Math 415 - Lecture 37

Singular Value Decomposition

Friday December 4th 2015

Textbook reading: Chapter 6.3

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Suggested practice exercises: Chapter 6.3, # 1, 2, 3, 5, 8, 9, 15

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Strang lecture: Lecture 29: Singular Value Decomposition

Review

- Spectral theorem: If A is an $n \times n$ symmetric matrix, then it has an orthonormal basis of eigenvectors $\mathbf{v}_1 \dots \mathbf{v}_n$, and all eigenvalues are real.

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  - Doesn't even have to be square!
  - The price we pay: different bases on the left and right sides.

Singular Value Decomposition

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Remember: for each A we get 4 subspaces

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- Output space $\mathbb{R}^m$ contains columns space $\mathrm{Col}(A)$ and left null space $\mathrm{Nul}(A^T)$. Dimensions are r and $m - r$.

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- We get $A\mathbf{v_i} = \sigma_i \mathbf{u_i}$ for $i = 1, 2, \ldots, r$. The stretch factors $\sigma_i > 0$, $i = 1, 2, \ldots, r$ are called the *Singular Values* of A

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- Extend the $\mathbf{u_i}$ basis of $\text{Col}(A)$ to a basis $\mathbf{u_{r+1}}, \ldots, \mathbf{u_m}$ of the output space.

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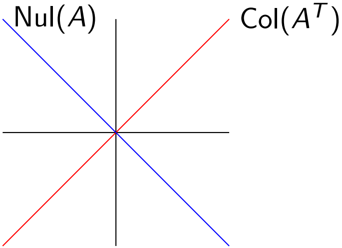

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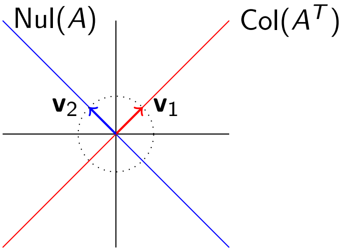

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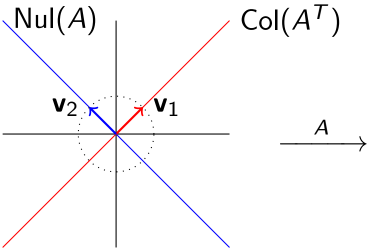

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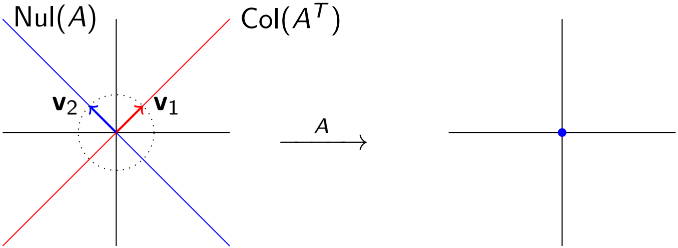

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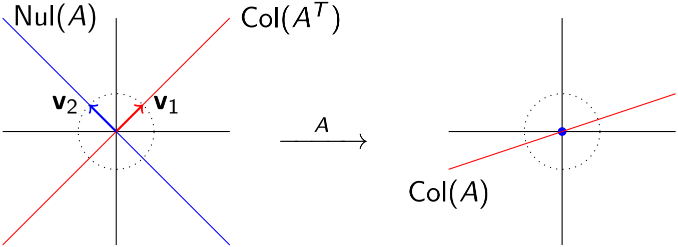

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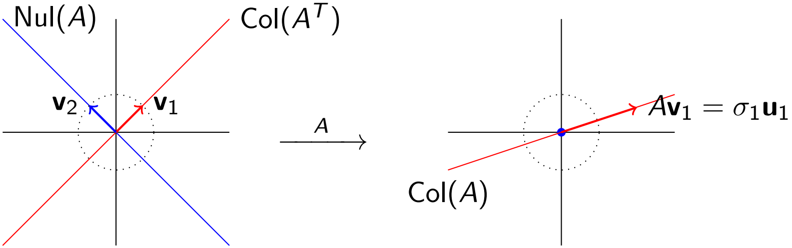

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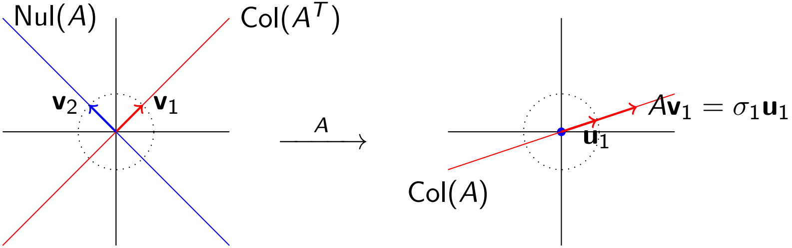
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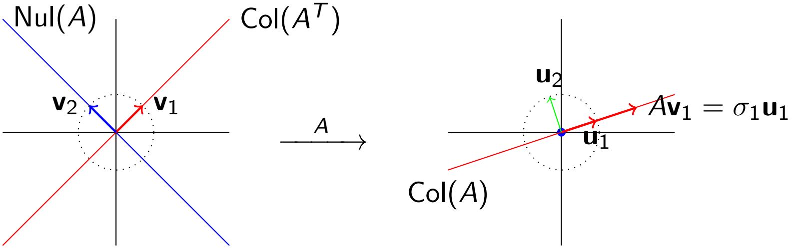

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What is SVD?

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In Linear Algebra everything is a matrix factorization.

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The complicated story with orthonormal basis and singular values for A gives a factorization, called Singular Value Decomposition:

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- Magic: $\lambda_1, \ldots, \lambda_r$ always positive!

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- Of $\mathbf{v_1}, \ldots, \mathbf{v_n}$, make the first ones, $\mathbf{v_1}, \ldots, \mathbf{v_r}$, be ones with **nonzero** eigenvalues $\lambda_1 \geq \ldots \geq \lambda_r > 0$.
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Now you have $A = U\Sigma V^T$!

Example

Compute the SVD of

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Solution

Example

Compute the SVD of
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Solution

Compute $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Make V: Basis of eigenvectors for $A^T A$: $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and

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Solution (continued)

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Note: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has no real eigenvalues.

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Note: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has no real eigenvalues. It's not diagonalizable with real matrices! But, it has an SVD! Wolfram Alpha

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Compute the SVD of

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution

Example

Compute the SVD of

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution

Compute $A^T A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

Make V: Basis of eigenvectors for $A^T A$:

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

with eigenvalues $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0$.

Solution (continued)

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$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2$$

Solution (continued)

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$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{1} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Solution (continued)

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Final result:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Notice how A behaves in the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

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$A\mathbf{v}_1 = \sigma_1 \mathbf{u}_1$:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} = \sqrt{3} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$A\mathbf{v}_2 = \sigma_2 \mathbf{u}_2$:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = 1 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$A\mathbf{v}_3 = 0$.

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Note the difference: for $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ the eigenvalues are $\lambda = i, -i$ but the singular values are $\sigma = 1, 1$.

Approximation

* To calculate matrix product AB we can use the **ROW** times **COLUMN** method: the ij component is the product $R_i B_j$, where R_i is row i of A and B_j is the jth column of B.

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix} =$$

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* This works for any matrix multiplication: AB is a sum of **COLUMN** times **ROW** matrices.

It turns out we can write A as a sum:

$$A = \underbrace{\begin{bmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_m \\ | & & | \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 & \\ 0 & \sigma_2 & \\ & & \ddots \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} - & \mathbf{v}_1^T & - \\ & \vdots & \\ - & \mathbf{v}_n^T & - \end{bmatrix}}_{V^T}$$

$$= \sigma_1 \mathbf{u_1} \mathbf{v_1}^T + \sigma_2 \mathbf{u_2} \mathbf{v_2}^T + \cdots + \sigma_r \mathbf{u_r} \mathbf{v_r}^T$$

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(Sanity check: An $m \times 1$ column vector times a $1 \times n$ row vector is an $m \times n$ matrix.)

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Idea

We can get a good approximation to A by taking the entries of the sum with the largest singular values! We'll see this when we talk about image compression later.

Example

If $\mathbf{u}, \mathbf{v}$ are non-zero, then the matrix $\mathbf{u}\mathbf{v}^T$ has rank 1. Why?

Approximation

Example

Use

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

to write $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ as a sum of rank 1 matrices.

Solution

SVD and the Four Fundamental Subspaces

The SVD of A gives orthonormal bases for all four fundamental subspaces of A.

Given $\{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$ and $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$,

- $Col(A^T) = \text{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$

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Practice Questions

Example

Suppose A is an invertible square matrix. Find a singular value decomposition of A^{-1}.

Example

If A is a square matrix, then $|\det(A)|$ is the product of the singular values of A. Why?

Example

Find the singular value decomposition of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$.