Math 415 - Lecture 19

Orthonormal basis, orthogonal complement

Friday October 9th 2015

Textbook reading: Ch 3.1

Suggested practice exercises: Ch 3.1: 7, 8, 9, 10, 11, 12, 14, 15, 17, 18, 19, 20, 22

Khan Academy videos: Introduction to orthonormal bases, Coordinates with respect to orthonormal bases

Strang lectures: Lec 10: The Four Fundamental Subspaces / Lec 14: Orthogonal Vectors and Subspaces

1 Review

- $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = v_1 w_1 + \cdots + v_n w_n$ is the **inner product** of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.
  - The **length** of $\mathbf{v}$, $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$.
  - The **distance** between points $\mathbf{v}, \mathbf{w}$ is $\|\mathbf{v} - \mathbf{w}\|$.
- $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are **orthogonal** if $\mathbf{v} \cdot \mathbf{w} = 0$.
  - This simple criterion is equivalent to Pythagoras' theorem.

2 Unit Vectors and Orthonormal basis

Definition. A vector $\mathbf{u} \in \mathbb{R}^n$ is called a *unit vector* if

- $\|\mathbf{u}\| = 1$, or, equivalently,
- $\mathbf{u} \cdot \mathbf{u} = 1$

Example 1. The standard basis vectors $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ of $\mathbb{R}^n$ are all unit vectors.

Example 2. If $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then is $\mathbf{x}$ a unit vector?

Solution. Since $\mathbf{x} \cdot \mathbf{x} = 5$ and $\|\mathbf{x}\| = \sqrt{5}$. However, $u = \frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{\mathbf{x}}{\sqrt{5}}$ *is* a unit vector. The unit vector $\mathbf{u}$ is called the *normalization* of $\mathbf{x}$.

Definition.
- A bunch of vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_p}$ is called *orthogonal* if they are all nonzero and $\mathbf{x_i} \cdot \mathbf{x_j} = 0$ for $i \neq j$.
- Orthogonal vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_p}$ are called *orthonormal* if they are all unit vectors.

Example 3. Let $\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then they are orthogonal but not orthonormal, since they are not unit vectors. We can normalize them to get a orthonormal set $\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Let $\mathcal{B} := (\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n})$ be an orthonormal basis for $\mathbb{R}^n$, so a basis consisting of unit vectors that are all perpendicular. Suppose we want to calculate the coordinates of $\mathbf{x} \in \mathbb{R}^n$:

$$\mathbf{x} = c_1 \mathbf{u_1} + c_2 \mathbf{u_2} + \cdots + c_n \mathbf{u_n}.$$

If this were an arbitrary basis, we would have to solve a system of equations to find the coordinates $c_1, \ldots, c_n$. Now we know that we have an orthonormal basis things are easier. Just calculate

$$\mathbf{u_1} \cdot \mathbf{x} = \mathbf{u_1} \cdot (c_1 \mathbf{u_1} + c_2 \mathbf{u_2} + \cdots + c_n \mathbf{u_n}) =$$

$$= c_1 \mathbf{u_1} \cdot \mathbf{u_1} + c_2 \mathbf{u_1} \cdot \mathbf{u_2} + \cdots + c_n \mathbf{u_1} \cdot \mathbf{u_n} = c_1$$

In the same way

$$\mathbf{u_2} \cdot \mathbf{x} = c_2, \ldots, \mathbf{u_n} \cdot \mathbf{x} = c_n$$

Example 4. $\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an orthonormal basis for $\mathbb{R}^2$. Let $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Solution. Then

$$\mathbf{x} = c_1 \mathbf{u_1} + c_2 \mathbf{u_2} = c_1 \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for

$$c_1 = \mathbf{u_1} \cdot \mathbf{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5}{\sqrt{2}}$$

$$c_2 = \mathbf{u_2} \cdot \mathbf{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{-1}{\sqrt{2}}$$

Theorem 1. *Let $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ be non-zero and mutually orthogonal. Then $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ are linearly independent.*

Solution. **Proof.** Suppose that

$$c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n} = \mathbf{0}.$$

Take the inner product of $\mathbf{v_1}$ on both sides.

$$\begin{aligned}
\mathbf{0} &= \mathbf{v_1} \cdot (c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n}) \\
&= c_1\mathbf{v_1}\cdot\mathbf{v_1} + c_2\mathbf{v_1}\cdot\mathbf{v_2} + \cdots + c_n\mathbf{v_1}\cdot\mathbf{v_n} \\
&= c_1\mathbf{v_1}\cdot\mathbf{v_1} = c_1\|\mathbf{v_1}\|^2
\end{aligned}$$

But $\|\mathbf{v_1}\| \neq 0$ and so $c_1 = 0$. Similarly, we find that $c_2 = 0, \ldots, c_n = 0$. Therefore, the vectors are independent.

3 Orthogonality and the Fundamental subspaces

Example 5. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. Find $Nul(A)$ and $Col(A^T)$.

Solution. Note that $Nul(A)$ and $Col(A^T)$ both are subspace of $\mathbb{R}^2$.

$$Nul(A) = span\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}, \quad Col(A^T) = span\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

The basis vectors for the null and row space are orthogonal.

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$$

Example 6. Repeat for $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix} \underset{RREF}{\longrightarrow} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Nul(A) = span\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}, \quad Col(A^T) = span\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Again, the basis for the null space is orthogonal to the basis for the row space.

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0.$$

Since $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is orthogonal to both basis vectors for the row space, it's orthogonal to *every* vector in the row space. It turns out this is true for the null and row space of any matrix A. That is, vectors in $Nul(A)$ are orthogonal to vectors in $Col(A^T)$ for *all* matrices A.

4 Fundamental Theorem of Linear Algebra (Revisited)

Definition. Let W be a subspace of $\mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$.

- $\mathbf{v}$ is **orthogonal** to W if $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in W$. ($\iff$ $\mathbf{v}$ is orthogonal to each vector in a basis for W.)
- Another subspace V is **orthogonal** to W if every vector in V is orthogonal to W.
- The **orthogonal complement** of W is the space $W^\perp$ of all vectors that are orthogonal to W. (Show that the orthogonal complement of any vector space is also a vector space.)

Example 7. Let $V = \text{Span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $W = \text{Span} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then

- $V \perp W$, because every vector of V is perp to each vector in W.
- It is not true that $V^\perp = W$ since $V^\perp$ consists of *all* vectors in $\mathbb{R}^3$ perp to V. Which vectors are missing?
- $V^\perp = \text{Span}(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})$

Example 8. In the last example, $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. We found that

$$Nul(A) = span \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad Col(A^T) = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are orthogonal subspaces. Indeed, $Nul(A)$ and $Col(A^T)$ are orthogonal complements. Why?

Solution. Because $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are orthogonal (so independent), and so they're a basis for all of $\mathbb{R}^3$.

Remark. In the last example, $Nul(A)$ and $Col(A)$ both happen to be subspaces of $\mathbb{R}^3$ (because A was a square 3×3 matrix).

$$Nul(A) = span \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad Col(A) = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

However, these spaces are **not** orthogonal. Why?

Solution.

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq 0$$

Theorem 2. *Let A be an $m \times n$ matrix of rank r.*

- $dim\ Col(A) = r$ *(subspace of $\mathbb{R}^m$)*
- $dim\ Col(A^T) = r$ *(subspace of $\mathbb{R}^n$)*
- $dim\ Nul(A) = n - r$ *(subspace of $\mathbb{R}^n$)*
- $dim\ Nul(A^T) = m - r$ *(subspace of $\mathbb{R}^m$)*
- $Nul(A)^\perp = Col(A^T)$ *(both subspaces of $\mathbb{R}^n$) Note that* $dim\ Nul(A) + dim\ Col(A^T) = n$.
- $Nul(A^T)^\perp = Col(A)$

Example 9. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

- $Col(A) = Span \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
- $Nul(A) = Span \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
- $Col(A^T) = Span \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- $Nul(A^T) = Span \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.