Math 415 - Lecture 19

Orthonormal basis, orthogonal complement

Friday October 9th 2015

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Textbook reading: Ch 3.1

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Suggested practice exercises: Ch 3.1: 7, 8, 9, 10, 11, 12, 14, 15, 17, 18, 19, 20, 22

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Khan Academy videos:

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Khan Academy videos:

Strang lectures: Lec 10: The Four Fundamental Subspaces / Lec 14: Orthogonal Vectors and Subspaces

Review

- $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = v_1 w_1 + \cdots + v_n w_n$ is the **inner product** of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

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  - This simple criterion is equivalent to Pythagoras' theorem.

Unit Vectors and Orthonormal basis

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The standard basis vectors $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ of $\mathbb{R}^n$ are all unit vectors.

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If $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then is $\mathbf{x}$ a unit vector?

Solution

Since $\mathbf{x} \cdot \mathbf{x} = 5$ and $\|\mathbf{x}\| = \sqrt{5}$.

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Since $\mathbf{x} \cdot \mathbf{x} = 5$ and $\|\mathbf{x}\| = \sqrt{5}$. However, $u = \frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{\mathbf{x}}{\sqrt{5}}$ *is* a unit vector. The unit vector $\mathbf{u}$ is called the *normalization* of $\mathbf{x}$.

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In the same way

$$\mathbf{u_2} \cdot \mathbf{x} = c_2, \ldots, \mathbf{u_n} \cdot \mathbf{x} = c_n$$

Example

$\mathbf{u_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ is an orthonormal basis for $\mathbb{R}^2$. Let $\mathbf{x} = \begin{bmatrix}2\\3\end{bmatrix}$.

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Theorem

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$$\begin{aligned} \mathbf{0} &= \mathbf{v_1} \cdot (c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n}) \\ &= c_1\mathbf{v_1} \cdot \mathbf{v_1} + c_2\mathbf{v_1} \cdot \mathbf{v_2} + \cdots + c_n\mathbf{v_1} \cdot \mathbf{v_n} \\ &= c_1\mathbf{v_1} \cdot \mathbf{v_1} = c_1\|\mathbf{v_1}\|^2 \end{aligned}$$

But $\|\mathbf{v_1}\| \neq 0$ and so $c_1 = 0$.

Theorem

Let $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ *be non-zero and mutually orthogonal. Then* $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ *are linearly independent.*

Solution

Proof. Suppose that

$$c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n} = \mathbf{0}.$$

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But $\|\mathbf{v_1}\| \neq 0$ and so $c_1 = 0$. Similarly, we find that $c_2 = 0, \ldots, c_n = 0$. Therefore, the vectors are independent.

Orthogonality and the Fundamental subspaces

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. Find $Nul(A)$ and $Col(A^T)$.

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The basis vectors for the null and row space are orthogonal.

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$$

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$$Nul(A) = span\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad Col(A^T) = span\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Again, the basis for the null space is orthogonal to the basis for the row space.

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0.$$

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Since $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ is orthogonal to both basis vectors for the row space, it's orthogonal to *every* vector in the row space.
It turns out this is true for the null and row space of any matrix A.
That is, vectors in $Nul(A)$ are orthogonal to vectors in $Col(A^T)$ for *all* matrices A.

Fundamental Theorem of Linear Algebra (Revisited)

Definition

Let W be a subspace of $\mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$.

- $\mathbf{v}$ is **orthogonal** to W if $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in W$. ($\iff$ $\mathbf{v}$ is orthogonal to each vector in a basis for W.)

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- The **orthogonal complement** of W is the space $W^{\perp}$ of all vectors that are orthogonal to W. (Show that the orthogonal complement of any vector space is also a vector space.)

Example

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- $V^\perp = \text{Span}(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix})$

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Why?

Solution

Because $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are orthogonal (so independent), and so they're a basis for all of $\mathbb{R}^3$.

Remark

In the last example, $Nul(A)$ and $Col(A)$ both happen to be subspaces of $\mathbb{R}^3$ (because A was a square 3×3 matrix).

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Solution

$$\begin{bmatrix}-2\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} \neq 0$$

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Let A be an $m \times n$ matrix of rank r.

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