Math 415 - Lecture 29

Determinants

Wednesday November 4th 2015

Textbook reading: Chapters 4.2, 4.3

Textbook reading: Chapters 4.2, 4.3

Suggested practice exercises: Chapter 4.2, # 1, 2, 4, 5, 10, 14, 15, 17, 18, 19, 20, 22, 23

Textbook reading: Chapters 4.2, 4.3

Suggested practice exercises: Chapter 4.2, # 1, 2, 4, 5, 10, 14, 15, 17, 18, 19, 20, 22, 23

Khan Academy video: 3×3 Determinant,
$n \times n$ Determinant,
Determinants along other rows/ columns,

Textbook reading: Chapters 4.2, 4.3

Suggested practice exercises: Chapter 4.2, # 1, 2, 4, 5, 10, 14, 15, 17, 18, 19, 20, 22, 23

Khan Academy video: 3×3 Determinant,
$n \times n$ Determinant,
Determinants along other rows/ columns,

Strang lecture: Lecture 18: Properties of determinants,
Lecture 19: Determinant formulas and cofactors

Determinants

For the next few lectures, all matrices are square!

For the next few lectures, all matrices are square!

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For the next few lectures, all matrices are square!
Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The determinant of

- a 2×2 matrix is $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$,

For the next few lectures, all matrices are square!
Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The determinant of

- a 2×2 matrix is $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$,
- a 1×1 matrix is $\det([a]) = a$.

For the next few lectures, all matrices are square!

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The determinant of

- a 2×2 matrix is $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$,
- a 1×1 matrix is $\det([a]) = a$.
- What is the determinant of an $n \times n$ matrix?

For the next few lectures, all matrices are square!

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The determinant of

- a 2×2 matrix is $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$,
- a 1×1 matrix is $\det([a]) = a$.
- What is the determinant of an $n \times n$ matrix?

Goal (Point of the determinant)

A is invertible $\iff \det(A) \neq 0$

For the next few lectures, all matrices are square!

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The determinant of

- a 2×2 matrix is $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$,
- a 1×1 matrix is $\det([a]) = a$.
- What is the determinant of an $n \times n$ matrix?

Goal (Point of the determinant)

A is invertible $\iff \det(A) \neq 0$

For the next few lectures, all matrices are square!

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The determinant of

- a 2×2 matrix is $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$,
- a 1×1 matrix is $\det([a]) = a$.
- What is the determinant of an $n \times n$ matrix?

Goal (Point of the determinant)

A is invertible $\iff \det(A) \neq 0$

Notation: We will write both $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ for the determinant.

We want to define for every $n \times n$ matrix A a number $\det(A)$.

We want to define for every $n \times n$ matrix A a number $\det(A)$.

Definition

The determinant is characterized by:

We want to define for every $n \times n$ matrix A a number $\det(A)$.

Definition

The determinant is characterized by:

We want to define for every $n \times n$ matrix A a number $\det(A)$.

Definition

The determinant is characterized by:

- the normalization $\det I_{n\times n} = 1$,

We want to define for every $n \times n$ matrix A a number $\det(A)$.

Definition

The determinant is characterized by:

- the normalization $\det I_{n \times n} = 1$,
- and how it is affected by elementary row operations:

We want to define for every $n \times n$ matrix A a number $\det(A)$.

Definition

The determinant is characterized by:

- the normalization $\det I_{n\times n} = 1$,
- and how it is affected by elementary row operations:
  - **(Replacement)** Add a multiple of one row to another row. *Does not change* the determinant.

We want to define for every $n \times n$ matrix A a number $\det(A)$.

Definition

The determinant is characterized by:

- the normalization $\det I_{n\times n} = 1$,
- and how it is affected by elementary row operations:
  - **(Replacement)** Add a multiple of one row to another row. *Does not change* the determinant.
  - **(Interchange)** Interchange two rows. *Reverses the sign* of the determinant.

We want to define for every $n \times n$ matrix A a number $\det(A)$.

Definition

The determinant is characterized by:

- the normalization $\det I_{n\times n} = 1$,
- and how it is affected by elementary row operations:
    - **(Replacement)** Add a multiple of one row to another row.
      Does not change the determinant.
    - **(Interchange)** Interchange two rows.
      Reverses the sign of the determinant.
    - **(Scaling)** Multiply all entries in a row by s.
      Multiplies the determinant by s.

We want to define for every $n \times n$ matrix A a number $\det(A)$.

Definition

The determinant is characterized by:

- the normalization $\det I_{n\times n} = 1$,
- and how it is affected by elementary row operations:
  - **(Replacement)** Add a multiple of one row to another row.
    Does not change the determinant.
  - **(Interchange)** Interchange two rows.
    Reverses the sign of the determinant.
  - **(Scaling)** Multiply all entries in a row by s.
    Multiplies the determinant by s.

We want to define for every $n \times n$ matrix A a number $\det(A)$.

Definition

The determinant is characterized by:

- the normalization $\det I_{n\times n} = 1$,
- and how it is affected by elementary row operations:
  - **(Replacement)** Add a multiple of one row to another row. *Does not change* the determinant.
  - **(Interchange)** Interchange two rows. *Reverses the sign* of the determinant.
  - **(Scaling)** Multiply all entries in a row by s. *Multiplies* the determinant by s.

This allows us to compute the determinant using just row operations!

Important Fact

The determinant of a triangular matrix is the product of the diagonal entries.

Important Fact

The determinant of a triangular matrix is the product of the diagonal entries.

Example

$$\det \begin{bmatrix} 2 & 3 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 6 \end{bmatrix} = 2 \cdot 4 \cdot 6.$$

Important Fact

The determinant of a triangular matrix is the product of the diagonal entries.

Example

$$\det \begin{bmatrix} 2 & 3 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 6 \end{bmatrix} = 2 \cdot 4 \cdot 6.$$

Important Fact

The determinant of a triangular matrix is the product of the diagonal entries.

Example

$$\det \begin{bmatrix} 2 & 3 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 6 \end{bmatrix} = 2 \cdot 4 \cdot 6.$$

Why?

Important Fact

The determinant of a triangular matrix is the product of the diagonal entries.

Example

$$\det \begin{bmatrix} 2 & 3 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 6 \end{bmatrix} = 2 \cdot 4 \cdot 6.$$

Why? Take out the diagonal entries, and then use row operations to get the identity matrix.

Example (Generic matrix)

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$

Example (Generic matrix)

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} \underset{R3 \to R3 - 2R1}{\overset{R2 \to R2 - 3R1}{=}}$$

Example (Generic matrix)

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} \overset{\substack{R2 \to R2 - 3R1 \\ R3 \to R3 - 2R1}}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & -4 & 1 \end{vmatrix}$$

Example (Generic matrix)

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} \overset{\substack{R2 \to R2 - 3R1 \\ R3 \to R3 - 2R1}}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & -4 & 1 \end{vmatrix}$$

$$\overset{R3 \to R3 - \frac{4}{7}R2}{=}$$

Example (Generic matrix)

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} \overset{\substack{R2\to R2-3R1 \\ R3\to R3-2R1}}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & -4 & 1 \end{vmatrix}$$

$$\overset{R3\to R3-\frac{4}{7}R2}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & -\frac{1}{7} \end{vmatrix}$$

Example (Generic matrix)

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} \overset{\substack{R2\to R2-3R1 \\ R3\to R3-2R1}}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & -4 & 1 \end{vmatrix}$$

$$\overset{R3\to R3-\frac{4}{7}R2}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & -\frac{1}{7} \end{vmatrix}$$

$$= 1\cdot(-7)\cdot\left(-\frac{1}{7}\right) =$$

Example (Generic matrix)

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} \overset{\substack{R2\to R2-3R1 \\ R3\to R3-2R1}}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & -4 & 1 \end{vmatrix}$$

$$\overset{R3\to R3-\frac{4}{7}R2}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & -\frac{1}{7} \end{vmatrix}$$

$$= 1\cdot(-7)\cdot(-\frac{1}{7}) = 1$$

Example (Reality check)

Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Example (Reality check)

Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solution

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Example (Reality check)

Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solution

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{R2 \to R2 - \frac{c}{a} R1}{=}$$

Example (Reality check)

Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solution

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{R2 \to R2 - \frac{c}{a}R1}{=} \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{vmatrix} =$$

Example (Reality check)

Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solution

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{R2 \to R2 - \frac{c}{a}R1}{=} \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{vmatrix} = a\left(d - \frac{c}{a}b\right) =$$

Example (Reality check)

Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solution

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{R2 \to R2 - \frac{c}{a} R1}{=} \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{vmatrix} = a\left(d - \frac{c}{a} b\right) = ad - bc$$

Example (Reality check)

Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solution

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{R2 \to R2 - \frac{c}{a} R1}{=} \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{vmatrix} = a\left(d - \frac{c}{a} b\right) = ad - bc$$

NB: this only works if $a \neq 0$. What do you do if $a = 0$?

Example (Larger matrix)

Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

Example (Larger matrix)

Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

Solution

The matrix looks complicated, but we can do this in two steps!

Example (Larger matrix)

Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

Solution

The matrix looks complicated, but we can do this in two steps!

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$$

Example (Larger matrix)

Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

Solution

The matrix looks complicated, but we can do this in two steps!

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix} \overset{R4 \to R4 - \frac{3}{2}R3}{=}$$

Example (Larger matrix)

Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

Solution

The matrix looks complicated, but we can do this in two steps!

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix} \overset{R4 \to R4 - \frac{3}{2}R3}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & \frac{7}{2} \end{vmatrix} =$$

Example (Larger matrix)

Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

Solution

The matrix looks complicated, but we can do this in two steps!

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix} \overset{R4 \to R4 - \frac{3}{2}R3}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & \frac{7}{2} \end{vmatrix} = 1 \cdot 2 \cdot 2 \cdot \frac{7}{2} =$$

Example (Larger matrix)

Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

Solution

The matrix looks complicated, but we can do this in two steps!

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix} \overset{R4 \to R4 - \frac{3}{2}R3}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & \frac{7}{2} \end{vmatrix} = 1 \cdot 2 \cdot 2 \cdot \frac{7}{2} = 14$$

The following important properties follow from behavior under row operations.

The following important properties follow from behavior under row operations.

Important properties

- $\det(A) = 0 \iff A$ is not invertible.
  Why?

The following important properties follow from behavior under row operations.

Important properties

- $\det(A) = 0 \iff A$ is not invertible.
  Why? Because $\det(A) = 0$, if and only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).

The following important properties follow from behavior under row operations.

Important properties

- $\det(A) = 0 \iff A$ is not invertible.
  Why? Because $\det(A) = 0$, if and only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).
- $\det(AB) = \det(A)\det(B)$

The following important properties follow from behavior under row operations.

Important properties

- $\det(A) = 0 \iff A$ is not invertible.
  Why? Because $\det(A) = 0$, if and only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).
- $\det(AB) = \det(A)\det(B)$
  Challenge: Figure out why! (Matrix multiplication can be seen as linear combinations of rows)

The following important properties follow from behavior under row operations.

Important properties

- $\det(A) = 0 \iff A$ is not invertible.
  Why? Because $\det(A) = 0$, if and only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).
- $\det(AB) = \det(A)\det(B)$
  Challenge: Figure out why! (Matrix multiplication can be seen as linear combinations of rows)
- $\det(A^{-1}) = \frac{1}{\det(A)}$
  Why?

The following important properties follow from behavior under row operations.

Important properties

- $\det(A) = 0 \iff A$ is not invertible.
  Why? Because $\det(A) = 0$, if and only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).
- $\det(AB) = \det(A)\det(B)$
  Challenge: Figure out why! (Matrix multiplication can be seen as linear combinations of rows)
- $\det(A^{-1}) = \frac{1}{\det(A)}$
  Why? Because $AA^{-1} = I$.

The following important properties follow from behavior under row operations.

Important properties

- $\det(A) = 0 \iff A$ is not invertible.
  Why? Because $\det(A) = 0$, if and only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).
- $\det(AB) = \det(A)\det(B)$
  Challenge: Figure out why! (Matrix multiplication can be seen as linear combinations of rows)
- $\det(A^{-1}) = \frac{1}{\det(A)}$
  Why? Because $AA^{-1} = I$. Since $\det(AA^{-1}) = \det(I) = 1$ and $\det(AA^{-1}) =$

The following important properties follow from behavior under row operations.

Important properties

- $\det(A) = 0 \iff A$ is not invertible.
  Why? Because $\det(A) = 0$, if and only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).
- $\det(AB) = \det(A)\det(B)$
  Challenge: Figure out why! (Matrix multiplication can be seen as linear combinations of rows)
- $\det(A^{-1}) = \frac{1}{\det(A)}$
  Why? Because $AA^{-1} = I$. Since $\det(AA^{-1}) = \det(I) = 1$ and $\det(AA^{-1}) = \det(A)\det(A^{-1})$, we have

The following important properties follow from behavior under row operations.

Important properties

- $\det(A) = 0 \iff A$ is not invertible.
  Why? Because $\det(A) = 0$, if and only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).
- $\det(AB) = \det(A)\det(B)$
  Challenge: Figure out why! (Matrix multiplication can be seen as linear combinations of rows)
- $\det(A^{-1}) = \frac{1}{\det(A)}$
  Why? Because $AA^{-1} = I$. Since $\det(AA^{-1}) = \det(I) = 1$ and $\det(AA^{-1}) = \det(A)\det(A^{-1})$, we have $\det(A)\det(A^{-1}) = 1$.

The following important properties follow from behavior under row operations.

Important properties

- $\det(A) = 0 \iff A$ is not invertible.
  Why? Because $\det(A) = 0$, if and only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).
- $\det(AB) = \det(A)\det(B)$
  Challenge: Figure out why! (Matrix multiplication can be seen as linear combinations of rows)
- $\det(A^{-1}) = \frac{1}{\det(A)}$
  Why? Because $AA^{-1} = I$. Since $\det(AA^{-1}) = \det(I) = 1$ and $\det(AA^{-1}) = \det(A)\det(A^{-1})$, we have $\det(A)\det(A^{-1}) = 1$.
- $\det(A^T) = \det(A)$.

The following important properties follow from behavior under row operations.

Important properties

- $\det(A) = 0 \iff A$ is not invertible.
  Why? Because $\det(A) = 0$, if and only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).
- $\det(AB) = \det(A)\det(B)$
  Challenge: Figure out why! (Matrix multiplication can be seen as linear combinations of rows)
- $\det(A^{-1}) = \frac{1}{\det(A)}$
  Why? Because $AA^{-1} = I$. Since $\det(AA^{-1}) = \det(I) = 1$ and $\det(AA^{-1}) = \det(A)\det(A^{-1})$, we have $\det(A)\det(A^{-1}) = 1$.
- $\det(A^T) = \det(A)$.
  (Think about why this works at home.)

Remark

$\det(A^T) = \det(A)$ means that everything you know about determinants in terms of *rows* of A is also true for the columns.

Remark

$\det(A^T) = \det(A)$ means that everything you know about determinants in terms of *rows* of A is also true for the columns.

Remark

$\det(A^T) = \det(A)$ means that everything you know about determinants in terms of *rows* of A is also true for the columns.For instance:

- If you exchange two columns in a determinant you get a minus sign.

Remark

$\det(A^T) = \det(A)$ means that everything you know about determinants in terms of *rows* of A is also true for the columns.For instance:

- If you exchange two columns in a determinant you get a minus sign.
- You can add a multiple of a column to another column without changing the determinant.

Remark

$\det(A^T) = \det(A)$ means that everything you know about determinants in terms of *rows* of A is also true for the columns.For instance:

- If you exchange two columns in a determinant you get a minus sign.
- You can add a multiple of a column to another column without changing the determinant.
- If your matrix has equal columns the determinant is zero.

Remark

$\det(A^T) = \det(A)$ means that everything you know about determinants in terms of *rows* of A is also true for the columns.For instance:

- If you exchange two columns in a determinant you get a minus sign.
- You can add a multiple of a column to another column without changing the determinant.
- If your matrix has equal columns the determinant is zero.
- If your matrix has a zero column the determinant is zero.

Example

Recall that $AB = \mathbf{0}$, then it does not follow that $A = \mathbf{0}$ or $B = \mathbf{0}$.

Example

Recall that $AB = \mathbf{0}$, then it does not follow that $A = \mathbf{0}$ or $B = \mathbf{0}$. However, show that $\det(A) = 0$ or $\det(B) = 0$.

Example

Recall that $AB = \mathbf{0}$, then it does not follow that $A = \mathbf{0}$ or $B = \mathbf{0}$. However, show that $\det(A) = 0$ or $\det(B) = 0$.

Solution

If $AB = \mathbf{0}$, then $\det(AB) = \det(\mathbf{0}) = 0$.

Example

Recall that $AB = \mathbf{0}$, then it does not follow that $A = \mathbf{0}$ or $B = \mathbf{0}$. However, show that $\det(A) = 0$ or $\det(B) = 0$.

Solution

If $AB = \mathbf{0}$, then $\det(AB) = \det(\mathbf{0}) = 0$. Follows from $\det(AB) = \det(A)\det(B)$.

A "bad" way to compute determinants, Cofactor expansion

Idea

The determinant is linear in each row or column.

Idea

The determinant is linear in each row or column.

Idea

The determinant is linear in each row or column.

(In some texts this is one of the basic assumptions about det.)

Idea

The determinant is linear in each row or column.

(In some texts this is one of the basic assumptions about det.)

Fact

$$\det\begin{bmatrix} a & b & c \\ * & * & * \\ * & * & * \end{bmatrix} = \det\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} + \det\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & c \\ * & * & * \\ * & * & * \end{bmatrix}$$

Idea

The determinant is linear in each row or column.

(In some texts this is one of the basic assumptions about det.)

Fact

$$\det\begin{bmatrix} a & b & c \\ * & * & * \\ * & * & * \end{bmatrix} = \det\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} + \det\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & c \\ * & * & * \\ * & * & * \end{bmatrix}$$

Idea

The determinant is linear in each row or column.

(In some texts this is one of the basic assumptions about det.)

Fact

$$\det \begin{bmatrix} a & b & c \\ * & * & * \\ * & * & * \end{bmatrix} = \det \begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} + \det \begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & c \\ * & * & * \\ * & * & * \end{bmatrix}$$

We can use this idea to calculate an $n \times n$ determinant in terms of n determinants of smaller matrices.

Example

What is the determinant $\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$? What about $\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$?

Example

What is the determinant $\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$? What about $\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$?

Solution

$$\det \begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ * & B \end{bmatrix} = a \det \begin{bmatrix} B \end{bmatrix},$$

where B is the 2×2 right lower block.

Example

What is the determinant $\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$? What about $\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$?

Solution

$$\det \begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ * & B \end{bmatrix} = a \det \begin{bmatrix} B \end{bmatrix},$$

where B is the 2×2 right lower block.

Example

What is the determinant $\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$? What about $\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$?

Solution

$$\det \begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ * & B \end{bmatrix} = a \det \begin{bmatrix} B \end{bmatrix},$$

where B is the 2×2 right lower block. Same way, with a twist:

Example

What is the determinant $\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$? What about $\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$?

Solution

$$\det \begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ * & B \end{bmatrix} = a \det \begin{bmatrix} B \end{bmatrix},$$

where B is the 2×2 right lower block. Same way, with a twist:

$$\det \begin{bmatrix} 0 & b & 0 \\ v_1 & v_2 & v_3 \end{bmatrix} =$$

Example

What is the determinant $\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$? What about $\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$?

Solution

$$\det \begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ * & B \end{bmatrix} = a \det \begin{bmatrix} B \end{bmatrix},$$

where B is the 2×2 right lower block. Same way, with a twist:

$$\det \begin{bmatrix} 0 & b & 0 \\ v_1 & v_2 & v_3 \end{bmatrix} = \boxed{-1} b \det \left(\begin{bmatrix} 1 & 0 & 0 \\ v_2 & v_1 & v_3 \end{bmatrix} \right) =$$

Example

What is the determinant $\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$? What about $\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$?

Solution

$$\det \begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ * & B \end{bmatrix} = a \det \begin{bmatrix} B \end{bmatrix},$$

where B is the 2×2 right lower block. Same way, with a twist:

$$\det \begin{bmatrix} 0 & b & 0 \\ v_1 & v_2 & v_3 \end{bmatrix} = \boxed{-1} b \det \left(\begin{bmatrix} 1 & 0 & 0 \\ v_2 & v_1 & v_3 \end{bmatrix} \right) = -b \det \begin{bmatrix} v_1 & v_3 \end{bmatrix}.$$

Example

What is the determinant $\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$? What about $\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$?

Solution

$$\det \begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ * & B \end{bmatrix} = a \det \begin{bmatrix} B \end{bmatrix},$$

where B is the 2×2 right lower block. Same way, with a twist:

$$\det \begin{bmatrix} 0 & b & 0 \\ v_1 & v_2 & v_3 \end{bmatrix} = \boxed{-1} b \det \left(\begin{bmatrix} 1 & 0 & 0 \\ v_2 & v_1 & v_3 \end{bmatrix} \right) = -b \det \begin{bmatrix} v_1 & v_3 \end{bmatrix}.$$

We can use this idea to calculate an $n \times n$ determinant in terms of n determinants of $(n-1) \times (n-1)$ matrices. Then repeat ....

Example

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution

Example

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution

We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} =$$

Example

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution

We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix}$$

Example

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution

We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix}$$

Example

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution

We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix}$$

Example

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution

We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix}$$

$\overset{\text{i.e.}}{=}$

Example

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution

We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix}$$

$$\overset{\text{i.e.}}{=} 1 \cdot \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} -$$

Example

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution

We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix}$$

$$\overset{\text{i.e.}}{=} 1 \cdot \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} +$$

Example

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution

We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix}$$

$$\overset{\text{i.e.}}{=} 1 \cdot \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} =$$

Example

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution

We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1\cdot \begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix} - 2\cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + 0\cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix}$$

$$\overset{\text{i.e.}}{=} 1\cdot \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0\cdot \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 1\times(-1) - 2\cdot(-1) + 0 =$$

Example

Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution

We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1\cdot \begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix} - 2\cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + 0\cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix}$$

$$\overset{\text{i.e.}}{=} 1\cdot \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0\cdot \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 1\times(-1) - 2\cdot(-1) + 0 = 1$$

Each term in the cofactor expansion is ± 1 times an entry times a smaller determinant (row and column of entry deleted).

Each term in the cofactor expansion is ± 1 times an entry times a smaller determinant (row and column of entry deleted).
The $\pm$ is assigned to each entry according to

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}.$$

Each term in the cofactor expansion is ± 1 times an entry times a smaller determinant (row and column of entry deleted).
The $\pm$ is assigned to each entry according to

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}.$$

There is nothing special about the first row. We can use any other row or column.

For example, let's use the second column:

For example, let's use the second column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} =$$

For example, let's use the second column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} +$$

For example, let's use the second column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & & 0 \\ & + & \\ 2 & & 1 \end{vmatrix} -$$

For example, let’s use the second column:

Solution

$$\begin{vmatrix}1&2&0\\3&-1&2\\2&0&1\end{vmatrix} = -2\cdot\begin{vmatrix} & - & \\ 3 & & 2\\ 2 & & 1\end{vmatrix} + (-1)\cdot\begin{vmatrix}1 & & 0\\ & + & \\ 2 & & 1\end{vmatrix} - 0\cdot\begin{vmatrix}1 & & 0\\ 3 & & 2\\ & - & \end{vmatrix}$$

For example, let's use the second column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2\cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + (-1)\cdot \begin{vmatrix} 1 & & 0 \\ & + & \\ 2 & & 1 \end{vmatrix} - 0\cdot \begin{vmatrix} 1 & & 0 \\ 3 & & 2 \\ & - & \end{vmatrix}$$

$$= -2\cdot(-1) + (-1)\cdot 1 - 0 =$$

For example, let’s use the second column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2\cdot\begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + (-1)\cdot\begin{vmatrix} 1 & & 0 \\ & + & \\ 2 & & 1 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & & 0 \\ 3 & & 2 \\ & - & \end{vmatrix}$$

$$= -2\cdot(-1) + (-1)\cdot 1 - 0 = 1$$

For example, let's use the second column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2\cdot\begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + (-1)\cdot\begin{vmatrix} 1 & & 0 \\ & + & \\ 2 & & 1 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & & 0 \\ 3 & & 2 \\ & - & \end{vmatrix}$$

$$= -2\cdot(-1) + (-1)\cdot 1 - 0 = 1$$

Same answer!

Let use the third column:

Let use the third column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} =$$

Let use the third column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix} -$$

Let use the third column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 & \\ & & - \\ 2 & 0 & \end{vmatrix} +$$

Let use the third column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 & \\ & & - \\ 2 & 0 & \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 & \\ 3 & -1 & \\ & & + \end{vmatrix}$$

Let use the third column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 & \\ & & - \\ 2 & 0 & \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 & \\ 3 & -1 & \\ & & + \end{vmatrix}$$

$$= 0 - 2 \cdot (-4) + 1 \cdot (-7) =$$

Let use the third column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 & \\ & & - \\ 2 & 0 & \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 & \\ 3 & -1 & \\ & & + \end{vmatrix}$$

$$= 0 - 2 \cdot (-4) + 1 \cdot (-7) = 1$$

Let use the third column:

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 & \\ & & - \\ 2 & 0 & \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 & \\ 3 & -1 & \\ & & + \end{vmatrix}$$

$$= 0 - 2 \cdot (-4) + 1 \cdot (-7) = 1$$

Same answer!

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)?

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)? Because to compute a large $n \times n$ matrix,

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)? Because to compute a large $n \times n$ matrix,

- one reduces to n determinants of size $(n-1) \times (n-1)$,

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)? Because to compute a large $n \times n$ matrix,

- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,

Why not cofactor expansion

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)? Because to compute a large $n \times n$ matrix,

- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,
- and so on.

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)? Because to compute a large $n \times n$ matrix,

- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,
- and so on.

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)? Because to compute a large $n \times n$ matrix,

- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,
- and so on.

In the end, we have $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ many numbers to add.

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)? Because to compute a large $n \times n$ matrix,

- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,
- and so on.

In the end, we have $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ many numbers to add. WAY TOO MUCH WORK!

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)? Because to compute a large $n \times n$ matrix,

- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,
- and so on.

In the end, we have $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ many numbers to add. WAY TOO MUCH WORK! Already

$$25! = 15511210043330985984000000 \approx 1.55 \cdot 10^{25}.$$

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)? Because to compute a large $n \times n$ matrix,

- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,
- and so on.

In the end, we have $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ many numbers to add. WAY TOO MUCH WORK! Already

$$25! = 15511210043330985984000000 \approx 1.55 \cdot 10^{25}.$$

Context: today's fastest computer, Tianhe-2, runs at 34 pflops ($3.4 \cdot 10^{16}$ operations per second).

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)? Because to compute a large $n \times n$ matrix,

- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,
- and so on.

In the end, we have $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ many numbers to add. WAY TOO MUCH WORK! Already

$$25! = 15511210043330985984000000 \approx 1.55 \cdot 10^{25}.$$

Context: today's fastest computer, Tianhe-2, runs at 34 pflops ($3.4 \cdot 10^{16}$ operations per second).
By the way: "fastest" is measured by computing LU decompositions!

Practice Problems

Example

Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$. Use your favorite method (or a mix of methods!)

Solution

The final answer should be -10.

- What's **wrong**?!

$$\det(A^{-1}) = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}(da-(-b)(-c) = 1$$

- What's **wrong**?!

$$\det(A^{-1}) = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}(da-(-b)(-c) = 1$$

The correct calculation is:

$$\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} =$$

- What's **wrong**?!

$$\det(A^{-1}) = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}(da-(-b)(-c) = 1$$

The correct calculation is:

$$\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(ad-bc)^2}\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}$$

- What's **wrong**?!

$$\det(A^{-1}) = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}(da-(-b)(-c)) = 1$$

The correct calculation is:

$$\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(ad-bc)^2}\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}$$

Example

Suppose A is a 3×3 matrix with $\det(A) = 5$. What is $\det(2A)$?

- What's **wrong**?!

$$\det(A^{-1}) = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}(da-(-b)(-c)) = 1$$

The correct calculation is:

$$\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(ad-bc)^2}\det \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}$$

Example

Suppose A is a 3×3 matrix with $\det(A) = 5$. What is $\det(2A)$?

Solution

A has three rows. Multiplying all 3 of them produces $2A$. Hence, $\det(2A) = 2^3 \det(A) = 40$.

Example

First off, say hello to our new friend: i, the **imaginary unit**.

Example

First off, say hello to our new friend: i, the **imaginary unit**.

Example

First off, say hello to our new friend: i, the **imaginary unit**. It is infamous for $\boxed{i^2 = -1}$.

Example

First off, say hello to our new friend: i, the **imaginary unit**. It is infamous for $\boxed{i^2 = -1}$. Let us calculate some determinants.

Example

First off, say hello to our new friend: i, the **imaginary unit**. It is infamous for $\boxed{i^2 = -1}$. Let us calculate some determinants.

$$|1| = 1$$

Example

First off, say hello to our new friend: i, the **imaginary unit**. It is infamous for $\boxed{i^2 = -1}$. Let us calculate some determinants.

$$\begin{aligned} |1| &= 1 \\ \begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} &= 1 - i^2 = 2 \end{aligned}$$

Example

First off, say hello to our new friend: i, the **imaginary unit**. It is infamous for $\boxed{i^2 = -1}$. Let us calculate some determinants.

$$\begin{aligned} |1| &= 1 \\ \begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} &= 1 - i^2 = 2 \\ \begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} &= 1 \begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} - i \begin{vmatrix} i & 0 \\ i & 1 \end{vmatrix} = 2 - i^2 = 3 \end{aligned}$$

Example

First off, say hello to our new friend: i, the **imaginary unit**. It is infamous for $\boxed{i^2 = -1}$. Let us calculate some determinants.

$$\begin{aligned}
\begin{vmatrix} 1 \end{vmatrix} &= 1 \\
\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} &= 1 - i^2 = 2 \\
\begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} &= 1 \begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} - i \begin{vmatrix} i & 0 \\ i & 1 \end{vmatrix} = 2 - i^2 = 3 \\
\begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} &= 1 \begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} - i \begin{vmatrix} i & 0 & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 3 - i^2 \begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} = 5
\end{aligned}$$

Example (continued)

$$\begin{vmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & i & \\ & & i & 1 & i \\ & & & i & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} - i^2 \begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 5 + 3 = 8$$

Example (continued)

$$\begin{vmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & i & \\ & & i & 1 & i \\ & & & i & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} - i^2 \begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 5 + 3 = 8$$

Example (continued)

$$\begin{vmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & i & \\ & & i & 1 & i \\ & & & i & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} - i^2 \begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 5 + 3 = 8$$

The Fibonacci numbers!

Example (continued)

$$\begin{vmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & i & \\ & & i & 1 & i \\ & & & i & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} - i^2 \begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 5 + 3 = 8$$

The Fibonacci numbers!

Example (continued)

$$\begin{vmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & i & \\ & & i & 1 & i \\ & & & i & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} - i^2 \begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 5 + 3 = 8$$

The Fibonacci numbers!

Do you know about the connection of Fibonacci numbers and rabbits? If not, Google is your friend.

Example (continued)

$$\begin{vmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & i & \\ & & i & 1 & i \\ & & & i & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} - i^2 \begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 5 + 3 = 8$$

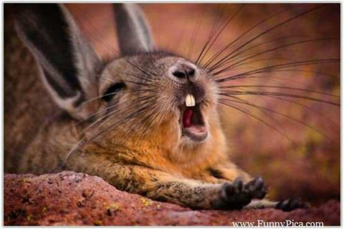
The Fibonacci numbers!

Do you know about the connection of Fibonacci numbers and rabbits? If not, Google is your friend.

Example (continued)

$$\begin{vmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & i & \\ & & i & 1 & i \\ & & & i & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} - i^2 \begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 5 + 3 = 8$$

The Fibonacci numbers!



Do you know about the connection of Fibonacci numbers and rabbits? If not, Google is your friend.