Math 415 - Lecture 34

Discrete dynamical systems, Spectral Theorem

Wednesday November 18th 2015

Textbook reading: Chapter 5.3, Chapter 5.6 p. 297-298

Suggested practice exercises: Chapter 5.3, 2, 3, 4, 7, 8, 9, 10, 12, 14

Strang lecture: Lecture 25: Symmetric Matrices and Positive Definiteness

1 Review

Diagonalization

Suppose that A is an $n \times n$ and has independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then A can be **diagonalized** as $A = PDP^{-1}$.

- the columns of P are eigenvectors
- the diagonal matrix D has the eigenvalues on the diagonal

Such a diagonalization is possible if and only if A has an eigenbasis.

Calculating Powers

If $A = PDP^{-1}$ for some diagonal matrix D, then $A^n = PD^nP^{-1}$ for every n. This is helpful, because calculating powers of diagonal matrices is very easy!

2 Application: Discrete Dynamical Systems

Suppose you want to describe the evolution of some part of the world. Describe the **state** of your part of the world at time $t = 0$ by a vector $\mathbf{x}_{t=0}$, the **state-vector.** For a simple system the state vector $\mathbf{x}_t$ might have 2 components, for a complicated system there might be thousands of components. Then you want to know what the state $\mathbf{x}_t$ at arbitrary time t is. How? Assume

$$\mathbf{x}_{t+1} = A\mathbf{x}_t.$$

In other words, time evolution by one time step is given by matrix multiplication by some matrix A. If we start with $\mathbf{x}_0$, we get $\mathbf{x}_1 = A\mathbf{x}_0$, $\mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0$, and more generally, the state of the system at arbitrary time $t = k$ is

$$\mathbf{x}_k = A^k\mathbf{x}_0.$$

So to solve our system we need to be able to calculate high powers of the matrix A. Use eigenbasis of A for this.

2.1 Golden ratio and Fibonacci numbers

Example 1. 'A certain man put a pair of rabbits in a place surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair from which the second month on becomes productive?' (Liber abbaci, chapter 12, p. 283-4)

Solution. Idea: use discrete dynamical system to produce the Fibonacci numbers.

- $F_{n+1} = F_n + F_{n-1} \implies \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$
- Hence $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} \qquad \left(\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$
- But we know how to compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$ or $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$!

Solution.

- The characteristic polynomial of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is $\lambda^2 - \lambda - 1$
- The eigenvalues are $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$ (the golden mean!) and $\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$
- Corresponding eigenvectors: $\mathbf{v}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$
- Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. ($c_1 = \frac{1}{\sqrt{5}}, c_2 = -\frac{1}{\sqrt{5}}$)
- $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1^n c_1 \mathbf{v}_1 + \lambda_2^n c_2 \mathbf{v}_2$
- Hence, $F_n = \lambda_1^n c_1 + \lambda_2^n c_2 = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$. That is **Binet's formula**.
- but $|\lambda_2| < 1$, so $F_n \approx \lambda_1^n c_1 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$. In fact, $F_n = \text{round} \left(\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \right)$.

Fibonacci numbers: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ Did you notice: $\frac{13}{8} = 1.625$, $\frac{21}{13} = 1.615$, $\frac{34}{21} = 1.619, \ldots$ The **golden ratio** $\varphi = 1.618...$ Where's that from? We just showed that $F_n = \text{round}\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$. Therefore

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \left(\frac{1+\sqrt{5}}{2}\right).$$

Definition 2. Let A be a $n \times n$-matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. The discrete dynamical system $\mathbf{x}_{t+1} = A\mathbf{x}_t$ is

- **stable** if all eigenvalues satisfy $|\lambda_i| < 1$,
- **neutrally stable** if some $|\lambda_i| = 1$ and all the other $|\lambda_i| < 1$,
- **unstable** if at least one eigenvalue has $|\lambda_i| > 1$.

Example 3.

1. The discrete dynamical system used to construct the Fibonacci numbers is unstable.
2. If A is a Markov matrix with positive entries, then $\mathbf{x}_{t+1} = A\mathbf{x}_t$ is neutrally stable.
3. If $A = \begin{bmatrix} 0 & 4 \\ 0 & \frac{1}{2} \end{bmatrix}$, is $\mathbf{x}_{t+1} = A\mathbf{x}_t$ stable?

3 Spectral Theorem

- Not every matrix A has a basis of eigenvectors ☹
- Special case:

Definition. A is symmetric if $A = A^T$

Theorem 1. *If A is symmetric, then it has an orthonormal basis of eigenvectors*

and all eigenvalues are real! ☺

If Q is the matrix of eigenvectors, then Q is orthogonal. So, $Q^{-1} = Q^T$. Thus,

$$A = QDQ^{-1} = QDQ^T$$

and

$$D = Q^{-1}AQ = Q^TAQ$$

Remark.

- The converse is also true: If A has an orthogonal basis of eigenvectors, then A is symmetric! Why?
- It is important that if A is symmetric the eigenvalues are always real. No complex eigenvalues!

Example 4. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Write A as QDQ^T.

Solution. We've seen this matrix before!

Find eigenvalues: We have seen that A has eigenvalues 2 and 4.

Find eigenbasis corresponding to eigenvalues:

$\lambda_1 = 2$**:** We have seen $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Normalized, we get $\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

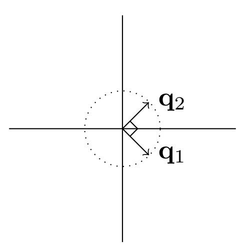
$\lambda_2 = 4$**:** We have seen $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Normalized, we get $\mathbf{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

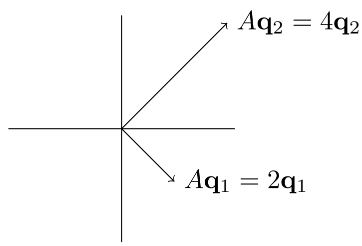
Write D**:** $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

Write Q**:** $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Get $A = QDQ^T$**:** $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

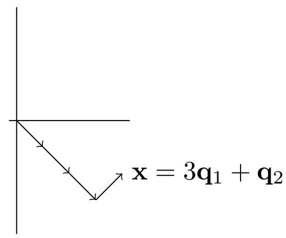
What does A do to the eigenvectors?

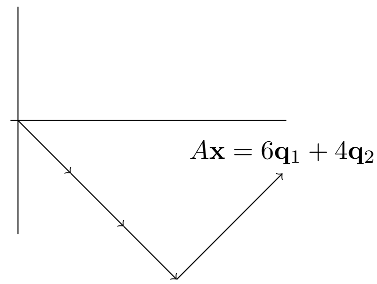




What happens to a vector $\mathbf{x}$?

Suppose $\mathbf{x} = 3\mathbf{q}_1 + \mathbf{q}_2$:





Why are symmetric matrices special? Why does spectral theorem work? If $A = A^T$, and if

$$A\mathbf{x} = \lambda_1 \mathbf{x} \text{ and } A\mathbf{y} = \lambda_2 \mathbf{y}$$

(for $\lambda_1 \neq \lambda_2$), then $\mathbf{x}$ and $\mathbf{y}$ **must** be orthogonal! Why?

Let's show $\mathbf{x} \cdot \mathbf{y} = 0$:

$$\begin{aligned}
\lambda_1(\mathbf{x} \cdot \mathbf{y}) &= (\lambda_1 \mathbf{x}) \cdot \mathbf{y} \\
&= (A\mathbf{x}) \cdot \mathbf{y} \\
&= (A\mathbf{x})^T \mathbf{y} \\
&= \mathbf{x}^T A^T \mathbf{y} \\
&= \mathbf{x}^T A \mathbf{y} \quad \longleftarrow \text{ because } A \text{ is symmetric!} \\
&= \mathbf{x} \cdot (A\mathbf{y}) \\
&= \lambda_2 (\mathbf{x} \cdot \mathbf{y})
\end{aligned}$$

Since $\lambda_1 \neq \lambda_2$, must have $\mathbf{x} \cdot \mathbf{y} = 0$! By a similar argument you can show that the eigenvalues of a symmetric matrix **must** be real.

Example 5. Let $A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$. Then

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \end{bmatrix}$. Find $A^3\mathbf{x}$.