Worksheet 4 (September 15th and 17th)

1. *Consider the matrix:*

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$$

Decompose the matrix A into LU, where L is a lower triangular matrix and U is an upper triangular matrix. Then use this factorization to solve:

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

That means, find a vector $\mathbf{c}$ in $\mathbb{R}^3$ such that:

$$L\mathbf{c} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

and then find a vector $\mathbf{x}$ in $\mathbb{R}^3$ such that:

$$U\mathbf{x} = \mathbf{c}$$

Solution. We start by bringing A to echelon form by multiplying A by elementary matrices. Let E be the matrix corresponding to subtracting row 1 three times from row 3, that is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Then

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}.$$

Since EA is already upper triangular, we set $U := EA$. Then $E^{-1}U = A$. To compute E^{-1} explicitly, this explicitly, note that the inverse operation to subtracting row 1 from row 3 three times is adding row 1 three times to row 3. Hence

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

Thus L is E^{-1} and the LU-decomposition of A is

$$\underbrace{\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}}_{U}.$$

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Midterm Dates: September 29th, October 22nd, November 19th (All Midterms 7-8:15 PM, see `learn.illinois.edu` **for locations)**

[Note: the above steps include the maximum amount of details (not necessary for the exam). Can you see how to get L and U directly from the one row operation that is performed here?]
We now solve the

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5. \end{bmatrix}.$$

We first need to solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5. \end{bmatrix}.$$

We bring in the augmented matrix to reduced echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 1 & 5 \end{array}\right] \xrightarrow{R3 \to R3 - 3R1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Hence

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

Now we solve

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1. \end{bmatrix}.$$

We bring in the augmented matrix to reduced echelon form:

$$\left[\begin{array}{ccc|c} 2 & 3 & 3 & 2 \\ 0 & 5 & 7 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow{R2 \to R2 + 7R1, R1 \to R1 + 3R3} \left[\begin{array}{ccc|c} 2 & 3 & 0 & -1 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{R2 \to 1/5R2, R3 \to -R3} \left[\begin{array}{ccc|c} 2 & 3 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R1 \to R1 - 3R2} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R1 \to 1/2R1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

Hence

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}. \qquad \square$$

2. *Let* $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}$, *and* $U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$.

(1) *Show that* $A = LU$.

(2) *Let* A_i *be the matrix introduced by the first* i *rows and the first* i *columns of* A, *for* $i = 1, 2, 3$, *i.e.,*

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \textit{and} \quad A_1 = \begin{bmatrix} 2 \end{bmatrix}.$$

What is an LU-decomposition of A_i, *for* $i = 1, 2, 3$*?*

Solution. (1) Easy to check (just do the multiplication $L \cdot U$ and check that the product is A).

(2) An LU-decomposition for A_i is $L_i U_i$ where L_i (respectively, U_i) is the matrix introduced by the first i rows and the first i columns of L (respectively U). More specifically,

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{A_3} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 0 \end{bmatrix}}_{L_3} \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{2}{3} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}}_{U_3},$$

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{A_2} = \underbrace{\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} 2 & -1 \\ 0 & \frac{3}{2} \end{bmatrix}}_{U_2}, \quad \text{and}$$

$$\underbrace{\begin{bmatrix} 2 \end{bmatrix}}_{A_1} = \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 2 \end{bmatrix}}_{U_1}. \quad \square$$

3. *Answer the following true-false questions. Justify your answers!* A *and* B *are arbitrary* $n \times n$ *square matrices.*

(1) *If* A *is invertible then* $A\mathbf{x} = \mathbf{0}$ *has exactly one solution,* $\mathbf{x} = \mathbf{0}$.

(2) *If* A *is invertible, then* AB *is also invertible.*

(3) *If* A *and* B *are invertible, then* $A + B$ *is also invertible.*

(4) *If* A *is invertible, then the reduced echelon form of* A *is equal to* I.

Solution. (1) True, $A\mathbf{x} = \mathbf{b}$ has *at least* one solution, $\mathbf{x} = A^{-1}\mathbf{b}$. To see that $A\mathbf{x} = \mathbf{x}$ has *at most* one solution, suppose that $\mathbf{x}_1, \mathbf{x}_2$ are solutions, i.e., $A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{b}$. Multiplying through by A^{-1} we see that $\mathbf{x}_1 = \mathbf{x}_2 = A^{-1}\mathbf{b}$, i.e., all solutions *must* be equal to $A^{-1}\mathbf{b}$ and to each other. Hence there is also *at most* one solution.

(2) False, let $A = I$ and $B = 0$.

(3) False, let $A = I$ and $B = -I$.

(4) True, see the first part of problem (8). $\square$

4. *Let* $A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. *Use the Gauss-Jordan method to either find the inverse of* A *or to show that* A *is not invertible.*

Solution.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2\to R2-2R1} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R3\to R3-R1} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R3\to R3-R2/3} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1/3 & -1/3 & 1 \end{array}\right]$$

We get a row of zeros in the left hand side, so it is not possible to transform left hand side to the identity matrix. Thus, A is not invertible. □

5. *If* $G = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, *find* G^{-1} *using as many methods as possible. Check that* $G^{-1}G = I$.

Solution. For 2×2 matrices we may use the following formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So, $G^{-1} = -\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$.

We may also compute G^{-1} via the Gauss-Jordan method:

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R1\leftrightarrow R2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R1\to R1-2R2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

Finally, we can solve the systems $G\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $G\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and then set $G^{-1} = [\mathbf{v}_1\mathbf{v}_2]$. Are there any other methods? □

6. *Calculate the inverse of the matrix:* $\begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

Solution.

$$\left[\begin{array}{cccc|cccc} 2 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3\leftrightarrow R4} \left[\begin{array}{cccc|cccc} 2 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R3\to R3-R4,R1\to R1+R4} \left[\begin{array}{cccc|cccc} 2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R2\leftrightarrow R1} \left[\begin{array}{cccc|cccc} 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R2\to R2-2R1} \left[\begin{array}{cccc|cccc} 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & -2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - R1} \left[\begin{array}{cccc|cccc} 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & -2 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R2 \to R2/3} \left[\begin{array}{cccc|cccc} 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & 1/3 & -2/3 & 1/3 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R1 \to R1 + R2} \left[\begin{array}{cccc|cccc} 1 & 0 & 1/3 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & -2/3 & 0 & 1/3 & -2/3 & 1/3 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - 2R2} \left[\begin{array}{cccc|cccc} 1 & 0 & 1/3 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & -2/3 & 0 & 1/3 & -2/3 & 1/3 & 0 \\ 0 & 0 & 4/3 & 0 & -2/3 & 1/3 & -5/3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R3 \to 3R3/4} \left[\begin{array}{cccc|cccc} 1 & 0 & 1/3 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & -2/3 & 0 & 1/3 & -2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & -2/4 & 1/4 & -5/4 & 3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R2 \to R2 + 2/3R3} \left[\begin{array}{cccc|cccc} 1 & 0 & 1/3 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -2/4 & 1/4 & -5/4 & 3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R1 \to R1 - 1/3R3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & 1/4 & 3/4 & -1/4 \\ 0 & 1 & 0 & 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -2/4 & 1/4 & -5/4 & 3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right].$$

So the inverse matrix is:

$$\begin{bmatrix} 1/2 & 1/4 & 3/4 & -1/4 \\ 0 & -1/2 & -1/2 & 1/2 \\ -2/4 & 1/4 & -5/4 & 3/4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

□

7. *Let*

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Such a matrix called ***band matrix****. Band matrices often appear in applications (see the next exercise) and they are one reason LU-decomposition is so important.*

(1) *Find the LU-decomposition of A.*
(2) *Determine A^{-1}.*

Given a vector $\boldsymbol{b}$ in $\mathbb{R}^5$, suppose we want to solve $A\boldsymbol{x} = \boldsymbol{b}$. In many applications, we probably don't want to do this for a single $\boldsymbol{b}$, but for many (maybe a million) different vectors $\boldsymbol{b}$. One way *for finding $\boldsymbol{x}$ is to solve the following two linear systems which arise from the LU-decomposition of A:*

$$L\boldsymbol{c} = \boldsymbol{b}, \text{ and } U\boldsymbol{x} = \boldsymbol{c}.$$

Another way is to calculate A^{-1}, and then we would get $\boldsymbol{u}$ by simply multiplying $\boldsymbol{b}$ by A^{-1}. Which way is more efficient?

(3) *Given the LU-decomposition of A, count the numbers of operations (addition and multiplication of real numbers) needed to find $\boldsymbol{x}$ using first method?*

(4) *Given the inverse of A, count the number of operations needed to calculate $A^{-1}\boldsymbol{b}$?*

(5) *Suppose A is not a 5×5 band matrix, but a 1000×1000 band matrix? Which of the two methods would you use?*

Solution. (1) Doing the row operations $R_2 \to R_2 + \frac{1}{2}R_1$, $R_3 \to R_3 + \frac{2}{3}R_2$,$R_4 \to R_4 + \frac{3}{4}R_3$, $R_5 \to R_5 + \frac{4}{5}R_4$, we get to the echelon form.

$$U = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & -1 \\ 0 & 0 & 0 & 0 & \frac{6}{5} \end{bmatrix}.$$

We get L by applying the inverses of these row operations, in reverse operation to the identity matrix. This then gives us that

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 & 0 \\ 0 & 0 & 0 & -\frac{4}{5} & 1 \end{bmatrix}$$

(2) A comptuer calculation (or using Gauss Jordan) gives that the inverse of this matrix is

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

(3) To understand the complexity of solving $Lc = b$, we need to know how many row operations to get L to the identity matrix. To get the second row right, we need to add $\frac{1}{2}$ of the first row to the second row, which requires four operations (two each for the 1 in the first column, and two for the augmented portion). Similarly for the third row, we need to add $\frac{2}{3}$ of the second row to the third row. Continuing this pattern, we need 16 operations to solve $Lc = b$.

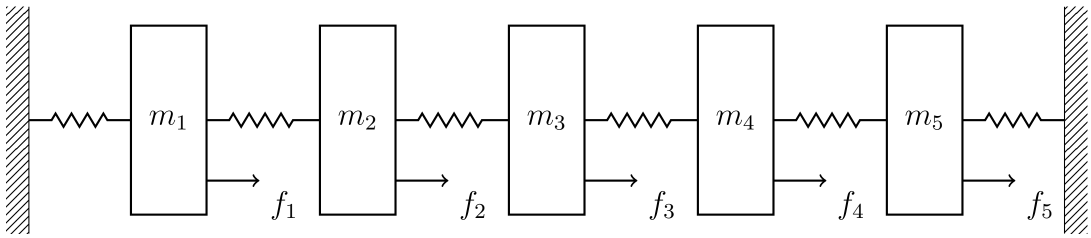
We now consider how many operations to calculate $Ux = c$. We first multiply the fifth row by $\frac{5}{6}$, which is two operations. Then we add the fifth row to the fourth, which is two operations. Continuing like this, we get that this takes 18 operations. Thus all together this solution takes 34 operations.

It is worth noting that the number of operations here scale linearly as the size of the matrix grows.

(4) We consider $A^{-1}b$. To calculate the each entry of this, we need to do five multiplications and five additions. This then takes 50 operations. Note that this scales with the square of the size of A.
(5) Looking at how these scale (linearly versus quadratically), the LU decomposition is the superior solution method.

□

8. *Consider the following spring-mass system, consisting of five masses and six springs fixed between two walls.*



For simplicity, the stiffness of the springs is assumed to be 1*. We denote*

- *by f_i a (steady) applied force on mass i,*
- *by u_i the displacement of the mass i.*

Note that positive values of u_i correspond to displacement away from the wall on left. We choose our reference such that in the absence of applied forces we have $u_i = 0$. We want to calculate the steady state of this system; that is we wish to determine the value of $u_1, \dots, u_5$ in the equilibrium.

(1) *In equilibrium the sum of the forces on mass i (the applied forces f_i and the forces due to the two springs next two it) must sum to zero. Using Hooke's law, this can be express as a linear equation in terms u_{i-1}, u_i and u_{i+1} for $i = 2, 3, 4$, in terms of u_1, u_2 for $i = 1$ and in terms of u_4, u_5 for $i = 5$. For each $i = 1, \dots, 5$, write down this linear equation. (Hint: the equation for mass 1 is $2u_1 - u_2 = f_1$. Why?)*
(2) *Write this five equations into one system of linear equations with unknowns $u_1, \dots, u_5$. The coefficient matrix of this system should be equal to the sparse matrix given in the previous exercise.*
(3) *Suppose $f_1 = \cdots = f_5 = 1$. What is $u_1, \dots, u_5$ in the equilibrium?*

Solution. (1) We first consider the first mass. The first mass will experience forces from each of the strings it attaches to. It will experience a force proportional to u_1 from both springs it is attached to (one for pulling away from the left wall, and one from compressing the spring on the other side). It will also experience a force in the other direction proportional to u_2 as this corresponds to a stretching in that spring. A similar argument will apply to the forces on u_5. So we know that $2u_1 - u_2 = f_1$ and $2u_5 - u_4 = f_5$. Now consider any mass in the middle, say the second mass. Then the displacement, u_2, of this mass will induce forces from both springs it is attached to in the same direction, while displacements of the other two masses will induce forces in the opposite direction. Thus in the middle we get $-u_1 + 2u_2 - u_3 = f_2$. This works similarly for all the masses in the middle.
(2) This is clear from the previous.

(3) We seek to solve the system

$$\left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 & 1 \\ 0 & -1 & 2 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 & -1 & 1 \\ 0 & 0 & 0 & -1 & 2 & 1 \end{array}\right].$$

Using whatever method you prefer, this will give a solution of

$$\begin{bmatrix} \frac{5}{2} \\ 4 \\ \frac{9}{2} \\ 4 \\ \frac{5}{2} \end{bmatrix}$$

Note that this solution also obeys the symmetries of the problem, that is if we swap the first and fifth mass and the second and fourth mass, this doesn't change the setup, and this should be reflected in any answer here.

□

9. *(Optional and more challenging) Let A and B be $n \times n$ matrices such that $AB = I$. (Be Careful! The assumption "$AB = I$" doesn't quite mean that A is invertible: that is what we will show in this problem!).*

(1) *What is the reduced echelon form of A? (Hint: Let F be the product of all elementary matrices used to reduce A, so FA is the reduced echelon form of A. How many pivots can/does FA have?)*

(2) *Show that $BA = I$. (Note that this means that if A has a "right inverse", then it also has a "left inverse" and also that these inverses are the same. Thus A is invertible in the usual sense).*

Solution. (1) If the reduced echelon form of A has less than n pivot positions, then we get a row of zeros in the echelon form. So, there is a matrix F (namely, the product of all the elementary matrices used to reduce A) such that FA has a row of zeros. Therefore, $FAB = FI = F$ also has a row of zeros; but this is impossible since such a matrix F cannot possibly correspond to a reversible sequence of row operations.

So, the echelon form of A has exactly n pivot positions and we have exactly one pivot position in each row and in each column. These pivots are exactly on the main diagonal. Therefore, in the reduced echelon form every element on the main diagonal

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The following may be useful in the above problems:

Definition. An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A.

Hooke's law is a principle of physics that states that the force F needed to extend or compress a spring by some distance u is proportional to that distance. That is: $F = -ku$, where k is a constant factor characteristic of the spring, called its **stiffness**.

is 1 and every other element is equal to 0. Thus, the reduced echelon form of A is I.

(2) From the first part, there is a matrix F such that $FA = I$. We have:

$$F = FI = FAB = IB = B$$

So:

$$BA = FA = I \qquad \square$$