Examples for 12/01/15, Part 2

Two-Way ANOVA with Replication

A two-factor analysis of variance experiment was performed with $I = 3,\ J = 2,$ and $K = 4$ (a 3×2 factorial experiment with 4 replicates).

Factor A	Factor B	
	1	2
1	23	20
	18	16
	17	15
	20	19
2	26	30
	23	24
	20	29
	27	27
3	23	27
	21	19
	24	21
	16	23

a) Test at the 5% significance level to determine if factors A and B interact.

b) Test at the 5% significance level to determine if differences exist among the levels of factor A.

c) Test at the 5% significance level to determine if differences exist among the levels of factor B.

$$y_{ijk} = \bar{y}_{\bullet\bullet\bullet} + \left(\bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet}\right) + \left(\bar{y}_{\bullet j\bullet} - \bar{y}_{\bullet\bullet\bullet}\right)$$
$$+ \left(\bar{y}_{ij\bullet} - \bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet j\bullet} + \bar{y}_{\bullet\bullet\bullet}\right) + \left(y_{ijk} - \bar{y}_{ij\bullet}\right)$$

Two factors are said to **interact** if the difference between levels (treatment) of one factor depends on the level of the other factor.

(some combinations of levels of factors A and B result in higher responses and some result in lower)

Factors that do not interact are called additive.

$$y_{ijk} = \bar{y}_{\bullet\bullet\bullet} + \left(\bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet}\right) + \left(\bar{y}_{\bullet j\bullet} - \bar{y}_{\bullet\bullet\bullet}\right)$$
$$+ \left(\bar{y}_{ij\bullet} - \bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet j\bullet} + \bar{y}_{\bullet\bullet\bullet}\right) + \left(y_{ijk} - \bar{y}_{ij\bullet}\right)$$

$$\begin{bmatrix} 23 & 20 \\ 18 & 16 \\ 17 & 15 \\ 20 & 19 \\ \\ 26 & 30 \\ 23 & 24 \\ 20 & 29 \\ 27 & 27 \\ \\ 23 & 27 \\ 21 & 19 \\ 24 & 21 \\ 16 & 23 \end{bmatrix} = \begin{bmatrix} 22 & 22 \\ 22 & 22 \\ 22 & 22 \\ 22 & 22 \\ \\ 22 & 22 \\ 22 & 22 \\ 22 & 22 \\ 22 & 22 \\ \\ 22 & 22 \\ 22 & 22 \\ 22 & 22 \\ 22 & 22 \end{bmatrix} + \begin{bmatrix} -3.5 & -3.5 \\ -3.5 & -3.5 \\ -3.5 & -3.5 \\ -3.5 & -3.5 \\ \\ 3.75 & 3.75 \\ 3.75 & 3.75 \\ 3.75 & 3.75 \\ 3.75 & 3.75 \\ \\ -0.25 & -0.25 \\ -0.25 & -0.25 \\ -0.25 & -0.25 \\ -0.25 & -0.25 \end{bmatrix} + \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \\ -0.5 & 0.5 \\ -0.5 & 0.5 \\ \\ -0.5 & 0.5 \\ -0.5 & 0.5 \\ -0.5 & 0.5 \\ -0.5 & 0.5 \\ \\ -0.5 & 0.5 \\ -0.5 & 0.5 \\ -0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$+ \begin{bmatrix} 1.5 & -1.5 \\ 1.5 & -1.5 \\ 1.5 & -1.5 \\ 1.5 & -1.5 \\ \\ -1.25 & 1.25 \\ -1.25 & 1.25 \\ -1.25 & 1.25 \\ -1.25 & 1.25 \\ \\ -0.25 & 0.25 \\ -0.25 & 0.25 \\ -0.25 & 0.25 \\ -0.25 & 0.25 \end{bmatrix} + \begin{bmatrix} 3.5 & 2.5 \\ -1.5 & -1.5 \\ -2.5 & -2.5 \\ 0.5 & 1.5 \\ \\ 2 & 2.5 \\ -1 & -3.5 \\ -4 & 1.5 \\ 3 & -0.5 \\ \\ 2 & 4.5 \\ 0 & -3.5 \\ 3 & -1.5 \\ -5 & 0.5 \end{bmatrix}$$

SST $\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(y_{ijk}-\bar{y}_{\bullet\bullet\bullet}\right)^2$ $IJK-1$ d.f.

SSA $\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\bar{y}_{i\bullet\bullet}-\bar{y}_{\bullet\bullet\bullet}\right)^2 = JK\sum_{i=1}^{I}\left(\bar{y}_{i\bullet\bullet}-\bar{y}_{\bullet\bullet\bullet}\right)^2$ $I-1$ d.f.

SSB $\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\bar{y}_{\bullet j\bullet}-\bar{y}_{\bullet\bullet\bullet}\right)^2 = IK\sum_{j=1}^{J}\left(\bar{y}_{\bullet j\bullet}-\bar{y}_{\bullet\bullet\bullet}\right)^2$ $J-1$ d.f.

SSAB $\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\bar{y}_{ij\bullet}-\bar{y}_{i\bullet\bullet}-\bar{y}_{\bullet j\bullet}+\bar{y}_{\bullet\bullet\bullet}\right)^2$ $(I-1)(J-1)$ d.f.

$= K\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\bar{y}_{ij\bullet}-\bar{y}_{i\bullet\bullet}-\bar{y}_{\bullet j\bullet}+\bar{y}_{\bullet\bullet\bullet}\right)^2$ $= IJ-I-J+1$ d.f.

SSR $\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(y_{ijk}-\bar{y}_{ij\bullet}\right)^2$ $IJ(K-1)$ d.f.

ANOVA table:

Source	SS	DF	MS	F
Factor A	211	2	105.5	11.72222222
Factor B	6	1	6	0.666666667
Interaction	31	2	15.5	1.722222222
Residuals	162	18	9	
Total	410	23		

$\bar{y}_{11\bullet} = 19.5$ $\bar{y}_{12\bullet} = 17.5$ $\bar{y}_{1\bullet\bullet} = 18.5$

$\bar{y}_{21\bullet} = 24$ $\bar{y}_{22\bullet} = 27.5$ $\bar{y}_{2\bullet\bullet} = 25.75$

$\bar{y}_{31\bullet} = 21$ $\bar{y}_{32\bullet} = 22.5$ $\bar{y}_{3\bullet\bullet} = 21.75$

$\bar{y}_{\bullet 1\bullet} = 21.5$ $\bar{y}_{\bullet 2\bullet} = 22.5$ $\bar{y}_{\bullet\bullet\bullet} = 22$

$$\text{SSA} = JK \sum_{i=1}^{I} \left(\bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet} \right)^2 = 2 \times 4 \times [\, (18.5 - 22)^2 + (25.75 - 22)^2 + (21.75 - 22)^2 \,]$$

$$= 8 \times [\, 12.25 + 14.0625 + 0.0625 \,] = 211.$$

$$\text{SSB} = IK \sum_{j=1}^{J} \left(\bar{y}_{\bullet j\bullet} - \bar{y}_{\bullet\bullet\bullet} \right)^2 = 3 \times 4 \times [\, (21.5 - 22)^2 + (22.5 - 22)^2 \,]$$

$$= 12 \times [\, 0.25 + 0.25 \,] = 6.$$

$$\text{SSAB} = K \sum_{i=1}^{I} \sum_{j=1}^{J} \left(\bar{y}_{ij\bullet} - \bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet j\bullet} + \bar{y}_{\bullet\bullet\bullet} \right)^2$$

$$= 4 \times [\, (19.5 - 18.5 - 21.5 + 22)^2 + (17.5 - 18.5 - 22.5 + 22)^2$$

$$+ (24 - 25.75 - 21.5 + 22)^2 + (27.5 - 25.75 - 22.5 + 22)^2$$

$$+ (21 - 21.75 - 21.5 + 22)^2 + (22.5 - 21.75 - 22.5 + 22)^2 \,] = 31.$$

$$\text{SSR} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{ijk} - \bar{y}_{ij\bullet} \right)^2 = (23 - 19.5)^2 + (18 - 19.5)^2 + \ldots + (23 - 22.5)^2 = 162.$$

$$\text{SST} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{ijk} - \bar{y}_{\bullet\bullet\bullet} \right)^2 = (23 - 22)^2 + (18 - 22)^2 + \ldots + (23 - 22)^2 = 410.$$

```r
Y <- c(23,18,17,20,26,23,20,27,23,21,24,16,20,
       16,15,19,30,24,29,27,27,19,21,23)
A <- c(1,1,1,1,2,2,2,2,3,3,3,3,1,1,1,1,2,2,2,2,3,3,3,3)
B <- c(1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2)
fit <- lm(Y ~ factor(A) * factor(B))
summary(aov(fit))
```

```
##                     Df Sum Sq Mean Sq F value  Pr(>F)
## factor(A)            2    211   105.5  11.722 0.00055 ***
## factor(B)            1      6     6.0   0.667 0.42489
## factor(A):factor(B)  2     31    15.5   1.722 0.20683
## Residuals           18    162     9.0
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}, \qquad i = 1, 2, 3, \quad j = 1, 2, \quad k = 1, 2, 3, 4.$$

ε_{ijk} are independent $N(0, \sigma^2)$ random variables,

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \qquad \beta_1 + \beta_2 = 0,$$

$$(\alpha\beta)_{1j} + (\alpha\beta)_{2j} + (\alpha\beta)_{3j} = 0, \qquad j = 1, 2,$$

$$(\alpha\beta)_{i1} + (\alpha\beta)_{i2} = 0, \qquad i = 1, 2, 3.$$

a) $H_0: (\alpha\beta)_{11} = (\alpha\beta)_{12} = (\alpha\beta)_{21} = (\alpha\beta)_{22} = (\alpha\beta)_{31} = (\alpha\beta)_{32} = 0$

```
qf(0.95,2,18)
```

```
## [1] 3.554557
```

F = 1.7222. **Do NOT Reject H_0** Interaction A × B is NOT significant.

b) $H_0: \alpha_1 = \alpha_2 = \alpha_3 = 0$

```
qf(0.95,2,18)
```

```
## [1] 3.554557
```

F = 11.7222. **Reject H_0** Factor A IS significant.

c) $H_0: \beta_1 = \beta_2 = 0$

```
qf(0.95,1,18)
```

```
## [1] 4.413873
```

F = 0.6667. **Do NOT Reject H_0** Factor B is NOT significant.

Fitting an additive model:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}, \qquad i = 1, 2, 3, \quad j = 1, 2, \quad k = 1, 2, 3, 4.$$

ε_{ijk} are independent $N(0, \sigma^2)$ random variables,

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \qquad \beta_1 + \beta_2 = 0.$$

```
fit2 <- lm(Y ~ factor(A) + factor(B))
summary(aov(fit2))
```

```
##             Df Sum Sq Mean Sq F value   Pr(>F)
## factor(A)    2    211  105.50  10.933 0.000619 ***
## factor(B)    1      6    6.00   0.622 0.439641
## Residuals   20    193    9.65
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

= =

Recall Examples for April 10:

```
Time <- c(12,2,8,1,7,20,14,17,12,17,13,7,13,8,14,11,5,10,3,6)
A <- c(1,1,1,1,1,2,2,2,2,2,3,3,3,3,3,4,4,4,4,4)
B <- c(1,2,3,4,5,1,2,3,4,5,1,2,3,4,5,1,2,3,4,5)
results <- lm(Time ~ factor(A) + factor(B))
summary(aov(results))
```

```
##             Df Sum Sq Mean Sq F value   Pr(>F)
## factor(A)    3    310   103.3   51.67 3.91e-07 ***
## factor(B)    4    184    46.0   23.00 1.49e-05 ***
## Residuals   12     24     2.0
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
results2 <- lm(Time ~ factor(A) * factor(B))
summary(aov(results2))
```

```
##                     Df Sum Sq Mean Sq
## factor(A)            3    310   103.3
## factor(B)            4    184    46.0
## factor(A):factor(B) 12     24     2.0
```