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1. Let $X_1, X_2, \dots, X_n$ be a random sample from the Poisson distribution with mean θ.
   a. Find the MLE $\hat{\theta}$ for θ.
   b. Find the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$.
2. Using the results in Problem 1, find an approximate 95% confidence interval for θ given a sample of size n. Using the large sample approximation theorems in the notes, explain why your confidence interval has approximate coverage equal to 95%.
3. Under the assumptions of Problem 1. Show that $Y = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.
4. Let $X_1, \dots, X_n$ be a random sample from $N(0, \theta)$, $0 < \theta < \infty$.
   a. Show that $Y = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ is an unbiased estimator of θ.
   b. Show that the variance of Y is $2\theta^2/n$.
5. Let $X \sim N(0, \theta)$, $0 < \theta < \infty$.
   a. Find the Fisher information $I(\theta)$.
   b. If $X_1, \dots, X_n$ is a random sample from this distribution, show that the mle of θ is an efficient estimator of θ.
   c. What is the asymptotic distribution of $\sqrt{n}\left(\hat{\theta} - \theta\right)$?
6. Let $X_1, \dots, X_n$ be a random sample from the distribution with pdf

$$f_X(x; \theta) = (\theta + 1)(1 - x)^{\theta}, \qquad 0 < x < 1, \qquad \theta > -1$$

   a. Find a sufficient statistic $Y = u(X_1, \dots, X_n)$ for θ.
   b. Determine the Fisher information $I(\theta)$.
7. If $X_1, \dots, X_n$ is a random sample from a distribution with pdf,

$$f_X(x; \theta) = \frac{3\theta^3}{(x + \theta)^4}, \qquad 0 < x < \infty, \qquad 0 < \theta < \infty$$

   Show that $Y = 2\bar{X}$ is an unbiased estimator of θ and determine its efficiency.

8. Let $X_1, \ldots, X_n$ be a random sample from a distribution with pmf $p(x;\theta) = \theta^x(1-\theta)^{1-x}, x = 0, 1$ where $0 < \theta < 1$. We wish to test $H_0: \theta = \frac{1}{3}$ versus $H_a: \theta \neq \frac{1}{3}$. Find expressions for the likelihood ratio Λ and $-2\ln\Lambda$ in simplified forms.
9. Consider again the pmf in Problem 8.
   a. Determine the form of the Wald-type test statistic for H_0 versus H_a.
   b. Determine the form of Rao's score statistic for testing H_0 versus H_a.
10. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with the following pdf:

$$f(x;\theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \qquad \text{zero elsewhere,}$$

where θ is an unknown fixed parameter in the range $0 < \theta < \infty$. Notice that $\theta = 1$ corresponds to the uniform distribution on the interval (0, 1). Therefore we can test for uniformity by testing $H_0: \theta = 1$ versus $H_a: \theta \neq 1$. Find simplified expressions for the likelihood ratio Λ and $-2\ln\Lambda$ for this testing problem.

Graduate Students

11. Let $\bar{X}_n$ be the mean of a random sample of size n from a $N(\theta, \sigma^2)$ distribution, $-\infty < \theta < \infty, \sigma^2 > 0$. Assume that σ^2 is known. Show that $n\,\bar{X}_n^2 - \sigma^2/n$ is an unbiased estimator of θ^2 and find its efficiency.
12. Consider the two pdfs

$$f_1(x) = \begin{cases} \sin(x), & 0 < x < \dfrac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} \cos(x), & 0 < x < \dfrac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

You will have just a single observation of X on which to base your choice between the hypotheses

$$H_0: X \text{ has pdf } f_1(x) \quad vs. \quad H_1: X \text{ has pdf } f_2(x).$$

Use the likelihood ratio to find the best rejection region with the significance level $\alpha = 0.10$, and find the power of the test.