1. a) $f_X(x) = \int_0^1 f_{XY}(x,y)\,dy = 1 \qquad f_Y(y) = \int_0^1 f_{XY}(x,y) = 1$

$E(X) = \int_0^1 x\,dx = \frac{1}{2} \qquad E(Y) = \int_0^1 y\,dy = \frac{1}{2}$

$Var(X) = \int_0^1 x^2\,dx - (E(X))^2 = \frac{1}{12}$

$Var(Y) = \int_0^1 y^2\,dy - (E(Y))^2 = \frac{1}{12}$

$Cov(X+Y, X) = Cov(X,X) + Cov(X,Y) = Var(X) + Cov(X,Y) = \frac{1}{12}$

$Cov(X,Y) = \int_0^1 \int_0^1 (x-\frac{1}{2})(y-\frac{1}{2})\,dx\,dy = 0$

$Var(U) = Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) = 0$

$$\rho_{U,X} = \frac{Cov(U,X)}{\sqrt{Var(U)Var(X)}} = \frac{\frac{1}{12}}{\sqrt{\frac{1}{6}\cdot\frac{1}{12}}} = \frac{\sqrt{2}}{2}$$

b) $Cov(U,V) = Cov(X+Y, X-Y) = Cov(X,X) - Cov(X,Y) + Cov(Y,X) - Cov(Y,Y)$

$= Var(X) - Var(Y) = 0$

$$\rho_{U,V} = \frac{Cov(U,V)}{\sqrt{Var(U)\,Var(U)}} = 0$$

c) $0<U<1$: $F_U(u) = \int_0^u \int_0^{u-x} dy\,dx = \frac{u^2}{2}$, $f_u(u) = F'(u) = u$

$1<U<2$: $F_U(u) = \int_{u-1}^1 \int_{u-x}^1 dy\,dx = -\frac{u^2}{2} + 2u - 1$

$f_u(u) = F'(u) = 2-u$

d) $-1<v<0$: $F_V(v) = \int_0^{v+1} \int_{x-v}^1 dx\,dy = \frac{v^2}{2} + v + \frac{1}{2}$

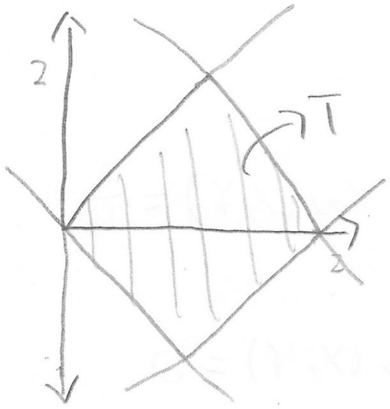
$f_V(v) = v+1$

$0<v<1$: $F_V(v) = 1 - \int_u^1 \int_0^{x-u} dy\,dx = -\frac{v^2}{2} + v + \frac{1}{2}$

$f_V(v) = 1-v$

2. (a) $U = X+Y$, $V = X-Y \Rightarrow X = \frac{1}{2}(U+V)$, $Y = \frac{1}{2}(U-V)$

Since $0<X<1$, $0<Y<1$, then $0<\frac{U+V}{2}<1$, $0<\frac{U-V}{2}<1$



(b) $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

$f_{U,V} = f_{XY}(x(u,v), y(u,v))\,|J| = 1 \cdot |\frac{1}{2}| = \frac{1}{2}$

$U, V \in T$

(c) $f_{UV}(u,v) \neq f_U(u) f_V(v) \Rightarrow$ not ind.

3. a) $0 < Z < 1$

b) $P(Z \leq z) = P(\frac{Y}{X} \leq z) = P(Y \leq Xz)$

$F_Z(z) = \int_0^1 \int_0^{xz} 2 \, dy \, dx = z$

$$F_Z(z) = \begin{cases} 0 & z \leq 0 \\ z & 0 < z < 1 \\ 1 & z \geq 1 \end{cases}$$

c) $f_Z(z) = F_Z'(z) = 1$

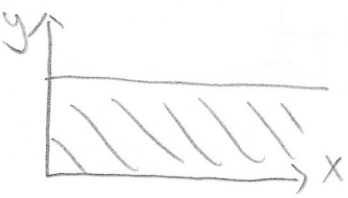
$$f_Z = \begin{cases} 1 & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$$

4. Let $X = X$, $Z = Y/X \Rightarrow X = X$, $Y = ZX$

$J = \begin{vmatrix} 1 & 0 \\ 2 & x \end{vmatrix} = x$

$f_{XZ}(x, z) = 2 \cdot |x| = 2x \quad x \in (0,1), z \in (0,1)$

5. a)



b) $C \int_0^\infty \int_0^1 e^{-x} \, du \, dx = 1$

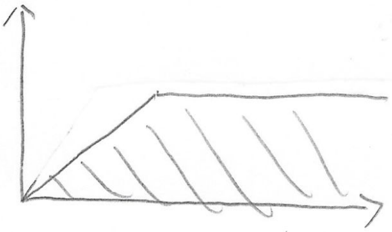
$\Rightarrow C = 1$

c) $F_Y(y) = P(Y \leq y) = P(\frac{X}{U} \leq y) = P(X \leq yU)$

$F_Y(y) = \int_0^1 \int_0^{uy} e^{-x} \, dx \, du = \frac{1}{y} e^{-y} - \frac{1}{y} + 1 \quad 0 < y < \infty$

d) $f_Y(y) = F'(y) = \frac{e^{-y}(-y+e^{y}-1)}{y^2} \quad 0 < y < \infty$

6. a)



b) $F_{XU}(x,u) = \int_0^1 \int_0^x Ce^{-x}\,du\,dx = C(-2e^{-1}+e^{-1})$

$\Rightarrow \quad C = \frac{e}{e-1}$

c) $F_Y(y) = P(Y \le y) = P(\frac{X}{U} \le y) = P(X \le uy)$

$= \frac{e}{e-1} \int_0^1 \int_0^{uy} e^{-x}\,dx\,du = \frac{e}{1-e}(-\frac{1}{y}e^{-y} + \frac{1}{e} + \frac{1}{y} - 1)$

$F_Y(y) = \begin{cases} 0 & y<1 \\ \frac{e}{1-e}(-\frac{1}{y}e^{-y} + \frac{1}{e} + \frac{1}{y} - 1) & y \ge 1 \end{cases}$

d) $f_Y(y) = \begin{cases} 0 & y<1 \\ \frac{e}{1-e}(y^2 e^{-y} + e^{-y}\cdot\frac{1}{y} - y^{-2}) & y \ge 1 \end{cases}$

7. a) $P(T_1 < t) = 1 - P(\min(X,Y) > t) = 1 - P(X>t)P(Y>t)$

$= 1 - \int_t^1 3(1-x)^2 dx \int_t^1 2(1-y)\,dy = 1-(1-t)^5$

$f_{T_1}(t) = 5(1-t)^4 \quad (0<t<1)$

b) $P(T_2 < t_2) = P(\max(X,Y) < t_2)$

$= P(X<t)P(Y<t)$

$= \int_0^t 3(1-X)^2 dx \int_0^t 2(1-y)\,dy$

$= 6t^2 - 6t^3 + 2t^4 - 3t^3 + 3t^4 - t^5$

$f_{T_2}(t) = -5t^4 + 20t^3 - 27t^2 + 12(t) \quad (0<t<1)$

(c) $E(T_2 - T_1) = E(T_2) - E(T_1) = \frac{1}{4}$

$E(T_1) = \int_0^1 t f_{T_1}(t)\, dt = \frac{1}{6}$

$E(T_2) = \int_0^1 t f_{T_2}(t)\, dt = \frac{5}{12}$

8. $f_{XY} = f_X f_Y = 6x^2 y \quad x \in (0,1),\ y \in (0,1)$

$0 \leq w < 1$: $F_W(w) = \int_0^w \int_0^{w-x} 6x^2 y\, dy\, dx = \frac{1}{10} w^5$

$1 \leq w \leq 2$: $F_W(w) = 1 - \int_{w-1}^1 \int_{w-x}^1 6x^2 y\, dy\, dx = \frac{1}{5} - \frac{w^5}{10} + w^3 - w^2$

So $f_W(w) = F'(w) = \begin{cases} \frac{1}{2} w^4 & 0 < w \leq 1 \\ -\frac{1}{2} w^4 + 3w^2 - 2w & 1 \leq w \leq 2 \\ 0 & \text{otherwise} \end{cases}$

9. a) $P(Y_3 = i) = P(X_1, X_2, X_3 \leq i) - P(X_1, X_2, X_3 \leq (i-1))$

$= P(X_1 \leq i) P(X_2 \leq i) P(X_3 \leq i) - P(X_1 \leq i-1,\ X_2 \leq i-1,\ X_3 \leq i-1)$

for $i = 1, 2, 3, 4$

$\Rightarrow f_{Y_3}(y_3) = \begin{cases} \frac{1}{1000} & y_3 = 1 \\ \frac{26}{1000} & y_3 = 2 \\ \frac{189}{1000} & y_3 = 3 \\ \frac{784}{1000} & y_3 = 4 \end{cases}$

b) $P(Y_1 = k) = P(X_1, X_2, X_3 \geq k) - P(X_1, X_2, X_3 \geq (k+1))$

$= P(X_1 \geq k) P(X_2 \geq k) P(X_3 \geq k) - P(X_1 \geq k+1) P(X_2 \geq k+1) P(X_3 \geq k+1)$

$\Rightarrow f_{Y_1}(y_1) = \begin{cases} \frac{271}{1000} & y_1 = 1 \\ \frac{386}{1000} & y_2 = 2 \\ \frac{279}{1000} & y_3 = 3 \\ \frac{64}{1000} & y_4 = 4 \end{cases}$

10. Refer to page 229 in textbook

a) $F(x) = x^2$

$$f_{Y_4}(y_4) = \frac{4!}{3!\,0!}(y_4^2)^3(1-y_4^2)\,2y_4 = 8y_4^7$$

b) $f_{Y_3 Y_4} = 48\, y_3^5\, y_4$

c) $$f_{Y_3|Y_4}(y_3|y_4) = \frac{f_{Y_3Y_4}(y_3, y_4)}{f_{Y_4}(y_4)} = \frac{48\,y_3^5\,y_4}{8y_4^7} = \frac{6Y_3^5}{Y_4^6}$$

d) $$E(Y_3 | Y_4 = y_4) = \int_0^{y_4} y_3\, f_{Y_3|Y_4}(y_3|y_4) = \frac{6}{7}y_4$$

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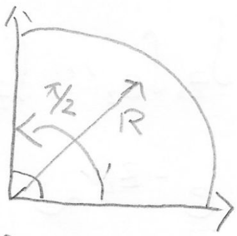
11. a) $X_1 = R\cos(w)$ $\quad X_2 = R\sin(w)$ $\quad 0<R<\infty \quad 0<w<\frac{\pi}{2}$

$$J = \begin{vmatrix} \cos(w) & -r\sin(w) \\ \sin(w) & r\cos(w) \end{vmatrix} = r$$



$$g_{R,w}(r,w) = \frac{2g\left(\sqrt{r^2\cos^2(w) + r^2\sin^2(w)}\right)}{\pi\sqrt{r^2\cos^2(w) + r^2\sin^2(w)}}|r| = \frac{2g(r)}{\pi}$$

b) $$g_R(r) = \int_0^{\pi/2} \frac{2}{\pi} g(r)\, dw = g(r)$$

$$E(r) = \int_0^\infty r g(r)\, dr = \theta$$

c) $$g_w(w) = \int_0^\infty \frac{2}{\pi} g(r)\, dr = \frac{2}{\pi}$$

$$\Rightarrow g_w(w) = \int_0^\infty \frac{2}{\pi} g(r)\, dr = \frac{2}{\pi}$$

$$\Rightarrow g(w) \cdot g(r) = g(r,w) \Rightarrow \text{ind.}$$

12. According to Theorem 4.4.1

$f(y_1,\cdots,y_n) = n!\, e^{-y_1} e^{-y_2} \cdots e^{-y_n} \quad 0<y_1<y_2<\cdots<y_n<\infty$

$Y_1 = Z_1/n$

$Y_2 = \frac{Z_1}{n} + \frac{Z_2}{n-1}$

$\vdots$

$Y_n = \frac{Z_1}{n} + \frac{Z_n}{n-1} + \cdots \frac{Z_{n-1}}{2} + Z_n$

$$J = \begin{vmatrix} \frac{1}{n} & 0 & 0 \cdots 0 \\ \frac{1}{n} & \frac{1}{n-1} & \\ \vdots & \vdots & \ddots \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & 1 \end{vmatrix} = \frac{1}{n!}$$

$f(z_1 z_2, \cdots z_n)\, n!\, e^{-z_1/n} e^{-(z_1/n + z_2/n-1)} \cdots e^{-(z_1/n + z_2/n-1 + \cdots + z_n)} \frac{1}{n!}$

$= e^{-z_1} e^{-z_2} \cdots e^{-z_n} \quad 0 < z_1, \cdots, z_n < \infty$

$f_{Z_i}(z_i) = \int_{z_1} \int_{z_2} \cdots \int_{z_{i-1}} \int_{z_{i+1}} \cdots \int_{z_n} e^{-z_1} e^{-z_2} \cdots e^{-z_n}\, dz_n \cdots dz_1$

$= e^{-z_i} \sim \exp(1)$

then $f(z_1, z_2, \cdots z_n) = f_{Z_1}(z_1)\, f_{Z_2}(z_2) \cdots f_{Z_n}(z_n)$

$Z_1, \cdots Z_n$ are independent and each has an exponential distribution.