Bivariate Transformations

- CDF
- Convolutions
- MGF - for independent r.v.'s

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1) X & Y independent

$$f_X(x) = \frac{2}{x^3},\ x > 1$$

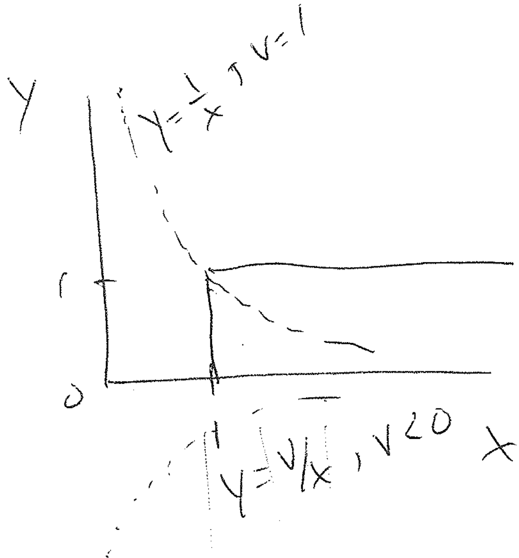
$$f_Y(y) = I(0 < y < 1)$$

Last time we found $F_W(w)$ where $w = x + y$

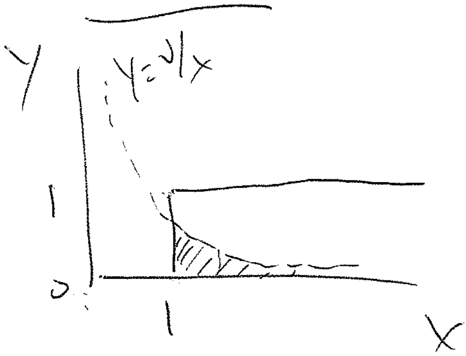
Today let's consider $U = \frac{Y}{X}$ and $V = XY$.

a) Find $f_V(v)$ for $V = XY$.

$$F_V(v) = P(V \le v)$$
$$= P(XY \le v)$$
$$= P(Y \le v/x) = P(X \le v/y)$$



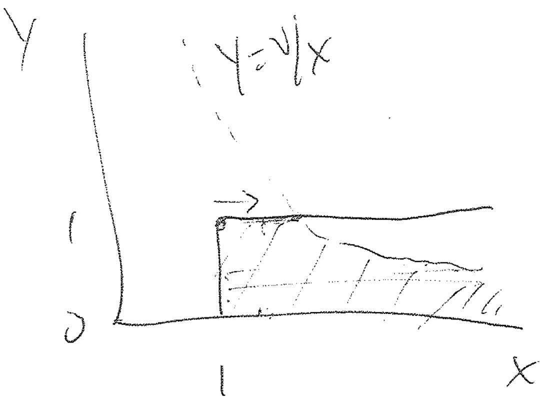
Case #2: $0 < v < 1$



$$F_V(v) = P(Y \le v/x)$$
$$= \int_1^\infty \int_0^{v/x} \frac{2}{x^3}\, dy\, dx = \frac{2}{3} v$$

$$F_V(v) = \begin{cases} 0 & , v < 0 \\ \frac{2}{3} v & , 0 \le v < 1 \\ 1 - \frac{1}{3v^2} & , v \ge 1 \end{cases}$$

<u>Case #3</u> : $v \ge 1$



$$F_V(v) = P(X \le v/y)$$

$$= \int_0^1 \int_1^{v/y} \frac{2}{x^3}\, dx\, dy$$

$$= 1 - \frac{1}{3v^2}$$

$$f_V(v) = F_V'(v) = \begin{cases} 0 & , v < 0 \\ \frac{2}{3} & , 0 < v < 1 \\ \frac{2}{3}\frac{1}{v^3} & , v > 1 \end{cases}$$

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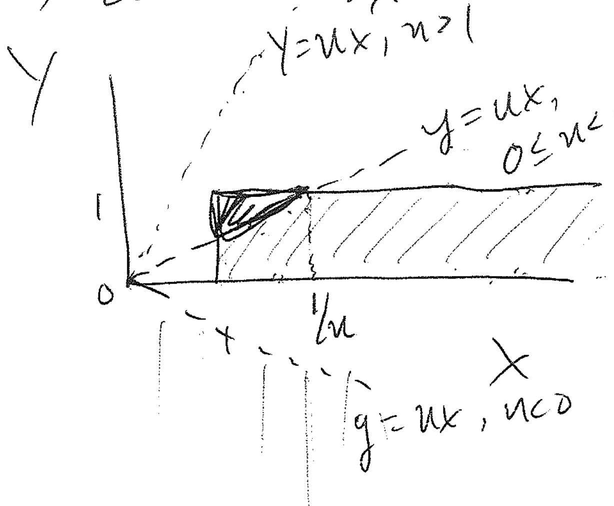
c) Let $U = Y/X$. Find $F_U(u)$

$F_U(u) = P(U \le u) = P(Y \le uX)$



$$F_U(u) = \begin{cases} 0 & , u < 0 \\ u(2-u) & , 0 \le u < 1 \\ 1 & , u \ge 1 \end{cases}$$

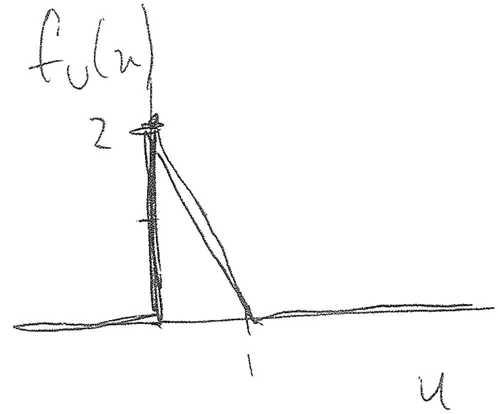
~~Case~~ #2: $0 \le u < 1$

$$F_U(u) = P(Y \le ux) = \cancel{\iint} \; 1 - P(Y > ux)$$

$$= 1 - \int_1^{1/u}\int_{ux}^{1} \frac{2}{x^3}\, dy\, dx = \cancel{2(1-u)}$$

$$2u - u^2 = u(2-u)$$

$$f_U(u) = \begin{cases} 0, & u < 0 \\ 2(1-u), & 0 < u < 1 \\ 0 & u > 1 \end{cases}$$



Sums of R.v.s

1) CDF
2) Convolution
3) Change of Variables
4) If r.v.s are independent, we can use MGFs to find the MGF of sum of r.v.s

Reconsider X & Y independent

$$f_X(x) = \frac{2}{x^3}, \quad x > 1, \quad Y \sim U(0,1), \quad W = X + Y$$

$$M_W(t) = E(e^{Wt}) = E(e^{(X+Y)t}) = E(e^{Xt}e^{Yt})$$

$$= E(e^{Xt})\,E(e^{Yt}) = M_X(t)\,M_Y(t) = M_W(t)$$

I am not sure what $M_W(t)$ corresponds to.

Ex. X & Y ~~indep~~ i.i.d. $\exp(\lambda = 1)$

independent & identically distributed

$X \sim Exp(1)$ & $Y \sim Exp(1)$

$W = X + Y$, we know $M_W(t) = M_X(t)\,M_Y(t)$

In general $X \sim Exp(\lambda)$

$$M_X(t) = \frac{1}{1 - \frac{t}{\lambda}} = \frac{1}{1-t} = M_Y(t)$$

$$M_W(t) = \frac{1}{(1-t)^2}, \quad Gamma(\alpha = 2, \lambda = 1)$$

$$\frac{1}{\left(1 - \frac{t}{\lambda}\right)^{\alpha}}$$

Convolutions

Let X and Y are cont. r.v.'s w/ joint pdf $f(x,y)$. Then for $W = X+Y$, $\Rightarrow Y = W - X$
$X = W - Y$

$$f_W(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx$$

or

$$f_W(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy$$

$\boxed{pf}$

$$F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{w-x} f(x,y)\, dy\, dx$$

let $u = x+y$, $du = dy$

$y \to -\infty \Rightarrow u \to -\infty$

$y \to w-x \Rightarrow u \to w$

$$F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{w} f(x, u-x)\, du\, dx = \int_{-\infty}^{w} \left(\int_{-\infty}^{\infty} f(x, u-x)\, dx \right) du$$

$$f_W(w) = F_W'(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx$$

By fundamental Θ^{m} of calc.
Differentiation removes integral.

If X & Y independent, the convolution result implies

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx$$

$$f_W(w) = \int_{-\infty}^{\infty} f_X(w-y) f_Y(y)\, dy$$

Ex. X & Y iid $Exp(\lambda=1)$, Find $f_W(w)$ $W=X+Y$

$f_X(x) = e^{-x}$, $\underline{x>0}$

$f_Y(y) = e^{-y}$, $y>0$ $\Rightarrow f_Y(w-x) = e^{-w+x}$,

$\underline{y>0 \Rightarrow w-x>0 \Rightarrow w>x}$

$$f_W(w) = \int f_X(x) f_Y(w-x)\, dx$$

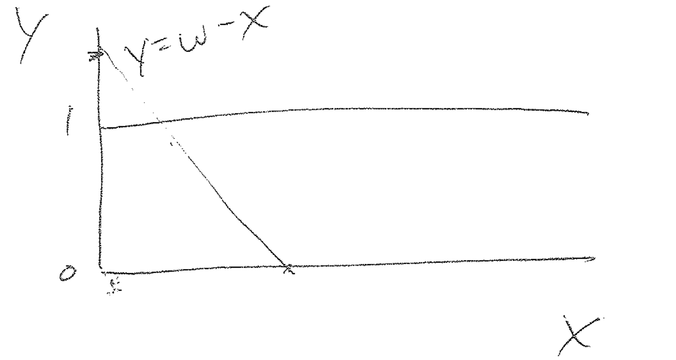
$$= \int_0^w e^{-x} e^{-w+x}\, dx = e^{-w}\int_0^w dx = \underline{\underline{we^{-w}, w>0}}$$

gamma$(\alpha=2, \lambda=1)$

2) X & Y independent Find $f_W(w)$ for $W = X+Y$

$f_X(x) = e^{-x}, \ x>0$

$f_Y(y) = 2y, \ 0<y<1$



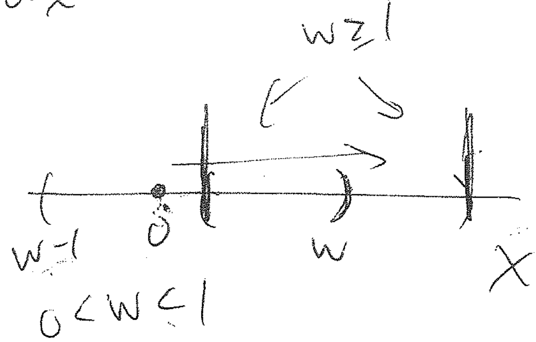
$f_Y(w-x) = 2(w-x), \ 0 < w-x < 1$

$\Rightarrow -1 < x-w < 0$

$\Rightarrow w-1 < x < w$

Bounds for x
1) $x>0$
2) $w-1 < x < w$

$f_W(w) = \int f_X(x) f_Y(w-x)\, dx$



Case #1 : $0 \le w < 1$

$$f_W(w) = \int_0^w e^{-x}\, 2(w-x)\, dx$$
$$= 2(e^{-w} - 1 + w)$$

Case #2 : $w \ge 1$

$$f_W(w) = \int_{w-1}^{w} e^{-x}\, 2(w-x)\, dx = 2e^{-w}$$

3) Let X and Y be independent Poisson r.v.'s w/ means λ_1 & λ_2. $W = X+Y$ find ~~$f_W(w)$~~ $p_W(W=w)$

a) Find pmf for W $\quad X+Y=n$

$$\cancel{p(W=n) = \sum_{X+Y=n} p(X,Y \mid X+Y=n)}$$

$$\cancel{= \sum_{k=0}^{n} p(X, n-X)} =$$

$$P_X(k) = \frac{\lambda_1^k e^{-\lambda_1}}{k!}$$

$$P_Y(k) = \frac{\lambda_2^k e^{-\lambda_2}}{k!}$$

$$p(W=n) = \sum_{X+Y=n} p(X,Y)$$

$$= \sum_{k=0}^{n} p(X=k, \overset{Y=n-k}{n-k}) = \sum_{k=0}^{n} \underline{p_X(X=k)\, P_Y(n-k)}$$

$$= \sum_{k=0}^{n} \frac{\lambda_1^k e^{-\lambda_1}}{k!} \frac{\lambda_2^{(n-k)} e^{-\lambda_2}}{(n-k)!} = \boxed{e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!}}$$

Binomial Theorem

$$\boxed{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^k = 1}$$

We need to simplify

$$= \frac{(\lambda_1+\lambda_2)^n}{n!} e^{-(\lambda_1+\lambda_2)} \underbrace{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}}_{1}$$

$$= \frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!} \sim \text{Poisson}(\lambda_1+\lambda_2)$$