Preparation problems for the discussion sections on August 25th and 27th

1. *For the following systems determine:*
(1) *the augmented matrix,*
(2) *an echelon form of the matrix,*
(3) *the reduced echelon form of the matrix,*
(4) *whether the system is consistent,*
(5) *a parametric description of the set of solutions,*
(6) *how many solutions the system has, and*
(7) *the geometric interpretation of the set of solutions.*

System A:

$$\begin{aligned} x_2 &= 3 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

System B:

$$\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 + 2x_2 &= 6 \end{aligned}$$

System C:

$$\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 + 2x_2 &= 7 \end{aligned}$$

Solution. **System A:** For (1): The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 2 & 4 \end{array}\right].$$

For (2): We reduce it to echelon form as follows:

$$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \end{array}\right].$$

Note that this is not the reduced echelon form of the augmented matrix, but it is enough to answer (4). The system is consistent, because in echelon form there is no row of the form

$$\left[\begin{array}{cc|c} 0 & 0 & x \end{array}\right],$$

where x is non-zero.
For (3): To get to the reduced echelon form:

$$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R1 \leftrightarrow R1 - 2R2} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right].$$

Tutoring Room : Time and Place TBA
Midterm Dates: September 29th, October 22nd, November 19th (All Midterms 7-8:15 PM, see `learn.illinois.edu` **for locations)**

For (5): By (3), we get that System A has the same solutions as

$$\begin{aligned} x_1 &= -2 \\ x_2 &= 3. \end{aligned}$$

Hence a parametric description of the set of solutions is simply

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

For (6): The system has exactly one solution.
For (7): The set of solutions is a point in $\mathbb{R}^2$.

System B: For (1): The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 6 \end{array}\right].$$

For (2): We reduce it to echelon form as follows:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 6 \end{array}\right] \xrightarrow{R2 \leftrightarrow R2-2R1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

Note that is also the reduced echelon form of the augmented matrix, answering (3).
For (4): The system is consistent, because in echelon form there is no row of the form

$$\left[\begin{array}{cc|c} 0 & 0 & x \end{array}\right],$$

where x is non-zero.
For (5): By (3), we get that System B has the same solutions as

$$x_1 + x_2 = 3.$$

Hence a parametric description of the set of solutions is simply

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 - x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

For (6): The system has infinitely many solutions, because it is consistent and has a free variable (x_2).
For (7): The set of solutions is a line in $\mathbb{R}^2$.

System C: For (1): The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 7 \end{array}\right].$$

For (2): We reduce it to echelon form as follows:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 7 \end{array}\right] \xrightarrow{R2 \leftrightarrow R2-2R1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right].$$

For (3): To get to the reduced echelon form:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{R1 \leftrightarrow R1-3R2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

For (4): The system is inconsistent, because in echelon form there is a row of the form

$$\left[\begin{array}{cc|c} 0 & 0 & x \end{array}\right],$$

where x is non-zero.
For (5): By (4), we get that System C is inconsistent and hence there is no solution.
For (6): No solution.
For (7): The set of solutions is empty. $\square$

2. *Find a parametric description of the set of solutions of*

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= -6 \end{aligned}$$

Solution. We bring the augmented matrix to reduced echelon form:

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right] \xrightarrow{R2\to R2-R1, R3\to R3+3R1} \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{array}\right]$$

$$\xrightarrow{R3\to R3-2R2} \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R1\to R1-3R2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence a parametric description of the set of solutions is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 - 4x_3 \\ 3 + 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}. \qquad \square$$

3. *For which values of h_1 and h_2 is the following system consistent?*

$$\begin{aligned} x_1 &= h_1 \\ x_2 &= 5 \\ x_1 + 2x_2 &= h_2 \end{aligned}$$

Solution. We bring the augmented matrix to echelon form:

$$\left[\begin{array}{cc|c} 1 & 0 & h_1 \\ 0 & 1 & 5 \\ 1 & 2 & h_2 \end{array}\right] \xrightarrow{R3\to R3-R1} \left[\begin{array}{cc|c} 1 & 0 & h_1 \\ 0 & 1 & 5 \\ 0 & 2 & h_2 - h_1 \end{array}\right] \xrightarrow{R3\to R3-2R2} \left[\begin{array}{cc|c} 1 & 0 & h_1 \\ 0 & 1 & 5 \\ 0 & 0 & h_2 - h_1 - 10 \end{array}\right].$$

Hence the augmented matrix contains a row of the form $\left[\begin{array}{cc|c} 0 & 0 & x \end{array}\right]$, where x is non-zero, if and only if $h_2 - h_1 - 10 \neq 0$. Hence the system is consistent if and only if $h_2 - h_1 = 10$. $\square$

4. *According to the* New York Times*, Sony's 2014 film* The Interview *grossed* \$15 *million in its first four days through* \$15 *purchases and* \$6 *rentals. Sony did not reveal what the numbers of each of these were, but they did reveal that there were roughly* 2 *million transactions in all. While Sony may not have told the number of each type of transactions, what can Math 415 tell the reporter?*

Solution. Suppose that r is the number of rentals and p is the number of purchases. There were 2 million transactions, so

$$r + p = 2,000,000$$

Then from the total amount earned we know that

$$6r + 15p = 15,000,000$$

If we put this as an augmented system, we get:

$$\left[\begin{array}{cc|c} 1 & 1 & 2,000,000 \\ 6 & 15 & 15,000,000 \end{array}\right] \xrightarrow{R2 \to R2 - 6R1} \left[\begin{array}{cc|c} 1 & 1 & 2,000,000 \\ 0 & 9 & 3,000,000 \end{array}\right] \xrightarrow{R2 \to \frac{1}{9}R2} \left[\begin{array}{cc|c} 1 & 1 & 2,000,000 \\ 0 & 1 & 333,333.33 \end{array}\right]$$

Now if we subtract the second row from the first to reach a reduced row echelon form, we see that there were approximately 333,333 purchases and 166,667 rentals of *The Interview* in that four day period.

$\square$

5. *Let $A = [a_{ij}]_{3\times 4}$, and let $B = [b_{ij}]_{3\times 4}$ be an echelon form of A.*

(1) *Is it true that, if $a_{11} = 0$, then $b_{11} = 0$?*
(2) *Is it true that, if A has a column of zeros, then B also has a column of zeros?*
(3) *Suppose B has a row of zeros. What can you say about rows of A? (Explain.)*
(4) *Suppose we form a new matrix using some columns of A, let's say the first and the third column. What is an echelon form corresponding to this new matrix?*

Solution. (1) False, consider $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(2) True, since row operations do not change a column of zeros.
(3) Each row of B is a linear combination of rows of A with at least one non-zero coefficient. Therefore, a linear combination of rows of A with at least one non-zero coefficient is equal to zero.
(4) The matrix consisting of the first and the third column of B has the same reduced echelon form as the new matrix introduced in the problem. So, instead of transforming the new matrix into echelon form you can transform the matrix consisting of the first and the third column of B into echelon form. $\square$

6. *Show that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.*

Solution. We can replace $R_i \leftrightarrow R_j$ with the following sequence of row operations: $(i \neq j)$

$$R_i \leftrightarrow R_i + R_j,\ R_j \leftrightarrow R_j - R_i,\ R_j \leftrightarrow -R_j,\ R_i \leftrightarrow R_i - R_j$$

$\square$

The following may be useful in the above problems:

Definition. A system of linear equations is **consistent** if there **exists** a solution.

Definition. A matrix is in **(row) echelon form** if

(1) All nonzero rows are above all zero rows.
(2) Each *leading entry* (i.e., leftmost nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
(3) All entries in a column below a leading entry are zero.

A matrix is in **reduced (row) echelon form** if in addition to (1), (2) and (3) above it also satisfies:

(4) The leading entry in each nonzero row is 1.
(5) Each leading 1 is the only nonzero entry in its column.

Definition. A **pivot position** is the position of a leading entry in an echelon form of the matrix. A **pivot** is a nonzero number that either is used in a pivot position to create zeros or is changed into a leading 1, which in turn is used to create zeros. A **pivot column** is a column that contains a pivot position.

Definition. An **elementary row operation** is one of the following:

- **Replacement:** Add a multiple of one row to another row (denoted $R_i \to R_i + cR_j$),
- **Interchange:** Interchange two rows (denoted $R_i \leftrightarrow R_j$), or
- **Scaling:** Multiply all entries in a row by a nonzero constant (denoted $R_i \to cR_i$).