Math 415 - Lecture 30

Eigenvectors and Eigenvalues

Friday November 6th 2015

Textbook reading: Chapter 5.1

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Suggested practice exercises: 12, 20, 21, 22, 36

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Khan Academy video: Introduction to Eigenvalues and Eigenvectors, Proof of formula for determining Eigenvalues, Finding Eigenvectors and Eigenspaces example

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Strang lecture: Lecture 21: Eigenvalues and eigenvectors

Review

Definition

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This allows us to compute the determinant using just row operations!. Bring A into echelon form= triangular form, keeping track how the determinant changes under the row operations you are using.

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A has three rows. Multiplying all 3 of them produces $2A$. Hence, $\det(2A) = 2^3 \det(A) = 40$.

Eigenvectors and eigenvalues

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Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$. Is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an eigenvector?

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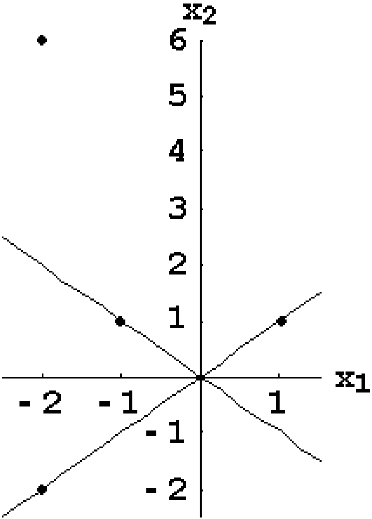
Hence, $\mathbf{x}$ is an eigenvector of A with eigenvalue -2.


x2
6
5
4
3
2
1
-1
-2
x1
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-1
1

Geometric interpretation

Example

Use your geometric understanding to find the eigenvectors and the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

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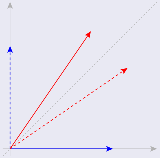

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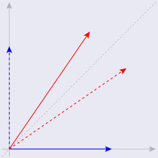

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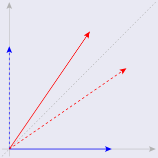

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* How to find eigenvectors? This uses null spaces.

Eigenspaces

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- The eigenspace of $\lambda = 0$ is $V^{\perp}$.

How to solve $A\mathbf{x} = \lambda\mathbf{x}$

Key observation:

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This $\mathbf{x}$ is a non trivial solution!

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The characteristic polynomial

Example

Find the eigenvectors and eigenvalues of

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Span $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is the **eigenspace** for the eigenvalue $\lambda = 2$.

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Triangular matrices

Example

Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$$

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- The characteristic polynomial is:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & 3 \\ 0 & 6-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda)(2-\lambda)$$

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  $\boxed{\text{The eigenvalues of a triangular matrix are its diagonal entries.}}$

Solution (continued)

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These three vectors are independent. By the next result, this is always so.

Independent eigenvectors

Theorem

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In other words, the matrix with columns $\mathbf{x}_1, \ldots, \mathbf{x}_m$ has one-dimensional null space. Now multiply this relation with A:

$$A(c_1\mathbf{x}_1 + \ldots + c_m\mathbf{x}_m) = c_1\lambda_1\mathbf{x}_1 + \ldots + c_m\lambda_m\mathbf{x}_m = \mathbf{0}$$

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Relations between eigenvalues

Product of Eigenvalues

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The characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ has constant term $\det(A)$. On the other hand $p(\lambda)$ factors, because the roots are the eigenvalues we get $p(\lambda) = (\pm 1)(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$, which has constant term $\lambda_1 \lambda_2 \ldots \lambda_n$. $\square$

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Example

Let $A = \begin{bmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{bmatrix}$. Then the eigenvalues are λ_1, λ_2 and $\det(A) = \lambda_1 \lambda_2$.

Sum of Eigenvalues

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Definition

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ be $n \times n$. Then the **TRACE** of A is the sum of the diagonal entries: $\mathrm{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

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Let A be $n \times n$. Then the trace of A is the **sum** *of eigenvalues:*

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The Characteristic Polynomial for 2×2

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Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Then the characteristic polynomial is*

$$p(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A).$$

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Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. What are the eigenvalues and what is the characteristic polynomial?

2×2 matrices are easy.

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$\mathrm{Tr}(A) = 6$, $\det(A) = 8$,

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Practice problems

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Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

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Example

What are the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 4 \end{bmatrix}$.

No calculations!