# Worksheet 7 for October 13th and 15th

**1.** *Let* $A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.

(a) *Find an echelon form* $U$ *of* $A$. *What are the columnspaces* Col($A$),Col($U$)*? Are they equal?*
(b) *Find a basis for* Col($U$) *and a basis for* Col($A$).
(c) *What are the row spaces* Col($A^T$), *and*Col($U^T$). *Are they equal?*
(d) *Find a basis for the row space of* $A$, Col($A^T$).

*Solution.* (a) We have:

$$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 2 \\ 1 & 2 & 5 \end{bmatrix} \xrightarrow{R2 \to R2-4R1, R3 \to R3-R1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{R3 \to R3+2R2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\mathrm{Col}(A)=\mathrm{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \right\}$ and $\mathrm{Col}(U) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}$. They are not equal since the third entry of any vector in Col($U$) is equal to 0 and in particular, the first column of $A$ is not in Col($U$).

(b) Since the first column and the third column are pivot columns, a basis for Col($A$) is $\left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \right\}$; and a basis for Col($U$) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}$.

(c) Col($A^T$) and Col($U^T$) are equal since each row of $U$ is a linear combination of rows of $A$ and vice versa. We have:

$$\mathrm{Col}(A^T) = \mathrm{Col}(U^T) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right\}$$

(the vectors given above are the transposes of the rows of $U$).

(d) Non-zero rows of $U$ (i.e., the rows that contain the pivots) form a basis for Col($A^T$). Hence, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right\}$ is a basis for Col($A^T$). □

**2.** *Let* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *be the linear transformation with*

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(i) *Consider the basis* $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ *of* $\mathbb{R}^2$ *and the basis* $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ *of* $\mathbb{R}^3$.*Determine the matrix* $A$ *which represents* $T$ *with respect to the bases* $\mathcal{B}_1$ *and* $\mathcal{B}_2$.*Do you have* $T(\boldsymbol{x}) = A\boldsymbol{x}$ *for all* $\boldsymbol{x} \in \mathbb{R}^2$*?*

(ii) *Consider the basis* $\mathcal{C}_1 := \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ *of* $\mathbb{R}^2$ *and the basis* $\mathcal{C}_2 = \left\{ \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ *of* $\mathbb{R}^3$.*Determine the matrix* $B$ *which represents* $T$ *with respect to the bases* $\mathcal{C}_1$ *and* $\mathcal{C}_2$. *Compute* $T(\mathbf{v})$ *where the coordinate vector of* $\mathbf{v}$ *with respect to the basis* $\mathcal{C}_1$ *is* $\mathbf{v}_{\mathcal{C}_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

*Solution.* (In these solutions, we use colors for scalars to help the reader keep track of them) For (i),

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$
$$= \underbrace{\frac{5}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{lin. combo. of vectors from } \mathcal{B}_2}.$$

and

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(-\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = -\frac{1}{2}T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = -\frac{1}{2}\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$
$$= \underbrace{-\frac{5}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{lin. combo. of vectors from } \mathcal{B}_2}.$$

Then

$$A = \begin{bmatrix} \frac{5}{2} & -\frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Yes, in this case $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$ (since both $\mathcal{B}_1$ and $\mathcal{B}_2$ are standard basis). For example,

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} \frac{5}{2} & -\frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

For (ii),

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}5\\0\\1\end{bmatrix} = \underbrace{1\begin{bmatrix}5\\0\\1\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{lin. combo. of vectors from } \mathcal{C}_2},$$

and

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix} = \underbrace{0\begin{bmatrix}5\\0\\1\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{lin. combo. of vectors from } \mathcal{C}_2}.$$

Then

$$B = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}.$$

No, $T(\mathbf{x}) \neq B\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^2$. For example,

$$\begin{bmatrix}5\\0\\1\end{bmatrix} = T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) \neq \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

□

**3.** *In this problem we consider the bases $\mathcal{B} = \{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ and $\mathcal{C} = \{1, t, t^2, t^3, t^4\}$ of $\mathbb{P}_4$.*

(a) *Let $I\colon \mathbb{P}_3 \to \mathbb{P}_4$ be the lineartransformation that maps a polynomial $p(t)$ to the polynomial*

$$I(p(t)) := \int_0^t p(s)ds,$$

*(e.g., $I(t^2 + 2t) = \int_0^t (s^2 + 2s)ds = [\frac{1}{3}s^3 + s^2]_0^t = \frac{1}{3}t^3 + t^2 \in \mathbb{P}_4$).Determine the matrix which represents $I$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.*

(b) *Let $J\colon \mathbb{P}^3 \to \mathbb{P}^4$ be the lineartransformation that maps a polynomial $p(t)$ to the polynomial*

$$J(p(t)) := tp(t) + p'(t).$$

*Determine the matrix which represents $J$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.*

*Solution.* (a) In our calculation below, we calculate going down $I(1), I(t), I(t^2), I(t^3)$ (i.e., $I$ applied to all elements of the basis $\mathcal{B}$ of $\mathbb{P}_3$) in terms of the basis $\mathcal{C}$ of $\mathbb{P}_4$ (going across). For the readers convenience, we put all scalars in color.

$$\begin{aligned}
I(1) = t \quad &= \quad 0 \cdot 1 + 1 \cdot t + 0 \cdot t^2 + 0 \cdot t^3 + 0 \cdot t^4\\
I(t) = \frac{1}{2}t^2 \quad &= \quad 0 \cdot 1 + 0 \cdot t + \frac{1}{2} \cdot t^2 + 0 \cdot t^3 + 0 \cdot t^4\\
I(t^2) = \frac{1}{3}t^3 \quad &= \quad 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 + \frac{1}{3} \cdot t^3 + 0 \cdot t^4\\
I(t^3) = \frac{1}{4}t^4 \quad &= \quad 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 + 0 \cdot t^3 + \frac{1}{4} \cdot t^4.
\end{aligned}$$

Hence the matrix which represents $I$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$, is

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

(b) We have:

$$\begin{aligned} J(1) &= t \cdot 1 - 0 = 0 \cdot 1 + 1 \cdot t + 0 \cdot t^2 + 0 \cdot t^3 + 0 \cdot t^4 \\ J(t) &= t^2 + 1 = 1 \cdot 1 + 0 \cdot t + 1 \cdot t^2 + 0 \cdot t^3 + 0 \cdot t^4 \\ J(t^2) &= t^3 + 2 \cdot t = 0 \cdot 1 + 2 \cdot t + 0 \cdot t^2 + 1 \cdot t^3 + 0 \cdot t^4 \\ J(t^3) &= t^4 + 3 \cdot t^2 = 0 \cdot 1 + 0 \cdot t + 3 \cdot t^2 + 0 \cdot t^3 + 1 \cdot t^4 \end{aligned}$$

Therefore, the matrix $A$ that represent $I$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ is: (we put coefficients of $J(1), J(t), J(t^2)$, and $J(t^3)$ respectively in the first, second, third, and forth column)

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \square$$

**4.** *Let* $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$*.Find the length of* $\mathbf{v}$*. Find avector* $\mathbf{u}$ *in the direction of* $\mathbf{v}$ *that has length* $1$ *. Find a vector* $\mathbf{w}$ *that isorthogonal to* $\mathbf{v}$*.*

*Solution.* The length of $\mathbf{v}$ is $\sqrt{1^2 + 1^2} = \sqrt{2}$. Since $\mathbf{u} = a\mathbf{v}$, we have to find $a$ so that length of $\mathbf{u}$ is 1. So:

$$\sqrt{a^2 + a^2} = 1$$

Thus, $a = \frac{1}{\sqrt{2}}$ and we have:

$$\mathbf{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For a vector $\mathbf{y}$ orthogonal to $\mathbf{u}$, we need to find $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ such that

$$0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1 + y_2$$

One pair $y_1, y_2$ that satisfies the equation is $1, -1$. So the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is orthogonal to $\mathbf{v}$. $\square$

**5.** *True or False? Justify your answers.*

(a) *The map* $T\colon \mathbb{R}^2 \to \mathbb{R}$ *given by* $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \sqrt{a^2 + b^2}$ *is a linear transformation.*

(b) *The map* $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} -b \\ a \end{bmatrix}$ *is a linear transformation.*

(c) *If* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *are such that* $\mathbf{u}\cdot\mathbf{v} = 0$ *then* $\mathbf{u}$ *and* $\mathbf{v}$ *are perpendicular (geometrically) to each other. (Hint: Plot* $\mathbf{u}$ *and* $\mathbf{v}$ *as rays coming out of the origin, and the "hypotenuse"* $\mathbf{u} - \mathbf{v}$*. The Pythagorean theorem will hold if this is a right triangle.)*

(d) *Let* $V$ *be a subspace of* $\mathbb{R}^n$ *and* $\mathbf{u}, \mathbf{v}$ *be two vectors in* $V$*, then* $\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$ *is orthogonal to* $\mathbf{u}$*.*

(e) *Let* $T : V \to W$ *be a linear transformation and* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *be vectors in* $V$*. If* $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ *are linearly independent then* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *are also linearly independent.*

(f) *Let* $T : V \to W$ *be a linear transformation and* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *be vectors in* $V$*. If* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *are linearly independent then* $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ *are also linearly independent.*

*Solution.* (a) False, since we have:

$$T\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) \neq -T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

(b) True. Since we have:

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} a' \\ b' \end{bmatrix}\right) = \begin{bmatrix} -b \\ a \end{bmatrix} + \begin{bmatrix} -b' \\ a' \end{bmatrix} = \begin{bmatrix} -(b+b') \\ a+a' \end{bmatrix} = T\left(\begin{bmatrix} a+a' \\ b+b' \end{bmatrix}\right),$$

$$T\left(\begin{bmatrix} ra \\ rb \end{bmatrix}\right) = \begin{bmatrix} -rb \\ ra \end{bmatrix} = r\begin{bmatrix} -b \\ a \end{bmatrix} = rT\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$$

(c) True, let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^2$. Then:

$$[length(\mathbf{a}-\mathbf{b})]^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} = [length(\mathbf{a})]^2 + [length(\mathbf{b})]^2$$

Thus by Pythagorean theorem, $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other.

(d) True, Since we have:

$$\mathbf{u}\cdot\left(\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) = \mathbf{u}\cdot\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\cdot\mathbf{u} = 0$$

(e) True, since if $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = 0$ then $x_1T(\mathbf{v}_1) + x_2T(\mathbf{v}_2) + \ldots + x_nT(\mathbf{v}_n) = 0$, but $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ are linearly independent so all $x_i$s are equal to 0.

(f) False, consider $T : \mathbb{R}^3 \to \mathbb{R}$ such that $T(\mathbf{v}) = 0$. □

**6.** *Let* $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, *and* $\mathbf{v} = \begin{bmatrix}2\\3\end{bmatrix}$. *Find real numbers* $c_1, c_2$ *such that*

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$$

*Solution.* Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal (i.e. $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$), we have that if

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$$

for some real number $c_1, c_2$, then

$$\mathbf{u}_1 \cdot \mathbf{v} = c_1\mathbf{u}_1 \cdot \mathbf{u}_1 + c_2\mathbf{u}_1 \cdot \mathbf{u}_2 = c_1\mathbf{u}_1 \cdot \mathbf{u}_1$$

and

$$\mathbf{u}_2 \cdot \mathbf{v} = c_1\mathbf{u}_2 \cdot \mathbf{u}_1 + c_2\mathbf{u}_2 \cdot \mathbf{u}_2 = c_2\mathbf{u}_2 \cdot \mathbf{u}_2.$$

Hence

$$c_1 = \frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1} = \frac{\begin{bmatrix}2\\3\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}}{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}} = \frac{5}{\sqrt{2}}.$$

and

$$c_2 = \frac{\mathbf{u}_2 \cdot \mathbf{v}}{\mathbf{u}_2 \cdot \mathbf{u}_2} = \frac{\begin{bmatrix}2\\3\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}}{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}} = \frac{-1}{\sqrt{2}}.$$

□

**7.** *Let* $B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

(a) *Find a basis for* Nul$(B)$.
(b) *Find two linear independent vectors that are orthogonal to* Nul$(B)$.
(c) *Is there a non-zero vector in* $\mathbb{R}^2$ *orthogonal to* Col$(B)$*?*

*Solution.* (a) We bring $B$ to reduced echelon form:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Hence if $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ is in Nul$(B)$, we have

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$$

Hence $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathrm{Nul}(B)$.

(b) The row space of $B$ is orthogonal to $\mathrm{Nul}(B)$. Hence it is enough to find a basis of $\mathrm{Row}(B)$.

$$B^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 - R1, R4 \to R4 - R1} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathrm{Row}(B)$. Thus $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent and each one is orthogonal to $\mathrm{Nul}(B)$.

(c) By part (a) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ are the pivot columns of $B$ and hence form a basis of $\mathrm{Col}(B)$. Hence $\dim \mathrm{Col}(B) = 2$ and so $\mathbb{R}^2 = \mathrm{Col}(B)$. Hence a vector $\mathbf{v}$ that is orthogonal to $\mathrm{Col}(B)$, is orthogonal to every vector in $\mathbb{R}^2$. In particular, $\mathbf{v}$ is orthogonal to itself. That is $\mathbf{v} \cdot \mathbf{v} = 0$. But then $\mathbf{v} = 0$. Hence there is no non-zero vector orthogonal to $\mathrm{Col}(B)$. $\square$

**8.** *Consider the three vectors*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}.$$

(a) *Check that* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *form an orthogonal set of vectors and conclude that they form a basis for* $\mathbb{R}^3$.

(b) *Construct an orthonormal basis* $\mathcal{B}$ *for* $\mathbb{R}^3$ *by normalizing the vectors in* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(c) *Compute the coordinate vector* $[\mathbf{v}]_{\mathcal{B}}$ *for the following vectors (hint: use the fact that* $\mathcal{B}$ *is an orthonormal basis):*

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

*Solution.* (a) We compute:

$$\mathbf{v}_1^T \mathbf{v}_2 = \mathbf{v}_1^T \mathbf{v}_3 = \mathbf{v}_2^T \mathbf{v}_3 = 0$$

Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form an orthogonal set of 3 nonzero vectors in $\mathbb{R}^3$. This implies that they are linearly independent and must also span all of $\mathbb{R}^3$ and so they are a basis.

(b) Normalizing $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ gives

$$\mathcal{B} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) We compute:

$$\begin{aligned}
\begin{bmatrix}1\\0\\0\end{bmatrix} &= \left(\begin{bmatrix}1\\0\\0\end{bmatrix}^T \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \left(\begin{bmatrix}1\\0\\0\end{bmatrix}^T \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \left(\begin{bmatrix}1\\0\\0\end{bmatrix}^T \begin{bmatrix}0\\0\\1\end{bmatrix}\right)\begin{bmatrix}0\\0\\1\end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\end{bmatrix} \quad \Rightarrow \left[\begin{bmatrix}1\\0\\0\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} \\
\begin{bmatrix}0\\1\\0\end{bmatrix} &= \left(\begin{bmatrix}0\\1\\0\end{bmatrix}^T \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \left(\begin{bmatrix}0\\1\\0\end{bmatrix}^T \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \left(\begin{bmatrix}0\\1\\0\end{bmatrix}^T \begin{bmatrix}0\\0\\1\end{bmatrix}\right)\begin{bmatrix}0\\0\\1\end{bmatrix} \\
&= -\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\end{bmatrix} \quad \Rightarrow \left[\begin{bmatrix}0\\1\\0\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} \\
\begin{bmatrix}0\\0\\1\end{bmatrix} &= \left(\begin{bmatrix}0\\0\\1\end{bmatrix}^T \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \left(\begin{bmatrix}0\\0\\1\end{bmatrix}^T \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \left(\begin{bmatrix}0\\0\\1\end{bmatrix}^T \begin{bmatrix}0\\0\\1\end{bmatrix}\right)\begin{bmatrix}0\\0\\1\end{bmatrix} \\
&= 0\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} + 0\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\end{bmatrix} \quad \Rightarrow \left[\begin{bmatrix}0\\0\\1\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}0\\0\\1\end{bmatrix} \\
\begin{bmatrix}1\\2\\3\end{bmatrix} &= \left(\begin{bmatrix}1\\2\\3\end{bmatrix}^T \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \left(\begin{bmatrix}1\\2\\3\end{bmatrix}^T \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \left(\begin{bmatrix}1\\2\\3\end{bmatrix}^T \begin{bmatrix}0\\0\\1\end{bmatrix}\right)\begin{bmatrix}0\\0\\1\end{bmatrix} \\
&= -\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \frac{3}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix} \quad \Rightarrow \left[\begin{bmatrix}1\\2\\3\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{3}{\sqrt{2}}\\3\end{bmatrix} \\
\begin{bmatrix}1\\1\\0\end{bmatrix} &= \left(\begin{bmatrix}1\\1\\0\end{bmatrix}^T \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \left(\begin{bmatrix}1\\1\\0\end{bmatrix}^T \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \left(\begin{bmatrix}1\\1\\0\end{bmatrix}^T \begin{bmatrix}0\\0\\1\end{bmatrix}\right)\begin{bmatrix}0\\0\\1\end{bmatrix} \\
&= 0\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} + \sqrt{2}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\end{bmatrix} \quad \Rightarrow \left[\begin{bmatrix}1\\1\\0\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}0\\\sqrt{2}\\0\end{bmatrix} \qquad \square
\end{aligned}$$

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## The following may be useful in the above problems:

**Definition.** Two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are **orthogonal** if $\mathbf{v}^T\mathbf{w} = 0$ (where $\mathbf{v}^T$ is the transpose of $\mathbf{v}$ as an $n \times 1$ matrix).

**Definition.** A vector $\mathbf{v} \in \mathbb{R}^n$ is **orthogonal to a subspace** $V$ of $\mathbb{R}^n$ if $\mathbf{v}$ is orthogonal to every $\mathbf{w} \in V$.