Math 415 - Lecture 9

Vector spaces and subspaces

Monday September 14th 2015

Textbook: Chapter 2.1.

Suggested practice exercises: Chapter 2.1: 1, 2, 10, 11, 17, 18.

Khan Academy video: Linear Subspaces

Theorem 1. *An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1}.*

A few questions. Assume that A is invertible.

There are how many pivot positions in A?

How many free variables has the equation $Ax = b$?

Is there a b such that $Ax = b$ is inconsistent?

Example 1. Use the Gauss Jordan method to compute the inverse of

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Solution.

Failure: the reduced row echelon form of A will not be I, so A has no inverse!

Practice Problems. Find the inverse of A:

- $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.
- $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$. Hint: What is $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?
- $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
- $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 8 & 0 \\ 9 & 0 & 1 & 0 \end{bmatrix}$.
- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

1 Vector Spaces and Subspaces

- The most important property of column vectors in $\mathbb{R}^n$ is that you can take *linear combinations* of them.
- There are many mathematical objects $X, Y, \ldots$ for which a linear combination $cX + dY$ make sense, and have the usual properties of linear combination in $\mathbb{R}^n$
- We are going to define a *vector space* in general as a collection of objects for which linear combinations make sense. The objects of such a set are called vectors.

Definition. A **vector space** is a non-empty set V of objects, called *vectors*, for which linear combinations make sense. More precisely: on V there are defined two operations, called *addition* and *multiplication* by scalars (real numbers), subject to the ten axioms below. The axioms must hold for all u, v, and w in V and for all scalars c and d.

1. $\mathbf{u} + \mathbf{v}$ is in V. (V is "closed under addition".)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V. (V is "closed under scalar multiplication".)
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $(cd)\mathbf{u} = c(d\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$.

2 Vector Space Examples

Example 2. Let $M_{2x2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$. This is a vector space.

In this context, note that the **0** vector is $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$.

Addition:

Multiplication:

Remarks

- We can take instead of matrices of size 2×2 matrices of any shape: you can check that the set $M_{m\times n}$ of $m \times n$ matrices is also a vector space, in the same way as we indicated above.
- Confusing: in the vector space $M_{2\times 2}$ the vectors are in fact 2×2 matrices!
- In the definition of the vector space $M_{2\times 2}$ the multiplication of matrices plays no role; matrix multiplication will show up when we study the connections *between* vector spaces.
- a "vector" $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ behaves very much like a column vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. A fancy person would say that the vector spaces $M_{2\times 2}$ and $\mathbb{R}^4$ are *isomorphic*.

Example 3. Let $n \geq 0$ be an integer and let

$$\mathbf{P}_n = \text{the set of all polynomials of degree at most } n.$$

Members of $\mathbf{P}_n$ have the form

$$\boxed{\mathbf{p}(t) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

where $a_0, a_1, \ldots, a_n$ are real numbers and t is a real variable.

The set $\mathbf{P}_n$ is a vector space. We will just verify 3 out of the 10 axioms here.

Let $\mathbf{p}(t) = a_0 + a_1 t + \cdots + a_n t^n$ and $\mathbf{q}(t) = b_0 + b_1 t + \cdots + b_n t^n$. Let c be a scalar.

Axiom 1:

The polynomial $\mathbf{p} + \mathbf{q}$ is defined as follows: $(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t)$. Therefore,

$$\begin{aligned}(\mathbf{p} + \mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t) \\ &= (\_\_\_\_\_\_\_\_\_\_) + (\_\_\_\_\_\_\_\_\_\_)\, t + \cdots + (\_\_\_\_\_\_\_\_\_\_)\, t^n\end{aligned}$$

which is also a ______________________ of degree at most _________. So $\mathbf{p} + \mathbf{q}$ is in $\mathbf{P}_n$.

Axiom 4:

$$\mathbf{0} = 0 + 0t + \cdots + 0t^n$$

is the zero vector in $\mathbf{P}_n$.

$$(\mathbf{p} + \mathbf{0})(t) = \mathbf{p}(t) + \mathbf{0}$$

and so $\mathbf{p} + \mathbf{0} = \mathbf{p}$.

Axiom 6:

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = (\_\_\_\_\_\_\_\_) + (\_\_\_\_\_\_\_\_)\,t + \cdots + (\_\_\_\_\_\_\_\_)\,t^n$$

which is in $\mathbf{P}_n$.

The other 7 axioms also hold, so $\mathbf{P}_n$ is a vector space.

3 Subspaces

Vector spaces may be formed from subsets of other vector spaces. These are called **subspaces**.

Definition. A *subspace* of a vector space V is a subset H of V that satisfies 3 properties:

- The zero vector (of V) belongs to H.
- If $\mathbf{u}, \mathbf{v}$ both belong to H also the sum $\mathbf{u} + \mathbf{v}$ belongs to H. (H is *closed* under vector addition).
- If $\mathbf{u}$ is in H and c is any scalar also $c\mathbf{u}$ belongs to H. (H is closed under scalar multiplication.)

Note that if the subset H satisfies these three properties, then H itself is a vector space.

Example 4. $Z = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^2$. Why?

Solution.

Example 5. $H = span\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ is a subspace of $\mathbb{R}^2$. Why?

Solution.

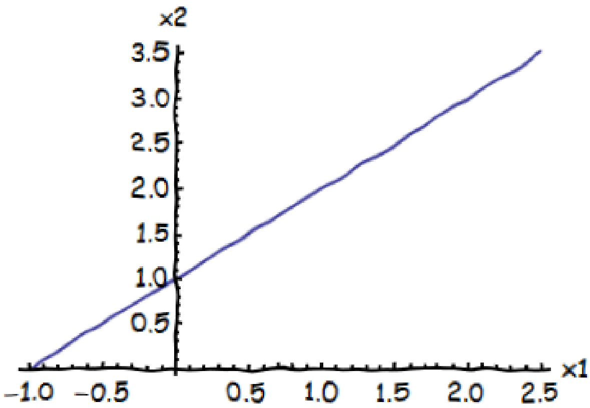
Example 6. Let $H = \left\{\begin{bmatrix}a\\0\\b\end{bmatrix} : a, b \in \mathbb{R}\right\}$. Show that H is a subspace of $\mathbb{R}^3$.

Solution.

Remark. Vectors $(a, 0, b)$ look and act like the points (a, b) in $\mathbb{R}^2$. But they are **not** the same!

Example 7. Is $H = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$? (i.e. does H satisfy the properties of a subspace?)

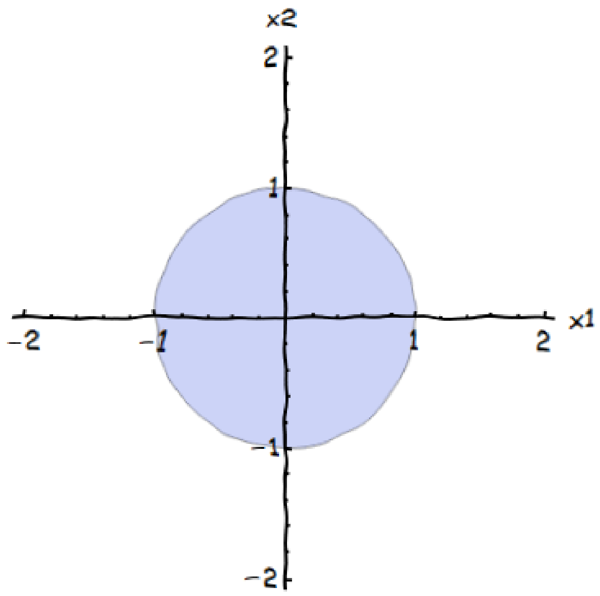
Solution.

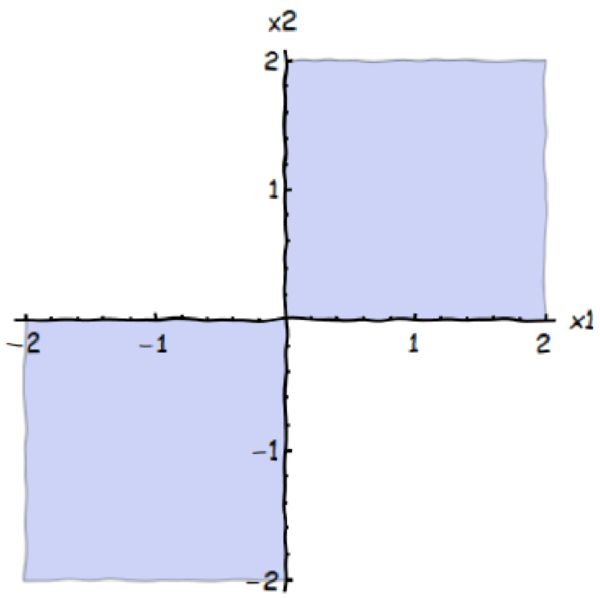


Problem 8. *Find as many subspaces in $\mathbb{R}^2$ as you can.*

Think of this at home.

Example 9. Is one of the following a subspace of $\mathbb{R}^2$?





Example 10. Is this a subspace of $\mathbb{R}^3$?