Math 415 - Lecture 13

Basis and Dimension

Wednesday September 23rd 2015

Textbook reading: Chapter 2.3

Suggested practice exercises: Chapter 2.3 Exercise 1, 2, 3, 5, 6, 9, 11, 16, 19, 20, 22, 27.

Khan Academy video: Introduction to Linear Independence, More on linear independence, Span and Linear Independence Example, Basis of a Subspace

Strang lecture: Independence, Basis, and Dimension

1 Review

- Vectors $\mathbf{v_1}, \dots, \mathbf{v_p}$ are *linearly **Dependent*** if

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_p\mathbf{v_p} = \mathbf{0},$$

  and not all the coefficients are zero.

- The columns of A are linearly **IN**dependent $\iff$ each column of A contains a pivot $\iff$ there are no free variables for $A\mathbf{x} = \mathbf{0}$.

- Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix}$ independent?

- Any set of 11 vectors in $\mathbb{R}^{10}$ is linearly dependent. Why?

Definition 1. In a list of vectors $(\mathbf{v_1}, \ldots, \mathbf{v_p})$ in a vector space V we call $\mathbf{v_k}$ *redundant* if v_k is a linear combination of the previous vectors. In this case $\text{Span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{k-1}}, \mathbf{v_k}) = \text{Span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{k-1}})$, i.e., you can delete the redundant vector and get the same span.

Example 2. Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$. Are there redunant vectors?

Solution.

2 A Basis of a Vector Space

Definition. A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in V is a **basis** of V if

- $V = \text{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$, and
- the vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are linearly independent.

Fact: $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in V is a basis of V if and only if every vector $\mathbf{w}$ in V can be uniquely expressed as $\mathbf{w} = c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p}$.

Fact: A basis is a *minimal spanning set*: the elements of the basis span V but you cannot delete any of these elements and still get all of V. There are no redundant vectors.

Example 3. Let $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Show that $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is a basis of $\mathbb{R}^3$. (It is called the **standard basis**.)

Solution.

Definition. V is said to have **dimension** p if it has a basis consisting of p vectors.

This definition makes sense because if V has a basis of p vectors, then every basis of V has p vectors. Why?

(Challenge: can you think of an argument that is more "rigorous"?)

Example 4. $\mathbb{R}^3$ has dimension 3. Indeed, the standard basis

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

has three elements. Likewise, $\mathbb{R}^n$ has dimension n.

Example 5. Not all vectors spaces have a finite basis. For instance, the vector space of all polynomials has *infinite dimension*. Its standard basis is $1, t, t^2, t^3, \ldots$ Why?

Solution.

Theorem 1. *Suppose that V has dimension d.*

- *A set of d vectors in V are a basis if they span V.*
- *A set of d vectors in V are a basis if they are linearly independent.*

Why?

Solution.

Example 6. Are the following sets a basis for $\mathbb{R}^3$?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

Solution.

Example 7. (c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$. Is this a basis?

Solution.

Example 8. Let P_2 be the space of polynomials of degree at most 2.

- What is the dimension of P_2?
- Is $\{t,\ 1-t,\ 1+t-t^2\}$ a basis of P_2?

Solution.

3 Shrinking and Exanding Sets of Vectors

We can find a basis for $V = \text{Span}\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ by discarding, if necessary, some of the vectors in the spanning set.

Example 9. Produce a basis of $\mathbb{R}^2$ from the vectors

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution.

Example 10. Produce a basis of $\mathbb{R}^2$ from the vector

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solution.

4 Checking Our Understanding

Example 11. Subspaces of $\mathbb{R}^3$ can have dimension 0, 1, 2, or 3.

- The only 0-dimensional subspace is $\{\mathbf{0}\}$.
- A 1-dimensional subspace is of the form Span $\{\mathbf{v}\}$ where $\mathbf{v} \neq \mathbf{0}$. These subspaces are lines through the origin.
- A 2-dimensional subspace is of the form Span $\{\mathbf{v}, \mathbf{w}\}$ where $\mathbf{v}$ and $\mathbf{w}$ are not multiples of each other. These subspaces are planes through the origin.
- The only 3-dimensional subspace is $\mathbb{R}^3$ itself.

True or false?

1. Suppose that V has dimension n. Then any set in V containing more than n vectors must be linearly dependent.

2. The space P_n of polynomials of degree at most n has dimension $n+1$.

3. The vector space of functions $f : \mathbb{R} \to \mathbb{R}$ is infinite-dimensional.

4. Consider $V = \text{Span}\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$. If one of the vectors, say $\mathbf{v_k}$, in the spanning set is a linear combination of the remaining ones, then the remaining vectors still span V.