Math 415 - Lecture 30

Eigenvectors and Eigenvalues

Friday November 6th 2015

Textbook reading: Chapter 5.1

Suggested practice exercises: 12, 20, 21, 22, 36

Khan Academy video: Introduction to Eigenvalues and Eigenvectors, Proof of formula for determining Eigenvalues, Finding Eigenvectors and Eigenspaces example

Strang lecture: Lecture 21: Eigenvalues and eigenvectors

1 Eigenvectors and eigenvalues

Throughout, A will be an $n \times n$ matrix.

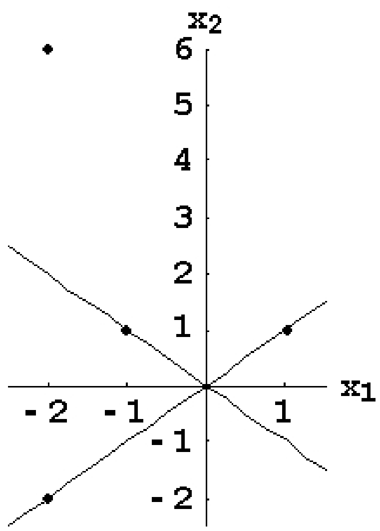
Definition. An **eigenvector** of A is a nonzero $\mathbf{x}$ such that

The scalar λ is the corresponding **eigenvalue**.

In words, eigenvectors are those $\mathbf{x}$, for which $A\mathbf{x}$ is parallel to $\mathbf{x}$.

Example 1. Verify that $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix}0 & -2\\-4 & 2\end{bmatrix}$. Is $\begin{bmatrix}-1\\1\end{bmatrix}$ an eigenvector?

Solution.



Example 2. Use your geometric understanding to find the eigenvectors and the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution.

Example 3. Use your geometric understanding to find the eigenvectors and the eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution.

Summary

* Eigenvectors $\mathbf{x}$ get stretched by eigenvalue λ under multiplication by A:

$$A\mathbf{x} = \lambda\mathbf{x}.$$

* Eigenvectors $\mathbf{x}$ **CANNOT** be zero. Why? $A\mathbf{0} = \lambda\mathbf{0}$ for any λ. Not useful!

* Eigenvalues λ **CAN** be zero. See the projection example.

Problems

* How to find possible eigenvalues for A? This uses determinants.

* How to find eigenvectors? This uses null spaces.

2 Eigenspaces

Definition. The **eigenspace** of A corresponding to λ is the set of all $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$. It consists of all the eigenvectors of A with eigenvalue λ, and also the zero vector.

Example 4. We saw the projection matrix P of the projection onto a subspace V has two eigenvalues $\lambda = 0, 1$.

- The eigenspace of $\lambda = 1$ is V.
- The eigenspace of $\lambda = 0$ is $V^{\perp}$.

3 How to solve $A\mathbf{x} = \lambda\mathbf{x}$

Key observation: $\mathbf{x} \neq 0$ is an eigenvector means:

This $\mathbf{x}$ is a non trivial solution! This can happen $\iff$ the square matrix $A - \lambda I$ is not invertible $\iff$ $\det(A - \lambda I) = 0$

Recipe
To find the eigenvectors and eigenvalues of A:

- First, find the eigenvalues using λ is an eigenvalue $\iff$ $\det(A - \lambda I) = 0$
- Then, for each eigenvalue λ, find the corresponding eigenvectors by solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$. So you need to find the null space $\mathrm{Nul}(A - \lambda I)$.

3.1 The characteristic polynomial

Example 5. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Solution.

3.2 Triangular matrices

Example 6. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution.

3.3 Independent eigenvectors

Theorem 1. *If $\mathbf{x}_1, \dots, \mathbf{x}_m$ are eigenvectors of A corresponding to different eigenvalues, then they are independent.*

Proof.

□

4 Relations between eigenvalues

4.1 Product of Eigenvalues

If A is $n \times n$ get in principle n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. How are these eigenvalues related?

Theorem 2. *The product of eigenvalues $\lambda_1 \lambda_2 \dots \lambda_n$ is equal to the determinant of A.*

Proof.

$\square$

Example 7. Let $A = \begin{bmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{bmatrix}$. Then the eigenvalues are λ_1, λ_2 and $\det(A) = \lambda_1 \lambda_2$.

4.2 Sum of Eigenvalues

What other relations are there between the eigenvalues?

Definition 8. Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ be $n \times n$. Then the **TRACE** of A is the sum of the diagonal entries: $\mathrm{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

Theorem 3. *Let A be $n \times n$. Then the trace of A is the* ***sum*** *of eigenvalues:*

$$\mathrm{Tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Example 9. Let $A = \begin{bmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{bmatrix}$. What are the eigenvalues and what is $\mathrm{Tr}(A)$?

Solution.

4.3 The Characteristic Polynomial for 2×2

2×2 matrices are easy.

Theorem 4. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Then the characteristic polynomial is*

$$p(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A).$$

Example 10. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. What are the eigenvalues and what is the characteristic polynomial?

Solution.

5 Practice problems

Example 11. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

Example 12. What are the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 4 \end{bmatrix}$. No calculations!