Math 415 - Lecture 20

Fundamental Theorem of Linear algebra, orthogonal complement of fundamental subspaces of a matrix

Monday October 12th 2015

Textbook reading: Chapter 3.1

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Suggested practice exercises: Chapter 2.6, 5,6,7,36,37

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Khan Academy video: Orthogonal complements

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Strang lecture: Lecture 14: Orthogonal vectors and subspaces

Review

Orthogonality and FTLA

- $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are **orthogonal** iff $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w} = v_1 w_1 \cdots + v_n w_n = 0$.
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- $\dim(V) + \dim(V^{\perp}) = \dim(\mathbb{R}^n) = n$

Example

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Why?

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- It means that the inner product of every row of A(transposed!) with $\mathbf{x}$ is zero. But that implies that $\mathbf{x}$ is **orthogonal to the row space.**

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FLTA in action

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$$Nul\left(\begin{bmatrix}1 & 0\\1 & 1\\1 & 1\end{bmatrix}^T\right) = Nul\left(\begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\end{bmatrix}\right)$$

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Basis for *Nul*: $\left\{\begin{bmatrix}0\\ -1\\ 1\end{bmatrix}\right\}$

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Final answer: the set of vectors orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is

$$span\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$$

Alternative solution.

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This is the same null space we obtained from the FTLA.

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Geometrically this makes sense: V is a plane with normal vector $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

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Interpret the above: V is actually defined as the orthogonal complement of

$$span \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

Example

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By FTLA the orthogonal complement is $Nul\left(\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \right)$

FLTA in action

Get RREF to compute Null space: $\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \rightarrow$

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Get RREF to compute Null space: $\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$

FLTA in action

Get RREF to compute Null space: $\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$

Basis for the Null space: $\left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$

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So the orthogonal complement to V is: $span \left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$

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By equations: If $V = Nul(B)$ then you know that any $\mathbf{v}$ in V satisfies the equations $\mathbf{R_i^T} \cdot \mathbf{v} = 0$, for all rows $\mathbf{R_i}$ of B.

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Both descriptions are useful, and we will often switch between them, to answer any particular question we want to answer.

A new perspective on $A\mathbf{x} = \mathbf{b}$

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The indirect approach means:

$$\text{if } \underbrace{\mathbf{y}^T A = \mathbf{0}}_{\mathbf{y} \in Nul(A^T)}, \text{ then } \underbrace{\mathbf{y}^T \mathbf{b} = 0}_{\mathbf{b} \perp \mathbf{y}}.$$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution?

Solution (old)

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Write augmented matrix, get Echelon form:

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When is this consistent?

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When is this consistent? Whenever $-3b_1 + b_2 + b_3 = 0$.

Solution (new)

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Find basis for $Nul(A^T)$:

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This is the same condition as before!

Motivation

How to find almost-solutions

Why do we care about orthogonality?

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Idea

Instead of giving up, we want the $\mathbf{x}$ which makes $A\mathbf{x}$ and $\mathbf{b}$ as close as possible.

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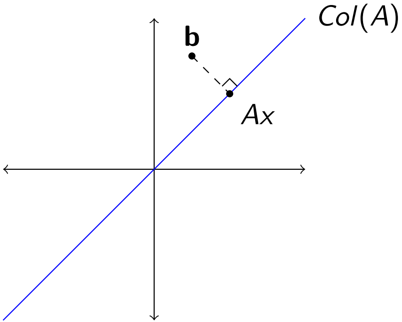
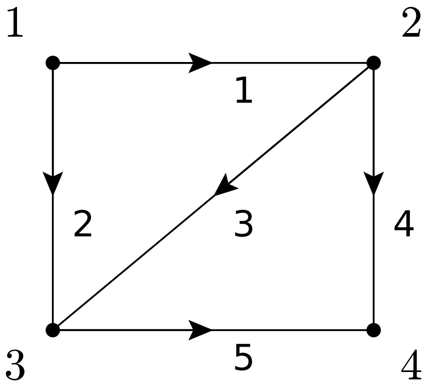
Such $\mathbf{x}$ is characterized by $A\mathbf{x}$ being **orthogonal** to the error $\mathbf{b} - A\mathbf{x}$.



Application: Directed graphs

Set up

- Graphs appear in network analysis (e.g. internet) or circuit analysis.



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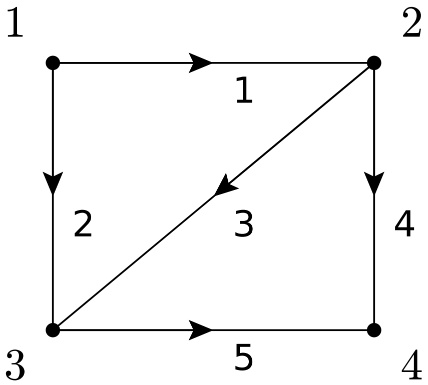

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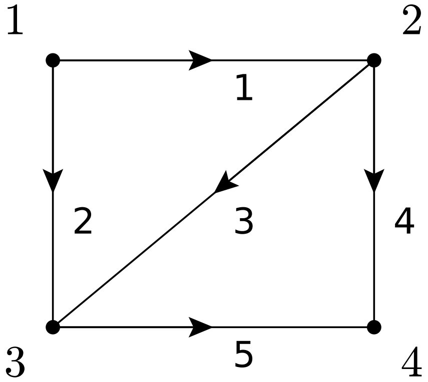
- Arrow indicates direction of flow
- No edges from a node to itself



Definition

Let G be a graph with m edges and n nodes.

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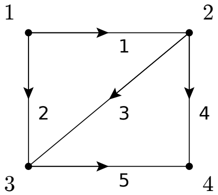
Let G be a graph with m edges and n nodes. The edge-node incidence matrix of G is the $m \times n$ matrix A with

$$A_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j \\ +1, & \text{if edge } i \text{ enters node } j \\ 0, & \text{otherwise} \end{cases}$$

Example

Give the edge-node incidence matrix of our graph.



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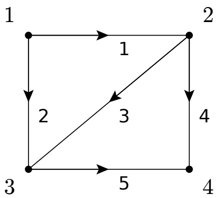

Solution

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- Each column represents a node

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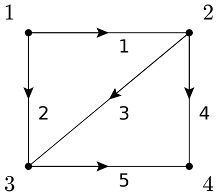

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- Each column represents a node
- Each row represents an edge

Set up

We are going to use linear algebra to study networks!