Math 415 - Lecture 5

Matrices and Linear Systems

Wednesday September 2nd 2015

Textbook: Chapter 1.4

Suggested Practice Exercise: Chapter 1.4 Exercise 1, 2, 10, 12, 13, 21, 30, 34, 45,

Khan Academy Video: Matrix multiplication (part I), Matrix multiplication (part II), Defined and undefined matrix operations

Matrix operations

Review Matrix Multiplication

Motto 1
A matrix is a machine.

A is a $m \times n$ matrix. So n columns, m rows. How is it a machine?

- Input: n-component vector $x \in \mathbb{R}^n$.
- Output: m-component vector $b = Ax \in \mathbb{R}^m$.

How defined?

Motto 2
Matrix Multiplication is Linear Combination.

$$Ax = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n, \quad \text{if } A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a_n} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Problem 1. *Consider the linear combination*

$$3\begin{bmatrix}1\\2\end{bmatrix} - 1\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} = b.$$

Write the linear combination b as a matrix multiplication $b = Ax$. What can you take for A, x?

Solution. $A = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$ and $x = \begin{bmatrix}3\\-1\end{bmatrix}$.

We know:

- Solving a Linear System is finding Linear Combinations.
- Linear Combination is matrix multiplication.

So there must be a relation between linear system and matrix multiplications. Let's spell it out:

Theorem 1.

- *A solution $(x_1, x_2, \ldots, x_n)$ of system with augmented matrix $\left[\begin{array}{c|c} A & b \end{array}\right]$ corresponds to*
- *linear combination $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = b$, which corresponds to*
- *matrix multiplication $Ax = b$*

From now on we will write $Ax = b$ for the system of equations with augmented matrix $\left[\begin{array}{c|c} A & b \end{array}\right]$.

The most important property of the machine corresponding to a matrix A is that it *plays nice* with linear combinations.

Theorem 2. *Let A be a matrix, $\mathbf{x}, \mathbf{y}$ vectors and c, d scalars. If the input vector is a linear combination then also the output vector is a linear combination:*

$$A(c\mathbf{x} + d\mathbf{y}) = cA\mathbf{x} + dA\mathbf{y}.$$

To see this write both sides out! This property of matrix multiplication is called *Linearity.*

Assume we have a linear system $Ax = b$. Suppose x and y are two distinct solutions.

This means $Ax = b, \quad Ay = b$. Let us subtract:

$$Ax - Ay = b - b = 0.$$

The LHS is a linear combinations of outputs of A, so it is A applied to a linear combination:

$$A(x - y) = 0.$$

Define then the difference vector $z = x - y$, so that $Az = 0$. This is not zero because x and y are distinct. Then we can use the vector z to produce many solutions: choose a scalar c, and calculate again using linearity

$$A(x + cz) = Ax + cAz = b + c0 = b.$$

So we see that we get infinitely many new solutions $x + cz$, if we have found just two solutions.

We know how to multiply a matrix and a vector (of the right size!). Now we want to define matrix times matrix.

- Let B be $n \times p$: input $x \in \mathbb{R}^p$, output $c = Bx \in \mathbb{R}^n$.
- Let A be $m \times n$: input $y \in \mathbb{R}^n$, output $b = Ay \in \mathbb{R}^m$.

We want to define the product AB. Notice that the output of B can be the input of A. We can chain the machines B and A together.

Definition. The **machine** AB takes as input $x \in \mathbb{R}^p$ and produces as output $A(Bx) \in \mathbb{R}^n$.

So given two matrices A and B (of the right size) we defined a *machine* that we call AB.

Theorem 3. *The machine AB is in fact a matrix of size $m \times p$ given explicitly by*

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}.$$

Example 2. Previous example, again

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

Example 3. Compute AB where

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$$

$$\begin{aligned} A\mathbf{b}_1 &= 2 \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 6 \\ 0 \end{bmatrix} + \begin{bmatrix} -12 \\ -30 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix} \end{aligned}$$

$$A\mathbf{b}_1 = \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix}$$

$$\begin{aligned} A\mathbf{b}_2 &= -3 \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} - 7 \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -12 \\ -9 \\ 0 \end{bmatrix} + \begin{bmatrix} 14 \\ 35 \\ -7 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix} \end{aligned}$$

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2] = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

Note that $A\mathbf{b}_1$ is a linear combination of the columns of A and $A\mathbf{b}_2$ is a linear combination of the columns of A. Each column of AB is a linear combination of the columns of A using weights from the corresponding columns of B.

Example 4. If A is 4×3 and B is 3×2, then what are the sizes of AB and BA?

Solution.

- AB is 4×2,
- BA is not defined

Row-Column Rule for Computing AB

When A and B have small sizes, the following method is more efficient when working by hand.

Method. If AB is defined, let $(AB)_{ij}$ denote the entry in the ith row and jth column of AB. Then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$\begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix} = \begin{bmatrix} & & \\ & (AB)_{ij} & \\ & & \end{bmatrix}$$

If you know about dot products you see that every entry in the product AB is the dot product of a row vector (of A) and a column vector (of B).

Example 5. $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined.

Solution.

$$\begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$$

Motto
Matrices are like numbers.

Theorem 4. *Let A be $m \times n$ and B and C have sizes for which the indicated sums and products are defined.*

(a) $A(BC) = (AB)C$ *(associative law of multiplication)*

(b) $A(B + C) = AB + AC$, $(B + C)A = BA + CA$ *(distributive laws)*

(d) $r(AB) = (rA)B = A(rB)$ *for any scalar* r

(e) $I_m A = A = AI_n$ *(identity for matrix multiplication)*

Here $I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$ is the identity matrix af size n.

WARNING. Properties above are analogous to properties of real numbers. But **NOT ALL** real number properties correspond to matrix properties.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $AB \neq BA$, because

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Powers of A

We write: $A^k = A \cdots A$, k-times. For which matrices A does this make sense? If A is $m \times n$ what can m, n be?

Example 6.

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}^3 &= \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 9 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 21 & 8 \end{bmatrix} \end{aligned}$$

Calculating high powers of large matrix is hard. We will later learn a clever way to do this efficiently.