Elementary proofs are nothing if not ways to think about something in all applicable cases so that repetitive testing on each possibility is not necessary.

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Note

I would like to present two problems which I think both partake in the general theme: "think generally, not specifically". The first is about tennis clubs, the second about Pascal's Triangle. There is another result which I think is too good to be left out, and for one reason or another, it may be proven quick enough for me to put it here: "7 is the only prime number to be followed by a perfect cube." The idea: formulate the problem in a general setting.

Theorem. *Suppose p is a prime number. Then $\exists n \in \mathbf{N} : p + 1 = n^3$ if and only if $p, n = 7, 2$.*

Proof. The necessity is trivial. Suppose $\exists n \in \mathbf{N} : p + 1 = n^3$. Then $p = n^3 - 1 = (n-1)\left(1 + n + n^2\right)$. Since p is a prime number, either $n - 1 = 1$ or $1 + n + n^2 = 1$. The latter has no solutions in $\mathbf{N}$. So $n = 2$, and so $p = 7$. $\square$

Tennis Club Problem. *Let $n \in \mathbf{N}$. Suppose a tennis club has n members and decides to hold a tournament to determine a winner. The rules of the tournament are as follows:*

(1) *In the first round, if the number of members is even, every member draws a lot to see who will play whom. If the number of members is odd, remove an odd one out, and proceed as above.*
(2) *Eliminate losers of the first round. (Re)Include the odd one out to the winners, and proceed as above.*
(3) *Follow this routine until only one person remains.*

How many matches will have to be played?

Let us work towards understanding what a "good" way to solve the problem might be. In many instances, there are multiple ways to solve a problem. I believe is valuable to trace each solution to understand its method, but also to acknowledge that not all solutions may be equal, even if correct. Namely, I think the *simplest* solution ought to have a significance. What simple actually means would depend on the context, and sometimes may not even be clear. Let's go on and see what this looks like here, by solving particular cases.

Case. *Suppose $n = 1025$.*

Solution. The games are played in pairs, and since losers are out, the number of people available to play in the next round will be halved. It is suspiciously convenient that $1025 = 2^{10} + 1$. The ten in the exponent means it will take division by two a total of ten times to arrive at the final player. So, there will be ten rounds, with an additional final round for the extra odd player. There are 512, or 2^9 matches in the first round, and this too keeps halving (the second round will hold 256 matches, and so on). To count the number of individual matches, use the sum of a geometric series (and add an additional 1 at the end) to get 1024 total matches played. □

1025 is just one away from a power of 2. Which lent itself well to our method. For other numbers, our method would cause trouble in dealing with odd ones out, as you should verify. You may also notice that the number of matches played is precisely one less than the number of members. Let us try another case, with a different number of players, to test if this initial observation is indeed a pattern, and look at what other methods we may use.

Case. *Suppose $n = 927$.*

Solution. Let us try to solve this using a "greedy algorithm". Define the following "arrow operations":

$$\begin{aligned} \rightarrow &:= -1; \\ \leftarrow &:= +1; \\ \downarrow &:= \div 2. \end{aligned}$$

Our greedy algorithm will use these arrows on the number of players at start of each round. Say that our algorithm is greedy in the sense that it wants to use the down arrow the most, but it may only use it if the result is still an integer. If the result is not, it may use the right or left arrows, but in a way that keeps its "horizontal displacement" from the start as least as possible. See that this algorithm puts into practise the rules in our problem. All that is left is to run it, a visual is offered below:

$$\begin{array}{rcr}
937 & \rightarrow & 936 \\
 & & \downarrow \\
 & & 468 \\
 & & \downarrow \\
 & & 234 \\
 & & \downarrow \\
118 & \leftarrow & 117 \\
\downarrow & & \\
59 & \rightarrow & 58 \\
 & & \downarrow \\
30 & \leftarrow & 29 \\
\downarrow & & \\
15 & \rightarrow & 14 \\
 & & \downarrow \\
8 & \leftarrow & 7 \\
\downarrow & & \\
4 & & \\
\downarrow & & \\
2 & & \\
\downarrow & & \\
1. & &
\end{array}$$

Now, counting the rounds is just the number of down arrows, in this case ten. Further, the number of matches in each round is the value below the down arrow. So, to determine the total number of matches, count every number below a down arrow, getting $468 + 234 + 117 + 59 + 29 + 15 + 7 + 4 + 2 + 1 = 936$ matches. □

Because the algorithm (in some sense) simulates our game, we may use it to solve the problem for any number now. We made progress — we now have an argument that could work for any given number, but is that satisfactory? Indeed, how may we run the algorithm through all natural numbers, and further, know whether the result will always be one less the number of members? The idea (again): think generally.

Theorem. *The solution to the tennis club problem is $n - 1$.*

Proof. The way the tournament is set up means that to win, you must not lose any games[1]. This means that to arrive at *one* winner, every member *but one* must lose. So, there must be as many matches as there are losers, since you can only lose

[1]This is precisely what a "cup" competition is, and in common examples, say the FA Cup, you always start out with a power-of-two number of competitors so there is no odd one out business.

once before you are eliminated. So, the number of matches required to determine a winner is one less than the number of members in the club. $\square$

Pascal's Triangle and Combinations

To be appended.