Math 415 - Lecture 8

Inverses.

Wednesday September 11th 2015

Textbook: Chapter 1.6

Suggested Practice Exercise: Chapter 1.6 Exercise 1, 2, 4, 6, 10, 11, 18, 35, 36, 37, 38, 40, 49, 50

Khan Academy Video: Inverse Matrix (part I), Inverse Matrix (part II)

1 Review

- Elementary matrices perform row operations:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ -2a+d & -2b+e & -2c+f \\ g & h & i \end{bmatrix}$$

- Gaussian elimination on A gives a decomposition $A = LU$:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

  U is the echelon form, L records the reverse of the row operations we did.

- LU decomposition lets us solve $A\mathbf{x} = \mathbf{b}$ quickly for many different $\mathbf{b}$.

1.1 Today's goal

- We know how to reverse a single row operation:

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Inverting a more complicated matrix is harder:

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & b & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab & -b & 1 \end{bmatrix}$$

  Goal today: how to find an "inverse" to any (square!) matrix. *Today A will be an $n \times n$ matrix*

2 The inverse of a matrix

The inverse of a real number a is denoted by a^{-1}. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1.$$

Remember that the identity matrix I_n is the $n \times n$-matrix

Definition. An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C satisfying

where I_n is the $n \times n$ identity matrix. We call C the inverse of A .

Example 1. We already know that an elementary matrix is invertible:

$$\begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In fact:

$$\begin{bmatrix}1&0&0\\8&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-8&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\-8&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\8&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Theorem 1. *Let A be an invertible matrix, then its inverse C is unique.*

Proof. Assume B and C are both inverses of A. Then

□

- We will write A^{-1} for the inverse of A. Multiplying by A^{-1} is like "dividing by A."
- Do not write $\frac{A}{B}$. Why?
- Fact: if $AB = I$ then $A^{-1} = B$ and so $BA = I$. (Not so easy to show at this stage.)

Remark. Not all $n \times n$ matrices are invertible. For example, the 2×2 matrix

is not invertible. Try to find an inverse!

Definition. A matrix which is *not* invertible is sometimes called a singular matrix. An invertible matrix is also called nonsingular matrix.

Theorem 2. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc \neq 0$, *then* A *is invertible and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, *then* A *is not invertible.*

Proof. Calculate

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc}$$

□

Theorem 3. *If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.*

Proof. The vector $A^{-1}\mathbf{b}$ is a solution, because

Suppose there is another solution $\mathbf{w}$, then

□

Example 2. Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$\begin{aligned} -7x_1 + 3x_2 &= 2 \\ 5x_1 - 2x_2 &= 1 \end{aligned}$$

Solution. Matrix form of the linear system:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} =$$

$$\mathbf{x} = A^{-1}\mathbf{b} =$$

3 Calculational rules

Theorem 4. *Suppose A and B are invertible. Then the following results hold:*

(a) A^{-1} *is invertible and* $\left(A^{-1}\right)^{-1} = A$ *(i.e. A is the inverse of A^{-1}).*

(b) AB *is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$

(c) A^T *is invertible and* $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

Proof.

$\square$

4 An algorithm for computing the inverse matrix

Idea:

- To solve $Ax = b$ we row reduce $[\ A \mid b\ ]$.
- To solve $AX = I_n$ we row reduce $[\ A \mid I\ ]$.

Theorem 5. *An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1}.*

So here is the algorithm:

- Place A and I side-by-side to form an augmented matrix $[\ A \mid I\ ]$. This is an $n \times 2n$ matrix (*B*ig *A*ugmented *M*atrix), instead of $n \times (n+1)$, for the usual augmented matrix.
- Then perform row operations on this matrix (which will produce identical operations on A and I).
- So by Theorem 5:

$$[\ A \mid I\ ] \text{ will row reduce to } \left[\ I \mid A^{-1}\ \right]$$

  or A is not invertible.

Example 3. Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$[A \quad I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$$

Example 4 (Let's do the previous example step by step.).

Remark. Why does this algorithm work?

- At each step, we get

$$[\ A \mid I\ ] \rightsquigarrow [\ E_1 A \mid E_1\ ] \rightsquigarrow [\ E_2 E_1 A \mid E_2 E_1\ ] \rightsquigarrow \dots$$

- So each step is of the form

$$[\ FA \mid F\ ], \quad F = E_r \dots E_3 E_2 E_1$$

- If we succeed in row reducing A to I, the final step is

$$[\ FA \mid F\ ] = [\ I \mid F\ ]$$

- So $FA = I$, which means that $A^{-1} = F$.