Math 415 - Lecture 27

An application of QR-decomposition, Change of basis

Friday October 30th 2015

Textbook reading: Chapter 3.4, Chapter 2.6

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Suggested practice exercises: Chapter 2.6: Exercises 36, 37, 38,39, 40,43

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Khan Academy video: Change of basis

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Khan Academy video: Change of basis

Strang lecture: Change of basis; image compression

Review

Theorem (QR decomposition)

Let A be a $m \times n$ matrix of rank n with linear independent columns. There is an orthogonal matrix $m \times n$-matrix Q and an upper triangular $n \times n$ invertible matrix R such that

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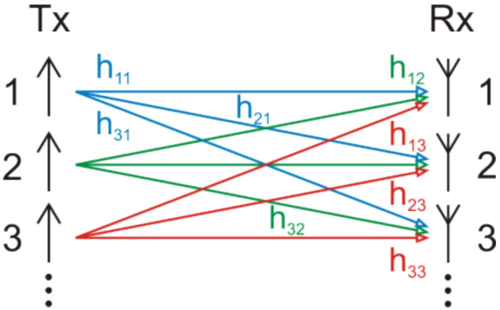
An application in wireless communication

In multiple-input and multiple-output (short: MIMO) systems, a transmitter sends multiple streams by multiple transmit antennas. Let us suppose there are n transmitters and m receivers. This can modelled using Linear Algebra:

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}}_{\text{received vector } \mathbf{y}} = \underbrace{\begin{bmatrix} h_{1,1} & \dots & h_{1,n} \\ \vdots & \ddots & \vdots \\ h_{m,1} & \dots & h_{m,n} \end{bmatrix}}_{\text{channel matrix } H} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\text{transmitted vector } \mathbf{x}} .$$

Suppose that the channel matrix H is known both to person A who sending information and to person B who is receiving the information.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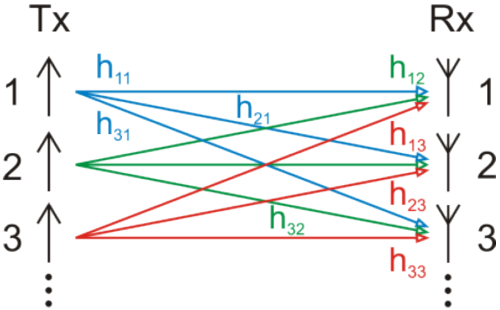

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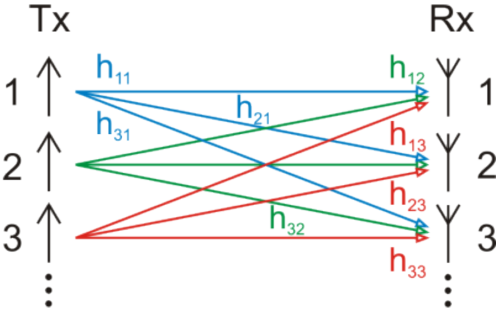

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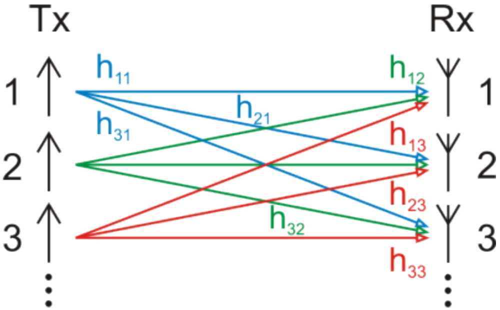
Remember: the x vector describes what the transmiter is sending out and y is the vector describing what is received.



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- In a well designed system you want Dim(Nul(H)) $=$?
- What is $Col(H)$?

When B receives the signal, she wants to reconstruct the vector $\mathbf{x}$. Optimally, she would just solve the linear system

$$H\mathbf{x} = \mathbf{y}.$$

Unfortunately, almost always B received $\mathbf{y} + \epsilon$ instead of $\mathbf{y}$, where $\epsilon \in \mathbb{R}^m$ is noise.

So B would try to solve

$$Hx = y + \epsilon.$$

instead. However, that system might not have a solution. So B has to find the least square solution! Because B receives many messages from A, she will have to find the least square solution many times. Luckily, H does not change, and has independent columns ($\mathrm{Nul}(H) = 0$). So B determines the QR-decomposition of H

$$H = QR,$$

once, and then just solves

$$R\mathbf{x} = Q^T(\mathbf{y} + \epsilon)$$

each time she receives a new message. This is easy to do, since R is upper triangular.

Linear transformation revisited

Recall the notion of coordinate vectors.

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$$x_{\mathcal{C}} = I_{\mathcal{C},\mathcal{B}} x_{\mathcal{B}}.$$

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Theorem

Let $\mathcal{B}$ be a basis of $\mathbb{R}^m$ and $\mathcal{C}$ be a basis of $\mathbb{R}^n$ and let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then there is a $n \times m$ matrix $T_{\mathcal{C},\mathcal{B}}$ such that for every $\mathbf{v} \in \mathbb{R}^m$

$$T(\mathbf{v})_{\mathcal{C}} = T_{\mathcal{C},\mathcal{B}} \mathbf{v}_{\mathcal{B}}.$$

and

$$T_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} T(\mathbf{v_1})_{\mathcal{C}} & T(\mathbf{v_2})_{\mathcal{C}} & \dots & T(\mathbf{v_m})_{\mathcal{C}} \end{bmatrix}$$

where $\mathcal{B} = \left(\mathbf{v_1}; \dots ; \mathbf{v_m}\right)$.

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where $\mathcal{B} = (\mathbf{v_1}; \dots; \mathbf{v_m})$.

We will use this first in the special case $T = I$, where $I(v) = v$ (seemingly boring!).

Example

Consider $\mathcal{E} := \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{B} := \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Let $I : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation (the Identity!)

$$I\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Find the matrix $I_{\mathcal{E},\mathcal{B}}$ that represents I with respect to the input basis $\mathcal{B}$ and output basis $\mathcal{E}$.

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Solution

By definition the matrix $I_{\mathcal{E},\mathcal{B}}$ has as first column b_1 expressed in the standard basis, and as second column b_2 also expressed in the standard basis.

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$$I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}.$$

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Solution

Let $\mathbf{v}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Then

$$I_{\mathcal{E},\mathcal{B}}\mathbf{v}_{\mathcal{B}} = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 b_1 + c_2 b_2 = v!$$

Suppose $\mathbf{v}$ is a vector in $\mathbb{R}^n$, and we have two bases in $\mathbb{R}^n$. so that we get two coordinate vectors $\mathbf{v}_{\mathcal{C}}$ and $\mathbf{v}_{\mathcal{B}}$.

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Let $\mathcal{B}$ be a basis of $\mathbb{R}^n$ and $\mathcal{C}$ be another basis of $\mathbb{R}^n$ and let $I : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation such that $I(\mathbf{v}) = \mathbf{v}$ for every $\mathbf{v} \in \mathbb{R}^n$. Then

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$$\mathbf{v}_\mathcal{C} = I_{\mathcal{C},\mathcal{B}} \mathbf{v}_\mathcal{B}.$$

We call the matrix $I_{\mathcal{C},\mathcal{B}}$ a **change of base matrix**, it transforms coordinate vectors from the $\mathcal{B}$ to the $\mathcal{C}$ basis.

Example

Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^n$ and $\mathcal{B}$ be another basis of $\mathbb{R}^n$. What is $I_{\mathcal{E},\mathcal{B}}$?

Solution

The columns of $I_{\mathcal{E},\mathcal{B}}$ are the basic vectors $b_1, b_2, \ldots$ expressed in the standard basis. So

$$I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix}$$

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$$v = v_{\mathcal{E}} = I_{\mathcal{E},\mathcal{B}} v_{\mathcal{B}}$$

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Solution

$I_{\mathcal{C},\mathcal{B}}$ is the matrix with columns the $\mathcal{B}$ basis vectors expressed in the $\mathcal{C}$ basis, and $I_{\mathcal{C},\mathcal{B}}^{-1}$ is the inverse of this matrix. These matrices have the property that

$$v_{\mathcal{C}} = I_{\mathcal{C},\mathcal{B}} v_{\mathcal{B}}, \quad v_{\mathcal{B}} = I_{\mathcal{C},\mathcal{B}}^{-1} v_{\mathcal{C}}.$$

Example

As before, let $\mathcal{E} := \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{B} := \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. What is $I_{\mathcal{B},\mathcal{E}}$?

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Solution

We know what $I_{\mathcal{E},\mathcal{B}}$ is, it is just $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Then $I_{\mathcal{B},\mathcal{E}}$ is the transition matrix going the other way, so it is the inverse of the easy matrix, so

$$I_{\mathcal{B},\mathcal{E}} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

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Example

Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^n$ and $\mathcal{C}$ be a orthonormal basis of $\mathbb{R}^n$. Then $I_{\mathcal{C},\mathcal{E}} = I_{\mathcal{E},\mathcal{C}}^T$. Why?

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Solution

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Example

Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^n$ and $\mathcal{C}$ be a orthonormal basis of $\mathbb{R}^n$. Then $I_{\mathcal{C},\mathcal{E}} = I^T_{\mathcal{E},\mathcal{C}}$. Why?

Solution

$I_{\mathcal{E},\mathcal{C}}$ the matrix with orthonormal columns, so it is an orthogonal matrix. $I_{\mathcal{C},\mathcal{E}}$ is the inverse. But the inverse of an orthogonal matrix is easy, just the transpose.

Theorem

Let $\mathcal{B} := (\mathbf{u_1}, \ldots, \mathbf{u_n})$ *be an orthonormal basis of* $\mathbb{R}^n$ *and* $U = \begin{bmatrix} \mathbf{u_1} & \ldots & \mathbf{u_n} \end{bmatrix}$. *Then for every* $\mathbf{v} \in \mathbb{R}^n$

$$v_{\mathcal{B}} = U^T v.$$

Change of basis

Theorem

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation and $\mathcal{A}$ and $\mathcal{B}$ be two bases of $\mathbb{R}^m$ and $\mathcal{C}, \mathcal{D}$ be two bases of $\mathbb{R}^n$. Then

$$T_{\mathcal{C},\mathcal{A}} = I_{\mathcal{C},\mathcal{D}} T_{\mathcal{D},\mathcal{B}} I_{\mathcal{B},\mathcal{A}}.$$

$$\begin{tikzcd}
(\mathbb{R}^m, \mathcal{A}) \arrow[r, "\text{apply } T_{\mathcal{C},\mathcal{A}}"] \arrow[d, "I_{\mathcal{B},\mathcal{A}}"'] & (\mathbb{R}^n, \mathcal{C}) \\
(\mathbb{R}^m, \mathcal{B}) \arrow[r, "\text{apply } T_{\mathcal{D},\mathcal{B}}"] & (\mathbb{R}^n, \mathcal{D}) \arrow[u, "I_{\mathcal{C},\mathcal{D}}"']
\end{tikzcd}$$

Example

Consider $\mathcal{B} := \mathcal{D} := \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{A} := \mathcal{C} := \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ as before. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be again the linear transformation that

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto A \begin{bmatrix} x \\ y \end{bmatrix}, \text{ where } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Determine $T_{\mathcal{C},\mathcal{A}}$.

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Determine $T_{\mathcal{C},\mathcal{A}}$.

Solution

Example

Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^n$, let $\mathcal{B} := (\mathbf{u_1}, \dots, \mathbf{u_n})$ be an orthonormal basis of $\mathbb{R}^n$ and $U = \begin{bmatrix} \mathbf{u_1} & \dots & \mathbf{u_n} \end{bmatrix}$ Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then

$$T_{\mathcal{B},\mathcal{B}} = U^T T_{\mathcal{E},\mathcal{E}} U.$$

Why?

Solution