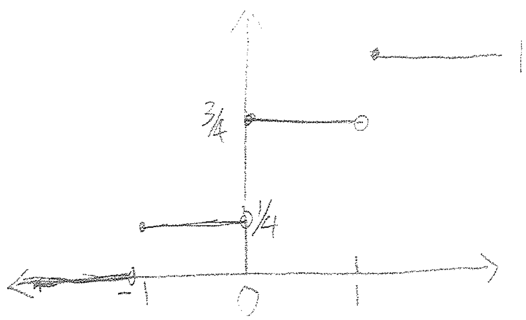
1.(a) $F(x) = P(X \le x) = \begin{cases} 0 & x < -1 \\ 1/4 & -1 \le x < 1 \\ 3/4 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$

(b)



(c) $E(X) = \sum x P(X=x)$

$= (-1)(\frac{1}{4}) + (0)(\frac{1}{2}) + (1)(\frac{1}{4}) = 0$

$V(X) = E(X^2) - [E(X)]^2$

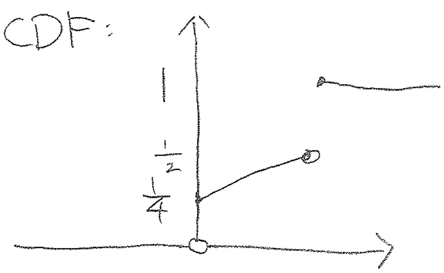
$= \sum x^2 p(X=x) - 0$

$= (-1)^2(\frac{1}{4}) + 0 + (1)^2(\frac{1}{4}) = \boxed{\frac{1}{2}}$

(d) $M_X(t) = \sum e^{tx} p(X=x) = e^{t(-1)} \cdot (\frac{1}{4}) + e^{t(0)} \cdot (\frac{1}{2}) + e^{t}(\frac{1}{4})$

$= \frac{1}{4}e^{-t} + \frac{1}{2} + \frac{1}{4}e^{t}$

2. CDF:



(a) $P(X=0) = \frac{1}{4}$, $P(X=1) = \frac{1}{2}$ } only this 2

$P(X=x) = 0$ for $x \in (0,1)$ since the distribution is cont.

(b) $P(-\frac{1}{2} < X \le \frac{1}{2}) = F(\frac{1}{2}) - F(-\frac{1}{2}) = \frac{\frac{1}{2}+1}{4} - 0 = \boxed{\frac{3}{8}}$

(c) $P(X > \frac{1}{2}) = 1 - P(X \le \frac{1}{2}) = 1 - F(\frac{1}{2}) = \boxed{\frac{5}{8}}$

(d) $E(X) = E_{Dis}(X) + E_{cont}(X)$ where

$f_X(x) = \begin{cases} \frac{1}{4} & x = 0 \\ 1/4 & 0 < x < 1 \\ 1/2 & x = 1 \\ 0 & \text{otherwise} \end{cases}$

$E_{Dis}(X) = (0)(\frac{1}{4}) + (1)(\frac{1}{2}) = \frac{1}{2}$

$E_{cont}(X) = \int x f_X(x)\, dx$

$= \int_0^1 \frac{x}{4}\, dx = \frac{1}{8}$

$E(X) = \frac{1}{2} + \frac{1}{8} = \boxed{\frac{5}{8}}$

3 (a) $f_X(x) = P(X=x) = \left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right)$

(b) $\sum\limits_{x=1}^{\infty} P(X=x) = \sum\limits_{x=0}^{\infty} \left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right) = \left(\frac{1}{6}\right)\sum\limits_{x=0}^{\infty}\left(\frac{5}{6}\right)^{x-1}$

$= \dfrac{1/6}{1-(5/6)} = 1$

(c) $\sum\limits_{t=1}^{x}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{t-1} = \frac{1}{6}\sum\limits_{t=1}^{x}\left(\frac{5}{6}\right)^{t-1} = \frac{1}{6}\,\dfrac{1-(5/6)^x}{1-5/6} = 1-(5/6)^x$

(d) $\sum\limits_{x=0}^{\infty} e^{tx}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{x-1} = \left(\frac{1}{6}e^t\right)\sum\limits_{x=1}^{\infty}\left(\frac{5}{6}e^t\right)^{x-1} = \dfrac{\frac{1}{6}e^t}{1-\frac{5}{6}e^t}$ if $\left|\frac{5}{6}e^t\right| < 1$

$m'_X(t) = \dfrac{\frac{1}{6}e^t\left(1-\frac{5}{6}e^t\right) - \frac{1}{6}e^t\left(-\frac{5}{6}e^t\right)}{\left(1-5/6\,e^t\right)^2} = \dfrac{1/6\,e^t}{\left(1-5/6\,e^t\right)^2}$

$E(X) = m'_X(0) = \dfrac{(1/6)(1/6)-(1/6)(-5/6)}{(1/6)^2} = \boxed{6}$

4. (a) $P(Y=y) = \left(\frac{5}{6}\right)^{\sqrt{y}-1}\left(\frac{1}{6}\right)$

(b) $E(Y) = E(X^2) = m''_X(0)$

$m''_X(t) = \dfrac{1/6\,e^t(1-5/6\,e^t)^2 - 1/6\,e^t(-5/6\,e^t)\,2(1-5/6\,e^t)}{(1-5/6\,e^t)^4} \Rightarrow m''_X(0) = \boxed{66}$

5. (a)

$f_U(u) = \begin{cases} 1 & 0<u<1 \\ 0 & \text{otherwise} \end{cases}$ $\quad X = 1-u \quad u = 1-X \quad f_X(x) = f_u(1-x)\left|\frac{du}{dx}\right| = f_u(1-x)$

pdf: $f_X(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$ $\quad F_X(x) = \int_0^x 1\,dt = t\Big|_0^x = x$ $\quad$ cdf: $\begin{cases} 0 & x<0 \\ x & 0\le x<1 \\ 1 & x\ge 1 \end{cases}$

(b) $Y = \frac{U}{1-U}$, $F_Y(y) = P(Y\le y) = P\left(\frac{U}{1-U}\le y\right) = P\left(U\le\frac{y}{1+y}\right) = F_u\left(\frac{y}{1+y}\right)$

cdf: $F_Y(y) = \begin{cases} 0 & y\le 0 \\ y/(1+y) & 0<y<\infty \end{cases}$ $\quad f_Y(y) = F'_Y(y) = \dfrac{1}{(1+y)^2}$ $\quad$ pdf: $\begin{cases} 1/(1+y)^2 & 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$

(c) $W = \ln(Y)$ $\quad F_W(w) = P(W\le w) = P(\ln(Y)\le w) = P(Y\le e^w) = F_Y(e^w) = \dfrac{e^w}{1+e^w}$

$-\infty < w < \infty$

$f_W(w) = F'_W(w) = \dfrac{e^w}{(1+e^w)^2} \quad -\infty < w < \infty$

6. a. $E(X) = \int_0^1 x\,dx = \frac{1}{2}$

b. $E(Y) = \int_0^1 \frac{x}{1-x}\,dx = \int_0^1 -1 \cdot dx + \int_0^1 \frac{1}{1-x}\,dx$

$= +\infty$

$\therefore$ $E(Y)$ does not exist

c. $E(W) = \int_0^1 \ln\left(\frac{x}{1-x}\right) dx$

$= \int_0^1 [\ln x - \ln(1-x)]\,dx$ $\quad\left(\int_0^1 \ln(1-x)\,dx = \int_0^1 \ln x\,dx\right)$

$= \int_0^1 \ln x\,dx - \int_0^1 \ln x\,dx$

$= 0$

7. $\frac{e^{\frac{x}{2}}}{1+e^{\frac{x}{2}}} = \frac{1}{2} \Rightarrow \pi_{0.5} = 0$

$\frac{e^{\frac{x}{2}}}{1+e^{\frac{x}{2}}} = \frac{1}{4} \Rightarrow \pi_{0.25} = 2\ln\frac{1}{3}$ $\qquad IQR = \pi_{0.75} - \pi_{0.25} = 4\ln 3$

$\frac{e^{\frac{x}{2}}}{1+e^{\frac{x}{2}}} = \frac{3}{4} \Rightarrow \pi_{0.75} = 2\ln 3$

8. a. $E(X) = \int_0^1 x \cdot 5x^4\,dx = \frac{5}{6}$

b. $E\left(\frac{1}{X}\right) = \int_0^1 \frac{1}{x} \cdot 5x^4\,dx = \frac{5}{4} \neq \frac{1}{E(X)}$

c. $Y = \frac{1}{X}$ $\quad 0 < x < 1 \Rightarrow y > 1$

$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| = \frac{5}{y^6}$ $\qquad y \in (1, +\infty)$

9. a. $\left.\begin{array}{l} p(x) \geq 0 \\ \sum_{x=0}^{\infty} p(x) = e^{-5} \sum_{x=0}^{\infty} \frac{5^x}{x!} = e^{-5} \cdot e^5 = 1 \end{array}\right\} \Rightarrow$ valid

b. $E(X) = \sum_{x=0}^{\infty} x \cdot e^{-5} \frac{5^x}{x!}$

$= 5 \sum_{x=1}^{\infty} e^{-5} \cdot \frac{5^{x-1}}{(x-1)!}$

$= 5$

c. $E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) e^{-5} \frac{5^x}{x!}$

$= 5^2 \sum_{x=2}^{\infty} e^{-5} \frac{5^{x-2}}{(x-2)!}$

$= 25$

d. $E((X-5)^2) = E(X^2 - 10X + 25)$

$= E(X(X-1) - 9X + 25)$

$= E(X(X-1)) - 9E(X) + 25$

$= 5$

10. a. $\int_0^{+\infty} f(x)\,dx = -e^{-\frac{x}{5}} \Big|_0^{+\infty}$

$= 1$

$\therefore$ valid

b. $E(X) = \int_0^{+\infty} x \cdot f(x)\,dx = -5e^{-\frac{x}{5}} \Big|_0^{+\infty}$

$= 5$

c. $P(X > x) = 1 - F_X(x)$

$= 1 - \int_0^x f(t)\,dt$

$= 1 - (-e^{-\frac{t}{5}} \Big|_0^x)$

$= e^{-\frac{x}{5}}$

d. $P(X > 10 + x \mid X > 10) = \frac{P(X > 10 + x \text{ and } X > 10)}{P(X > 10)}$

$= \frac{e^{-\frac{10+x}{5}}}{e^{-\frac{10}{5}}}$

$= e^{-\frac{x}{5}}$