Math 415 - Lecture 13

Basis and Dimension

Wednesday September 23rd 2015

Textbook reading: Chapter 2.3

Suggested practice exercises: Chapter 2.3 Exercise 1, 2, 3, 5, 6, 9, 11, 16, 19, 20, 22, 27.

Khan Academy video: Introduction to Linear Independence, More on linear independence, Span and Linear Independence Example, Basis of a Subspace

Strang lecture: Independence, Basis, and Dimension

* Exam 1 (7-8:15 pm Tuesday September 29):

* Rooms:

  - 213 Gregory Hall: AD3, ADG, ADU
  - 151 Loomis: ADC, ADD, ADL, ADM
  - 100 Gregory Hall: ADE, ADF, ADN, ADO
  - 66 Library: ADH, ADP, ADQ, ADX
  - 141 Loomis: AD1, AD2, ADS, ADT, ADW, ADZ
  - 100 MSEB: AD4, ADV, ADY, ADI, ADR
  - 150 ASL: ADA, ADB, ADJ, ADK

  MSEB is the Materials Science and Engineering Building. ASL is the Animal Science Lab.

* Conflicts: You should have signed up for a conflict exam by now.

* No Discussion Sections next week.

* No Class on Wednesday next week.

1 Review

- Vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are *linearly* ***Dependent*** if

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_p\mathbf{v_p} = \mathbf{0},$$

  and not all the coefficients are zero.

- The columns of A are linearly **IN**dependent $\iff$ each column of A contains a pivot $\iff$ there are no free variables for $A\mathbf{x} = \mathbf{0}$.

- Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix}$ independent?

$$\begin{bmatrix}1 & 1 & -1\\1 & 2 & 1\\1 & 3 & 3\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 1 & -1\\0 & 1 & 2\\0 & 2 & 4\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 1 & -1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$$

  So no, they are dependent! (Coeff's for instance $x_3 = 1, x_2 = -2, x_1 = 3$)

- Any set of 11 vectors in $\mathbb{R}^{10}$ is linearly dependent. Why?

Definition 1. In a list of vectors $(\mathbf{v_1}, \ldots, \mathbf{v_p})$ in a vector space V we call $\mathbf{v_k}$ *redundant* if v_k is a linear combination of the previous vectors. In this case $\mathrm{Span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{k-1}}, \mathbf{v_k}) = \mathrm{Span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{k-1}})$, i.e., you can delete the redundant vector and get the same span.

Example 2. Let $\mathbf{v_1} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix}1\\0\\2\end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix}2\\0\\3\end{bmatrix}$. Are there redunant vectors?

Solution. Since $\mathbf{v_3} = \mathbf{v_1} + \mathbf{v_2}$, $\mathbf{v_3}$ is redundant and $\mathrm{Span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \mathrm{Span}(\mathbf{v_1}, \mathbf{v_2})$.

Today we are going to study sets of vectors without redundant elements.

2 A Basis of a Vector Space

Definition. A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in V is a **basis** of V if

- $V = \mathrm{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$, and
- the vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are linearly independent.

Fact: $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in V is a basis of V if and only if every vector $\mathbf{w}$ in V can be uniquely expressed as $\mathbf{w} = c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p}$.

Fact: A basis is a *minimal spanning set*: the elements of the basis span V but you cannot delete any of these elements and still get all of V. There are no redundant vectors.

Example 3. Let $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Show that $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is a basis of $\mathbb{R}^3$. (It is called the **standard basis**.)

Solution.

- Clearly, $\operatorname{Span}\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\} = \mathbb{R}^3$.
- $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ are independent, because $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has a pivot in each column, no free variables. Note that we can not delete one of $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ and still get all of $\mathbb{R}^3$.

Definition. V is said to have **dimension** p if it has a basis consisting of p vectors.

This definition makes sense because if V has a basis of p vectors, then every basis of V has p vectors. Why? (Think of $V = \mathbb{R}^3$.) A basis of $\mathbb{R}^3$ cannot

have more than 3 vectors, because any set of 4 or more vectors in $\mathbb{R}^3$ is linearly dependent. A basis of $\mathbb{R}^3$ cannot have less than 3 vectors,because 2 vectors

span at most a plane. (Challenge: can you think of an argument that is more "rigorous"?)

Example 4. $\mathbb{R}^3$ has dimension 3. Indeed, the standard basis

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

has three elements. Likewise, $\mathbb{R}^n$ has dimension n.

Example 5. Not all vectors spaces have a finite basis. For instance, the vector space of all polynomials has *infinite dimension*. Its standard basis is $1, t, t^2, t^3, \ldots$

Why?

Solution. This is indeed a basis, because any polynomial can be written as a unique linear combination:

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

for some n.

Recall that vectors in V form a **basis** of V if

- They span V.
- They are linearly independent.

These are two conditions. If we know the dimension of V, we only need to check *one* of these two conditions:

Theorem 1. *Suppose that V has dimension d.*

- *A set of d vectors in V are a basis if they span V.*
- *A set of d vectors in V are a basis if they are linearly independent.*

Why?

Solution.

- If the d vectors were not independent, then $d-1$ of them would still span V. In the end, we would find a basis of less than d vectors.
- If the d vectors would not span V, then we could add another vector to the set and have $d+1$ independent ones.

Example 6. Are the following sets a basis for $\mathbb{R}^3$?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

Solution.

(a) No, the set has less than 3 elements.

(b) No, the set has more than 3 elements.

Example 7. (c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$. Is this a basis?

Solution.

(c) The set has 3 elements. Hence, it is a basis if and only if the vectors are independent.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

Since each column contains a pivot, the three vectors are independent. Hence, this is a basis for $\mathbb{R}^3$.

Example 8. Let P_2 be the space of polynomials of degree at most 2.

- What is the dimension of P_2?
- Is $\left\{t,\ 1-t,\ 1+t-t^2\right\}$ a basis of P_2?

Solution.

- The standard basis for P_2 is $\left\{1, t, t^2\right\}$. This is indeed a basis because every polynomial

$$a_0 + a_1 t + a_2 t^2$$

can clearly be written as a linear combination of $1, t, t^2$ in a unique way. Hence, P_2 has dimension 3.

- The set $\left\{t,\ 1-t,\ 1+t-t^2\right\}$ has 3 elements. Hence, it is a basis if and only if the three polynomials are linearly independent. We need to check whether

  $$x_1 t + x_2(1-t) + x_3(1+t-t^2) = 0$$

  has only the trivial solution $x_1 = x_2 = x_3 = 0$. We get the equations

  $$\begin{aligned} x_2 + x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ -x_3 &= 0 \end{aligned}$$

  which clearly only have the trivial solution. (If you don't see it, solve the system!) Hence, $\left\{t, 1-t, 1+t-t^2\right\}$ is a basis of P_2.

3 Shrinking and Exanding Sets of Vectors

We can find a basis for $V = \operatorname{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ by discarding, if necessary, some of the vectors in the spanning set.

Example 9. Produce a basis of $\mathbb{R}^2$ from the vectors

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution. Here, we notice that $\mathbf{v_2} = -2\mathbf{v_1}$. The remaining vectors $\{\mathbf{v_1}, \mathbf{v_3}\}$ are a basis for $\mathbb{R}^2$, because the two vectors are clearly linearly independent.

Example 10. Produce a basis of $\mathbb{R}^2$ from the vector

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solution. $\mathbf{v_1}$ is independent. But is does not span $\mathbb{R}^2$. For instance $\mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not in the span of $\mathbf{v_1}$. Let's add it! Then $\operatorname{Span}(\mathbf{v_1}, \mathbf{v_2})$ is all of $\mathbb{R}^2$ and we found a basis.

4 Checking Our Understanding

Example 11. Subspaces of $\mathbb{R}^3$ can have dimension 0, 1, 2, or 3.

- The only 0-dimensional subspace is $\{\mathbf{0}\}$.
- A 1-dimensional subspace is of the form $\operatorname{Span}\{\mathbf{v}\}$ where $\mathbf{v} \neq \mathbf{0}$. These subspaces are lines through the origin.
- A 2-dimensional subspace is of the form $\operatorname{Span}\{\mathbf{v}, \mathbf{w}\}$ where $\mathbf{v}$ and $\mathbf{w}$ are not multiples of each other. These subspaces are planes through the origin.

- The only 3-dimensional subspace is $\mathbb{R}^3$ itself.

True or false?

1. Suppose that V has dimension n. Then any set in V containing more than n vectors must be linearly dependent. True.

2. The space P_n of polynomials of degree at most n has dimension $n+1$. True. A basis is $\left\{1, t, t^2, \ldots, t^n\right\}$.

3. The vector space of functions $f : \mathbb{R} \to \mathbb{R}$ is infinite-dimensional. True. A still-infinite-dimensional subspace are the polynomials.

4. Consider $V = \operatorname{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$. If one of the vectors, say $\mathbf{v_k}$, in the spanning set is a linear combination of the remaining ones, then the remaining vectors still span V. True. $\mathbf{v_k}$ is not adding anything new.