Math 415 - Lecture 15

The Four Fundamental Subspaces, the Fundamental Theorem of Linear Algebra,

Linear Transformations

Monday September 28th 2015

Textbook: Chapter 2.4, 2.6

Suggested Practice Exercise: Chapter 2.4 Exercise 1, 2, 3, 4, 7, 10, 18, 20, 21, 22, 27, 32, 37 Chapter 2.6 Exercise 5, 6, 7, 36, 37

Khan Academy Video: Linear Transformation, Linear Transformations as Matrix Vector Products, Linear Transformation Examples: Rotations in $\mathbb{R}^2$

Strang lectures: Lecture 9: Independence, Basis, and Dimension Lecture 10: The Four Fundamental Subspaces Lecture 30: Linear Transformations

1 Review

1.1 Basis for the Null Space

- $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a basis of V if the vectors span V and are independent.
- To find a basis for $Nul(A)$, solve $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 3 & 6 & 6 & 3 \\ 6 & 12 & 15 & 0 \end{bmatrix} \underset{RREF}{\longrightarrow} \begin{bmatrix} \boxed{1} & 2 & 0 & 5 \\ 0 & 0 & \boxed{1} & -2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} -2x_2 - 5x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

So a basis for $Nul(A)$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

1.2 Basis for the Column space.

- To find a basis for $Col(A)$, take the pivot columns of A.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{\text{-1}} & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So a basis for $Col(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

Question. Why do we take columns of A and not columns of the Echelon form?

Solution.

2 Rank and Dimensions

2.1 Dimension of Column and Null Space

Definition. The rank of a matrix A is the number of pivots it has.

Theorem 1. ***Rank-Nullity Theorem*** *Let A be an $m \times n$ matrix of rank r. Then*

$dim\ Col(A) = r$ Why?

$dim\ Nul(A) = n - r$ is the number of free variables of A. Why?

$dim\ Col(A) + dim\ Nul(A) = n$ Why?

3 The Four Fundamental Subpaces

Let A be a matrix. We already know two fundamental subspaces:

- The column space of A and
- The null space of A

There are two more!

Definition.
- The row space of A is the column space of A^T.
- The left null space of A is the null space of A^T.

Remark. Why is it called the "left" null space?

Example 1. Find a basis for $Col(A)$ and $Col(A^T)$ if

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

Solution.

3.1 Fundamental Theorem of Linear Algebra (Part 1)

Theorem 2. *Let A be an $m \times n$ matrix with rank r.*

- *dim $Col(A) = r$* *(subspace of $\mathbb{R}^m$)*
- *dim $Col(A^T) = r$* *(subspace of $\mathbb{R}^n$)*
- *dim $Nul(A) = n - r$* *(subspace of $\mathbb{R}^n$)*
- *dim $Nul(A^T) = m - r$* *(subspace of $\mathbb{R}^m$)*

Remark. The column and row space always have the same dimension. In other words, A and A^T have the same rank. (i.e. same number of pivots). Why?

Solution.

4 Coordinates

Definition. If $w \in V$ and $\mathcal{B} = (\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p})$ is a basis for V, the **coordinate vector** of w with respect to the basis $\mathcal{B}$ is

So w is a vector in some vector space, but it's coordinate vector is always a column vector in $\mathbb{R}^p$, if $\dim(V) = p$.

Example 2. Let $V = \mathbb{R}^2$, $\mathcal{B} = (\mathbf{b_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. What is the coordinate vector of $\mathbf{w}$?

Solution.

Example 3. Let $V = P_2$, the vector space of polynomials of the form $a_0 + a_1 t + a_2 t^2$. Let $\mathcal{B} = (\mathbf{b_1} = 1, \mathbf{b_2} = t, \mathbf{b_3} = t^2)$ be the obvious basis of P_2. Let $\mathbf{w} = 1 + 2t + 3t^2$. What is the coordinate vector of $\mathbf{w}$ with respect to basis $\mathcal{B}$?

Solution.

Example 4. Let $V = \mathbb{R}^3$ and let $E = (\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ be the standard basis. If $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ what is the coordinate vector with respect to the standard basis?

Solution.

5 Linear Transformations

Let V and W be vector spaces.

Definition. A map $T : V \to W$ is a linear transformation if

Remark. It follows immediately that

- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- $T(c\mathbf{x}) = cT(\mathbf{x})$

- $T(\mathbf{0}) = \mathbf{0}$ (because $T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}$)

Example 5. Let $V = \mathbb{R}, W = \mathbb{R}$. Then the map $f(x) = 3x$ is linear. Why?

Solution.

Example 6. Let A be an $m \times n$ matrix. Then the map $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. Why?

Solution.

Example 7. Let P_n be the vector space of all polynomials of degree at most n. Consider the map $T : P_n \to P_{n-1}$ given by

$$T(p(t)) = \frac{d}{dt} p(t).$$

This map is linear! Why?

Solution.