Examples for 1.1

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1. Suppose a 6-sided die is rolled. The sample space, S, is $\{ 1, 2, 3, 4, 5, 6 \}$.
Consider the following events:

$A = \{ \text{the outcome is even} \}$,

$B = \{ \text{the outcome is greater than 3} \}$,

a) List outcomes in A, B, A', $A \cap B$, $A \cup B$.

$A = \{ \text{the outcome is even} \} = \{ 2, 4, 6 \}$,

$B = \{ \text{the outcome is greater than 3} \} = \{ 4, 5, 6 \}$,

$A' = \{ 1, 3, 5 \}$,

$A \cap B = \{ 4, 6 \}$,

$A \cup B = \{ 2, 4, 5, 6 \}$.

b) Find the probabilities $P(A)$, $P(B)$, $P(A')$, $P(A \cap B)$, $P(A \cup B)$
if the die is balanced (fair).

$P(A) = P(2) + P(4) + P(6) = 1/6 + 1/6 + 1/6 = \mathbf{3/6}$,

$P(B) = P(4) + P(5) + P(6) = 1/6 + 1/6 + 1/6 = \mathbf{3/6}$,

$P(A') = 1 - P(A) = 1 - 3/6 = \mathbf{3/6}$,

$P(A \cap B) = P(4) + P(6) = 1/6 + 1/6 = \mathbf{2/6}$,

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 3/6 + 3/6 - 2/6 = \mathbf{4/6}$.

c) Suppose the die is loaded so that the probability of an outcome is proportional to the outcome, i.e.

$$\mathrm{P}(1) = p, \quad \mathrm{P}(2) = 2p, \quad \mathrm{P}(3) = 3p, \quad \mathrm{P}(4) = 4p, \quad \mathrm{P}(5) = 5p, \quad \mathrm{P}(6) = 6p.$$

i) Find the value of p that would make this a valid probability model.

$\mathrm{P}(1) + \mathrm{P}(2) + \mathrm{P}(3) + \mathrm{P}(4) + \mathrm{P}(5) + \mathrm{P}(6) = 1.$

$p + 2p + 3p + 4p + 5p + 6p = 21p = 1. \qquad \Rightarrow \qquad p = \mathbf{1/21}.$

ii) Find the probabilities $\mathrm{P}(A)$, $\mathrm{P}(B)$, $\mathrm{P}(A')$, $\mathrm{P}(A \cap B)$, $\mathrm{P}(A \cup B)$.

$\mathrm{P}(A) = \mathrm{P}(2) + \mathrm{P}(4) + \mathrm{P}(6) = 2/21 + 4/21 + 6/21 = \mathbf{12/21}.$

$\mathrm{P}(B) = \mathrm{P}(4) + \mathrm{P}(5) + \mathrm{P}(6) = 4/21 + 5/21 + 6/21 = \mathbf{15/21}.$

$\mathrm{P}(A') = 1 - \mathrm{P}(A) = 1 - 12/21 = \mathbf{9/21}.$

OR

$\mathrm{P}(A') = \mathrm{P}(1) + \mathrm{P}(3) + \mathrm{P}(5) = 1/21 + 3/21 + 5/21 = \mathbf{9/21}.$

$\mathrm{P}(A \cap B) = \mathrm{P}(4) + \mathrm{P}(6) = 4/21 + 6/21 = \mathbf{10/21}.$

$\mathrm{P}(A \cup B) = \mathrm{P}(2) + \mathrm{P}(4) + \mathrm{P}(5) + \mathrm{P}(6) = 2/21 + 4/21 + 5/21 + 6/21 = \mathbf{17/21}.$

OR

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 12/21 + 15/21 - 10/21 = \mathbf{17/21}.$

2. Consider a "thick" coin with three possible outcomes of a toss (Heads, Tails, and Edge) for which Heads and Tails are equally likely, but Heads is five times as likely than Edge. What is the probability of Heads ?

$\mathrm{P}(\text{ Heads }) = \mathrm{P}(\text{ Tails }) = p$ for some p. $\mathrm{P}(\text{ Edge }) = \frac{1}{5}\,p$.

$\mathrm{P}(\text{ Heads }) + \mathrm{P}(\text{ Tails }) + \mathrm{P}(\text{ Edge }) = 1.$

$p + p + \frac{1}{5}\,p = 1.$ $\frac{11}{5}\,p = 1.$

$\mathrm{P}(\text{ Heads }) = p = \mathbf{\frac{5}{11}}.$

3. The probability that a randomly selected student at Anytown College owns a bicycle is 0.55, the probability that a student owns a car is 0.30, and the probability that a student owns both is 0.10.

$\mathrm{P}(\text{ B }) = 0.55,$ $\mathrm{P}(\text{ C }) = 0.30,$ $\mathrm{P}(\text{ B} \cap \text{C }) = 0.10.$

a) What is the probability that a student selected at random does not own a bicycle?

$\mathrm{P}(\text{ B}') = 1 - \mathrm{P}(\text{ B }) = 1 - 0.55 = \mathbf{0.45}.$

	C	C′	
B	0.10	0.45	0.55
B′	0.20	0.25	0.45
	0.30	0.70	1.00

b) What is the probability that a student selected at random owns either a car or a bicycle, or both?

$P(B \cup C) = P(B) + P(C) - P(B \cap C) = 0.55 + 0.30 - 0.10 = \mathbf{0.75}$.

OR

$P(B \cup C) = P(B \cap C) + P(B' \cap C) + P(B \cap C') = 0.10 + 0.20 + 0.45 = \mathbf{0.75}$.

OR

$P(B \cup C) = 1 - P(B' \cap C') = 1 - 0.25 = \mathbf{0.75}$.

c) What is the probability that a student selected at random has neither a car nor a bicycle?

$P(B' \cap C') = 1 - P(B \cup C) = \mathbf{0.25}$.

4. During the first week of the semester, 80% of customers at a local convenience store bought either beer or potato chips (or both). 60% bought potato chips. 30% of the customers bought both beer and potato chips. What proportion of customers bought beer?

$P(B \cup PC) = 0.80, \qquad P(PC) = 0.60, \qquad P(B \cap PC) = 0.30.$

$P(B \cup PC) = P(B) + P(PC) - P(B \cap PC).$

$0.80 = P(B) + 0.60 - 0.30. \qquad \Rightarrow \qquad P(B) = \mathbf{0.50}.$

5. Suppose

$P(A) = 0.22,$ $P(B) = 0.25,$ $P(C) = 0.28,$

$P(A \cap B) = 0.11,$ $P(A \cap C) = 0.05,$

$P(B \cap C) = 0.07,$ $P(A \cap B \cap C) = 0.01.$

Find the following:

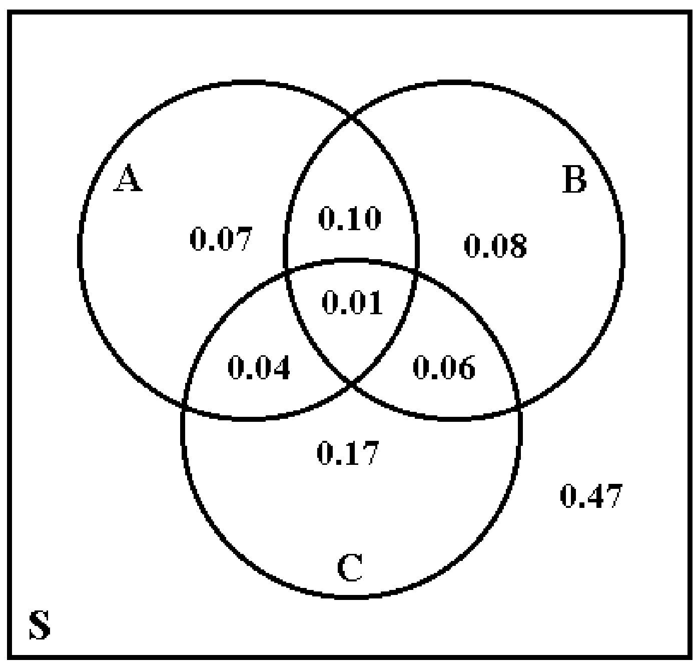
a) $P(A \cup B)$ b) $P(A' \cap B')$

c) $P(A \cup B \cup C)$ d) $P(A' \cap B' \cap C')$

e) $P(A' \cap B' \cap C)$ f) $P((A' \cap B') \cup C)$

g) $P((A \cup B) \cap C)$ h) $P((B \cap C') \cup A')$

a) $P(A \cup B) = \mathbf{0.36}.$



b) $P(A' \cap B') = \mathbf{0.64}.$

c) $P(A \cup B \cup C) = \mathbf{0.53}.$

d) $P(A' \cap B' \cap C') = \mathbf{0.47}.$

e) $P(A' \cap B' \cap C) = \mathbf{0.17}.$

f) $P((A' \cap B') \cup C) = \mathbf{0.75}.$

g) $P((A \cup B) \cap C) = \mathbf{0.11}.$

h) $P((B \cap C') \cup A') = \mathbf{0.88}.$

6. Let $a > 2$. Suppose S = { 0, 1, 2, 3, … } and

$$P(0) = c, \qquad P(k) = \frac{1}{a^k}, \quad k = 1, 2, 3, \ldots .$$

a) Find the value of c (c will depend on a) that makes this is a valid probability distribution.

Must have $\sum\limits_{\text{all } x} p(x) = 1.$ $\quad\Rightarrow\quad$ $c + \sum\limits_{k=1}^{\infty} \frac{1}{a^k} = 1.$

$$\sum_{k=0}^{\infty} b^k = \frac{1}{1-b}, \qquad |b| < 1.$$

$$\sum_{k=1}^{\infty} \frac{1}{a^k} = \left[\sum_{k=0}^{\infty} \frac{1}{a^k}\right] - 1 = \frac{1}{1 - 1/a} - 1 = \frac{1}{a-1}.$$

OR

$$\sum_{k=1}^{\infty} \frac{1}{a^k} = \frac{1}{a} \cdot \sum_{k=0}^{\infty} \frac{1}{a^k} = \frac{1}{a} \cdot \frac{1}{1 - 1/a} = \frac{1}{a-1}.$$

$$c + \frac{1}{a-1} = 1. \qquad\qquad c = 1 - \frac{1}{a-1} = \frac{a-2}{a-1} = 2 - \frac{a}{a-1}.$$

b) Find the probability of an odd outcome.

$$P(\text{odd}) = p(1) + p(3) + p(5) + \ldots = \frac{1}{a^1} + \frac{1}{a^3} + \frac{1}{a^5} + \ldots$$

$$= \frac{\textit{first term}}{1 - \textit{base}} = \frac{\frac{1}{a}}{1 - \frac{1}{a^2}} = \frac{a}{a^2 - 1}.$$

7. Suppose S = { 0, 1, 2, 3, … } and

$$P(0) = p, \qquad P(k) = \frac{1}{2^k \cdot k!}, \quad k = 1, 2, 3, \ldots .$$

Find the value of p that would make this a valid probability model.

Must have $\displaystyle\sum_{\text{all } x} p(x) = 1.$ $\quad\Rightarrow\quad$ $\displaystyle p + \sum_{k=1}^{\infty} \frac{1}{2^k \cdot k!} = 1.$

Since $\displaystyle\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a,$ $\quad$ $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k \cdot k!} = \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} - 1 = e^{1/2} - 1.$

Therefore, $p + (e^{1/2} - 1) = 1$ and $p = \mathbf{2 - e^{1/2}}$.