Math 415 - Lecture 16

Linear Transformations

Friday October 2nd 2015

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Textbook reading: Chapter 2.6

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Suggested practice exercises: Chapter 2.6: 5, 6, 7, 36, 37

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Khan Academy videos: Linear Transformations / Linear Transformations as Matrix Vector Products / Linear Transformation Examples: Rotations in $\mathbb{R}^2$

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Strang lecture: Lecture 30: Linear Transformations

Review

If $\mathcal{B} = (\mathbf{b_1}, \ldots, b_p)$ is a basis for a vector space V then the coordinate vector of a vector $\mathbf{w} \in V$ is the column vector

$$\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}, \quad \text{if } \mathbf{w} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \cdots + c_p\mathbf{b_p}$$

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Example

Let $V = \mathbb{R}^2$, $\mathcal{B} = (\mathbf{b_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

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Solution

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Geometrically: this means that to reach $\mathbf{w}$ walk 1 unit along the $\mathbf{b_1}$ basis vector and 2 units along the $\mathbf{b_2}$ basis vector.

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Remark

Translating to the standard basis is always easy. To go from the standard basis to a new basis requires solving a system of equations, so is generally harder.

Linear Transformations

Definition

Let V and W be vector spaces.

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A map $T : V \to W$ is a linear transformation if

$$T(c\mathbf{x} + d\mathbf{y}) = cT(\mathbf{x}) + dT(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in V$ and all $c, d \in \mathbb{R}$.

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Some examples

First example

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What about the function $g(x) = 2x - 2$? Is this a linear transformation?

Matrices are linear transformations!

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Let A be an $m \times n$ matrix. Then the map $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. Why?

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We will argue that all linear transformations are essentially matrix multiplication!

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Important Geometric Examples

Let's consider some linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ which are defined by matrix multiplication ($\mathbf{x} \mapsto A\mathbf{x}$).

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In fact, it turns out that all linear maps $\mathbb{R}^n \to \mathbb{R}^m$ are given by $\mathbf{x} \mapsto A\mathbf{x}$ for some $m \times n$ matrix A.

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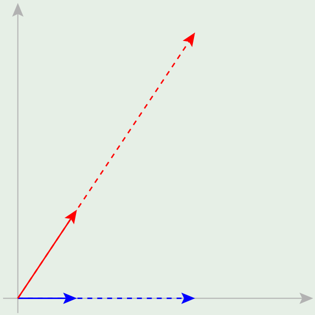

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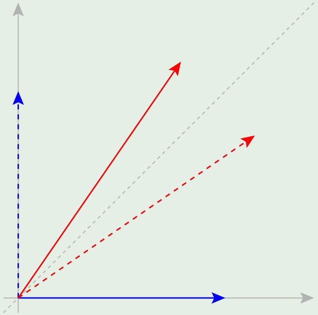

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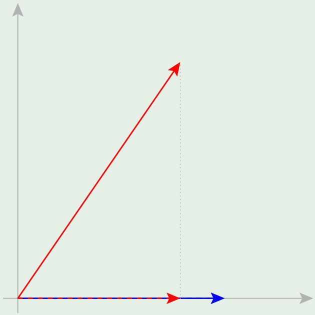

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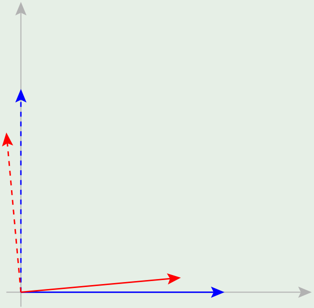

Representing linear maps by matrices

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  A linear map $T : V \to W$ is determined by the values $T(\mathbf{x_1}), \ldots, T(\mathbf{x_n})$.
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It can be written as $\mathbf{v} = c_1\mathbf{x_1} + \cdots + c_n\mathbf{x_n}$ because

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  A linear map $T : V \to W$ is determined by the values $T(\mathbf{x_1}), \ldots, T(\mathbf{x_n})$.
- Why?

Take any $\mathbf{v} \in V$.

It can be written as $\mathbf{v} = c_1\mathbf{x_1} + \cdots + c_n\mathbf{x_n}$ because $\{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$ is a basis and hence spans V.

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So we know how to write $T(\mathbf{v})$ as long as we know $T(\mathbf{x_1}), \ldots, T(\mathbf{x_n})$!

Standard Basis Coordinates

Example

Suppose $T : R^2 \to \mathbb{R}^3$ is a linear map so that

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} \text{ and } T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\-3\end{bmatrix}$$

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$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1&0\\2&0\\3&-3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}.$$

Hence calculating T is multiplying by the matrix $A=\begin{bmatrix}1&0\\2&0\\3&-3\end{bmatrix}$.

Summary: The linear transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^3, \quad T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

is the same as multiplying by the matrix

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We say that the linear transformation T is represented by the matrix A, or that A is the *coordinate matrix* of the linear transformation T, (with respect to the standard bases).

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$$T_\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}, \quad T_\alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{bmatrix},$$

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Theorem (Linear Transformation is Matrix Multiplication, Standard basis)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n.$$

Explicitly,

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix},$$

where $e_1, e_2, \dots, e_n$ is the standard basis of $\mathbb{R}^n$.

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Proof.

We can write $\mathbf{x} = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$. Then

$$\begin{aligned} T(\mathbf{x}) &= T(x_1 e_1 + x_2 e_2 + \dots + x_n e_n) = \\ &= x_1 T(e_1) + x_2 T(e_2) + \dots + x_n T(e_n) = \\ &= A\mathbf{x}. \end{aligned}$$

Example

Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a + 3b - c \\ -a + b + 2c \end{bmatrix}$. What is the matrix representing T (with respect to the standard bases)?

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Nonstandard Bases

Untill now we have used the standard bases to describe $T\colon \mathbb{R}^n \to \mathbb{R}^m$. Often it is useful to use other bases.

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But let us use, instead of the standard basis, another basis adapted to T. Put

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What is the coordinate matrix for T with respect to $\mathcal{B} = (\mathbf{b_1}, \mathbf{b_2})$?

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$$B = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

Summary: The linear transformation $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a + 1b \\ 1a + 3b \end{bmatrix}$ has with respect to the standard basis the coordinate matrix A, but with respect to the other basis $\mathcal{B}$ the coordinate B:

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The linear transformation T is geometrically clear in the $\mathcal{B}$ basis: T is just stretching vectors by a factor 2 along $\mathbf{b_1}$ and by a factor 4 along $\mathbf{b_2}$. So using the standard basis T is an obscure operation on vectors, but using the basis $\mathcal{B}$ it becomes clear. You can say that $\mathcal{B}$ is a basis adapted to T.

Additional Problems

- Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 7 & 8 & 1 \end{bmatrix}$. Find the dimensions and a basis for all four fundamental subspaces of A.

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- Suppose A is 5×5 and $\mathbf{v}$ is a vector in $\mathbb{R}^5$ which is not a linear combination of the columns of A. What can you say about the number of solutions to $A\mathbf{x} = \mathbf{0}$?

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- Let T be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

What is $T\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right)$?