Worksheet 2 (September 1st and 3rd)

1. *Some questions to check your understanding:*

a) *What is the largest possible number of pivots a* 4×6*matrix can have? Why?*

b) *What is the largest possible number of pivots a* 6×4 *matrix canhave? Why?*

c) *How many solutions does a consistent linear system of 3 equations and 4unknowns have? Why?*

d) *Suppose the coefficient matrix corresponding to a linearsystem is* 4×6 *and has 3 pivot columns. How many pivot columns doesthe augmented matrix have if the linear system is inconsistent?*

Solution. For (a): 4. Each row can have at most one leading entry and hence at most one pivot.
For (b): Again 4. Each column has at most one pivot.
For (c): The augmented matrix of this system has 3 rows and 4 columns. Hence it has at most 3 pivots. So one variable can not correspond to a pivot column. Hence the system has one free variable. Any consistent system with a free variable has infinitely many solutions.
For (d): 4. In order to be inconsistent, the augmented matrix has a row $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & x \end{array}\right]$, where $x \neq 0$. This row gives the augmented matrix a fourth pivot. □

2. *Determine if the vector* $\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ *is a linear combination of* $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$, *and* $\begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$.

Solution. We check whether there are x_1, x_2, x_3 in $\mathbb{R}$ such that

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}.$$

For this, it is enough to check whether the systems of linear equations with the following augmented matrix is consistent:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right].$$

We bring the augmented matrix in echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right] \xrightarrow{R2 \to R2+2R1, R3 \to R3-2R1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{array}\right]$$

$$\xrightarrow{R3 \to R3-R2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The system is inconsistent, because in echelon form there is a row of the form

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & x \end{array}\right],$$

where x is non-zero. Hence the vector $\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ is *not* a linear combination of

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \text{ and } \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}.$$

□

3. *Give a geometric description of* **span** $\left\{ \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \right\}$.

Solution. Two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ span a plane iff there is no real number c such that $c\mathbf{v}_1 = \mathbf{v}_2$. Suppose there is c such that

$$c \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}.$$

By the first entry of the two vectors, we have $c3 = -2$. So $c = -\frac{2}{3}$. But by the third entry, we get $c2 = 3$. So $c = \frac{3}{2}$. This is impossible since $-\frac{2}{3} \neq \frac{3}{2}$. Hence **Span**$\left\{ \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \right\}$ is a plane. □

4. *True or false? Justify your answers!*

(a) *Let A be an $m \times n$-matrix and B be an $m \times l$-matrix, where l, m, n are all distinct. Then the product AB is defined.*

(b) *The weights $c_1, ..., c_p \in \mathbb{R}$ in a linear combination $c_1\boldsymbol{v}_1 + ... + c_p\boldsymbol{v}_p$ cannot all be zero.*

(c) *Given nonzero vectors* $\mathbf{u}, \mathbf{v}$ *in* $\mathbb{R}^n$, **span**$\{\boldsymbol{u}, \boldsymbol{v}\}$ *contains the line through* $\boldsymbol{u}$ *and the origin. Hint: can you describe this line as a set of vectors?*

(d) *Asking whether the linear system corresponding to* $\left[\begin{array}{ccc|c} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{b} \end{array} \right]$ *is consistent, is the same as asking whether* $\boldsymbol{b}$ *is a linear combination of* $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$.

Solution. (a) This is false. Let A be a $m_1 \times m_2$ matrix and B be a $n_1 \times n_2$ matrix. Then AB is defined if and only if $m_2 = n_1$.
(b) This is false. The weights can be zero. Check the definition in the lecture notes!
(c) This is correct. The line through $\mathbf{u}$ and the origin is the set $\{s\mathbf{u} : s \in \mathbb{R}\}$ and **Span**$\{\mathbf{u}, \mathbf{v}\}$ is the set $\{s\mathbf{u} + t\mathbf{v} : s, t \in \mathbb{R}\}$. Since vectors on the line can be written as $s\mathbf{u} + 0\mathbf{v}$, it is clear that the line is contained in the span.
(d) This is correct. Check the definition of being a linear combination in the lecture notes. □

5. *Determine whether* $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ *is a linear combination of the columns of* $\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}$.

Solution. We check whether there are x_1, x_2, x_3 in $\mathbb{R}$ such that

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}.$$

For this, it is enough to check whether the systems of linear equations with the following augmented matrix is consistent:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right].$$

We bring the augmented matrix in echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \xrightarrow{R2 \to R2 + 2R1,} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - R2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent, because in echelon form there is a row of the form

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & x \end{array}\right],$$

where x is non-zero. Hence the vector $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is a linear combination of the columns of $\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}$. □

6. *Compute AB by the definition, where $A\boldsymbol{b}_1$ and $A\boldsymbol{b}_2$ are calculated separately.*

(a) $A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 5 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$

Solution. For (a), if we calculate $A\mathbf{b}_1$ and $A\mathbf{b}_2$ separately, we have

$$\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 13 \end{bmatrix}$$

and

$$\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ -9 \\ 4 \end{bmatrix}.$$

For (b), if we calculate $A\mathbf{b}_1$ and $A\mathbf{b}_2$ separately, we have

$$\begin{bmatrix} 5 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 30 \\ 36 \end{bmatrix}$$

and

$$\begin{bmatrix} 5 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

□

7. *Let* $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

(a) *If* $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, *what is* Ax*?*

(b) *If* $x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, *what is* Ax*?*

(c) *Is* $Ax = b$ *uniquely solvable: is there for a given* b *alwaysexactly one* x*? Hint: use parts (a) and (b).*

(d) *Put* A *into echelon form. Are there any "missing" pivots?*

Solution. For (a):

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For (b):

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For (c): As we have shown in (1) and (2), if $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, there are at least two vectors we can plug in for x, namely $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, such that $Ax = b$.
For (d), we put A into echelon form:

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R_1} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

We observe that there is only one pivot. Since the echelon form of A does not have a "full" set of pivots (which would be two pivots in this case) and A is a square matrix, we would expect the equation $Ax = b$ not to be uniquely solvable. □

8. *(Some interesting matrices) Find matrices* A, B, C, D *(what size!) such that:*

(a) $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

(b) $B \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y + 3x \end{bmatrix}$.

(c) $C \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

(d) $D \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$

Solution. All matrices must be 2×2 matrices (why?).
For (a):

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For (b):

$$B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

For (c):

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For (d):

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

□

9. *Colors on a computer are usually based on the RGB model. In this model colors are represented by the percentages of the primary colors red (R), green (G) and blue (B) they contain. That means a **color** c is a vector $\begin{bmatrix} r \\ g \\ b \end{bmatrix}$, where r is the percentage of red, g the percentage of green and b the percentage of blue in the mix c. Obviously in this system*

$$red = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad green = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad blue = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Moreover, the color yellow is given by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and the color purple by $\begin{bmatrix} .5 \\ 0 \\ .5 \end{bmatrix}$.
*We say a **mix of colors** $c_1, \dots, c_n$ is a color c that is of the from $c = a_1c_1 + \cdots + a_nc_n$ for some $a_1, \dots, a_n$ between 0 and 1.*

(a) *Is every color a mix of red, green, and blue?*
(b) *Is green a mix of yellow and purple?*

Solution. (a) Every color will be a mix of these three since

$$\begin{bmatrix} r \\ g \\ b \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + g \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) To see whether green is a mix of yellow and purple we need to check whether the corresponding system is consistent. $\left[\begin{array}{cc|c} 1 & .5 & 0 \\ 1 & 0 & 1 \\ 0 & .5 & 0 \end{array}\right] \xrightarrow{R2 \to R2 - R1} \left[\begin{array}{cc|c} 1 & .5 & 0 \\ 0 & -.5 & 1 \\ 0 & .5 & 0 \end{array}\right] \xrightarrow{R3 \to R3 + R2}$ $\left[\begin{array}{cc|c} 1 & .5 & 0 \\ 0 & -.5 & 1 \\ 0 & 0 & 1 \end{array}\right]$ Since our last row of our augmented matrix is $\left[\begin{array}{cc|c} 0 & 0 & 1 \end{array}\right]$, we see that green is not a mix of yellow and purple.

□

The following may be useful in the above problems:

Definition. Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ using weights $c_1, \ldots, c_p$.

Definition. Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$, the **span of** $\mathbf{v}_1, \ldots, \mathbf{v}_p$ **in** $\mathbb{R}^n$ is the set

$$\mathbf{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \{c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p : c_1, \ldots, c_p \in \mathbb{R}\},$$

i.e., the collection of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$.