Math 415 - Lecture 31

Markov matrices and Google

Monday November 9th 2015

Textbook reading: Chapter 5.3

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Suggested practice exercises: Chapter 5.3: 8, 9, 12, 14, 10.

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Khan Academy video: Finding Eigenvectors and Eigenspaces example

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Khan Academy video: Finding Eigenvectors and Eigenspaces example

Strang lecture: Lecture 21: Eigenvalues and eigenvectors Lecture 24: Markov Matrices and Fourier Series.

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Review

Properties of eigenvectors and eigenvalues

- If $A\mathbf{x} = \lambda\mathbf{x}$ then $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ.

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  By the way: this means that the eigenspace of λ is just $\mathrm{Nul}(A - \lambda I)$.
- E.g. if $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$ then
  $\det(A - \lambda I) = (3 - \lambda)(6 - \lambda)(2 - \lambda)$.

If $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$ then the eigenvalues are $2, 3, 6$

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These three vectors are independent. By the next result, this is always so.

Independent eigenvectors

Theorem

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By kicking out some vectors, we may assume that there is (up to multiples) only one linear relation: $c_1\mathbf{x}_1 + \ldots + c_m\mathbf{x}_m = \mathbf{0}$.

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In other words, the matrix with columns $\mathbf{x}_1, \ldots, \mathbf{x}_m$ has one-dimensional null space. Now multiply this relation with A:

$$A(c_1\mathbf{x}_1 + \ldots + c_m\mathbf{x}_m) = c_1\lambda_1\mathbf{x}_1 + \ldots + c_m\lambda_m\mathbf{x}_m = \mathbf{0}$$

This is a second independent relation!

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Relations between eigenvalues

Product of Eigenvalues

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The characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ has constant term $\det(A)$. On the other hand $p(\lambda)$ factors, because the roots are the eigenvalues we get $p(\lambda) = (\pm 1)(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$, which has constant term $\lambda_1 \lambda_2 \ldots \lambda_n$. $\square$

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Example

Let $A = \begin{bmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{bmatrix}$. Then the eigenvalues are λ_1, λ_2 and $\det(A) = \lambda_1 \lambda_2$.

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Definition

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ be $n \times n$. Then the **TRACE** of A is the sum of the diagonal entries: $\mathrm{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

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Theorem

Let A be $n \times n$. Then the trace of A is the **sum** *of eigenvalues:*

$$\mathrm{Tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

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Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Then the characteristic polynomial is*

$$p(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A).$$

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$\mathrm{Tr}(A) = 6$, $\det(A) = 8$,

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Practice problems

Example

Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

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Example

What are the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 4 \end{bmatrix}$.

No calculations!

Example

Find the eigenvalues of A as well as a basis for the corresponding eigenspaces, where

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

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$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -1 & 3-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix}$$

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Solution (continued)

- A has eigenvalues $2, 2, 4$ $\qquad \left(A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \right)$

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- $\lambda_1 = 2$:

$$(A - \lambda_1 I)\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

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  Since $\lambda = 2$ is a double root, it has **(algebraic) multiplicity** 2.

- $\lambda_1 = 2$:

$$(A - \lambda_1 I)\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

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- In summary, A has eigenvalues 2 and 4:
  - eigenspace for $\lambda_1 = 2$ has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$,
  - eigenspace for $\lambda_2 = 4$ has basis $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Markov matrices

Definition

An $n \times n$ matrices A is **Markov matrix** if has non negative entries, and the entries in each column add to 1.

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Example

Let A be

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}.$$

Is A a Markov matrix?

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Let A be an $n \times n$-Markov matrix with only positive entries and let $\mathbf{v} \in \mathbb{R}^n$. Then

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and $A\mathbf{v}_\infty = \mathbf{v}_\infty$. In this case $\mathbf{v}_\infty$ is often called the **steady state**.

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if the eigenspace of $\lambda = 1$ is 1-dimensional. $\square$

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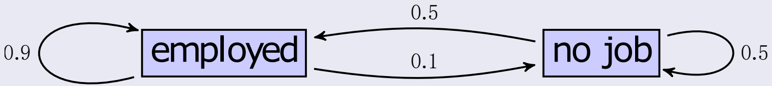

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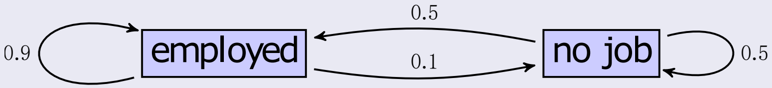

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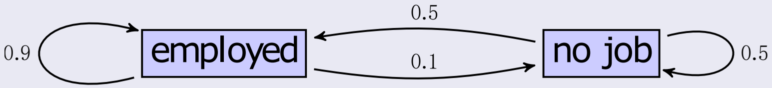

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$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.9x_t + 0.5y_t \\ 0.1x_t + 0.5y_t \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

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The matrix $\begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$ is a **Markov matrix**. Its columns add to 1 and it has no negative entries.

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Hence, the unemployment rate in the long term equilibrium is $1/6$

Page rank (or: the 25000000000 $ eigenvector)

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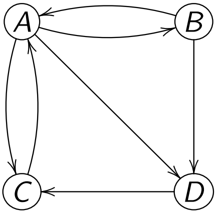

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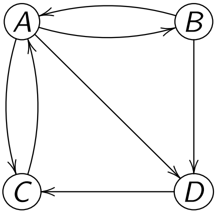

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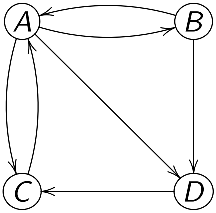

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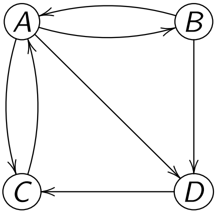

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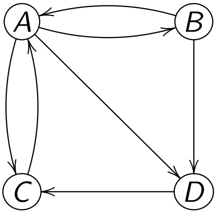

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Definition

The **PageRank vector** is the long-term equilibrium.
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Example (continued)

Let's call the Markov matrix with the probabilities T:

- $$T - 1I = \begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  $$\implies \text{ eigenspace of } \lambda = 1 \text{ is spanned by } \begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}.$$

- Now we need to make the entries add up to 1.

  $$\begin{bmatrix} PR(A) \\ PR(B) \\ PR(C) \\ PR(D) \end{bmatrix} = \frac{3}{16} \begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix}.$$

  This is the PageRank vector.

- The corresponding ranking of the webpages is A, C, D, B.

Remark

In practical situations the system might be too large for finding the eigenvalues by row operations.

- Google reports having met 60 trillion webpages.
  Google's search index is over $100,000,000$ gigabytes.
  Number of Google's servers is secret: about $2,500,000$
  More than 1,000,000,000 websites (i.e. hostnames; about 75% not active)
- Thus we have a gigantic but very sparse matrix.

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An alternative to row operations is the **power method** (see Theorem 2):

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An alternative to row operations is the **power method** (see Theorem 2):

Power method

If T is an (acyclic and irreducible) Markov matrix, then for any $\mathbf{v}_0$ the vectors $T^n\mathbf{v}_0$ converge to an eigenvector with eigenvalue 1

Here:

$$T = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

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Start with an arbitrary state vector, hit it with powers of T.

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$$\left(\begin{bmatrix} PR(A) \\ PR(B) \\ PR(C) \\ PR(D) \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix} \right),$$

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$$\left(\begin{bmatrix} PR(A) \\ PR(B) \\ PR(C) \\ PR(D) \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix} \right), T \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.083 \\ 0.333 \\ 0.208 \end{bmatrix}$$

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. Note that the ranking of the webpages is already A, C, D, B if we stop here.

$$T \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.083 \\ 0.333 \\ 0.208 \end{bmatrix}, \qquad T^2 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.333 \\ 0.167 \end{bmatrix},$$

Here:

$$T = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

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- If all entries of T are positive (no zero entries!), then the power method is guaranteed to work.
- In the context of PageRank, we can make sure that this is the case by replacing T with

$$(1-p)\cdot\begin{bmatrix}0 & \frac{1}{2} & 1 & 0\\ \frac{1}{3} & 0 & 0 & 0\\ \frac{1}{3} & 0 & 0 & 1\\ \frac{1}{3} & \frac{1}{2} & 0 & 0\end{bmatrix}+p\cdot\begin{bmatrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\end{bmatrix}.$$

Just to make sure: still a Markov matrix, now with positive entries

Google used to use $p = 0.15$.

Practice problems

Problem

Can you see why 1 *is an eigenvalue for every Markov matrix?*

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Problem (just for fun)

The real web contains pages which have no outgoing links. In that case, our random surfer would get "stuck" (the transition matrix is not a Markov matrix). Do you have an idea how to deal with this issue?