Math 415 - Lecture 21

Introduction

Wednesday October 14th 2015

Textbook reading: Chapter 2.5.

Suggested practice exercises: Chapter 2.5: 1, 2, 6.

Strang lecture: Lecture 12: Graphs, Networks, Incidence Matrices

1 Review

Recall that if $V \subset \mathbb{R}^n$ is a subspace, $V^\perp$ is the *orthogonal complement* of V, the subspace of all vectors $\mathbf{x}$ perp to all vectors of V.

Theorem 1. ***Fundamental Theorem of Linear Algebra.***

- $\dim(V) + \dim(V^\perp) = \dim(\mathbb{R}^n) = n$.
- $\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$.
- $\mathrm{Nul}(A)^\perp = \mathrm{Col}(A^T)$.

2 Directions and Equations

Directions and Equations. Let V be a subspace of $\mathbb{R}^n$. Then there are *two* ways of describing V.

By directions: If $V = Col(A)$ then you know that any vector $\mathbf{v}$ in V is a linear combination of the columns of A, so you know in which directions $\mathbf{v}$ can point.

By equations: If $V = \mathrm{Nul}(B)$ then you know that any $\mathbf{v}$ in V satisfies the equations $\mathbf{R_i^T} \cdot \mathbf{v} = 0$, for all rows $\mathbf{R_i}$ of B.

Both descriptions are useful, and we will often switch between them, to answer any particular question we want to answer.

3 A new perspective on $A\mathbf{x} = \mathbf{b}$

To see if $A\mathbf{x} = \mathbf{b}$ has a solution, check that

Direct approach: $\mathbf{b} \in Col(A)$

Indirect approach: $\mathbf{b} \perp Nul(A^T)$

The indirect approach means:

$$\text{if } \underbrace{\mathbf{y}^T A = \mathbf{0}}_{\mathbf{y} \in Nul(A^T)}, \text{ then } \underbrace{\mathbf{y}^T \mathbf{b} = 0}_{\mathbf{b} \perp \mathbf{y}}.$$

Example 2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution?

Solution (old)**. Write augmented matrix, get Echelon form:**

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 1 & b_2 \\ 0 & 5 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -5 & -3b_1 + b_2 \\ 0 & 0 & -3b_1 + b_2 + b_3 \end{array}\right]$$

When is this consistent? Whenever $-3b_1 + b_2 + b_3 = 0$.

Solution (new)**. Indirect approach says:** $A\mathbf{x} = \mathbf{b}$ solvable $\iff$ $\mathbf{b} \perp Nul(A^T)$.

Find basis for $Nul(A^T)$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 1 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

so $Nul(A^T)$ has basis $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

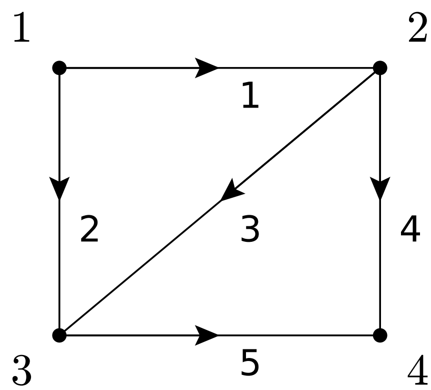
Need $\mathbf{b} \perp Nul(A^T)$: $A\mathbf{x} = \mathbf{b}$ is solvable $\iff$ $\mathbf{b} \cdot \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = 0$

This is the same condition as before!

4 Application: Directed graphs

4.1 Set up

- Graphs appear in network analysis (e.g. internet) or circuit analysis.
- Arrow indicates direction of flow
- No edges from a node to itself

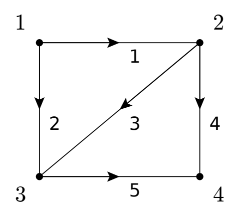


Definition 3. Let G be a graph with m edges and n nodes. The edge-node incidence matrix of G is the $m \times n$ matrix A with

$$A_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j \\ +1, & \text{if edge } i \text{ enters node } j \\ 0, & \text{otherwise} \end{cases}$$

So each row (describing an edge=arrow) contains a single -1 (the tail of the arrow), a single $+1$ (the head of the arrow), and for the rest zeroes.

Example 4. Give the edge-node incidence matrix of our graph.



Solution.

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

4.2 Meaning of the null space

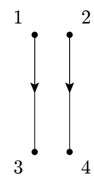
The $\mathbf{x}$ in $A\mathbf{x}$ assigns values to each node. (Think: assigning potentials)

$$A\mathbf{x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ -x_2 + x_3 \\ -x_2 + x_4 \\ -x_3 + x_4 \end{bmatrix}$$

Idea. $Ax = 0 \iff$ nodes connected by an edge are assigned the same value.

For our graph, $Nul(A)$ has basis $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ (i.e. $x_1 = x_2 = x_3 = x_4$.) This always happens as long as G is connected.

Example 5. Give a basis for $Nul(A)$ for this graph:



Solution. If $A\mathbf{x} = 0$, then

$$\underbrace{x_1 = x_3}_{\text{connected by an edge}} \qquad \text{and} \qquad \underbrace{x_2 = x_4}_{\text{connected by an edge}}$$

So, $Nul(A)$ has basis: $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$

Just to make sure, the edge-node incidence matrix is:

$$A = \begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}$$

Theorem 6. *$dim(Nul(A))$ is the number of connected subgraphs.*

- For large graphs, disconnection may not be visually apparent
- But, we can always find out by computing $dim(Nul(A))$ using Gaussian elimination!

4.3 Meaning of left null space

The $\mathbf{y}$ in $\mathbf{y}^T A$ is assigning values to each edge. (Think: assigning currents to edges, so that $\mathbf{y}$ describes a *flow pattern.*)

$$A = \begin{bmatrix}-1 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & -1 & 1 & 0\\ 0 & -1 & 0 & 1\\ 0 & 0 & -1 & 1\end{bmatrix}, A^T = \begin{bmatrix}-1 & -1 & 0 & 0 & 0\\ 1 & 0 & -1 & -1 & 0\\ 0 & 1 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & 1\end{bmatrix}$$

$$A^T\mathbf{y} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -y_1 - y_2 \\ y_1 - y_3 - y_4 \\ y_2 + y_3 - y_5 \\ y_4 + y_5 \end{bmatrix}$$

$$A^T\mathbf{y} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -y_1 - y_2 \\ y_1 - y_3 - y_4 \\ y_2 + y_3 - y_5 \\ y_4 + y_5 \end{bmatrix}$$

Idea. So: $A^T\mathbf{y} = 0 \iff$ at each node, (directed) values assigned to edges add to zero.

When thinking of currents, this is Kirchhoff's first law: at each node, incoming and outgoing currents balance. **Flow in = Flow out.**

What is the simplest way to balance current?

Assign current in a loop! We have two loops:

$$edge_1 \to edge_3 \to -edge_2 \text{ and } edge_3 \to edge_5 \to -edge_4$$

Example 7. Solve $A^T\mathbf{y} = 0$ for our graph. Recall

$$A^T = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution. Get RREF:

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \overset{RREF}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The parametric solution is: $\begin{bmatrix} y_3 - y_5 \\ -y_3 + y_5 \\ y_3 \\ -y_5 \\ y_5 \end{bmatrix}$

So a basis for $Nul(A^T)$ is: $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

Observation: These two basis vectors correspond to loops.

Note: get the "simpler" loop $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ as $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

Theorem 8. *In general, $dim(Nul(A^T))$ is the number of (independent) loops.*

For large graphs, we now have a nice way to computationally find all loops.

5 Summary/Outlook

* We described a network by using a matrix A.

* The Null space $Nul(A)$ has as dimension the number of connected components of the network.

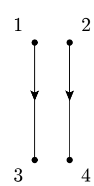
* The Left Null Space $Nul(A^T)$ has as dimension the number of independent loops.

* The column space $Col(A)$ and row space $Col(A^T)$ also have "geometric" meaning in terms of the network, see the book and Strang's lecture.

6 Practice problems

6.1 Problem 1

Example 9. Give a basis for $Nul(A^T)$ for the following graph:



Solution. This graph contains no loops, so $Nul(A^T) =$??. $Nul(A^T)$ has the empty set as basis.

To check, the incidence matrix is :

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Indeed, $Nul(A^T) = \{0\}$.

6.2 Problem 2

Example 10. Draw the graph with edge-node incidence matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Give a basis for $Nul(A)$ and $Nul(A^T)$.

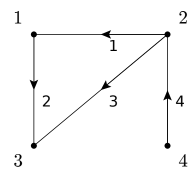


Figure 1: The graph

Solution.

If $A\mathbf{x} = 0$, then $x_1 = x_2 = x_3 = x_4$ (all connected by edges.)

So, $Nul(A)$ has basis $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(This graph is connected, so only 1 connected subgraph, so $dim(Nul(A)) = 1$.)

Loops: This graph has one loop: $edge_1 \to edge_2 \to -edge_3$. Assign values $y_1 = 1, y_2 = 1, y_3 = -1$ along the edges of that loop.

$Nul(A^T)$ has basis $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

(The graph has 1 loop, so $dim(Nul(A^T)) = 1$.)