Worksheet 5 (September 22nd and 24th)

1. *Determine which of the following sets are subspaces of the indicated vector spaces and* **give reasons**. *For any sets that are subspaces, find a matrix A such that $W_i = Null(A)$ or $W_i = Col(A)$.*

(a) $W_1 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 2b = c, 4a + 2c = 0 \right\} \subseteq \mathbb{R}^3$,

(b) $W_2 = \left\{ \begin{bmatrix} a-b \\ c \\ a+c \\ a-2b-c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \subseteq \mathbb{R}^4$,

(c) $W_3 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a \cdot b \geq 0 \right\} \subseteq \mathbb{R}^2$.

(d) $W_4 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a^2 + b^2 \leq 1 \right\} \subseteq \mathbb{R}^2$.

Also sketch the sets W_3 and W_4 and give geometric reasons why W_3 and W_4 either are or are not subspaces of $\mathbb{R}^2$.

Solution. (a) W_1 is a subspace of $\mathbb{R}^3$, since it is the Nullspace of the matrix

$$\begin{bmatrix} 1 & -2 & -1 \\ 4 & 0 & 2 \end{bmatrix}$$

(b) Since

$$\left\{ \begin{bmatrix} a-b \\ c \\ a+c \\ a-2b-c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and every span of a set of vectors in $\mathbb{R}^4$ is a subspace of $\mathbb{R}^4$.We can also show this by seeing that W_2 is the column space of

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & -1 \end{bmatrix}$$

(c) W_3 is not a subspace of $\mathbb{R}^2$. Consider the two vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Both are in W_3, because $1 \cdot 0 = 0 \cdot (-1) \geq 0$. But

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Midterm Dates: September 29th, October 22nd, November 19th (All Midterms 7-8:15 PM, see `learn.illinois.edu` **for locations)**

and $1 \cdot (-1) = -1 < 0$. Hence $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not in W_3. Hence W_3 is not closed under addition.A sketch of W_3 in $\mathbb{R}^2$ will contain the x-axis, the y-axis and the first and third quadrants. Plotting the above two vectors and their sum on your sketch will show visually how vector addition will leave W_3. Furthermore, the only subspaces of $\mathbb{R}^2$ are (1) the origin, (2) a single line through the origin or (3) the entire plane; its clear from the sketch that W_3 is none of these. [Note: on a quiz or exam, simply sketching W_3 is **not** sufficient justification for why it is not a subspace of $\mathbb{R}^2$, you must give an **explicit** counterexample and state which rule for subspaces is violated (closure under addition in the case of W_3). However, it may be a good idea to use a sketch to help find a counterexample.]

(d) This set is not a subspace of $\mathbb{R}^2$. Consider the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since $1^2 \leq 1$, we have that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in W_4. However, $2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is not in W_4, since $2^2 = 4 > 1$. Hence W_4 is not closed under scalar multiplication. A sketch of W_4 is the disc of radius 1 in $\mathbb{R}^2$ (i.e., the "filled-in" circle of radius 1). By plotting our counterexample, we can see visually that W_4 is not closed under scalar multiplication. Also note that W_4 is not closed under vector addition - can you find a counterexample? $\square$

2. *Is* $H = \left\{ \begin{bmatrix} a+1 \\ a \end{bmatrix} : a \text{ in } \mathbb{R} \right\}$ *a subspace of* $\mathbb{R}^2$*? Why or why not?*

Is $K = \left\{ \begin{bmatrix} a+1 \\ b \end{bmatrix} : a \text{ and } b \text{ in } \mathbb{R} \right\}$ *a subspace of* $\mathbb{R}^2$*? Why or why not?*

Solution. The set H is not a subspace of $\mathbb{R}^2$, because it does not contain the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (Why? Because if there is a in $\mathbb{R}$ such that

$$\begin{bmatrix} a+1 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then $a + 1 = 0$ and $a = 0$. Such an a cannot exist). While H is not a subspace of $\mathbb{R}^2$, K **is** a subspace of $\mathbb{R}^2$. It is enough to realize that

$$\left\{ \begin{bmatrix} a+1 \\ b \end{bmatrix} : a \text{ and } b \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} c \\ b \end{bmatrix} : c \text{ and } b \text{ in } \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2.$$

Note that the first equality holds, because you can take c to be $a - 1$. Equivalently, you can show directly that $K = \mathbb{R}^2$ (i.e., K and $\mathbb{R}^2$ are **the same set of vectors**!), and $\mathbb{R}^2$ is a subspace of itself. $\square$

3. *Are the following subspaces of* $M_{2\times 2}$*, the set of all* 2×2 *matrices?*

(a) *S, the set of all* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$*, such that* $ad - bc = 0$

(b) *V, the set of all* 2×2 *matrices such that* $B^T = B$

Solution. (a) S is not a subspace of $M_{2\times 2}$ since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in S$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in S$, but $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin S$. Thus S is not closed under addition.

(b) Yes, V is a subspace of $M_{2\times 2}$. We have to check that it contains $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and is closed under addition and scalar multiplication. First note that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in V, because it is obviously symmetric.

Now take two matrices A, B in V. So we have $A^T = A$ and $B^T = B$. Then we have

$$(A+B)^T = A^T + B^T = A + B.$$

Hence $A + B$ is in V. So V is closed under addition.

Now take a matrix A in V and a scalar r. Since A is in V, we have $A^T = A$. Then we have

$$(rA)^T = rA^T = rA.$$

Hence rA is in V. So V is closed under scalar multiplication.

Equivalently, you can justify that every symmetric matrix is of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$ (why?). Thus

$$V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} = \operatorname{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a subspace of $M_{2\times 2}$ since it is the span of the "vectors" $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. $\square$

4. *For the* 3×5 *matrix*

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

determine the nullspace, $\mathrm{Nul}(A)$*, of* A*. Write your answer as the span of a set of vectors.*

Solution. $\mathrm{Nul}(A)$ is the set of all solutions to the homogeneous equation

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

As A is already in reduced row echelon form, we identify x_2 and x_4 as pivot variables and x_1, x_3 and x_5 as the free variables. Thus we can write the general solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_3 - 2x_5 \\ x_3 \\ -3x_5 \\ x_5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

where $x_1, x_3, x_5 \in \mathbb{R}$ are free.We conclude that

$$\mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-2\\0\\-3\\1\end{bmatrix} \right\}.$$

$\square$

5. *Consider the* 2×2 *matrix*

$$A = \begin{bmatrix}2 & -6\\-5 & 15\end{bmatrix}.$$

Determine which of the following vectors

$$\begin{bmatrix}-6\\2\end{bmatrix}, \begin{bmatrix}2\\15\end{bmatrix}, \begin{bmatrix}4\\-10\end{bmatrix}$$

belong to the column space, $\mathrm{Col}(A)$*, of* A*. Also, find* B *such that* $\mathrm{Col}(A) = \mathrm{Nul}(B)$*.*

Solution. By definition,

$$\mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix}2\\-5\end{bmatrix}, \begin{bmatrix}-6\\-15\end{bmatrix} \right\} = \mathrm{span}\left\{ \begin{bmatrix}2\\-5\end{bmatrix} \right\} = \left\{ \alpha \begin{bmatrix}2\\-5\end{bmatrix} : \alpha \in \mathbb{R} \right\},$$

i.e., in this case the column space of A is just the set of all scalar multiples of the vector $\begin{bmatrix}2\\-5\end{bmatrix}$. Thus by inspection we see that

$$\begin{bmatrix}-6\\2\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}2\\15\end{bmatrix}$$

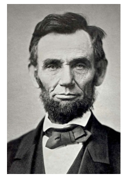
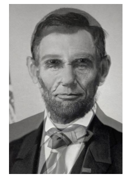
are not in $\mathrm{Col}(A)$ (because they are not scalar multiples of $\begin{bmatrix}2\\-5\end{bmatrix}$), whereas

$$\begin{bmatrix}4\\-10\end{bmatrix} = 2\begin{bmatrix}2\\-5\end{bmatrix}$$

is a member of the column space of A.Since $\mathrm{Col}(A)$ consists of all multiples of $\begin{bmatrix}2\\-5\end{bmatrix}$, we look for B such that $B\begin{bmatrix}2\\-5\end{bmatrix} = 0$.

$B = \begin{bmatrix}5 & 2\end{bmatrix}$ works. $\square$

6. *Let us consider the vector space* $M_{m\times n}$ *of* $m \times n$*-matrices. We can think of a* ***grayscale picture*** *consisting of* $m \times n$ *many pixels as a matrix* $(a_{ij})_{1\le i\le m, 1\le j\le n}$ *in this vector space where all the* a_{ij}*'s are between* 0 *and* 1*. Here the entry* a_{ij} *of a grayscale picture represents the grayscale value of the pixel at position* i, j *in this picture. So the value* 1 *means the pixel is white and* 0 *means the pixel is black. Many operations on this vector space correspond to functions your favorite image manipulation software can carry out. For example, let* P_1 *and* P_2 *be two grayscale pictures. Then taking the linear combination* $\frac{1}{2}P_1 + \frac{1}{2}P_2$ *is the same as blending the two pictures together. For example,*

$0.5\cdot$  $+0.5\cdot$  $=$  .

In this exercise we will look at a few other operations.

(a) *Which vector space operation of $M_{m\times n}$ corresponds to changing the brightness of a grayscale picture?*

(b) *Let P be a grayscale picture and let B be the $m \times n$-matrix all whose entries are 1. Calculating $B - P$ correspond to which function of your favorite image manipulation software?*

(c) *Let suppose that $m = n$. Let P be a grayscale picture and let C be the $m \times m$-matrix of the form*

$$\begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & 0 & 1 & 0 \\ 0 & \dots & 0 & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}.$$

What happens to the picture P if you multiply it with C from left (ie. calculate CP)? What happens to P if you multiply it with C from right?

Solution. (a) This will correspond with scalar multiplication on $M_{m\times n}$.

(b) This will invert the brightness of a picture. At brighter pixels, we will now have darkness (because if a_{ij} is near to 1, then $B - P$ will have an entry near to 0). Similarly dark pixels will become brighter.

(c) If we multiply from the left, this swaps the first and last row, the second and second to last, and so on. This will flip the picture along the horizontal axis. Thus the top and bottom of the picture will be swapped.If instead we multiply from the right, then this will swap columns instead of rows, thus this will give a flip about the vertical axis, and will mirror the image.

□

The following may be useful in the above problems:

Definition. A **subspace** of a vector space V is a subset H of V that has three properties:

(1) The zero vector V is in H.

(2) For each $\mathbf{u}$ and $\mathbf{v}$ in H, $\mathbf{u} + \mathbf{v}$ is in H. (In this case, we say H is **closed under vector addition**.)

(3) For each $\mathbf{u}$ in H and each scalar $c \in \mathbb{R}$, $c\mathbf{u}$ is in H. (In this case, we say H is **closed under scalar multiplication**.)

Theorem. If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V, the the **subset** span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of V, is also a **subspace** of V.

Theorem (Zero Test)**.** If H is a **subset** of the vector space V, and the zero vector $\mathbf{0}$ is **not** in H, then H is **not** a subspace of V. (Caution: The converse of the zero test is not always true!)

Definition. The **nullspace** of an $m \times n$ matrix A, written Nul(A), is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In other words,

$$\text{Nul}(A) = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

Definition. The **column space** of an $m \times n$ matrix A, written $\mathrm{Col}(A)$, is the set of all linear columns of A. In other words, if $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$, then

$$\mathrm{Col}(A) = \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}.$$