Math 415 - Lecture 17

Linear Transformations

Monday October 5th 2015

Textbook reading: Chapter 2.6

Suggested practice exercises: same as lecture 16

1 Review

- A map $T : V \to W$ between vector spaces is **linear** if
  - $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$.
  - $T(c\mathbf{x}) = cT(\mathbf{x})$.
- If $\mathbf{x_1}, \dots, \mathbf{x_n}$ is a basis for V, then T is determined by the values $T(\mathbf{x_1}), \dots, T(\mathbf{x_n})$:

$$T(\mathbf{v}) = T(c_1\mathbf{x_1} + \cdots + c_n\mathbf{x_n}) = c_1T(\mathbf{x_1}) + \cdots + c_nT(\mathbf{x_n}).$$

- Let A be an $m \times n$ matrix.
  - $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is linear.
  - *Every* linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is of the form $T(\mathbf{x}) = A\mathbf{x}$.
- $T : \mathbb{P}_n \to \mathbb{P}_{n-1}$ defined by $T(p(t)) = p'(t)$ is linear. What is its "matrix"?

2 Nonstandard Bases

Until now we have used the standard bases to describe $T : \mathbb{R}^n \to \mathbb{R}^m$. Often it is useful to use other bases.

Theorem 1 (Linear Transformation is Matrix Multiplication)**.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\mathcal{B} := (\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ be a basis of $\mathbb{R}^n$ and let $\mathcal{C} := (\boldsymbol{w}_1, \dots, \boldsymbol{w}_m)$ be a basis of $\mathbb{R}^m$. Then there is a matrix B such that*

$$T(\mathbf{x})_{\mathcal{C}} = B\mathbf{x}_{\mathcal{B}}, \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n.$$

Explicitly,

$$B = \begin{bmatrix} T(\boldsymbol{v}_1)_{\mathcal{C}} & \dots & T(\boldsymbol{v}_n)_{\mathcal{C}}. \end{bmatrix},$$

Example 1. Let $T\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}3a+1b\\1a+3b\end{bmatrix}$. Then the matrix of T is $A=\begin{bmatrix}3&1\\1&3\end{bmatrix}$. But let us use, instead of the standard basis, another basis adapted to T. Put

$$\mathbf{b_1}=\begin{bmatrix}1\\-1\end{bmatrix},\quad \mathbf{b_2}=\begin{bmatrix}1\\1\end{bmatrix}.$$

What is the coordinate matrix for T with respect to $\mathcal{B}=(\mathbf{b_1},\mathbf{b_2})$?

Solution. What do we want? We want to find a matrix B that relates the coordinate vectors (w.r.t. basis $\mathcal{B}$) of input vector $\mathbf{x}$ and and output vector $T(x)$:

$$T(x)_{\mathcal{B}}=Bx_{\mathcal{B}}.$$

This matrix B has columns $T(\mathbf{b_1})_{\mathcal{B}}$ and $T(\mathbf{b_2})_{\mathcal{B}}$. So let us calculate

$$T(\mathbf{b_1})=T\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}3&1\\1&3\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=2\begin{bmatrix}1\\-1\end{bmatrix}=2\mathbf{b_1},$$

$$T(\mathbf{b_2})=T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3&1\\1&3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=4\begin{bmatrix}1\\1\end{bmatrix}=4\mathbf{b_2}$$

This means that the coordinate matrix with respect to $\mathcal{B}$ is simply

$$B=\begin{bmatrix}2&0\\0&4\end{bmatrix}$$

Summary: The linear transformation $T\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}3a+1b\\1a+3b\end{bmatrix}$ has with respect to the standard basis the coordinate matrix A, but with respect to the other basis $\mathcal{B}$ the coordinate B:

$$A=\begin{bmatrix}3&1\\1&3\end{bmatrix},\quad B=\begin{bmatrix}2&0\\0&4\end{bmatrix}$$

The linear transformation T is geometrically clear in the $\mathcal{B}$ basis: T is just stretching vectors by a factor 2 along $\mathbf{b_1}$ and by a factor 4 along $\mathbf{b_2}$. So using the standard basis T is an obscure operation on vectors, but using the basis $\mathcal{B}$ it becomes clear. You can say that $\mathcal{B}$ is a basis adapted to T.

3 Matrices for... Polynomials?

Let P_n be the vector space of polynomials of degree at most n.

Example 2. Consider the map $T:P_2\to P_1$ given by

$$T(p(t))=\frac{d}{dt}p(t).$$

Describe T by a matrix.

Solution. Wait, what?! We can't multiply a polynomial by a matrix! Use coordinate vectors instead.

Pick bases $\mathcal{A} = (1, t, t^2)$ for P_2 and $\mathcal{B} = (1, t)$ for P_1. Find a matrix D that does to the coordinate vectors what T does to the polynomials.

$$T(2 + 3t + 4t^2) = 3 + 8t$$

$$D \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$T(t^2) = 2t$$

$$D \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Formally,

$$D \cdot (f_{\mathcal{A}}) = T(f)_{\mathcal{B}}$$

From the equation

$$D \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The third column of D is $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$. What are the remaining two columns?

$$T(1) = 0 \implies D \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$T(t) = 1 \implies D \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$T(t^2) = 2t \implies D \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Hence $D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Check Take $f(t) = 2 - t + 3t^2$. Then the coordinate vector for $f(t)$ is

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Then

$$D\begin{bmatrix}2\\-1\\3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}2\\-1\\3\end{bmatrix} = \begin{bmatrix}-1\\6\end{bmatrix}.$$

On the other hand $T(f(t)) = f'(t) = -1 + 6t$, with coordinate vector $\begin{bmatrix}-1\\6\end{bmatrix}$.

4 Matrices for Linear Transformations

Let's organize this. Let $T : V \to W$ be a linear transformation, $\mathcal{A} = \{\mathbf{x_1}, \dots, \mathbf{x_n}\}$ be an *input basis* for V, and $\mathcal{B} = \{\mathbf{y_1}, \dots, \mathbf{y_m}\}$ an *output basis* for W. Each vector in V has a coordinate vector in $\mathbb{R}^n$, each vector in W has a coordinate vector in $\mathbb{R}^m$. T now corresponds to a matrix from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$\begin{tikzcd}
\text{vector in } V \arrow[r, "\text{apply } T"] \arrow[d, "\text{write in coordinates}"'] & \text{vector in } W \arrow[d, "\text{write in coordinates}"] \\
\text{coordinate vector in } \mathbb{R}^n \arrow[r, "\text{multiply by } A"] & \text{coordinate vector in } \mathbb{R}^m
\end{tikzcd}$$

In the last example this was

$$T(2 + 3t + 4t^2) = 3 + 8t$$

$$A\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}3\\8\end{bmatrix}$$

Definition. Let $\mathcal{A} = \{\mathbf{x_1}, \dots, \mathbf{x_n}\}$ be a basis for V, and $\mathcal{B} = \{\mathbf{y_1}, \dots, \mathbf{y_m}\}$ a basis for W. The matrix $T_{\mathcal{B}\mathcal{A}}$ representing T with respect to these bases

- has n columns (one for each of the $\mathbf{x_j}$),
- the j-th column is the coordinate vector of $T(\mathbf{x_j})$ in the basis $\mathcal{B}$.

$$T_{\mathcal{B}\mathcal{A}} = \begin{bmatrix} T(\mathbf{x_1})_{\mathcal{B}} & T(\mathbf{x_2})_{\mathcal{B}} & \dots & T(\mathbf{x_n})_{\mathcal{B}} \end{bmatrix}$$

Example 3. Give the matrix for $T : P_2 \to P_1$ given by

$$T(p(t)) = \frac{d}{dt}p(t).$$

in the bases $\mathcal{A} = (1, t, t^2)$ and $\mathcal{B} = (1, t)$.

Solution.

$$T_{\mathcal{B}\mathcal{A}} = \begin{bmatrix} T(1)_{\mathcal{B}} & T(t)_{\mathcal{B}} & T(t^2)_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 2\end{bmatrix}$$

Example 4. Recall the map T given by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$. (It reflects every vector in $\mathbb{R}^2$ across the line $y = x$.)

(a) Which matrix A represents T with respect to the standard bases?

(b) Which matrix B represents T with respect to the basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$?

Solution. (a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So $A = \begin{bmatrix} 0 & * \\ 1 & * \end{bmatrix}$. $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So $B = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$. $T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Draw a picture!

Remark. If a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is represented by the matrix A with respect to the standard bases, then $T(\mathbf{x}) = A\mathbf{x}$. Matrix multiplication corresponds to function composition! That is, if T_1, T_2 are represented by A_1, A_2, then $T_1(T_2(\mathbf{x})) = (A_1 A_2)\mathbf{x}$.

Example 5. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}.$$

What is the matrix B representing T with respect to the following bases?

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ for } \mathbb{R}^2, \ \mathbf{y_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{y_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{y_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ for } \mathbb{R}^3.$$

Solution.

$$\begin{aligned}
T(\mathbf{x_1}) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) &= T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 10 \end{bmatrix} \\
&= 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\implies B &= \begin{bmatrix} 5 & * \\ -3 & * \\ 5 & * \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
T(\mathbf{x_2}) = T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) &= -T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\
&= -\begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}4\\0\\7\end{bmatrix} = \begin{bmatrix}7\\-2\\11\end{bmatrix} \\
&= 7\begin{bmatrix}1\\1\\1\end{bmatrix} - 9\begin{bmatrix}0\\1\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\end{bmatrix} \\
\implies B &= \begin{bmatrix}5 & 7\\-3 & -9\\5 & 4\end{bmatrix}
\end{aligned}
$$

Remark. A matrix representing T encodes in column j the coefficients of $T(\mathbf{x_j})$ expressed as a linear combination of $\mathbf{y_1}, \dots, \mathbf{y_m}$.

5 Recap

What is the Point? Why write $T\colon V \to W$ as a matrix?

- Replace obscure computations in V and W by transparent computations with matrices.
- Even if $T : \mathbb{R}^n \to \mathbb{R}^m$ (already have standard coordinates), T may be simpler in a different coordinate system.

Summary: Given $\mathbf{v}$ in V, want to calculate $T(\mathbf{v})$ in W. Take an input basis $\mathcal{A} = (\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_n})$ and and output basis $\mathcal{B} = (\mathbf{y_1}, \mathbf{y_2}, \dots, \mathbf{y_m})$.

- We know $\mathbf{v}$ if we know the coordinate vector $\mathbf{v}_{\mathcal{A}}$.
- We know $T(\mathbf{v})$ if we know the coordinate vector $T(\mathbf{v})_{\mathcal{B}}$.
- So we know T if we know the matrix $T_{\mathcal{B}\mathcal{A}}$:

$$T(\mathbf{v})_{\mathcal{B}} = T_{\mathcal{B}\mathcal{A}}\mathbf{v}_{\mathcal{A}}.$$

[-.5cm]The output coordinate vector equals the matrix for T times the input coordinate vector.

Example 6. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let T be the linear map such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\0\\7\end{bmatrix}.$$

What is the matrix A representing T with respect to the standard bases? Use that to calculate $T\begin{bmatrix}2\\3\end{bmatrix}$.

Solution. The standard bases are

$$\mathbf{x_1} = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{x_2} = \begin{bmatrix}0\\1\end{bmatrix}, \quad \mathbf{y_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{y_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{y_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{aligned}
T(\mathbf{x_1}) = \begin{bmatrix}1\\2\\3\end{bmatrix} &= 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix} \\
&= 1\mathbf{y_1} + 2\mathbf{y_2} + 3\mathbf{y_3} \\
T(\mathbf{x_2}) = \begin{bmatrix}4\\0\\7\end{bmatrix} &= 4\mathbf{y_1} + 0\mathbf{y_2} + 7\mathbf{y_3} \\
\implies A &= \begin{bmatrix}1 & 4\\2 & 0\\3 & 7\end{bmatrix}
\end{aligned}$$

So $T\begin{bmatrix}2\\3\end{bmatrix} = A\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}14\\4\\27\end{bmatrix}$

6 Additional Problems

- Suppose $A = \begin{bmatrix}1 & 2 & 3 & 4 & 1\\2 & 4 & 7 & 8 & 1\end{bmatrix}$. Find the dimensions and a basis for all four fundamental subspaces of A.

- Suppose A is 5×5 and $\mathbf{v}$ is a vector in $\mathbb{R}^5$ which is not a linear combination of the columns of A. What can you say about the number of solutions to $A\mathbf{x} = \mathbf{0}$?

- Let T be the linear map such that

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\4\end{bmatrix}, \quad T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-2\\0\end{bmatrix}.$$

  What is $T\left(\begin{bmatrix}0\\4\end{bmatrix}\right)$?