Math 415 - Lecture 39

Review

Wednesday December 6th 2015

Final Information:

- Thursday December 17th, 8:00-11:00AM.
  - 101 Armory: AD3,ADG,ADU,ADW
  - 180 Bevier: ADH,ADP,ADQ,ADX
  - 100 Gregory: ADA,ADB,ADJ,ADK,ADV,ADY
  - 151 Loomis: AD4,AD7,AD8,ADI,ADR
  - 103 Mumford: AD9,ADE,ADF,ADN,ADO
  - 100 MSEB: AD1,AD2,ADS,ADT,ADZ
  - 135 THBH: ADC,ADD,ADL,ADM (THBH is Temple Hoyne Buell Hall)
- Conflict Tuesday, December 15th, 8:00-11:00AM.

Bring university ID, pencils and erasers, there will be a part multiple choice.

1 After Exam 3

After Exam 3

- Diagonalization,
- Discrete Dynamical Systems.
- Spectral Theorem and Quadratic forms: each symmetric matrix A gives a quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, and conversely. The eigenvalues of A (real!) determine if the quadratic form is always positive.
- Critical points of functions $f\colon \mathbb{R}^n \to \mathbb{R}$ are described by a quadratic form (Hessian) containing the second derivatives of f. Minima, maxima, saddle points. Constrained optimization.

- Singular Value Decomposition of A from spectral theorem for $A^T A$, and AA^T.
- Approximation of a matrix A according to the singular values: image compression.

2 Big Topics

- Solving Systems $A\mathbf{x} = \mathbf{b}$
  - Augmented matrix.
  - Row Operations, Reduced Row echelon form.
  - Pivots, free variables, parametric form of general solution.
  - Inconsistent system, unique solution or infinitely many solutions.

- Vectors and Matrices
  - Linear Combinations
  - Matrix multiplication is linear combination
  - Row/column calculation of matrix multiplication
  - Transpose, symmetric matrices.
  - Elementary row operations and elementary matrices.
  - LU factorization, solving $Ax = b$ by $Lc = b$, $Ux = c$.
  - Inverse of a square matrix, Gauss-Jordan calculation of A^{-1} (Big Augmented Matrix).

- Vector Spaces.
  - Linear combinations.
  - Subspace.
  - Spanning set, independence.
  - Basis and dimension.
  - Coordinates with respect to a basis.

- Linear Transformations
  - Linear transformation determined by basis.

- Coordinate matrix with respect to input/output bases.

- Orthogonality
  - Dot product=inner product.
  - Length of vector.
  - angle between vectors.
  - Orthogonal complement $W^\perp$, dimensions add: $\dim(W)+\dim(W^\perp) = \dim(\mathbb{R}^n)$.
  - Orthogonal and orthonormal basis.

- Fundamental thm of Linear Algebra.
  - Four fundamental subspaces of A: $\mathrm{Col}(A), \mathrm{Col}(A^T), \mathrm{Nul}(A), \mathrm{Nul}(A^T)$.
  - $\mathrm{Nul}(A)$ and uniquess of solutions of $A\mathbf{x} = \mathbf{b}$.
  - $\mathrm{Col}(A)$ and existence of solutions of $A\mathbf{x} = \mathbf{b}$.
  - 4 subspaces pairwise orthogonal.
  - Dimensions of the subspaces and bases, from echelon form.
  - Networks and fundamental subspaces.

- Projections
  - Projection on a line.
  - Orthogonal basis makes projection easy.
  - Projection matrix.
  - Orthogonal decomposition: x can be written as $x = x_W + x_{W^\perp}$ for $x_W \in W$, $x_{W^\perp} \in W^\perp$.

- Least Squares
  - Approximate solutions of $A\mathbf{x} = \mathbf{b}$: make $\|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|$ as small as possible.
  - Least square solution is $\hat{\mathbf{x}}$ satisfying the normal equations $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$.

- The projection of $\mathbf{b}$ on the subspace $\mathrm{Col}(A)$ is $A\hat{\mathbf{x}}$.
- Data Fitting

- Gram-Schmidt
  - From arbitrary basis get orthonormal basis.
  - $A = QR$ factorization.
  - Orthogonal matrix Q: $Q^T Q = I$.

- Determinants
  - Definition through elementary row operations.
  - $\det(AB) = \det(A)\det(B)$, $\det(A^T) = \det(A)$.
  - Cofactor expansion.

- Eigenvalues and eigenvectors: $Ax = \lambda x$
  - Characteristic polynomial.
  - Eigenspace.
  - Eigenbasis and diagonalization.
  - Sum and product of eigenvectors and trace and det of A.
  - Powers of A.
  - Discrete Dynamical systems: state vector $\mathbf{x}_t$ evolves in time by $\mathbf{x}_{t+1} = A\mathbf{x}_t$.

- Symmetric matrices and spectral theorem.
  - if $A = A^T$ then eigenvalues of A are real
  - A has an orthonormal basis of eigenvectors.

3 Random Examples

Example 1. Let

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$$

Is b a linear combination of a_1, a_2, a_3? Explain!

Solution 2. *We need to solve a system $Ax = b$, with augmented matrix* $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 1 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right]$. *So b is/is not a linear combination?*

Example 3. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 3 \\ 6 & 3 & 2 \end{bmatrix}$.

- Find the LU factorization.
- Find two descriptions of the column space: $\mathrm{Col}(A)$ is the span of which vectors, and if $b \in \mathrm{Col}(A)$ give equations for b.
- If $b \in \mathrm{Col}(A)$ is the general set of solutions $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of $Ax = b$ a point, a line, a plane or all of $\mathbb{R}^3$?

Solution 4. $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 3 \\ 6 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. *So* $\mathrm{Col}(A) = \mathrm{Span}(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix})$. *This is the description by directions. We can also give equations for $b \in \mathrm{Col}(A)$: such b is perpendicular to what space? What is $\dim(\mathrm{Nul}(A^T))$? So we need to find a single equation for b, for instance $5b_1 - b_2 - b_3 = 0$. (If you don't see this immediately, do row operations on* $\left[\begin{array}{ccc|c} 2 & 1 & 1 & b_1 \\ 4 & 2 & 3 & b_2 \\ 6 & 3 & 2 & b_3 \end{array}\right]$*)If $b \in \mathrm{Col}(A)$ how many solutions of $Ax = b$, how many free variables? Get point, line, plane....?*

Example 5. Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and let $V = \mathrm{Span}(v_1, v_2)$. If $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ we can write $x = x_V + x_{V^\perp}$.

- Explain why $x_V = \frac{\langle x, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle x, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$ is not correct.
- Find an orthonormal basis for V.

- Calculate $x_{V^\perp}$.

Solution 6.

- *The basis is not orthogonal, so we can not use the formula!*
- *Take* $q_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\\0\end{bmatrix}$. *Now* q_2 *must be perpendicular to* q_1 *and belong to* V.

  So $q_2 = \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\end{bmatrix}$. *(Gram-Schmidt.)[-.5cm]*

- *Now write* $x_V = \dfrac{\langle \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix} \rangle}{\langle \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix} \rangle} \begin{bmatrix}1\\2\\0\end{bmatrix} + \dfrac{\langle \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-2\\1\\0\end{bmatrix} \rangle}{\langle \begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\end{bmatrix} \rangle} \begin{bmatrix}-2\\1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix}$. *Hence*

  $x_{V^\perp} = \begin{bmatrix}0\\0\\3\end{bmatrix}$

Example 7.

- Give an example of a 2×2 matrix A that is not invertible.
- Give an example of a 2×3 matrix A that has rank 0, or explain that that is not possible.
- Give an example of a 2×3 matrix A that has rank 1, but non of the entries are zero, or explain that that is not possible.
- Give an example of a 2×3 matrix A that has rank 2.
- Is the equation $Ax = 0$ always solvable?
- If A is the 2×3 zero matrix, then $\mathrm{Nul}(A) = \{ \begin{bmatrix}0\\0\\0\end{bmatrix} \}$. True or false?

Example 8. Let $W = \{ \begin{bmatrix}x\\y\\0\end{bmatrix} \}$, subspace of $\mathbb{R}^3$. If possible:

- Find 3 dependent vectors in W.
- Find 1 dependent vector in W.
- Find 2 independent vectors in W.
- Find 3 independent vectors in W.
- Find a spanning set of W containing 3 vectors.
- Find a spanning set of W containing 2 vectors.

- Find a spanning set of W containing 1 vectors.
- Find 2 bases for W.
- Find 2 independent vectors in $W^{\perp}$.