Math 415 - Lecture 22

Orthogonal projection

Friday October 16th 2015

Textbook reading: Chapter 3.2.

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Suggested practice exercises: Chapter 3.2: 9, 10, 17, 19.

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Strang lecture: Lecture 15: Projections onto Subspaces

Review/Outlook

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$$v_1 \cdot (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 v_1 \cdot v_1.$$

Orthogonal Bases

Orthogonal Basis

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A basis $v_1, v_2, \ldots, v_n$ of $\mathbb{R}^n$ is called *orthogonal* if the vectors are pairwise orthogonal, $v_i \cdot v_j = 0$ if $i \neq j$.

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Example

The standard basis $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthogonal basis for $\mathbb{R}^3$.

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The standard basis $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthogonal basis for $\mathbb{R}^3$. Similarly, the standard basis $e_1, e_2, \ldots, e_n$ is an orthogonal basis for $\mathbb{R}^n$.

Example

Are the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ an orthogonal basis for $\mathbb{R}^3$?

Solution

Just check:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} =$$

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So this *is* an orthogonal basis. Note that we don't have to check it is a basis: orthogonality implies independence, and 3 independent vectors form a basis in $\mathbb{R}^3$.

Example

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$$w = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

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$$v_1 \cdot w = v_1 \cdot \left(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n\right) = c_1 v_1 \cdot v_1.$$

Hence $c_1 = \frac{v_1 \cdot w}{v_1 \cdot v_1}$

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Hence $c_1 = \frac{v_1 \cdot w}{v_1 \cdot v_1}$ and more generally $c_i = \frac{v_i \cdot w}{v_i \cdot v_i}$.

Easy (and Important) Formula

If $v_1, v_2, \ldots, v_p$ form an orthogonal basis of $V \subset \mathbb{R}^n$, $w \in V$, then $w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$, with

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Special Case

If $v_1, v_2, \ldots, v_p$ is orthonormal then

$$c_i = v_i \cdot w.$$

Example

Express $w = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in the basis $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

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Solution

$$\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =$$

We use the formula for the coordinates:

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We use the formula for the coordinates:

$$c_1 = \frac{v_1 \cdot w}{v_1 \cdot v_1} = \frac{-4}{2}, c_2 = \frac{v_2 \cdot w}{v_2 \cdot v_2} = \frac{10}{2}, c_3 = \frac{v_3 \cdot w}{v_3 \cdot v_3} = \frac{4}{1},$$

Warning

The easy formula for the coordinates only works for orthogonal bases.

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Example

Take the basis $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}$ and the vector $w = \begin{bmatrix}4\\9\end{bmatrix}$. Then

$$\begin{bmatrix}4\\9\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\3\end{bmatrix},$$

and the coefficients are not the numbers you get from the easy formula.

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and the coefficients are not the numbers you get from the easy formula. To find them you need to solve a system of equations.

Example

The standard basis $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ is orthonormal. Find the coordinates of $\begin{bmatrix}3\\7\\4\end{bmatrix}$ in the standard basis.

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Solution

This is trivial of course,

$$\begin{bmatrix}3\\7\\4\end{bmatrix} = 3\begin{bmatrix}1\\0\\0\end{bmatrix} + 7\begin{bmatrix}0\\1\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\end{bmatrix}$$

But ....

Solution (continued)

$$\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

but note that the coordinates are dot products with orthonormal vectors:

$$\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 3, \quad \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 7, \quad \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 4.$$

Example

The vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form an orthogonal basis. Produce from it an *orthonormal* basis.

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Solution

We just divide by the lengths of these vectors (this will keep them orthogonal).

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ has length } \sqrt{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} = \sqrt{2}, \text{ normalized: } \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

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and

$$\begin{bmatrix}0\\0\\1\end{bmatrix}$$

is already normalized. So we get as orthonormal basis

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Example

Express $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in the orthonormal basis

$$\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right).$$

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Solution

Just calculate dot products:

$$c_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = \frac{-4}{\sqrt{2}}, \quad c_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = \frac{10}{\sqrt{2}},$$

Solution (continued)

$$c_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = 4$$

so that

$$\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = \frac{-4}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{10}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =$$

$$= -2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

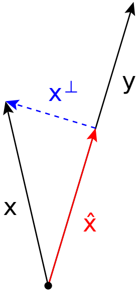
Orthogonal Projection

Definition (Orthogonal Projection)

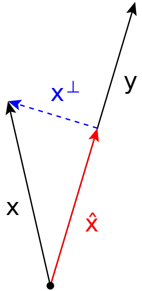
The **orthogonal projection** of vector $\mathbf{x}$ on vector $\mathbf{y}$ is

$$\hat{\mathbf{x}} = \frac{\mathbf{y} \cdot \mathbf{x}}{\mathbf{y} \cdot \mathbf{y}} \mathbf{y}.$$

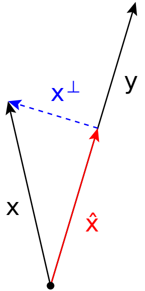
The **error** is $\mathbf{x}^{\perp} = \mathbf{x} - \hat{\mathbf{x}}$.



- The projection $\hat{\mathbf{x}}$ is the *closest point* to $\mathbf{x}$ on the line through $\mathbf{y}$.



- The projection $\hat{\mathbf{x}}$ is the *closest point* to $\mathbf{x}$ on the line through $\mathbf{y}$.
- The error $\mathbf{x}^\perp = \mathbf{x} - \hat{\mathbf{x}}$ is characterized by the property that it is orthogonal to $Span(\mathbf{y})$.



- The projection $\hat{\mathbf{x}}$ is the *closest point* to $\mathbf{x}$ on the line through $\mathbf{y}$.
- The error $\mathbf{x}^{\perp} = \mathbf{x} - \hat{\mathbf{x}}$ is characterized by the property that it is orthogonal to $\mathit{Span}(\mathbf{y})$.
- We have a decomposition $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{x}^{\perp}$. The **projection** $\hat{\mathbf{x}}$ is in $\mathit{Span}(\mathbf{y})$ and $\mathbf{x}^{\perp}$ is orthogonal to $\mathit{Span}(\mathbf{y})$.

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- We know $\hat{\mathbf{x}}$ is in the direction of $\mathbf{y}$, so $\hat{\mathbf{x}} = c\mathbf{y}$ for some constant c.

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- We know $\hat{\mathbf{x}}$ is in the direction of $\mathbf{y}$, so $\hat{\mathbf{x}} = c\mathbf{y}$ for some constant c.
- The error $\mathbf{x} - \hat{\mathbf{x}}$ orthogonal to $\mathbf{y}$.

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- The error $\mathbf{x} - \hat{\mathbf{x}}$ orthogonal to $\mathbf{y}$.
- So $0 = \mathbf{y} \cdot (\mathbf{x} - \hat{\mathbf{x}}) = \mathbf{y} \cdot (\mathbf{x} - c\mathbf{y}) = \mathbf{y} \cdot \mathbf{x} - c\mathbf{y} \cdot \mathbf{y}$.

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- The error $\mathbf{x} - \hat{\mathbf{x}}$ orthogonal to $\mathbf{y}$.
- So $0 = \mathbf{y} \cdot (\mathbf{x} - \hat{\mathbf{x}}) = \mathbf{y} \cdot (\mathbf{x} - c\mathbf{y}) = \mathbf{y} \cdot \mathbf{x} - c\mathbf{y} \cdot \mathbf{y}$.
- Solving for c gives $c = \frac{\mathbf{y} \cdot \mathbf{x}}{\mathbf{y} \cdot \mathbf{y}}$.

Example

Find the orthogonal projection of $\mathbf{x} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ onto $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

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Solution

$$\hat{\mathbf{x}} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}} \mathbf{y} = \frac{-8.3 + 4.1}{3^2 + 1^2} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}.$$

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Find the orthogonal projection of $\mathbf{x} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ onto $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Solution

$$\hat{\mathbf{x}} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}} \mathbf{y} = \frac{-8.3 + 4.1}{3^2 + 1^2} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}.$$

The error is

$$\mathbf{x}^{\perp} = \mathbf{x} - \hat{\mathbf{x}} = \begin{bmatrix} -8 \\ 4 \end{bmatrix} - \begin{bmatrix} -6 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

Example

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Note that vector $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and error $\mathbf{x}^{\perp} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ are orthogonal.

Example

What is the projection of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ onto each of the vectors

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

Solution

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ onto } \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} : \frac{2.1 + 1.(-1) + 1.0}{1^2 + (-1)^2 + 0^2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Solution

[Continued.]

$$\begin{bmatrix}2\\1\\1\end{bmatrix} \text{ onto } \begin{bmatrix}1\\1\\0\end{bmatrix} : \frac{2.1+1.1+1.0}{1^2+1^2+0^2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}.$$

$$\begin{bmatrix}2\\1\\1\end{bmatrix} \text{ onto } \begin{bmatrix}0\\0\\1\end{bmatrix} : \frac{2.0+1.0+1.1}{0^2+0^2+1^2}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Note that these sum up to $\frac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} + \frac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\1\end{bmatrix} = \mathbf{x}.$

Why?

Solution

[Continued.]

$$\begin{bmatrix}2\\1\\1\end{bmatrix} \text{ onto } \begin{bmatrix}1\\1\\0\end{bmatrix} : \frac{2.1+1.1+1.0}{1^2+1^2+0^2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}.$$

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Why? because ...

Theorem

If $v_1, \ldots, v_n$ is orthogonal basis of V and $w \in V$ then

$$w = c_1 v_1 + \cdots + c_n v_n, \quad \text{with } c_j = \frac{w \cdot v_j}{v_j \cdot v_j}.$$

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So the terms in this sum are precisely the projections onto each basis vector.

Projection Matrix

If $\mathbf{y}$ is a fixed nonzero vector, we get from any vector $\mathbf{x}$ the projection $\hat{\mathbf{x}}$.

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where $P = \frac{1}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}\mathbf{y}^T$. P is called the projection matrix on the subspace $Span(\mathbf{y})$.

Example

Let $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find the projection matrix P for $\mathbf{y}$ and use it to calculate the projections of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ on $\mathbf{y}$.

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Solution

$$P = \frac{1}{\mathbf{y} \cdot \mathbf{y}} \mathbf{y}\mathbf{y}^T = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

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