Poisson(λ) pmf $\quad \dfrac{e^{-\lambda}\lambda^x}{x!}$

1. $\Lambda(x) = \dfrac{L(\lambda_0;x)}{L(\lambda_1;x)} = \dfrac{e^{-\lambda_0}\lambda_0^x}{e^{-\lambda_1}\lambda_1^x} \le k$

$\iff x \ln\dfrac{\lambda_0}{\lambda_1} \le \ln\left(k\cdot\dfrac{e^{\lambda_0}}{e^{\lambda_1}}\right)$

$\iff X \le c = \ln\left(k\cdot\dfrac{e^{\lambda_0}}{e^{\lambda_1}}\right) \Big/ \ln\dfrac{\lambda_0}{\lambda_1}$

(a) $P_0(X_7 \le c) = 0.05 \Rightarrow c = 5$

Reject H_0 if $X_7 \le 5$

(b) $P_1(X_7 \le 5) = P(\text{Poisson}(7) \le 5) = 0.301$

2. (a) $\alpha = P_0(X_7 \le 6) = P(\text{Poisson}(10.5) \le 6) = 0.102$

(b) $\text{p-value} = P_0(X_7 \le 4) = P(\text{Poisson}(10.5) \le 4) = 0.021$

3. $\Lambda(t) = \dfrac{L(\lambda_0;t)}{L(\lambda_1;t)} = \dfrac{\frac{1}{\Gamma(4)}\frac{1}{(\frac{1}{\lambda_0})^4} t^{4-1} e^{-t/\frac{1}{\lambda_0}}}{\frac{1}{\Gamma(4)}\frac{1}{(\frac{1}{\lambda_1})^4} t^{4-1} e^{-t/\frac{1}{\lambda_1}}} \le k$

$\iff (\lambda_1 - \lambda_0)t \le \ln\left[k\cdot\left(\dfrac{\lambda_1}{\lambda_0}\right)^4\right]$

$\iff t \ge c = \dfrac{1}{\lambda_1-\lambda_0}\ln\left[k\cdot\left(\dfrac{\lambda_1}{\lambda_0}\right)^4\right]$

$0.05 = P_0(T_4 \ge c) = P_0(2T_4\cdot\lambda_0 \ge 2\lambda_0 c)$

$= P(\chi^2(8) \ge 3c)$

$\Rightarrow 3c = 15.51 \Rightarrow c = 5.17$

Reject H_0 if $T_4 \ge 5.17$

4. (a) $\alpha = P_0(T_4 \ge 5) = P_0(\text{Poisson}(5\cdot 1.5) \le 4-1)$

$= P(\text{Poisson}(7.5) \le 3)$

$= 0.059$

(b) Power $= P_1(T_4 \geq 5) = P(\text{Poisson}(5) \leq 3) = 0.265$

(c) p-value $= P_0(T_4 \geq 6) = P(\text{Poisson}(6 \cdot 1.5) \leq 3)$

$= P(\text{Poisson}(9) \leq 3)$

$= 0.021$

5. (a) $L(x|\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$

$h(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}$

$k(\theta|x) \propto L(x|\theta)h(\theta) \propto \theta^{x+a-1}(1-\theta)^{n-x+b-1}$

$\Rightarrow \theta|x \sim Beta(x+a, n-x+b)$

$Beta(x+135, n-x+135)$

(b) $E(\theta|x) = \frac{x+135}{n+270} = \frac{n}{n+270}\hat{\theta} + \frac{270}{n+270} \cdot \frac{1}{2}$ $\quad (\hat{\theta} = \frac{x}{n})$

pbeta

6. $\theta | x=255 \sim Beta(390, 380)$

$H_0: \theta > 0.5 \quad H_1: \theta < 0.5$

$P(\theta > 0.5 | x=255) = 0.6408$

$P(\theta < 0.5 | x=255) = 0.3592$

$\Rightarrow$ Accept H_0

qbeta

7. $\int_0^{\pi} k(\theta|x=255)\, d\theta = p$

$(\pi_{0.025}, \pi_{0.975}) = (0.4712, 0.5418)$

8. (a) $g(y_n|\theta) = n\left(\frac{y_n}{\theta}\right)^{n-1}\frac{1}{\theta} \qquad 0 < y_n < \theta$

$h(\theta) = \beta\alpha^\beta / \theta^{\beta+1} \qquad \alpha < \theta < \infty$

$k(\theta|y_n) \propto \theta^{-n} 1_{(0,\theta)}(y_n) \cdot \theta^{-\beta-1} \propto \theta^{-n-\beta-1} 1_{(0,\theta)}(y_n)$

if $y_n < \alpha \qquad \int_\alpha^\infty k(\theta|y_n)\, d\theta = \frac{1}{n+\beta}\alpha^{-n-\beta}$

if $y_n \geq \alpha \qquad \int_\alpha^\infty k(\theta|y_n)\, d\theta = \frac{1}{n+\beta} y_n^{-n-\beta}$

$$k(\theta|y_n) = \begin{cases} (n+\beta)\alpha^{n+\beta}\theta^{-n-\beta-1} & \alpha < \theta < \infty \quad \text{if } y_n < \alpha \\ (n+\beta)y_n^{n+\beta}\theta^{-n-\beta-1} & y_n < \theta < \infty \quad \text{if } y_n \geq \alpha \end{cases}$$

(b) $$\delta(y_n) = E(\Theta \mid y_n) = \begin{cases} \dfrac{(n+\beta)\alpha}{n+\beta-1} & \text{if } y_n < \alpha \\[2ex] \dfrac{(n+\beta)y_n}{n+\beta-1} & \text{if } y_n \geq \alpha \end{cases}$$

9. if $y_n < \alpha$

$$\int_{\alpha}^{\pi_{0.025}} k(\theta \mid y_n)\, d\theta = 0.025 \Rightarrow \pi_{0.025} = \frac{\alpha}{0.975^{\frac{1}{n+\beta}}}$$

$$\int_{\alpha}^{\pi_{0.975}} k(\theta \mid y_n)\, d\theta = 0.975 \Rightarrow \pi_{0.975} = \frac{\alpha}{0.025^{\frac{1}{n+\beta}}}$$

if $y_n \geq \alpha$

$$\int_{y_n}^{\pi_{0.025}} k(\theta \mid y_n)\, d\theta = 0.025 \Rightarrow \pi_{0.025} = \frac{y_n}{0.975^{\frac{1}{n+\beta}}}$$

$$\int_{y_n}^{\pi_{0.975}} k(\theta \mid y_n)\, d\theta = 0.975 \Rightarrow \pi_{0.975} = \frac{y_n}{0.025^{\frac{1}{n+\beta}}}$$

10. $$g(y \mid \theta) = \frac{y^{r-1} e^{-y \cdot \theta}}{\Gamma(r)\left(\frac{1}{\theta}\right)^r} \qquad y > 0$$

$$h(\theta) = \frac{\theta^{s-1} e^{-\theta \cdot \mu}}{\Gamma(s)\left(\frac{1}{\mu}\right)^s} \qquad \theta > 0$$

$$k(\theta \mid y) \propto e^{-(\mu+y)\theta} \cdot \theta^{r+s-1} \Rightarrow \Theta \mid Y=y \sim \text{Gamma}(\alpha = r+s, \lambda = y+\mu)$$

11. $$g_1(x) = \frac{L(x \mid \theta) h(\theta)}{k(\theta \mid x)} = \frac{\binom{n}{x}\theta^x (1-\theta)^{n-x} \cdot \theta^{a-1}(1-\theta)^{b-1} / B(a,b)}{\theta^{x+a-1} \cdot (1-\theta)^{n-x+b-1} / B(x+a, n-x+b)}$$

$$= \frac{\binom{n}{x} B(x+a, n-x+b)}{B(a,b)}$$

$$= \frac{\binom{n}{x} \cdot \frac{(x+a-1)!}{(a-1)!} \frac{(n-x+b-1)!}{(b-1)!}}{\frac{(n+a+b-1)!}{(a+b-1)!}}$$

$$= \frac{\frac{(x+a-1)!}{x!(a-1)!} \frac{(n-x+b-1)!}{(n-x)!(b-1)!}}{\frac{(n+a+b-1)!}{n!(a+b-1)!}}$$

$$= \frac{\binom{x+a-1}{x}\binom{n-x+b-1}{n-x}}{\binom{n+a+b-1}{n}}$$

$$B(x+a, n-x+b) = B(x+a-1, n-x+b) \cdot \frac{x+a-1}{n+a+b-1}$$

$$= \cdots$$

$$= B(a, n-x+b)\frac{x+a-1}{n+a+b-1} \cdots\cdots \frac{a}{n+a+b-x}$$

$$B(a, n-x+b) = B(a, n-x+b-1)\frac{n-x+b-1}{n-x+a+b-1}$$

$$= B(a,b)\frac{n-x+b-1}{n-x+a+b-1} \cdots\cdots \frac{b}{a+b}$$

12. $L(\underset{\sim}{x}|\theta) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(\sum_{i=1}^{n} -(x_i-\theta)^2/2\sigma^2\right)$

$$h(\theta) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-(\theta-\mu)^2/2\tau^2\right)$$

$$k(\theta|x) \propto \exp\left(-(\theta-\mu)^2/2\tau^2 + \sum_{i=1}^{n} -(x_i-\theta)^2/2\sigma^2\right)$$

$$\propto \exp\left[-\left(\frac{1}{2\tau^2} + \frac{n}{2\sigma^2}\right)\theta^2 + \left(\frac{\mu}{\tau^2} + \frac{\sum x_i}{\sigma^2}\right)\theta\right]$$

$\Rightarrow k(\theta|x) \sim N(\mu', \sigma'^2)$

$$\mu' = \frac{\frac{\mu}{\tau^2} + \frac{\sum x_i}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}} = \frac{\frac{\sigma^2\mu}{n\tau^2} + \frac{\sum x_i}{n}}{\frac{\sigma^2}{n\tau^2} + 1} = \frac{1}{1+\frac{\sigma^2}{n\tau^2}}\bar{X} + \frac{\frac{\sigma^2}{n\tau^2}}{1+\frac{\sigma^2}{n\tau^2}}\mu = \tilde{X}$$

$$\sigma'^2 = \frac{1}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}} = \frac{1}{1+\frac{\sigma^2}{n\tau^2}} \cdot \frac{\sigma^2}{n} = U$$