Worksheet 4 (September 15th and 17th)

1. *Consider the matrix:*

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$$

Decompose the matrix A into LU, where L is a lower triangular matrix and U is an upper triangular matrix. Then use this factorization to solve:

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

That means, find a vector $\mathbf{c}$ in $\mathbb{R}^3$ such that:

$$L\mathbf{c} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

and then find a vector $\mathbf{x}$ in $\mathbb{R}^3$ such that:

$$U\mathbf{x} = \mathbf{c}$$

2. *Let* $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}$, *and* $U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$.

(1) *Show that $A = LU$.*

(2) *Let A_i be the matrix introduced by the first i rows and the first i columns of A, for $i = 1, 2, 3$, i.e.,*

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \text{and} \quad A_1 = \begin{bmatrix} 2 \end{bmatrix}.$$

What is an LU-decomposition of A_i, for $i = 1, 2, 3$?

3. *Answer the following true-false questions. Justify your answers! A and B are arbitrary $n \times n$ square matrices.*

(1) *If A is invertible then $A\mathbf{x} = \mathbf{0}$ has exactly one solution, $\mathbf{x} = \mathbf{0}$.*
(2) *If A is invertible, then AB is also invertible.*
(3) *If A and B are invertible, then $A + B$ is also invertible.*
(4) *If A is invertible, then the reduced echelon form of A is equal to I.*

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Midterm Dates: September 29th, October 22nd, November 19th (All Midterms 7-8:15 PM, see `learn.illinois.edu` **for locations)**

4. *Let* $A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. *Use the Gauss-Jordan method to either find the inverse of A or to show that A is not invertible.*

5. *If* $G = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, *find G^{-1} using as many methods as possible. Check that $G^{-1}G = I$.*

6. *Calculate the inverse of the matrix:* $\begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

7. *Let*

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

*Such a matrix called **band matrix**. Band matrices often appear in applications (see the next exercise) and they are one reason LU-decomposition is so important.*

(1) *Find the LU-decomposition of A.*
(2) *Determine A^{-1}.*

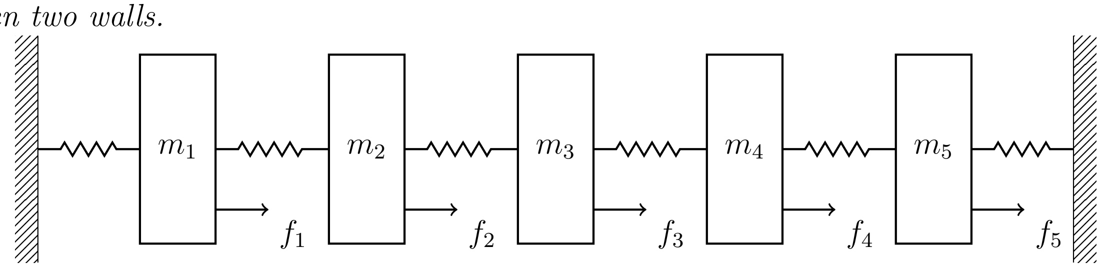
Given a vector $\boldsymbol{b}$ in $\mathbb{R}^5$, suppose we want to solve $A\boldsymbol{x} = \boldsymbol{b}$. In many applications, we probably don't want to do this for a single $\boldsymbol{b}$, but for many (maybe a million) different vectors $\boldsymbol{b}$. One way for finding $\boldsymbol{x}$ is to solve the following two linear systems which arise from the LU-decomposition of A:

$$L\boldsymbol{c} = \boldsymbol{b}, \text{ and } U\boldsymbol{x} = \boldsymbol{c}.$$

Another way is to calculate A^{-1}, and then we would get $\boldsymbol{u}$ by simply multiplying $\boldsymbol{b}$ by A^{-1}. Which way is more efficient?

(3) *Given the LU-decomposition of A, count the numbers of operations (addition and multiplication of real numbers) needed to find $\boldsymbol{x}$ using first method?*
(4) *Given the inverse of A, count the number of operations needed to calculate $A^{-1}\boldsymbol{b}$?*
(5) *Suppose A is not a 5×5 band matrix, but a 1000×1000 band matrix? Which of the two methods would you use?*

8. *Consider the following spring-mass system, consisting of five masses and six springs fixed between two walls.*



For simplicity, the stiffness of the springs is assumed to be 1. We denote

- *by f_i a (steady) applied force on mass i,*
- *by u_i the displacement of the mass i.*

Note that positive values of u_i correspond to displacement away from the wall on left. We choose our reference such that in the absence of applied forces we have $u_i = 0$. We want to calculate the steady state of this system; that is we wish to determine the value of $u_1, \ldots, u_5$ in the equilibrium.

(1) *In equilibrium the sum of the forces on mass i (the applied forces f_i and the forces due to the two springs next two it) must sum to zero. Using Hooke's law, this can be express as a linear equation in terms u_{i-1}, u_i and u_{i+1} for $i = 2, 3, 4$, in terms of u_1, u_2 for $i = 1$ and in terms of u_4, u_5 for $i = 5$. For each $i = 1, \ldots, 5$, write down this linear equation. (Hint: the equation for mass 1 is $2u_1 - u_2 = f_1$. Why?)*

(2) *Write this five equations into one system of linear equations with unknowns $u_1, \ldots, u_5$. The coefficient matrix of this system should be equal to the sparse matrix given in the previous exercise.*

(3) *Suppose $f_1 = \cdots = f_5 = 1$. What is $u_1, \ldots, u_5$ in the equilibrium?*

9. *(Optional and more challenging) Let A and B be $n \times n$ matrices such that $AB = I$. (Be Careful! The assumption "$AB = I$" doesn't quite mean that A is invertible: that is what we will show in this problem!).*

(1) *What is the reduced echelon form of A? (Hint: Let F be the product of all elementary matrices used to reduce A, so FA is the reduced echelon form of A. How many pivots can/does FA have?)*

(2) *Show that $BA = I$. (Note that this means that if A has a "right inverse", then it also has a "left inverse" and also that these inverses are the same. Thus A is invertible in the usual sense).*

The following may be useful in the above problems:

Definition. An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A.

Hooke's law is a principle of physics that states that the force F needed to extend or compress a spring by some distance u is proportional to that distance. That is: $F = -ku$, where k is a constant factor characteristic of the spring, called its **stiffness**.