STAT 410

Practice Problems 22

Summer 2015
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1. Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x) = 3\theta x^2 e^{-\theta x^3} \qquad x > 0 \qquad \theta > 0.$$

a) Obtain the maximum likelihood estimator of θ, $\hat{\theta}$.

b) Find a sufficient statistic $Y = u(X_1, X_2, \dots, X_n)$ for θ.

c) Find the probability distribution of Y from part (b).

d) Suppose $n = 5$, and

$$x_1 = 0.2, \qquad x_2 = 1.2, \qquad x_3 = 0.2, \qquad x_4 = 0.9, \qquad x_5 = 0.3.$$

Use part (c) to construct a 95% confidence interval for θ.

e) If $n = 5$, find a uniformly most powerful rejection region of size $\alpha = 0.10$ for testing

$$H_0: \theta = 3 \quad \text{vs.} \quad H_1: \theta < 3.$$

f) Consider the rejection region "Reject H_0 if $\sum_{i=1}^{5} x_i^3 \ge 3$". Find the significance level of this test.

g) Consider the rejection region "Reject H_0 if $\sum_{i=1}^{5} x_i^3 \ge 3$". Find the power of this test at $\theta = 2$ and $\theta = 1$.

h) Suppose $n = 5$, and

$$x_1 = 0.2, \qquad x_2 = 1.2, \qquad x_3 = 0.2, \qquad x_4 = 0.9, \qquad x_5 = 0.3.$$

Find the p-value of the test.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n = 19$ from the normal distribution $N(\mu, \sigma^2)$.

a) Find a rejection region of size $\alpha = 0.05$ for testing

$$H_0: \sigma^2 = 30 \text{ vs. } H_1: \sigma^2 > 30.$$

For which values of the sample variance s^2 should the null hypothesis be rejected?

b) What is the probability of Type II Error for the rejection region in part (a) if $\sigma^2 = 80$?

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(0, \sigma^2)$.

a) Show that $\{ (x_1, x_2, \ldots, x_n): \sum_{i=1}^{n} x_i^2 \ge c \}$ is the best rejection region for testing $H_0: \sigma^2 = 4$ vs. $H_1: \sigma^2 = 16$.

b) If $n = 15$, find the value of c so that $\alpha = 0.05$.

c) If $n = 15$ and c is the value found in part (b), find the probability of Type II Error.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with mean θ.

a) Find a uniformly most powerful rejection region for testing

$$H_0: \theta = 3 \text{ vs. } H_1: \theta > 3$$

that is based on the statistic $\sum_{i=1}^{n} X_i$.

That is, find a rejection region that is most powerful for testing

$$H_0: \theta = 3 \text{ vs. } H_1: \theta = \theta_1 \quad \text{for all } \theta_1 > 3.$$

b) If $n = 12$, use the fact that $\frac{2}{\theta} \cdot \sum_{i=1}^{12} X_i$ is $\chi^2(24)$ to find a uniformly most powerful rejection region for testing $H_0: \theta = 3$ vs. $H_1: \theta > 3$ of size $\alpha = 0.10$.

c) If $\theta = 7$, what is the power of the rejection region from part (b)?

5. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be independent random variables, each with the probability density function

$$f(x) = \begin{cases} \dfrac{\lambda}{x^{\lambda+1}} & x \ge 1 \\ 0 & x < 1 \end{cases}.$$

We wish to test $H_0 : \lambda = 1$ vs. $H_1 : \lambda > 1$.

a) Find a sufficient statistic for λ.

b) Find a uniformly most powerful rejection region.
That is, find a rejection region that is most powerful for testing
$H_0 : \lambda = 1$ vs. $H_1 : \lambda = \lambda_1$ for all $\lambda_1 > 1$.

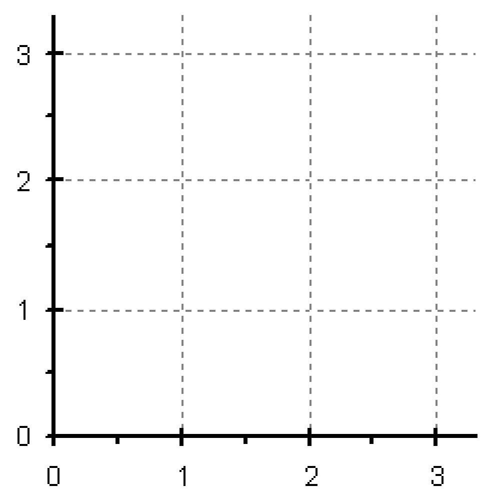
Hint: It should look like "Reject H_0 if $Y \le c$" or "Reject H_0 if $Y \ge c$", where $Y = u(X_1, X_2, \ldots, X_n)$ is a sufficient statistic for λ.

6. **5.** (continued)

Let X_1, X_2 be a random sample of size $n = 2$ from a probability distribution with p.d.f. $f(x)$.

c) Sketch a typical rejection region obtained in part **7**(b).

Hint: Recall that
$x_1 \ge 1,\ x_2 \ge 1,$
so $c > 1$
(if you are using $\prod_{i=1}^{n} x_i$).



d) Find the power function (as a function of c and λ) of the test in part (b). ($n = 2$)

Answers:

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x) = 3\theta x^2 e^{-\theta x^3} \qquad x > 0 \qquad \theta > 0.$$

a) Obtain the maximum likelihood estimator of θ, $\hat{\theta}$.

$$L(\theta) = \prod_{i=1}^{n} \left(3\theta x_i^2 e^{-\theta x_i^3} \right)$$

$$\ln L(\theta) = n \cdot \ln\theta + \sum_{i=1}^{n} \ln\left(3 x_i^2\right) - \theta \cdot \sum_{i=1}^{n} x_i^3$$

$$(\ln L(\theta))' = \frac{n}{\theta} - \sum_{i=1}^{n} x_i^3 = 0 \qquad \Rightarrow \qquad \hat{\theta} = \frac{n}{\sum_{i=1}^{n} X_i^3}$$

b) Find a sufficient statistic $Y = u(X_1, X_2, \ldots, X_n)$ for θ.

$$f(x_1, x_2, \ldots x_n; \theta) = f(x_1;\theta)\, f(x_2;\theta) \ \ldots\ f(x_n;\theta)$$

$$= \left[3^n \theta^n e^{-\theta \sum_{i=1}^{n} x_i^3} \right] \left(\prod_{i=1}^{n} x_i^2 \right).$$

By Factorization Theorem, $Y = \sum_{i=1}^{n} X_i^3$ is a sufficient statistic for θ.

OR

$$f(x;\lambda) = \exp\left\{ -\theta \cdot x^3 + \ln\theta + \ln 3 + 2\ln x \right\}. \qquad \Rightarrow \qquad K(x) = x^3.$$

$\Rightarrow \qquad Y = \sum_{i=1}^{n} X_i^3$ is a sufficient statistic for λ.

c) Find the probability distribution of Y from part (b).

$$F_X(x) = 1 - e^{-\theta x^3}, \qquad x > 0. \qquad \text{Let } V = X^3.$$

$$F_V(v) = P(V \le v) = P(X \le v^{1/3}) = 1 - e^{-\theta v}, \qquad v > 0.$$

V has an Exponential distribution with mean "usual θ" $= \dfrac{1}{\theta}$.

$$\Rightarrow \qquad Y = \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} X_i^3 \ \text{ has a Gamma distribution with } \alpha = n$$

$$\text{and "usual } \theta\text{"} = \frac{1}{\theta} \qquad (\text{"usual } \lambda\text{"} = \theta).$$

d) Suppose $n = 5$, and

$x_1 = 0.2, \qquad x_2 = 1.2, \qquad x_3 = 0.2, \qquad x_4 = 0.9, \qquad x_5 = 0.3.$

Use part (c) to construct a 95% confidence interval for θ.

If T has a Gamma$(\alpha, \beta = 1/\lambda)$ distribution, where α is an integer, then $2T/\beta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$$\Rightarrow \qquad 2Y/\beta = 2\theta \sum_{i=1}^{n} X_i^3 \ \text{ has a chi-square distribution with } r = 2\alpha = 2n \text{ d.f.}$$

$$\Rightarrow \qquad P\left(\chi^2_{1-\alpha/2}(2n) < 2\theta \sum_{i=1}^{n} X_i^3 < \chi^2_{\alpha/2}(2n) \right) = 1 - \alpha.$$

$$\Rightarrow \qquad P\left(\frac{\chi^2_{1-\alpha/2}(2n)}{2\sum_{i=1}^{n} X_i^3} < \theta < \frac{\chi^2_{\alpha/2}(2n)}{2\sum_{i=1}^{n} X_i^3} \right) = 1 - \alpha.$$

A $(1-\alpha)\,100\,\%$ confidence interval for θ: $$\left(\frac{\chi^2_{1-\alpha/2}(2n)}{2\sum\limits_{i=1}^{n} X_i^3}, \frac{\chi^2_{\alpha/2}(2n)}{2\sum\limits_{i=1}^{n} X_i^3} \right).$$

$\chi^2_{0.975}(10) = 3.247$, $\quad \chi^2_{0.025}(10) = 20.48$. $\quad \sum\limits_{i=1}^{n} x_i^3 = 2.5$.

$\left(\frac{3.247}{2 \cdot 2.5}, \frac{20.48}{2 \cdot 2.5} \right)$ $\quad$ **(0.6494, 4.096)**

e) If n = 5, find a uniformly most powerful rejection region of size $\alpha = 0.10$ for testing

$$H_0 : \theta = 3 \quad \text{vs.} \quad H_1 : \theta < 3.$$

Let $\theta < 3$.

$$\lambda(x_1, x_2, \ldots, x_n) = \frac{L(H_0; x_1, x_2, \ldots, x_n)}{L(H_1; x_1, x_2, \ldots, x_n)} = \frac{\prod\limits_{i=1}^{n} \left(9\, x_i^2\, e^{-3\, x_i^3} \right)}{\prod\limits_{i=1}^{n} \left(3\theta\, x_i^2\, e^{-\theta\, x_i^3} \right)}$$

$$= \left(\frac{3}{\theta} \right)^n \exp\left\{ (\theta - 3) \sum_{i=1}^{n} x_i^3 \right\}.$$

$$\lambda(x_1, x_2, \ldots, x_n) \le k \quad \Leftrightarrow \quad (\theta - 3) \sum_{i=1}^{n} x_i^3 \le k_1$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} x_i^3 \ge c \qquad (\text{since } \theta < 3).$$

Reject H_0 if $\sum\limits_{i=1}^{n} x_i^3 \ge c$.

$\sum_{i=1}^{n} X_i^3$ has a Gamma distribution with $\alpha = n = 5$ and "usual θ" $= \dfrac{1}{\theta}$.

If T has a Gamma$(\alpha, \beta = 1/\lambda)$ distribution, where α is an integer, then $2T/\beta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$\Rightarrow \quad \dfrac{2}{\beta} \sum_{i=1}^{n} X_i^3 = 2\theta \sum_{i=1}^{n} X_i^3$ has a $\chi^2(2\alpha = 10$ degrees of freedom$)$ distribution.

$$0.10 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{n} X_i^3 \ge c \mid \theta = 3\right)$$

$$= P\left(6\sum_{i=1}^{5} X_i^3 \ge 6c \mid \theta = 3\right) = P\left(\chi^2(10) \ge 6c\right).$$

$\Rightarrow \quad 6c = \chi^2_{0.10}(10) = 15.99. \qquad \Rightarrow \quad c = 2.665.$

Reject H_0 if $\sum_{i=1}^{5} x_i^3$ **≥ 2.665**.

f) Consider the rejection region "Reject H_0 if $\sum_{i=1}^{5} x_i^3 \ge 3$". Find the significance level of this test.

If T has a Gamma$(\alpha, 1/\lambda)$ distribution, where α is an integer, then $P(T > t) = P(Y \le \alpha - 1)$, where Y has a Poisson$(\lambda t)$ distribution.

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{5} X_i^3 \ge 3 \mid \theta = 3\right)$$

$= P(\text{Poisson}(3 \times 3) \le 5 - 1) = P(\text{Poisson}(9) \le 4) =$ **0.055**.

g) Consider the rejection region "Reject H_0 if $\sum_{i=1}^{5} x_i^3 \ge 3$". Find the power of this test at $\theta = 2$ and $\theta = 1$.

$$\text{Power}(\theta = 2) = P(\text{Reject } H_0 \mid \theta = 2) = P(\sum_{i=1}^{5} X_i^3 \ge 3 \mid \theta = 2)$$

$$= P(\text{Poisson}(3 \times 2) \le 5 - 1) = P(\text{Poisson}(6) \le 4) = \mathbf{0.285}.$$

$$\text{Power}(\theta = 1) = P(\text{Reject } H_0 \mid \theta = 1) = P(\sum_{i=1}^{5} X_i^3 \ge 3 \mid \theta = 1)$$

$$= P(\text{Poisson}(3 \times 1) \le 5 - 1) = P(\text{Poisson}(3) \le 4) = \mathbf{0.815}.$$

h) Suppose $n = 5$, and

$x_1 = 0.2, \quad x_2 = 1.2, \quad x_3 = 0.2, \quad x_4 = 0.9, \quad x_5 = 0.3.$

Find the p-value of the test.

$\sum_{i=1}^{5} x_i^3 = 2.5.$ $\qquad$ $\sum_{i=1}^{n} X_i^3$ has a Gamma distribution with $\alpha = n$ and $\beta =$ "usual θ" $= \frac{1}{\theta}$

If T has a Gamma($\alpha, \beta = 1/\lambda$) distribution, where α is an integer, then $P(T > t) = P(Y \le \alpha - 1)$, where Y has a Poisson($\lambda t$) distribution.

$$\text{p-value} = P(\sum_{i=1}^{5} X_i^3 \ge 2.5 \mid \theta = 3) = P(\text{Poisson}(2.5 \times 3) \le 5 - 1) = \mathbf{0.132}.$$

2. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n = 19$ from the normal distribution $N(\mu, \sigma^2)$.

a) Find a rejection region of size $\alpha = 0.05$ for testing

$$H_0: \sigma^2 = 30 \text{ vs. } H_1: \sigma^2 > 30.$$

For which values of the sample variance s^2 should the null hypothesis be rejected?

$H_0: \sigma^2 = 30$ vs. $H_1: \sigma^2 > 30$. Right – tailed.

Recall: If $X_1, X_2, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, then $\dfrac{(n-1)\cdot S^2}{\sigma^2}$ is $\chi^2(n-1)$.

Test Statistic: $\chi^2 = \dfrac{(n-1)\cdot s^2}{\sigma_0^2} = \dfrac{18\cdot s^2}{30}$.

Reject H_0 if $\chi^2 > \chi^2_\alpha(n-1) = \chi^2_{0.05}(18) = 28.87$.

$\dfrac{18\cdot s^2}{30} > 28.87 \quad \Leftrightarrow \quad s^2 > \mathbf{48.116667}$.

b) What is the probability of Type II Error for the rejection region in part (a) if $\sigma^2 = 80$?

$$\begin{aligned}
P(\text{Type II Error}) &= P(\text{Accept } H_0 \mid H_0 \text{ is not true}) \\
&= P(S^2 < 48.116667 \mid \sigma^2 = 80) \\
&= P\left(\frac{(n-1)\cdot S^2}{\sigma^2} < \frac{18\cdot 48.116667}{80} \mid \sigma^2 = 80\right) \\
&= P(\chi^2(18) < 10.82625) \\
&\approx \mathbf{0.10}, \qquad \text{since } \chi^2_{0.90}(18) = 10.86.
\end{aligned}$$

EXCEL: =1-CHIDIST(CHIINV(0.05,18)*30/80,18) $\Rightarrow$ **0.0984**.

3. Let $X_1, X_2, \dots, X_n$ be a random sample from $N(0, \sigma^2)$.

a) Show that $\{ (x_1, x_2, \dots, x_n): \sum_{i=1}^{n} x_i^2 \ge c \}$ is the best rejection region for testing $H_0: \sigma^2 = 4$ vs. $H_1: \sigma^2 = 16$.

$$\lambda(x_1, x_2, \dots, x_n) = \frac{L\left(\sigma^2 = 4; x_1, x_2, \dots, x_n\right)}{L\left(\sigma^2 = 16; x_1, x_2, \dots, x_n\right)} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\, 2} \exp\left\{ -x_i^2 / 8 \right\}}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\, 4} \exp\left\{ -x_i^2 / 32 \right\}}$$

$$= 2^n \exp\left\{ \left(\frac{1}{32} - \frac{1}{8} \right) \sum_{i=1}^{n} x_i^2 \right\} = 2^n \exp\left\{ -\frac{3}{32} \sum_{i=1}^{n} x_i^2 \right\}.$$

$$\lambda(x_1, x_2, \dots, x_n) \le k \quad \Leftrightarrow \quad n \cdot \ln 2 - \frac{3}{32} \sum_{i=1}^{n} x_i^2 \le \ln k$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} x_i^2 \ge c.$$

b) If $n = 15$, find the value of c so that $\alpha = 0.05$.

$n = 15 \qquad \alpha = 0.05.$

$\dfrac{\sum \left(X_i - \mu \right)^2}{\sigma^2}$ is $\chi^2(n)$; here $\mu = 0$.

$$\Rightarrow \quad P\left(\sum_{i=1}^{n} X_i^2 \ge c \right) = P\left(\chi^2(n) \ge \frac{c}{\sigma^2} \right).$$

$$0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{n} X_i^2 \ge c \mid \sigma^2 = 4 \right)$$

$$= P\left(\frac{1}{4} \cdot \sum_{i=1}^{15} X_i^2 \ge \frac{c}{4} \mid \sigma^2 = 4 \right) = P\left(\chi^2(15) \ge \frac{c}{4} \right).$$

$$\chi^2_{0.05}(15) = 25.00. \qquad \Rightarrow \qquad c = 4\chi^2_{0.05}(15) = 4 \times 25.00 = \mathbf{100.00}.$$

c) If $n = 15$ and c is the value found in part (b), find the probability of Type II Error.

$$\beta = \mathrm{P(\,Type\ II\ Error\,)} = \mathrm{P(\,Accept\ H_0 \mid H_0\ is\ not\ true\,)}$$

$$= \mathrm{P}\Big(\sum_{i=1}^{n} \mathrm{X}_i^{\,2} < c \mid \sigma^2 = 16\Big) = \mathrm{P}\Big(\sum_{i=1}^{15} \mathrm{X}_i^{\,2} < 100.00 \mid \sigma^2 = 16\Big)$$

$$= \mathrm{P}\Big(\frac{1}{16}\cdot\sum_{i=1}^{15} \mathrm{X}_i^{\,2} < \frac{100.00}{16} \mid \sigma^2 = 16\Big)$$

$$= \mathrm{P}(\chi^2(15) < 6.25).$$

$$\mathrm{P}(\chi^2(15) < 6.262) = 0.025.$$

$$\Rightarrow \qquad \beta = \mathrm{P(\,Type\ II\ Error\,)} \approx \mathbf{0.025}.$$

4. Let $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n$ be a random sample from an exponential distribution with mean θ.

$$f_{\mathrm{X}}(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 < x < \infty.$$

a) Find a uniformly most powerful rejection region for testing

$$\mathrm{H}_0: \theta = 3 \ \text{ vs. } \ \mathrm{H}_1: \theta > 3$$

that is based on the statistic $\sum_{i=1}^{n} \mathrm{X}_i$.

That is, find a rejection region that is most powerful for testing

$$\mathrm{H}_0: \theta = 3 \ \text{ vs. } \ \mathrm{H}_1: \theta = \theta_1 \quad \text{for all } \theta_1 > 3.$$

Let $\theta_1 > 3$.

$$\lambda(x_1, x_2, \ldots, x_n) = \frac{\mathrm{L}(\theta = 3;\, x_1, x_2, \ldots, x_n)}{\mathrm{L}(\theta = \theta_1;\, x_1, x_2, \ldots, x_n)} = \frac{\prod_{i=1}^{n} \frac{1}{3} e^{-x_i/3}}{\prod_{i=1}^{n} \frac{1}{\theta_1} e^{-x_i/\theta_1}}.$$

$$= \left(\frac{\theta_1}{3}\right)^n \exp\left\{\left(-\frac{1}{3}+\frac{1}{\theta_1}\right)\sum_{i=1}^{n} x_i\right\} = \left(\frac{\theta_1}{3}\right)^n \exp\left\{-\frac{\theta_1-3}{3\theta_1}\sum_{i=1}^{n} x_i\right\}.$$

If $\theta_1 > 3$, $\qquad \lambda(x_1, x_2, \ldots, x_n) < k \qquad \Leftrightarrow \qquad \sum_{i=1}^{n} x_i > c.$

$\Rightarrow$ Same rejection region for all $\theta_1 > 3$.

$\Rightarrow$ Uniformly most powerful rejection region for $\mathrm{H}_0 : \theta = 3$ vs. $\mathrm{H}_1 : \theta > 3$.

b) If $n = 12$, use the fact that $\frac{2}{\theta} \cdot \sum_{i=1}^{12} X_i$ is $\chi^2(24)$ to find a uniformly most powerful rejection region for testing $\mathrm{H}_0 : \theta = 3$ vs. $\mathrm{H}_1 : \theta > 3$ of size $\alpha = 0.10$.

$$\mathbf{0.10} = \alpha = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is true}) = \mathrm{P}\left(\sum_{i=1}^{12} x_i > c \mid \theta = 3\right)$$

$$= \mathrm{P}\left(\frac{2}{3}\sum_{i=1}^{n} x_i > \frac{2}{3}\, c \mid \theta = 3\right) = \mathrm{P}\left(\chi^2(24) > \frac{2}{3}\, c\right).$$

$\Rightarrow \qquad \frac{2}{3}\, c = \chi^2_{0.10}(24) = 33.20. \qquad \Rightarrow \qquad c = \mathbf{49.8}.$

Reject H_0 if $\sum_{i=1}^{n} x_i > \mathbf{49.8}. \qquad (\Leftrightarrow \quad \bar{x} > 4.15 \,)$

c) If $\theta = 7$, what is the power of the rejection region from part (b)?

$$\text{Power}(\theta = 7) = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \theta = 7) = \mathrm{P}\left(\sum_{i=1}^{12} x_i > 49.8 \mid \theta = 7\right)$$

$$= \mathrm{P}\left(\frac{2}{7}\sum_{i=1}^{n} x_i > \frac{2}{7}\, 49.8 \mid \theta = 3\right) = \mathrm{P}(\chi^2(24) > 14.23)$$

is **between 0.90 and 0.95**.

EXCEL: =CHIDIST(49.8*2/7,24) $\Rightarrow$ **0.94132**.

5. Let $\lambda > 0$ and let $X_1, X_2, \dots, X_n$ be independent random variables, each with the probability density function

$$f(x) = \begin{cases} \dfrac{\lambda}{x^{\lambda+1}} & x \ge 1 \\ \\ 0 & x < 1 \end{cases}.$$

We wish to test $H_0 : \lambda = 1$ vs. $H_1 : \lambda > 1$.

a) Find a sufficient statistic for λ.

$$f(x_1;\lambda)\, f(x_2;\lambda)\, \dots\, f(x_n;\lambda) = \frac{\lambda^n}{\left(\prod_{i=1}^{n} x_i\right)^{\lambda+1}}. \quad \Rightarrow \quad \prod_{i=1}^{n} X_i \text{ is sufficient for } \lambda.$$

OR

$$f(x;\lambda) = \exp\{-\lambda \ln x + \ln \lambda - \ln x\}. \quad \Rightarrow \quad K(x) = \ln x.$$

$$\Rightarrow \quad \sum_{i=1}^{n} \ln X_i \quad \text{is a sufficient statistic for } \lambda.$$

b) Find a uniformly most powerful rejection region.
That is, find a rejection region that is most powerful for testing
$H_0 : \lambda = 1$ vs. $H_1 : \lambda = \lambda_1$ for all $\lambda_1 > 1$.

Hint: It should look like "Reject H_0 if $Y \le c$" or "Reject H_0 if $Y \ge c$",
where $Y = u(X_1, X_2, \dots, X_n)$ is a sufficient statistic for λ.

$$\lambda(x_1, x_2, \dots, x_n) = \frac{L(1; x_1, x_2, \dots, x_n)}{L(\lambda; x_1, x_2, \dots, x_n)} = \frac{x_1^{-2}\, x_2^{-2} \dots x_n^{-2}}{\lambda^n\, x_1^{-\lambda-1}\, x_2^{-\lambda-1} \dots x_n^{-\lambda-1}}$$

$$= \frac{x_1^{\lambda-1}\, x_2^{\lambda-1} \dots x_n^{\lambda-1}}{\lambda^n} = \frac{1}{\lambda^n}\left(\prod_{i=1}^{n} x_i\right)^{\lambda-1}.$$

Since $\lambda > 1$, $\quad \lambda(x_1, x_2, \ldots, x_n) \le k \quad \Leftrightarrow \quad \prod_{i=1}^{n} x_i \le c.$

Uniformly most powerful rejection region is given by

$$C = \{ (x_1, x_2, \ldots, x_n) : \prod_{i=1}^{n} x_i \le c \}.$$

6. **5.** (continued)

Let X_1, X_2 be a random sample of size $n = 2$ from a probability distribution with p.d.f. $f(x)$.

c) Sketch a typical rejection region obtained in part **7**(b).

Hint: Recall that

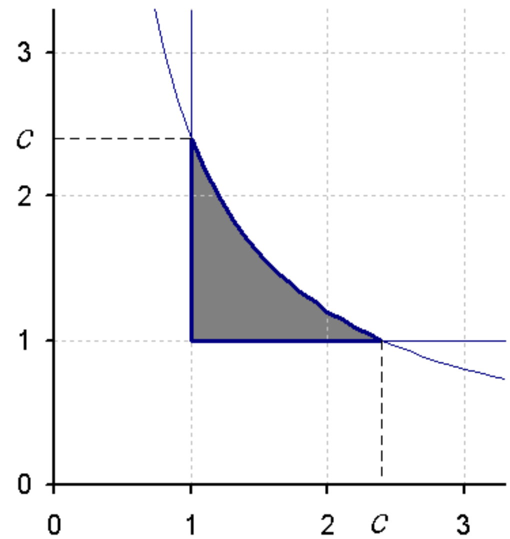
$x_1 \ge 1,\ x_2 \ge 1,$

so $c > 1$

(if you are using $\prod_{i=1}^{n} x_i$).

$x_1 x_2 \le c$

$\Rightarrow \quad x_2 \le {}^{c}/_{x_1}$



d) Find the power function (as a function of c and λ) of the test in part (b). $(n = 2)$

$$\gamma(\lambda) = P(X_1 X_2 \le c \mid \lambda) = \int_1^c \left(\int_1^{c/x_1} \lambda^2 x_1^{-\lambda-1} x_2^{-\lambda-1}\, dx_2 \right) dx_1$$

$$= \int_1^c \lambda x_1^{-\lambda-1} \left(1 - \frac{x_1^{\lambda}}{c^{\lambda}} \right) dx_1 = \int_1^c \left(\lambda x_1^{-\lambda-1} - \frac{\lambda x_1^{-1}}{c^{\lambda}} \right) dx_1$$

$$= 1 - c^{-\lambda} - \lambda c^{-\lambda} \ln c.$$