to elucidate the general processes. Running obliging to the mathematical process, this is very much in the spirit of its teaching.

2 Solving the cubic

2.1 Completing the square: a precursor

The quadratic formula is the archetypal example cited by students to demonstrate the lack of applicability of school mathematics in real-life. Rarely is any explanation given on what it actually means to use it to solve equations. The link between algebra and geometry is withheld. Completing the square is used to derive the quadratic formula, and not only does it help in finding solutions, it also leaves the expression in a form that provides some insight as to its graphical representation - the coordinates of the vertex.

The benefits of this geometric perspective is best appreciated through an example:

$$x^2 + 26x = 27. \tag{1}$$

Considering and manipulating areas, Figure 1 (overleaf) shows a square with area $(x + 13)^2$. Since 13^2 has been added to the LHS, the same needs to be done to the RHS, leaving, $27 + 169 = 196$. The side lengths of a square with area 196 is just $\sqrt{196} = 14$, and now it is clear why $x = 1$ is a solution.

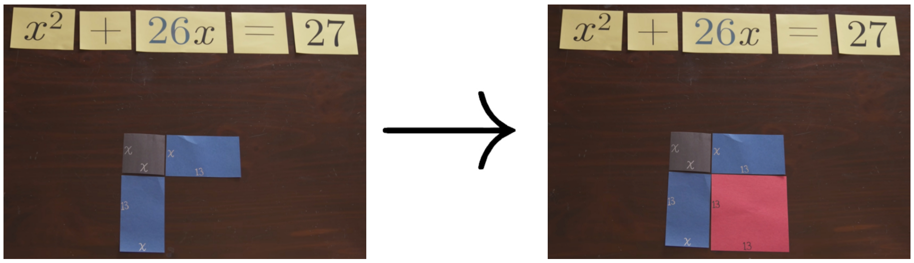

Figure 1: Literally completing the square, extracted from (Derek Last accessed 04/02/2022).

From a pedagogical perspective, this fosters the genuine “doing” of mathematics, rather than passively being at mercy to its methods.

A shortcoming of this geometrical picture is its limitations in dealing with negatives - one solution of which has not been found here: $x = -27$. This failure can be attributed to the method purely dealing with the tangible - what would it mean, or look like, for the x in Figure 1 to have negative length[3]? The nature of this tactile approach encourages students to question the validity in generalising the idea of completing the square into higher dimensions to determine solutions to cubic equations.

2.2 Completing the cube

Gerolamo Cardano devised a general solution for the cubic in *Ars Magna*, attributing a significant portion of this to Niccolò Tartaglia’s ingenious method of “completing the cube”. This method was used, in particular, to solve *de-*

[3] An ancient method, completing the square demonstrates the repulsion of ancient mathematicians from negative numbers, due to their apparent nonexistence in the real world. Many now have a similar reaction when introduced to complex numbers - revisited in Section 2.2.1.

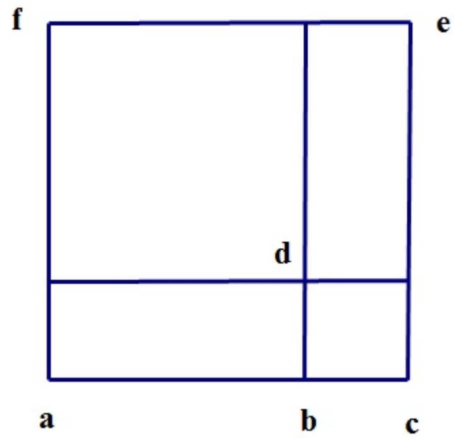

Figure 2: Cardano's figure for cubes, (Branson Last accessed 04/02/2022)

pressed cubics, a cubic polynomial with no x^2 term. Reproduced in Figure 2, by Branson[4], is Cardano's interpretation of it. Branson describes it as "the familiar [binomial expansion] $(x+y)^3$, (Branson Last accessed 04/02/2022)". In the following, I use Tartaglia's tactile approach - the idea of completing the square for cubes - and in the process, make this significant link between number, algebra and geometry self-evident.

Derek Muller, in his YouTube explainer, *How Imaginary Numbers Were Invented*, was able to present this beautifully, mimicking the *three dimensional* thought processes of Tartaglia. He asks us to imagine a cube with side lengths x, and extend the sides by a length of y - Figure 3 shows this.

[4](Branson Last accessed 04/02/2022) and (Cardano et al. 2007, Chapter IV)".

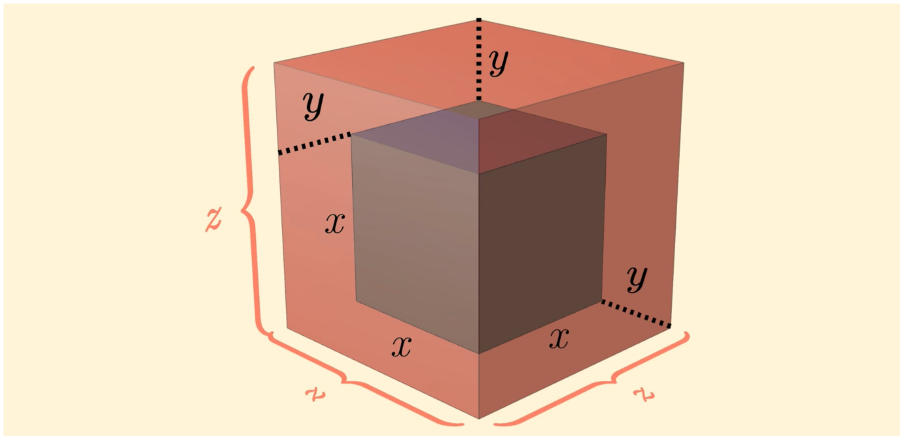

Figure 3: Completing the cube, step 1, (Derek Last accessed 04/02/2022)

This new cube now has side lengths $x + y$, defined as,

$$z := x + y. \tag{2}$$

This new cube can now be *decomposed* into its constituent parts. Figure 4 (overleaf) shows the cube of Figure 3 after decomposition. Consider the volume of the new $(x + y)^3 = z^3$ cube (on the left-hand side) in terms of the new shapes that are created out of the decomposition (on the right hand side). The right hand side shows the original purple x^3 cube, added with a composite cuboid (formed by three each of the green and blue cuboids) whose volume is,

$$V = x \cdot \overbrace{(x + y)}^{z} \cdot 3y = 3yzx. \tag{3}$$

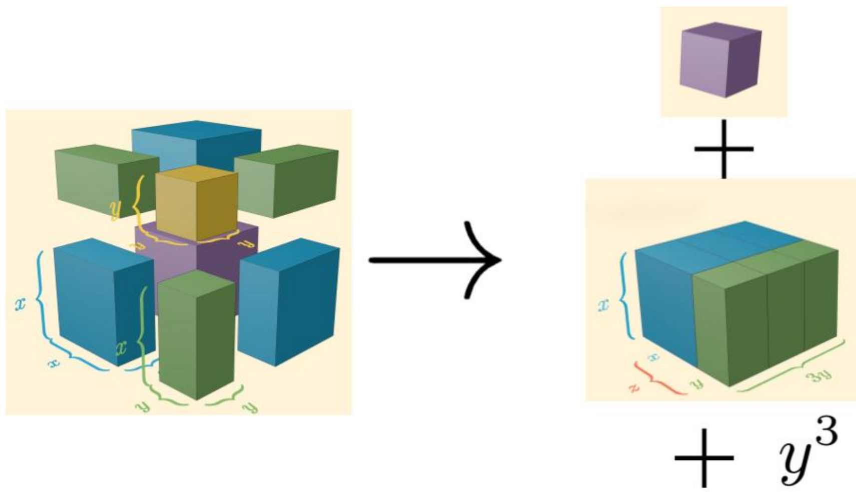

Figure 4: Decomposition, graphical shapes provided by (Derek Last accessed 04/02/2022)

There is one final cube that had to be added to the right hand side to "complete" this cube, and that is the yellow cube with volume y^3. In sum, this leaves the total volume of the cube in terms of its decomposed parts,

$$x^3 + 3yzx + y^3 = z^3. \tag{4}$$

Equation (4) is useful because it can then be compared to one form of the depressed cubic,

$$x^3 + bx = c. \tag{5}$$

In particular, completing the cube, or adding a y^3 term to both sides of (5),

$$\begin{aligned} & x^3 + bx + y^3 = c + y^3 \\ \Leftrightarrow \qquad & z^3 = c + y^3 \iff b = 3yz. \end{aligned}$$

As such, a general method to solve the depressed cubics of the form in (5) has been found from completing the cube. As instantiating offers clarity, consider (6), which is (5) with $b = 9$ and $c = 26$,

$$x^3 + 9x = 26. \tag{6}$$

Completing the cube with the addition of y^3 to both sides of (6),

$$\begin{aligned} \overbrace{x^3 + 9x + y^3}^{z^3} &= 26 + y^3 \\ \Rightarrow z^3 &= 26 + y^3 \iff 3yz = 9, \end{aligned}$$

leaves the following simultaneous equations,

$$3yz = 9 \Leftrightarrow z = \frac{3}{y} \tag{7}$$

$$z^3 = 26 + y^3. \tag{8}$$

(7) can be substituted into (8), which gives,

$$\begin{aligned}
& \left(\frac{3}{y}\right)^3 = 26 + y^3 \\
\Leftrightarrow \quad & \frac{27}{y^3} = 26 + y^3 \\
\Leftrightarrow \quad & 27 = 26y^3 + y^6 \\
\Leftrightarrow \quad & 27 = 26\left(y^3\right) + \left(y^3\right)^2 . \qquad (9)
\end{aligned}$$

Equation (9) is just the quadratic (1), in y^3. Hence, completing the square, as was done in Figure 1, would give the solution for y,

$$\begin{aligned}
& y^3 = 1 \\
\Leftrightarrow \; & y = 1.
\end{aligned}$$

Now this value for y can be back substituted into (7) to determine z,

$$z = \frac{3}{1} = 3.$$

Finally, back substituting the values for z and y into (2) gives,

$$\begin{aligned}
& 3 = x + 1 \\
\Leftrightarrow \; & x = 2, \qquad (10)
\end{aligned}$$

as the solution to the original cubic equation, (6).

After heuristically examining the three dimensional representation, and developing it into algebraic form, a little more work can, in fact, reveal the link to the binomial expansion:

$$
\begin{aligned}
z^3 &= x^3 + 3yzx + y^3 \\
\Leftrightarrow (x+y)^3 &= x^3 + 3y\,(x+y)\,x + y^3 \\
\Leftrightarrow (x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 = \sum_{i=0}^{3} \binom{3}{i} x^{3-i} y^i.
\end{aligned}
$$

Dealing in familiar three dimensional space made the abstract link with the binomial expansion self-evident. This is useful from a teaching standpoint because it is exemplary of “self-discovery” - a motivation for the use of Pascal’s Triangle in binomial expansions has been provided.

2.2.1 Completing the numbers

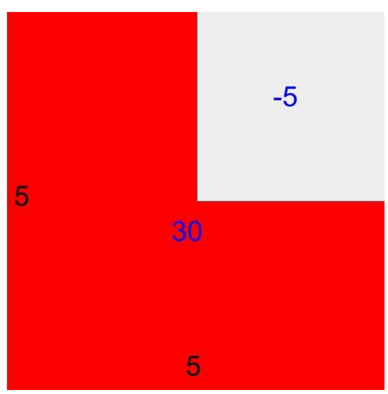

Figure 5: Adding negative area: the birth of complex numbers?

Through a similar heuristic enquiry into the solution of one particular cubic, Cardano ran into a peculiar instance of completing the square; he encountered part of a square with area 30, and side lengths 5, and to "complete the square", he realised he must add -5 units of area. Figure 5 shows the geometric paradox that entailed. As mentioned earlier, a geometrical perspective, while nonsensical, does provide students with an explanation as to the necessity of complex numbers in the abstract - here they are required to fulfill the Fundamental Theorem of Algebra in providing solutions to polynomials, thereby completing our numbering system.

3 Counting squares and cubes

The preceding discussion is evidence of the utility in the "divide and conquer" method to handle multifaceted problems. Reducing the question into simpler ones (complete the square $\rightarrow$ complete the cube $\rightarrow$ solve the cubic) develops the process of applying logic for algebraic deduction, as part of *The Axiomatic Method*, and the "Lemma $\rightarrow$ Theorem $\rightarrow$ Proof" model. In other words, teaching problems in the fashion so far discussed forces the student to actively engage in what are the fundamentals of mathematics, without carrying their burden of formality. We ask in the following, "how many cuboids are there in a Rubik's Cube?". The question is another perfect example of a problem that unifies all of the fundamentals whilst also drawing together ideas from Combinatorics and Pascal's Triangle, as well as helping to distinguish between Sequences and Series.

3.1 Chessboard problem

As was done previously, thinking about the 2-D problem lends itself as a good starting point. Considering the number of squares in a chessboard: the dimensions of a chessboard is 8×8, therefore there must be $8 \cdot 8 = 64$ individual squares. This is true only if squares of dimension 1×1 are considered - the question becomes far more tricky when squares of any dimension are allowed. In the spirit of dividing and conquering, start by counting the number of squares in a 1×1 chessboard - there is clearly only one. Next, consider the 2×2 chessboard. There are four individual 1×1 squares, along with the 2×2 square itself, therefore, there are a total of $4 + 1 = 5$ squares. One more time, for the 3×3 chessboard, there are nine 1×1 squares, four 2×2 squares and one 3×3 square. So, there are $9 + 4 + 1 = 14$ total squares. Using the sequence notation, s_j, for the number of squares in a j sized chessboard:

$$\{s_1,\ s_2,\ s_3\} \equiv \{1,\ 5,\ 14\} \equiv \left\{ \left(1^2\right),\ \left(1^2 + 2^2\right),\ \left(1^2 + 2^2 + 3^2\right) \right\}.$$

To generalise, the sequence is just the j^{th} partial sum of the squares:

$$s_j = \sum_{i=1}^{j} i^2.$$

Returning to the original question, it follows that the number of squares in a traditional chessboard would be,

$$s_8 = \sum_{i=1}^{8} i^2 = 204.$$

Although the original question has been answered, another question naturally arises out of it[5] "how many *rectangles* in a chessboard?".

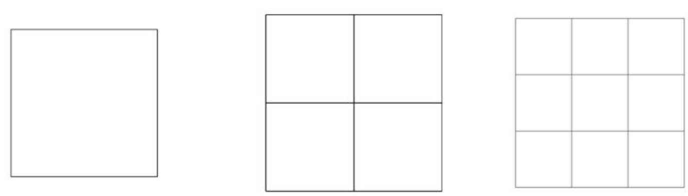

Figure 6: Different size grids - for reference

Using Figure 6 for reference, and the knowledge that there is clearly only one rectangle in the 1×1 chessboard, begin counting for the 2×2 board (note that a square is counted as a rectangle). First, it will help to delineate the sizes of the different rectangles that will be counted, so as to avoid any double counting. A pattern develops, one which will become more evident when dealing with the 3×3 board. In any case, the sizes of all the rectangles that need to be considered here are:

[5]This is the beauty of teaching mathematics in terms of problems like this; not only does it encourage the development of problem-solving skills, it promotes a continuous line of inquiry.

1. $1 \times 1 \rightarrow 4$
2. $2 \times 1 \rightarrow 4$
3. $2 \times 2 \rightarrow 1.$

Now, there are four 1×1 rectangles, four 2×1s (two horizontal, two vertical) and one 2×2. That gives a total of $4 + 4 + 1 = 9$ rectangles. Moving on to the 3×3 board, the list of all the rectangles that need to be considered and the number of them inside the board are:

1. $1 \times 1 \rightarrow 9$
2. $2 \times 1 \rightarrow 12$
3. $2 \times 2 \rightarrow 4$
4. $3 \times 1 \rightarrow 6$
5. $3 \times 2 \rightarrow 4$
6. $3 \times 3 \rightarrow 1.$

This leaves a total of $9 + 12 + 4 + 6 + 4 + 1 = 36$ rectangles. As was done before, sequencing these values makes it easier to see the pattern. Using the capitalised S_j for rectangles:

$$
\begin{aligned}
\{S_1,\ S_2,\ S_3\} &\equiv \{1,\ 9,\ 36\} \\
&\equiv \left\{\left(1^3\right),\ \left(1^3 + 2^3\right),\ \left(1^3 + 2^3 + 3^3\right)\right\} \\
&\equiv \left\{(1)^2,\ (1+2)^2,\ (1+2+3)^2\right\}.
\end{aligned}
$$

Hence, it can be deduced that (and subsequently proved by mathematical induction), for a $j \times j$ board,

$$S_j = \sum_{i=1}^{j} i^3 = \left(\sum_{i=1}^{j} i \right)^2 , \tag{11}$$

and for the number of rectangles in the traditional chessboard,

$$S_8 = \sum_{i=1}^{8} i^3 = \left(\sum_{i=1}^{8} i \right)^2 = 1296.$$

So, not only has a pattern emerged, but an identity has been established,

$$\sum_{i=1}^{j} i^3 = \left(\sum_{i=1}^{j} i \right)^2 . \tag{12}$$

Now we can "import" this relationship, and the others, into the higher dimension to determine the sequences produced by counting cubes and cuboids.

3.2 Rubik's Cube: the problem in three dimensions

3.2.1 Of cubes in cubes

It was found that the number squares was the j^{th} finite sum of the squares, so, we would expect the number of cubes in a $j \times j \times j$ Rubik's cube, c_j, to be the *ansatz*:

$$s_j = \sum_{i=1}^{j} i^2 \xrightarrow{\text{in 3D?}} c_j = \sum_{i=1}^{j} i^3. \tag{13}$$

Applying mathematical induction to prove the supposition (13) would not allow any intuition to be built whatsoever. This is due to the nature of mathematical induction; verifying for the base case - the sequence for the number of squares in a $j \times j$ chessboard - and proceeding to induct for sequences of the Rubik's Cubes and their higher dimensional counterparts, gives no indication as to whether of the sequence postulated is *actually* appropriate in describing the sequences produced. In which case, it seems a good idea to replicate the "divide and conquer" method - to manually count the cubes, so as to establish whether the assertion is valid, before attempting to prove it. In other words, not only is a proof sufficient, it needs to be accompanied by some verification that it accurately relates to the counting of squares and cubes - a bottom-up test for the plausibility of the proof.

Manually counting then, consider the number of cubes in a $2 \times 2 \times 2$ cube (appreciating the trivial fact that the number of cubes in a $1 \times 1 \times 1$ cube is just one). There are eight individual $1 \times 1 \times 1$ cubes, and one $2 \times 2 \times 2$ cube: $8 + 1 = 9$ total cubes. For a $3 \times 3 \times 3$ cube, there are twenty seven individual cubes, eight $2 \times 2 \times 2$ cubes, and the cube itself. So the total number of cubes are $27 + 8 + 1 = 36$. Sequencing these values gives $\{1,\ 9,\ 36\}$. This sequence is the same as the one for the number of rectangles in the chessboard, that which was determined to be the finite sum up to j, of i^3 - equation (11). So, it seems that the number of cubes in a cube is just the j^{th} partial sum of the cubes, (12), following the pattern of the number of squares in the $j \times j$ board. Intuiting this slightly further for the $3 \times 3 \times 3$ cube, there are $3 \cdot 3 \cdot 3 = 3^3$

individual cubes, $2 \cdot 2 \cdot 2 = 2^3$ for the $2 \times 2 \times 2$ cubes, and $1 \cdot 1 \cdot 1 = 1^3$ for the cube itself.

Hence, it has been shown heuristically (through investigation) that the postulation (13) is, in fact, true in the instance of the third dimension. It is only now that a proof by mathematical induction is pedagogically appropriate, since investigating the tangible instances - squares and cubes - is suggestive that the system satisfies the postulated sequence, allowing the assertion *itself* to be made for the Tesseract and indeed any higher dimension.

3.2.2 Of cuboids in cubes

More contentious is hypothesising the sequence that counts the number of cuboids in a $j \times j \times j$ Rubik's cube. This is because there are multiple patterns the sequence, C_j, could follow. The heuristic argument from Section 3.1 suggests a choice between the following three possibilities:

1. $$S_j = \sum_{i=1}^{j} i^3 \xrightarrow{\text{in 3D?}} C_j = \sum_{i=1}^{j} i^4 \tag{14}$$

2. $$S_j = \left(\sum_{i=1}^{j} i\right)^2 \xrightarrow{\text{in 3D?}} C_j = \left(\sum_{i=1}^{j} i^2\right)^2 \tag{15}$$

3. $$S_j = \left(\sum_{i=1}^{j} i\right)^2 \xrightarrow{\text{in 3D?}} C_j = \left(\sum_{i=1}^{j} i\right)^3 \tag{16}$$

Here, three distinct sequences have been postulated, therefore, the nature of mathematical induction becomes even more multiplex[6]. In particular, something is needed to correctly determine which of the three assertions *actually describes* the sequence produced by the number of cuboids in a cube. This selection can only be made using heuristics.

Of *that* triangle and its squares Looking back to Pascal's Triangle, Figure 7, for guidance, I will formally prove, transitively,

$$\sum_{i=1}^{j} i^3 = \binom{j+1}{2}^2.$$

Indeed, through squaring the diagonal of triangular numbers, this appears

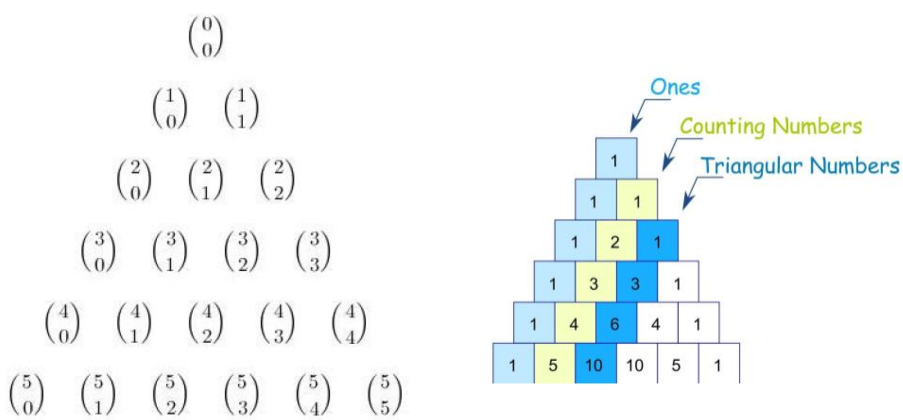

Figure 7: Pascal's Triangle, (MathsIsFun Last accessed 04/02/2022)

[6]Formal proofs will always be necessary in mathematics, because only in them is the deduction based on rational axiomatic manipulation (algebra), self evident. Despite this, one cannot ask "how?". The question "how many cubes are there in an $j \times j \times j$ cube?", demonstrates the difficulties of using formal proof without heuristic understanding.

to be true[7]. One characteristic that jumps out from the 2D problem is the discovery of the same identity (12) - that the square of the j^{th} partial sum of the natural numbers, is equal to the j^{th} partial sum of the cubes. This prompts the *ansatz* to be made that the selection of (16) correctly describes the link between sequencing rectangles and cuboids. (12) is a particularly rich identity because it is, in essence, defined by Pascal's Triangle.

To see this, start with the formal combinatorial definition of the triangle:

$$\binom{j}{r} + \binom{j}{r+1} = \binom{j+1}{r+1} + \binom{j-1}{r}; \tag{17}$$

with,

$$\binom{j}{1} = j. \tag{18}$$

So, by way of example:

$$\begin{aligned}
& \binom{5}{2} = \binom{4}{1} + \binom{4}{2} \\
\Leftrightarrow \quad & \binom{5}{2} = \binom{4}{1} + \binom{3}{1} + \binom{3}{2} \\
\Leftrightarrow \quad & \binom{5}{2} = \binom{4}{1} + \binom{3}{1} + \binom{2}{1} + \binom{2}{2} \\
\Leftrightarrow \quad & \binom{5}{2} = \binom{4}{1} + \binom{3}{1} + \binom{2}{1} + \binom{1}{1} + 0
\end{aligned}$$

[7] Note that even upon the provision of formal proof, intuition as to *why* this is the case would still lack. This is again representative as to the limitations of formal proofs in fostering understanding. In fact, a heuristic inquiry, of which is outside the scope here, shows that: a square with the side length of a triangular number partitions itself into squares and half-squares whose areas add up to cubes.

$$\Leftrightarrow \qquad \binom{5}{2} = 4+3+2+1$$
$$\Leftrightarrow \binom{4+1}{2} = \sum_{i=1}^{4} i.$$

Now, this can be put into context of (12), to show how each *Triangular Number* is the sum of all the natural numbers preceding it - which is useful to understand (12). The diagonal of triangular numbers all have $r = 2$ - they are all in the form $\binom{j}{2}$. With this in mind, the triangular number on row $j+1$ of Pascal's Triangle represents the $(j-1)^{\text{th}}$ partial sum of the natural numbers:

$$\sum_{i=1}^{j} i = \binom{j+1}{2}.$$

Squaring both sides of the above allows the transitive addition of this binomial coefficient to our identity (12):

$$\sum_{i=1}^{j} i^3 = \left(\sum_{i=1}^{j} i\right)^2 = \binom{j+1}{2}^2. \tag{19}$$

Back to counting cuboids Now, pausing to reason what (19) actually means, reveals that $\binom{j+1}{2} \cdot \binom{j+1}{2}$ does, in fact, count the number of rectangles in a $j \times j$ chessboard:

$$\left\{\binom{2}{2}^2,\ \binom{3}{2}^2,\ \binom{4}{2}^2\right\} = \{1,\ 9,\ 36\} = \{S_1,\ S_2,\ S_3\}.$$

Considering the binomial coefficient from this alternative combinatorial perspective shows the *logic* behind why it counts the number of rectangles. This can also be thought of as the *number of ways* to choose two out of $j+1$ objects, *independently*. So, Figure 8 demonstrates an arbitrary choosing of two out of the nine lines, *consecutively* - one in each dimension, and the resulting rectangle that is produced in an 8×8 chessboard. In this way, the *number of possible rectangles* that can be made is identical to the number of them present in the board. This can be confirmed, since $\binom{9}{2}^2 = 1296$, which was the same value obtained earlier.

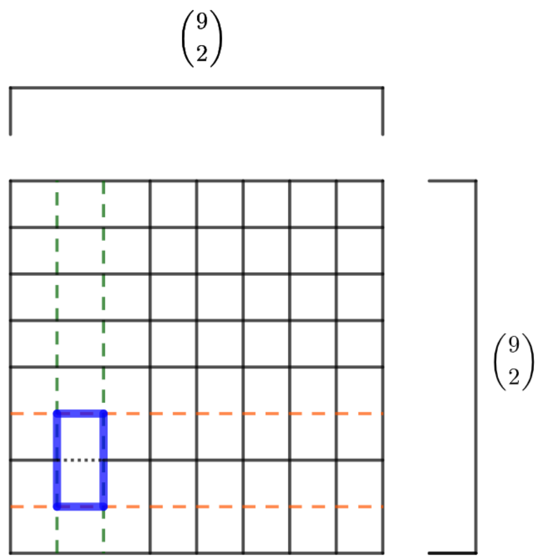

Figure 8: Counting the number of ways to choose a rectangle

From here, the deduction follows that for a Rubik's Cube of any size, the choice of two lines must be made *thrice*, one for each dimension - the length, width, and depth. This would correspond to the number of cuboids in a $j \times j \times j$ Rubik's cube just being $\binom{j+1}{2}^3$, or the cube of the $(j+1)^{\text{th}}$ triangular number, or the cube of the $(j+1)^{\text{th}}$ partial sum of the natural numbers. Carefully counting the cuboids in a $2 \times 2 \times 2$ Rubik's cube confirms $3^3 = 27$ total cuboids, and $6^3 = 216$ in a $3 \times 3 \times 3$ Rubik's Cube.

4 Conclusion

A geometric outlook provides an understanding of the validity of algebraically using completing the square to solve quadratics. In doing so, it also supplements the algebra by demonstrating the actual mechanics and reasoning behind the method. Additionally, the contradiction it reveals - of adding negative area - is evidence of the basis of a mathematical paradigm shift: the detachment of algebra from geometry. This algebraic storytelling has the power to take mathematics to a world of infinite dimensions, if need be.

Many make the ironic argument that it is only after this detachment from the real-world that mathematics became a truly useful tool in describing physical phenomena. The physicist Freeman Dyson argues, "Schrödinger put the square root of minus one into the [wave] equation, and suddenly it made sense[8]". What is neglected however, is how a real-world geometric ex-

[8]from (Freeman 2008)

planation, like Figure 5, allows teachers to show their students the reasons behind the *need* for complex numbers. By encountering them through actively "doing" mathematics, in completing the cube and square, no longer do they seem like an arbitrary definition.

Coleman and Hartshorn point out, "one of the joys of mathematics is the serendipitous meeting of seemingly different ideas[9]" - the process of heuristic investigation of the 2D chessboard revealed a useful identity (19), the extent of which can now be more fully appreciated, since it has unmasked another link in the web of mathematics. The problem-solving provided a concrete example - counting rectangles and cuboids. This revealed the intuition behind Pascal's Triangle's inherent combinatorial descriptions. Rather than commencing with it as a prerequisite and attempting to carry out explanations using its properties, actively dealing with these discrete problems has allowed the synthesis to be made by the problem-solver. The same can be said of the binomial theorem and the decomposition of the cube.

Within the classroom, repeated variations of computation tend to hinder engagement, while do nothing to enhance conceptual understanding. Searching for more formal solutions, through mathematical induction, is not pedagogically effective. Throughout this essay, the deductions made, built bottom-up on the concrete heuristic examples, have been shown to serve better in building a foundation of conceptual understanding.

[9] (Coleman & Hartshorn 2012)

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