This is new for me !! ☺

1) Jensen's Inequality
2) Chebyshev's Inequality
   - LLN
3) Dists of 2 R.V.'s

M-R
5 - 7:20
Illini Hall

$$\bar{Y}_n = \frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{P} E(X) = \mu$$

$X_1, \ldots, X_n$ independent
$E(X_i) = \mu \;\; \forall i$
$Var(X_i) = \sigma^2 \;\; \forall i$

- Political Polls
- Medical Research

$$\underline{\frac{\sum g(X_i)}{n}} \xrightarrow{P} E\, g(X) \quad \text{as } \underline{n \to \infty}$$

Chebyshev's Inequality

$U(X) = (X-\mu)^2$, using Markov

- $P(|X-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$
- $P(|X-\mu| \geq \underline{\varepsilon\sigma}) \leq \frac{1}{\varepsilon^2}$

$$U(X) = \frac{(X-\mu)^2}{\sigma^2}$$

$$E[g(X)] \ge E[g(\mu)] + E[g'(\mu)(X-\mu)]$$

$$= g(\mu) + g'(\mu)E(X-\mu) = g(\mu) = g[E(X)]$$

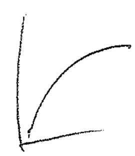
Ex

$\tilde{\theta} = \frac{\Sigma \ln X_i}{n}$, Maybe we can't evaluate $E(\tilde{\theta})$

- $E(e^{tX}) \ge e^{tEX}$
- $E\left(\frac{1}{X}\right) \ge \frac{1}{E(X)}$

For Concave functions, $\frac{d^2 g(X)}{dx^2} \le 0 \quad \forall x \in I$

$$E(g(X)) \le g(E(X))$$

- $E[\ln(X)] \le \ln(E(X))$
- For a nonnegative X, $E(\sqrt{X}) \le \sqrt{E(X)}$ 

- $P(|X-\mu| < \varepsilon) \geq 1 - \frac{\sigma^2}{\varepsilon^2}$
- $P(|X-\mu| < \varepsilon\sigma) \geq 1 - \frac{1}{\varepsilon^2}$

1.6σ units

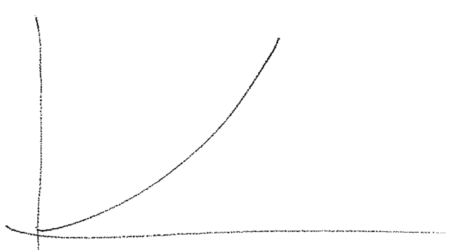
Ex $\mu = EX = 17$, $\sigma = 5$ $\quad (17-8,\ 17+8)$

$$P(9 < X < 25) = P(|X-\mu| < 1.6\sigma) \geq 1 - \frac{1}{1.6^2} \approx .609$$

Jensen's Inequality

If $g(\cdot)$ that is convex on an open interval I and a r.v. X whose support is contained in I and has finite expectation,

$$E[g(X)] > g[E(X)] = g(\mu)$$



convexity $\Rightarrow \dfrac{d^2(g(x))}{dx^2} \geq 0 \quad \forall\, x \in I$

$\boxed{pf}$ Use a 2nd order taylor series of $g(x)$ at μ.

$$g(x) = g(\mu) + g'(\mu)(x-\mu) + \frac{g''(\xi)}{2}(x-\xi)^2$$

$$g(x) = g(\mu) + g'(\mu)(x-\mu) + \frac{g''(\xi)}{2}(x-\xi)^2$$

$$\geq g(\mu) + g'(\mu)(x-\mu)$$

$\cancel{\Rightarrow E(X) \geq}$

Bivariate PMF's and PDF's

- Discrete
- PMF . $P(X=x, Y=y) = P(x,y)$
- A is any set of pairs (x,y). Then

$$P((X,Y) \in A) = \sum_{(x,y)\in A}\sum p(x,y)$$

Ex.

		Y		
X	0	1	2	
1	.15	.10	0	.25
2	.25	.30	.20	.75
	.40	.40	.20	

$P(X>Y)$
$= P(X=1, Y=0) + P(X=2, Y=0)$
$+ P(X=2, Y=1)$
$= .7$

- Marginal Pmfs

$$P_X(x) = \sum_{\text{all } y} p(x,y) \qquad P_Y(y) = \sum_{\text{all } x} p(x,y)$$

- MGFs $M_{X,Y}(t_1, t_2) = E(e^{t_1 X + t_2 Y}) = \sum_{\text{all } y}\sum_{\text{all } x} e^{t_1 x + t_2 y} p(x,y)$

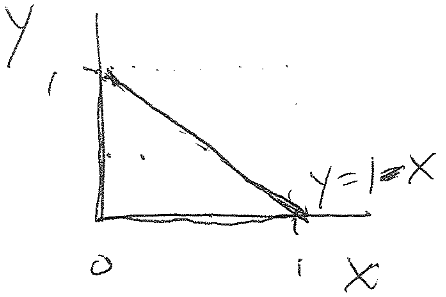
$= .15 e^{t_1 \cdot 1 + t_2 \cdot 0} + .10 e^{t_1 + t_2} + 0 e^{t_1 + 2t_2} + .25 e^{2t_1 + 0 \cdot t_2}$
$+ .30 e^{2t_1 + t_2} + .20 e^{2t_1 + 2t_2}$

Continuous

- $f(x,y)$ is joint pdf.
- $P((X,Y)\in A) = \displaystyle\iint_{(X,Y)\in A} f(x,y)\,dx\,dy$
- $f_X(x) = \int f(x,y)\,dy$; $f_Y(y) = \int f(x,y)\,dx$

Ex $f(x,y) = \begin{cases} 60x^2y, & 0\le x\le 1,\ 0\le y\le 1,\ x+y\le 1 \\ 0 & \text{o.w.} \end{cases}$

$y \le 1-x$

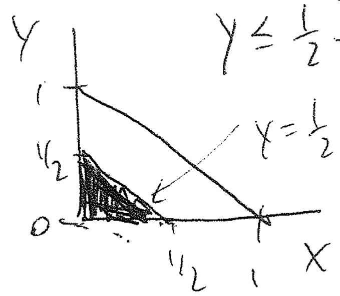


a) Is it valid?

$$\int_0^1\int_0^{1-y} 60x^2y\,dx\,dy = 1 = \int_0^1\int_0^{1-x} 60x^2y\,dy\,dx$$

b) $P(X+Y\le 1/2) = \displaystyle\int_0^{1/2}\int_0^{1/2-x} 60x^2y\,dy\,dx = \frac{1}{32}$

$y \le \frac{1}{2} - x$



c) $P(X\ge Y) = \displaystyle\int_0^{1/2}\int_y^{1-y} 60x^2y\,dx\,dy = \frac{11}{16}$

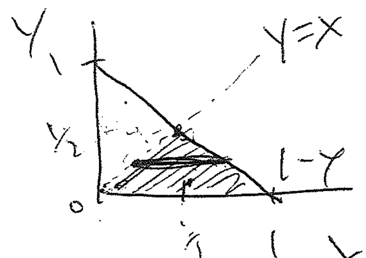
$y=x$, $y=1-x$ $\Rightarrow x=\frac{1}{2}$

OR

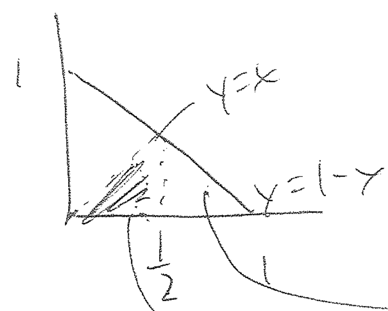
$= 1 - P(X<Y) = 1 - \displaystyle\int_0^{1/2}\int_x^{1-x} 60x^2y\,dy\,dx$

OR

=



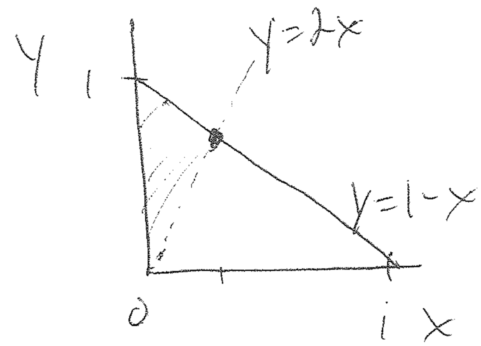
OR



$$= \int_0^{1/2}\int_0^{x} 60x^2y \, dy\,dx + \int_{1/2}^{1}\int_0^{1-x} 60x^2y \, dy\,dx$$

D) $P(Y \geq 2X) = \int_0^{1/3}\int_{2x}^{1-x} 60x^2y \, dy\,dx$

$y = 2x$, $y = 1-x \Rightarrow x = 1/3$

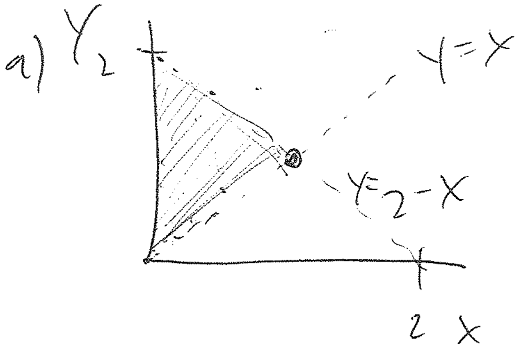


$= \frac{1}{9}$

$f_{(x,y)} = Cx^2y, \quad 0 < x \leq y, \quad x+y < 2$

$y < 2-x$

a)

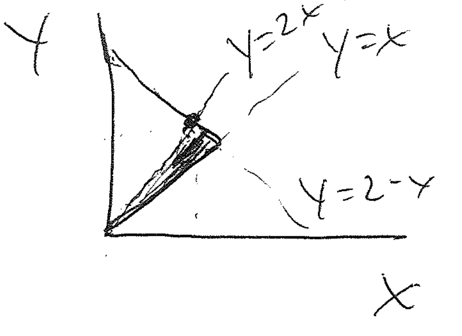


a) Find C

$$\int_0^{1}\int_x^{2-x} Cx^2y \, dy\,dx = 1$$

$\Rightarrow C = 6$

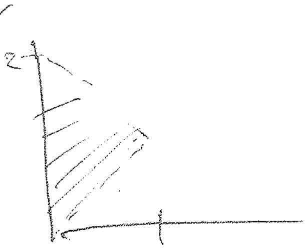
b) $P(Y < 2X) = 1 - P(Y \geq 2X)$



$$= 1 - \int_0^{2/3}\int_{2x}^{2-x} 6x^2y \, dy\,dx = \frac{87}{135}$$

$y = 2x$
$y = 2-2x \quad x = 2/3$

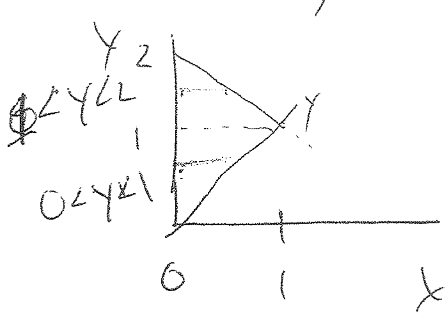
$f(x,y) = 6x^2 y \quad 0<x<y,\ x+y<2$



c) $f_X(x)$?

$$= \int f(x,y)\,dy = \int_x^{2-x} 6x^2 y\,dy = 12x^2(1-x),\ 0<x<1$$

d) $f_Y(y) = \int f(x,y)\,dx$



$$f_Y(y) = \begin{cases} \int_0^y 6x^2 y\,dx, & 0<y<1 \\ \int_0^{2-y} 6x^2 y\,dx, & 1<y<2 \end{cases}$$

$$= \begin{cases} 2y^4, & 0<y<1 \\ 2y(2-y)^3, & 1<y<2 \end{cases}$$

Independence of R.v.s

If X and Y are independent

- $P(X=x, Y=y) = P(X=x)P(Y=y)$
- $f_{XY}(x,y) = f_X(\cancel{x}) f_Y(y)$ [constructing likelihood functions]
- $f_{XY}(x,y) = \dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$
- $E[g(X)\,h(Y)] = E[g(X)]\,E[h(Y)]$

pf
$$= \iint g(x)\,h(y)\,\underline{f(x,y)}\,dx\,dy$$
$$= \iint g(x)\,h(y)\,f_X(x)\,f_Y(y)\,dx\,dy$$
$$= \left(\int g(x)\,f_X(x)\,dx\right)\left(\int h(y)\,f_Y(y)\,dy\right)$$
$$= E(g(X))\,E(h(Y))$$

- $M_{X,Y}(t_1,t_2) = M_X(t)\,M_Y(t)$

pf $g(x) = e^{xt_1}$
$h(y) = e^{yt_2}$

- $M_{X,Y}(t_1,0) = M_X(t)$
$M_{X,Y}(0,t_2) = M_{\cancel{Y}}(t)$