MA(1) $\quad Y_t = e_t - \theta e_{t-1}$

$E(e_t) = 0, \qquad Var(e_t) = \sigma_\epsilon^2 \qquad$ for all t

$E(e_t e_s) = 0, \qquad$ for $t \neq s$

$E(e_t Y_s) = 0, \qquad$ for $s < t$

$\gamma(0) = Var(Y_t) = Cov(Y_t, Y_t) = \ldots$

$$\begin{array}{lcll} Y_t & = & e_t & -\theta e_{t-1} \\ Y_t & = & e_t & -\theta e_{t-1} \end{array}$$

$$\ldots = \sigma_\epsilon^2\,(1+\theta^2).$$

$\gamma(1) = Cov(Y_t, Y_{t-1}) = \ldots$

$$\begin{array}{lcllll} Y_t & = & e_t & -\theta e_{t-1} & \\ Y_{t-1} & = & & e_{t-1} & -\theta e_{t-2} \end{array}$$

$$\ldots = -\theta\sigma_\epsilon^2.$$

$\gamma(k) = Cov(Y_t, Y_{t-k}) = \ldots \qquad k > 1.$

$$\begin{array}{lcllll} Y_t & = & e_t & -\theta e_{t-1} & & \\ Y_{t-k} & = & & & e_{t-k} & -\theta e_{t-k-1} \end{array}$$

$$\ldots = 0.$$

Therefore, $\quad \rho_0 = 1, \qquad \rho_1 = \dfrac{-\theta}{1+\theta^2},$

$$\rho_k = 0, \qquad k > 1.$$

MA(q) $$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \dots - \theta_q e_{t-q}$$

$\gamma(0) = \mathrm{Var}(Y_t) = \mathrm{Cov}(Y_t, Y_t) = \dots$

$$\begin{array}{lcccccc}
Y_t & = & e_t & -\theta_1 e_{t-1} & -\theta_2 e_{t-2} & \dots & -\theta_q e_{t-q} \\
Y_t & = & e_t & -\theta_1 e_{t-1} & -\theta_2 e_{t-2} & \dots & -\theta_q e_{t-q}
\end{array}$$

$$\dots = \sigma_e^2 \, (1 + \theta_1^2 + \theta_2^2 + \dots + \theta_q^2).$$

$\gamma(1) = \mathrm{Cov}(Y_t, Y_{t-1}) = \dots$

$$\begin{array}{lccccccc}
Y_t & = & e_t & -\theta_1 e_{t-1} & -\theta_2 e_{t-2} & \dots & -\theta_q e_{t-q} & \\
Y_{t-1} & = & & e_{t-1} & -\theta_1 e_{t-2} & \dots & -\theta_{q-1} e_{t-q} & -\theta_q e_{t-q-1}
\end{array}$$

$$\dots = \sigma_e^2 \, (-\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \dots + \theta_{q-1}\theta_q).$$

$\gamma(k) = \mathrm{Cov}(Y_t, Y_{t-k}) = \dots$ $\qquad k = 1, \dots, q-1$

$$\dots = \sigma_e^2 \, (-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \dots + \theta_{q-k}\theta_q).$$

$\gamma(q) = \mathrm{Cov}(Y_t, Y_{t-q}) = -\theta_q \, \sigma_e^2.$

$\gamma(k) = \mathrm{Cov}(Y_t, Y_{t-k}) = 0,$ $\qquad k > q.$

Therefore,

$$\rho_k = \frac{-\theta_k + \theta_1\theta_{k+1} + \dots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \dots + \theta_q^2}, \qquad k = 1, \dots, q-1,$$

$$\rho_q = \frac{-\theta_q}{1 + \theta_1^2 + \dots + \theta_q^2},$$

$$\rho_k = 0, \qquad k > q.$$

PACF $\quad \mathrm{Corr}(Y_t, Y_{t+k} \mid Y_{t+1}, \dots, Y_{t+k-1})$

Consider a regression type model

$$Y_{t+k} = \phi_{k1} Y_{t+k-1} + \phi_{k2} Y_{t+k-2} + \dots + \phi_{kk} Y_t + e_{t+k}$$

$$\downarrow$$

Consider $\quad \mathrm{Cov}(\dots, Y_{t+k-j}), \quad j = 1, 2, \dots, k. \quad$ Divide by γ_0:

$$\Rightarrow \quad \begin{aligned} \rho_1 &= \phi_{k1}\rho_0 + \phi_{k2}\rho_1 + \dots + \phi_{kk}\rho_{k-1} \\ \rho_2 &= \phi_{k1}\rho_1 + \phi_{k2}\rho_0 + \dots + \phi_{kk}\rho_{k-2} \\ &\dots \\ \rho_k &= \phi_{k1}\rho_{k-1} + \phi_{k2}\rho_{k-2} + \dots + \phi_{kk}\rho_0 \end{aligned}$$

$$\mathrm{PACF}(k) = \mathrm{Corr}(Y_t, Y_{t+k} \mid Y_{t+1}, \dots, Y_{t+k-1}) = \phi_{kk}.$$

<u>$k = 1$</u>:

$$\rho_1 = \phi_{11}\rho_0 = \phi_{11} \qquad\qquad \phi_{11} = \rho_1$$

<u>$k = 2$</u>:

$$\rho_1 = \phi_{21}\rho_0 + \phi_{22}\rho_1 \qquad\qquad \phi_{21} = \frac{\rho_1 - \rho_1\rho_2}{1 - \rho_1^2}$$

$$\rho_2 = \phi_{21}\rho_1 + \phi_{22}\rho_0 \qquad\qquad \phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

and so on…

AR(1)	MA(1)
$\rho_k = \phi^k$	$\rho_1 = \dfrac{-\theta}{1+\theta^2}$ $\rho_k = 0, \quad k > 1$
$\phi_{11} = \rho_1 = \phi$ $\phi_{kk} = 0, \quad k > 1$	$\phi_{kk} = \dfrac{-\theta^k}{1+\theta^2+\theta^4+\dots+\theta^{2k}}$ $= \dfrac{-\theta^k\left(1-\theta^2\right)}{1-\theta^{2(k+1)}}$

Determine whether the following processes are stationary:

a) $Y_t + 0.2\, Y_{t-1} = e_t - 0.8\, e_{t-1} + 0.5\, e_{t-2}$

$$\Phi(B) = 1 + 0.2\, B$$

The root of $\Phi(z) = 0$ is $z = -5$, it is outside the unit circle.

$\Rightarrow$ This process is stationary.

b) $Y_t - 0.1\, Y_{t-1} - 0.2\, Y_{t-2} = e_t - 0.6\, e_{t-1}$

$$\Phi(B) = 1 - 0.1\, B - 0.2\, B^2 = (1 - 0.5\, B)(1 + 0.4\, B)$$

The roots of $\Phi(z) = 0$ are $z_1 = 2$ and $z_2 = -2.5$.
All roots of $\Phi(z) = 0$ are outside the unit circle.

$\Rightarrow$ This process is stationary.

OR

An AR(2) model is stationary if

$-1 < \phi_2 < 1$, $\quad \phi_2 + \phi_1 < 1$, $\quad \phi_2 - \phi_1 < 1$.

$-1 < 0.2 < 1$, $\quad 0.2 + 0.1 < 1$, $\quad 0.2 - 0.1 < 1$.

$\Rightarrow$ This process is stationary.

c) $Y_t - 0.7\, Y_{t-1} - 0.6\, Y_{t-2} = e_t + 0.9\, e_{t-1} - 0.7\, e_{t-2}$

$$\Phi(B) = 1 - 0.7\, B - 0.6\, B^2 = (1 - 1.2\, B)(1 + 0.5\, B)$$

The roots of $\Phi(z) = 0$ are $z_1 = 5/6$ and $z_2 = -2$. $z_1 = 5/6$ is NOT outside the unit circle, it is inside the unit circle. The roots of $\Phi(z) = 0$ must ALL be outside the unit circle for the process to be stationary.

$\Rightarrow$ This process is NOT stationary.

OR

An AR(2) model is stationary if

$$-1 < \phi_2 < 1, \qquad \phi_2 + \phi_1 < 1, \qquad \phi_2 - \phi_1 < 1.$$

$$-1 < 0.6 < 1, \qquad 0.6 + 0.7 > 1, \qquad 0.6 - 0.7 < 1.$$

$\times$

$\Rightarrow$ This process is NOT stationary.

d) $Y_t - 0.8\, Y_{t-1} + 0.25\, Y_{t-2} = e_t$

$\Phi(B) = 1 - 0.8\, B + 0.25\, B^2$

The roots of $\Phi(z) = 0$ are $z_{1,2} = \dfrac{0.8 \pm \sqrt{0.8^2 - 4 \cdot 0.25 \cdot 1}}{2 \cdot 0.25} = 1.6 \pm 1.2\, i.$

All roots of $\Phi(z) = 0$ are outside the unit circle.

$\Rightarrow$ This process is stationary.

OR

An AR(2) model is stationary if

$$-1 < \phi_2 < 1, \qquad \phi_2 + \phi_1 < 1, \qquad \phi_2 - \phi_1 < 1.$$

$$-1 < -0.25 < 1, \qquad -0.25 + 0.8 < 1, \qquad -0.25 - 0.8 < 1.$$

$\Rightarrow$ This process is stationary.

e) $Y_t = e_t - 0.6\, e_{t-1} - 0.4\, e_{t-2} - 0.2\, e_{t-3}$

$\Phi(B) = 1$

No roots of $\Phi(z) = 0$ inside or on the unit circle.

$\Rightarrow$ This process is stationary.

f) $\quad \mathrm{Y}_t - \frac{14}{13}\,\mathrm{Y}_{t-1} + \frac{10}{13}\,\mathrm{Y}_{t-2} = e_t + 0.7\,e_{t-1}$

$$\Phi(B) = 1 - \frac{14}{13}\,B + \frac{10}{13}\,B^2$$

The roots of $\Phi(z) = 0$ are $z_{1,2} = \dfrac{14 \pm \sqrt{14^2 - 4 \cdot 10 \cdot 13}}{2 \cdot 10} = 0.7 \pm 0.9\,i$.

$|z_{1,2}|^2 = 0.7^2 + 0.9^2 = 1.3$. All roots of $\Phi(z) = 0$ are outside the unit circle.

$\Rightarrow$ This process is stationary.

OR

An AR(2) model is stationary if

$-1 < \phi_2 < 1, \qquad \phi_2 + \phi_1 < 1, \qquad \phi_2 - \phi_1 < 1.$

$-1 < -\frac{10}{13} < 1, \qquad -\frac{10}{13} + \frac{14}{13} < 1, \qquad -\frac{10}{13} - \frac{14}{13} < 1.$

$\Rightarrow$ This process is stationary.

g) $\quad \mathrm{Y}_t - 0.7\,\mathrm{Y}_{t-1} - 0.3\,\mathrm{Y}_{t-2} = e_t + 0.5\,e_{t-1}$

$$\Phi(B) = 1 - 0.7\,B - 0.3\,B^2 = (1 - B)(1 + 0.3\,B)$$

The roots of $\Phi(z) = 0$ are $z_1 = 1$ and $z_2 = -3\,{}^1\!/_3$. $z_1 = 1$ is NOT outside the unit circle, it is on the unit circle. The roots of $\Phi(z) = 0$ must ALL be outside the unit circle for the process to be stationary.

$\Rightarrow$ This process is NOT stationary. $\qquad$ ARIMA$(1, 1, 1)$

OR

An AR(2) model is stationary if

$-1 < \phi_2 < 1, \qquad \phi_2 + \phi_1 < 1, \qquad \phi_2 - \phi_1 < 1.$

$-1 < 0.3 < 1, \qquad 0.3 + 0.7 = 1, \qquad 0.3 - 0.7 < 1.$

$\times$

$\Rightarrow$ This process is NOT stationary.

1. Consider the ARMA(1, 1) process

$$(Y_t - \quad) = \phi(Y_{t-1} - \quad) + e_t - \theta e_{t-1}$$

or

$$(1 - \phi B)\,\dot{Y}_t = (1 - \theta B)\, e_t, \qquad \text{where } \dot{Y}_t = Y_t - \quad .$$

Suppose $\quad E(e_t) = 0, \qquad \mathrm{Var}(e_t) = \sigma_e^2 \qquad$ for all t

$\qquad E(e_t e_s) = 0, \qquad$ for $t \neq s$

$\qquad E(e_t Y_s) = 0, \qquad$ for $s < t$

Derive the expression for ρ_k in terms of ϕ and θ.

$$\begin{aligned}
\mathrm{Cov}(Y_t, e_t) &= E[(Y_t - \quad)e_t] \\
&= \phi E[(Y_{t-1} - \quad)e_t] + E[e_t^2] - \theta E[e_{t-1}e_t] \\
&= \sigma_e^2 .
\end{aligned}$$

$$\begin{aligned}
\gamma(0) &= \mathrm{Var}(Y_t) = E[(Y_t - \quad)^2] \\
&= \phi^2 E[(Y_{t-1} - \quad)^2] + E[e_t^2] + \theta^2 E[e_{t-1}^2] \\
&\qquad + 2\phi E[(Y_{t-1} - \quad)e_t] - 2\phi\theta E[(Y_{t-1} - \quad)e_{t-1}] \\
&\qquad - 2\theta E[e_t e_{t-1}] \\
&= \phi^2 \gamma(0) + \sigma_e^2 + \theta^2 \sigma_e^2 - 2\phi\theta\sigma_e^2 .
\end{aligned}$$

$$\Rightarrow \qquad \gamma(0) = \sigma_e^2 \cdot \frac{\left(1 - 2\phi\theta + \theta^2\right)}{1 - \phi^2}.$$

$$\gamma(1) = \mathrm{Cov}(\mathrm{Y}_t, \mathrm{Y}_{t-1}) = \mathrm{E}[(\mathrm{Y}_t - \quad)(\mathrm{Y}_{t-1} - \quad)]$$

$$= \phi \mathrm{E}[(\mathrm{Y}_{t-1} - \quad)^2] + \mathrm{E}[e_t(\mathrm{Y}_{t-1} - \quad)] - \theta \mathrm{E}[e_{t-1}(\mathrm{Y}_{t-1} - \quad)]$$

$$= \phi\gamma(0) - \theta\,\sigma_e^2$$

$$= \sigma_e^2 \cdot \frac{\phi\left(1-2\phi\theta+\theta^2\right)-\theta\left(1-\phi^2\right)}{1-\phi^2}$$

$$= \sigma_e^2 \cdot \frac{\phi-2\phi^2\theta+\phi\theta^2-\theta+\phi^2\theta}{1-\phi^2}$$

$$= \sigma_e^2 \cdot \frac{\phi-\theta-\phi^2\theta+\phi\theta^2}{1-\phi^2}$$

$$= \sigma_e^2 \cdot \frac{(\phi-\theta)(1-\phi\theta)}{1-\phi^2}$$

$$\Rightarrow \qquad \rho_1 = \gamma(1)\big/\gamma(0) = \frac{(\phi-\theta)(1-\phi\theta)}{1-2\phi\theta+\theta^2}.$$

$k > 1$

$$\gamma(k) = \mathrm{Cov}(\mathrm{Y}_t, \mathrm{Y}_{t-k}) = \mathrm{E}[(\mathrm{Y}_t - \quad)(\mathrm{Y}_{t-k} - \quad)]$$

$$= \phi \mathrm{E}[(\mathrm{Y}_{t-1} - \quad)(\mathrm{Y}_{t-k} - \quad)] + \mathrm{E}[e_t(\mathrm{Y}_{t-k} - \quad)]$$
$$- \theta \mathrm{E}[e_{t-1}(\mathrm{Y}_{t-k} - \quad)]$$

$$= \phi\gamma(k-1).$$

$$\Rightarrow \qquad \rho_k = \frac{\phi^{k-1}(\phi-\theta)(1-\phi\theta)}{1-2\phi\theta+\theta^2}, \qquad\qquad k \geq 1.$$