Math 415 - Lecture 10

Span is a subspace, Null Space

Wednesday September 16th 2015

Textbook: Chapter 2.1, 2.2.

Suggested practice exercises: Chapter 2.1: 3, 21, 28. Chapter 2.2: 33 and additional exercises in this lecture note.

Khan Academy videos: Linear Subspaces, Introduction to the Null Space of a Matrix, Calculating the Null Space of a Matrix

1 Review of vector space and subspace

- A vector space is a set of vectors which can be added and scaled (without leaving the space!); subject to the "usual" rules.
- The set of all polynomials of degree up to 2 is a vector space. Why?

$$[a_0 + a_1t + a_2t^2] + [b_0 + b_1t + b_2t^2] = [(a_0+b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2]$$
$$r[a_0 + a_1t + a_2t^2] = [(ra_0) + (ra_1)t + (ra_2)t^2]$$

  Note how it "works" just like $\mathbb{R}^3$.

- The set of all polynomials of degree exactly 2 is **not** a vector space. Why?

$$\underbrace{[1 + 4t + t^2]}_{\text{degree 2}} + \underbrace{[3 - t - t^2]}_{\text{degree 2}} = \underbrace{[4 + 3t]}_{\text{NOT degree 2}}$$

- Easy test: Is the zero vector in the set? (If not, then it's **not** a vector space.)

Example 1. Let V be the set of all function $f : \mathbb{R} \to \mathbb{R}$. Is V a vector space?

Solution. Yes! Adding of functions f and g:

$$f(x) + g(x) = (f + g)(x)$$

so $f(x) + g(x)$ is in V.

Note that, once more, this definition is "component-wise". Scalar multiplication works the same way.

Definition. A **subspace** of a vector space V is a subset H of V that has three properties:

1. The zero vector of V is in H.
2. For each $\mathbf{u}$ and $\mathbf{v}$ are in H, $\mathbf{u}+\mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)
3. For each $\mathbf{u}$ in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

Problem 2. *Find as many subspaces in $\mathbb{R}^2$ as you can.*

2 A Shortcut for Determining Subspaces

Definition. Recall that *span* $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$ is the collection of all vectors that can be written as

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_p\mathbf{v_p},$$

where $x_1, x_2, \ldots, x_p$ are scalars.

Theorem 1. *If $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$ are in a vector space V, then span $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$ is a subspace of V.*

Example 3. Is $V = \left\{ \begin{bmatrix} a + 2b \\ 2a - 3b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$? Why or why not?

Solution. Write vectors in V as:

$$\begin{bmatrix} a + 2b \\ 2a - 3b \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} 2b \\ -3b \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

So $V = \mathit{span}\,\{\mathbf{v_1}, \mathbf{v_2}\}$ where

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

and therefore V is a subspace of $\mathbb{R}^2$ by the previous theorem.

Example 4. Is $H = \left\{ \begin{bmatrix} a + 2b \\ a + 1 \\ a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^3$? Why or why not?

Solution. No! H does not contain the zero vector. In other words,

$$\begin{bmatrix} a+2b \\ a+1 \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

cannot equal the zero vector for any choice of a or b.

Example 5. Is the set H of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of M_{2x2}?

Solution. Yes!

$$H = span \left\{ \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \right\}.$$

Since H can be written as a span, it's a subspace of M_{2x2}.

Problem 6. *Determine which of the following sets are subspaces and give reasons:*

1. $W_1 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 2b = c, 4a + 2c = 1 \right\}.$

2. $W_2 = \left\{ \begin{bmatrix} a-b \\ c \\ a+c \\ a-2b-c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$

3. $W_3 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a \cdot b \geq 0 \right\}.$

3 Null Spaces

Definition. The **nullspace** of an $m \times n$ matrix A, written as $\mathrm{Nul}(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\mathrm{Nul}(A) = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

Theorem 2. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.*

Proof: $\mathrm{Nul}(A)$ is a subset of $\mathbb{R}^n$ since A has n columns. We have to verify properties (a), (b), and (c) of the definition of a subspace.

Property (a): Show that $\mathbf{0}$ is in $\text{Nul}(A)$.

$$A\mathbf{0} = \mathbf{0}.$$

and

$$A \underset{\text{in } \mathbb{R}^n}{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}} = \underset{\text{in } \mathbb{R}^m}{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}$$

Property (b): If $\mathbf{u}$ and $\mathbf{v}$ are in $\text{Nul}(A)$, show that $\mathbf{u} + \mathbf{v}$ is also in $\text{Nul}(A)$. Suppose $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Then

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Property (c): If $\mathbf{u}$ is in $\text{Nul}(A)$ and c is a scalar, show that $c\mathbf{u}$ is also in $\text{Nul}(A)$. Suppose $A\mathbf{u} = \mathbf{0}$. Then

$$A(c\mathbf{u}) = c(A\mathbf{u}) = c(\mathbf{0}) = \mathbf{0}.$$

Let's restate the theorem.

Theorem 3. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.*

Remark.

- Since properties (a), (b), and (c) hold, $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$.
- Since $\text{Nul}(A)$ is a subspace, it is closed under linear combinations. You can add solutions of $A\mathbf{x} = 0$ and get a new solution! This is very important. Not true for $A\mathbf{x} = \mathbf{b}$ for $b \neq 0$. Here you cannot add solutions!
- Solving $A\mathbf{x} = \mathbf{0}$ yields an explicit description of $\text{Nul}(A)$.

Example 7. Find and explicit description of $\text{Nul}(A)$ where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

Solution. We want to find all the solutions to $A\mathbf{x} = \mathbf{0}$. So we need to do Gaussian elimination on the augmented matrix $\begin{bmatrix} A \mid \mathbf{0} \end{bmatrix}$.

$$\begin{bmatrix} A \mid \mathbf{0} \end{bmatrix} = \left[\begin{array}{ccccc|c} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 3 & 6 & 6 & 3 & 9 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccccc|c} 3 & 6 & 0 & 39 & 99 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{array}\right] = \begin{bmatrix} U \mid \mathbf{0} \end{bmatrix}.$$

$$[U \mid \mathbf{0}\ ] = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{array}\right].$$

This corresponds to the solution:

$$\begin{aligned} x_1 &= -2x_2 - 13x_4 - 33x_5 \\ x_3 &= 6x_4 + 15x_5. \end{aligned}$$

Write this as a linear combination:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}.$$

So each vector in $\text{Nul}(A)$ looks like:

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$\text{Nul}(A) = \textit{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

In other words,

$$\text{Nul}\left(\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \right) = \textit{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Remark. If $\text{Nul}(A) \neq \{\mathbf{0}\}$, then the number of vectors in the spanning set for $\text{Nul}(A)$ equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.
In this example, we had 3 free variables (x_2, x_4, and x_5) so there were 3 vectors in the spanning set for $\text{Nul}(A)$. More about this later!