# Math 415 - Lecture 1

Introduction

Monday August 24 2015

- Textbook: Chapter 1.3, Chapter 2.2 (just the pages 78 and 79)
- Suggested Practice Exercise: in Chapter 1.3, Exercise 1,3, 5, 6, 11
- Khan Academy Video: Matrices: Reduced Row Echelon Form 1

## 1 Systems of Linear Equations

**Definition.** A **linear equation** is a equation of the form

$$a_1x_1 + \ldots + a_nx_n = b$$

where $a_1, ..., a_n, b$ are numbers and $x_1, ..., x_n$ are variables.

*Example* 1. Which of the following equations are linear equations (or can be rearranged to become linear equations)?

| | | |
|---|---|---|
| $4x_1 - 5x_2 + 2 = x_1$ | $3x_1 - 5x_2 = -2$ | *Linear.* |
| $x_2 = 2(\sqrt{6} - x_1) + x_3$ | $2x_1 + x_2 - x_3 = 2\sqrt{6}$ | *Linear.* |
| $4x_1 - 6x_2 = x_1x_2$ | $4x_1 - 6x_2 = \underline{x_1x_2}$ | *Not linear.* |
| $x_2 = 2\sqrt{x_1} - 7$ | $x_2 = \underline{2\sqrt{x_1}} - 7$ | *Not linear.* |

This course will focus on linear equations.

**Definition.** A **system of linear equations (**or a **linear system)** is a collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, ..., x_n$.

**Definition.** A **solution** of a linear system is a list $(s_1, s_2, ..., s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, ..., s_n$ are substituted for $x_1, x_2, ..., x_n$, respectively.

**Definition.** The **solution set** of a system of linear equations is the set of all possible solutions of a linear system.

*Example* 2. Two equations in two variables:

$$
\begin{aligned}
x_1 + x_2 &= 1 \\
-x_1 + x_2 &= 0.
\end{aligned}
$$

What is a solution for this system of linear equations?

**Add them.** $2x_2 = 1 \quad \Rightarrow \quad x_2 = 5$

**Plug into first equation.** $x_1 + .5 = 1 \quad \Rightarrow \quad x_1 = .5$

$(x_1, x_2) = (.5, .5)$ is the only solution.

*Example* 3. Does every system of linear equation have a solution?

$$
\begin{aligned}
x_1 - 2x_2 &= -3 \\
2x_1 - 4x_2 &= 8.
\end{aligned}
$$

**Multiply first equation by 2.** $2x_1 - 4x_2 = -6$

**Subtract from second equation.** $0 = 14$

The equation $0 = 14$ is always false, so no solutions exist.

*Example* 4. How many solutions are there to the following system?

$$
\begin{aligned}
x_1 + x_2 &= 3 \\
-2x_1 - 2x_2 &= -6
\end{aligned}
$$

**Multiply first equation by 2.** $2x_1 + 2x_2 = 6$

**Add to second equation.** $0 = 0$

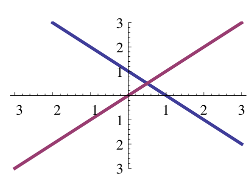
*Any* value of $x_1$ works. $x_2 = 3 - x_1$. Infinitely many solutions.

**Theorem 1.** *This is all there is: A linear system has either*

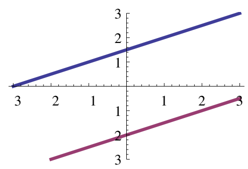
***one unique solution*** *or* ***no solution*** *or* ***infinitely many solutions****.*

Can you draw the set of solutions of the above equations?

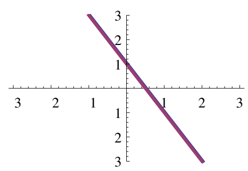
$$
\begin{aligned}
x_1 + x_2 &= 1 \\
-x_1 + x_2 &= 0.
\end{aligned}
$$



$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8. \end{aligned}$$




$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$




(The numbers in the graphs are not quite right.)

**Take away:** Whenever you have a linear system with $n$ equations, then the set of solutions of this system is precisely the intersection of the sets of solutions of each of the $n$ equations on its own.

## 1.1 Strategies for solving systems of linear equations

**Definition.** Two systems are **equivalent** if they have the same solution set.

The general strategy is to replace one system with an equivalent system that is easier to solve.

*Example* 5. Consider

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array}$$

$$R2 \to R2 + R1 \quad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ 0 & + & x_2 & = & 2 \end{array}$$

$x_2 = 2$, so $x_1 = 3$.

## 1.2 Matrix Notation

Matrix Notation

From a system of equations, we can get:

**Coefficient Matrix**

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

**Augmented matrix**

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

$R2 \to R2 + R1$

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

$R1 \to R1 + 2R2$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

**Solution:** $x_1 = 3,\ x_2 = 2$

**Definition.** An **elementary row operation** is one of the following

**(Replacement)** Add a multiple of one row to another row,

**(Interchange)** Interchange two rows, or

**(Scaling)** Multiply all entries in a row by a nonzero constant.

**Definition.** Two matrices are **row equivalent**, if one matrix can be transformed into the other matrix by a sequence of elementary row operations.

**Theorem 2.** *If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.*

*Example* 6. Solve the following system (or show there is no solution):

$$\begin{aligned} x_1 - 2x_2 + x_3 &= \quad 0 \\ 2x_2 - 8x_3 &= \quad 8 \\ -4x_1 + 5x_2 + 9x_3 &= \quad -9 \end{aligned}$$

$$
\begin{array}{rr|c}
x_1 - 2x_2 + x_3 = & 0 & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \\
2x_2 - 8x_3 = & 8 & \\
-4x_1 + 5x_2 + 9x_3 = & -9 & \\
\hline
x_1 - 2x_2 + x_3 = & 0 & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \\
2x_2 - 8x_3 = & 8 & \\
-3x_2 + 13x_3 = & -9 & \\
\hline
\end{array}
$$

$$
\begin{array}{rr|c}
x_1 - 2x_2 + x_3 = & 0 & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \\
x_2 - 4x_3 = & 4 & \\
-3x_2 + 13x_3 = & -9 & \\
\hline
x_1 - 2x_2 + x_3 = & 0 & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \\
x_2 - 4x_3 = & 4 & \\
x_3 = & 3 & \\
\hline
x_1 - 2x_2 = & -3 & \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \\
x_2 = & 16 & \\
x_3 = & 3 & \\
\hline
x_1 = & 29 & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \\
{}_2 = & 16 & \\
x_3 = & 3 &
\end{array}
$$

**Solution:** $(29, 16, 3)$
**Check:** Is $(29, 16, 3)$ a solution of the ***original*** system?

$$
\begin{array}{rcrcrcr}
x_1 & - & 2x_2 & + & x_3 & = & 0 \\
 & & 2x_2 & - & 8x_3 & = & 8 \\
-4x_1 & + & 5x_2 & + & 9x_3 & = & -9
\end{array}
$$

$$
\begin{array}{rcrcrcrl}
29 & - & 32 & + & 3 & = & 0 & \checkmark \\
 & & 32 & - & 24 & = & 8 & \checkmark \\
-116 & + & 80 & + & 27 & = & -9 & \checkmark
\end{array}
$$

## 2 Two Fundamental Questions (Existence and Uniqueness)

Two Fundamental Questions (Existence and Uniqueness)
There are two fundamental question about linear equation:

(1) Is the system consistent? (I.e. does a solution **exist**?)

(2) If a solution exists, is it **unique**? (I.e. is there one only one solution?)

*Example* 7. Is this system consistent? If so, is the solution unique?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

In the last example, this system was reduced to the triangular form:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This is sufficient to see that the system is consistent and unique. Why?

- The last row determines $x_3$ uniquely.
- Knowing $x_3$, the second row determines $x_2$ uniquely.
- Knowing $x_2$ and $x_3$, the first row determines $x_1$ uniquely.
- So, exactly one possible solution $(x_1, x_2, x_3)$.

*Example* 8. Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \left[\begin{array}{rrr|r} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right]$$

**Solution:**

$$\xrightarrow[R1 \leftrightarrow R2]{} \left[\begin{array}{rrr|r} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 5 & -7 & 9 & 0 \end{array}\right]$$

$$\xrightarrow[R3 \to R3 - 5R1]{} \left[\begin{array}{rrr|r} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{array}\right]$$

$$\xrightarrow[R3 \to R3 - R2]{} \left[\begin{array}{rrr|r} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Equation notation of triangular form:

$$\begin{array}{rrrcr} x_1 & -2x_2 & +3x_3 & = & -1 \\ & 3x_2 & -6x_3 & = & 8 \\ & & 0 & = & -3 \end{array}$$

The original system is inconsistent!

*Example* 9. For what values of $h$ will the following system be consistent?

$$\begin{array}{rcrcl} 3x_1 & - & 9x_2 & = & 4 \\ -2x_1 & + & 6x_2 & = & h \end{array}$$

**Solution:**

$$\begin{aligned} & \left[\begin{array}{rr|r} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \\ \xrightarrow[R1 \to \frac{1}{3}R1]{} & \left[\begin{array}{rr|r} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array}\right] \\ \xrightarrow[R2 \to R2 + 2R1]{} & \left[\begin{array}{rr|r} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h + \frac{8}{3} \end{array}\right] \end{aligned}$$

System is consistent if and only if $h = -\frac{8}{3}$.