Worksheet 7 for October 13th and 15th

1. *Let* $A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.

(a) *Find an echelon form U of A. What are the column spaces* $\mathrm{Col}(A)$, $\mathrm{Col}(U)$*? Are they equal?*

(b) *Find a basis for* $\mathrm{Col}(U)$ *and a basis for* $\mathrm{Col}(A)$.

(c) *What are the row spaces* $\mathrm{Col}(A^T)$, *and* $\mathrm{Col}(U^T)$. *Are they equal?*

(d) *Find a basis for the row space of A,* $\mathrm{Col}(A^T)$.

2. *Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation with*

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(i) *Consider the basis* $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ *of* $\mathbb{R}^2$ *and the basis* $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ *of* $\mathbb{R}^3$. *Determine the matrix A which represents T with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$. Do you have $T(\boldsymbol{x}) = A\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^2$?*

(ii) *Consider the basis* $\mathcal{C}_1 := \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ *of* $\mathbb{R}^2$ *and the basis* $\mathcal{C}_2 = \left\{ \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ *of* $\mathbb{R}^3$. *Determine the matrix B which represents T with respect to the bases $\mathcal{C}_1$ and $\mathcal{C}_2$. Compute $T(\mathbf{v})$ where the coordinate vector of $\mathbf{v}$ with respect to the basis $\mathcal{C}_1$ is* $\mathbf{v}_{\mathcal{C}_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

3. *In this problem we consider the bases $\mathcal{B} = \{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ and $\mathcal{C} = \{1, t, t^2, t^3, t^4\}$ of $\mathbb{P}_4$.*

(a) *Let $I \colon \mathbb{P}_3 \to \mathbb{P}_4$ be the linear transformation that maps a polynomial $p(t)$ to the polynomial*

$$I(p(t)) := \int_0^t p(s)ds,$$

(e.g., $I(t^2 + 2t) = \int_0^t (s^2 + 2s)ds = [\frac{1}{3}s^3 + s^2]_0^t = \frac{1}{3}t^3 + t^2 \in \mathbb{P}_4$). Determine the matrix which represents I with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.

(b) *Let $J \colon \mathbb{P}^3 \to \mathbb{P}^4$ be the linear transformation that maps a polynomial $p(t)$ to the polynomial*

$$J(p(t)) := tp(t) + p'(t).$$

Determine the matrix which represents J with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Midterm Dates: September 29th, October 22nd, November 19th (All Midterms 7-8:15 PM, see `learn.illinois.edu` **for locations)**

4. *Let* $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *Find the length of* $\mathbf{v}$. *Find a vector* $\mathbf{u}$ *in the direction of* $\mathbf{v}$ *that has length* 1 . *Find a vector* $\mathbf{w}$ *that is orthogonal to* $\mathbf{v}$.

5. *True or False? Justify your answers.*

(a) *The map* $T\colon \mathbb{R}^2 \to \mathbb{R}$ *given by* $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \sqrt{a^2 + b^2}$ *is a linear transformation.*

(b) *The map* $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} -b \\ a \end{bmatrix}$ *is a linear transformation.*

(c) *If* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *are such that* $\mathbf{u} \cdot \mathbf{v} = 0$ *then* $\mathbf{u}$ *and* $\mathbf{v}$ *are perpendicular (geometrically) to each other. (Hint: Plot* $\mathbf{u}$ *and* $\mathbf{v}$ *as rays coming out of the origin, and the "hypotenuse"* $\mathbf{u} - \mathbf{v}$. *The Pythagorean theorem will hold if this is a right triangle.)*

(d) *Let* V *be a subspace of* $\mathbb{R}^n$ *and* $\mathbf{u}, \mathbf{v}$ *be two vectors in* V, *then* $\mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}$ *is orthogonal to* $\mathbf{u}$.

(e) *Let* $T : V \to W$ *be a linear transformation and* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *be vectors in* V. *If* $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ *are linearly independent then* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *are also linearly independent.*

(f) *Let* $T : V \to W$ *be a linear transformation and* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *be vectors in* V. *If* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *are linearly independent then* $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ *are also linearly independent.*

6. *Let* $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, *and* $\mathbf{v} = \begin{bmatrix}2\\3\end{bmatrix}$. *Find real numbers* c_1, c_2 *such that*

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$$

7. *Let* $B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

(a) *Find a basis for* $\mathrm{Nul}(B)$.
(b) *Find two linear independent vectors that are orthogonal to* $\mathrm{Nul}(B)$.
(c) *Is there a non-zero vector in* $\mathbb{R}^2$ *orthogonal to* $\mathrm{Col}(B)$*?*

8. *Consider the three vectors*

$$\mathbf{v}_1 = \begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}2\\2\\0\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}0\\0\\5\end{bmatrix}.$$

(a) *Check that* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *form an orthogonal set of vectors and conclude that they form a basis for* $\mathbb{R}^3$.
(b) *Construct an orthonormal basis* $\mathcal{B}$ *for* $\mathbb{R}^3$ *by normalizing the vectors in* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
(c) *Compute the coordinate vector* $[\mathbf{v}]_{\mathcal{B}}$ *for the following vectors (hint: use the fact that* $\mathcal{B}$ *is an orthonormal basis):*

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$$

The following may be useful in the above problems:

Definition. Two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are **orthogonal** if $\mathbf{v}^T\mathbf{w} = 0$ (where $\mathbf{v}^T$ is the transpose of $\mathbf{v}$ as an $n \times 1$ matrix).

Definition. A vector $\mathbf{v} \in \mathbb{R}^n$ is **orthogonal to a subspace** V of $\mathbb{R}^n$ if $\mathbf{v}$ is orthogonal to every $\mathbf{w} \in V$.