Name Solutions

Netid ____________________

- Show all your work in this exam booklet; your partial credit might depend on it.
- Put your final answers at the end of your work and mark them clearly.
- If the answer is a function, its support must be included.
- No credit will be given without supporting work.
- The exam is closed book and closed notes.
- You are allowed to use one 8½" x 11" sheet with notes.
- Turn in all scratch paper with your exam.

Academic Integrity

The University statement on your obligation to maintain academic integrity is:

If you engage in an act of academic dishonesty, you become liable to severe disciplinary action. Such acts include cheating; falsification or invention of information or citation in an academic endeavor; helping or attempting to help others commit academic infractions; plagiarism; offering bribes, favors, or threats; academic interference; computer related infractions; and failure to comply with research regulations.

Rule 33 of the Code of Policies and Regulations Applying to All Students gives complete details of rules governing academic integrity for all students. You are responsible for knowing and abiding by these rules.

1. Let $X_1, \ldots, X_n$ be iid such that $X_i \sim Gamma(\alpha = 3, \theta)$ with pdf

$$f(x;\theta) = \frac{1}{2}\frac{1}{\theta^3}x^2 e^{-\frac{x}{\theta}}, x > 0$$

and mean and variance,

$$E(X) = 3\theta, \qquad Var(X) = 3\theta^2.$$

a) Show that the method of moments estimator $\tilde{\theta}$, of θ is

$$\tilde{\theta} = \frac{\bar{X}_n}{3}.$$

MoM implies $E(X) = \bar{X} \Rightarrow 3\theta = \bar{X}$

$\Rightarrow \tilde{\theta} = \frac{\bar{X}}{3}$

b) Is $\tilde{\theta}$ an efficient estimator of θ?

$Var(\tilde{\theta}) = \frac{1}{9} Var(\bar{X}) = \frac{1}{9}\frac{3\theta^2}{n} = \frac{1}{3n}\theta^2$

Rao-Cramer Lower Bound (R.C.L.B.)

$\ln f(x;\theta) = -\ln 2 - 3\ln\theta + 2\ln x - x/\theta$

$\frac{\partial \ln f(x;\theta)}{\partial\theta} = \frac{-3}{\theta} + \frac{x}{\theta^2}$

$I(\theta) = Var\left(\frac{\partial \ln(f(x;\theta))}{\partial\theta}\right) = \frac{1}{\theta^4}Var(x) = \frac{3\theta^2}{\theta^4} = \frac{3}{\theta^2}$

$E(\tilde{\theta}) = k(\theta) = \frac{1}{3}E(\bar{x}) = \frac{1}{3}E(x) = \theta$

$\Rightarrow k'(\theta) = 1$

$R.C.L.B = \frac{1}{n I(\theta)} = \frac{\theta^2}{3n} = Var(\tilde{\theta})$, so $\tilde{\theta}$ is an efficient estimator of θ.

2. Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x;\lambda) = \frac{\lambda}{x^{\lambda+1}}, \qquad 1 < x < \infty, \qquad 2 < \lambda < \infty.$$

Note that,

$$E(X) = \frac{\lambda}{\lambda - 1}, \ Var(X) = \frac{\lambda}{(\lambda-2)(\lambda-1)^2}$$

Find the limiting distribution of the method of moments estimator

$$\tilde{\lambda} = \frac{\bar{X}_n}{\bar{X}_n - 1}.$$

Note

$$\sqrt{n}(\bar{X}_n - E(X)) \xrightarrow{D} N(0, Var(X))$$

$$g(x) = \frac{x}{x-1}, \quad g'(x) = \frac{(x-1) - x}{(x-1)^2} = -\frac{1}{(x-1)^2}$$

$$g(\bar{X}_n) = \tilde{\lambda}$$

$$g'(E(X)) = \frac{-1}{\left(\frac{\lambda}{\lambda-1} - 1\right)^2} = \frac{-1}{\left(\frac{\lambda - \lambda + 1}{\lambda - 1}\right)^2} = -(\lambda-1)^2$$

$$g(E(X)) = \frac{\frac{\lambda}{\lambda-1}}{\frac{\lambda}{\lambda-1} - 1} = \lambda$$

the Δ method implies

$$\sqrt{n}\left(g(\bar{X}_n) - g(EX)\right) \xrightarrow{D} N\left(0, [g'(E(X))]^2 Var(X)\right)$$

$$\Rightarrow \sqrt{n}(\tilde{\lambda} - \lambda) \xrightarrow{D} N\left(0, \frac{\lambda(\lambda-1)^2}{\lambda-2}\right)$$

$$\tilde{\lambda} \xrightarrow{D} N\left(\lambda, \frac{\lambda(\lambda-1)^2}{(\lambda-2)n}\right)$$

3. Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x;\lambda) = \frac{\lambda}{x^{\lambda+1}}, \qquad 1 < x < \infty, \qquad 2 < \lambda < \infty.$$

Find $I(\lambda)$.

$$\ln f(x;\lambda) = \ln \lambda - (\lambda+1)\ln x$$

$$\frac{\partial \ln f(x;\lambda)}{\partial \lambda} = \frac{1}{\lambda} - \ln x$$

(Note: need to know $\mathrm{Var}(\ln x)$ to use score function. Taking another derivative)

$$\frac{\partial^2 \ln f(x;\lambda)}{\partial \lambda^2} = -\frac{1}{\lambda^2}$$

$$\Rightarrow I(\lambda) = -E\left(-\frac{1}{\lambda^2}\right) = \frac{1}{\lambda^2}$$

4. Let $X_1, \dots, X_n$ be iid with pdf

$$f(x;\lambda) = \frac{\lambda}{x^{\lambda+1}}, \qquad 1 < x < \infty, \qquad 2 < \lambda < \infty.$$

a) Show that $Y_i = \ln(X_i) \sim Exponential(\lambda)$.

$$Y = \ln x \Rightarrow x = e^Y \Rightarrow \frac{dx}{dy} = e^Y$$

$$f(y;\lambda) = \frac{\lambda}{(e^Y)^{\lambda+1}} \left| e^Y \right| = \cancel{\lambda e^{(y+1)+}}$$

$$= \lambda e^{-Y(\lambda+1)} e^Y = \lambda e^{-\lambda Y}, \quad Y > 0$$

b) Show that the maximum likelihood estimator

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^n \ln x_i},$$

is a consistent estimator of λ.

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^n \ln x_i} = \frac{1}{\frac{1}{n}\sum \ln x_i} = \frac{1}{\overline{\ln x}} \xrightarrow{P} \frac{1}{E(\ln x)} = \frac{1}{1/\lambda} = \lambda$$

Note: $\ln x_i \sim Exp(\lambda) \Rightarrow E(\ln x) = 1/\lambda$

$\Rightarrow \cancel{\sum_{i=1}^n \ln x_i \sim Gamma(\alpha = n, \theta = 1/\lambda)}$

$\Rightarrow \cancel{\frac{1}{n}\sum_{i=1}^n \ln x_i \sim Gamma(\alpha = n}$

1) WLLN $\Rightarrow \bar{X}_n \xrightarrow{P} E(x)$

2) $X_n \xrightarrow{D} X$, and $g(\cdot)$ cont

$\Rightarrow g(x_n) \xrightarrow{D} g(x)$

In this case $g(x) = 1/x$, which is continuous for $\ln x > 0$.

5. Let $X_1, \dots, X_n$ be iid with pdf

$$f(x;\lambda) = \frac{\lambda^2}{x^3}\exp\left(-\frac{\lambda}{x}\right), \qquad 0 < y < \infty, \qquad 0 < \lambda < \infty.$$

Show that,

$$Y = \sum_i^n \frac{1}{X_i}$$

is a sufficient statistic for λ.

Factorization Theorem

$$L(\lambda; \underline{x}) = \prod_{i=1}^{n} \frac{\lambda^2}{x_i^3} e^{-\frac{\lambda}{x_i}} = \lambda^{2n} e^{-\lambda \sum \frac{1}{x_i}} \left(\prod 1/x_i\right)^3$$

$g(Y;\lambda) = \lambda^{2n} e^{-\lambda \sum 1/x_i}$, which depends upon

$x_1, \dots, x_n$ only through $\sum_{i=1}^{n} 1/x_i$, so $Y = \sum_{i=1}^{n} 1/x_i$

Exp Family Models

$$f(x;\lambda) = e^{2\ln\lambda} e^{-3\ln x} e^{-\lambda/x} = \exp\left[-\lambda \frac{1}{x} + 2\ln\lambda - 3\ln x\right]$$

Clearly $k(x) = 1/x$, so $Y = \sum_{i=1}^{n} \frac{1}{x_i} = \sum_{i=1}^{n} k(x_i)$

6. Let $X_1, \dots, X_n$ be iid with pdf

$$f(x;\theta) = \frac{1}{\theta}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{1}{2}\frac{x^2}{\theta^2}\right), \qquad 0 < x < \infty, \qquad 0 < \theta < \infty.$$

a) Show that the maximum likelihood estimator is,

$$\hat{\theta} = \sqrt{\frac{\sum_{i=1}^n x_i^2}{n}}$$

$$L(\theta;x) = \prod_{i=1}^{n} \frac{1}{\theta}\left(\frac{2}{\pi}\right)^{1/2} e^{-\frac{1}{2}\frac{x_i^2}{\theta^2}} = \frac{1}{\theta^n}\left(\frac{2}{\pi}\right)^{n/2} e^{-\frac{1}{2}\frac{1}{\theta^2}\sum x_i^2}$$

$$\ell(\theta;x) = -n\ln\theta + n/2 \ln(2/\pi) - \frac{1}{2}\frac{1}{\theta^2}\sum x_i^2$$

$$\ell'(\theta;x) = \frac{-n}{\theta} + \frac{1}{\theta^3}\sum x_i^2 = 0 \quad \Rightarrow \quad \theta^2 = \frac{\sum x_i^2}{n}$$

$$\Rightarrow \hat{\theta} = \sqrt{\sum x_i^2 / n}$$

b) Find the likelihood ratio statistic Λ for, $H_0: \theta = 1 \quad vs. \quad H_1: \theta \neq 1$.

$$\Lambda = \frac{L(\theta=1;x)}{L(\hat{\theta};x)} = \frac{\frac{1}{1^n}\left(\frac{2}{\pi}\right)^{n/2} e^{-\frac{1}{2}\frac{1}{1^2}\sum x_i^2}}{\frac{1}{\hat{\theta}^n}\left(\frac{2}{\pi}\right)^{n/2} e^{-\frac{1}{2}n}} = \hat{\theta}^n e^{n/2} e^{-\frac{1}{2}\sum x_i^2}$$

$$= \left(\frac{\sum x_i^2}{n}\right)^{n/2} e^{n/2} e^{-\frac{1}{2}\sum x_i^2}$$

Let $Y = \sum x_i^2$ be the sufficient statistic

$$\Lambda = \left(\frac{Y}{n}\right)^{n/2} e^{n/2} e^{-\frac{nY}{2}}$$