Numerical Methods (CS 357)

Worksheet

Part 1. Objectives

- Know about reduced vs full SVD
- Have an intuition for low-rank approximation
- Understand the PCA
- Understand the relationship between norm, condition number, and SVD

Part 2. SVD and condition number

A matrix A has the Singular Value Decomposition

$$A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

What is the 2-norm condition number of A?

Part 3. SVD and nullspace

A matrix A has the Singular Value Decomposition

$$A = \begin{bmatrix} | & | & \cdots & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ \vdots & \vdots & \vdots \\ - & v_n & - \end{bmatrix}$$

with $\sigma_1 \neq 0$.

What is the nullspace of A?

(A) $N(A) = \text{span}(\{u_1\})$

(B) $N(A) = \text{span}(\{u_2, \ldots, u_m\})$

(C) $N(A) = \text{span}(\{v_1\})$

(D) $N(A) = \text{span}(\{v_2, \ldots, v_m\})$

Part 4. Finding the 'dominant direction'

You are given n points in the plane in an array `X` of shape `(2,n)`.

Find (as `direction`) a vector so that the distance of the points to the line spanned by `direction` in the 2-norm is minimized. Make sure that the 2-norm of `direction` is 1.

Plot the points in `X` and the direction you found.

INPUT: `X`, an $2 \times n$ matrix

OUTPUT: `direction`, an array of shape `(2,)`

```
import numpy as np
import numpy.linalg as la
import matplotlib.pyplot as pt
```