Math 415 - Lecture 15

The Four Fundamental Subspaces, the Fundamental Theorem of Linear Algebra, Linear Transformations

Monday September 28th 2015

Textbook: Chapter 2.4, 2.6

Suggested Practice Exercise: Chapter 2.4 Exercise 1, 2, 3, 4, 7, 10, 18, 20, 21, 22, 27, 32, 37 Chapter 2.6 Exercise 5, 6, 7, 36, 37

Khan Academy Video: Linear Transformation, Linear Transformations as Matrix Vector Products, Linear Transformation Examples: Rotations in $\mathbb{R}^2$

Strang lectures: Lecture 9: Independence, Basis, and Dimension Lecture 10: The Four Fundamental Subspaces Lecture 30: Linear Transformations

* Exam 1 (7-8:15 pm Tuesday September 29):

* Rooms: look on Moodle.

* Conflicts: if you have a conflict you should have received an email about it. If not, talk to me after class.

* No Discussion Sections next week.

* No Class on Wednesday next week.

* The Exam will be part multiple choice. Bring pencils and erasers! Also bring ID.

* The material for the exam covers the lectures upto and including Lecture 12 (last Monday), and this weeks worksheet and quiz.

1 Review

1.1 Basis for the Null Space

- $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a basis of V if the vectors span V and are independent.

- To find a basis for $Nul(A)$, solve $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 3 & 6 & 6 & 3 \\ 6 & 12 & 15 & 0 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} \boxed{1} & 2 & 0 & 5 \\ 0 & 0 & \boxed{1} & -2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} -2x_2 - 5x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

So a basis for $Nul(A)$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

1.2 Basis for the Column space.

- To find a basis for $Col(A)$, take the pivot columns of A.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{\text{-1}} & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So a basis for $Col(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

Question. Why do we take columns of A and not columns of the Echelon form?

1.3 The Column spaces of A and U.

Question. Why do we take columns of A and not columns of the Echelon form?

- Row operations do not preserve the column space. For example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow[R1 \leftrightarrow R2]{} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- On the other hand, row operations do preserve the null space. Why? Remember, we can do row operations to solve systems like $A\mathbf{x} = \mathbf{0}$.

2 Rank and Dimensions

2.1 Dimension of Column and Null Space

Definition. The rank of a matrix A is the number of pivots it has.

Theorem 1. ***Rank-Nullity Theorem*** *Let A be an $m \times n$ matrix of rank r. Then*

dim $Col(A) = r$ *Why?*

> *A basis for $Col(A)$ is given by the pivot columns of A.*

dim $Nul(A) = n - r$ *is the number of free variables of A. Why?*

> *In our method for finding a basis for $Nul(A)$, each free variable corresponds to an element in the basis.*

dim $Col(A) +$ *dim* $Nul(A) = n$ *Why?*

> *Each of the n columns of A either contains a pivot or corresponds to a free variable.*

3 The Four Fundamental Subpaces

3.1 Two Spaces we know

Let A be a matrix. We already know two fundamental subspaces:

- The column space of A and
- The null space of A

There are two more!

3.2 Row Space and Left Null Space

Definition.
- The row space of A is the column space of A^T. $Col(A^T)$ is spanned by the columns of A^T and these are the rows of A (but transposed, to turn into columns!).
- The left null space of A is the null space of A^T. Why is it called the "left" null space?

	Suppose $\mathbf{x} \in Nul(A^T)$.	Thus,
	$\iff A^T\mathbf{x} = \mathbf{0}$.	Take transposes of both sides:
	$\iff (A^T\mathbf{x})^T = \mathbf{0}^T$.	So,
	$\iff \mathbf{x}^T A = \mathbf{0}$.	

Therefore, $\mathbf{x} \in Nul(A^T) \iff \mathbf{x}^T A = \mathbf{0}$.

Example 1. Find a basis for $Col(A)$ and $Col(A^T)$ if

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

Solution. We need to compute an echelon form of A to find a basis for $Col(A)$. Then we might compute an echelon form of A^T to find a basis for $Col(A^T)$. However, an echelon form of A will allow us to find a basis for both $Col(A)$ and $Col(A^T)$.

Instead of doing twice the work, we only need to find an echelon form of A.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We identify the pivot columns:

$$\longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{\text{-1}} & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

So $r = 2$ for A and a basis for $Col(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$.

Remark. Key idea: The row space is preserved by elementary row operations.

Remember, $Col(A) \neq Col(U)$ because we did row operations. However, the row spaces are the same! i.e.

$$Col(A^T) = Col(U^T)$$

$$U = \begin{bmatrix} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{\text{-1}} & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 4 & -5 & 0 & 0 \end{bmatrix}$$

In particular, a basis for $Col(A^T)$ is given by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ -5 \end{bmatrix} \right\}$.

3.3 Fundamental Theorem of Linear Algebra (Part 1)

Theorem 2. *Let A be an $m \times n$ matrix with rank r.*

- *dim* $Col(A) = r$ *(subspace of* $\mathbb{R}^m$*)*
- *dim* $Col(A^T) = r$ *(subspace of* $\mathbb{R}^n$*)*
- *dim* $Nul(A) = n - r$ *(subspace of* $\mathbb{R}^n$*)*
- *dim* $Nul(A^T) = m - r$ *(subspace of* $\mathbb{R}^m$*)*

Remark. The column and row space always have the same dimension. In other words, A and A^T have the same rank. (i.e. same number of pivots). Why?

It's easy to see this for a matrix in echelon form.

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 7 \end{bmatrix}$$

(3 pivot columns in A, 3 non-zero columns in A^T.) But it's not as obvious for a random matrix.

4 Coordinates

4.1 Why Bases?

What is the point of having a *basis* for a vector space V?

- **Dimension!** If you have a basis $\mathcal{B} = (\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p})$ for V, you know that the dimension of V is p, so that you have an idea of the Size of V. In particular, if V has dimension 0 V is just the zero vector space.
- **Coordinates!** If $w \in V$ and $\mathcal{B} = (\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p})$ is a basis for V, we can express w in this basis. This means that we can write (uniquely!)

  $$w = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_p\mathbf{v_p}.$$

  We call the scalars $c_1, c_2, \ldots, c_p$ the *coordinates* of w with respect to the basis $\mathcal{B}$.

We are going to organize the coordinates in a convenient package.

4.2 Coordinate Vectors

Definition. If $w \in V$ and $\mathcal{B} = (\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p})$ is a basis for V, the **coordinate vector** of w with respect to the basis $\mathcal{B}$ is

$$w_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_p \end{bmatrix}, \quad \text{if } w = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_p\mathbf{v_p}.$$

So w is a vector in some vector space, but it's coordinate vector is always a column vector in $\mathbb{R}^p$, if $\dim(V) = p$. Why is the coordinate vector useful? Computations in V can be translated in computations in the familiar vector space $\mathbb{R}^p$.

Let $V = \mathbb{R}^2$, $\mathcal{B} = (\mathbf{b_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. What is the coordinate vector of $\mathbf{w}$? Express in the basis:

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = c_1\mathbf{b_1} + c_2\mathbf{b_2} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore

$$\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Geometrically: this means that to reach $\mathbf{w}$ walk 1 unit along the $\mathbf{b_1}$ basis vector and 2 units along the $\mathbf{b_2}$ basis vector.

4.3 Example with polynomials

Let $V = P_2$, the vector space of polynomials of the form $a_0 + a_1t + a_2t^2$. Let $\mathcal{B} = (\mathbf{b_1} = 1, \mathbf{b_2} = t, \mathbf{b_3} = t^2)$ be the obvious basis of P_2. Let $\mathbf{w} = 1 + 2t + 3t^2$. What is the coordinate vector of $\mathbf{w}$ with respect to basis $\mathcal{B}$? Express $\mathbf{w}$ in terms of the basis:

$$\mathbf{w} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + c_3\mathbf{b_3} = c_1 1 + c_2 t + c_3 t^2 = 1 + 2t + 3t^2.$$

Hence

$$\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

What if we take another basis? Say take $\bar{\mathcal{B}} = (t^2, t, 1)$. (Different order!). Then

$$\mathbf{w}_{\bar{\mathcal{B}}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

4.4 Standard Coordinate Vectors

Let $V = \mathbb{R}^3$ and let $E = (\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ be the standard basis. If $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ what is the coordinate vector with respect to the standard basis? Express in the basis:

$$\mathbf{w} = c_1\mathbf{e_1} + c_2\mathbf{e_2} + c_3\mathbf{e_3} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence

$$\mathbf{w}_E = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = w!$$

So the coordinate vector with respect to the standard basis is just the vector itself!

5 Linear Transformations

Let V and W be vector spaces.

Definition. A map $T : V \to W$ is a linear transformation if

$$T(c\mathbf{x} + d\mathbf{y}) = cT(\mathbf{x}) + dT(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in V$ and all $c, d \in \mathbb{R}$. In other words, a linear transformation respects addition and scaling.

Remark. It follows immediately that

- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- $T(c\mathbf{x}) = cT(\mathbf{x})$
- $T(\mathbf{0}) = \mathbf{0}$ (because $T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}$)

5.1 Some examples

Example 2. Let $V = \mathbb{R}, W = \mathbb{R}$. Then the map $f(x) = 3x$ is linear. Why?

If $x, y \in \mathbb{R}$, then $f(ax + by) = 3(ax + by) = a \cdot 3x + b \cdot 3y = af(x) + bf(y)$. What about the function $g(x) = 2x - 2$? Is this a linear transformation?

Example 3. Let A be an $m \times n$ matrix. Then the map $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. Why? Because matrix multiplication is linear.

$$A(c\mathbf{x} + d\mathbf{y}) = cA\mathbf{x} + dA\mathbf{y}$$

The left-hand side is $T(c\mathbf{x} + d\mathbf{y})$ and the right-hand side is $cT(\mathbf{x}) + dT(\mathbf{y})$.

Example 4. Let P_n be the vector space of all polynomials of degree at most n. Consider the map $T : P_n \to P_{n-1}$ given by

$$T(p(t)) = \frac{d}{dt}p(t).$$

This map is linear! Why?

Because differentiation is linear:

$$\frac{d}{dt}\left[ap(t) + bq(t)\right] = a\frac{d}{dt}p(t) + b\frac{d}{dt}q(t).$$

The left-hand side is $T(ap(t) + bq(t))$ and the right-hand side is $aT(p(t)) + bT(q(t))$.