Math 415 - Lecture 3

Existence and Uniqueness, linear combinations

Friday August 28 2015

Textbook: Chapter 1.2

Suggested Practice Exercise: Read section 1.2, do problem 1.3:9 (drawing optional)

Khan Academy Video: Linear Combinations and Span

1 Review

Existence and Uniqueness Theorem

Theorem 1 (Existence and Uniqueness Theorem)**.** *A linear system is **consistent** if and only if an echelon form of the augmented matrix has **no** row of the form*

$$\left[\begin{array}{ccc|c} 0 & \dots & 0 & b \end{array}\right]$$

*where b is nonzero.**If** a linear system is consistent, then the solution contains either*

- *a unique solution (when there are no free variables) or*
- *infinitely many solutions (when there is at least one free variable).*

Example 1. A consistent system can have 1 or ∞ many solutions. Look at the system with augmented matrix

$$\left[\begin{array}{cc|c} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{array}\right]$$

How many pivot variables can this matrix have? Do you expect the system to be consistent? Well, there are at most 2 pivots, so the last row of an echelon form should be $\left[\begin{array}{cc|c} 0 & 0 & b \end{array}\right]$. We cannot predict the value of b without doing some work. We need an echelon form.

The (reduced) echelon form of

$$\left[\begin{array}{cc|c} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{array}\right] \quad \text{is} \quad \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

- So what is b? Is the system consistent? $b = 0$ so consistent.
- So how many pivots? 2 pivots.
- How many free variables? No free variables.
- How many solutions? Exactly one!

Look now at the system with augmented matrix

$$\left[\begin{array}{cc|c} 3 & 4 & -3 \\ 3 & 4 & -3 \\ 6 & 8 & -6 \end{array}\right]$$

- How many free variables can this matrix have? One or two.Need to calculate.
- What is the Echelon form? $\begin{bmatrix} 3 & 4 & | & -3 \\ 0 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$.
- Is the system consistent? Yes!
- How many free variables? Exactly one free variable!
- How many solutions? ∞ly many!

1.1 Recap

Recap: Using Row Reduction to Solve Linear Systems

Use the following algorithm:

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If not, stop; otherwise go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. State the solution by expressing each pivot variable in terms of the free variables and declare the free variables.

1.2 Questions

Questions to check understanding

- On an exam, you are asked to find all solutions to a system of linear equations. You find exactly two solutions. Should you be worried? **YES!**
- True or false?
  - There is no more than one pivot in any row. **TRUE!**
  - There is no more than one pivot in any column. **TRUE!**
  - There cannot be more free variables than pivot variables. **FALSE!**
  - Why? Look at the equation

$$x_1 + x_2 + x_3 = 0.$$

    How many pivot variables? Free variables?

2 Geometry of Linear Equations

Definition. A **vector** in $\mathbb{R}^n$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

i.e., a column with n numbers $x_1, x_2, \ldots, x_n$ in it.

Definition. The **Sum** of $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$.

Let c be a real number. Then we define the **Scalar Multiple** $c\mathbf{u}$ by

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$$

Example 2. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Then $\mathbf{u} + \mathbf{v}$ is $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $2\mathbf{u}$ and $-\frac{3}{2}\mathbf{u}$ are $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} -3/2 \\ -\frac{9}{2} \end{bmatrix}$.

2.1 Linear Combinations

Definition. Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given **scalars** $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ using scalars (or weights) $c_1, c_2, \ldots, c_p$.

Example 3. Linear combinations don't all look the same. The following are linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$:

- $3\mathbf{v}_1 + 2\mathbf{v}_2$,
- $\frac{1}{3}\mathbf{v}_1$,
- $\mathbf{v}_1 - 2\mathbf{v}_2$,
- $\mathbf{0}$.

Example 4. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

Solution. Try first $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{a}$ or $c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Staring at this you see that $c_1 = c_2$ and hence $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{a}$ or $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Try the others for your selves.

Example 5. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$.

Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Solution. Vector $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ if can we find scalars (weights) x_1, x_2, x_3 such that $x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$. It is easy to check that $x_1 = 1, x_2 = -2, x_3 = 2$ works, so $\mathbf{b}$ is indeed a linear combination. How to find these numbers?

How to find these numbers?: $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}$ are columns of the augmented matrix

$$
\begin{array}{cccc}
\multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{array}\right]} \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b}
\end{array}
$$

Solution to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

is found by solving the linear system whose augmented matrix is

$$\left[\begin{array}{ccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{array}\right].$$

2.2 Linear combinations and linear systems

Motto
Solving linear systems is the same as finding linear combinations!

Theorem 2. *A vector equation*

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

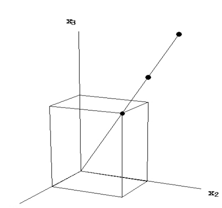
has the same solution set as the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{array}\right]$$

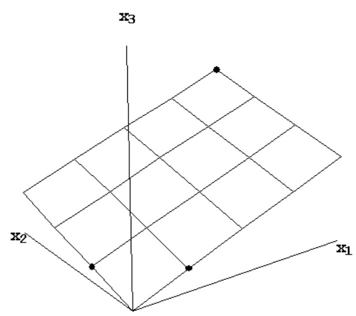
In particular, $\mathbf{b}$ *can be generated by a linear combination of* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *if and only if there is a solution to the linear system corresponding to the augmented matrix.*

2.3 Span

Example 6. Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. The origin $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ together with $\mathbf{v}$, $2\mathbf{v}$ and $1.5\mathbf{v}$ all lie on the same line.



Span$\{\mathbf{v}\}$ is the set of all vectors of the form $c\mathbf{v}$. Here, **Span**$\{\mathbf{v}\}$ = a line through the origin.



Example 7. Label $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$ and $3\mathbf{u}$+$4\mathbf{v}$ on the graph below. $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$ and $3\mathbf{u}$+$4\mathbf{v}$ all lie in the same plane. $\mathbf{Span}\{\mathbf{u},\mathbf{v}\}$ is the set of all vectors of the form $x_1\mathbf{u}+x_2\mathbf{v}$. Here, $\mathbf{Span}\{\mathbf{u},\mathbf{v}\}$ = a plane through the origin.

Definition. Suppose $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p$ are in $\mathbb{R}^n$; then the $\mathbf{Span}\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p\}$ is defined as the set of all linear combinations of $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p$.

Stated another way: $\mathbf{Span}\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p\}$ is the collection of all vectors that can be written as

$$x_1\mathbf{v}_1+x_2\mathbf{v}_2+\cdots+x_p\mathbf{v}_p$$

where $x_1,x_2,\ldots,x_p$ are scalars.

Example 8. Let $\mathbf{v}_1=\begin{bmatrix}2\\1\end{bmatrix}$ and $\mathbf{v}_2=\begin{bmatrix}4\\2\end{bmatrix}$.

(a) Find a vector in $\mathbf{Span}\{\mathbf{v}_1,\mathbf{v}_2\}$.

(b) Describe $\mathbf{Span}\{\mathbf{v}_1,\mathbf{v}_2\}$ geometrically.

Solution. (a) For instance $2\mathbf{v}_1=\mathbf{v}_2$.

(b) *Span*$\{\mathbf{v}_1,\mathbf{v}_2\}$ is the collection of all vectors in the direction of $\mathbf{v}_1$ (or $\mathbf{v}_2$!). It is a line through the origin.

So the **Span** of two vectors is a plane if and only if they don't point in the same direction.

Example 9. Let $\mathbf{v}_1=\begin{bmatrix}4\\2\\2\end{bmatrix}$ and $\mathbf{v}_2=\begin{bmatrix}6\\3\\2\end{bmatrix}$. Is $\mathbf{Span}\{\mathbf{v}_1,\mathbf{v}_2\}$ a line or a plane?

Is v_1 a multiple of v_2? Do they point in the same direction?

Example 10. Let $A=\begin{bmatrix}1&2\\3&1\\0&5\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}8\\3\\17\end{bmatrix}$. Is $\mathbf{b}$ in the plane spanned by the columns of A?

Solution.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$$

Do x_1 and x_2 exist such that

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}?$$

pause Try and find the answer at home.