Midterm 1
CS 373: Theory of Computation

Date: Thursday, October 4, 2012.

Instructions:

- This is a closed book exam. No notes, cheat sheets, textbook, or printed material allowed.
- You have 90 minutes to solve this exam.
- This exam has 5 problems each worth 10 points. However, not all problems are of equal difficulty.
- Please write your name on the top of *every* page in the space provided.
- If your solution does not fit in the space provided, and continues onto one of the back sheets, please indicate clearly where we should look for the solution.
- Unless otherwise stated, "prove that", "show that" for a problem means you need to formally prove what you are claiming.
- You may use, without proof, any result that you were asked to prove in the homework or was proved in the lecture. If you use such a result, please explicitly state the result (like " 'Perfect shuffle of regular languages is regular' was proved in a homework", instead of "this was shown in a homework").

Name	
Netid	

Discussion: T 2:00–2:50 T 3:00–3:50 W 1:00–1:50 W 4:00–4:50 W 5:00–5:50

Problem	Maximum Points	Points Earned	Grader
1	10		
2	10		
3	10		
4	10		
5	10		
Total	50		

Problem 1. [Category: Comprehension] **True/False.** Decide for each statement whether it is true or false. Circle **T** if the statement is *necessarily true*; circle **F** if it it is not necessarily true. Each correct answer is worth **1 point**.

(a) Let Σ and Δ be two alphabets. For a set A, let $|A|$ denote the number of elements in A. Then for all n, $|\Sigma^n| = |\Delta^n|$.
T **F**

(b) Suppose M is a DFA such that $\epsilon \in \mathbf{L}(M)$. Then the initial state of M must be a final state.
T **F**

(c) For language L_1 and L_2 over the alphabet Σ, $L_1 \setminus L_2$ denotes the difference between the two sets, i.e., it is the set of all strings that belong to L_1 but not L_2. If L_1 and L_2 are regular then $L_1 \setminus L_2$ is regular.
T **F**

(d) There is an NFA N with n states, such that any DFA recognizing $\mathbf{L}(N)$ has at least 2^n states.
T **F**

(e) If $L \subseteq \{0\}^*$ then L is regular.
T **F**

(f) Let $L_1 \subseteq \Sigma^*$ and $L_2 \subseteq \Sigma^*$ be languages. Then $L_1^* \cap \Sigma^0 = L_2^* \cap \Sigma^0$.
T **F**

(g) Suppose R_1, R_2 are regular expressions such that $\mathbf{L}(R_1) = \mathbf{L}(R_2)$. Then R_1 and R_2 have the same number of operators.
T **F**

(h) Since regular languages are closed under homomorphism, *non-regular* languages are also closed under homomorphisms. That is, if L is *not* regular and h is a homomorphism then $h(L)$ is not regular.
T **F**

(i) The following is correct proof showing that the language $A = \{a^n b^n \mid n \geq 0\}$ is not regular: Let $h : \{a, b\}^* \rightarrow \{0, 1\}^*$ be a homomprphism given by $h(a) = 0$ and $h(b) = 1$. Then since $A = h^{-1}(L_{0n1n})$, A is not regular. (Recall that $L_{0n1n} = \{0^n 1^n \mid n \geq 0\}$.)
T **F**

(j) There is a non-regular language L that satisfies the pumping lemma.
T **F**

Problem 2. [Category: Comprehension+Proof] For a binary string $w \in \{0,1\}^*$, let $[\![w]\!]$ denote the number

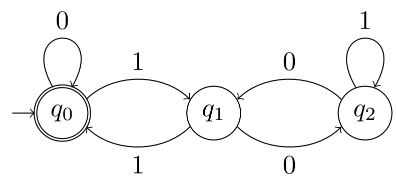

Figure 1: DFA A recognizing L_3

whose binary representation is given by w; here we will assume that the rightmost symbol is the least significant bit. We could define this inductively as

$$[\![\epsilon]\!] = 0 \quad [\![w0]\!] = 2 \times [\![w]\!] \quad [\![w1]\!] = 2 \times [\![w]\!] + 1$$

Thus, for example, $[\![10]\!] = (2^1 \times 1) + (2^0 \times 0) = 2$ and $[\![101]\!] = (2^2 \times 1) + (2^1 \times 0) + (2^0 \times 1) = 5$. Let $L_3 = \{w \in \{0,1\}^* \mid [\![w]\!] \bmod 3 = 0\}$ is the collection of all binary strings w that are multiples of 3. (Recall $a \bmod b = c$ means that c is the remainder when a is divided by b.)

The DFA A (shown in Figure 1) recognizes the language L_3. The states of A keep track of the remainder when the input string is divided by 3; thus, reaching state q_i means that the remainder is i. The transitions of A are defined based on the observation that

$$[\![wa]\!] \bmod 3 = (2([\![w]\!] \bmod 3) + a) \bmod 3$$

(a) Answer the following:

$\hat{\delta}_A(q_0, 111) =$ ______________________________ **[1 point]**

$\hat{\delta}_A(q_2, 101) =$ ______________________________ **[1 point]**

(b) Let us define

$$L_A(q_0, q_1) = \{w \in \{0,1\}^* \mid \hat{\delta}_A(q_0, w) = \{q_1\}\}$$
$$L_A(q_1, q_0) = \{w \in \{0,1\}^* \mid \hat{\delta}_A(q_1, w) = \{q_0\}\}$$

Answer the following questions:

(i) Is $100 \in L_A(q_0, q_1)$? ______________________________ **[1 point]**

(ii) Is $100 \in L_A(q_1, q_0)$? ______________________________ **[1 point]**

(iii) Is $1000 \in L_A(q_0, q_1)$? ______________________________ **[1 point]**

(iv) Is $1000 \in L_A(q_1, q_0)$? ______________________________ **[1 point]**

(c) Describe formally the strings that belong to $L_A(q_0, q_1)$ and $L_A(q_1, q_0)$. (Don't repeat the definitions in part(b) but rather come up with a description based on how the automaton A works.) **[2 points]**

(d) Let M with initial state q_0 be *any* DFA that recognizes L_3. Prove that $\hat{\delta}_M(q_0, \epsilon) \neq \hat{\delta}_M(q_0, 1)$. **[2 points]**

Problem 3. [Category: Comprehension+Design] For a string $w = a_1a_2 \cdots a_n \in \Sigma^*$ where each $a_i \in \Sigma$, $w^R = a_n a_{n-1} \cdots a_1$ is the "reverse" of w. For a language $A \subseteq \Sigma^*$, $A^R = \{w^R \mid w \in A\}$.

(a) For $L_1 = \{\epsilon, 01, 11, 100\}$ what is L_1^R? **[1 point]**

(b) For $L_2 = \mathbf{L}(0^*(10)^*(0 \cup 1)^*)$, give a regular expression describing L_2^R. **[1 point]**

(c) Regular languages are closed under the "reversing" operation. That is, if A is regular then A^R is regular. This can be shown by constructing an NFA M^R recognizing A^R, given a DFA M recognizing A. Essentially, the NFA M^R "reverses" the direction of the transitions of M and has a new initial state that has ϵ-transitions to the final states of M. Complete the formal definition of M^R based on this intuition.

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing A. The NFA $M^R = (Q^R, \Sigma, \delta^R, q_0^R, F^R)$ where

(i) $Q^R =$ ______________________ **[2 points]**

(ii) $q_0^R =$ ______________________ **[1 point]**

(iii) $F^R =$ ______________________ **[1 point]**

(iv) Describe the transition function δ^R. **[3 points]**

Problem 4. [Category: Proof] Complete the following proof by induction that $\mathbf{L}(M^R) = (\mathbf{L}(M))^R$, where the DFA M and NFA M^R are as defined in Problem 3.

(a) The correctness can be established by capturing the relationship between computations of M and computation of M^R. The statement to be proved by induction is **[2 points]**

$$\forall w \in \Sigma^*.\ \forall q \in Q.\ q_0 \xrightarrow{w}_M q \text{ iff } \underline{\hspace{10cm}}$$

The proof of this statement by induction on $|w|$ is as follows.

(b) Prove the base case. **[2 points]**

(c) State the induction hypothesis. **[1 point]**

(d) Prove the induction step. **[3 points]**

(e) Using the statement in part (a), prove that $\mathbf{L}(M^R) = (\mathbf{L}(M))^R$. **[2 points]**

Problem 5. [Category: Proof] As in Problem 2, for a binary string $w \in \{0,1\}^*$, let $[\![w]\!]$ denote the number whose binary representation is given by w where the rightmost symbol is the least significant bit; the formal inductive definition of $[\![w]\!]$ is given in Problem 2. Let $L_{m3} \subseteq \{0,1,\#\}^*$ be the language

$$L_{m3} = \{x\#y \mid x, y \in \{0,1\}^* \text{ and } [\![y]\!] = 3 \times [\![x]\!]\}$$

Prove that L_{m3} is not regular. You may use any of the proof techniques discussed in class. **[10 points]**