Math 415 - Lecture 33

Diagonalization

Monday November 16th 2015

Textbook reading: Chapter 5.2

Suggested practice exercises: Chapter 5.2: 1, 2, 3, 4, 5, 6, 7, 15, 16, 17, 18, 19, 20, 25, 26, 29, 30, 31, 32, 33

Strang lecture: Lecture 22: Diagonalization and powers of A

1 Review

- Eigenvector equation: $A\mathbf{x} = \lambda \mathbf{x} \iff (A \lambda I)\mathbf{x} = \mathbf{0}$ λ is an eigenvalue of $A \iff \det(A \lambda I) = 0$.
- An $n \times n$ matrix A has up to n eigenvectors for λ .
 - The **eigenspace** of λ is $Nul(A-\lambda I)$. That is, all eigenvectors of A with eigenvalues λ (plus the zero vector).
 - If λ has **multiplicity** m, then A has up to m (independent) eigenvectors for λ . At least one eigenvector is guaranteed (because $\det(A \lambda I) = 0$).
 - An **Eigenbasis** for an $n \times n$ matrix A is a basis $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ of \mathbb{R}^n so that each $\mathbf{v_i}$ is also an eigenvector: $A\mathbf{v_i} = \lambda_i \mathbf{v_i}$.
- Test yourself! What are the eigenvectors and eigenvalues? Is there an eigen basis?
 - $-\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $\lambda=1,1$ (i.e. multiplicity 2), eigenspace is \mathbb{R}^2 . Any basis is eigen basis.
 - $-\begin{bmatrix}0&0\\0&0\end{bmatrix}$, $\lambda=0,0$, eigenspace is \mathbb{R}^2 . Again any basis is an eigenbasis.

These are trivial cases. Is there always an eigenbasis?

Example 1. To solve $A\mathbf{x} = \mathbf{b}$ we use row operations. If we want to find eigenvectors, $A\mathbf{x} = \lambda \mathbf{x}$, can we also use row operations? Try $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$.

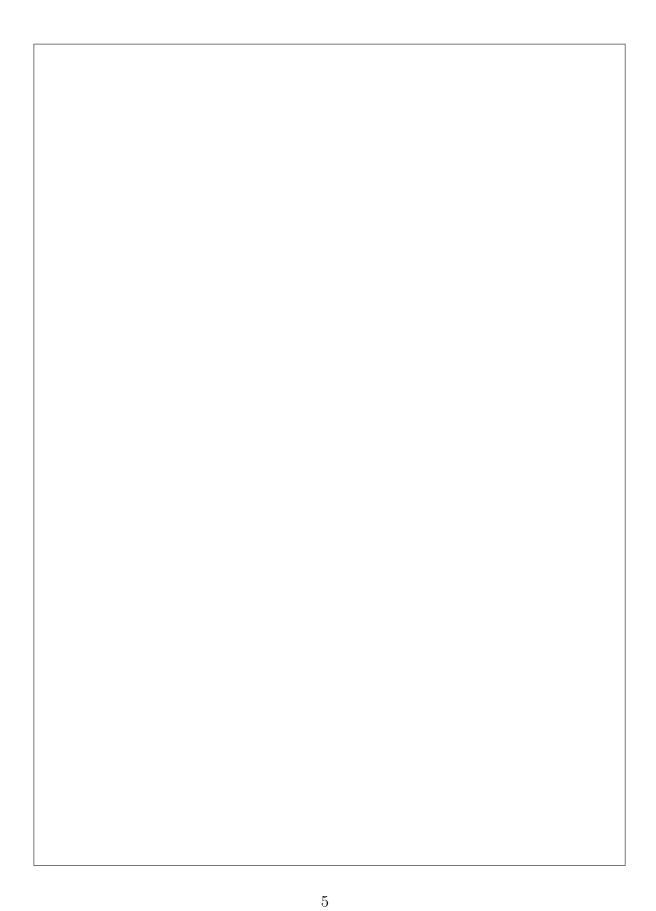
- What is the echelon form U of A?
- What are the characteristic polynomials $\det(A \lambda I)$ and $\det(U \lambda I)$? Roots?
- ullet Do A and U have the same eigenvalues? Eigenvectors?

| S | Solution. | | | | | | | |
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Upshot: Don't use row operations to deal with eigenvalues and eigenvectors! (Can use row operations to calculate determinants, though.)

| Example 2. | Find the eige | nvectors and | d eigenvalues | s of $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ | $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Wha | at is the trouble? |
|------------|---|--------------|-----------------|---|--|--------------------|
| Solution. | | | | | | |
| | | | | | | |
| | agonaliza Let $A = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ | | at is A^2 ? W | That is A^{100} | ? | |
| | | | | | | |

Example 4. If $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$, then $A^{100} = ?$ Solution.



The key idea of previous example is to work with respect to an *Eigenbasis*, a basis given by eigenvectors.

• Put the eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ as columns into a matrix P.

$$A\mathbf{x}_{i} = \lambda \mathbf{x}_{i} \implies A \begin{bmatrix} | & & | \\ \mathbf{x}_{1} & \cdots & \mathbf{x}_{n} \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_{1}\mathbf{x}_{1} & \cdots & \lambda_{n}\mathbf{x}_{n} \\ | & & | \end{bmatrix}$$
$$= \begin{bmatrix} | & & | \\ \mathbf{x}_{1} & \cdots & \mathbf{x}_{n} \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_{1} & & & \\ & \ddots & & \\ & & & \lambda_{n} \end{bmatrix}$$

• In summary AP = PD. Such a diagonalization is possible if and only if A has enough eigenvectors.

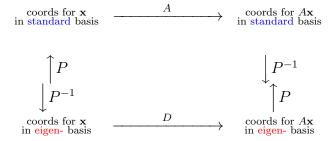
So we are going to use eigenvalues and eigenvectors for A to factor A and A^{100} in a useful way. This is called *diagonalization*.

Definition. A square matrix A is said to be **diagonalizable** if there is a invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

Theorem 1. An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

We can express the relation between A and D in terms of change of base matrices.



P changes from eigenbasis coordinates to standard coordinates, and P^{-1} goes the other way! Let \mathcal{E} be the standard basis of \mathbb{R}^n and \mathcal{B} the basis of eigenvectors of A, then

$$P = I_{\mathcal{E},\mathcal{B}}$$
 and $P^{-1} = I_{\mathcal{B},\mathcal{E}}$.

3 Application: Large powers

If A has an eigenbasis, then we can raise it to large powers easily!

Theorem 5. If $A = PDP^{-1}$, where D is a diagonal matrix, then for any m,

$$A^m = PD^mP^{-1}$$

Proof.

Finding D^m is easy!

$$D^{m} = \begin{bmatrix} \lambda_{1} & & \\ & \ddots & \\ & & \lambda_{n} \end{bmatrix}^{m} = \begin{bmatrix} (\lambda_{1})^{m} & & \\ & & \ddots & \\ & & & (\lambda_{n})^{m} \end{bmatrix}$$

Example 6. Let $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$. A has eigenvectors and eigenvalues

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{with} \quad \lambda_1 = \frac{1}{2}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{with} \quad \lambda_2 = 1$$

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} \quad \text{with} \quad \lambda_3 = 2$$

| Find A^{100} . | | | | | | |
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| Solution. | | | | | | |
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