Worksheet 3 for September 8th and 10th

1. (1) *True or False: If A and x are real numbers such that $Ax = 0$, then either $A = 0$ or $x = 0$*

(2) *Find a nonzero matrix A and vector x such that $Ax = 0$ but x is nonzero.*

(3) *Show that if A is a 2×2 matrix with pivots in each column, then $Ax = 0$ implies that $x = 0$.*

Solution. (1) Suppose A is a nonzero real number, and $Ax = 0$. Then if we divide by A (which we can do as A is nonzero), we get that $x = 0$.

(2) Consider the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and the vector

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(3) Since A has pivots in both columns, the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus if we consider the augmented system for $Ax = 0$, we'll get reduced row echelon form of $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ so $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

□

2. *Let θ be a real number. Then consider the following matrix:*

$$A_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

(1) *What does A_θ do to $\mathbb{R}^2$ geometrically? For a hint: consider what A_θ does to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.*

(2) *Using the geometric intuition from the previous part, show that $A_{\theta_1+\theta_2} = A_{\theta_1} A_{\theta_2}$*

(3) *Using the previous, we see that $A_{2\theta} = A_\theta^2$. Considering both sides of this equality, what trigonometric identity do you discover?*

Solution. (1) This matrix acts geometrically by rotating $\mathbb{R}^2$ by θ radians in the clockwise direction.

(2) If we rotate first by θ_2 radians clockwise, and then rotate by θ_1 radians clockwise, this is the same as rotating by $\theta_1 + \theta_2$ radians clockwise.

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Midterm Dates: September 29th, October 22nd, November 19th (All Midterms 7-8:15 PM, see `learn.illinois.edu` **for locations)**

(3)

$$A_{2\theta} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{bmatrix} = \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & 2\sin(\theta)\cos(\theta) \\ -2\sin(\theta)\cos(\theta) & -\cos^2(\theta) + \sin^2(\theta) \end{bmatrix}$$

If we match the top left entries we get that $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ and matching the top right entries we see that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, which are the double angle identities.

□

3. *Let*

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

(1) *Is* $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *of the form Ax for some x? Set up a system and solve.*

(2) *Now do the same as in (1), but by thinking of vectors of the form Ax as linear combinations of the columns of A. What form do such linear combinations take?*

Solution. (1) We set up the system $Ax = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and row reduce. $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 1 \end{array}\right] \xrightarrow{R2 \to R2 - 2R1}$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$ We see a row of $\left[\begin{array}{cc|c} 0 & 0 & 1 \end{array}\right]$ so this system is inconsistent, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not of the form Ax for any x.

(2) If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $Ax = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (x_1 + x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus any vector of the form Ax is a multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Looking at the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we see it is not of this form.

□

4. *Let*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

(1) *What is A^{100}?*

(2) *Can you calculate B^{100} by hand?*

Solution. (1) We claim that

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

This is true for $n = 1$ and

$$A^n A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}$$

and thus is true for all n.

(2) Actually later on in this course we will be able to, but for now this is a problem for a computer.

□

5. *The processors of a supercomputer are inspected weekly in order todetermine their condition. The conditionof a processor can either be perfect, good or bad. A perfect processoris still perfect after oneweek with probability 0.7, with probability 0.2 the state is good andwith probability 0.1 it is bad.A processor in good condition is still good after one week withprobability 0.6 and bad with probability 0.4. A bad processor stays bad.*

(a) *What is the probability that a processor in perfectcondition is in bad condition after two weeks?*

(b) *Let us see how this connects to matrix multiplication. Writedown the matrix*

$$T = \begin{bmatrix} T_{p,p} & T_{g,p} & T_{b,p} \\ T_{p,g} & T_{g,g} & T_{b,g} \\ T_{p,b} & T_{g,b} & T_{b,b} \end{bmatrix},$$

where the entries of T correspond to the probability that thecondition of a processor changes from one week to the next. So forexample, $T_{p,g}$ is the probability that a processor in perfectcondition is in good condition after one week. Calculate T^2! Theentry in the third row and the first column of T^2 should be equal toyour result in (a). Why is that?

(c) *How can you use matrix multiplication to determine theprobability that a processor in perfect condition is in bad conditionafter n weeks?*

Solution. (a) There are three possible paths for the processor to go from good to bad in three weeks. Either it goes perfect→perfect→bad, perfect→good→bad, or perfect→bad→bad. The probability of the first is $(.7)(.1) = .07$, the probability of the second is $(.2)(.4) = .08$, and the probability of the third is $.1$. Thus adding these up we see the total probability is $.25$.

(b) We get that

$$T = \begin{bmatrix} .7 & 0 & 0 \\ .2 & .6 & 0 \\ .1 & .4 & 1 \end{bmatrix}$$

and

$$T^2 = \begin{bmatrix} .49 & 0 & 0 \\ .26 & .36 & 0 \\ .25 & .64 & 1 \end{bmatrix}$$

If we use the row column rule for multiplying matrices, we see that the bottom left entry comes from using the bottom row and left column of T. Thus the bottom left entry is $T_{p,p}T_{p,b} + T_{p,g}T_{g,b} + T_{p,b}T_{b,b}$. Each of these corresponds to one of the three paths from [(a)].

(c) This will be the bottom left entry of T^n.

□

6. (1) *Find a matrix E such that:*

$$E \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 - 2R_1 \\ R_3 \end{bmatrix}$$

Which matrix E' undoes the row operation implemented by$E$? What is $E'E$? Is E invertible, and if so, what is E^{-1}?

(2) *Find a matrix F such that:*

$$F \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_2 \\ R_1 \\ R_3 \end{bmatrix}$$

Which matrix F' undoes the row operation implemented by$F$? What is $F'F$?Is F invertible, and if so, what is F^{-1}?

(3) *Find a matrix G such that:*

$$G \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ 3R_2 \\ R_3 \end{bmatrix}$$

Which matrix G' undoes the row operation implemented by$G$? What is $G'G$?Is G invertible, and if so, what is G^{-1}?

Solution. (1) $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To undo this row operation we have to replace $R_2 \to R_2 + 2R_1$, so $E' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is easy to check that $E'E = I = EE'$. Thus E is invertible and $E^{-1} = E'$.

(2) $F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To undo this row operation we have to replace $R_2 \leftrightarrow R_1$, so $F' = F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is easy to check that $F'F = I = FF'$. Thus F is invertible and $F^{-1} = F$.

(3) $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To undo this row operation we have to replace $R_2 \to \frac{1}{3}R_2$, so $G' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is easy to check that $G'G = I = GG'$. Thus G is invertible and $G^{-1} = G'$.

Note: each row operation on A is equivalent to multiplying A from left by an invertible matrix. $\square$

The following may be useful in the above problems:

Definition. An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A.