Math 415 - Lecture 33

Diagonalization

Monday November 16th 2015

Textbook reading: Chapter 5.2

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Suggested practice exercises: Chapter 5.2: 1, 2, 3, 4, 5, 6, 7, 15, 16, 17, 18, 19, 20, 25, 26, 29, 30, 31, 32, 33

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Strang lecture: Lecture 22: Diagonalization and powers of A

Review

- **Eigenvector** equation: $A\mathbf{x} = \lambda\mathbf{x} \iff (A - \lambda I)\mathbf{x} = \mathbf{0}$

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These are trivial cases. Is there always an eigenbasis?

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  $= \lambda^2 - 0\lambda + (-4) = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2)$,

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  These are solutions to $(A - \lambda I)^2 \mathbf{x} = \mathbf{0}, (A - \lambda I)^3 \mathbf{x} = \mathbf{0}, \ldots$

Diagonalization

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  For A^{100}, we need $A^{100}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A^{100}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

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- We find the second column of A^{100} likewise. Left as exercise!

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\begin{aligned}
A\mathbf{x}_i = \lambda \mathbf{x}_i \implies A \begin{bmatrix} | & & | \\ \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ | & & | \end{bmatrix} &= \begin{bmatrix} | & & | \\ \lambda_1 \mathbf{x}_1 & \cdots & \lambda_n \mathbf{x}_n \\ | & & | \end{bmatrix} \\
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- In summary $AP = PD$. Such a diagonalization is possible if and only if A has enough eigenvectors.

Motto

Everything in Linear Algebra is a matrix factorization.

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A square matrix A is said to be **diagonalizable** if there is a invertible matrix P and a diagonal matrix D such that

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Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

We can express the relation between A and D in terms of change of base matrices.

$$\begin{array}{ccc} \text{coords for } \mathbf{x} & \xrightarrow{\quad A \quad} & \text{coords for } A\mathbf{x} \\ \text{in standard basis} & & \text{in standard basis} \\ \\ \text{coords for } \mathbf{x} & \xrightarrow{\quad D \quad} & \text{coords for } A\mathbf{x} \\ \text{in eigen- basis} & & \text{in eigen- basis} \end{array}$$

,

P changes from eigenbasis coordinates to standard coordinates, and P^{-1} goes the other way! Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^n$ and $\mathcal{B}$ the basis of eigenvectors of A, then

$$P = I_{\mathcal{E},\mathcal{B}} \text{ and } P^{-1} = I_{\mathcal{B},\mathcal{E}}.$$

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P changes from eigenbasis coordinates to standard coordinates, and P^{-1} goes the other way! Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^n$ and $\mathcal{B}$ the basis of eigenvectors of A, then

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Application: Large powers

If A has an eigenbasis, then we can raise it to large powers easily!

Theorem

If $A = PDP^{-1}$, where D is a diagonal matrix, then for any m,

$$A^m = PD^mP^{-1}$$

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Only the outside P and P^{-1} remain! $\square$

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Why?

Example

Let $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$.

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Find A^{100}. Hint: Write $A = PDP^{-1}$.

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$$\underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}}_{P^{-1}}$$

Solution (continued)

Take power

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{100} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2^{100}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2^{100}} & 0 & 0 \\ (\frac{1}{2^{100}} - 1) & 1 & (6 \cdot 2^{100} - 6) \\ 0 & 0 & 2^{100} \end{bmatrix}$$