# Math 415 - Lecture 16

Linear Transformations

Friday October 2nd 2015

**Textbook reading:** Chapter 2.6

**Suggested practice exercises:** Chapter 2.6: 5, 6, 7, 36, 37

**Khan Academy videos:** Linear Transformations / Linear Transformations as Matrix Vector Products / Linear Transformation Examples: Rotations in $\mathbb{R}^2$

**Strang lecture:** Lecture 30: Linear Transformations

## 1 Review

If $\mathcal{B} = (\mathbf{b_1}, \dots, b_p)$ is a basis for a vector space $V$ then the coordinate vector of a vector $\mathbf{w} \in V$ is the column vector

$$\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}, \quad \text{if } \mathbf{w} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \cdots + c_p\mathbf{b_p}$$

*Example* 1. Let $V = \mathbb{R}^2$, $\mathcal{B} = (\mathbf{b_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

**Solution.** Then

$$\mathbf{w} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \mathbf{w}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Geometrically: this means that to reach $\mathbf{w}$ walk 1 unit along the $\mathbf{b_1}$ basis vector and 2 units along the $\mathbf{b_2}$ basis vector.

*Example* 2. Still $V = \mathbb{R}^2$, $\mathcal{B} = (\mathbf{b_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ a basis for $V$. Suppose $\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ is a coordinate vector with respect to the basis $\mathcal{B}$. What is the vector $\mathbf{w}$, with respect to the standard basis?

**Solution.** $\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ means that you reach $\mathbf{w}$ by walking 4 units along $\mathbf{b_1}$ and 5 units along $\mathbf{b_2}$. So

$$\mathbf{w} = 4\mathbf{b_1} + 5\mathbf{b_2} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$$

**Remark.** Translating to the standard basis is always easy. To go from the standard basis to a new basis requires solving a system of equations, so is generally harder.

# 2 Linear Transformations

Let $V$ and $W$ be vector spaces.

**Definition.** A map $T : V \to W$ is a linear transformation if

$$T(c\mathbf{x} + d\mathbf{y}) = cT(\mathbf{x}) + dT(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in V$ and all $c, d \in \mathbb{R}$. In other words, a linear transformation respects addition and scaling.

- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- $T(c\mathbf{x}) = cT(\mathbf{x})$

It also sends the zero vector in $V$ to the zero vector in $W$:

- $T(\mathbf{0}) = \mathbf{0}$ (because $T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}$)

## 2.1 Some examples

*Example* 3. Let $V = \mathbb{R}, W = \mathbb{R}$. Then the map $f(x) = 3x$ is linear. Why?

**Solution.** If $x, y \in \mathbb{R}$, then $f(ax + by) = 3(ax + by) = a \cdot 3x + b \cdot 3y = af(x) + bf(y)$. What about the function $g(x) = 2x - 2$? Is this a linear transformation?

*Example* 4. Let $A$ be an $m \times n$ matrix. Then the map $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. Why?

**Solution.** Because matrix multiplication is linear.

$$A(c\mathbf{x} + d\mathbf{y}) = cA\mathbf{x} + dA\mathbf{y}$$

The left-hand side is $T(c\mathbf{x} + d\mathbf{y})$ and the right-hand side is $cT(\mathbf{x}) + dT(\mathbf{y})$.

We will argue that all linear transformations are essentially matrix multiplication!

*Example* 5. Let $P_n$ be the vector space of all polynomials of degree at most $n$. Consider the map $T : P_n \to P_{n-1}$ given by

$$T(p(t)) = \frac{d}{dt}p(t).$$

This map is linear! Why?

**Solution.** Because differentiation is linear:

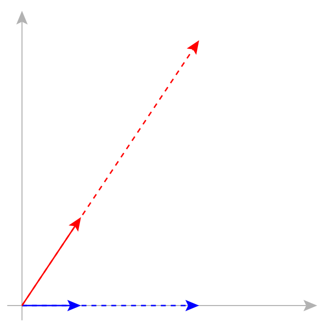
$$\frac{d}{dt}\left[ap(t) + bq(t)\right] = a\frac{d}{dt}p(t) + b\frac{d}{dt}q(t).$$

The left-hand side is $T(ap(t) + bq(t))$ and the right-hand side is $aT(p(t)) + bT(q(t))$.

# 3 Important Geometric Examples

Let's consider some linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ which are defined by matrix multiplication ($\mathbf{x} \mapsto A\mathbf{x}$). In fact, it turns out that all linear maps $\mathbb{R}^n \to \mathbb{R}^m$ are given by $\mathbf{x} \mapsto A\mathbf{x}$ for some $m \times n$ matrix $A$.

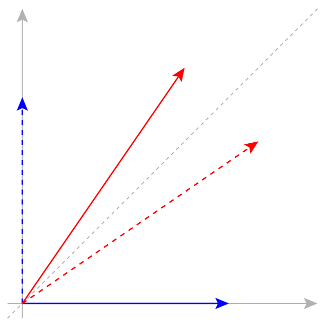
*Example* 6 (Stretching). The matrix $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ gives the map $x \mapsto c\mathbf{x}$, It stretches every vector in $\mathbb{R}^2$ by a factor $c$.

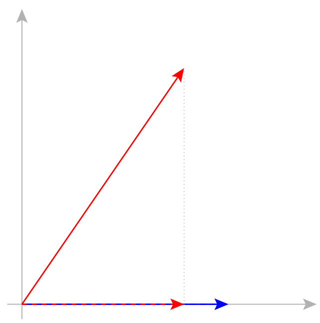


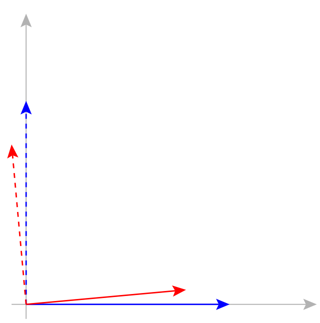
*Example* 7 (Reflection). The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$. It reflects every vector in $\mathbb{R}^2$ across the line $y = x$.

*Example* 8 (Projection.). The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$. It projects every vector in $\mathbb{R}^2$ onto the x-axis.

*Example* 9 (Rotation by 90°.). The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$. It rotates every vector in $\mathbb{R}^2$ counter-clockwise by 90 degrees.







## 4 Representing linear maps by matrices

**Motto**
If you know $T$ on a basis, you know $T$ everywhere.

- Let $\mathbf{x_1}, \dots, \mathbf{x_n}$ be an input basis, a basis for $V$. A linear map $T : V \to W$ is determined by the values $T(\mathbf{x_1}), \dots, T(\mathbf{x_n})$.

- Why?

Take any $\mathbf{v} \in V$. It can be written as $\mathbf{v} = c_1\mathbf{x_1} + \cdots + c_n\mathbf{x_n}$ because $\{\mathbf{x_1}, \dots, \mathbf{x_n}\}$ is a basis and hence spans $V$. Hence by the linearity of $T$:

$$T(\mathbf{v}) = T(c_1\mathbf{x_1} + \cdots + c_n\mathbf{x_n}) = c_1T(\mathbf{x_1}) + \cdots + c_nT(\mathbf{x_n}).$$

So we know how to write $T(\mathbf{v})$ as long as we know $T(\mathbf{x_1}), \ldots, T(\mathbf{x_n})$!

## 4.1 Standard Basis Coordinates

*Example* 10. Suppose $T : R^2 \to \mathbb{R}^3$ is a linear map so that

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} \text{ and } T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\-3\end{bmatrix}$$

What is

$$T\begin{bmatrix}1\\2\end{bmatrix}?$$

**Solution.**

$$T\begin{bmatrix}1\\2\end{bmatrix} = T\begin{bmatrix}1\\0\end{bmatrix} + 2T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}0\\0\\-3\end{bmatrix} = \begin{bmatrix}1\\2\\-3\end{bmatrix}$$

Let us look at the example again. The linear transformation was given on the standard basis by

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} \text{ and } T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\-3\end{bmatrix}$$

Let's take a general input vector for $T$:

$$T\begin{bmatrix}x\\y\end{bmatrix} = xT\begin{bmatrix}1\\0\end{bmatrix} + yT\begin{bmatrix}0\\1\end{bmatrix} = x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}0\\0\\-3\end{bmatrix}$$

A linear combination! Linear combination is matrix multiplication!

$$T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1 & 0\\2 & 0\\3 & -3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}.$$

Hence calculating $T$ is multiplying by the matrix $A = \begin{bmatrix}1 & 0\\2 & 0\\3 & -3\end{bmatrix}$.

**Summary:** The linear transformation

$$T\colon \mathbb{R}^2 \to \mathbb{R}^3, \quad T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\-3\end{bmatrix}$$

is the same as multiplying by the matrix

$$A = \begin{bmatrix} T\begin{bmatrix}1\\0\end{bmatrix} & T\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix} \begin{bmatrix}1\\2\\3\end{bmatrix} & \begin{bmatrix}0\\0\\-3\end{bmatrix}\end{bmatrix}$$

We say that the linear transformation $T$ is represented by the matrix $A$, or that $A$ is the *coordinate matrix* of the linear transformation $T$, (with respect to the standard bases).

*Example* 11. Let $T_\alpha\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the "rotation over $\alpha$ radians (counterclockwise)" map. So $T_\alpha(\mathbf{x})$ is the vector obtained by rotating $\mathbf{x}$ over angle $\alpha$. Can you find a matrix so that $T_\alpha(\mathbf{x}) = A_\alpha\mathbf{x}$?

**Solution.** We just need to find what happens under rotation to the standard basic vectors. If you draw a picture you see that

$$T_\alpha\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos(\alpha)\\ \sin(\alpha)\end{bmatrix}, \quad T_\alpha\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin(\alpha)\\ \cos(\alpha)\end{bmatrix},$$

So our matrix is $A_\alpha = \begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}$. This is called the rotation matrix for angle $\alpha$. It allows you to calculate the rotation of any vector!

**Theorem 1** (Linear Transformation is Matrix Multiplication, Standard basis)**.** *Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a matrix $A$ such that*

$$T(\mathbf{x}) = A\mathbf{x}, \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n.$$

*Explicitly,*

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n)\end{bmatrix},$$

*where $e_1, e_2, \dots, e_n$ is the standard basis of $\mathbb{R}^n$.*

*Proof.* We can write $\mathbf{x} = x_1e_1 + x_2e_2 + \cdots + x_ne_n$. Then

$$\begin{aligned} T(\mathbf{x}) &= T(x_1e_1 + x_2e_2 + \cdots + x_ne_n) = \\ &= x_1T(e_1) + x_2T(e_2) + \cdots + x_nT(e_n) = \\ &= A\mathbf{x}. \end{aligned}$$

$\square$

*Example* 12. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}2a+3b-c\\ -a+b+2c\end{bmatrix}$. What is the matrix representing $T$ (with respect to the standard bases)?

**Solution.** First think about the size of $A$. It must be $2 \times 3$.Then calculate the columns of $A$:

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}, \quad \text{Why? } a = 1, b = c = 0,$$

$$T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix},\quad \text{Why? } a=0, b=1, c=0,$$

**Example continued.**

$$T\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}2a+3b-c\\-a+b+2c\end{bmatrix},\quad T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix},\quad T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix},$$

$$T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix},\quad \text{Why? } a=0=b, c=1,$$

So $A=\begin{bmatrix}2 & 3 & -1\\-1 & 1 & 2\end{bmatrix}$. Check:

$$A=\begin{bmatrix}2 & 3 & -1\\-1 & 1 & 2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}2a+3b-c\\-a+b+2c\end{bmatrix}=T\begin{bmatrix}a\\b\\c\end{bmatrix}$$

# 5 Nonstandard Bases

Untill now we have used the standard bases to describe $T\colon \mathbb{R}^n \to \mathbb{R}^m$. Often it is useful to use other bases.

*Example* 13. Let $T\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}3a+1b\\1a+3b\end{bmatrix}$. Then the matrix of $T$ is $A=\begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}$. But let us use, instead of the standard basis, another basis adapted to $T$. Put

$$\mathbf{b_1}=\begin{bmatrix}1\\-1\end{bmatrix},\quad \mathbf{b_2}=\begin{bmatrix}1\\1\end{bmatrix}.$$

What is the coordinate matrix for $T$ with respect to $\mathcal{B}=(\mathbf{b_1},\mathbf{b_2})$?

**Solution.** What do we want? We want to find a matrix $B$ that relates the coordinate vectors (w.r.t. basis $\mathcal{B}$) of input vector $\mathbf{x}$ and and output vector $T(x)$:

$$T(x)_{\mathcal{B}}=Bx_{\mathcal{B}}.$$

This matrix $B$ has columns $T(\mathbf{b_1})_{\mathcal{B}}$ and $T(\mathbf{b_2})_{\mathcal{B}}$. So let us calculate

$$T(\mathbf{b_1})=T\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=2\begin{bmatrix}1\\-1\end{bmatrix}=2\mathbf{b_1},$$

$$T(\mathbf{b_2})=T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=4\begin{bmatrix}1\\1\end{bmatrix}=4\mathbf{b_2}$$

This means that the coordinate matrix with respect to $\mathcal{B}$ is simply

$$B=\begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}$$

**Summary:** The linear transformation $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a+1b \\ 1a+3b \end{bmatrix}$ has with respect to the standard basis the coordinate matrix $A$, but with respect to the other basis $\mathcal{B}$ the coordinate $B$:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

The linear transformation $T$ is geometrically clear in the $\mathcal{B}$ basis: $T$ is just stretching vectors by a factor 2 along $\mathbf{b_1}$ and by a factor 4 along $\mathbf{b_2}$. So using the standard basis $T$ is an obscure operation on vectors, but using the basis $\mathcal{B}$ it becomes clear. You can say that $\mathcal{B}$ is a basis adapted to $T$.

# 6 Additional Problems

- Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 7 & 8 & 1 \end{bmatrix}$. Find the dimensions and a basis for all four fundamental subspaces of $A$.

- Suppose $A$ is $5 \times 5$ and v is a vector in $\mathbb{R}^5$ which is not a linear combination of the columns of $A$. What can you say about the number of solutions to $A\mathbf{x} = \mathbf{0}$?

- Let $T$ be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

  What is $T\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right)$?