> *The wonderful thing about standards is that there are so many of them to choose from.*
>
> — Real Admiral Grace Murray Hopper

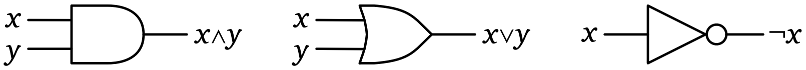
> *If a problem has no solution, it may not be a problem, but a fact — not to be solved, but to be coped with over time.*
>
> — Shimon Peres

29 NP-Hard Problems

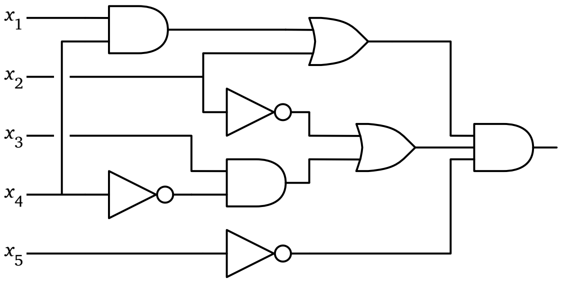
29.1 'Efficient' Problems

A generally-accepted minimum requirement for an algorithm to be considered 'efficient' is that its running time is polynomial: $O(n^c)$ for some constant c, where n is the size of the input.[1] Researchers recognized early on that not all problems can be solved this quickly, but we had a hard time figuring out exactly which ones could and which ones couldn't. there are several so-called *NP-hard* problems, which most people believe *cannot* be solved in polynomial time, even though nobody can prove a super-polynomial lower bound.

Circuit satisfiability is a good example of a problem that we don't know how to solve in polynomial time. In this problem, the input is a *boolean circuit*: a collection of AND, OR, and NOT gates connected by wires. We will assume that there are no loops in the circuit (so no delay lines or flip-flops). The input to the *circuit* is a set of m boolean (TRUE/FALSE) values $x_1, \ldots, x_m$. The output is a single boolean value. Given specific input values, we can calculate the output of the circuit in polynomial (actually, *linear*) time using depth-first-search, since we can compute the output of a k-input gate in $O(k)$ time.



An AND gate, an OR gate, and a NOT gate.



A boolean circuit. inputs enter from the left, and the output leaves to the right.

The circuit satisfiability problem asks, given a circuit, whether there is an input that makes the circuit output TRUE, or conversely, whether the circuit *always* outputs FALSE. Nobody knows how to solve

[1] This notion of efficiency was independently formalized by Alan Cobham (The intrinsic computational difficulty of functions. *Logic, Methodology, and Philosophy of Science (Proc. Int. Congress)*, 24–30, 1965), Jack Edmonds (Paths, trees, and flowers. *Canadian Journal of Mathematics* 17:449–467, 1965), and Michael Rabin (Mathematical theory of automata. *Proceedings of the 19th ACM Symposium in Applied Mathematics*, 153–175, 1966), although similar notions were considered more than a decade earlier by Kurt Gödel and John von Neumann.

this problem faster than just trying all 2^m possible inputs to the circuit, but this requires exponential time. On the other hand, nobody has ever proved that this is the best we can do; maybe there's a clever algorithm that nobody has discovered yet!

29.2 P, NP, and co-NP

A *decision problem* is a problem whose output is a single boolean value: YES or NO.[2] Let me define three classes of decision problems:

- ***P*** is the set of decision problems that can be solved in polynomial time.[3] Intuitively, P is the set of problems that can be solved quickly.
- ***NP*** is the set of decision problems with the following property: If the answer is YES, then there is a *proof* of this fact that can be checked in polynomial time. Intuitively, NP is the set of decision problems where we can verify a YES answer quickly if we have the solution in front of us.
- ***co-NP*** is the opposite of NP. If the answer to a problem in co-NP is NO, then there is a proof of this fact that can be checked in polynomial time.

For example, the circuit satisfiability problem is in NP. If the answer is YES, then any set of m input values that produces TRUE output is a proof of this fact; we can check the proof by evaluating the circuit in polynomial time. It is widely believed that circuit satisfiability is *not* in P or in co-NP, but nobody actually knows.

Every decision problem in P is also in NP. If a problem is in P, we can verify YES answers in polynomial time recomputing the answer from scratch! Similarly, any problem in P is also in co-NP.

Perhaps the single most important open questions in theoretical computer science, if not all of mathematics, is whether the complexity classes P and NP are actually different. Intuitively, it seems obvious to most people that P $\neq$ NP; the homeworks and exams in this class and others have (I hope) convinced you that problems can be incredibly hard to solve, even when the solutions are obvious in retrospect. The prominent physicist Richard Feynman, one of the early pioneers of quantum computing, apparently had trouble accepting that P $\neq$ NP was actually an open question. But nobody knows how to prove it. The Clay Mathematics Institute lists P versus NP as the first of its seven Millennium Prize Problems, offering a $1,000,000 reward for its solution.

A more subtle but still open question is whether the complexity classes NP and co-NP are different. Even if we can verify every YES answer quickly, there's no reason to believe we can also verify NO answers quickly. For example, as far as we know, there is no short proof that a boolean circuit is *not* satisfiable. It is generally believed that NP $\neq$ co-NP, but nobody knows how to prove it.

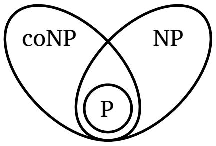
29.3 NP-hard, NP-easy, and NP-complete

A problem Π is ***NP-hard*** if a polynomial-time algorithm for Π would imply a polynomial-time algorithm for *every problem in NP*. In other words:

$$\boxed{\textbf{Π is NP-hard} \iff \textbf{If Π can be solved in polynomial time, then P=NP}}$$

[2] Technically, I should be talking about *languages*, which are just sets of bit strings. The language associated with a decision problem is the set of bit strings for which the answer is YES. For example, for the problem is 'Is the input graph connected?', the corresponding language is the set of connected graphs, where each graph is represented as a bit string (for example, its adjacency matrix).

[3] More formally, P is the set of languages that can be recognized in polynomial time by a single-tape Turing machine. If you want to learn more about Turing machines, take CS 579.

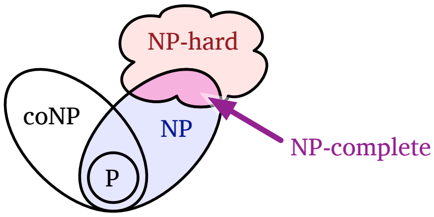


What we *think* the world looks like.

Intuitively, if we could solve one particular NP-hard problem quickly, then we could quickly solve *any* problem whose solution is easy to understand, using the solution to that one special problem as a subroutine. NP-hard problems are at least as hard as any problem in NP.

Calling a problem is NP-hard is like saying 'If I own a dog, then it can speak fluent English.' You probably don't know whether or not I own a dog, but you're probably pretty sure that I don't own a *talking* dog. Nobody has a mathematical *proof* that dogs can't speak English—the fact that no one has ever heard a dog speak English is evidence, as are the hundreds of examinations of dogs that lacked the proper mouth shape and braiNPower, but mere evidence is not a mathematical proof. Nevertheless, no sane person would believe me if I said I owned a dog that spoke fluent English. So the statement 'If I own a dog, then it can speak fluent English' has a natural corollary: No one in their right mind should believe that I own a dog! Likewise, if a problem is NP-hard, no one in their right mind should believe it can be solved in polynomial time.

Finally, a problem is ***NP-complete*** if it is both NP-hard and an element of NP (or 'NP-easy'). NP-complete problems are the hardest problems in NP. If anyone finds a polynomial-time algorithm for even one NP-complete problem, then that would imply a polynomial-time algorithm for *every* NP-complete problem. Literally *thousands* of problems have been shown to be NP-complete, so a polynomial-time algorithm for one (and therefore all) of them seems incredibly unlikely.



More of what we *think* the world looks like.

It is not immediately clear that *any* decision problems are NP-hard or NP-complete. NP-hardness is already a lot to demand of a problem; insisting that the problem also have a nondeterministic polynomial-time algorithm seems almost completely unreasonable. The following remarkable theorem was first published by Steve Cook in 1971 and independently by Leonid Levin in 1973.[4] I won't even sketch the proof, since I've been (deliberately) vague about the definitions.

The Cook-Levin Theorem. *Circuit satisfiability is NP-complete.*

*29.4 Formal Definition (*HC SVNT DRACONES*)

More formally, a problem Π is defined to be NP-hard if and only if, for every problem Π' in NP, there is a polynomial-time ***Turing reduction*** from Π' to Π—a Turing reduction just means a reduction that can

[4] Levin first reported his results at seminars in Moscow in 1971; news of Cook's result did not reach the Soviet Union until at least 1973, after Levin's announcement of his results had been published. In accordance with Stigler's Law, this result is often called 'Cook's Theorem'. Cook won the Turing award for his proof. Levin was denied his PhD at Moscow University for political reasons; he emigrated to the US in 1978 and earned a PhD at MIT a year later.

be executed on a Turing machine. Polynomial-time Turing reductions are also called *oracle reductions* or *Cook reductions*. It is elementary, but *extremely* tedious, to prove that any algorithm that can be executed on a random-access machine[5] in time $T(n)$ can be simulated on a single-tape Turing machine in time $O(T(n)^2)$, so polynomial-time Turing reductions don't actually need to be described using Turing machines. We'll simply take that step on faith.

Complexity-theory researchers prefer to define NP-hardness in terms of polynomial-time ***many-one reductions***, which are also called *Karp reductions*. Karp reductions are defined over *languages*: sets of strings over a fixed alphabet Σ. (Without loss of generality, we can assume that $\Sigma = \{0,1\}$.) A *many-one* reduction from one language $\Pi' \subseteq \Sigma^*$ to another language $\Pi \subseteq \Sigma^*$ is an function $f : \Sigma^* \to \Sigma^*$ such that $x \in \Pi'$ if and only if $f(x) \in \Pi$. Then we could define a *language* Π to be NP-hard if and only if, for any language $\Pi' \in \text{NP}$, there is a many-one reduction from Π' to Π that can be computed in polynomial time.

Every Karp reduction is a Cook reduction, but not vice versa; specifically, any Karp reduction from Π to Π' is equivalent to transforming the input to Π into the input for Π', invoking an oracle (that is, a subroutine) for Π', and then returning the answer verbatim. However, as far as we know, not every Cook reduction can be simulated by a Karp reduction.

Complexity theorists prefer Karp reductions primarily because NP is closed under Karp reductions, but is *not* closed under Cook reductions (unless NP=co-NP, which is considered unlikely). There are natural problems that are (1) NP-hard with respect to Cook reductions, but (2) NP-hard with respect to Karp reductions only if P=NP. As a trivial example, consider the problem UNSAT: Given a boolean formula, is it *always false*? On the other hand, many-one reductions apply *only* to decision problems (or more formally, to languages); formally, no optimization or construction problem is Karp-NP-hard.

To make things even more confusing, both Cook and Karp originally defined NP-hardness in terms of ***logarithmic-space*** reductions. Every logarithmic-space reduction is a polynomial-time reduction, but (we think) not vice versa. It is an open question whether relaxing the set of allowed (Cook or Karp) reductions from logarithmic-space to polynomial-time changes the set of NP-hard problems.

Fortunately, none of these subtleties raise their ugly heads in practice, so you can wake up now.

29.5 Reductions and SAT

To prove that a problem is NP-hard, we use a *reduction argument*. Reducing problem A to another problem B means describing an algorithm to solve problem A under the assumption that an algorithm for problem B already exists. You're already used to doing reductions, only you probably call it something else, like writing subroutines or utility functions, or modular programming. To prove something is NP-hard, we describe a similar transformation between problems, but not in the direction that most people expect.

You should tattoo the following rule of onto the back of your hand, right next to your Mom's birthday and the *actual* rules of Monopoly.[6]

[5] Random-access machines are a model of computation that more faithfully models physical computers. A random-access machine has unbounded random-access memory, modeled as an array $M[0..\infty]$ where each address $M[i]$ holds a single w-bit integer, for some fixed integer w, and can read to or write from any memory addresses in constant time. RAM algorithms are formally written in assembly-like language, using instructions like ***ADD*** i, j, k (meaning "$M[i] \leftarrow M[j] + M[k]$"), ***INDIR*** i, j (meaning "$M[i] \leftarrow M[M[j]]$"), and ***IFZGOTO*** i, ℓ (meaning "if $M[i] = 0$, go to line ℓ"). But in practice, RAM algorithms can be faithfully described in higher-level pseudocode, as long as we're careful about arithmetic precision.

[6] If a player lands on an available property and declines (or is unable) to buy it, that property is immediately auctioned off to the highest bidder; the player who originally declined the property may bid, and bids may be arbitrarily higher or lower than the list price. Players in Jail can still buy and sell property, buy and sell houses and hotels, and collect rent. The game has 32 houses and 12 hotels; once they're gone, they're gone. In particular, if all houses are already on the board, you cannot downgrade a hotel to four houses; you must sell all three hotels in the group. Players can sell/exchange undeveloped properties, but not buildings or cash. A player landing on Free Parking does not win anything. A player landing on Go gets \$200, no more. Railroads are not magic transporters. Finally, Jeff *always* gets the car.

To prove that problem A is NP-hard, reduce a known NP-hard problem to A.

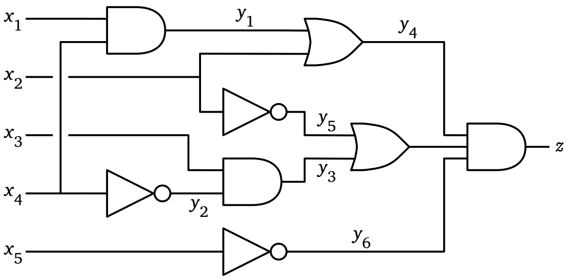
In other words, to prove that your problem is hard, you need to describe an algorithm to solve a *different* problem, which you already know is hard, using a mythical algorithm for *your* problem as a subroutine. The essential logic is a proof by contradiction. Your reduction shows implies that if your problem were easy, then the other problem would be easy, too. Equivalently, since you know the other problem is hard, your problem must also be hard.

For example, consider the *formula satisfiability* problem, usually just called ***SAT***. The input to SAT is a boolean *formula* like

$$(a \vee b \vee c \vee \bar{d}) \Leftrightarrow ((b \wedge \bar{c}) \vee \overline{(\bar{a} \Rightarrow d)} \vee (c \neq a \wedge b)),$$

and the question is whether it is possible to assign boolean values to the variables $a, b, c, \dots$ so that the formula evaluates to TRUE.

To show that SAT is NP-hard, we need to give a reduction from a known NP-hard problem. The only problem we know is NP-hard so far is circuit satisfiability, so let's start there. Given a boolean circuit, we can transform it into a boolean formula by creating new output variables for each gate, and then just writing down the list of gates separated by ANDS. For example, we can transform the example circuit into a formula as follows:



$$(y_1 = x_1 \wedge x_4) \wedge (y_2 = \overline{x_4}) \wedge (y_3 = x_3 \wedge y_2) \wedge (y_4 = y_1 \vee x_2) \wedge$$
$$(y_5 = \overline{x_2}) \wedge (y_6 = \overline{x_5}) \wedge (y_7 = y_3 \vee y_5) \wedge (z = y_4 \wedge y_7 \wedge y_6) \wedge z$$

A boolean circuit with gate variables added, and an equivalent boolean formula.

Now the original circuit is satisfiable if and only if the resulting formula is satisfiable. Given a satisfying input to the circuit, we can get a satisfying assignment for the formula by computing the output of every gate. Given a satisfying assignment for the formula, we can get a satisfying input the the circuit by just ignoring the internal gate variables y_i and the output variable z.

We can transform any boolean circuit into a formula in linear time using depth-first search, and the size of the resulting formula is only a constant factor larger than the size of the circuit. Thus, we have a polynomial-time reduction from circuit satisfiability to SAT:

$$\begin{tikzcd}
\boxed{\text{boolean circuit}} \arrow[r, "O(n)"] & \boxed{\text{boolean formula}} \arrow[d, Rightarrow, "\text{SAT}"] \\
\boxed{\text{TRUE or FALSE}} & \boxed{\text{TRUE or FALSE}} \arrow[l, "\text{trivial}"']
\end{tikzcd}$$

$$T_{\text{CSAT}}(n) \leq O(n) + T_{\text{SAT}}(O(n)) \quad \Longrightarrow \quad T_{\text{SAT}}(n) \geq T_{\text{CSAT}}(\Omega(n)) - O(n)$$

The reduction implies that if we had a polynomial-time algorithm for SAT, then we'd have a polynomial-time algorithm for circuit satisfiability, which would imply that P=NP. So SAT is NP-hard.

To prove that a boolean formula is satisfiable, we only have to specify an assignment to the variables that makes the formula TRUE. We can check the proof in linear time just by reading the formula from left to right, evaluating as we go. So SAT is also in NP, and thus is actually NP-complete.

29.6 3SAT (from SAT)

A special case of SAT that is particularly useful in proving NP-hardness results is called *3SAT*.

A boolean formula is in *conjunctive normal form* (CNF) if it is a conjunction (AND) of several *clauses*, each of which is the disjunction (OR) of several *literals*, each of which is either a variable or its negation. For example:

$$\overbrace{(a \vee b \vee c \vee d)}^{\text{clause}} \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b})$$

A *3CNF* formula is a CNF formula with exactly three literals per clause; the previous example is not a 3CNF formula, since its first clause has four literals and its last clause has only two. 3SAT is just SAT restricted to 3CNF formulas: Given a 3CNF formula, is there an assignment to the variables that makes the formula evaluate to TRUE?

We could prove that 3SAT is NP-hard by a reduction from the more general SAT problem, but it's easier just to start over from scratch, with a boolean circuit. We perform the reduction in several stages.

1. *Make sure every* AND *and* OR *gate has only two inputs.* If any gate has $k > 2$ inputs, replace it with a binary tree of $k - 1$ two-input gates.

2. *Write down the circuit as a formula, with one clause per gate.* This is just the previous reduction.

3. *Change every gate clause into a CNF formula.* There are only three types of clauses, one for each type of gate:

$$\begin{aligned} a = b \wedge c &\quad\longmapsto\quad (a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee b) \wedge (\bar{a} \vee c) \\ a = b \vee c &\quad\longmapsto\quad (\bar{a} \vee b \vee c) \wedge (a \vee \bar{b}) \wedge (a \vee \bar{c}) \\ a = \bar{b} &\quad\longmapsto\quad (a \vee b) \wedge (\bar{a} \vee \bar{b}) \end{aligned}$$

4. *Make sure every clause has exactly three literals.* Introduce new variables into each one- and two-literal clause, and expand it into two clauses as follows:

$$\begin{aligned} a &\quad\longmapsto\quad (a \vee x \vee y) \wedge (a \vee \bar{x} \vee y) \wedge (a \vee x \vee \bar{y}) \wedge (a \vee \bar{x} \vee \bar{y}) \\ a \vee b &\quad\longmapsto\quad (a \vee b \vee x) \wedge (a \vee b \vee \bar{x}) \end{aligned}$$

For example, if we start with the same example circuit we used earlier, we obtain the following 3CNF formula. Although this may look a lot more ugly and complicated than the original circuit at first glance, it's actually only a constant factor larger—every binary gate in the original circuit has been transformed into at most five clauses. Even if the formula size were a large *polynomial* function (like n^{573}) of the circuit size, we would still have a valid reduction.

$$\begin{aligned}
&(y_1 \vee \overline{x_1} \vee \overline{x_4}) \wedge (\overline{y_1} \vee x_1 \vee z_1) \wedge (\overline{y_1} \vee x_1 \vee \overline{z_1}) \wedge (\overline{y_1} \vee x_4 \vee z_2) \wedge (\overline{y_1} \vee x_4 \vee \overline{z_2})\\
\wedge\, &(y_2 \vee x_4 \vee z_3) \wedge (y_2 \vee x_4 \vee \overline{z_3}) \wedge (\overline{y_2} \vee \overline{x_4} \vee z_4) \wedge (\overline{y_2} \vee \overline{x_4} \vee \overline{z_4})\\
\wedge\, &(y_3 \vee \overline{x_3} \vee \overline{y_2}) \wedge (\overline{y_3} \vee x_3 \vee z_5) \wedge (\overline{y_3} \vee x_3 \vee \overline{z_5}) \wedge (\overline{y_3} \vee y_2 \vee z_6) \wedge (\overline{y_3} \vee y_2 \vee \overline{z_6})\\
\wedge\, &(\overline{y_4} \vee y_1 \vee x_2) \wedge (y_4 \vee \overline{x_2} \vee z_7) \wedge (y_4 \vee \overline{x_2} \vee \overline{z_7}) \wedge (y_4 \vee \overline{y_1} \vee z_8) \wedge (y_4 \vee \overline{y_1} \vee \overline{z_8})\\
\wedge\, &(y_5 \vee x_2 \vee z_9) \wedge (y_5 \vee x_2 \vee \overline{z_9}) \wedge (\overline{y_5} \vee \overline{x_2} \vee z_{10}) \wedge (\overline{y_5} \vee \overline{x_2} \vee \overline{z_{10}})\\
\wedge\, &(y_6 \vee x_5 \vee z_{11}) \wedge (y_6 \vee x_5 \vee \overline{z_{11}}) \wedge (\overline{y_6} \vee \overline{x_5} \vee z_{12}) \wedge (\overline{y_6} \vee \overline{x_5} \vee \overline{z_{12}})\\
\wedge\, &(\overline{y_7} \vee y_3 \vee y_5) \wedge (y_7 \vee \overline{y_3} \vee z_{13}) \wedge (y_7 \vee \overline{y_3} \vee \overline{z_{13}}) \wedge (y_7 \vee \overline{y_5} \vee z_{14}) \wedge (y_7 \vee \overline{y_5} \vee \overline{z_{14}})\\
\wedge\, &(y_8 \vee \overline{y_4} \vee \overline{y_7}) \wedge (\overline{y_8} \vee y_4 \vee z_{15}) \wedge (\overline{y_8} \vee y_4 \vee \overline{z_{15}}) \wedge (\overline{y_8} \vee y_7 \vee z_{16}) \wedge (\overline{y_8} \vee y_7 \vee \overline{z_{16}})\\
\wedge\, &(y_9 \vee \overline{y_8} \vee \overline{y_6}) \wedge (\overline{y_9} \vee y_8 \vee z_{17}) \wedge (\overline{y_9} \vee y_8 \vee \overline{z_{17}}) \wedge (\overline{y_9} \vee y_6 \vee z_{18}) \wedge (\overline{y_9} \vee y_6 \vee \overline{z_{18}})\\
\wedge\, &(y_9 \vee z_{19} \vee z_{20}) \wedge (y_9 \vee \overline{z_{19}} \vee z_{20}) \wedge (y_9 \vee z_{19} \vee \overline{z_{20}}) \wedge (y_9 \vee \overline{z_{19}} \vee \overline{z_{20}})
\end{aligned}$$

This process transforms the circuit into an equivalent 3CNF formula; the output formula is satisfiable if and only if the input circuit is satisfiable. As with the more general SAT problem, the formula is only a constant factor larger than any reasonable description of the original circuit, and the reduction can be carried out in polynomial time. Thus, we have a polynomial-time reduction from circuit satisfiability to 3SAT:

$$\begin{tikzcd}
\boxed{\text{boolean circuit}} \arrow[r, "O(n)"] & \boxed{\text{3CNF formula}} \arrow[d, Rightarrow, "\text{3SAT}"] \\
\boxed{\text{True or False}} & \boxed{\text{True or False}} \arrow[l, "\text{trivial}"']
\end{tikzcd}$$

$$T_{\text{CSAT}}(n) \le O(n) + T_{\text{3SAT}}(O(n)) \quad \Longrightarrow \quad T_{\text{3SAT}}(n) \ge T_{\text{CSAT}}(\Omega(n)) - O(n)$$

We conclude 3SAT is NP-hard. And because 3SAT is a special case of SAT, it is also in NP. Therefore, 3SAT is NP-complete.

29.7 Maximum Independent Set (from 3SAT)

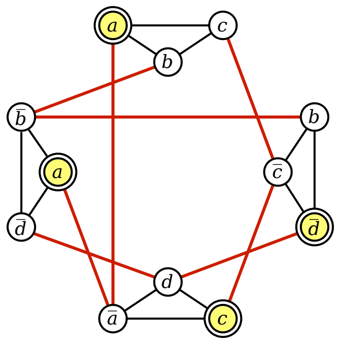
For the next few problems we consider, the input is a simple, unweighted graph, and the problem asks for the size of the largest or smallest subgraph satisfying some structural property.

Let G be an arbitrary graph. An ***independent set*** in G is a subset of the vertices of G with no edges between them. The *maximum independent set* problem, or simply **MaxIndSet**, asks for the size of the largest independent set in a given graph.

I'll prove that MaxIndSet is NP-hard (but not NP-complete, since it isn't a decision problem) using a reduction from 3SAT. I'll describe a reduction from a 3CNF formula into a graph that has an independent set of a certain size if and only if the formula is satisfiable. The graph has one node for each instance of each literal in the formula. Two nodes are connected by an edge if (1) they correspond to literals in the same clause, or (2) they correspond to a variable and its inverse. For example, the formula $(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$ is transformed into the following graph.

Now suppose the original formula had k clauses. Then I claim that the formula is satisfiable if and only if the graph has an independent set of size k.

1. **independent set $\Longrightarrow$ satisfying assignment:** If the graph has an independent set of k vertices, then each vertex must come from a different clause. To obtain a satisfying assignment, we assign



A graph derived from a 3CNF formula, and an independent set of size 4.
Black edges join literals from the same clause; red (heavier) edges join contradictory literals.

the value True to each literal in the independent set. Since contradictory literals are connected by edges, this assignment is consistent. There may be variables that have no literal in the independent set; we can set these to any value we like. The resulting assignment satisfies the original 3CNF formula.

2. **satisfying assignment $\Longrightarrow$ independent set:** If we have a satisfying assignment, then we can choose one literal in each clause that is True. Those literals form an independent set in the graph.

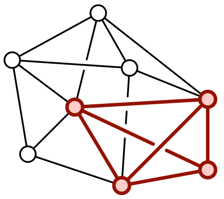
Thus, the reduction is correct. Since the reduction from 3CNF formula to graph takes polynomial time, we conclude that MaxIndSet is NP-hard. Here's a diagram of the reduction:

$$\begin{tikzcd}
\boxed{\text{3CNF formula with } k \text{ clauses}} \arrow[r, "O(n)"] & \boxed{\text{graph with } 3k \text{ nodes}} \arrow[d, Rightarrow, "\textsc{MaxIndSet}"] \\
\boxed{\textsc{True} \text{ or } \textsc{False}} & \boxed{\text{maximum independent set size}} \arrow[l, "O(1)"']
\end{tikzcd}$$

$$T_{\text{3SAT}}(n) \le O(n) + T_{\textsc{MaxIndSet}}(O(n)) \quad \Longrightarrow \quad T_{\textsc{MaxIndSet}}(n) \ge T_{\text{3SAT}}(\Omega(n)) - O(n)$$

29.8 Clique (from Independent Set)

A *clique* is another name for a complete graph, that is, a graph where every pair of vertices is connected by an edge. The *maximum clique size* problem, or simply MaxClique, is to compute, given a graph, the number of nodes in its largest complete subgraph.



A graph with maximum clique size 4.

There is an easy proof that MaxClique is NP-hard, using a reduction from MaxIndSet. Any graph G has an *edge-complement* $\overline{G}$ with the same vertices, but with exactly the opposite set of edges—(u, v) is an edge in $\overline{G}$ if and only if it is *not* an edge in G. A set of vertices is independent in G if and only if the same vertices define a clique in $\overline{G}$. Thus, we can compute the largest independent in a graph simply by computing the largest clique in the complement of the graph.

$$\begin{tikzcd}
\boxed{\text{graph } G} \arrow[r, "O(n)"] & \boxed{\text{complement graph } \overline{G}} \arrow[d, Rightarrow, "\textsc{MaxClique}"] \\
\boxed{\text{largest independent set}} & \boxed{\text{largest clique}} \arrow[l, "\text{trivial}"']
\end{tikzcd}$$

29.9 Vertex Cover (from Independent Set)

A *vertex cover* of a graph is a set of vertices that touches every edge in the graph. The MinVertexCover problem is to find the smallest vertex cover in a given graph.

Again, the proof of NP-hardness is simple, and relies on just one fact: If I is an independent set in a graph $G = (V, E)$, then $V \setminus I$ is a vertex cover. Thus, to find the *largest* independent set, we just need to find the vertices that aren't in the *smallest* vertex cover of the same graph.

$$\begin{tikzcd}
\boxed{\text{graph } G = (V,E)} \arrow[r, "\text{trivial}"] & \boxed{\text{graph } G = (V,E)} \arrow[d, Rightarrow, "\textsc{MinVertexCover}"] \\
\boxed{\text{largest independent set } V \setminus S} & \boxed{\text{smallest vertex cover } S} \arrow[l, "O(n)"']
\end{tikzcd}$$

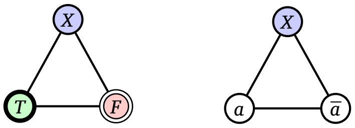
29.10 Graph Coloring (from 3SAT)

A *k-coloring* of a graph is a map $C : V \to \{1, 2, \ldots, k\}$ that assigns one of k 'colors' to each vertex, so that every edge has two different colors at its endpoints. The graph coloring problem is to find the smallest possible number of colors in a legal coloring. To show that this problem is NP-hard, it's enough to consider the special case 3Colorable: Given a graph, does it have a 3-coloring?

To prove that 3Colorable is NP-hard, we use a reduction from 3SAT. Given a 3CNF formula Φ, we produce a graph G_Φ as follows. The graph consists of a *truth* gadget, one *variable* gadget for each variable in the formula, and one *clause* gadget for each clause in the formula.

- The truth gadget is just a triangle with three vertices T, F, and X, which intuitively stand for True, False, and Other. Since these vertices are all connected, they must have different colors in any 3-coloring. For the sake of convenience, we will *name* those colors True, False, and Other. Thus, when we say that a node is colored True, all we mean is that it must be colored the same as the node T.

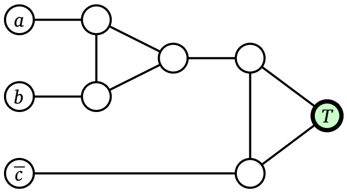
- The variable gadget for a variable a is also a triangle joining two new nodes labeled a and $\overline{a}$ to node X in the truth gadget. Node a must be colored either True or False, and so node $\overline{a}$ must be colored either False or True, respectively.



The truth gadget and a variable gadget for a.

- Finally, each clause gadget joins three literal nodes to node T in the truth gadget using five new unlabeled nodes and ten edges; see the figure below. A straightforward case analysis implies that if all three literal nodes in the clause gadget are colored False, then some edge in the gadget must be monochromatic. Since the variable gadgets force each literal node to be colored either True or

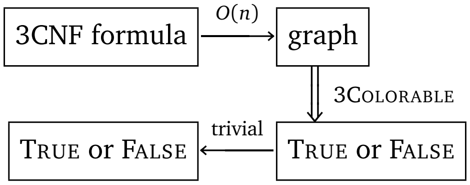
FALSE, in any valid 3-coloring, at least one of the three literal nodes is colored TRUE. On the other hand, for any coloring of the literal nodes where at least one literal node is colored TRUE, there is a valid 3-coloring of the clause gadget.



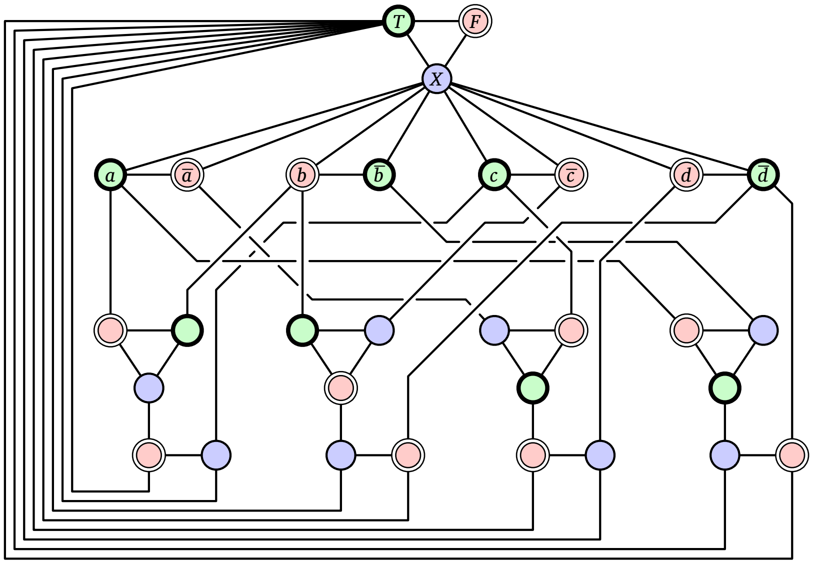
A clause gadget for $(a \vee b \vee \bar{c})$.

I need to emphasize here that the final graph G_Φ contains exactly *one* node T, exactly *one* node F, and exactly *two* nodes a and $\bar{a}$ for each variable.

Now the proof of correctness is just brute force case analysis. If the graph is 3-colorable, then we can extract a satisfying assignment from any 3-coloring—at least one of the three literal nodes in every clause gadget is colored TRUE. Conversely, if the formula is satisfiable, then we can color the graph according to any satisfying assignment.



For example, the formula $(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$ that I used to illustrate the MAXCLIQUE reduction would be transformed into the following graph. The 3-coloring is one of several that correspond to the satisfying assignment $a = c =$ TRUE, $b = d =$ FALSE.



A 3-colorable graph derived from a satisfiable 3CNF formula.

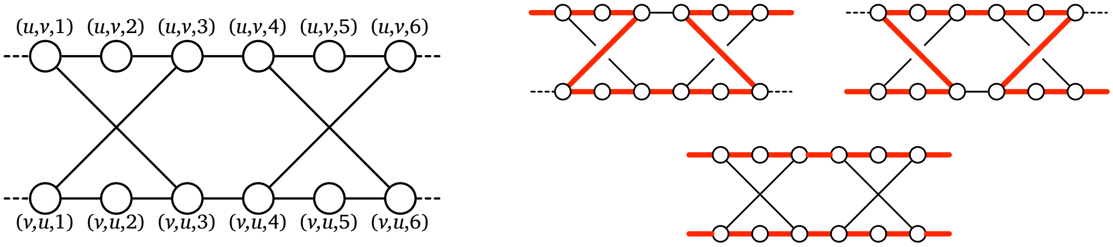
We can easily verify that a graph has been correctly 3-colored in linear time: just compare the endpoints of every edge. Thus, 3COLORING is in NP, and therefore NP-complete. Moreover, since 3COLORING is a special case of the more general graph coloring problem—What is the minimum number of colors?—the more problem is also NP-hard, but *not* NP-complete, because it's not a decision problem.

29.11 Hamiltonian Cycle (from Vertex Cover)

A *Hamiltonian cycle* in a graph is a cycle that visits every vertex exactly once. This is very different from an *Eulerian cycle*, which is actually a closed *walk* that traverses every *edge* exactly once. Eulerian cycles are easy to find and construct in linear time using a variant of depth-first search. Determining whether a graph contains a Hamiltonian cycle, on the other hand, is NP-hard.

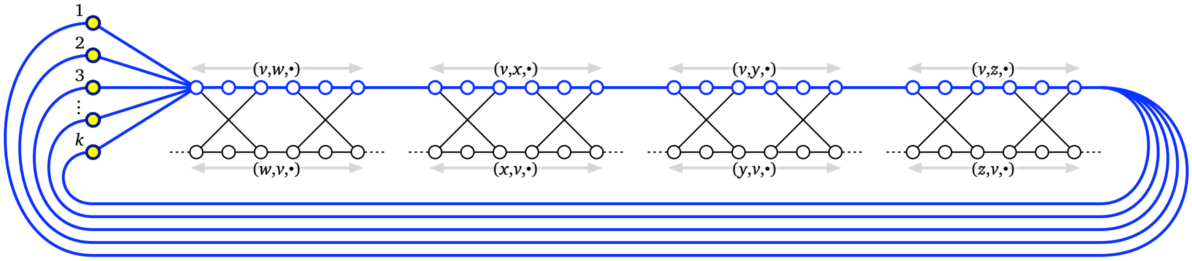
To prove this, we describe a reduction from the vertex cover problem. Given a graph G and an integer k, we need to transform it into another graph G', such that G' has a Hamiltonian cycle if and only if G has a vertex cover of size k. As usual, our transformation uses several gadgets.

- For each edge uv in G, we have an *edge gadget* in G' consisting of twelve vertices and fourteen edges, as shown below. The four corner vertices $(u, v, 1)$, $(u, v, 6)$, $(v, u, 1)$, and $(v, u, 6)$ each have an edge leaving the gadget. A Hamiltonian cycle can only pass through an edge gadget in only three ways. Eventually, these options will correspond to one or both vertices u and v being in the vertex cover.



An edge gadget for (u, v) and the only possible Hamiltonian paths through it.

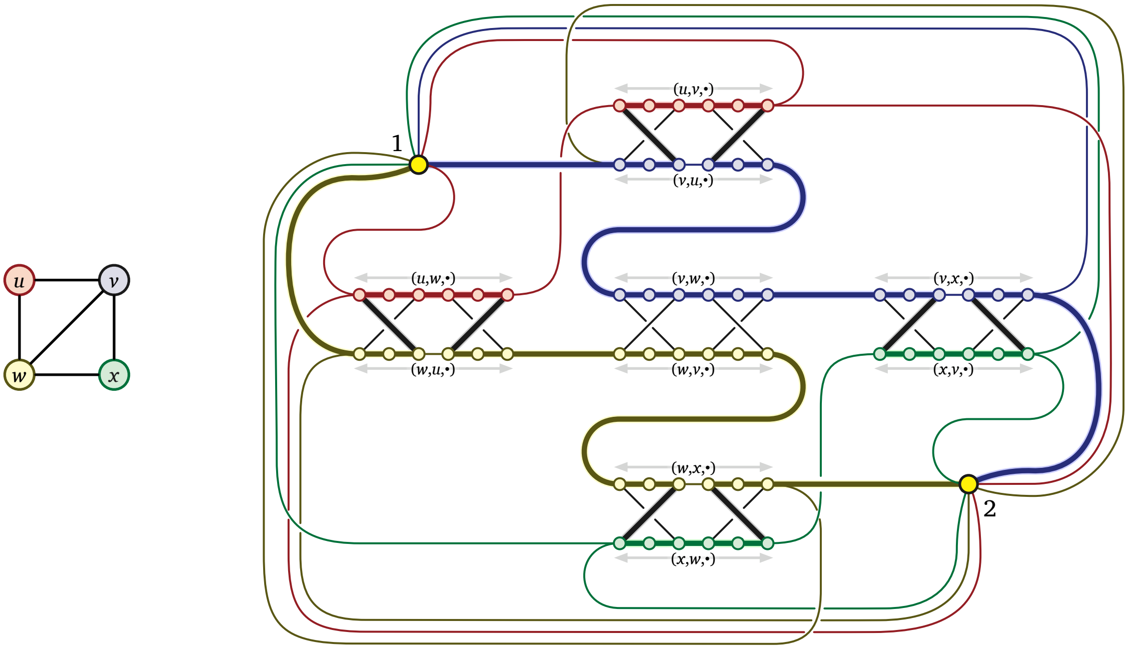
- G' also contains k *cover vertices*, simply numbered 1 through k.
- Finally, for each vertex u in G, we string together all the edge gadgets for edges (u, v) into a single *vertex chain*, and then connect the ends of the chain to all the cover vertices. Specifically, suppose vertex u has d neighbors $v_1, v_2, \dots, v_d$. Then G' has $d - 1$ edges between $(u, v_i, 6)$ and $(u, v_{i+1}, 1)$, plus k edges between the cover vertices and $(u, v_1, 1)$, and finally k edges between the cover vertices and $(u, v_d, 6)$.



The vertex chain for v: all edge gadgets involving v are strung together and joined to the k cover vertices.

An example of the output from our reduction is shown on the next page.

It is now straightforward but tedious to prove that if $\{v_1, v_2, \dots, v_k\}$ is a vertex cover of G, then G' has a Hamiltonian cycle—start at cover vertex 1, through traverse the vertex chain for v_1, then visit cover vertex 2, then traverse the vertex chain for v_2, and so forth, eventually returning to cover vertex 1. Conversely, any Hamiltonian cycle in G' alternates between cover vertices and vertex chains, and the vertex chains correspond to the k vertices in a vertex cover of G. (This is a little harder to prove.) Thus, G has a vertex cover of size k if and only if G' has a Hamiltonian cycle.



The original graph G with vertex cover $\{v, w\}$, and the transformed graph G' with a corresponding Hamiltonian cycle. Vertex chains are colored to match their corresponding vertices.

The transformation from G to G' takes at most $O(n^2)$ time; we conclude that the Hamiltonian cycle problem is NP-hard. Moreover, since we can easily verify a Hamiltonian cycle in linear time, the Hamiltonian cycle problem is in NP, and therefore is NP-complete.

A closely related problem to Hamiltonian cycles is the famous *traveling salesman problem*—Given a *weighted* graph G, find the shortest cycle that visits every vertex. Finding the shortest cycle is obviously harder than determining if a cycle exists at all, so the traveling salesman problem is also NP-hard.

29.12 Subset Sum (from Vertex Cover)

The last problem that we will prove NP-hard is the SubsetSum problem considered in the very first lecture on recursion: Given a set X of positive integers and an integer t, determine whether X has a subset whose elements sum to t.

To prove this problem is NP-hard, we apply another reduction from the vertex cover problem. Given a graph G and an integer k, we need to transform it into set of integers X and an integer t, such that X has a subset that sums to t if and only if G has an vertex cover of size k. Our transformation uses just two 'gadgets'; these are *integers* representing vertices and edges in G.

Number the *edges* of G arbitrarily from 0 to $m-1$. Our set X contains the integer $b_i := 4^i$ for each edge i, and the integer

$$a_v := 4^m + \sum_{i \in \Delta(v)} 4^i$$

for each vertex v, where $\Delta(v)$ is the set of edges that have v has an endpoint. Alternately, we can think of each integer in X as an $(m+1)$-digit number written in base 4. The mth digit is 1 if the integer represents a vertex, and 0 otherwise; and for each $i < m$, the ith digit is 1 if the integer represents edge i or one of its endpoints, and 0 otherwise. Finally, we set the target sum

$$t := k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i.$$

Now let's prove that the reduction is correct. First, suppose there is a vertex cover of size k in the original graph G. Consider the subset $X_C \subseteq X$ that includes a_v for every vertex v in the vertex cover, and b_i for every edge i that has *exactly one* vertex in the cover. The sum of these integers, written in base 4, has a 2 in each of the first m digits; in the most significant digit, we are summing exactly k 1's. Thus, the sum of the elements of X_C is exactly t.

On the other hand, suppose there is a subset $X' \subseteq X$ that sums to t. Specifically, we must have

$$\sum_{v \in V'} a_v + \sum_{i \in E'} b_i = t$$

for some subsets $V' \subseteq V$ and $E' \subseteq E$. Again, if we sum these base-4 numbers, there are no carries in the first m digits, because for each i there are only three numbers in X whose ith digit is 1. Each edge number b_i contributes only one 1 to the ith digit of the sum, but the ith digit of t is 2. Thus, for each edge in G, at least one of its endpoints must be in V'. In other words, V is a vertex cover. On the other hand, only vertex numbers are larger than 4^m, and $\lfloor t/4^m \rfloor = k$, so V' has at most k elements. (In fact, it's not hard to see that V' has *exactly* k elements.)

For example, given the four-vertex graph used on the previous page to illustrate the reduction to Hamiltonian cycle, our set X might contain the following base-4 integers:

$$\begin{array}{rlrl}
a_u &:= 111000_4 = 1344 & b_{uv} &:= 010000_4 = 256 \\
a_v &:= 110110_4 = 1300 & b_{uw} &:= 001000_4 = 64 \\
a_w &:= 101101_4 = 1105 & b_{vw} &:= 000100_4 = 16 \\
a_x &:= 100011_4 = 1029 & b_{vx} &:= 000010_4 = 4 \\
 & & b_{wx} &:= 000001_4 = 1
\end{array}$$

If we are looking for a vertex cover of size 2, our target sum would be $t := 222222_4 = 2730$. Indeed, the vertex cover $\{v, w\}$ corresponds to the subset $\{a_v, a_w, b_{uv}, b_{uw}, b_{vx}, b_{wx}\}$, whose sum is $1300 + 1105 + 256 + 64 + 4 + 1 = 2730$.

The reduction can clearly be performed in polynomial time. Since VertexCover is NP-hard, it follows that SubsetSum is NP-hard.

There is one subtle point that needs to be emphasized here. Way back at the beginning of the semester, we developed a dynamic programming algorithm to solve SubsetSum in time $O(nt)$. Isn't this a polynomial-time algorithm? idn't we just prove that P=NP? Hey, where's our million dollars? Alas, life is not so simple. True, the running time is polynomial in n and t, but in order to qualify as a true polynomial-time algorithm, the running time must be a polynomial function *of the size of the input*. The *values* of the elements of X and the target sum t could be exponentially larger than the number of input bits. Indeed, the reduction we just described produces a value of t that is exponentially larger than the size of our original input graph, which would force our dynamic programming algorithm to run in exponential time.

Algorithms like this are said to run in ***pseudo-polynomial time***, and any NP-hard problem with such an algorithm is called ***weakly NP-hard***. Equivalently, a weakly NP-hard problem is one that can be solved in polynomial time when all input numbers are represented in *unary* (as a sum of 1s), but becomes NP-hard when all input numbers are represented in *binary*. If a problem is NP-hard even when all the input numbers are represented in unary, we say that the problem is ***strongly NP-hard***.

29.13 A Frivolous Example

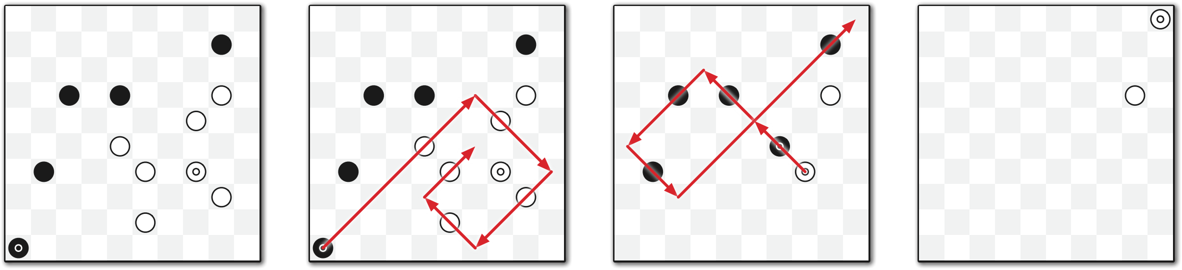
Draughts is a family of board games that have been played for thousands of years. Most American students are familiar with the version called *checkers* or *English draughts*, but the most common variant worldwide, known as ***international draughts*** or ***Polish draughts***, originated in the Netherlands in the 16th century. For a complete set of rules, the reader should consult Wikipedia; here a few important differences from the Anglo-American game:

- **Flying kings:** As in checkers, a piece that ends a move in the row closest to the opponent becomes a *king* and gains the ability to move backward. Unlike in checkers, however, a king in international draughts can move any distance along a diagonal line in a single turn, as long as the intermediate squares are empty or contain exactly one opposing piece (which is captured).

- **Forced maximum capture:** In each turn, the moving player must capture as many opposing pieces as possible. This is distinct from the forced-capture rule in checkers, which requires only that each player must capture if possible, and that a capturing move ends only when the moving piece cannot capture further. In other words, checkers requires capturing a *maximal* set of opposing pieces on each turn; whereas, international draughts requires a *maximum* capture.

- **Capture subtleties:** As in checkers, captured pieces are removed from the board only at the end of the turn. Any piece can be captured at most once. Thus, when an opposing piece is jumped, that piece remains on the board *but cannot be jumped again* until the end of the turn.

For example, in the first position shown below, each circle represents a piece, and doubled circles represent kings. Black must make the indicated move, capturing five white pieces, because it is not possible to capture more than five pieces, and there is no other move that captures five. Black cannot extend his capture further northeast, because the captured White pieces are still on the board.



Two forced(!) moves in international draughts.

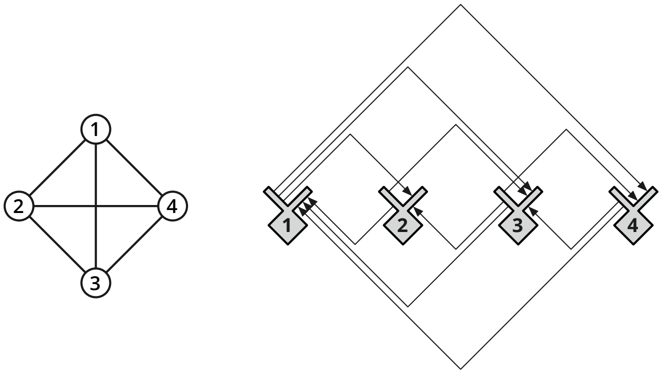
The actual game, which is played on a 10 × 10 board with 20 pieces of each color, is computationally trivial; we can precompute the optimal move for both players in every possible board configuration and hard-code the results into a lookup table of constant size. Sure, it's a *big* constant, but it's still just a constant!

But consider the natural generalization of international draughts to an $n \times n$ board. In this setting, ***finding a legal move is actually NP-hard!*** The following reduction from the Hamiltonian cycle problem in directed graphs was discovered by Bob Hearn in 2010.[7] In most two-player games, finding the *best* move is NP-hard (or worse); this is the only example I know of a game where just *following the rules* is an intractable problem!

Given a graph G with n vertices, we construct a board configuration for international draughts, such that White can capture a certain number of black pieces in a single move if and only if G has a Hamiltonian cycle. We treat G as a directed graph, with two arcs $u \to v$ and $v \to u$ in place of each undirected edge uv. Number the vertices arbitrarily from 1 to n. The final draughts configuration has several gadgets.

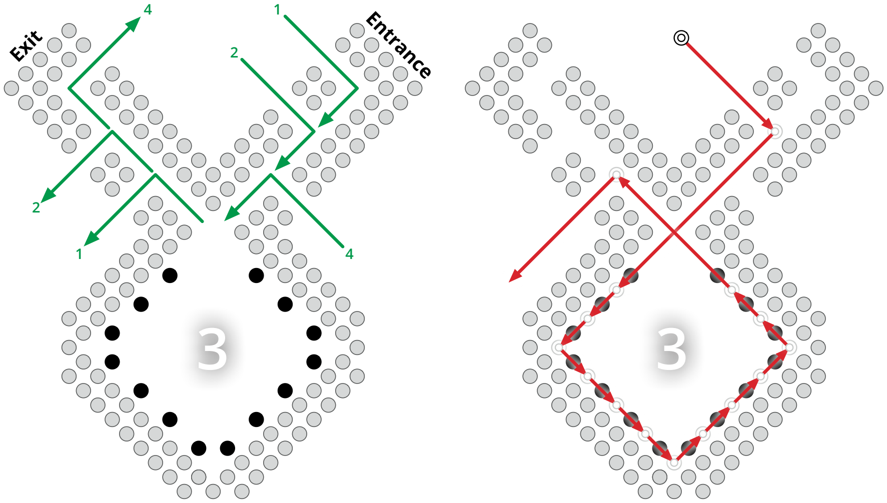
[7] Posted on Theoretical Computer Science Stack Exchange: http://cstheory.stackexchange.com/a/1999/111.

- The vertices of G are represented by rabbit-shaped *vertex gadgets*, which are evenly spaced along a horizontal line. Each arc $i \to j$ is represented by a path of two diagonal line segments from the "right ear" of vertex gadget i to the "left ear" of vertex gadget j. The path for arc $i \to j$ is located above the vertex gadgets if $i < j$, and below the vertex gadgets if $i > j$.



A high level view of the reduction from Hamiltonian cycle to international draughts.

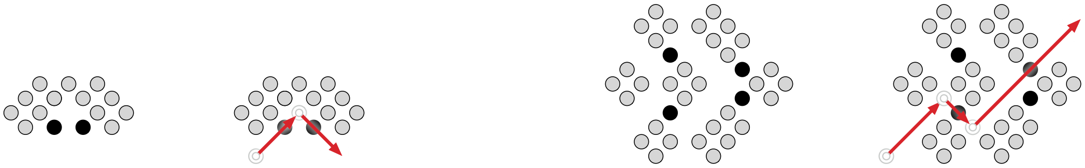
- The bulk of each vertex gadget is a diamond-shaped region called a *vault*. The walls of the vault are composed of two solid layers of black pieces, which cannot be captured; these pieces are drawn as gray circles in the figures. There are N capturable black pieces inside each vault, for some large integer N to be determined later. A white king can enter the vault through the "right ear", capture every internal piece, and then exit through the "left ear". Both ears are hallways, again with walls two pieces thick, with gaps where the arc paths end to allow the white king to enter and leave. The lengths of the "ears" can be adjusted easily to align with the other gadgets.



Left: A vertex gadget. Right: A white king emptying the vault.
Gray circles are black pieces that cannot be captured.

- For each arc $i \to j$, we have a *corner gadget*, which allows a white king leaving vertex gadget i to be redirected to vertex gadget j.

- Finally, wherever two arc paths cross, we have a *crossing gadget*; these gadgets allow the white king to traverse either arc path, but forbid switching from one arc path to the other.

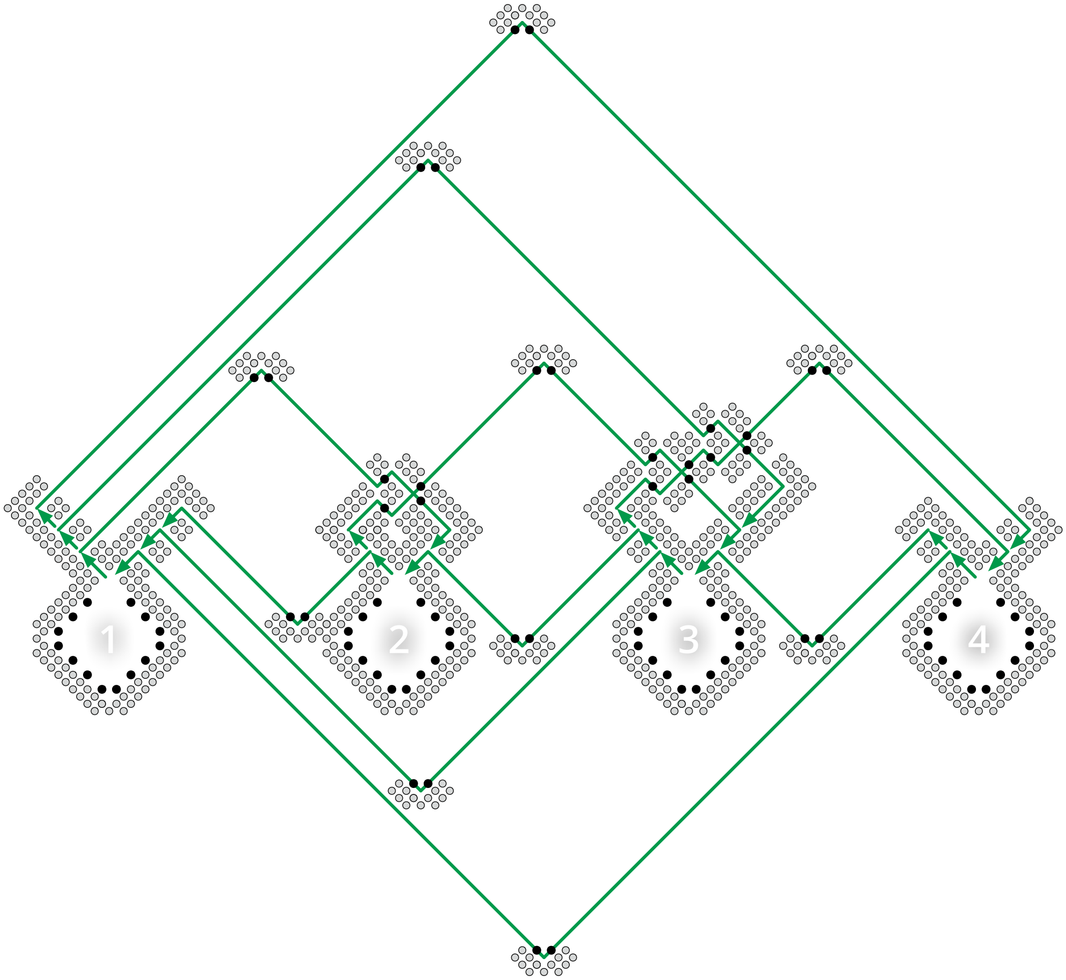


A corner gadget and a crossing gadget.

A single white king starts at the bottom corner of one of the vaults. In any legal move, this king must alternate between traversing entire arc paths and clearing vaults. The king can traverse the various gadgets backward, entering each vault through the exit and vice versa. But the reversal of a Hamiltonian cycle in G is another Hamiltonian cycle in G, so walking backward is fine.

If there is a Hamiltonian cycle in G, the white king can capture at least nN black pieces by visiting each of the other vaults and returning to the starting vault. On the other hand, if there is no Hamiltonian cycle in G, the white king can can capture at most half of the pieces in the starting vault, and thus can capture at most $(n-1/2)N + O(n^3)$ enemy pieces altogether. The $O(n^3)$ term accounts for the corner and crossing gadgets; each edge passes through one corner gadget and at most $n^2/2$ crossing gadgets.

To complete the reduction, we set $N = n^4$. Summing up, we obtain an $O(n^5) \times O(n^5)$ board configuration, with $O(n^5)$ black pieces and one white king. We can clearly construct this board configuration in polynomial time. A complete example is illustrated below.



The final draughts configuration for the example graph.

It is still open whether the following related question is NP-hard: Given an $n \times n$ board configuration for international draughts, can (and therefore *must*) White capture *all* the black pieces in a single turn?

29.14 Other Useful NP-hard Problems

Literally thousands of different problems have been proved to be NP-hard. I want to close this note by listing a few NP-hard problems that are useful in deriving reductions. I won't describe the NP-hardness proofs for these problems in detail, but you can find most of them in Garey and Johnson's classic *Scary Black Book of NP-Completeness*.[8]

- PLANARCIRCUITSAT: Given a boolean circuit that can be embedded in the plane so that no two wires cross, is there an input that makes the circuit output TRUE? This problem can be proved NP-hard by reduction from the general circuit satisfiability problem, by replacing each crossing with a small series of gates. (The details of the reduction are an easy[9] exercise.)
- NOTALLEQUAL3SAT: Given a 3CNF formula, is there an assignment of values to the variables so that every clause contains at least one TRUE literal *and* at least one FALSE literal? This problem can be proved NP-hard by reduction from the usual 3SAT.
- PLANAR3SAT: Given a 3CNF boolean formula, consider a bipartite graph whose vertices are the clauses and variables, where an edge indicates that a variable (or its negation) appears in a clause. If this graph is planar, the 3CNF formula is also called planar. The PLANAR3SAT problem asks, given a planar 3CNF formula, whether it has a satisfying assignment. This problem can be proved NP-hard by reduction from PLANARCIRCUITSAT.[10]
- EXACT3DIMENSIONALMATCHING or X3M: Given a set S and a collection of three-element subsets of S, called *triples*, is there a sub-collection of disjoint triples that exactly cover S? This problem can be proved NP-hard by a reduction from 3SAT.
- PARTITION: Given a set S of n integers, are there subsets A and B such that $A \cup B = S$, $A \cap B = \varnothing$, and
$$\sum_{a \in A} a = \sum_{b \in B} b?$$
This problem can be proved NP-hard by a simple reduction from SUBSETSUM. Like SUBSETSUM, the PARTITION problem is only weakly NP-hard.
- 3PARTITION: Given a set S of $3n$ integers, can it be partitioned into n disjoint three-element subsets, such that every subset has exactly the same sum? Despite the similar names, this problem is *very* different from PARTITION; sorry, I didn't make up the names. This problem can be proved NP-hard by reduction from X3M. Unlike PARTITION, the 3PARTITION problem is *strongly* NP-hard, that is, it remains NP-hard even if the input numbers are less than some polynomial in n.
- SETCOVER: Given a collection of sets $\mathscr{S} = \{S_1, S_2, \dots, S_m\}$, find the smallest sub-collection of S_i's that contains all the elements of $\bigcup_i S_i$. This problem is a generalization of both VERTEXCOVER and X3M.

[8] Michael Garey and David Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman and Co., 1979.

[9] or at least *nondeterministically* easy

[10] Surprisingly, PLANARNOTALLEQUAL3SAT is solvable in polynomial time!

- HittingSet: Given a collection of sets $\mathscr{S} = \{S_1, S_2, \ldots, S_m\}$, find the minimum number of elements of $\bigcup_i S_i$ that hit every set in $\mathscr{S}$. This problem is also a generalization of VertexCover.

- LongestPath: Given a non-negatively weighted graph G and two vertices u and v, what is the longest simple path from u to v in the graph? A path is *simple* if it visits each vertex at most once. This problem is a generalization of the HamiltonianPath problem. Of course, the corresponding *shortest* path problem is in P.

- SteinerTree: Given a weighted, undirected graph G with some of the vertices marked, what is the minimum-weight subtree of G that contains every marked vertex? If *every* vertex is marked, the minimum Steiner tree is just the minimum spanning tree; if exactly two vertices are marked, the minimum Steiner tree is just the shortest path between them. This problem can be proved NP-hard by reduction to HamiltonianPath.

In addition to these dry but useful problems, most interesting puzzles and solitaire games have been shown to be NP-hard, or to have NP-hard generalizations. (Arguably, if a game or puzzle isn't at least NP-hard, it isn't interesting!) Some familiar examples include Minesweeper (by reduction from CircuitSAT)[11], Tetris (by reduction from 3Partition)[12], and Sudoku (by reduction from 3SAT)[13].

⋆29.15 On Beyond Zebra

P and NP are only the first two steps in an enormous hierarchy of complexity classes. To close these notes, let me describe a few more classes of interest.

Polynomial Space. ***PSPACE*** is the set of decision problems that can be solved using polynomial *space*. Every problem in NP (and therefore in P) is also in PSPACE. It is generally believed that NP $\neq$ PSPACE, but nobody can even prove that P $\neq$ PSPACE. A problem Π is ***PSPACE-hard*** if, for any problem Π' that can be solved using polynomial *space*, there is a polynomial-*time* many-one reduction from Π' to Π. A problem is ***PSPACE-complete*** if it is both PSPACE-hard and in PSPACE. If any PSPACE-hard problem is in NP, then PSPACE=NP; similarly, if any PSPACE-hard problem is in P, then PSPACE=P.

The canonical PSPACE-complete problem is the *quantified boolean formula* problem, or ***QBF***: Given a boolean formula Φ that may include any number of universal or existential quantifiers, but no free variables, is Φ equivalent to True? For example, the following expression is a valid input to QBF:

$$\exists a\colon \forall b\colon \exists c\colon (\forall d\colon a \vee b \vee c \vee \bar{d}) \Longleftrightarrow ((b \wedge \bar{c}) \vee (\exists e\colon \overline{(\bar{a} \Rightarrow e)} \vee (c \neq a \wedge e))).$$

SAT is provably equivalent the special case of QBF where the input formula contains only existential quantifiers. QBF remains PSPACE-hard even when the input formula must have all its quantifiers at the beginning, the quantifiers strictly alternate between $\exists$ and $\forall$, and the quantified proposition is in conjunctive normal form, with exactly three literals in each clause, for example:

$$\exists a\colon \forall b\colon \exists c\colon \forall d\colon \big((a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})\big)$$

[11] Richard Kaye. Minesweeper is NP-complete. *Mathematical Intelligencer* 22(2):9–15, 2000. http://www.mat.bham.ac.uk/R.W.Kaye/minesw/minesw.pdf

[12] Ron Breukelaar*, Erik D. Demaine, Susan Hohenberger*, Hendrik J. Hoogeboom, Walter A. Kosters, and David Liben-Nowell*. Tetris is hard, even to approximate. *International Journal of Computational Geometry and Applications* 14:41–68, 2004. The reduction was *considerably* simplified between its discovery in 2002 and its publication in 2004.

[13] Takayuki Yato and Takahiro Seta. Complexity and completeness of finding another solution and its application to puzzles. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences* E86-A(5):1052–1060, 2003. http://www-imai.is.s.u-tokyo.ac.jp/~yato/data2/MasterThesis.pdf.

This restricted version of QBF can also be phrased as a two-player strategy question. Suppose two players, Alice and Bob, are given a 3CNF predicate with free variables $x_1, x_2, \ldots, x_n$. The players alternately assign values to the variables in order by index—Alice assigns a value to x_1, Bob assigns a value to x_2, and so on. Alice eventually assigns values to every variable with an odd index, and Bob eventually assigns values to every variable with an even index. Alice wants to make the expression TRUE, and Bob wants to make it FALSE. Assuming Alice and Bob play perfectly, who wins this game? Not surprisingly, most two-player games[14] like tic-tac-toe, reversi, checkers, go, chess, and mancala—or more accurately, appropriate generalizations of these constant-size games to arbitrary board sizes—are PSPACE-hard.

Another canonical PSPACE-hard problem is *NFA totality*: Given a non-deterministic finite-state automaton M over some alphabet Σ, does M accept every string in Σ^*? The closely related problems *NFA equivalence* (Do two given NFAs accept the same language?) and *NFA minimization* (Find the smallest NFA that accepts the same language as a given NFA) are also PSPACE-hard, as are the corresponding questions about regular expressions. (The corresponding questions about *deterministic* finite-state automata are all solvable in polynomial time.)

Exponential time. The next significantly larger complexity class, ***EXP*** (also called EXPTIME), is the set of decision problems that can be solved in exponential time, that is, using at most 2^{n^c} steps for some constant $c > 0$. Every problem in PSPACE (and therefore in NP (and therefore in P)) is also in EXP. It is generally believed that PSPACE $\subsetneq$ EXP, but nobody can even prove that NP $\neq$ EXP. A problem Π is ***EXP-hard*** if, for any problem Π' that can be solved in *exponential* time, there is a *polynomial*-time many-one reduction from Π' to Π. A problem is ***EXP-complete*** if it is both EXP-hard and in EXP. If any EXP-hard problem is in PSPACE, then EXP=PSPACE; similarly, if any EXP-hard probelm is in NP, then EXP=NP. We *do* know that P $\neq$ EXP; in particular, no EXP-hard problem is in P.

Natural generalizations of many interesting 2-player games—like checkers, chess, mancala, and go—are actually EXP-hard. The boundary between PSPACE-complete games and EXP-hard games is rather subtle. For example, there are three ways to draw in chess (the standard 8×8 game): stalemate (the player to move is not in check but has no legal moves), repeating the same board position three times, or moving fifty times without capturing a piece. The $n \times n$ generalization of chess is either in PSPACE or EXP-hard depending on how we generalize these rules. If we declare a draw after (say) n^3 capture-free moves, then every game must end after a polynomial number of moves, so we can simulate all possible games from any given position using only polynomial space. On the other hand, if we ignore the capture-free move rule entirely, the resulting game can last an exponential number of moves, so there no obvious way to detect a repeating position using only polynomial space; indeed, this version of $n \times n$ chess is EXP-hard.

Excelsior! Naturally, even exponential time is not the end of the story. ***NEXP*** is the class of decision problems that can be solve in *nondeterministic* exponential time; equivalently, a decision problem is in NEXP if and only if, for every YES instance, there is a *proof* of this fact that can be checked in exponential time. ***EXPSPACE*** is the set of decision problems that can be solved using exponential *space*. Even these larger complexity classes have hard and complete problems; for example, if we add the intersection operator $\cap$ to the syntax of regular expressions, deciding whether two such expressions describe the same language is EXPSPACE-hard. Beyond EXPSPACE are complexity classes with *doubly*-exponential

[14] For a good (but now slightly dated) overview of known results on the computational complexity of games and puzzles, see Erik D. Demaine and Robert Hearn's survey "Playing Games with Algorithms: Algorithmic Combinatorial Game Theory" at http://arxiv.org/abs/cs.CC/0106019.

resource bounds (EEXP, NEEXP, and EEXPSPACE), then *triply* exponential resource bounds (EEEXP, NEEEXP, and EEEXPSPACE), and so on ad infinitum.

All these complexity classes can be ordered by inclusion as follows:

$$\text{P} \subseteq \text{NP} \subseteq \text{PSPACE} \subseteq \text{EXP} \subseteq \text{NEXP} \subseteq \text{EXPSPACE} \subseteq \text{EEXP} \subseteq \text{NEEXP} \subseteq \text{EEXPSPACE} \subseteq \text{EEEXP} \subseteq \cdots,$$

Most complexity theorists strongly believe that every inclusion in this sequence is strict; that is, no two of these complexity classes are equal. However, the strongest result that has been proved is that every class in this sequence is strictly contained in the class *three* steps later in the sequence. For example, we have proofs that P≠EXP and PSPACE≠EXPSPACE, but not whether P≠PSPACE or NP≠EXP.

The limit of this series of increasingly exponential complexity classes is the class ***ELEMENTARY*** of decision problems that can be solved using time or space bounded by any so-called elementary function of n. A function of n is *elementary* if it is equal to $2 \uparrow^k p(n)$ for some integer k and some polynomial $p(n)$, where

$$2 \uparrow^k x := \begin{cases} x & \text{if } k = 0, \\ 2^{2 \uparrow^{k-1} x} & \text{otherwise.} \end{cases}$$

You might be tempted to conjecture that every natural decidable problem can be solved in elementary time, but then you would be wrong. Consider the *extended regular expressions* defined by recursively combining (possibly empty) strings over some finite alphabet by concatenation, union (+), Kleene closure (∗), and negation. For example, the extended regular expression $\overline{(0+1)^*00(0+1)^*}$ represents the set of strings in $\{0,1\}^*$ that do *not* contain two 0s in a row. It is possible to determine algorithmically whether two extended regular expressions describe identical languages, by recursively converting each expression into an equivalent NFA, converting each NFA into a DFA, and then minimizing the DFA. Unfortunately, however, this equivalence problem cannot be decided in only elementary time!!

Exercises

1. Consider the following problem, called BoxDepth: Given a set of n axis-aligned rectangles in the plane, how big is the largest subset of these rectangles that contain a common point?

   (a) Describe a polynomial-time reduction from BoxDepth to MaxClique.

   (b) Describe and analyze a polynomial-time algorithm for BoxDepth. *[Hint: $O(n^3)$ time should be easy, but $O(n \log n)$ time is possible.]*

   (c) Why don't these two results imply that P=NP?

2. (a) Describe a polynomial-time reduction from Partition to SubsetSum.

   (b) Describe a polynomial-time reduction from SubsetSum to Partition.

3. (a) Describe and analyze and algorithm to solve Partition in time $O(nM)$, where n is the size of the input set and M is the sum of the absolute values of its elements.

   (b) Why doesn't this algorithm imply that P=NP?

4. A boolean formula is in *disjunctive normal form* (or *DNF*) if it consists of a *disjunction* (Or) or several *terms*, each of which is the conjunction (And) of one or more literals. For example, the formula

$$(\overline{x} \wedge y \wedge \overline{z}) \vee (y \wedge z) \vee (x \wedge \overline{y} \wedge \overline{z})$$

is in disjunctive normal form. DNF-SAT asks, given a boolean formula in disjunctive normal form, whether that formula is satisfiable.

(a) Describe a polynomial-time algorithm to solve DNF-SAT.

(b) What is the error in the following argument that P=NP?

Suppose we are given a boolean formula in conjunctive normal form with at most three literals per clause, and we want to know if it is satisfiable. We can use the distributive law to construct an equivalent formula in disjunctive normal form. For example,

$$(x \vee y \vee \overline{z}) \wedge (\overline{x} \vee \overline{y}) \iff (x \wedge \overline{y}) \vee (y \wedge \overline{x}) \vee (\overline{z} \wedge \overline{x}) \vee (\overline{z} \wedge \overline{y})$$

Now we can use the algorithm from part (a) to determine, in polynomial time, whether the resulting DNF formula is satisfiable. We have just solved 3SAT in polynomial time. Since 3SAT is NP-hard, we must conclude that P=NP!

5. (a) Prove that PlanarCircuitSat is NP-complete.

(b) Prove that NotAllEqual3SAT is NP-complete.

(c) Prove that the following variant of 3SAT is NP-complete: Given a boolean formula Φ in conjunctive normal form where each clause contains at most 3 literals and each variable appears in at most 3 clauses, does Φ have a satisfying assignment?

6. There's something special about the number 3.

(a) Describe and analyze a polynomial-time algorithm for 2Partition. Given a set S of $2n$ positive integers, your algorithm will determine in polynomial time whether the elements of S can be split into n disjoint pairs whose sums are all equal.

(b) Describe and analyze a polynomial-time algorithm for 2Color. Given an undirected graph G, your algorithm will determine in polynomial time whether G has a proper coloring that uses only two colors.

(c) Describe and analyze a polynomial-time algorithm for 2SAT. Given a boolean formula Φ in conjunctive normal form, with exactly *two* literals per clause, your algorithm will determine in polynomial time whether Φ has a satisfying assignment.

7. There's nothing special about the number 3.

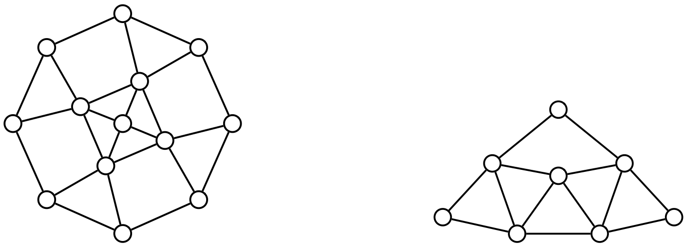
(a) The problem 12Partition is defined as follows: Given a set S of $12n$ positive integers, determine whether the elements of S can be split into n subsets of 12 elements each whose sums are all equal. Prove that 12Partition is NP-hard. *[Hint: Reduce from* 3Partition. *It may be easier to consider multisets first.]*

(b) The problem 12Color is defined as follows: Given an undirected graph G, determine whether we can color each vertex with one of twelve colors, so that every edge touches two different colors. Prove that 12Color is NP-hard. *[Hint: Reduce from* 3Color.*]*

(c) The problem 12SAT is defined as follows: Given a boolean formula Φ in conjunctive normal form, with exactly twelve literals per clause, determine whether Φ has a satisfying assignment. Prove that 12Sat is NP-hard. *[Hint: Reduce from 3SAT.]*

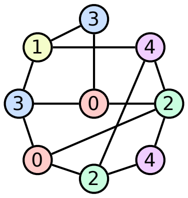
8. (a) Using the gadget on the right below, prove that deciding whether a given *planar* graph is 3-colorable is NP-complete. *[Hint: Show that the gadget can be 3-colored, and then replace any crossings in a planar embedding with the gadget appropriately.]*

   (b) Using part (a) and the middle gadget below, prove that deciding whether a planar graph *with maximum degree 4* is 3-colorable is NP-complete. *[Hint: Replace any vertex with degree greater than 4 with a collection of gadgets connected so that no degree is greater than four.]*



(a) Gadget for planar 3-colorability. (b) Gadget for degree-4 planar 3-colorability.

9. Recall that a 5-coloring of a graph G is a function that assigns each vertex of G an 'color' from the set $\{0, 1, 2, 3, 4\}$, such that for any edge uv, vertices u and v are assigned different 'colors'. A 5-coloring is *careful* if the colors assigned to adjacent vertices are not only distinct, but differ by more than 1 (mod 5). Prove that deciding whether a given graph has a careful 5-coloring is NP-complete. *[Hint: Reduce from the standard* 5Color *problem.]*



A careful 5-coloring.

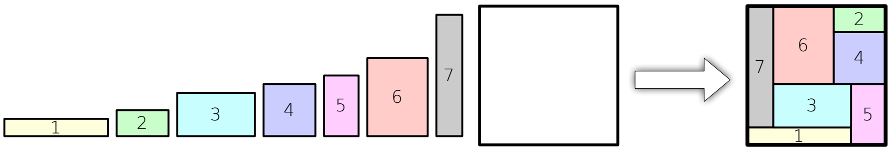
10. Prove that the following problems are NP-complete.

    (a) Given two undirected graphs G and H, is G isomorphic to a subgraph of H?

    (b) Given an undirected graph G, does G have a spanning tree in which every node has degree at most 17?

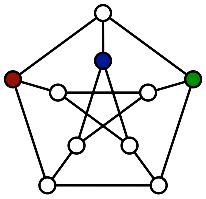
    (c) Given an undirected graph G, does G have a spanning tree with at most 42 leaves?

11. The RectangleTiling problem is defined as follows: Given one large rectangle and several smaller rectangles, determine whether the smaller rectangles can be placed inside the large rectangle with no gaps or overlaps. Prove that RectangleTiling is NP-complete.



A positive instance of the RectangleTiling problem.

12. Let $G = (V, E)$ be a graph. A *dominating set* in G is a subset S of the vertices such that every vertex in G is either in S or adjacent to a vertex in S. The DOMINATINGSET problem asks, given a graph G and an integer k as input, whether G contains a dominating set of size k. Either prove that this problem is NP-hard or describe a polynomial-time algorithm to solve it.



A dominating set of size 3 in the Peterson graph.

13. *Pebbling* is a solitaire game played on an undirected graph G, where each vertex has zero or more *pebbles*. A single *pebbling move* consists of removing two pebbles from a vertex v and adding one pebble to an arbitrary neighbor of v. (Obviously, the vertex v must have at least two pebbles before the move.) The PEBBLEDESTRUCTION problem asks, given a graph $G = (V, E)$ and a pebble count $p(v)$ for each vertex v, whether is there a sequence of pebbling moves that removes all but one pebble. Prove that PEBBLEDESTRUCTION is NP-complete.

14. (a) A *tonian path* in a graph G is a path that goes through at least half of the vertices of G. Show that determining whether a graph has a tonian path is NP-complete.

    (b) A *tonian cycle* in a graph G is a cycle that goes through at least half of the vertices of G. Show that determining whether a graph has a tonian cycle is NP-complete. *[Hint: Use part (a).]*

15. For each problem below, either describe a polynomial-time algorithm or prove that the problem is NP-complete.

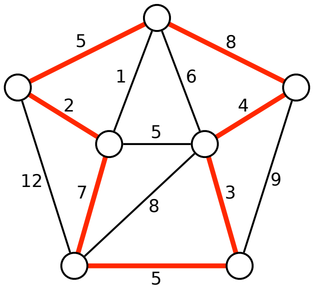
    (a) A *double-Eulerian circuit* in an undirected graph G is a closed walk that traverses every edge in G exactly twice. Given a graph G, does G have a double-Eulerian circuit?

    (b) A *double-Hamiltonian circuit* in an undirected graph G is a closed walk that visits every vertex in G exactly twice. Given a graph G, does G have a double-Hamiltonian circuit?

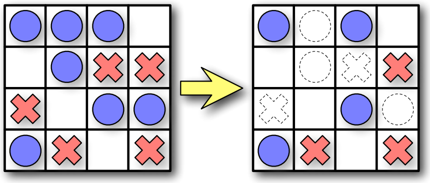
16. Let G be an undirected graph with weighted edges. A *heavy Hamiltonian cycle* is a cycle C that passes through each vertex of G exactly once, such that the total weight of the edges in C is at least half of the total weight of all edges in G. Prove that deciding whether a graph has a heavy Hamiltonian cycle is NP-complete.

17. Consider the following solitaire game. The puzzle consists of an $n \times m$ grid of squares, where each square may be empty, occupied by a red stone, or occupied by a blue stone. The goal of the puzzle is to remove some of the given stones so that the remaining stones satisfy two conditions: (1) every row contains at least one stone, and (2) no column contains stones of both colors. For some initial configurations of stones, reaching this goal is impossible.

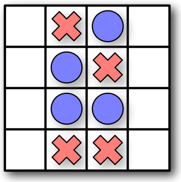
    Prove that it is NP-hard to determine, given an initial configuration of red and blue stones, whether the puzzle can be solved.



A heavy Hamiltonian cycle. The cycle has total weight 34; the graph has total weight 67.



A solvable puzzle and one of its many solutions.



An unsolvable puzzle.

18. A boolean formula in *exclusive-or conjunctive normal form* (XCNF) is a conjunction (AND) of several *clauses*, each of which is the *exclusive*-or of several literals; that is, a clause is true if and only if it contains an odd number of true literals. The XCNF-SAT problem asks whether a given XCNF formula is satisfiable. Either describe a polynomial-time algorithm for XCNF-SAT or prove that it is NP-hard.

19. Jeff tries to make his students happy. At the beginning of class, he passes out a questionnaire that lists a number of possible course policies in areas where he is flexible. Every student is asked to respond to each possible course policy with one of "strongly favor", "mostly neutral", or "strongly oppose". Each student may respond with "strongly favor" or "strongly oppose" to at most five questions. Because Jeff's students are very understanding, each student is happy if (but only if) he or she prevails in just one of his or her strong policy preferences. Either describe a polynomial-time algorithm for setting course policy to maximize the number of happy students, or show that the problem is NP-hard.

20. You're in charge of choreographing a musical for your local community theater, and it's time to figure out the final pose of the big show-stopping number at the end. ("Streetcar!") You've decided that each of the n cast members in the show will be positioned in a big line when the song finishes, all with their arms extended and showing off their best spirit fingers.

    The director has declared that during the final flourish, each cast member must either point both their arms up or point both their arms down; it's your job to figure out who points up and who points down. Moreover, in a fit of unchecked power, the director has also given you a list of arrangements that will upset his delicate artistic temperament. Each forbidden arrangement is a subset of the cast members paired with arm positions; for example: "Marge may not point her arms up while Ned, Apu, and Smithers point their arms down."

    Prove that finding an acceptable arrangement of arm positions is NP-hard.

21. The next time you are at a party, one of the guests will suggest everyone play a round of Three-Way Mumbledypeg, a game of skill and dexterity that requires three teams and a knife. The official

Rules of Three-Way Mumbledypeg (fixed during the Holy Roman Three-Way Mumbledypeg Council in 1625) require that (1) each team *must* have at least one person, (2) any two people on the same team *must* know each other, and (3) everyone watching the game *must* be on one of the three teams. Of course, it will be a really *fun* party; nobody will want to leave. There will be several pairs of people at the party who don't know each other. The host of the party, having heard thrilling tales of your prowess in all things algorithmic, will hand you a list of which pairs of party-goers know each other and ask you to choose the teams, while he sharpens the knife.

Either describe and analyze a polynomial time algorithm to determine whether the party-goers can be split into three legal Three-Way Mumbledypeg teams, or prove that the problem is NP-hard.

22. The party you are attending is going great, but now it's time to line up for ***The Algorithm March (アルゴリズムこうしん)***! This dance was originally developed by the Japanese comedy duo Itsumo Kokokara (いつもここから) for the children's television show PythagoraSwitch (ピタゴラスイッチ). The Algorithm March is performed by a line of people; each person in line starts a specific sequence of movements one measure later than the person directly in front of them. Thus, the march is the dance equivalent of a musical round or canon, like "Row Row Row Your Boat".

    Proper etiquette dictates that each marcher must know the person directly in front of them in line, lest a minor mistake during lead to horrible embarrassment between strangers. Suppose you are given a complete list of which people at your party know each other. ***Prove*** that it is NP-hard to determine the largest number of party-goers that can participate in the Algorithm March. You may assume there are no ninjas at your party.

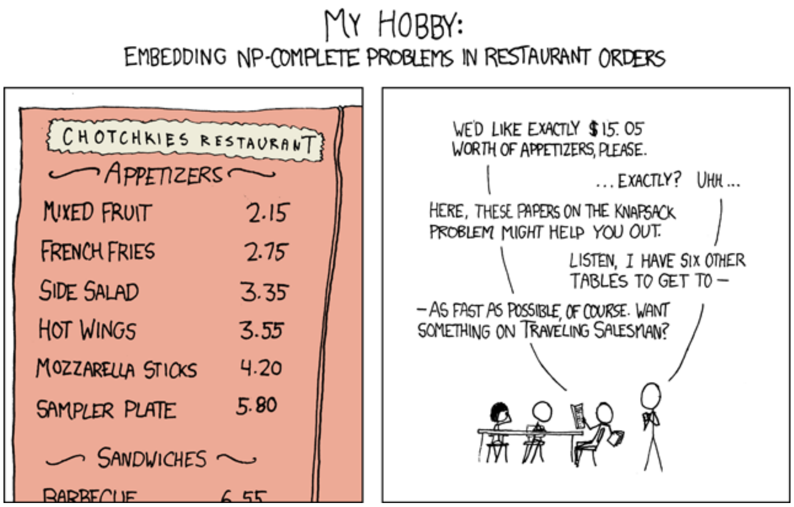
23. (a) Suppose you are given a magic black box that can determine **in polynomial time**, given an arbitrary weighted graph G, the length of the shortest Hamiltonian cycle in G. Describe and analyze a **polynomial-time** algorithm that computes, given an arbitrary weighted graph G, the shortest Hamiltonian cycle in G, using this magic black box as a subroutine.

    (b) Suppose you are given a magic black box that can determine **in polynomial time**, given an arbitrary graph G, the number of vertices in the largest complete subgraph of G. Describe and analyze a **polynomial-time** algorithm that computes, given an arbitrary graph G, a complete subgraph of G of maximum size, using this magic black box as a subroutine.

    (c) Suppose you are given a magic black box that can determine **in polynomial time**, given an arbitrary weighted graph G, whether G is 3-colorable. Describe and analyze a **polynomial-time** algorithm that either computes a proper 3-coloring of a given graph or correctly reports that no such coloring exists, using the magic black box as a subroutine. *[Hint: The input to the magic black box is a graph. Just a graph. Vertices and edges. Nothing else.]*

    (d) Suppose you are given a magic black box that can determine **in polynomial time**, given an arbitrary boolean formula Φ, whether Φ is satisfiable. Describe and analyze a **polynomial-time** algorithm that either computes a satisfying assignment for a given boolean formula or correctly reports that no such assignment exists, using the magic black box as a subroutine.

    *(e) Suppose you are given a magic black box that can determine **in polynomial time**, given an initial Tetris configuration and a finite sequence of Tetris pieces, whether a perfect player can play every piece. (This problem is NP-hard.) Describe and analyze a **polynomial-time** algorithm that computes the shortest Hamiltonian cycle in a given weighted graph in polynomial time, using this magic black box as a subroutine. *[Hint: Use part (a). You do not need to know anything about Tetris to solve this problem.]*


MY HOBBY:
EMBEDDING NP-COMPLETE PROBLEMS IN RESTAURANT ORDERS
CHOTCHKIES RESTAURANT
APPETIZERS
MIXED FRUIT 2.15
FRENCH FRIES 2.75
SIDE SALAD 3.35
HOT WINGS 3.55
MOZZARELLA STICKS 4.20
SAMPLER PLATE 5.80
SANDWICHES
BARBECUE 6.55
WE'D LIKE EXACTLY $15.05 WORTH OF APPETIZERS, PLEASE.
...EXACTLY? UHH...
HERE, THESE PAPERS ON THE KNAPSACK PROBLEM MIGHT HELP YOU OUT.
LISTEN, I HAVE SIX OTHER TABLES TO GET TO –
– AS FAST AS POSSIBLE, OF COURSE. WANT SOMETHING ON TRAVELING SALESMAN?

[General solutions give you a 50% tip.]

— Randall Munroe, *xkcd* (http://xkcd.com/287/)
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