1) Conditional Dists.　　　　　　　　　　　9/15/15

- Expectations
- Probabilities
- Properties

2) Bivariate Transformations

Properties

1) $E(a_1 X_1 + a_2 X_2 \mid Y) = E(a_1 X_1 \mid Y) + E(a_2 X_2 \mid Y)$

$= a_1 E(X_1 \mid Y) + a_2 E(X_2 \mid Y)$

2) $E[g(Y) \mid Y] = g(Y)$

3) $E[g(Y) X \mid Y] = g(Y) E(X \mid Y)$

4) Law of iterated expectations

$$E[\underbrace{E(X \mid Y)}_{\text{a function of } Y}] = E(X)$$

5) X and Y independent

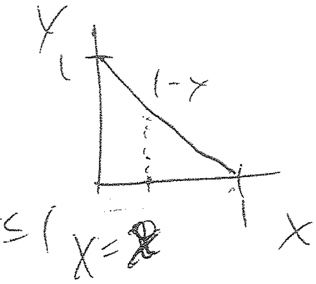
$E(Y \mid X) = E(Y)$

pf $E[E(X \mid Y)] = \int E(X \mid y) f_Y(y)\, dy$

$= \int \left\{ \int x f_{X|Y}(x \mid y)\, dx \right\} f_Y(y)\, dy$　　　Note: $f(x|y) f_Y(y) = f(x,y)$]*

$= \int x \left\{ \int f(x,y)\, dy \right\} dx = \int x f_X(x)\, dx = E(X)$

1) $f_{X,Y}(x,y) = \begin{cases} 60x^2 y & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0 & \text{o.w.} \end{cases}$



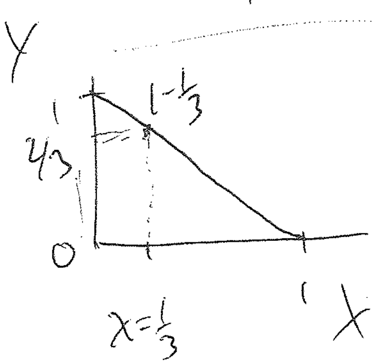
$$f_X(x) = \int_0^{1-x} 60x^2 y \, dy = 30x^2(1-x)^2, \quad 0 \le x \le 1$$

a) Find $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{60x^2 y}{30x^2(1-x)^2}, \quad 0 < y < 1-x$

$= \cancel{60x^2}$

$$= \frac{2y}{(1-x)^2}, \quad 0 < y < 1-x$$

b) $P(Y > \frac{1}{4} \mid X = \frac{1}{3}) = \displaystyle\int_{1/4}^{2/3} \frac{2y}{(\frac{2}{3})^2} \, dy = \frac{55}{64}$



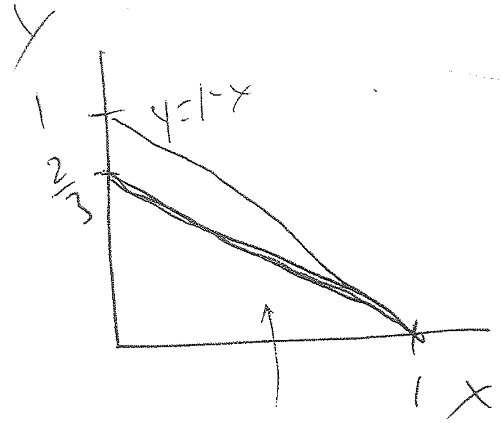
c) $E(Y|X=x) = \displaystyle\int_0^{1-x} y f_{Y|X}(y|x) \, dy = \int_0^{1-x} \frac{2y^2}{(1-x)^2} \, dy$

$$= \frac{2}{3}(1-x), \quad 0 \le x \le 1$$

$E(Y|X)$ is linear in x.

We can use simple linear regression to relate X to Y.

$$E(Y|X) = \mu_Y + \underbrace{\rho_{XY}\frac{\sigma_Y}{\sigma_X}}_{\text{Slope}}(X - \mu_X)$$



$$E(Y|X) = \frac{2}{3} - \frac{2}{3}X$$

$\boxed{\text{Check}}$

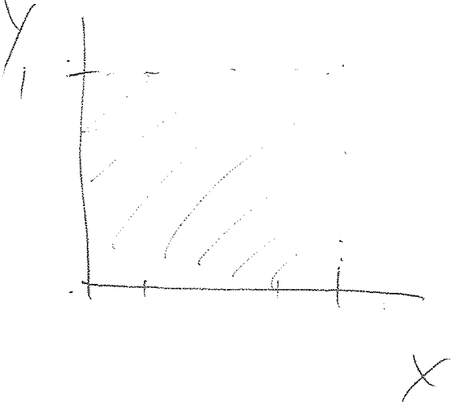
$$\mu_X = \frac{1}{2}, \quad \mu_Y = \frac{1}{3} \qquad \rho_{XY} = -\frac{1}{\sqrt{2}}$$

$$\sigma_X^2 = \frac{9}{252}, \quad \sigma_Y^2 = \frac{8}{252}$$

$$E(Y|X) = \frac{1}{3} + \left(-\frac{1}{\sqrt{2}}\right)\frac{\sqrt{\frac{8}{252}}}{\sqrt{\frac{9}{252}}}\left(X - \frac{1}{2}\right)$$

$$= \frac{1}{3} - \frac{2}{3}\left(X - \frac{1}{2}\right) = \frac{2}{3} - \frac{2}{3}X$$

2) $f(x,y) = \begin{cases} x+y & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$

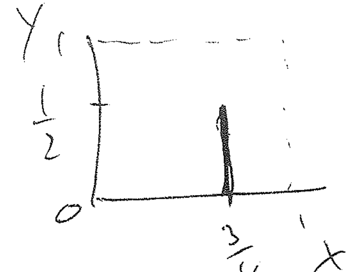


Recall: $f_X(x) = x + \frac{1}{2},\ 0 \le x \le 1$

$f_Y(y) = y + \frac{1}{2},\ 0 \le y \le 1$

a) $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f(x)} = \dfrac{x+y}{x+\frac{1}{2}},\ 0 \le y \le 1$

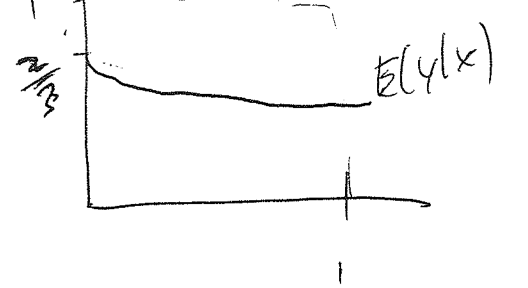
b) Find $P(Y \le \frac{1}{2} \mid X = \frac{3}{4})$



$$= \int_0^{\frac{1}{2}} \left(\frac{\frac{3}{4} + y}{\frac{3}{4} + \frac{1}{2}} \right) dy = \frac{2}{5}$$

c) $E(Y|x) = \displaystyle\int_0^1 y\, f_{Y|X}(y|x)\, dy = \int_0^1 y \left(\frac{x+y}{x+\frac{1}{2}} \right) dy$

$$= \frac{3x+2}{6x+3},\ 0 \le x \le 1 \quad \text{not linear in } X$$

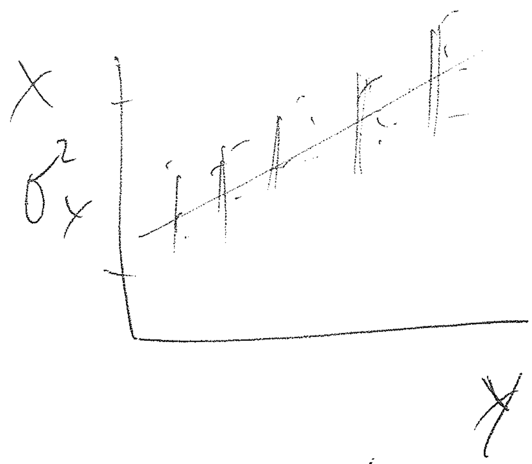


Linear relationship is missspecified.

Conditional Variance Formula

$$\underset{\substack{\uparrow \\ \text{unconditional} \\ \text{variance}}}{Var(X)} = \underbrace{Var[E(X|Y)]}_{\substack{\text{Variability in } X \\ \text{that is explained} \\ \text{by } Y.}} + \underbrace{E[Var(X|Y)]}_{\substack{\text{Variability in } X \\ \text{that is not explained} \\ \text{by } Y.}}$$

- political opinions
- sales
- behavior



Ex. Hierarchical Model

$$X|\lambda \sim Exp(\lambda)$$
$$\lambda \sim U(0,1).$$

$$\begin{aligned}
Var(X) &= Var[E(X|\lambda)] + E[Var(X|\lambda)] \\
&= Var(\lambda) + E(\lambda) \\
&= \frac{1}{12} + \frac{1}{2} = \frac{7}{12}
\end{aligned}$$

$$f_{xy} = \begin{cases} 60x^2y & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu_x = \frac{1}{2},\ \mu_y = \frac{1}{3}$$

$$\sigma_x^2 = \frac{9}{252},\ \sigma_y^2 = \frac{8}{252}$$

a) Find $E(Var(Y|X)) \leftarrow$ expected residual variance

We know $E(Var(Y|X)) = Var(Y) - Var(E(Y|X))$

$$= \frac{8}{252} - Var\left\{\frac{2}{3}(1-x)\right\}$$

$$= \frac{8}{252} - \left[Var\left(\frac{2}{3}\right) + var\left(-\frac{2}{3}x\right)\right]$$

$$= \frac{8}{252} - \frac{4}{9}Var(X) = \frac{8}{252} - \frac{4}{9}\frac{9}{252} = \frac{4}{252}$$

b) law of iterated expectation

$$E[E(Y|X)] = E(Y)$$

$$E\left[\frac{2}{3}(1-x)\right] = \frac{2}{3} - \frac{2}{3}Ex = \frac{2}{3} - \frac{2}{3}\frac{1}{2} = \frac{2}{3} - \frac{1}{3}$$

$$= \frac{1}{3}$$

Bivariate Transformations

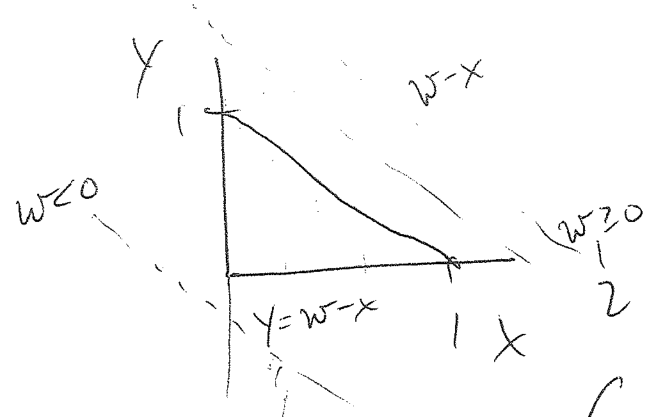
1) $f(x,y) = \begin{cases} 60x^2y & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0 & \text{o.w.} \end{cases}$

a) Let $W = X+Y$. Find $f_W(w)$!

i) Find $F_W(w) = P(W \le w) = P(X+Y \le w)$

$P(Y \le w-x)$

2+ii) $f_W(w) = F_W'(w)$



Need $F_W(w)\ \forall\ w$

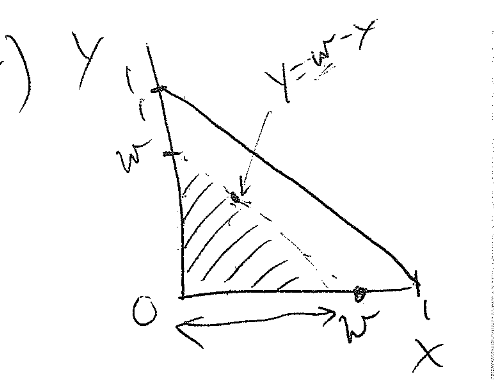
$$F_W(w) = \begin{cases} 0 & , w < 0 \\ w^5 & , 0 \le w < 1 \\ 1 & , w \ge 1 \end{cases}$$

$w \ge 2$

For $0 \le w < 1$,

$$F_W(w) = P(W \le w) = P(X+Y \le w)$$

$$= \int_0^w \int_0^{w-x} 60x^2y\, dy\, dx = w^5,\ 0 \le w < 1$$



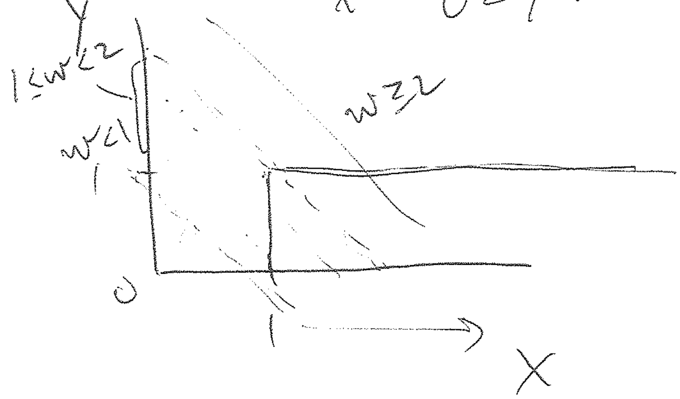
$$\Rightarrow f_W(w) = F_W'(w) = 5w^4,\ 0 \le w < 1 \quad //$$

2) Let X and Y be independent

$Y \sim U(0,1)$

$f(x) = 2/x^3, \ x>1$

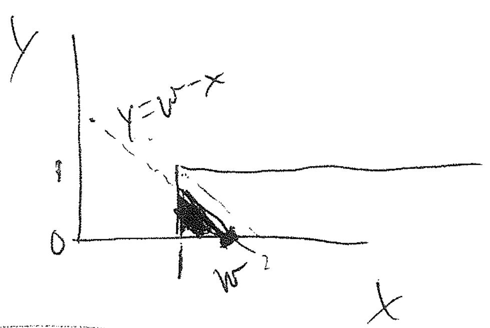
$\Rightarrow f(x,y) = \dfrac{2}{x^3}, \ x>1, \ 0<y<1$



a) $W = X+Y$. Find $f_W(w)$.

$$F_W(w) = \begin{cases} 0, & w<1 \\ \frac{1}{w}+w-2, & 1 \le w < 2 \\ 1-\frac{1}{w-1}+\frac{1}{w}, & w \ge 2 \end{cases}$$

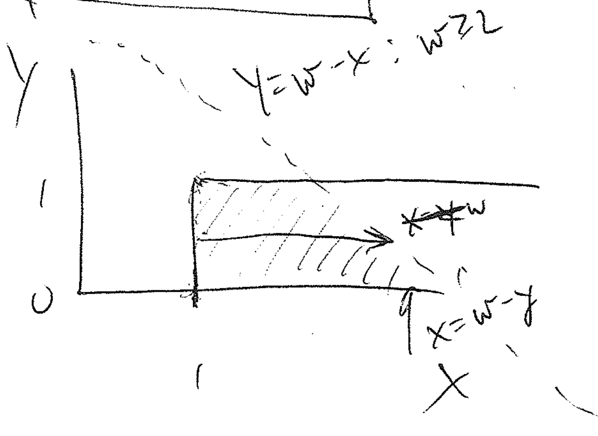
Case: $1 \le w < 2$ $\quad F_W(w) = P(W \le w) = P(X+Y \le w)$



$$= \int_1^w \int_0^{w-x} \frac{2}{x^3}\,dy\,dx = \frac{1}{w}+w-2,$$

$1 \le w < 2$

Case: $w \ge 2$:



$$F_W(w) = P(X+Y \le w) = \int_0^1 \int_1^{w-y} \frac{2}{x^3}\,dx\,dy = 1-\frac{1}{w-1}+\frac{1}{w},$$

$w \ge 2$