Transformation!

- MGFs vs Convolution
- Bivariate change of variables

Last time we considered $X \sim \text{Poisson}(\lambda_1)$
$Y \sim \text{Poisson}(\lambda_2)$

For X & Y indep. we found that for $W = X + Y$, $W \sim \text{Poisson}(\lambda_1 + \lambda_2)$

- we had to know the binomial theorem.

Let's use MGFs to find the dist for W.

Recall $M_X(t) = \exp(\lambda_1(e^t - 1))$
and

$$M_Y(t) = \exp(\lambda_2(e^t - 1))$$

Independence implies $M_W(t) = M_X(t)\, M_Y(t)$

$$= \exp\left[\underbrace{(\lambda_1 + \lambda_2)}(e^t - 1)\right]$$

$\Rightarrow W \sim \text{Poisson}(\lambda_1 + \lambda_2)$

a) Consider $X_1 \sim \text{Bernoulli}(p)$ $\Rightarrow X_1 = \begin{cases} 0 & 1-p \\ 1 & p \end{cases}$

Let X_1 and X_2 be independent. $X_2 \sim \text{Bernoulli}(p)$

Find $W = X_1 + X_2$.

$$\Rightarrow M_1(t) = (1-p)e^{0t} + pe^t$$
$$= 1-p+pe^t$$

$$M_W(t) = M_1(t)\, M_2(t) = (1-p+pe^t)^2$$

Binomial $(n=2, p)$

For Binomial $M_X(t) = (1-p+pe^t)^n$

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b) Suppose $X_1 \sim \text{Binomial}(n_1, p)$ X_1 & X_2 are independent

$X_2 \sim \text{Binomial}(n_2, p)$

Let $W = X_1 + X_2$. What is the dist of W?

$$M_W(t) = M_{X_1}(t)\, M_{X_2}(t) =$$
$$= (1-p+pe^t)^{n_1}(1-p+pe^t)^{n_2}$$
$$= (1-p+pe^t)^{\underline{(n_1+n_2)}} \equiv \text{Binomial}(n = n_1+n_2, p)$$

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c) Suppose $X_i \sim N(\mu_i, \sigma_i^2)$ and X_i are indep.

$W = X_1 + X_2$. Find $f_W(w)$. $M_i(t) = \exp\left(-\mu_i t + \sigma_i^2 \frac{t^2}{2}\right)$

$$M_W(t) = M_1(t)\, M_2(t) = \exp\left[-\mu_1 t + \sigma_1^2 \frac{t^2}{2}\right] \exp\left[-\mu_2 t + \sigma_2^2 \frac{t^2}{2}\right]$$
$$= \exp\left\{-(\mu_1+\mu_2)t + (\sigma_1^2+\sigma_2^2)\frac{t^2}{2}\right] \equiv N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

Let X_1 and X_2 be independent χ^2's r.v.'s

$X_1 \sim \chi^2(m)$, $X_2 \sim \chi^2(n)$

χ^2 is a special case of Gamma dist

$\chi^2(m) \equiv \text{Gamma}(\alpha = \frac{m}{2}, \lambda = \frac{1}{2})$

Suppose ~~$W = X_1 + W$~~ $W = X_1 + X_2$. Find $f_W(w)$

We should expect $\chi^2(m+n)$.

a) Convolution approach $W = X_1 + X_2 \Rightarrow X_2 = W - X_1$

$$f_1(x_1) = \frac{1}{\Gamma(\frac{m}{2}) 2^{\frac{m}{2}}} x_1^{\frac{m}{2}-1} e^{-\frac{x_1}{2}}, \quad x_1 > 0$$

gamma function

$$f_2(x_2) = \frac{1}{\Gamma(\frac{n}{2}) 2^{n/2}} x_2^{\frac{n}{2}-1} e^{-\frac{x_2}{2}}, \quad x_2 > 0$$

$w - x_1 > 0 \Rightarrow w > x_1$

$$f_W(w) = \int_0^w f_1(x_1) f_2(w - x_1)\, dx_1$$

$$= \int_0^w \frac{1}{\Gamma(\frac{m}{2}) 2^{\frac{m}{2}}} x_1^{\frac{m}{2}-1} e^{-\frac{x_1}{2}} \frac{1}{\Gamma(\frac{n}{2}) 2^{\frac{n}{2}}} (w - x_1)^{\frac{n}{2}-1} e^{-\frac{(w-x_1)}{2}}\, dx_1$$

* Write the ~~integral~~ integrand as a known pdf w/ support b/w 0 and ~~w~~, we are done.

- Use Beta dist !!

$$\int_0^1 \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} y^{\frac{m}{2}-1} (1-y)^{\frac{n}{2}-1}\, dy = 1$$

1) Multiply by $\frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m+n}{2}\right)}$

2) $X = wy \Rightarrow Y = \frac{X}{w}$ $\quad X \to 0 \Rightarrow Y \to 0$

$dx = w\,dy \qquad X \to w \Rightarrow Y \to 1$

$$\Rightarrow \int_0^w \frac{1}{2^{\frac{m+n}{2}}\Gamma\left(\frac{m+n}{2}\right)} \int_0^1 \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} (wy)^{\frac{m}{2}-1} e^{-\frac{w}{2}} (w-wy)^{\frac{n}{2}-1} w\,dy$$

$$= \frac{w^{\frac{m}{2}-1}\, w^{\frac{n}{2}-1}\, w}{\Gamma\left(\frac{m+n}{2}\right) 2^{\left(\frac{m+n}{2}\right)}} e^{-\frac{w}{2}} \underbrace{\int_0^1 \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} Y^{\frac{m}{2}-1}(1-Y)^{\frac{n}{2}-1}\,dy}_{=\,1 \text{ by def. of Beta pdf.}}$$

$$= \underline{\frac{w^{\frac{m+n}{2}-1} e^{-\frac{w}{2}}}{\Gamma\left(\frac{m+n}{2}\right) 2^{\left(\frac{m+n}{2}\right)}}, \ w > 0} \equiv \chi^2(m+n)$$

b) MGF's $\quad M_1(t) = \left(\frac{1}{1-2t}\right)^{m/2}, \ M_2(t) = \left(\frac{1}{1-2t}\right)^{\frac{n}{2}}$

$$M_w(t) = M_1(t)\,M_2(t) = \left(\frac{1}{1-2t}\right)^{\frac{m+n}{2}}$$

Suppose X & Y are independent discrete r.v.'s

$p_X(1) = .2$, $p_X(2) = .4$, $p_X(3) = .3$, $p_X(4) = .1$

$p_Y(1) = .3$, $p_Y(3) = .5$, $p_Y(5) = .2$

Let $W = X + Y$. Find $p_W(w)$ using MGFs.

$$M_W(t) = M_X(t)\, M_Y(t)$$

$$= \left(.2e^{t} + .4e^{2t} + .3e^{3t} + .1e^{4t}\right) \cdot \left(.3e^{t} + .5e^{3t} + .2e^{5t}\right)$$

$$= .06e^{2t} + .12e^{3t} + .19e^{4t} + .23e^{5t} + .19e^{6t} + .13e^{7t} + .06e^{8t} + .02e^{9t}$$

w	$P(W=w)$
2	.06
3	.12
4	.19
5	.23
6	.19
7	.13
8	.06
9	.02

	Y		
X	0	1	2
1	.15	.10	0
2	.25	.30	.20

Find the joint pmf of $U = X+Y$ and $V = XY$.

X	$Y=0$	$Y=1$	$Y=2$
1	.15 $u=1$ $v=0$	.10 $u=2$ $v=1$	0 $u=1$ $v=2$
2	.25 $u=2$ $v=0$	.30 $u=3$ $v=2$	.20 $u=4$ $v=4$

	V			
u	0	1	2	4
1	.15	0	0	0
2	.25	.10	0	0
3	0	0	.30	0
4	0	0	0	.20

Bivariate Change of Variable,

Let X_1 and X_2 have joint pdf $f(x_1, x_2)$

valid support $S = \{(x_1, x_2) : f(x_1, x_2) > 0\}$

Let $Y_1 = u_1(X_1, X_2)$ & $Y_2 = u_2(X_1, X_2)$

u_1 and u_2 to one-to-one transformations.

$\Rightarrow$ $x_1 = w_1(y_1, y_2)$ & $x_2 = w_2(y_1, y_2)$

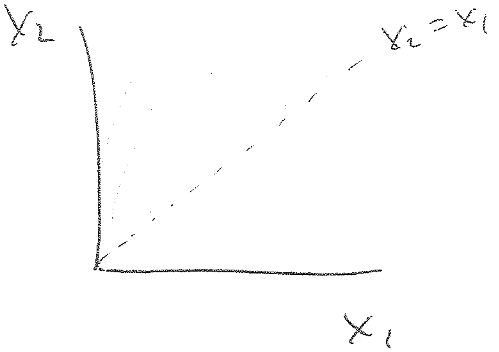
$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\[2ex] \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix}$$

the joint pdf of Y_1 & Y_2 is,

$$g(y_1, y_2) = f\big(\underbrace{w_1(y_1, y_2)}_{X_1}, \underbrace{w_2(y_1, y_2)}_{X_2}\big) \, |J|$$

Let X_1 and X_2 have joint pdf

$$f(x_1, x_2) = 2e^{-(x_1+x_2)}, \quad 0 < x_1 < x_2$$



a) $Y_1 = X_2 - X_1$, $Y_2 = X_1$

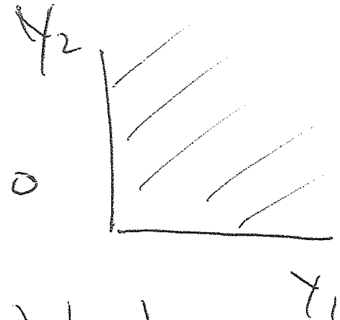
i) find w_1 and w_2

ii) Find J

iii) Identify the support for Y_1 and Y_2

i) $X_1 = Y_2$
$X_2 = Y_1 + Y_2$

ii) $J = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$



iii) • $X_1 > 0 \Rightarrow Y_2 > 0$
• $X_2 > X_1 \Rightarrow Y_1 + Y_2 > Y_2 \Rightarrow Y_1 > 0$

Finally, $g(y_1, y_2) = f(w_1(y_1, y_2), w_2(y_1, y_2))\,|J|$

$$= 2e^{-(Y_1+2Y_2)}\,|-1|, \quad Y_1 > 0,\ Y_2 > 0$$

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Are Y_1 and Y_2 independent?

$g_1(y_1) = e^{-y_1}, y_1 > 0$ and $g_2(y_2) = 2e^{-2y_2}, y_2 > 0$

Y_1 ∴ Y_2 are independent

b) Let $z_1 = x_1 + x_2$, $z_2 = x_2/x_1$

i) Find inverses

$$x_1 = \frac{z_1}{1+z_2}$$

$$x_2 = \frac{z_1 z_2}{1+z_2}$$

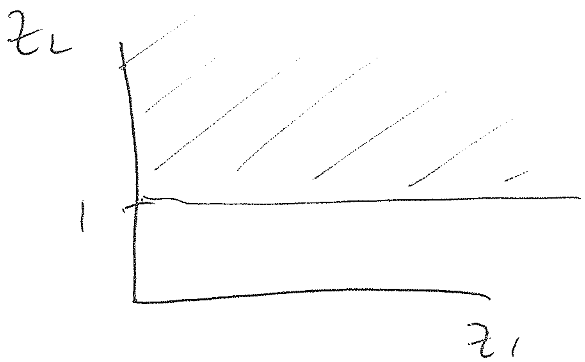
ii) $J = \begin{vmatrix} \frac{1}{1+z_2} & \frac{-z_1}{(1+z_2)^2} \\ \frac{z_2}{1+z_2} & \frac{z_1(1+z_2) - z_1 z_2}{(1+z_2)^2} \end{vmatrix}$

$$J = \frac{z_1}{(1+z_2)^2}$$

iii) $0 < x_1 < x_2$

- $x_1 > 0 \Rightarrow \frac{z_1}{1+z_2} > 0 \Rightarrow z_1 > 0$

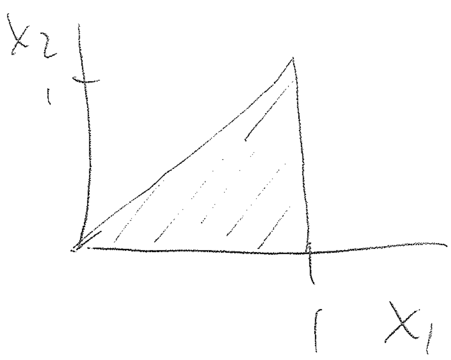
- $x_2 > x_1 \Rightarrow \frac{z_1 z_2}{1+z_2} > \frac{z_1}{1+z_2} \Rightarrow z_2 > 1$



$$g(z_1, z_2) = 2 e^{-z_1} \frac{z_1}{(1+z_2)^2} , \quad z_1 > 0 , \quad z_2 > 1$$

2) X_1 and X_2 w/ joint pdf

$$f(x_1, x_2) = 15 x_1 x_2^2, \quad 0 < x_2 < x_1 < 1$$



a) $Y_1 = X_1 + X_2$, $Y_2 = X_1$

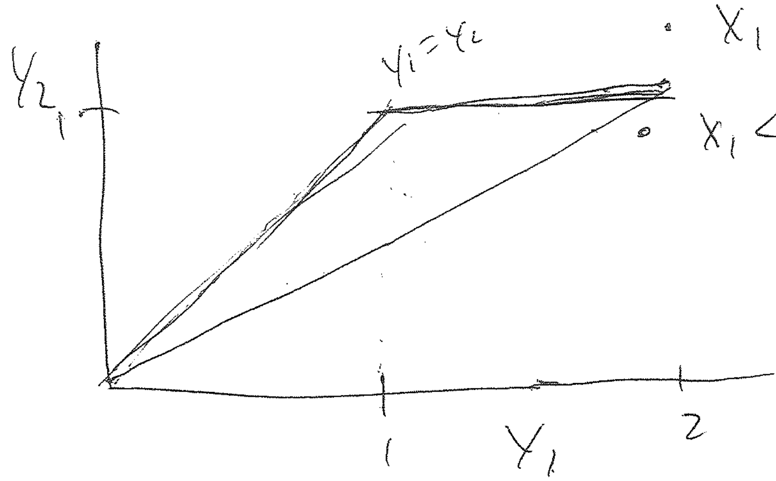
a i) $x_1 = y_2$

$x_2 = y_1 - y_2$

ii) $J = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$

iii) $0 < x_2 < x_1 < 1$

- $x_2 > 0 \Rightarrow y_1 - y_2 > 0 \Rightarrow \boxed{y_1 > y_2}$
- $x_1 > x_2 \Rightarrow y_2 > y_1 - y_2 \Rightarrow \boxed{y_2 > \frac{1}{2} y_1}$
- $x_1 < 1 \Rightarrow \boxed{y_2 < 1}$



$$g(y_1, y_2) = 15 y_2 (y_1 - y_2)^2 |-1|, \quad \cancel{y_1 > y_2 > \tfrac{1}{2}y_2}$$

$$y_1 > y_2 > \tfrac{1}{2} y_1, \; y_2 < 1$$

$$g(y_1) = \begin{cases} \int_{\frac{1}{2}y_1}^{y_1} g(y_1, y_2)\, dy_2 & 0 < y_1 < 1 \\ \int_{\frac{1}{2}y_1}^{1} g(y_1, y_2)\, dy_2 & 1 < y_1 < 2 \end{cases}$$