Worksheet 6 for October 6th and 8th

1. *Determine a basis for each of the following subspaces:*

(i) $H = \left\{ \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} : s, t \in \mathbb{R} \right\}$,

(ii) $K = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \ : \ a - 3b + c = 0 \right\}$,

(iii) $\mathrm{Col}\left(\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right)$,

(iv) $\mathrm{Nul}\left(\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right)$.

2. *Determine the rank of A and the dimension of* $\mathrm{Nul}(A)$, $\mathrm{Col}(A)$, $\mathrm{Nul}(A^T)$, *and* $\mathrm{Col}(A^T)$ *where*

$$A := \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}.$$

3. *Let A, B be two 4×3 matrices. Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ be the columns of A and let $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ be the columns of B.*

(i) *Suppose that $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are linearly independent. Find a basis for* $\mathrm{Col}(A)$ *and a basis for* $\mathrm{Nul}(A)$. *What are the dimensions of* $\mathrm{Col}(A)$ *and* $\mathrm{Nul}(A)$*?*

(ii) *Suppose that $\boldsymbol{b}_1, \boldsymbol{b}_2$ are linearly independent and $\boldsymbol{b}_3 = 2\boldsymbol{b}_1 + 7\boldsymbol{b}_2$. Find a basis for* $\mathrm{Col}(B)$ *and a basis for* $\mathrm{Nul}(B)$.

4. *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *and let* $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

(i) *Let* $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. *Express* $\mathbf{v}$ *in terms of the basis* $\mathcal{B}$ *(i.e., realize* $\mathbf{v}$ *as a linear combination of the vectors from* $\mathcal{B}$*). What is the coordinate vector* $\mathbf{v}_{\mathcal{B}}$ *of* $\mathbf{v}$ *in terms of the basis* $\mathcal{B}$.

(ii) *Let* $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *Express* $\mathbf{w}$ *in terms of the basis* $\mathcal{B}$. *What is the coordinate vector* $\mathbf{w}_{\mathcal{B}}$ *of* $\mathbf{w}$ *in terms of the basis* $\mathcal{B}$.

5. *Consider the vector space $\mathbb{P}_2$ of polynomials of degree at most 2. Let* $\mathcal{B} = \{t^2 + t + 1, t + 1, 1\}$.

(a) *Check that $\mathcal{B}$ is a basis for $\mathbb{P}_2$.*

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Midterm Dates: September 29th, October 22nd, November 19th (All Midterms 7-8:15 PM, see `learn.illinois.edu` **for locations)**

(a) *Suppose* $p(t) \in \mathbb{P}_2$ *has coordinate vector* $p(t)_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. *What is* $p(t)$*?*

6. *Which of the following mappings* T *are linear? Justify your answer!*

(a) $T : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 x_2$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ *defined by* $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ z \end{bmatrix}$.

(c) $T : \mathbb{R} \to \mathbb{R}$ *defined by* $T(x) = e^x$.

(d) $T : \mathbb{P}_3 \to \mathbb{P}_3$ *defined by* $T(f(t)) = \frac{d}{dt} f(t)$, *where* $\mathbb{P}_3$ *is the space of all polynomials of degree at most* 3.

(e) $T : \mathbb{R}^n \to \mathbb{R}^n$ *defined by* $T(\mathbf{v}) = \mathbf{v} + \mathbf{v}_0$, *for some fixed* $\mathbf{v}_0 \neq \mathbf{0} \in \mathbb{R}^n$.

7. *Let* $L: \mathbb{R}^2 \to \mathbb{R}^3$ *be a linear transformation such that*

$$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

What is $L\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$*?*

8. *Consider the following eight vectors in $\mathbb{R}^8$:*

$$\mathcal{B} = \left(\begin{bmatrix}1\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\-1\\1\end{bmatrix} \right), \quad \boldsymbol{w} = \begin{bmatrix}100\\200\\44\\50\\20\\20\\4\\2\end{bmatrix}.$$

(1) *Show that $\mathcal{B}$ is a basis of $\mathbb{R}^8$.*
(2) *Calculate $\boldsymbol{w}_{\mathcal{B}}$.*
(3) *Suppose Alice needs to inform Bob about $\boldsymbol{w}$. However, Alice can transmit only four non-zero numbers (and not eight) to Bob; that is Alice can only send Bob a vector in $\mathbb{R}^8$ which has at most four non-zero entries. Luckily Bob does not need to know the exact values in $\boldsymbol{w}$, a good approximation suffices. Using your result in (2), which vector should Alice communicate to Bob?*

The following may be useful in the above problems:

Definition. The **nullspace** of an $m \times n$ matrix A, written $\mathrm{Nul}(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In other words, $\mathrm{Nul}(A) = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$.

Definition. The **column space** of an $m \times n$ matrix A, written $\mathrm{Col}(A)$, is the set of all linear combinations of columns of A. In other words, if $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_n]$, then $\mathrm{Col}(A) = \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$.

Definition. A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in a vector space V is a **basis** if

(1) $V = \mathrm{span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$, and
(2) the vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ are linearly independent.

Definition. A vector space has **dimension** d if it has a basis consisting of d vectors.

Definition. Let V and W be vector spaces. A map $T : V \to W$ is a **linear transformation** if

$$T(c\mathbf{x} + d\mathbf{y}) = cT(\mathbf{x}) + dT(\mathbf{y})$$

for all $c, d \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$.