Math 415 - Lecture 36

Minima, maxima and saddle points, Constrained Optimization

Wednesday December 2nd 2015

Textbook reading: Chapter 6.1

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Suggested practice exercises: Chapter 6.1, # 1, 16

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Strang lecture: Lecture 27: Positive definite matrices and minima

Review

Spectral theorem:

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- So, we can write $A = QDQ^T$ where

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  3. If some $\lambda_i > 0$, some $\lambda_j < 0$, $\mathbf{x}^T A\mathbf{x}$ will have both positive and negative values.

2nd derivative test

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Definition

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$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x_1}(\mathbf{0}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{0}) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{0}) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{0}) & \frac{\partial^2 f}{\partial^2 x_2}(\mathbf{0}) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{0}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{0}) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{0}) & \dots & \frac{\partial^2 f}{\partial^2 x_n}(\mathbf{0}) \end{bmatrix}$$

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If $f:\mathbb{R}^n \to \mathbb{R}$ is twice differentiable and $\mathbf{0}$ is a critical point, then $f(\mathbf{x}) \approx f(\mathbf{0}) + \frac{1}{2}\mathbf{x} \cdot H\mathbf{x}$.

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- Turns out: $q(\mathbf{x})$ is determined by eigenvectors and eigenvalues of H!

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(An example of such a function is $f(x, y) = \frac{1}{2}x^2 + 2xy + \frac{1}{2}y^2$).

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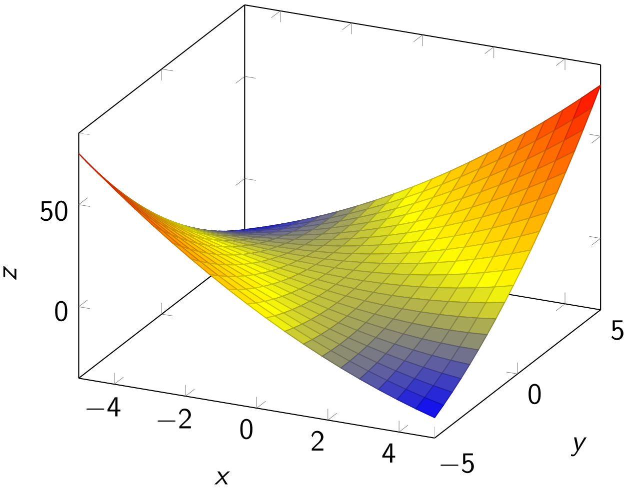

Figure : Graph of the function $f(x, y) = \frac{1}{2}x^2 + 2xy + \frac{1}{2}y^2$

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2nd derivative test says: $f(\mathbf{0})$ is local min.

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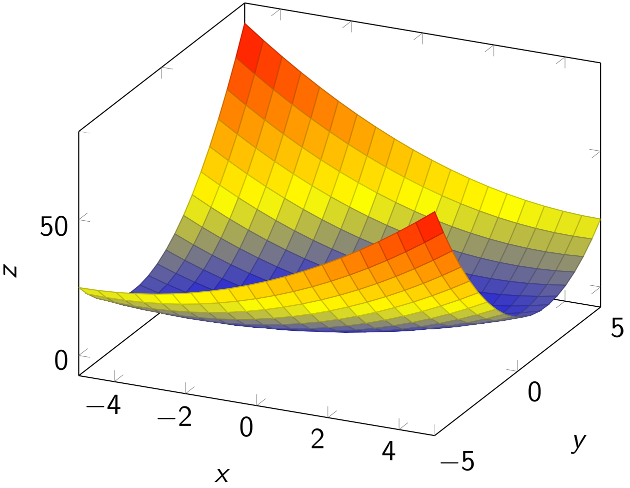

Figure : Graph of the function $f(x, y) = \frac{1}{2}x^2 - xy + \frac{3}{2}y^2$

Constrained optimization

Problem: Given a quadratic from $q(x)$, find the maximum or minimum value $q(x)$ for $\mathbf{x}$ in some specified set. Typically, the problem can be arranged such that $\mathbf{x}$ varies over the set of vectors with $\mathbf{x}^T\mathbf{x} = 1$.

Example

Let $A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find the maximum and minimum values of $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$.

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The quadratic form is $q(x_1, x_2, x_3) = 9x_1^2 + 4x_2^2 + 3x_3^2$. We are interested in the maximal value for q when (x_1, x_2, x_3) is such that $x_1^2 + x_2^2 + x_3^2 = 1$.

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$$q(\mathbf{x}) \leq 9x_1^2 + 9x_2^2 + 9x_3^2 = 9$$

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So $q(\mathbf{x})$ can not be bigger than 9, for any $\mathbf{x}$. Can we get $q(\mathbf{x}) = 9$ for some $\mathbf{x}$?

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Solution

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Theorem

Let A be a symmetric matrix and let λ_m be the least eigenvalue and λ_M be the greatest eigenvalue of A. Then

$$\lambda_m = \min\{\mathbf{x}^T A\mathbf{x} \ : \ \mathbf{x}^T\mathbf{x} = 1\},$$

moreover if $\mathbf{u}_m$ is a unit eigenvector corresponding to λ_m, then $\mathbf{u}_m^T A\mathbf{u}_m = \lambda_m$.
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Proof.

We know by the spectral theorem that $A = QDQ^T$, and so we can write $q(\mathbf{x}) = \mathbf{x}^T QDQ^T \mathbf{x} = u^T Du = \lambda_M u_1^2 + \cdots + \lambda_m u_m^2$, where $u = Q^T x$.

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Example

Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. Find the maximum and minimum values of $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$.

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