STAT 410

Summer 2014
A. Stepanov

# Practice Problems 2

**1 – 4.** Let X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{x}, \qquad x > 1, \quad 0 < y < \frac{1}{x}, \qquad \text{zero elsewhere.}$$

**1.**

a) Find $f_X(x)$. *Be sure to include its support.*

b) Find $E(X)$.

c) Find $f_Y(y)$. *Be sure to include its support.*

d) Find $E(Y)$.

**2.**

a) Find $f_{X|Y}(x|y)$. *Be sure to include its support.*

b) Find $f_{Y|X}(y|x)$. *Be sure to include its support.*

c) Find $E(X \mid Y = y)$.

d) Find $E(Y \mid X = x)$.

**3.**

a) Let $U = XY$. Find the p.d.f. of U, $f_U(u)$.

b) Let $V = Y/X$. Find the p.d.f. of V, $f_V(v)$.

c)* Let $W = X + Y$. Find the p.d.f. of W, $f_W(w)$.

**4.** Let $U = Y$ and $V = Y/X$.

Find the joint probability density function of $(U, V)$, $f_{U,V}(u, v)$.

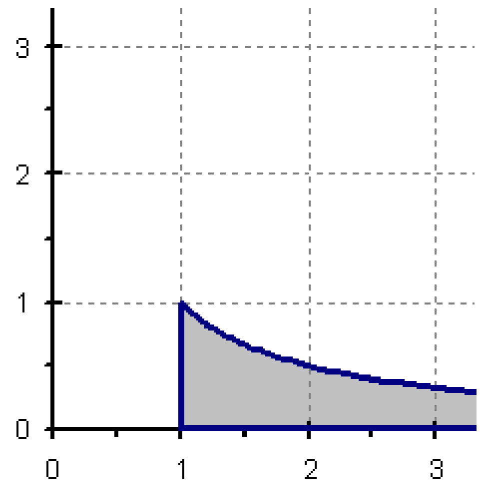
Sketch the support of $(U, V)$.

**1.** Let X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{x},$$

$$x > 1, \quad 0 < y < \frac{1}{x},$$

zero elsewhere.




a) Find $f_X(x)$.
*Be sure to include its support.*

$$f_X(x) = \int_0^{1/x} \frac{1}{x}\,dy = \frac{1}{x^2}, \qquad x > 1.$$

b) Find $E(X)$.

Since $\displaystyle\int_1^{\infty} x \cdot \frac{1}{x^2}\,dx = \int_1^{\infty} \frac{1}{x}\,dx = (\ln x)\Big|_1^{\infty}$ is not finite,

$E(X)$ **is not finite**.

c) Find $f_Y(y)$. *Be sure to include its support.*

$$f_Y(y) = \int_1^{1/y} \frac{1}{x}\,dx = (\ln x)\Big|_1^{1/y} = \ln\frac{1}{y} - \ln 1 = -\ln y, \qquad 0 < y < 1.$$

d) Find $E(Y)$.

$$E(Y) = \int_0^1 y(-\ln y)\,dy = \left(-\frac{y^2}{2}\ln y + \frac{y^2}{4}\right)\Bigg|_0^1 = \mathbf{\frac{1}{4}}.$$

**2.** Let X and Y have the joint probability density function

$$f_{X,Y}(x, y) = \frac{1}{x}, \qquad x > 1, \quad 0 < y < \frac{1}{x}, \qquad \text{zero elsewhere.}$$

a) Find $f_{X|Y}(x|y)$. *Be sure to include its support.*

$$f_Y(y) = \int_1^{1/y} \frac{1}{x}\,dx = \left(\ln x\right)\Big|_1^{1/y} = \ln\frac{1}{y} - \ln 1 = -\ln y, \qquad 0 < y < 1.$$

$$f_{X|Y}(x|y) = \frac{1/x}{-\ln y}, \qquad 1 < x < \frac{1}{y}, \qquad 0 < y < 1.$$

b) Find $f_{Y|X}(y|x)$. *Be sure to include its support.*

$$f_X(x) = \int_0^{1/x} \frac{1}{x}\,dy = \frac{1}{x^2}, \qquad x > 1.$$

$$f_{Y|X}(y|x) = \frac{1/x}{1/x^2} = x, \qquad 0 < y < \frac{1}{x}, \qquad x > 1.$$

c) Find $E(X \mid Y = y)$.

$$E(X \mid Y = y) = \int_1^{1/y} x \cdot \frac{1/x}{-\ln y}\,dx = \int_1^{1/y} \frac{1}{-\ln y}\,dx = \frac{\frac{1}{y} - 1}{-\ln y}, \qquad 0 < y < 1.$$

d) Find $E(Y \mid X = x)$.

$$E(Y \mid X = x) = \int_0^{1/x} y \cdot x\,dy = \frac{1}{2x}, \qquad x > 1.$$

OR

$Y \mid X = x$ has Uniform distribution on $\left(0, \frac{1}{x}\right)$. $\Rightarrow$ $E(Y \mid X = x) = \dfrac{1}{2x}$.

**3.** Let X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{x}, \qquad x > 1, \quad 0 < y < \frac{1}{x}, \qquad \text{zero elsewhere.}$$

a) Let $U = XY$. Find the p.d.f. of U, $f_U(u)$.

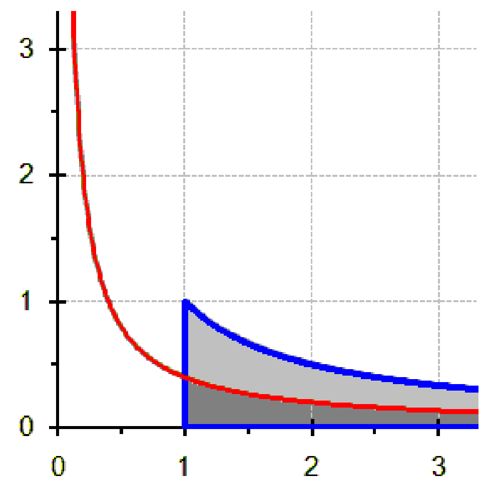
$$F_U(u) = P(U \le u)$$

$$= P(XY \le u)$$

$$= \int_1^{\infty} \left( \int_0^{u/x} \frac{1}{x}\, dy \right) dx$$

$$= \int_1^{\infty} \frac{u}{x^2}\, dx$$

$$= u, \qquad 0 < u < 1.$$




$$f_U(u) = F_U'(u) = 1, \qquad 0 < u < 1.$$

U has a Uniform distribution on interval $(0, 1)$.

b) Let $V = Y \big/ X$. Find the p.d.f. of V, $f_V(v)$.

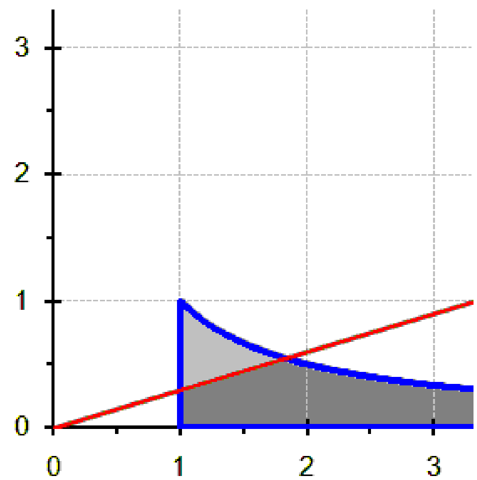
$$F_V(v) = P(V \le v)$$

$$= P(Y \le v\,X)$$

$$= 1 - \int_1^{1/\sqrt{v}} \left( \int_{v\,x}^{1/x} \frac{1}{x}\,dy \right) dx$$

$$= 1 - \int_1^{1/\sqrt{v}} \left( \frac{1}{x^2} - v \right) dx$$

$$= 2\sqrt{v} - v, \qquad 0 < v < 1.$$




$$f_V(v) = F_V'(v) = \frac{1}{\sqrt{v}} - 1, \qquad\qquad 0 < v < 1.$$

c)* Let W = X + Y. Find the p.d.f. of W, $f_{W}(w)$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\,dx.$$

$x > 1$

$0 < w - x \quad \Rightarrow \quad x < w$

$w - x < \dfrac{1}{x}$ :

$1 < w < 2 \quad \Rightarrow \quad w - x < \dfrac{1}{x}$ for all $x > 1$;

$w > 2 \quad \Rightarrow \quad x^2 - wx + 1 > 0 \quad \Rightarrow \quad x > \dfrac{w}{2} + \sqrt{\dfrac{w^2}{4} - 1} = w^* > 1.$

Case 1. $1 < w < 2$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\,dx = \int_{1}^{w} \frac{1}{x}\,dx = \ln w.$$

Case 2. $w > 2$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\,dx = \int_{w^*}^{w} \frac{1}{x}\,dx = \ln w - \ln w^*$$

$$= \ln w - \ln\left(\frac{w}{2} + \sqrt{\frac{w^2}{4} - 1}\right).$$

**4.** Let X and Y have the joint probability density function

$$f_{X,Y}(x, y) = \frac{1}{x}, \qquad x > 1, \quad 0 < y < \frac{1}{x}, \qquad \text{zero elsewhere.}$$

Let U = Y and V = Y / X.

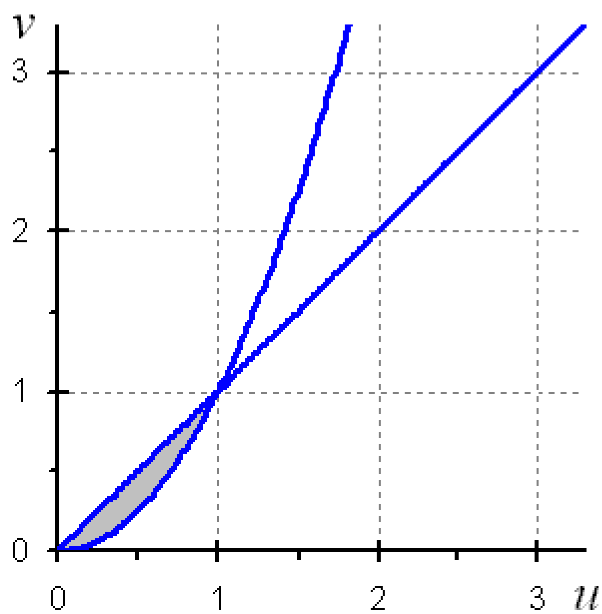
Find the joint probability density function of ( U, V ), $f_{U,V}(u, v)$.

Sketch the support of ( U, V ).

Y = U

$\Rightarrow \quad V = U / X$

$\Rightarrow \quad X = U / V$




$$J = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 1 & 0 \end{vmatrix} = \frac{u}{v^2}.$$

$x > 1 \quad \Rightarrow \quad v < u$

$y > 0 \quad \Rightarrow \quad u > 0$

$y < \frac{1}{x} \quad \Rightarrow \quad u < \frac{v}{u} \quad \Rightarrow \quad v > u^2$

$$f_{U,V}(u, v) = f_{X,Y}\left(\frac{u}{v}, u\right) \times |J| = \frac{v}{u} \times \frac{u}{v^2} = \frac{1}{v}, \qquad 0 < u < 1, \quad u^2 < v < u.$$