Math 415 - Lecture 15

The Four Fundamental Subspaces, the Fundamental Theorem of Linear Algebra, Linear Transformations

Monday September 28th 2015

Textbook: Chapter 2.4, 2.6

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Suggested Practice Exercise: Chapter 2.4 Exercise 1, 2, 3, 4, 7, 10, 18, 20, 21, 22, 27, 32, 37 Chapter 2.6 Exercise 5, 6, 7, 36, 37

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Khan Academy Video: Linear Transformation, Linear Transformations as Matrix Vector Products, Linear Transformation Examples: Rotations in $\mathbb{R}^2$

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Strang lectures: Lecture 9: Independence, Basis, and Dimension
Lecture 10: The Four Fundamental Subspaces
Lecture 30: Linear Transformations

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* The material for the exam covers the lectures upto and including Lecture 12 (last Monday), and this weeks worksheet and quiz.

Review

Basis for the Null Space

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Question

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The Column spaces of A and U.

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  For example, $\begin{bmatrix}1\\0\end{bmatrix} \xrightarrow[R1\leftrightarrow R2]{} \begin{bmatrix}0\\1\end{bmatrix}$.
- On the other hand, row operations do preserve the null space. Why?

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- On the other hand, row operations do preserve the null space. Why?
  Remember, we can do row operations to solve systems like $A\mathbf{x} = \mathbf{0}$.

Rank and Dimensions

Dimension of Column and Null Space

Definition

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$\dim Nul(A) = n - r$ *is the number of free variables of A. Why?*

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$\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) = n$ *Why?*

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Each of the n columns of A either contains a pivot or corresponds to a free variable.

The Four Fundamental Subpaces

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There are two more!

Row Space and Left Null Space

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  $\iff (A^T\mathbf{x})^T = \mathbf{0}^T$. So,
  $\iff \mathbf{x}^T A = \mathbf{0}$.
  Therefore, $\mathbf{x} \in Nul(A^T) \iff \mathbf{x}^T A = \mathbf{0}$.

Example

Find a basis for $Col(A)$ and $Col(A^T)$ if

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

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Solution. We need to compute an echelon form of A to find a basis for $Col(A)$. Then we might compute an echelon form of A^T to find a basis for $Col(A^T)$. However, an echelon form of A will allow us to find a basis for both $Col(A)$ and $Col(A^T)$.

Instead of doing twice the work, we only need to find an echelon form of A.

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So $r = 2$ for A and a basis for $Col(A)$ is $\left\{ \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\-1\\2\\0\end{bmatrix} \right\}$.

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Key idea: The row space is preserved by elementary row operations.

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In particular, a basis for $Col(A^T)$ is given by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ -5 \end{bmatrix} \right\}$.

Fundamental Theorem of Linear Algebra (Part 1)

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- *dim Col$(A) = r$* *(subspace of $\mathbb{R}^m$)*

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- *dim Col$(A^T) = r$* *(subspace of $\mathbb{R}^n$)*

Theorem

Let A be an $m \times n$ matrix with rank r.

- $\dim \mathrm{Col}(A) = r$ (subspace of $\mathbb{R}^m$)
- $\dim \mathrm{Col}(A^T) = r$ (subspace of $\mathbb{R}^n$)
- $\dim \mathrm{Nul}(A) = n - r$ (subspace of $\mathbb{R}^n$)

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- $\dim Nul(A) = n - r$ (subspace of $\mathbb{R}^n$)
- $\dim Nul(A^T) = m - r$ (subspace of $\mathbb{R}^m$)

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The column and row space always have the same dimension.
In other words, A and A^T have the same rank. (i.e. same number of pivots). Why?

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It's easy to see this for a matrix in echelon form.

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(3 pivot columns in A, 3 non-zero columns in A^T.)

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(3 pivot columns in A, 3 non-zero columns in A^T.) But it's not as obvious for a random matrix.

Coordinates

Why Bases?

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We are going to organize the coordinates in a convenient package.

Coordinate Vectors

Definition

If $w \in V$ and $\mathcal{B} = (\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p})$ is a basis for V, the **coordinate vector** of w with respect to the basis $\mathcal{B}$ is

$$w_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_p \end{bmatrix}, \quad \text{if } w = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_p\mathbf{v_p}.$$

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So w is a vector in some vector space, but it's coordinate vector is always a column vector in $\mathbb{R}^p$, if $\dim(V) = p$. Why is the coordinate vector useful? Computations in V can be translated in computations in the familiar vector space $\mathbb{R}^p$.

Let $V = \mathbb{R}^2$, $\mathcal{B} = (\mathbf{b_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

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Geometrically: this means that to reach $\mathbf{w}$ walk 1 unit along the $\mathbf{b_1}$ basis vector and 2 units along the $\mathbf{b_2}$ basis vector.

Example with polynomials

Let $V = P_2$, the vector space of polynomials of the form $a_0 + a_1 t + a_2 t^2$. Let $\mathcal{B} = (\mathbf{b_1} = 1, \mathbf{b_2} = t, \mathbf{b_3} = t^2)$ be the obvious basis of P_2. Let $\mathbf{w} = 1 + 2t + 3t^2$.

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Standard Coordinate Vectors

Let $V = \mathbb{R}^3$ and let $E = (\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ be the standard basis. If $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ what is the coordinate vector with respect to the standard basis?

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So the coordinate vector with respect to the standard basis is just the vector itself!

Linear Transformations

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Let V and W be vector spaces.

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A map $T : V \to W$ is a linear transformation if

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Some examples

First example

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Let $V = \mathbb{R}$, $W = \mathbb{R}$. Then the map $f(x) = 3x$ is linear. Why?

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$f(ax + by) = 3(ax + by) = a \cdot 3x + b \cdot 3y = af(x) + bf(y)$.
What about the function $g(x) = 2x - 2$? Is this a linear transformation?

Matrices are linear transformations!

Example

Let A be an $m \times n$ matrix. Then the map $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. Why?

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$$A(c\mathbf{x} + d\mathbf{y}) = cA\mathbf{x} + dA\mathbf{y}$$

The left-hand side is $T(c\mathbf{x} + d\mathbf{y})$ and the right-hand side is $cT(\mathbf{x}) + dT(\mathbf{y})$.

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