# Math 415 - Lecture 8

Inverses.

Wednesday September 11th 2015

**Textbook:** Chapter 1.6

**Suggested Practice Exercise:** Chapter 1.6 Exercise 1, 2, 4, 6, 10, 11, 18, 35, 36, 37, 38, 40, 49, 50

**Khan Academy Video:** Inverse Matrix (part I), Inverse Matrix (part II)

# 1 Review

- Elementary matrices perform row operations:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ -2a+d & -2b+e & -2c+f \\ g & h & i \end{bmatrix}$$

- Gaussian elimination on $A$ gives a decomposition $A = LU$:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

  $U$ is the echelon form, $L$ records the reverse of the row operations we did.

- $LU$ decomposition lets us solve $A\mathbf{x} = \mathbf{b}$ quickly for many different $\mathbf{b}$.

## 1.1 Today's goal

- We know how to reverse a single row operation:

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Inverting a more complicated matrix is harder:

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & b & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab & -b & 1 \end{bmatrix}$$

Goal today: how to find an "inverse" to any (square!) matrix. *Today $A$ will be an $n \times n$ matrix*

## 2 The inverse of a matrix

The inverse of a real number $a$ is denoted by $a^{-1}$. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1.$$

Remember that the identity matrix $I_n$ is the $n \times n$-matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

**Definition.** An $n \times n$ matrix $A$ is said to be invertible if there is an $n \times n$ matrix $C$ satisfying

$$CA = AC = I_n$$

where $I_n$ is the $n \times n$ identity matrix. We call $C$ the inverse of $A$ .

*Example* 1. We already know that an elementary matrix is invertible:

$$\begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In fact:

$$\begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Check this at home!) So the definition works!

**Theorem 1.** *Let $A$ be an invertible matrix, then its inverse $C$ is unique.*

*Proof.* Assume $B$ and $C$ are both inverses of $A$. Then

$$B = BI_n = BAC = I_nC = C$$

$\square$

- We will write $A^{-1}$ for the inverse of $A$. Multiplying by $A^{-1}$ is like "dividing by $A$."
- Do not write $\frac{A}{B}$. Why?

  It is unclear whether this means $AB^{-1}$ or $B^{-1}A$, and these two matrices are *different.*
- Fact: if $AB = I$ then $A^{-1} = B$ and so $BA = I$. (Not so easy to show at this stage.)

**Remark.** Not all $n \times n$ matrices are invertible. For example, the $2 \times 2$ matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is not invertible. Try to find an inverse!

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \neq I_2$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} \neq I_2$$

**Definition.** A matrix which is *not* invertible is sometimes called a singular matrix. An invertible matrix is also called nonsingular matrix.

**Theorem 2.** *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc \neq 0$, *then* $A$ *is invertible and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

*If* $ad - bc = 0$, *then* $A$ *is not invertible.*

*Proof.* Calculate

$$\begin{aligned} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \frac{1}{ad - bc} \begin{bmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$\square$

Quick question: when is the $1 \times 1$ matrix $[\ a\ ]$ invertible?
When $a \neq 0$. Its inverse is $[\ \frac{1}{a}\ ]$.

**Theorem 3.** *If $A$ is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.*

*Proof.* The vector $A^{-1}\mathbf{b}$ is a solution, because

$$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I_n\mathbf{b} = \mathbf{b}$$

Suppose there is another solution $\mathbf{w}$, then

$$\begin{aligned} A\mathbf{w} &= \mathbf{b} \\ A^{-1}A\mathbf{w} &= A^{-1}\mathbf{b} \\ I_n\mathbf{w} &= A^{-1}\mathbf{b} \\ \mathbf{w} &= A^{-1}\mathbf{b} \end{aligned}$$

□

*Example* 2. Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$\begin{aligned} -7x_1 + 3x_2 &= 2 \\ 5x_1 - 2x_2 &= 1 \end{aligned}$$

**Solution. Matrix form of the linear system:**

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14-15} \begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

Check this works!

# 3 Calculational rules

**Theorem 4.** *Suppose $A$ and $B$ are invertible. Then the following results hold:*

*(a)* $A^{-1}$ *is invertible and* $\left(A^{-1}\right)^{-1} = A$ *(i.e. $A$ is the inverse of $A^{-1}$).*

*(b)* $AB$ *is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$

*(c)* $A^T$ *is invertible and* $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

Note: the inverse of the product is the product of inverses *in opposite order*. Think about putting on socks and shoes. How do you undo those two operations?

*Proof.* (a) $A^{-1}$ is invertible and $\left(A^{-1}\right)^{-1} = A$

$$AA^{-1} = I = A^{-1}A \quad \checkmark$$

(b) $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

$$(B^{-1}A^{-1})(AB) = B^{-1}IB = B^{-1}B = I \quad \checkmark$$
$$(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I \quad \checkmark$$

(c) $A^T$ is invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I \quad \checkmark$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I \quad \checkmark$$

□

# 4 An algorithm for computing the inverse matrix

Idea:

- To solve $Ax = b$ we row reduce $[\ A \mid b\ ]$.
- To solve $AX = I_n$ we row reduce $[\ A \mid I\ ]$.

**Theorem 5.** *An $n \times n$ matrix $A$ is invertible if and only if $A$ is row equivalent to $I_n$, and in this case, any sequence of elementary row operations that reduces $A$ to $I_n$ will also transform $I_n$ to $A^{-1}$.*

So here is the algorithm:

- Place $A$ and $I$ side-by-side to form an augmented matrix $[\ A \mid I\ ]$. This is an $n \times 2n$ matrix (*B*ig *A*ugmented *M*atrix), instead of $n \times (n+1)$, for the usual augmented matrix.
- Then perform row operations on this matrix (which will produce identical operations on $A$ and $I$).
- So by Theorem 5:

$$[\ A \mid I\ ] \text{ will row reduce to } \left[\ I \mid A^{-1}\ \right]$$

or $A$ is not invertible.

*Example* 3. Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

*Solution:*

$$[A \quad I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$$

*Example* 4 (Let's do the previous example step by step.).

$$[\ A \mid I\ ] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R1 \to \frac{1}{2}R1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R2 \to R2+3R1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R2 \leftrightarrow R3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

Check at home that $AA^{-1} = I_3$.

**Remark.** Why does this algorithm work?

- At each step, we get

$$[\ A \mid I\ ] \rightsquigarrow [\ E_1A \mid E_1\ ] \rightsquigarrow [\ E_2E_1A \mid E_2E_1\ ] \rightsquigarrow \dots$$

- So each step is of the form

$$[\ FA \mid F\ ], \quad F = E_r \dots E_3E_2E_1$$

- If we succeed in row reducing $A$ to $I$, the final step is

$$[\ FA \mid F\ ] = [\ I \mid F\ ]$$

- So $FA = I$, which means that $A^{-1} = F$.

**Practice Problems.** Find the inverse of $A$:

- $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.
- $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$. Hint: What is $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?
- $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
- $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 8 & 0 \\ 9 & 0 & 1 & 0 \end{bmatrix}$.
- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.