KEY

Name ____

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- Show all your work in this exam booklet; your partial credit might depend on it.
- Put your final answers at the end of your work and mark them clearly.
- If the answer is a function, its support must be included.
- No credit will be given without supporting work.
- The exam is closed book and closed notes.
- You are allowed to use one 8½" x 11" sheet with notes.
- Turn in all scratch paper with your exam.

Academic Integrity

The University statement on your obligation to maintain academic integrity is:

If you engage in an act of academic dishonesty, you become liable to severe disciplinary action. Such acts include cheating; falsification or invention of information or citation in an academic endeavor; helping or attempting to help others commit academic infractions; plagiarism; offering bribes, favors, or threats; academic interference; computer related infractions; and failure to comply with research regulations.

Rule 33 of the Code of Policies and Regulations Applying to All Students gives complete details of rules governing academic integrity for all students. You are responsible for knowing and abiding by these rules.

1. Let X_1, X_2, X_3 be iid with common mgf $M(t) = \left[\frac{3}{4} + \frac{1}{4}e^t\right]^2$, for all $t \in R$.
   a. Determine the probabilities, $P(X_1 = k)$, $k = 0, 1, 2$.

$$M(t) = \left[\frac{3}{4} + \frac{1}{4}e^t\right]^2 = \frac{9}{16} + \frac{3}{8}e^t + \frac{1}{16}e^{2t}, t \in R$$

We see $P(X = 0) = \frac{9}{16}, P(X = 1) = \frac{3}{8}, P(X = 2) = \frac{1}{16}$.

   b. Find the mgf of $Y = X_1 + X_2 + X_3$.

Independence implies ,

$$M_Y(t) = E(e^{Yt}) = E(e^{X_1 t}e^{X_2 t}e^{X_3 t}) = E(e^{X_1 t})E(e^{X_1 t})E(e^{X_1 t})$$

$$= \left[\frac{3}{4} + \frac{1}{4}e^t\right]^6, t \in R$$

2. The joint pdf for random variables X and Y is $f(x, y) = 4e^{-2(x+y)}, x > 0, y > 0$, zero otherwise. Find the pdf of $W = X + Y$. (Hint: X and Y are independent exponential random variables; be sure to specify the support)

If X and $Y$are independent $Exp(\lambda = 2)$, we immediately know $W \sim Gamma(\alpha = 2, \lambda = 2)$. Alternatively, you could use convolutions,

$$y > 0 \Rightarrow w > x$$

$$f_W(w) = \int_0^w 4e^{2w} dx = 4we^{2w}, w > 0$$

3. Let X_1, X_2, and X_3 be independent uniform random variables with pdfs $X_1 \sim U(0,1), X_2 \sim U\left(0, \frac{1}{2}\right)$, and $X_3 \sim U\left(0, \frac{1}{3}\right)$. Find the pdf of $Y_3 = \max(X_1, X_2, X_3)$. (be sure to specify the support)

The cdfs are defined as, $F_{X_i}(x) = \begin{cases} ix, & 0 \leq x < \frac{1}{i} \\ 1, & x \geq \frac{1}{i} \end{cases}$, for $i = 1,2,3$. The cdf of Y_3 is,

$$F_{Y_3}(x) = P(Y_3 < x) = P(X_1 < x, X_2 < x, X_3 < x) = \begin{cases} 0, & x < 0 \\ 6x^3, & 0 \leq x < \frac{1}{3} \\ 2x^2, & \frac{1}{3} \leq x < \frac{1}{2} \\ x, & \frac{1}{2} \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

The pdf is,

$$f_{Y_3}(x) = F'_{Y_3}(x) = \begin{cases} 18x^2, & 0 \leq x < \frac{1}{3} \\ 4x, & \frac{1}{3} \leq x < \frac{1}{2} \\ 1, & \frac{1}{2} \leq x < 1 \\ 0 & o.w. \end{cases}$$

4. X_1, X_2, X_3 are iid random variables with pdf $f(x) = 2x, 0 < x < 1$, zero elsewhere. Let $Y_1 = \min(X_1, X_2, X_3)$, $Y_2 = 2^{\text{nd}}$ smallest of (X_1, X_2, X_3), and $Y_3 = \max(X_1, X_2, X_3)$. Find the joint pdf of Y_1 and Y_3. (be sure to specify the support)

By Theorem 4.4.1,

$$g(y_1, y_2, y_3) = 48y_1y_2y_3, 0 < y_1 < y_2 < y_3 < 1$$

$$g(y_1, y_3) = \int_{y_1}^{y_3} 48y_1y_2y_3dy_2 = 24y_1y_3(y_3^2 - y_1^2), 0 < y_1 < y_3 < 1$$

5. Let X and Y be continuous random variables. What is the determinant of the Jacobian of transformation for $U = XY$ and $V = \frac{Y}{X}$? (be sure to specify the support)

$$X = \sqrt{\frac{U}{V}}, Y = \sqrt{UV}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}\sqrt{\frac{1}{UV}} & \frac{1}{2}\frac{\sqrt{U}}{V^{\frac{3}{2}}} \\ \frac{1}{2}\sqrt{\frac{V}{U}} & \frac{1}{2}\sqrt{\frac{U}{V}} \end{vmatrix} = \frac{1}{2}\frac{1}{v}$$

6. Let $f(x,y) = 2, 0 < x < 1, x < y < 2x$. Find $f(y|x)$, $E(Y|X)$, and $Var(Y|X)$.

$$f(x) = \int_{x}^{2x} 2dy = 2x \Rightarrow f(y|x) = \frac{1}{x}, 0 < x < 1, x < y < 2x$$

$$E(Y|X) = \frac{1}{x}\int_{x}^{2x} ydy = \frac{3}{2}x, 0 < x < 1$$

$$E(Y^2|X) = \frac{1}{x}\int_{x}^{2x} y^2dy = \frac{7}{3}x^2, 0 < x < 1$$

$$Var(Y|X) = E(Y^2|X) - [E(Y|X)]^2 = \frac{x}{12}$$

7. Consider random variables X_1 and X_2 with $E(X_1) = 1$, $E(X_2) = 2$, $Var(X_1) = 2$, $Var(X_2) = 3$, and $Cov(X_1, X_2) = 1$. Let $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$. Find the correlation between Y_1 and Y_2.

$$Var(Y_1) = 2 + 3 - 2 = 3$$

$$Var(Y_2) = 2 + 3 + 2 = 7$$

$$Cov(Y_1, Y_2) = Var(X_1) - Var(X_2) = -1$$

$$Cor(Y_1, Y_2) = -\frac{1}{\sqrt{21}}$$

8. Let X_1 and X_2 be independent random variables with $E(X_1) = 0$, $E(X_2) = 1$, $Var(X_1) = 1$, and $Var(X_2) = 2$. Let $Z = X_1 X_2$. Find $Var(Z)$.

$$\begin{aligned} Var(Z) = E(Z^2) - [E(Z)]^2 &= E(X_1^2 X_2^2) - [E(X_1 X_2)]^2 \\ &= E(X_1^2)E(X_2^2) - [E(X_1)E(X_2)]^2 = (1 + 0^2)(2 + 1^2) = 3 \end{aligned}$$

9. Show that $Cov[X, Y - E(Y|X)] = 0$. (Hint: Recall $E(U) = E[E(U|V)]$).

$$Cov[X, Y - E(Y|X)] = E\big[X\big(Y - E(Y|X)\big)\big] - E(X)E[Y - E(Y|X)]$$

Note that $E[Y - E(Y|X)] = E(Y) - E[E(Y|X)] = E(Y) - E(Y) = 0$. The first term similarly reduces as,

$$\begin{aligned} E\big[X\big(Y - E(Y|X)\big)\big] &= E(XY) - E[XE(Y|X)] = E(XY) - E[E(XY|X)] \\ &= E(XY) - E(XY) = 0 \end{aligned}$$