Math 415 - Lecture 19

Orthonormal basis, orthogonal complement

Friday October 9th 2015

Textbook reading: Ch 3.1

Suggested practice exercises: Ch 3.1: 7, 8, 9, 10, 11, 12, 14, 15, 17, 18, 19, 20, 22

Khan Academy videos:

Strang lectures: Lec 10: The Four Fundamental Subspaces / Lec 14: Orthogonal Vectors and Subspaces

1 Review

- $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = v_1 w_1 + \cdots + v_n w_n$ is the **inner product** of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.
  - The **length** of $\mathbf{v}$, $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$.
  - The **distance** between points $\mathbf{v}, \mathbf{w}$ is $\|\mathbf{v} - \mathbf{w}\|$.
- $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are **orthogonal** if $\mathbf{v} \cdot \mathbf{w} = 0$.
  - This simple criterion is equivalent to Pythagoras' theorem.

2 Unit Vectors and Orthonormal basis

Definition. A vector $\mathbf{u} \in \mathbb{R}^n$ is called a *unit vector* if

Example 1. The standard basis vectors $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ of $\mathbb{R}^n$ are all unit vectors.

Example 2. If $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then is $\mathbf{x}$ a unit vector?

Solution.

Definition.

- A bunch of vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_p}$ is called *orthogonal* if they are

- Orthogonal vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_p}$ are called *orthonormal*

Example 3. Let $\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Are they orthonormal?

Let $\mathcal{B} := (\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n})$ be an orthonormal basis for $\mathbb{R}^n$, so a basis consisting of unit vectors that are all perpendicular. Let $\mathbf{x}$ in $\mathbb{R}^n$. Is there an easy way to find the coordinate vector $\mathbf{x}_{\mathcal{B}}$: ie. find $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_n\mathbf{u_n}.$$

Example 4. $\mathbf{u_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ is an orthonormal basis for $\mathbb{R}^2$. Let $\mathbf{x} = \begin{bmatrix}2\\3\end{bmatrix}$.

Solution.

Theorem 1. *Let* $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ *be non-zero and mutually orthogonal. Then* $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ *are linearly independent.*

Solution.

3 Orthogonality and the Fundamental subspaces

Example 5. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. Find $Nul(A)$ and $Col(A^T)$.

Solution.

Example 6. Repeat for $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$.

Solution.

4 Fundamental Theorem of Linear Algebra (Revisited)

Definition. Let W be a subspace of $\mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$.

- $\mathbf{v}$ is **orthogonal** to W if $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in W$. ($\iff$ $\mathbf{v}$ is orthogonal to each vector in a basis for W.)
- Another subspace V is **orthogonal** to W if every vector in V is orthogonal to W.
- The **orthogonal complement** of W is the space $W^{\perp}$

Example 7. Let $V = \text{Span}\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $W = \text{Span}\begin{bmatrix}0\\1\\0\end{bmatrix}$. Is $V \perp W$? Is $V^{\perp} = W$?

Example 8. In the last example, $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. We found that

$$Nul(A) = span \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad Col(A^T) = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are orthogonal subspaces. Indeed, $Nul(A)$ and $Col(A^T)$ are orthogonal complements. Why?

Solution.

Remark. In the last example, $Nul(A)$ and $Col(A)$ both happen to be subspaces of $\mathbb{R}^3$ (because A was a square 3×3 matrix).

$$Nul(A) = span \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad Col(A) = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

However, these spaces are **not** orthogonal. Why?

Solution.

Theorem 2. *Let A be an $m \times n$ matrix of rank r.*

- *$dim\ Col(A) = r$* *(subspace of $\mathbb{R}^m$)*
- *$dim\ Col(A^T) = r$* *(subspace of $\mathbb{R}^n$)*
- *$dim\ Nul(A) = n - r$* *(subspace of $\mathbb{R}^n$)*
- *$dim\ Nul(A^T) = m - r$* *(subspace of $\mathbb{R}^m$)*
- *$Nul(A)^\perp = Col(A^T)$* *(both subspaces of $\mathbb{R}^n$) Note that $dim\ Nul(A) + dim\ Col(A^T) = n$.*
- *$Nul(A^T)^\perp = Col(A)$*

Example 9. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.