Preparation problems for the discussion sections on October 27th and 29th

1. *Let* $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$ *be a subspace of* $\mathbb{R}^4$.

(i) *Find the closest point to* $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ *on the subspace* W.

(ii) *Find the projection matrix* P *corresponding to the projection onto* W.

(iii) *Use the projection matrix* P *to find the projection of* $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ *onto the subspace* W.

Solution. (i)

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}_W = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

(ii) We have:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_W = \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix},$$

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Midterm Date: November 19th 7-8:15 PM, Final Date: December 17 8am - 11am

$$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}_W = \frac{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{-1}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}-\frac{1}{4}\\\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix},$$

$$\begin{bmatrix}0\\0\\1\\0\end{bmatrix}_W = \frac{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 0\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix},$$

$$\begin{bmatrix}0\\0\\0\\1\end{bmatrix}_W = \frac{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 0\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix},$$

Hence, the projection matrix is:

$$P = \begin{bmatrix}\frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\end{bmatrix}$$

(iii) Using P, we have:

$$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}_W = \begin{bmatrix}\frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$ $\square$

2. *Let* $W = \operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right\}$ *and* $V = \operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix}\right\}$ *be subspaces of* $\mathbb{R}^3$.

(i) *Find the projection matrices, P and Q, corresponding to the projections onto W and V, respectively.*

(ii) *Check that $PQ = QP$. Can you interpret PQ as a projection matrix?*

Solution. (i)

$$\begin{bmatrix}1\\0\\0\end{bmatrix}_W = \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}\frac{5}{6}\\-\frac{1}{6}\\\frac{1}{3}\end{bmatrix},$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix}_W = \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}-\frac{1}{6}\\\frac{5}{6}\\\frac{1}{3}\end{bmatrix},$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix}_W = \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + 0\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix},$$

Hence, the projection matrix corresponding to the orthogonal projection onto W is:

$$P = \begin{bmatrix}\frac{5}{6} & -\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{5}{6} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix}$$

Also,

$$\begin{bmatrix}1\\0\\0\end{bmatrix}_V = \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}{\begin{bmatrix}1\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{bmatrix},$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix}_V = \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}{\begin{bmatrix}1\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{bmatrix},$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix}_V = \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}{\begin{bmatrix}1\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

Hence, the projection matrix corresponding to the orthogonal projection onto V is:

$$P = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}$$

(ii) $PQ = QP$ is the matrix corresponding to the orthogonal projection onto $W \cap V = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.[Note: since $\begin{bmatrix}1\\-1\\0\end{bmatrix}^T\begin{bmatrix}1\\1\\-2\end{bmatrix} = 0$ if you compute orthogonal projection onto W and then onto V the answer will be same as computing orthogonal projection onto V and then onto W] $\square$

3. *Let* $A = \begin{bmatrix}1 & -1\\ 1 & 1\\ 0 & 0\end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix}4\\5\\6\end{bmatrix}$.

- **a.** *Does* $\mathbf{b}$ *belong to the columnspace of* A*? Can you solve* $A\mathbf{x} = \mathbf{b}$*?*
- **b.** *What do you expect the projection of* $\mathbf{b}$ *onto*$W = \text{Col}(A)$ *to be?*
- **c.** *Find the projection* $\hat{\mathbf{b}}$ *of* $\mathbf{b}$ *onto* $\text{Col}(A)$*, and then solve* $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$*.(The vector* $\hat{\mathbf{x}}$ *is called* the least square solution *of* $A\mathbf{x} = \mathbf{b}$*.)*
- **d.** *Solve the equation*$A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$*.Compare with your result of the previous part! (This equation is called the* normal equation *of* $A\mathbf{x} = \mathbf{b}$*.)*
- **e.** *Answer these questions for* A *as above but with* $\mathbf{b} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ *(and then*$\mathbf{b} = \begin{bmatrix}0\\0\\4\end{bmatrix}$*).*

Solution. **a.** No, $\mathbf{b}$ does not belong to the column space of A, because it is not a linear combination of $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\0\end{bmatrix}$. Hence there is no solution to $A\mathbf{x} = \mathbf{b}$.

b. W is the span of $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\0\end{bmatrix}$. It can easily be seen that W is the set of vectors in $\mathbb{R}^3$

whose third entry is 0. Hence $\begin{bmatrix}4\\5\\0\end{bmatrix}$ is in W. Note

$$\begin{bmatrix}0\\0\\6\end{bmatrix} = \mathbf{b} - \begin{bmatrix}4\\5\\0\end{bmatrix}$$

is orthogonal to W. Hence $\begin{bmatrix}4\\5\\0\end{bmatrix}$ should be the orthogonal projection of $\mathbf{b}$ onto W.

c. Step 1: Find the orthogonal projection of $\mathbf{b}$ onto W.

$$\hat{\mathbf{b}} = \frac{\begin{bmatrix}4\\5\\6\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix}} \begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{\begin{bmatrix}4\\5\\6\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\0\end{bmatrix}}{\begin{bmatrix}-1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\0\end{bmatrix}} \begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}4.5\\4.5\\0\end{bmatrix} + \begin{bmatrix}-.5\\.5\\0\end{bmatrix} = \begin{bmatrix}4\\5\\0\end{bmatrix}.$$

Step 2: Solve $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$.

$$\left[\begin{array}{cc|c}1 & -1 & 4\\1 & 1 & 5\\0 & 0 & 0\end{array}\right] \xrightarrow{R2 \to R2 - R_1} \left[\begin{array}{cc|c}1 & -1 & 4\\0 & 2 & 1\\0 & 0 & 0\end{array}\right] \xrightarrow{R1 \to R1 + .5R2} \left[\begin{array}{cc|c}1 & 0 & 4.5\\0 & 2 & 1\\0 & 0 & 0\end{array}\right].$$

Hence,

$$\hat{\mathbf{x}} = \begin{bmatrix}4.5\\.5\end{bmatrix}.$$

d. We first calculate A^TA and $A^T\mathbf{b}$:

$$A^TA = \begin{bmatrix}1 & 1 & 0\\-1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix},$$

$$A^T\mathbf{b} = \begin{bmatrix}1 & 1 & 0\\-1 & 1 & 0\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}9\\1\end{bmatrix}.$$

Now we have to solve

$$\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\hat{\mathbf{x}} = \begin{bmatrix}9\\1\end{bmatrix}.$$

It is easy to check that then

$$\hat{\mathbf{x}} = \begin{bmatrix}4.5\\.5\end{bmatrix}.$$

e. I leave the details to you, but here are the solutions. For $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$:

$$\hat{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Note that in this case $\hat{\mathbf{b}} = \mathbf{b}$ and $\hat{\mathbf{x}}$ is a solution (not just a least square solution) of $A\mathbf{x} = \mathbf{b}$.
For $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$:

$$\hat{\mathbf{b}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \hat{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

□

4. *Let* $A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \\ 2 & 2 \end{bmatrix}$ *and* $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$*.Find the least squares solution* $\hat{\mathbf{x}}$ *of* $A\mathbf{x} = \mathbf{b}$*.*

Solution. We first calculate $A^T A$ and $A^T\mathbf{b}$:

$$A^T A = \begin{bmatrix} 0 & -2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 9 \end{bmatrix},$$

$$A^T\mathbf{b} = \begin{bmatrix} 0 & -2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Now we have to solve

$$\begin{bmatrix} 8 & 0 \\ 0 & 9 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

It is easy to check that then

$$\hat{\mathbf{x}} = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{3} \end{bmatrix}.$$

□

5. *A scientist tries to find therelation between the mysterious quantities x and y. She measuresthe following values:*

x	1	2	3	4
y	2	5	9	17

(i) *Suppose that y is a linear function of x of the form $a+bx$. Setup the system of equations to find the coefficients a and b.*
(ii) *Find the best estimate for the coefficients a and b.*
(iii) *Same question if we suppose that y is aquadratic function of the $a + bx + cx^2$.*

Solution. (i) We set up the equation as follows:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix}.$$

(ii) We calculate

$$\begin{bmatrix}1&1\\1&2\\1&3\\1&4\end{bmatrix}^T\begin{bmatrix}1&1\\1&2\\1&3\\1&4\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&2&3&4\end{bmatrix}\begin{bmatrix}1&1\\1&2\\1&3\\1&4\end{bmatrix}=\begin{bmatrix}4&10\\10&30\end{bmatrix}$$

and

$$\begin{bmatrix}1&1\\1&2\\1&3\\1&4\end{bmatrix}^T\begin{bmatrix}2\\5\\9\\17\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&2&3&4\end{bmatrix}\begin{bmatrix}2\\5\\9\\17\end{bmatrix}=\begin{bmatrix}33\\107\end{bmatrix}$$

Now we solve

$$\left[\begin{array}{cc|c}4&10&33\\10&30&107\end{array}\right]\xrightarrow{R2\to R2-2.5R_1}\left[\begin{array}{cc|c}4&10&33\\0&5&24.5\end{array}\right]\xrightarrow{R1\to R1-2R2}\left[\begin{array}{cc|c}4&0&-16\\0&5&24.5\end{array}\right].$$

Hence $a=-4$ and $b=4.9$.
(iii) We set up the equation as follows:

$$\begin{bmatrix}1&1&1\\1&2&4\\1&3&9\\1&4&16\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}2\\5\\9\\17\end{bmatrix}.$$

We calculate

$$\begin{bmatrix}1&1&1\\1&2&4\\1&3&9\\1&4&16\end{bmatrix}^T\begin{bmatrix}1&1&1\\1&2&4\\1&3&9\\1&4&16\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&4&9&16\end{bmatrix}\begin{bmatrix}1&1&1\\1&2&4\\1&3&9\\1&4&16\end{bmatrix}=\begin{bmatrix}4&10&30\\10&30&100\\30&100&354\end{bmatrix}$$

and

$$\begin{bmatrix}1&1&1\\1&2&4\\1&3&9\\1&4&16\end{bmatrix}^T\begin{bmatrix}2\\5\\9\\17\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&4&9&16\end{bmatrix}\begin{bmatrix}2\\5\\9\\17\end{bmatrix}=\begin{bmatrix}33\\107\\375\end{bmatrix}.$$

One can row reduce

$$\left[\begin{array}{ccc|c}4&10&30&33\\10&30&100&107\\30&100&354&375\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&0&2.25\\0&1&0&-1.35\\0&0&1&1.25\end{array}\right].$$

So $a=2.25$, $b=-1.35$ and $c=1.25$. □

6. *The system of the equations* $A\mathbf{x}=\mathbf{b}$ *with*

$$A=\begin{bmatrix}1&-1\\1&0\\1&1\\1&2\end{bmatrix}\quad \text{and}\quad \mathbf{b}=\begin{bmatrix}5\\0\\5\\10\end{bmatrix},$$

is not consistent.

(i) *Find the least squares solution* $\hat{\mathbf{x}}$ *for the equation* $A\mathbf{x}=\mathbf{b}$.

(ii) *Determine the least squares line for the data points* $(-1,5),(0,0),(1,5),(2,10)$.

Solution. (i) We first calculate A^TA and $A^T\mathbf{b}$:

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix},$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}.$$

Now we have to solve

$$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}.$$

We have

$$\left[\begin{array}{cc|c} 4 & 2 & 20 \\ 2 & 6 & 20 \end{array}\right] \xrightarrow{R2 \to R2 - 1/2R1} \left[\begin{array}{cc|c} 4 & 2 & 20 \\ 0 & 5 & 10 \end{array}\right].$$

Hence,

$$\hat{\mathbf{x}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

(ii) Denoting the least squares line as $y = ax + b$, we have to find $\begin{bmatrix} b \\ a \end{bmatrix}$ so that $\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix}$ is the closest possible value to $\begin{bmatrix} 5 \\ 0 \\ 5 \\ 10 \end{bmatrix}$. From the first part, $\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Hence, the least squares line is $y = ax + b = 2x + 4$. $\square$