Worksheet 13 for December 3rd and 8th

1. *Let* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. *Find a diagonal matrix D and an orthogonal matrix Q such that $A = QDQ^{-1}$.*

Solution. We have to find the eigenvalues and corresponding eigenspaces. We have:

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) - 1 = -\lambda(2-\lambda)$$

Hence, the eigenvalues of A are 0 and 2. For $\lambda = 0$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$.

For $\lambda = 2$:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{R2 \to R2 + R1, R1 \to -R1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$.

Columns of Q are (linearly independent) eigenvectors of A and D is the diagonal matrix with eigenvalues of A on the main diagonal in the appropriate order(corresonding to columns of P). Therefore:

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

□

2. *Find the limiting values of y_k and z_k (for $k \to \infty$) if*

$$y_{k+1} = .8y_k + .3z_k \qquad y_0 = 0$$
$$z_{k+1} = .2y_k + .7z_k \qquad z_0 = 5.$$

Also find formulas for y_k and z_k from $A^k = S\Lambda^k S^{-1}$.

Solution. We begin by diagonalizing the matrix:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

First we compute the eigenvalues:

$$p(\lambda) = \begin{vmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{vmatrix} (.8-\lambda)(.7-\lambda) - .06 = \lambda^2 - 1.5\lambda + .5 = (\lambda - 1)(\lambda - .5)$$

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Final Date: December 17 8-11AM, Conflict December 15, 8-11AM. You are allowed to take the conflict exam if you have more than two examination within 24 hours. Conflict sign up deadline: November 30

So we conclude the eigenvalues are $\lambda = .5, 1$.Next we find eigenvectors.($\lambda = 1$) We compute a basis of $\mathrm{Nul}(A - I)$:

$$\begin{bmatrix} -.2 & .3 \\ .2 & -.3 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -.2 & .3 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to -5R_1} \begin{bmatrix} 1 & -1.5 \\ 0 & 0 \end{bmatrix}$$

and so we conclude that the vector $\begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 1$.($\lambda = .5$) We compute a basis of $\mathrm{Nul}(A - .5I)$:

$$\begin{bmatrix} .3 & .3 \\ .2 & .2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - \frac{2}{3}R_1} \begin{bmatrix} .3 & .3 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{10}{3}R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and so we conclude that the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = .5$.Next, we must invert the matrix $S = \begin{bmatrix} 1.5 & -1 \\ 1 & 1 \end{bmatrix}$:

$$\begin{bmatrix} 1.5 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{2}{3}R_1} \begin{bmatrix} 1 & -\frac{2}{3} & \frac{2}{3} & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & -\frac{2}{3} & \frac{2}{3} & 0 \\ 0 & \frac{5}{3} & -\frac{2}{3} & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{3}{5}R_2} \begin{bmatrix} 1 & -\frac{2}{3} & \frac{2}{3} & 0 \\ 0 & 1 & -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \xrightarrow{R_1 \to R_1 + \frac{2}{3}R_2} \begin{bmatrix} 1 & 0 & \frac{2}{5} & \frac{2}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{3}{5} \end{bmatrix}.$$

Thus $S^{-1} = \begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix}$ and we have the factorization:

$$\underbrace{\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1.5 & -1 \\ 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix}}_{S^{-1}}$$

Next we compute explicit formulas for y_k, z_k by:

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = A^k \begin{bmatrix} 0 \\ 5 \end{bmatrix} = S\Lambda^k S^{-1} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 1.5 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^{-k} \end{bmatrix} \begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3(1 - 2^{-k}) \\ 2 + 3 \cdot 2^{-k} \end{bmatrix}$$

Take $k \to \infty$ in the above formulas for y_k, z_k, we get the limiting values:

$$\begin{bmatrix} y_\infty \\ z_\infty \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

□

3. *If* $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, *find* A^{100} *by diagonalizing* A.

Solution. We begin by computing the eigenvalues for A:

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$$

and so the eigenvalues are $\lambda = 1$ and $\lambda = 5$.Next we compute eigenvectors for these eigenvalues.($\lambda = 1$)

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{3}R_1, R_1 \to \frac{1}{3}R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

so we get the eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix} (\lambda = 5)$

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to -R_1} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

and so we get the eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ Next we need to invert the matrix $S = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$:

$$\begin{bmatrix} -1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to -R_1} \begin{bmatrix} 1 & -3 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & -3 & -1 & 0 \\ 0 & 4 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{4} R_2} \begin{bmatrix} 1 & -3 & -1 & 0 \\ 0 & 1 & 1/4 & 1/4 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 3R_2} \begin{bmatrix} 1 & 0 & -1/4 & 3/4 \\ 0 & 1 & 1/4 & 1/4 \end{bmatrix}$$

Thus we get the following factorization:

$$\underbrace{\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} -1/4 & 3/4 \\ 1/4 & 1/4 \end{bmatrix}}_{S^{-1}}$$

Finally, we compute:

$$A^{100} = S\Lambda^{100} S^{-1} = \frac{1}{4} \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5^{100} \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 + 3 \cdot 5^{100} & -3 + 3 \cdot 5^{100} \\ -1 + 5^{100} & 3 + 5^{100} \end{bmatrix} \qquad \square$$

4. *Decide for or against the positive definiteness of these matrices, and write out the corresponding $f = x^T Ax$:*

(a) $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$.

The determinant in (b) is zero; along what line is $f(x, y) = 0$?

Solution. (a) We want to know whether this matrix has a negative eigenvalue. The determinant of this matrix is -4, and also the determinant is always the product of the eigenvalues. Thus this matrix has a negative eigenvalue, and is not positive definite. In this case, $f(x, y) = x^2 + 6xy + 5y^2$.

(b) This matrix is not positive definite since the determinant is 0 (so zero is an eigenvalue). In this case, $f(x, y) = x^2 - 2xy + y^2$. Also, setting $f(x, y) = 0$, we get that $f(x, y) = (x - y)^2 = 0$ and so $x - y = 0$, i.e., $f(x, y) = 0$ along the line $y - x$.

(c) This matrix is positive definite since the determinant is positive $(= 10 - 9 = 1 > 0)$, so the eigenvalues both have the same sign. The trace (the sum of the eigenvalues) is $2 + 5 = 7 > 0$ which is positive, so both eigenvalues are positive. In this case, $f(x, y) = 2x^2 + 6xy + 10y^2$.

(d) This matrix is not positive definite since the determinant is positive $(= 8 - 4 = 4 > 0)$ but the trace is negative $(= -9 < 0)$. Thus both eigenvalues are negative. In this case, $f(x, y) = -x^2 + 4xy - 8y^2$.

$\square$