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1. Let X have the pdf

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, & x > 0 \\ 0, & otherwise \end{cases}$$

   where $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$ is defined for $t > 0$. Show that $E(X^k) = \Gamma(\alpha + k)/\Gamma(\alpha)$ for any $k > -\alpha$. Show that this can be written as $\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)$ if $k \geq 1$is an integer.

2. If a random variable X has the moment generating function $(1-2t)^{-2}$ for $t < 1/2$ find $P(X > 7.779)$.
3. A student purchased a laptop computer at Joe's Discount Store. She also purchased the "Lucky 7" warranty plan that would replace the laptop at no cost if it needs 7 or more repairs in 3 years. Suppose the laptop requires repairs according to Poisson process with the average rate of one repair per 4 month.
   a. Find the probability that the laptop would not need to be replaced. That is, find the probability that the seventh time the laptop needs repair will be after 3 years, when the warranty expires.
   b. Find the probability that the seventh time the laptop needs repair will be during the second year of warranty.
4. Let three random variables X_1, X_2, and X_3 have a multivariate normal distribution with mean vector $\boldsymbol{\mu} = (1,2,3)$ and variance-covariance matrix:

$$\Sigma = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

   a. Let $Y = X_1 - X_2 + 2X_3$. Find $E(Y)$ and $Var(Y)$.
   b. Find $P(X_1 > X_2 + X_3 - 4)$.

5. Suppose $Y = \mathbf{1}(X > 0)$ and $X \sim N(\eta, 1)$ where $\mathbf{1}(A) = 1$ if A is true; $= 0$ if A is false. Show that $P(Y = 1) = \Phi(\eta)$ where $\Phi(\cdot)$ denotes the standard normal cdf.

6. Let $Z \sim N(0,1)$ and let S be a random sign independent of Z:

$$P(S = -1) = P(S = 1) = \frac{1}{2}$$

   a. Show that $A = SZ \sim N(0,1)$. Hint: use the law of iterated expectations to find $M_A(t)$.
   b. Determine whether or not $Z + A$ is normally distributed.

7. Suppose $\ln(X) \sim N(\mu, \sigma^2)$. Find $E(X)$, $E(X^2)$ and $Var(X)$. Hint: first consider the moment generating function of $Y = \ln(X)$.
8. Let $\mathbf{X} \sim N_n(\boldsymbol{\mu}, \mathbf{I}_n)$, an n-dimensional normal random vector, and let $\mathbf{a}' = (1,1,\dots,1)$ so $\mathbf{a}$ is an n dimensional column vector of 1's. Find the distribution of $\bar{X} = \mathbf{a}'\mathbf{X}/n$.
9. Let $\mathbf{X} \sim N_n(\boldsymbol{\mu}, (1-\rho)\mathbf{I}_n + \rho\mathbf{J}_n)$, where $0 < \rho < 1$ and $\mathbf{J}_n$ is an n by n matrix of 1's.
   a. Find $Var(X_i), i = 1,2,\dots,n$.
   b. Find $Correlation(X_i, X_j)$ for $1 \le i < j \le n$.
10. Let $\mathbf{X} \sim N_n(\boldsymbol{\mu}, (1-\rho)\mathbf{I}_n + \rho\mathbf{J}_n)$, where $0 < \rho < 1$ and $\mathbf{J}_n$ is an n by n matrix of 1's. Find the distribution of $\bar{X} = \mathbf{a}'\mathbf{X}/n$, where $\mathbf{a}' = (1,1,\dots,1)$.

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11. Suppose $X \sim N(0,1)$ and $\Phi(\cdot)$ denotes the standard normal cdf. Find $E[\Phi(a - bX)]$.
12. Show that the constant c can be selected so that $f(x) = c2^{-x^2}, -\infty < x < \infty$, satisfies the conditions of a normal pdf. (Hint: write $2 = e^{\ln(2)}$)
   a. Suppose X has the pdf above. Find $P(0.4 < X < 1.3)$.
   b. Evaluate $\int_{0.5}^{1.8} 2^{-x^2}\,dx$. Use a standard normal cumulative probability table. Do NOT use a calculator or Wolfram Mathematica.