Linear Maps between (Quasi)Normed Spaces

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Outline

- Normed Spaces, Linear Maps, Continuity, Boundedness
- Spaces of Bounded Linear Maps, Completeness
- p-norms $(0 < p < 1)$, Quasinorms
- Our Theory in p-normed and Quasinormed Spaces

Normed Spaces and Linear Maps

Definition

(i) Let X be a linear space. An $\mathbb{R}$-valued map on X is a *norm* if it is positive definite, absolutely homogeneous, and satisfies the triangle inequality.

(ii) A *normed space* is a pair $(Y, \|\cdot\|)$, where Y is a linear space, and $\|\cdot\|$ is a norm on Y.

(iii) A *Banach space* is a complete normed space.

Notation

(i) Let X and Y be linear spaces. Denote by $\mathcal{L}(X,Y)$ the set of linear maps between X and Y.

(ii) From now on, let X, Y be normed spaces, and $T \in \mathcal{L}(X,Y)$.

Continuity of Linear Maps between Normed Spaces

Lemma

If T is continuous at 0, then T is continuous.

Proof.

Suppose T is continuous at 0. Let $x \in X$, $\{x_n\}_{n\in\mathbb{N}} \subset X$, and suppose $x_n \to x$. Since T is linear, $T(x_n) - T(x) = T(x_n - x)$, $\forall n \in \mathbb{N}$. So:

$$
\begin{aligned}
\lim_{n\to\infty}(T(x_n) - T(x)) &= \lim_{n\to\infty} T(x_n - x) \\
&= T\left(\lim_{n\to\infty}(x_n - x)\right) \\
&= T(0) \\
&= 0
\end{aligned}
$$

So T is continuous. □

Bounded Linear Maps

Definition

T is *bounded* if $\exists c > 0 : \forall x \in X \ \|T(x)\|_Y \leq c\|x\|_X$.

Examples

$Z = \{\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R} \mid \exists N \in \mathbb{N} : n > N \implies x_n = 0\} \leq \ell^1,\ \ell^\infty$;
$F \in \mathcal{L}(Z, \mathbb{R})$, $F : x \mapsto \sum_{n\in\mathbb{N}} x_n$.

(i) F is bounded on $(Z, \|\cdot\|_1)$.

(ii) F is not bounded on $(Z, \|\cdot\|_\infty)$.

(iii) Suppose X is finite dimensional. Then any $T \in \mathcal{L}(X, Y)$ is bounded.

Boundedness on the Unit Sphere

Lemma

T is bounded if and only if $\exists c > 0 : \sup_{\|x\|_X = 1} \|T(x)\|_Y \leq c$.

Proof.

(i) Suppose T is bounded. Let $x \in X$, and suppose $\|x\|_X = 1$. Then $\|T(x)\|_Y \leq c$, for some $c > 0$. So $\sup_{\|x\|_X = 1} \|T(x)\|_Y \leq c$.

(ii) Suppose $\exists c > 0 : \sup_{\|x\|_X = 1} \|T(x)\|_Y \leq c$. Let $x \in X$. Then:

$$\begin{aligned} \|T(x)\|_Y = \frac{\|T(x)\|_Y}{\|x\|_X} \|x\|_X &= \left\| T\left(\frac{x}{\|x\|_X}\right) \right\|_Y \|x\|_X \\ &\leq c\|x\|_X. \end{aligned}$$

□

Equivalence of Continuity and Boundedness

Theorem

T is continuous if and only if it is bounded.

Proof.

(i) Suppose T is continuous. Let $x \in X$, and suppose $\|x\|_X = 1$. In particular, T is continuous at 0, so, $\exists \delta > 0 : \forall \chi \in X \; \|\chi\|_X < \delta \implies \|T(\chi)\|_Y < 1$. So, $\left\|T\left(\frac{\delta}{2}x\right)\right\|_Y < 1$. So, $\|T(x)\|_Y < \frac{2}{\delta}$. So, $\sup\limits_{\|x\|_X=1} \|T(x)\|_Y \le \frac{2}{\delta}$. So T is bounded, by the preceding lemma.

(ii) Suppose T is bounded. Let $x \in X$. Then, $\exists c > 0 : \|T(x)\|_Y \le c\|x\|_X$. So T is Lipschitz continuous at 0, so continuous, by the (pre)preceding lemma. $\square$

The Space of Bounded Linear Maps

Notation

Denote by $\mathcal{B}(X, Y)$ the set of bounded linear maps between X and Y.

Definition

The *operator norm* of $T \in \mathcal{B}(X, Y)$ is: $\|T\| = \sup\limits_{\|x\|_X = 1} \|T(x)\|_Y$.

Remark

$\forall x \in X \ \|T(x)\|_Y \leq \|T\| \|x\|_X$.

Theorem

$(\mathcal{B}(X, Y), \|\cdot\|)$ *is a normed space.*

Bounded Linear Maps into a Banach Space

Theorem

If Y is complete, then $\mathcal{B}(X,Y)$ is complete.

Proof.

Suppose Y is complete. Let $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(X,Y)$, and suppose $\{T_n\}$ is Cauchy. If $x \in X$, $\|T_n(x) - T_m(x)\|_Y \le \|T_n - T_m\|\|x\|_X \to 0$. So $\{T_n(x)\}$ is Cauchy, and since Y is complete, converges to some $T(x)$. Define $T : X \to Y$; $x \mapsto T(x)$, then $T \in \mathcal{L}(X,Y)$. Let $\epsilon > 0$. Choose N: if n, $m > N$, $\|T_n - T_m\| < \frac{\epsilon}{4}$. Let $x \in X \setminus \{0\}$. Choose $M > N$: if $n > M$, $\|T_n(x) - T(x)\|_Y < \frac{\epsilon}{4}\|x\|_X$. Now:

$$\begin{aligned}\|T_n(x) - T(x)\|_Y &\le \|T_n(x) - T_{M+1}(x)\|_Y + \|T_{M+1}(x) - T(x)\|_Y \\ &< \|T_n - T_{M+1}\|\|x\|_X + \frac{\epsilon}{4}\|x\|_X < \frac{\epsilon}{2}\|x\|_X,\end{aligned}$$

whenever $n > N$. So if $n > N$,
$\|T_n - T\| = \sup\limits_{\|x\|_X=1} \|T_n(x) - T(x)\|_Y \le \frac{\epsilon}{2} \sup\limits_{\|x\|_X=1} \|x\|_X = \frac{\epsilon}{2} < \epsilon$.
So $T_n \to T$, and $T \in \mathcal{B}(X,Y)$, so $\mathcal{B}(X,Y)$ is complete. $\square$

p-norms $(0 < p < 1)$

Recall the definition of p-norms on sequences of real numbers $x = \{x_n\}_{n\in\mathbb{N}}$:

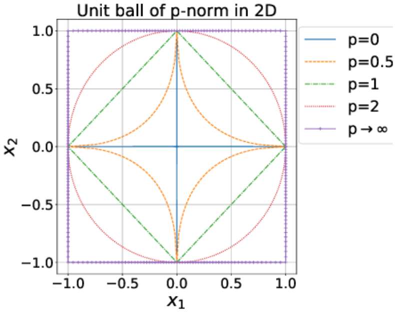
$$\|x\|_p = \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{\frac{1}{p}}.$$

Examples

(i) $\mathbb{R}^n$ with a p-norm, $1 \leq p \leq \infty$, is a Banach Space.

(ii) ℓ^p, $1 \leq p \leq \infty$, is a Banach Space.

(iii) p-norms, $0 < p < 1$, **do not** define norms on sequences of real numbers.



Quasinorms

Definition

An $\mathbb{R}$-valued map on X is a *quasinorm*, denote by $\|\cdot\|_q$, if it is positive definite, absolutely homogeneous, and
$\exists k > 0 : \forall x,\, x' \in X \;\; \|x + x'\|_q \leq k\,(\|x\|_q + \|x'\|_q)$.

Example

$\|\cdot\|_{\frac{1}{2}}$ defines a quasinorm on $\mathbb{R}^2$.

Questions

(i) Does $\|\cdot\|_p$, $0 < p < 1$, define a quasinorm on $\mathbb{R}^n$?

(ii) Is ℓ^p, $0 < p < 1$, a quasinormed space?

What about our two main results here?

Remark

All the work done on the boundedness and continuity of linear maps will hold in the more general setting of quasinormed spaces, since the triangle inequality was never used.

Question

If P and Q are quasinormed spaces, and Q is complete, then is $\mathcal{B}(P, Q)$ complete?