Solutions for Problem Set 5
CS 373: Theory of Computation

Assigned: February 21, 2013   Due on: February 28, 2013

Problem 1. [Category: Proof] Let $C = \{1^k x \mid x \in \{0,1\}^*,\ k \geq 1,\ \text{and } x \text{ contains at most } k \text{ 1s}\}$. Using the pumping lemma, prove that C is not regular. **[10 points]**

Solution: Let (for contradiction) C be a regular language with p as the pumping lemma constant. Consider the string $w = 1^p 0 1^p$. Observe that $w \in C$. Let x, y, z be any division of w such that $w = xyz$, $|y| > 0$ and $|xy| \leq p$. Thus, we can assume that $x = 1^i$, $y = 1^j$, and $z = 1^k 0 1^p$, where $i + j + k = p$, and $j > 0$. Consider $w' = xy^0z = 1^{i+k}01^p$. Since w' has $< p$ 1s before the first 0, and so the number of 1s after the first 0 cannot be more that $p - 1$. But w' has p 1s after the first 0 and so $w' \notin C$. Thus, C does not satisfy the pumping lemma and hence is not regular. ∎

Problem 2. [Category: Comprehension+Design] Let $L = \mathbf{L}(1^*0(00 \cup 01 \cup 1)(0 \cup 1)^*)$.

1. List all the equivalence classes of $\equiv_L$. Prove that your answer is correct. **[5 points]**
2. Draw the minimum state DFA M^L accepting L. **[5 points]**

Solution:

1. There are 4 equivalence classes for $\equiv_L$.

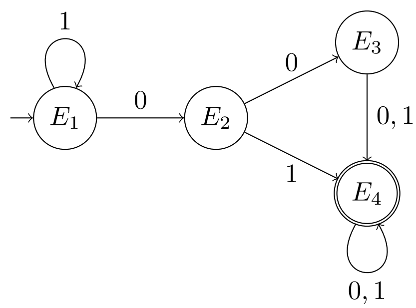
$$\begin{aligned} E_1 &= \mathbf{L}(1^*) \\ E_2 &= \mathbf{L}(1^*0) \\ E_3 &= \mathbf{L}(1^*00) \\ E_4 &= \mathbf{L}(1^*0(00 \cup 01 \cup 1)(0 \cup 1)^*) \end{aligned}$$

Observe that $E_1 \cup E_2 \cup E_3 \cup E_4 = \{0,1\}^*$ and for any $i, j \in \{1,2,3,4\}$ with $i \neq j$, $E_i \cap E_j = \emptyset$. Thus, if we prove that each E_i is an equivalence class, then this is an exhaustive list of all the equivalence classes. Note, that an equivalence class is a set E such that for every $x, y \in E$, we have $x \equiv_L y$, and for every $z \notin E$, $x \not\equiv_L z$. So this is what we will prove for each E_i.

- **Case** E_1**:** Let $x, y \in E_1$. Without loss of generality, we can take $x = 1^i$ and $y = 1^j$ for $i, j \geq 0$. Now $xw = 1^i w \in L$ iff $w \in \mathbf{L}(1^*0(00 \cup 01 \cup 1)(1 \cup 0)^*)$ iff $1^j w = yw \in L$. Now consider $z \notin E_1$; we will consider each case depending on whether z belongs to E_2, E_3 or E_4. If $z \in E_2$, then $z1 \in L$ but $x1, y1 \in \mathbf{L}(11^*)$ are not in L. If $z \in E_3$ then again $z1 \in L$ and $x1, y1$ are not in L. Finally, if $z \in E_4$ then $z\epsilon \in L$ but $x, y \notin L$.
- **Case** E_2**:** Let $x, y \in E_2$. Without loss of generality, we can take $x = 1^i0$ and $y = 1^j0$ for $i, j \geq 0$. Now $xw = 1^i0w \in L$ iff $w \in \mathbf{L}((00 \cup 01 \cup 1)(1 \cup 0)^*)$ iff $1^j0w = yw \in L$. Now consider $z \notin E_2$; we don't need to consider the case of $z \in E_1$ because we already established that in the previous case. If $z \in E_3$ then again $z0 \in L$ and $x0, y0$ are not in L. Finally, if $z \in E_4$ then $z\epsilon \in L$ but $x, y \notin L$.
- **Case** E_3**:** Let $x, y \in E_3$. Without loss of generality, we can take $x = 1^i00$ and $y = 1^j00$ for $i, j \geq 0$. Now $xw = 1^i00w \in L$ iff $w \in \mathbf{L}((0 \cup 1)(1 \cup 0)^*)$ iff $1^j00w = yw \in L$. Now consider $z \notin E_2$; we don't need to consider the case of $z \in E_1 \cup E_2$ because we already established that in the previous parts. If $z \in E_4$ then again $z\epsilon \in L$ and $x, y \notin L$.

- **Case** E_4**:** Let $x, y \in E_4$. Now $xw \in L$ iff $w \in \mathbf{L}((1 \cup 0)^*)$ iff $yw \in L$. The previous parts have already established that if $z \notin E_4$ then $x \not\equiv_L z$.

2. The automaton M^L has as states the equivalence classes E_1, E_2, E_3, E_4. Since $\epsilon \in E_1$, E_1 is the initial state, and since $E_4 = L$, E_4 is the only final state. The automaton can be drawn as follows.



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Problem 3. [Category: Comprehension+Proof] For a language $L \subseteq \Sigma^*$, define an equivalence $\simeq_L$ on Σ^* as follows

$$x \simeq_L y \text{ iff } \forall z.\ zx \in L \leftrightarrow zy \in L$$

Notice that this is a slightly different equivalence than $\equiv_L$ defined in Lecture 11. Prove that L is regular iff $\simeq_L$ has finitely many equivalence classes. *Hint:* Can you see a connection between $\simeq_L$ and $\equiv_{L^R}$, where L^R refers to the reverse of L? **[10 points]**

Solution: For a string $x \in \Sigma^*$, let $x^R \in \Sigma^*$ denote the reverse of string x. For a language $L \subseteq \Sigma^*$, let $L^R = \{x^R \mid x \in L\}$, be the "reverse" of language L. The crux of the proof is the following observation: $x \simeq_L y$ iff $x^R \equiv_L y^R$. This can be shown as follows.

$$\begin{array}{rll} x \simeq_L y & \text{iff} & \forall z.zx \in L \leftrightarrow zy \in L \\ & \text{iff} & \forall z.(zx)^R \in L^R \leftrightarrow (zy)^R \in L^R \\ & \text{iff} & \forall z^R.x^R z^R \in L^R \leftrightarrow y^R z^R \in L^R \\ & \text{iff} & x^R \equiv_{L^R} y^R \end{array}$$

Now this means that $[x]_{\simeq_L} = ([x^R]_{\equiv_{L^R}})^R$. Thus, $\#(\simeq_L) = \#(\equiv_{L^R})$.

We can use the Myhill-Nerode theorem and previous observations about closure properties to complete the proof. Observe that $(L^R)^R = L$, and we showed in Homework 4 Problem 2, that if L is regular L^R is regular. Therefore, we have L is regular if and only if L^R is regular. The following sequence of observations then completes the proof. L is regular if and only if L^R is regular if and only if $\equiv_{L^R}$ has finitely many equivalence classes (Myhill-Neorde theorem) if and only if $\simeq_L$ has finitely many equivalence classes (since $\#(\simeq_L) = \#(\equiv_{L^R})$). ■