Math 415 - Lecture 1

Introduction

Monday August 24 2015

- Textbook: Chapter 1.3, Chapter 2.2 (just the pages 78 and 79)
- Suggested Practice Exercise: in Chapter 1.3, Exercise 1,3, 5, 6, 11
- Khan Academy Video: Matrices: Reduced Row Echelon Form 1

1 Systems of Linear Equations

Definition. A **linear equation** is a equation of the form

where $a_1, ..., a_n, b$ are numbers and $x_1, ..., x_n$ are variables.

Example 1. Which of the following equations are linear equations (or can be rearranged to become linear equations)?

$4x_1 - 5x_2 + 2 = x_1$		Linear/Nonlinear?
$x_2 = 2(\sqrt{6} - x_1) + x_3$		Linear/Nonlinear?
$4x_1 - 6x_2 = x_1 x_2$		Linear/Nonlinear?
$x_2 = 2\sqrt{x_1} - 7$		Linear/Nonlinear?

Definition. A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, ..., x_n$.

Definition. A **solution** of a linear system is a list $(s_1, s_2, ..., s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, ..., s_n$ are substituted for $x_1, x_2, ..., x_n$, respectively.

Definition. The **solution set** of a system of linear equations is the set of all possible solutions of a linear system.

Example 2. Two equations in two variables:

$$\begin{aligned} x_1 + x_2 &= 1 \\ -x_1 + x_2 &= 0. \end{aligned}$$

What is a solution for this system of linear equations?

Example 3. Does every system of linear equation have a solution?

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8. \end{aligned}$$

Example 4. How many solutions are there to the following system?

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$

Theorem 1. *This is all there is: A linear system has either*

one unique solution *or* ***no solution*** *or* ***infinitely many solutions****.*

Can you draw the set of solutions of the above equations?

$$\begin{aligned} x_1 + x_2 &= 1 \\ -x_1 + x_2 &= 0. \end{aligned}$$

$$
\begin{aligned}
x_1 - 2x_2 &= -3 \\
2x_1 - 4x_2 &= 8.
\end{aligned}
$$

$$
\begin{aligned}
x_1 + x_2 &= 3 \\
-2x_1 - 2x_2 &= -6
\end{aligned}
$$

1.1 Strategies for solving systems of linear equations

Definition. Two systems are **equivalent** if they have the same solution set.

The general strategy is to replace one system with an equivalent system that is easier to solve.

Example 5. Consider

$$
\begin{array}{rcrcr}
x_1 & - & 2x_2 & = & -1 \\
-x_1 & + & 3x_2 & = & 3
\end{array}
$$

1.2 Matrix Notation

From a system of equations, we can get:

Coefficient Matrix

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

Augmented matrix

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

Definition. An **elementary row operation** is one of the following

(Replacement) Add a multiple of one row to another row,

(Interchange) Interchange two rows, or

(Scaling) Multiply all entries in a row by a nonzero constant.

Definition. Two matrices are **row equivalent**, if one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Theorem 2. *If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.*

Example 6. Solve the following system (or show there is no solution):

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 &= 29 \\ {}_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Solution: $(29, 16, 3)$
Check: Is $(29, 16, 3)$ a solution of the ***original*** system?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{rcrcrcrl} 29 & - & 32 & + & 3 & = & 0 & \checkmark \\ & & 32 & - & 24 & = & 8 & \checkmark \\ -116 & + & 80 & + & 27 & = & -9 & \checkmark \end{array}$$

2 Two Fundamental Questions (Existence and Uniqueness)

There are two fundamental question about linear equation:

(1) Is the system consistent? (I.e. does a solution **exist**?)

(2) If a solution exists, is it **unique**? (I.e. is there one only one solution?)

Example 7. Is this system consistent? If so, is the solution unique?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

In the last example, this system was reduced to the triangular form:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This is sufficient to see that the system is consistent and unique. Why?

Example 8. Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \left[\begin{array}{rrr|r} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right]$$

Solution:

Example 9. For what values of h will the following system be consistent?

$$\begin{array}{rcrcr} 3x_1 & - & 9x_2 & = & 4 \\ -2x_1 & + & 6x_2 & = & h \end{array}$$

Solution:
Reduce to triangular form.

$$\left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \to \left[\begin{array}{cc|c} \quad & \quad & \quad \\ \quad & \quad & \quad \end{array}\right] \to \left[\begin{array}{cc|c} \quad & \quad & \quad \\ \quad & \quad & \quad \end{array}\right]$$

System is consistent if and only if h is $\boxed{}$.