STAT 410

Gamma, Chi-square and Poisson Distributions

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Gamma Distribution:

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \quad 0 \le x < \infty$$

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad 0 \le x < \infty$$

$$\mathrm{E}(\mathrm{X}) = \alpha\,\theta \qquad\qquad \mathrm{E}(\mathrm{X}) = \alpha/\lambda,$$

$$\mathrm{Var}(\mathrm{X}) = \alpha\,\theta^2 \qquad\qquad \mathrm{Var}(\mathrm{X}) = \alpha/\lambda^2$$

Gamma functions – definition and useful properties:

$$\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1} e^{-u}\, du, \qquad \alpha > 0 \qquad\qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1) \qquad\qquad \Gamma(n) = (n-1)! \quad \text{if } n \text{ is an integer}$$

Incomplete Gamma Functions

$$\Gamma(\alpha, t) = \int_t^{\infty} u^{\alpha-1} e^{-u} du$$

Show using integration by parts:

$$\Gamma(\alpha, t) = (\alpha-1)\Gamma(\alpha-1, t) + t^{\alpha-1}e^{-t}$$

As a special case

$$\Gamma(1, t) = \int_t^{\infty} e^{-u} du = e^{-t}$$

Finally, notice that $\Gamma(\alpha) = \Gamma(\alpha, 0)$ so the incomplete Gamma recursion gives the complete Gamma recursion as a special case.

If X ~ Gamma(α, $\theta = 1/\lambda$) then, by a change of variables,

$$P(X > x) = 1 - F_X(x) = \frac{\int_x^\infty u^{\alpha-1} e^{-u/\theta} du}{\Gamma(\alpha)\theta^\alpha} = \frac{\int_{x/\theta}^\infty v^{\alpha-1} e^{-v}\, dv}{\Gamma(\alpha)}$$

$$= \frac{\Gamma(\alpha, \frac{x}{\theta})}{\Gamma(\alpha)} = \frac{\Gamma(\alpha, \lambda x)}{\Gamma(\alpha)}$$

Relating Gamma probabilities to Poisson probabilities:

If T_k has a Gamma$(k, \theta = {}^1\!/_\lambda)$ distribution, **where k is an integer**, then

$$F_{T_k}(t) = P(T_k \leq t) = P(X_t \geq k),$$

$$P(T_k > t) = P(X_t \leq k-1),$$

where X_t has a Poisson$(\lambda t = \frac{t}{\theta})$ distribution.

Proof: Using the incomplete Gamma function recursion given above

$$P(T_k > t) = \frac{\Gamma(k, \lambda t)}{\Gamma(k)} = \frac{(k-1)\Gamma(k-1, \lambda t) + (\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!}$$

$$= \frac{\Gamma(k-1, \lambda t)}{(k-2)!} + \frac{(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!}$$

$$= \frac{\Gamma(k-2, \lambda t)}{(k-3)!} + \frac{(\lambda t)^{k-2} e^{-\lambda t}}{(k-2)!} + \frac{(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!}$$

$$\vdots$$

$$= \frac{\Gamma(1, \lambda t)}{0!} + \frac{(\lambda t)e^{-\lambda t}}{1} + \cdots + \frac{(\lambda t)^{k-2} e^{-\lambda t}}{(k-2)!} + \frac{(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!}$$

$$= e^{-\lambda t} + (\lambda t)e^{-\lambda t} + \cdots + \frac{(\lambda t)^{k-2} e^{-\lambda t}}{(k-2)!} + \frac{(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!}$$

$$= P(X_t \leq k-1).$$

Deriving the moment generating function

If X ~ Gamma(α, $\theta = 1/\lambda$) then, for any $t < \lambda = 1/\theta$,

$$M_X(t) = \int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x} e^{tx}}{\Gamma(\alpha)} dx$$

$$= \frac{\lambda^\alpha}{(\lambda - t)^\alpha} \int_0^\infty \frac{(\lambda - t)^\alpha x^{\alpha-1} e^{-(\lambda - t)x}}{\Gamma(\alpha)} dx, \lambda > t$$

$$= \frac{\lambda^\alpha}{(\lambda - t)^\alpha} = \frac{1}{(1 - \frac{t}{\lambda})^\alpha} = \frac{1}{(1 - \theta t)^\alpha}, \lambda = \frac{1}{\theta} > t$$

Exponential distribution as a special case

If X ~ Exp($\theta = 1/\lambda$) then it has the pdf

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

This is the same as Gamma(1, $\theta = 1/\lambda$) and has the moment generating function

$$M(t) = \frac{1}{1 - \theta t}, \qquad t < \frac{1}{\theta}.$$

Sum of independent exponential waiting times

Suppose events occur randomly over time, and the time intervals between successive events are independently distributed as Exp(θ = 1/λ). The times between events are

$$X_1, X_2, \ldots, X_n, \ldots \;\; iid\ \mathrm{Exp}(\theta)$$

How long do we have to wait until the kth event, i.e, what is the distribution of the total time until the kth event? Let $W_k = X_1 + X_2 + \cdots + X_k$. We compute the MGF

$$M_{W_k}(t) = E\left(e^{t(X_1+X_2+\cdots+X_k)}\right) = E(e^{tX_1})E(e^{tX_2})\cdots E(e^{tX_k})$$
$$= \frac{1}{(1-\theta t)^k}$$

We conclude that the time until the kth event,

$$W_k = \sum_{i=1}^{k} X_i \sim \mathrm{Gamma}(k, \theta)$$

Poisson process interpretation

When the intervals between successive events are independent exponential waiting times the number of events by a fixed time point t follows a Poisson distribution. Why? Let Y_t denote the number of events occurring before time t. Then for k=0,1,2,3…, and using our previous results,

$$F_{Y_t}(k) = P(Y_t \le k) = 1 - P(Y_t \ge k+1) = 1 - P(W_{k+1} \le t)$$

$$= P(W_{k+1} > t) = P(X_t \le k) = F_{X_t}(k)$$

where $X_t \sim Poisson(\lambda t = \frac{t}{\theta})$.

The Chi-Square Distribution $\chi^2(r)$

$$f(y) = \frac{1}{\Gamma(r/2)2^{r/2}} y^{r/2-1} e^{-y/2}, \qquad 0 \le y < \infty$$

$E(Y) = r$ $\qquad$ $Var(Y) = 2r$

Chi-square as a special type of Gamma distribution:

Comparing with the form of the Gamma pdf we see that $\chi^2(r)$ is a special case of Gamma with $\alpha = r/2, \lambda = 1/2, \theta = 2 = 1/\lambda$,

and mgf given by

$$M(t) = \frac{1}{(1-2t)^{r/2}}$$

Chi-square(r): $$f(y) = \frac{1}{\Gamma(r/2)2^{r/2}} y^{r/2-1} e^{-y/2}, \qquad 0 \le y < \infty$$

Gamma(α, θ): $$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \qquad 0 \le x < \infty$$

Examples:

1. Alex is told that he needs to take bus #5 to the train station. He misunderstands the directions and decides to wait for the fifth bus. Suppose that the buses arrive to the bus stop according to Poisson process with the average rate of one bus per 20 minutes.

X_t = number of buses in t hours.	Poisson (t)
T_k = arrival time of the k th bus.	Gamma, $\alpha = k$.

One bus per 20 minutes $\Rightarrow$ $\lambda = 3$.

a) Find the probability that Alex would have to wait longer than 1 hour for the fifth bus to arrive.

$$P(T_5 > 1) = P(X_1 \le 4) = P(\text{Poisson}(3) \le 4) = \mathbf{0.815}.$$

OR

$$P(T_5 > 1) = \int_1^{\infty} \frac{3^5}{\Gamma(5)}\, t^{5-1} e^{-3t}\, dt = \int_1^{\infty} \frac{3^5}{4!}\, t^4 e^{-3t}\, dt = \ldots$$

b) Find the probability that the fifth bus arrives during the second hour.

$$\begin{aligned} P(1 < T_5 < 2) &= P(T_5 > 1) - P(T_5 > 2) \\ &= P(X_1 \le 4) - P(X_2 \le 4) \\ &= P(\text{Poisson}(3) \le 4) - P(\text{Poisson}(6) \le 4) \\ &= 0.815 - 0.285 = \mathbf{0.530}. \end{aligned}$$

OR

$$P(1 < T_5 < 2) = \int_1^{2} \frac{3^5}{\Gamma(5)}\, t^{5-1} e^{-3t}\, dt = \int_1^{2} \frac{3^5}{4!}\, t^4 e^{-3t}\, dt = \ldots$$

c) Find the probability that the fifth bus arrives during the third hour.

$$P(2 < T_5 < 3) = P(T_5 > 2) - P(T_5 > 3)$$

$$= P(X_2 \le 4) - P(X_3 \le 4)$$

$$= P(\text{Poisson}(6) \le 4) - P(\text{Poisson}(9) \le 4)$$

$$= 0.285 - 0.055 = \mathbf{0.230}.$$

OR

$$P(2 < T_5 < 3) = \int_2^3 \frac{3^5}{\Gamma(5)} t^{5-1} e^{-3t}\, dt = \int_2^3 \frac{3^5}{4!} t^4 e^{-3t}\, dt = \ldots$$

2. During a radio trivia contest, the radio station receives phone calls according to Poisson process with the average rate of five calls per minute.

X_t = number of phone calls in t minutes. Poisson (λt)

T_k = time of the k th phone call. Gamma, $\alpha = k$.

five calls per minute $\Rightarrow$ $\lambda = 5$.

a) Find the probability that we would have to wait less than two minutes for the ninth phone call.

$$P(T_9 < 2) = P(X_2 \geq 9) = 1 - P(X_2 \leq 8)$$
$$= 1 - P(\text{Poisson}(10) \leq 8)$$
$$= 1 - 0.333 = \mathbf{0.667}.$$

OR

$$P(T_9 < 2) = \int_0^2 \frac{5^9}{\Gamma(9)} t^{9-1} e^{-5t}\, dt = \int_0^2 \frac{5^9}{8!} t^8 e^{-5t}\, dt = \ldots$$

b) Find the probability that the ninth phone call would arrive during the third minute.

$$P(2 < T_9 < 3) = P(T_9 > 2) - P(T_9 > 3)$$
$$= P(X_2 \leq 8) - P(X_3 \leq 8)$$
$$= P(\text{Poisson}(10) \leq 8) - P(\text{Poisson}(15) \leq 8)$$
$$= 0.333 - 0.037 = \mathbf{0.296}.$$

OR

$$P(2 < T_9 < 3) = \int_2^3 \frac{5^9}{\Gamma(9)}\, t^{9-1} e^{-5t}\, dt = \int_2^3 \frac{5^9}{8!}\, t^{8} e^{-5t}\, dt = \dots$$

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Example: transforming Gamma to Chi-square for integer α

3. Let Y be a random variable with a Gamma distribution with parameters α and $\theta = {}^1\!/\!_\lambda$. Assume α is an integer. Show that ${}^{2\,Y}\!/\!_\theta$ has a chi-square distribution. What is the number of degrees of freedom?

$$M_Y(t) = M_{\text{Gamma}(\alpha,\theta)}(t) = \frac{1}{(1-\theta t)^{\alpha}}, \quad t < \frac{1}{\theta}.$$

If $W = a\,Y + b$, then $M_W(t) = e^{b\,t} M_Y(a\,t)$.

$$M_{2\,Y/\theta}(t) = M_Y({}^{2\,t}\!/\!_\theta) = \frac{1}{(1-2t)^{\alpha}}, \quad t < \frac{1}{2}.$$

${}^{2\,Y}\!/\!_\theta$ has a chi-square distribution with $r = 2\,\alpha$ degrees of freedom.

More examples relating Gamma, Chi-square and Poisson

4. Let Y be a random variable with a Gamma distribution with $\alpha = 5$ and $\theta = 3$.

a) Find the probability P (Y > 18) …

i) … by integrating the p.d.f. of the Gamma distribution;

$$P(Y > 18) = \int_{18}^{\infty} \frac{1}{\Gamma(5) \cdot 3^5} \cdot x^{5-1} \cdot e^{-x/3} dx = \int_{18}^{\infty} \frac{1}{5{,}832} \cdot x^4 \cdot e^{-x/3} dx = \ldots$$

ii) … by using the relationship between Gamma and Poisson distributions;

$P(Y > 18) = P(X_{18} \le 4) = \mathbf{0.285}$ where X_{18} is Poisson ($18/\theta = 6$).

EXCEL:	= POISSON(x , λ , 0)	gives	P(X = x)
	= POISSON(x , λ , 1)	gives	P(X ≤ x)

	A	B
1	=POISSON(4,18/3,1)	
2		

⇒

	A	B
1	0.285057	
2		

iii) … by using the relationship between Gamma and Chi-square distribution.

$$P(Y > 18) = P\left(\frac{2}{3}Y > \frac{2}{3}\cdot 18\right) = P(X > 12) \qquad \text{where } X \text{ is } \chi^2(5\cdot 2 = 10).$$

= CHIINV (α , v) gives $\chi^2_\alpha(v)$ for χ^2 distribution with v degrees of freedom, x s.t. $P(\chi^2(v) > x) = \alpha$.

= CHIDIST (x , v) gives the upper tail probability for χ^2 distribution with v degrees of freedom, $P(\chi^2(v) > x)$.

	A	B
1	=CHIDIST(12,10)	

$\Rightarrow$

	A	B
1	0.285057	
2		

b) Find a and b such that $P(a < Y < b) = 0.90$.

$\frac{2}{3}Y$ is χ^2 ($5\cdot 2 = 10$ degrees of freedom).

$$P(3.940 < \chi^2(10) < 18.31) = 0.95 - 0.05 = 0.90.$$

$$\Rightarrow \quad P\left(3.940\times\frac{3}{2} < Y < 18.31\times\frac{3}{2}\right) = P(\mathbf{5.91} < Y < \mathbf{27.465}) = 0.90.$$

$$P(0 < \chi^2(10) < 15.99) = 0.90 - 0.00 = 0.90.$$

$$\Rightarrow \quad P\left(0\times\frac{3}{2} < Y < 15.99\times\frac{3}{2}\right) = P(\mathbf{0} < Y < \mathbf{23.985}) = 0.90.$$

$$P(4.865 < \chi^2(10) < \infty) = 1.00 - 0.10 = 0.90.$$

$$\Rightarrow \quad P\left(4.865\times\frac{3}{2} < Y < \infty\times\frac{3}{2}\right) = P(\mathbf{7.2975} < Y < \boldsymbol{\infty}) = 0.90.$$

5. Let T_7 have a Gamma distribution with $\alpha = 7$ and $\theta = 5$.

Find the probability $P(20 < T_7 < 30)$.

[E.g., Text messages arrive according to Poisson process, on average once every 5 minutes. Find the probability that we would have to wait more than 20 minutes but less than 30 minutes for the 7th text message.]

$$P(20 < T_7 < 30) = \int_{20}^{30} \frac{1}{\Gamma(7) \cdot 5^7} \cdot t^{7-1} \cdot e^{-t/5} dt = \int_{20}^{30} \frac{1}{6! \cdot 5^7} \cdot t^6 \cdot e^{-t/5} dt =$$

…

OR

$$P(20 < T_7 < 30) = P(T_7 > 20) - P(T_7 > 30)$$

$$= P(X_{20} \le 6) - P(X_{30} \le 6)$$

$$= P(\text{Poisson}(4) \le 6) - P(\text{Poisson}(6) \le 6)$$

$$= 0.889 - 0.606 = \mathbf{0.283}.$$

[If the 7th text message arrives after 20 minutes, then we could have received at most 6 text messages during the first 20 minutes. If the average time between the text messages is 5 minutes, then the expected number of text messages in 20 minutes is 4.

If the 7th text message arrives after 30 minutes, then we could have received at most 6 text messages during the first 30 minutes. If the average time between the text messages is 5 minutes, then the expected number of text messages in 30 minutes is 6.]

	A	B
1	=POISSON(6,4,1)	
2	=POISSON(6,6,1)	
3	=A1-A2	
4		

⇒

	A	B
1	0.889326	
2	0.606303	
3	0.283023	
4		

OR

$$P(20 < T_7 < 30) = \left(\frac{2}{5}\cdot 20 < \frac{2}{5}\cdot T_7 < \frac{2}{5}\cdot 30\right)$$

$$= P(8 < \chi^2(2\cdot 7 = 14) < 12)$$

	A	B
1	=CHIDIST(8,14)	
2	=CHIDIST(12,14)	
3	=A1-A2	
4		

$\Rightarrow$

	A	B
1	0.889326	
2	0.606303	
3	0.283023	
4		

6. Let Y have a Gamma distribution with $\alpha = 5$ and $\theta = 4$ (i.e., $\lambda = 0.25$). Find the probability $P(Y \le 15)$ …

a) … by integrating the p.d.f. of the Gamma distribution;

$$P(Y \le 15) = \int_0^{15} \frac{1}{\Gamma(5)\cdot 4^5}\cdot x^{5-1}\cdot e^{-x/4}dx = \int_0^{15} \frac{1}{24{,}576}\cdot x^4\cdot e^{-x/4}dx = \ldots$$

b) … by using the relationship between Gamma and Poisson distributions;

$P(Y \le 15) = P(X_{15} \ge 5) = 1 - P(X_{15} \le 4)$ where X_{15} is Poisson ($15\lambda = 3.75$).

EXCEL: = POISSON(x , λ , 0) gives P($X = x$)

= POISSON(x , λ , 1) gives P($X \le x$)

	A	B
1	=1-POISSON(4,15/4,1)	
2		

$\Rightarrow$

	A	B
1	0.322452	
2		

c) … by using the relationship between Gamma and Chi-square distribution.

$$P\,(\,Y \le 15\,) \;=\; P\left(\frac{2}{4}Y \le \frac{2}{4}\cdot 15\right) \;=\; P\,(\,X \le 7.5\,)$$ where X is χ^2($5 \cdot 2 = 10$).

= CHIINV (α , v) gives $\chi^2_\alpha(v)$ for χ^2 distribution with v degrees of freedom, x s.t. $P\,(\,\chi^2(v) > x\,) = \alpha$.

= CHIDIST (x , v) gives the upper tail probability for χ^2 distribution with v degrees of freedom, $P\,(\,\chi^2(v) > x\,)$.

	A	B
1	=1-CHIDIST(7.5,10)	

$\Rightarrow$

	A	B
1	0.322452	
2		

7. Let X have a Gamma distribution with $\alpha = 3$ and $\theta = 5$ (i.e., $\lambda = 0.2$). Find the probability $P(X > 31.48)$ …

a) … by integrating the p.d.f. of the Gamma distribution;

$$P(X > 31.48) = \int_{31.48}^{\infty} \frac{1}{2 \cdot 5^3} \cdot x^2 \cdot e^{-x/5} dx$$

$$= \frac{1}{2 \cdot 5^3} \cdot \left(-5 \cdot x^2 \cdot e^{-x/5} - 2 \cdot 5^2 \cdot x \cdot e^{-x/5} - 2 \cdot 5^3 \cdot e^{-x/5} \right) \Bigg|_{31.48}^{\infty}$$

$= \mathbf{0.04999}$.

b) … by using the relationship between Gamma and Poisson distributions;

Hint: If X has a Gamma (α , $\theta = {}^1/_\lambda$) distribution, where α is an integer, then
$F_X(t) = P(X \le t) = P(Y \ge \alpha)$ and $P(X > t) = P(Y \le \alpha - 1)$,
where
Y has a Poisson ($\lambda\, t$) distribution.

$P(X > 31.48) = 1 - P(X \le 31.48) = 1 - P(Y \ge 3) = P(Y \le 2)$

where Y has a Poisson (${}^{31.48}/_5 = 6.296$) distribution.

$$= \frac{6.296^0 \cdot e^{-6.296}}{0!} + \frac{6.296^1 \cdot e^{-6.296}}{1!} + \frac{6.296^2 \cdot e^{-6.296}}{2!}$$

$= 0.00184 + 0.01161 + 0.03654 = \mathbf{0.04999}$.

c) … by using the relationship between Gamma and Chi-square distribution.

Hint: If X has a Gamma (α , $\theta = {}^1/_\lambda$) distribution, where α is an integer, then

${}^{2X}/_{\theta}$ has a chi-square distribution with 2α degrees of freedom.

$P(X > 31.48) = P({}^{2X}/_{5} > 12.592) = P(\chi^2(6) > 12.592) = \mathbf{0.05}$.

8. a) Mistakes that Doug makes in class occur according to Poisson process with the average rate of one mistake per 10 minutes. Find the probability that the third mistake Doug makes occurs during the last 15 minutes of a 50-minute class.

Notations: X_t = number of server failures in t days.
T_k = time of the k th server failure.

1 min $\theta = 10, \quad \lambda = \frac{1}{10} = 0.10.$

$P(35 < T_3 < 50) = P(T_3 > 35) - P(T_3 > 50) = P(X_{35} \le 2) - P(X_{50} \le 2) = \ldots$

5 min $\theta = 2, \quad \lambda = \frac{1}{2} = 0.50.$

$P(7 < T_3 < 10) = P(T_3 > 7) - P(T_3 > 10) = P(X_7 \le 2) - P(X_{10} \le 2) = \ldots$

10 min $\theta = 1, \quad \lambda = 1.$

$P(3.5 < T_3 < 5) = P(T_3 > 3.5) - P(T_3 > 5) = P(X_{3.5} \le 2) - P(X_5 \le 2) = \ldots$

$\ldots = P(\text{Poisson}(3.5) \le 2) - P(\text{Poisson}(5.0) \le 2) = 0.3208 - 0.1247 =$ **0.1961**.

OR

1 min $\quad P(35 < T_3 < 50) = \int_{35}^{50} \frac{0.10^3}{\Gamma(3)} t^{3-1} e^{-0.10t}\, dt = \int_{35}^{50} \frac{1}{2{,}000} t^2 e^{-0.10t}\, dt$

$= \ldots$

5 min $\quad P(7 < T_3 < 10) = \int_{7}^{10} \frac{0.50^3}{\Gamma(3)} t^{3-1} e^{-0.50t}\, dt = \int_{7}^{10} \frac{1}{16} t^2 e^{-0.50t}\, dt = \ldots$

10 min $\quad P(3.5 < T_3 < 5) = \int_{3.5}^{5.0} \frac{1}{\Gamma(3)} t^{3-1} e^{-t}\, dt = \int_{3.5}^{5.0} \frac{1}{2} t^2 e^{-t}\, dt =$

$\ldots$

b) Students ask questions in class according to Poisson process with the average rate of one question per 20 minutes. Find the probability that the third question is asked during the last 10 minutes of a 50-minute class.

1 min $\theta = 20, \quad \lambda = \frac{1}{20} = 0.05.$

$P(40 < T_3 < 50) = P(T_3 > 40) - P(T_3 > 50) = P(X_{40} \le 2) - P(X_{50} \le 2) = \ldots$

5 min $\theta = 4, \quad \lambda = \frac{1}{4} = 0.25.$

$P(8 < T_3 < 10) = P(T_3 > 8) - P(T_3 > 10) = P(X_8 \le 2) - P(X_{10} \le 2) = \ldots$

10 min $\theta = 2, \quad \lambda = \frac{1}{2} = 0.50.$

$P(4 < T_3 < 5) = P(T_3 > 4) - P(T_3 > 5) = P(X_4 \le 2) - P(X_5 \le 2) = \ldots$

20 min $\theta = 1, \quad \lambda = 1.$

$P(2 < T_3 < 2.5) = P(T_3 > 2) - P(T_3 > 2.5) = P(X_2 \le 2) - P(X_{2.5} \le 2) = \ldots$

$\ldots = P(\text{Poisson}(2.0) \le 2) - P(\text{Poisson}(2.5) \le 2) = 0.6767 - 0.5438 = \mathbf{0.1329}.$

OR

1 min $\quad P(40 < T_3 < 50) = \int_{40}^{50} \frac{0.05^3}{\Gamma(3)} t^{3-1} e^{-0.05t}\,dt = \int_{40}^{50} \frac{1}{16{,}000} t^2 e^{-0.05t}\,dt$

= …

5 min $\quad P(8 < T_3 < 10) = \int_{8}^{10} \frac{0.25^3}{\Gamma(3)} t^{3-1} e^{-0.25t}\,dt = \int_{8}^{10} \frac{1}{128} t^2 e^{-0.25t}\,dt =$

…

10 min $\quad P(4 < T_3 < 5) = \int_{4}^{5} \frac{0.50^3}{\Gamma(3)} t^{3-1} e^{-0.50t}\,dt = \int_{4}^{5} \frac{1}{16} t^2 e^{-0.50t}\,dt$

= …

20 min $\quad P(2 < T_3 < 2.5) = \int_{2.0}^{2.5} \frac{1}{\Gamma(3)} t^{3-1} e^{-t}\,dt = \int_{2.0}^{2.5} \frac{1}{2} t^2 e^{-t}\,dt =$

…