# Linear Maps between Normed Spaces: What's up with that?

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## A little preface

You don't have to be an artist to visit the National Gallery, nor an archaeologist to appreciate the British Museum. I'd like to put on display a tiny fraction of our museum of mathematics; wherever your interests lie, I hope you could find something here.

## Some motivation

I would like to offer some motivation to deal with abstract theory, where I know there is sometimes a risk of aimlessness, especially at first. Consider the question of distance between two places. An answer could be the length of the shortest route between them. I think you could just as well ask the same question for objects which don't obviously live in "space". How far is Man Utd from winning the Premier League again? Another example: two UCL students could be considered "close" based on any number of factors: the modules they share, grades, age, mutual friends.... A general theory regarding proximity could provide us a rigorous framework to draw conclusions across all classes of objects. Being independent of specifics, such theories can teach us the value of abstraction: although our examples were different, they share a common structure.

## Setting the stage

I began my project by dealing with theory concerning things called *linear maps* between things called *normed spaces*. Don't worry if you are not familiar with these, or any other words, I'm going to do my best to offer the general ideas behind them. In *Linear spaces*, we are able to perform certain arithmetic between things, as well as being able to scale them, and in *Normed (linear) spaces* we also have some way of making sense of size. *Metric space* theory can give us a way to talk about a set on which there is a notion of distance. It may not be difficult to conceive that a notion of size can give rise to a notion of distance, which would make every normed space a metric space.

For the reader with an eye for rigour, assume most of the fundamental theory of Linear Spaces and Metric Spaces. In fact, just a little familiarity thereof would be enough to prove many of our results, though I will keep the poster proofless. If you can, I do encourage you to try them yourself, you could see clearer how things fit together (and why things are called "Lemma").

## The juicy bits

Let $X$, $Y$ be normed spaces. You can use maps that preserve the structure of linear spaces (linear maps) to study the ways in which $X$ and $Y$ are related — this is a subject of Linear Algebra. Here, we will study analytic properties of these maps, and the spaces they form when collected together. Let $T$ be a linear map taking $X$ into $Y$.

**Lemma**
*If $T$ is continuous at $0$, then $T$ is continuous.*

So it suffices to assume continuity just at a single point, to deduce it anywhere else! $T$'s good behaviour is owed entirely to its respect for linear space structure (the homomorphism property).

**Definition**
$T$ is *bounded* if $\exists c > 0 : \forall x \in X \ \|T(x)\|_Y \leq c\|x\|_X$.

The $\|x\|_X$ on the right-hand side makes it different to any similar notion met in early analysis classes... so how should we interpret this? You could think of $c$ as a bound on scaling factors: $T$ will not scale the size of any $x$ by a factor more than $c$.

**Lemma**
*$T$ is bounded if and only if $\exists c > 0 : \sup\limits_{\|x\|_X=1} \|T(x)\|_Y \leq c$.*

So much for that definition! It turns out its enough just to check this more familiar notion of boundedness only on the unit sphere!

**Theorem**
*$T$ is continuous if and only if it is bounded.*

Two somewhat different ideas are now brought together with little effort. Collect all such maps into a set, call it $\mathcal{B}(X, Y)$.

**Definition**
The *operator norm* of $T \in \mathcal{B}(X, Y)$ is: $\|T\| = \sup\limits_{\|x\|_X=1} \|T(x)\|_Y$.

To each linear map we attribute a size: the sharpest bound on its scaling factors — the boundedness definition returned!

**Theorem**
*$(\mathcal{B}(X, Y), \|\cdot\|)$ is a normed space.*

It's not too good to be true, the natural definition does indeed define a norm! Now with a normed space of our own to play with, we move to studying such spaces of our maps.

## A moment of appreciation

**Theorem**
*If $Y$ is complete, then $\mathcal{B}(X, Y)$ is complete.*

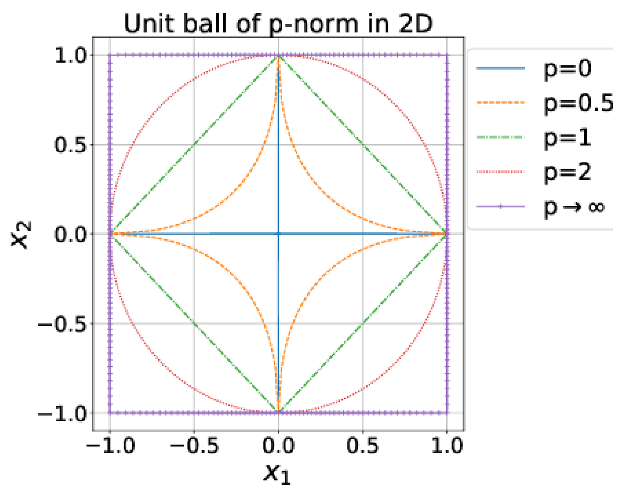
This result is probably the most interesting so far, and I think its proof is the least trivial (it took me many failed attempts). You could think of *completeness* as a way of capturing when a space isn't "missing" any points. Complete normed spaces are named Banach spaces, after the Polish mathematician Stefan Banach. His name embodies his parents — Stefan was his father's given name, and Banach his mother's family name — but it is documented that their poverty meant Banach had to be raised by relatives and friends. He is said to have engaged in part-time work, alongside his studies, to finance his education. One of his major works was functional analysis. Everything here is a testament to him.

A young Stefan Banach[1]

To me, his story is one of unstoppable curiosity and inspiration.

## What about extensions?

Examples of normed spaces are $p$-normed spaces, for $p$ a real number greater or equal to 1. You could think of them as generalisations of the Euclidean Distance. For $0 < p < 1$, it still makes sense to think about $p$-"norms", even though they don't actually define norms. I think you can interpret their failure to satisfy the triangle inequality in terms of the convexity of their unit balls. I would like to think more about our theory in this more general setting, and the adjustments that would be need to be made.




$\mathbb{R}^2$-unit balls with $p$-norms[2]

## Concluding remark

I found out that making a poster is not easy, and I had to make many compromises. If you would like more material, have any comments, or would like to argue with me, send me an email at ahaan.saini.22@ucl.ac.uk.

[1] [The Nowodworski Foundation, 2015]; [2] [Jun Lu and Christine P. Chai, 2022, ResearchGate]

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