Worksheet 2 (September 1st and 3rd)

1. *Some questions to check your understanding:*

a) *What is the largest possible number of pivots a* 4×6 *matrix can have? Why?*
b) *What is the largest possible number of pivots a* 6×4 *matrix can have? Why?*
c) *How many solutions does a consistent linear system of 3 equations and 4 unknowns have? Why?*
d) *Suppose the coefficient matrix corresponding to a linear system is* 4×6 *and has 3 pivot columns. How many pivot columns does the augmented matrix have if the linear system is inconsistent?*

2. *Determine if the vector* $\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ *is a linear combination of* $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$, *and* $\begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$.

3. *Give a geometric description of* $\textbf{span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \right\}$.

4. *True or false? Justify your answers!*

(a) *Let* A *be an* $m \times n$*-matrix and* B *be an* $m \times l$*-matrix, where* l, m, n *are all distinct. Then the product* AB *is defined.*
(b) *The weights* $c_1, ..., c_p \in \mathbb{R}$ *in a linear combination* $c_1\boldsymbol{v}_1 + ... + c_p\boldsymbol{v}_p$ *cannot all be zero.*
(c) *Given nonzero vectors* $\mathbf{u}, \mathbf{v}$ *in* $\mathbb{R}^n$, $\textbf{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ *contains the line through* $\boldsymbol{u}$ *and the origin. Hint: can you describe this line as a set of vectors?*
(d) *Asking whether the linear system corresponding to* $\left[\begin{array}{ccc|c} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{b} \end{array}\right]$ *is consistent, is the same as asking whether* $\boldsymbol{b}$ *is a linear combination of* $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$.

5. *Determine whether* $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ *is a linear combination of the columns of* $\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}$.

6. *Compute* AB *by the definition, where* $A\boldsymbol{b}_1$ *and* $A\boldsymbol{b}_2$ *are calculated separately.*

(a) $A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 5 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 6 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$

7. *Let* $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM
Midterm Dates: September 29th, October 22nd, November 19th (All Midterms 7-8:15 PM, see `learn.illinois.edu` for locations)

(a) *If* $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, *what is* Ax*?*

(b) *If* $x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, *what is* Ax*?*

(c) *Is* $Ax = b$ *uniquely solvable: is there for a given* b *always exactly one* x*? Hint: use parts (a) and (b).*

(d) *Put* A *into echelon form. Are there any "missing" pivots?*

8. *(Some interesting matrices) Find matrices* A, B, C, D *(what size!) such that:*

(a) $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

(b) $B \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y + 3x \end{bmatrix}$.

(c) $C \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

(d) $D \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$

9. *Colors on a computer are usually based on the RGB model. In this model colors are represented by the percentages of the primary colors red (R), green (G) and blue (B) they contain. That means a* ***color*** *c is a vector* $\begin{bmatrix} r \\ g \\ b \end{bmatrix}$, *where r is the percentage of red, g the percentage of green and b the percentage of blue in the mix c. Obviously in this system*

$$red = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad green = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad blue = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Moreover, the color yellow is given by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ *and the color purple by* $\begin{bmatrix} .5 \\ 0 \\ .5 \end{bmatrix}$.

We say a ***mix of colors*** *$c_1, \ldots, c_n$ is a color c that is of the from $c = a_1c_1 + \cdots + a_nc_n$ for some $a_1, \ldots, a_n$ between 0 and 1.*

(a) *Is every color a mix of red, green, and blue?*
(b) *Is green a mix of yellow and purple?*

The following may be useful in the above problems:

Definition. Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ using weights $c_1, \ldots, c_p$.

Definition. Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$, the **span of** $\mathbf{v}_1, \ldots, \mathbf{v}_p$ **in** $\mathbb{R}^n$ is the set

$$\mathbf{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \{c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p : c_1, \ldots, c_p \in \mathbb{R}\},$$

i.e., the collection of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$.