STAT 410

Practice Problems 3

Summer 2014
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1 – 7. Let the joint probability density function for (X, Y) be

$$f(x, y) = \frac{x+y}{3}, \quad 0 < x < 2, \quad 0 < y < 1, \qquad \text{zero otherwise.}$$

1. a) Find the probability $P(X > Y)$.

b) Find the marginal probability density function of X, $f_X(x)$.

c) Find the marginal probability density function of Y, $f_Y(y)$.

2. d) Are X and Y independent? If not, find $\operatorname{Cov}(X, Y)$.

3. e) Find $P(Y > 0.5 \mid X = 0.75)$. f) Find $P(Y > 0.5 \mid X < 0.75)$.

g) Find $E(X \mid Y = y)$.

4. Find and sketch the p.d.f. of $W = X + Y$, $f_W(w) = f_{X+Y}(w)$.

5. Find and sketch the p.d.f. of $V = X \times Y$, $f_V(v) = f_{X \times Y}(v)$.

6. Find and sketch the p.d.f. of $U = {}^{Y}\!/_{X}$, $f_U(u) = f_{Y/X}(u)$.

7. Let $U = X + Y$ and $V = X - Y + 1$.

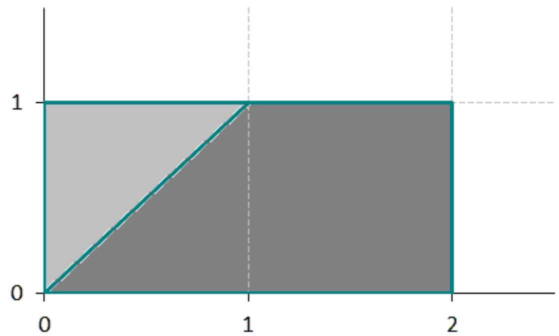
Find the joint probability density function of (U, V), $f_{U,V}(u, v)$.

Sketch the support of (U, V).

1 – 7. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{3}, \qquad 0 < x < 2, \quad 0 < y < 1, \qquad \text{zero otherwise.}$$

1. a) Find $P(X > Y)$.



$$P(X > Y) = 1 - \int_0^1 \left(\int_0^y \frac{x+y}{3}\,dx \right) dy$$

$$= 1 - \int_0^1 \left(\frac{y^2}{6} + \frac{y^2}{3} \right) dy$$

$$= 1 - \int_0^1 \frac{y^2}{2}\,dy = 1 - \frac{1}{6} = \mathbf{\frac{5}{6}}.$$

OR $$P(X > Y) = \int_0^1 \left(\int_y^2 \frac{x+y}{3}\,dx \right) dy = \ldots$$

OR $$P(X > Y) = \int_0^1 \left(\int_0^x \frac{x+y}{3}\,dy \right) dx + \int_1^2 \left(\int_0^1 \frac{x+y}{3}\,dy \right) dx = \ldots$$

b) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^1 \frac{x+y}{3}\,dy = \left(\frac{xy}{3} + \frac{y^2}{6} \right) \Bigg|_0^1 = \frac{2x+1}{6}, \qquad 0 < x < 2.$$

c) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^2 \frac{x+y}{3}\,dx = \left(\frac{x^2}{6} + \frac{xy}{3} \right) \Bigg|_0^2 = \frac{2+2y}{3}, \qquad 0 < y < 1.$$

2. d) Are X and Y independent? If not, find $\mathrm{Cov}(X, Y)$.

Since $f(x, y) \neq f_X(x) \cdot f_Y(y)$, X and Y are **NOT independent**.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^2 x \cdot \frac{2x+1}{6}\,dx = \left(\frac{x^3}{9} + \frac{x^2}{12}\right)\Bigg|_0^2 = \frac{11}{9}.$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_0^1 y \cdot \frac{2+2y}{3}\,dy = \left(\frac{y^2}{3} + \frac{y^3}{9}\right)\Bigg|_0^1 = \frac{5}{9}.$$

$$E(XY) = \int_0^2 \left(\int_0^1 x\,y \cdot \frac{x+y}{3}\,dy\right) dx = \int_0^2 \left(\frac{x^2}{6} + \frac{x}{9}\right) dx = \left(\frac{x^3}{18} + \frac{x^2}{18}\right)\Bigg|_0^2 = \frac{2}{3}.$$

$$\mathrm{Cov}(X, Y) = E(XY) - E(X) \times E(Y) = \frac{2}{3} - \frac{11}{9} \cdot \frac{5}{9} = \mathbf{-\frac{1}{81}} \approx -0.012345679.$$

3. e) Find $P(Y > 0.5 \mid X = 0.75)$.

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{x+y}{3}}{\frac{2x+1}{6}} = \frac{2x+2y}{2x+1}, \qquad 0 < y < 1.$$

$$f_{Y|X}(y \mid 0.75) = \frac{1.5+2y}{2.5}, \qquad 0 < y < 1.$$

$$P(Y > 0.5 \mid X = 0.75) = \int_{0.5}^{1} \frac{1.5+2y}{2.5}\,dy = \mathbf{0.6}.$$

f) Find $P(Y > 0.5 \mid X < 0.75)$.

Def $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, provided $P(B) > 0$.

$$P(B) = P(X < 0.75) = \int_0^{0.75} \frac{2x+1}{6}\,dx = \left.\frac{x^2+x}{6}\right|_0^{0.75} = \frac{21}{96} = \frac{14}{64}.$$

$$P(A \cap B) = P(Y > 0.5 \cap X < 0.75) = \int_0^{0.75}\left(\int_{0.5}^{1} \frac{x+y}{3}\,dy\right)dx$$

$$= \int_0^{0.75}\left(\left.\frac{xy}{3}+\frac{y^2}{6}\right)\right|_{0.5}^{1} dx = \int_0^{0.75}\left(\frac{x}{6}+\frac{1}{8}\right)dx = \left.\left(\frac{x^2}{12}+\frac{x}{8}\right)\right|_0^{0.75} = \frac{9}{64}.$$

$$P(Y > 0.5 \mid X < 0.75) = \frac{9/64}{14/64} = \mathbf{\frac{9}{14}} \approx 0.642857.$$

g) Find $E(X \mid Y = y)$.

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{x+y}{3}}{\frac{2+2y}{3}} = \frac{x+y}{2+2y}, \qquad 0 < x < 2.$$

$$E(X \mid Y = y) = \int_0^{2} x \cdot \frac{x+y}{2+2y}\,dx = \frac{4+3y}{3+3y}, \qquad 0 < y < 1.$$

4. Find and sketch the p.d.f. of $W = X + Y$, $f_W(w) = f_{X+Y}(w)$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy$$

$0 < x < 2 \quad \Rightarrow \quad 0 < w - y < 2 \quad \Rightarrow \quad w - 2 < y < w$

$0 < y < 1$

Case 1: $0 < w < 1$. Then $w - 2 < 0 < w$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy = \int_{0}^{w} \frac{w}{3}\, dy = \frac{w^2}{3}, \qquad 0 < w < 1.$$

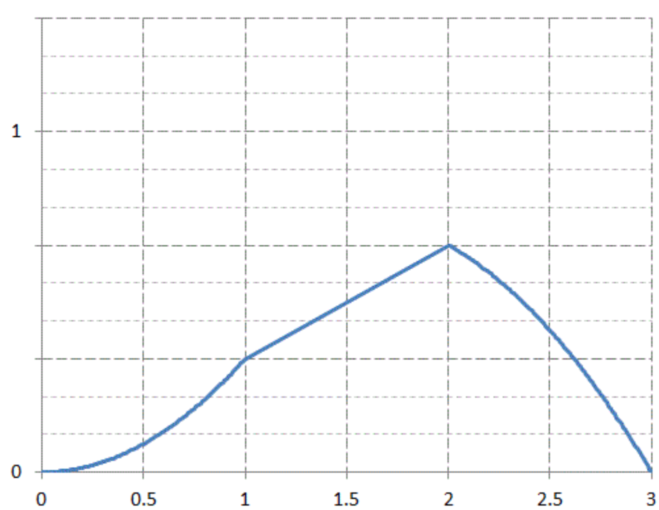
Case 2: $1 < w < 2$. Then $w - 2 < 0 < 1 < w < 2$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy = \int_{0}^{1} \frac{w}{3}\, dy = \frac{w}{3}, \qquad 1 < w < 2.$$

Case 3: $2 < w < 3$. Then $0 < w - 2 < 1 < w$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy = \int_{w-2}^{1} \frac{w}{3}\, dy = \frac{w}{3}(3-w) = \frac{3w - w^2}{3},$$

$2 < w < 3$.



OR

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx$$

$0 < x < 2$

$0 < y < 1 \qquad \Rightarrow \qquad 0 < w - x < 1 \qquad \Rightarrow \qquad w - 1 < x < w$

Case 1: $\quad 0 < w < 1. \qquad$ Then $\; w - 1 < 0 < w < 2.$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx = \int_0^w \frac{w}{3}\, dx = \frac{w^2}{3}, \qquad 0 < w < 1.$$

Case 2: $\quad 1 < w < 2. \qquad$ Then $\; 0 < w - 1 < w < 2.$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx = \int_{w-1}^{w} \frac{w}{3}\, dx = \frac{w}{3}, \qquad 1 < w < 2.$$

Case 3: $\quad 2 < w < 3. \qquad$ Then $\; 0 < w - 1 < 2 < w.$

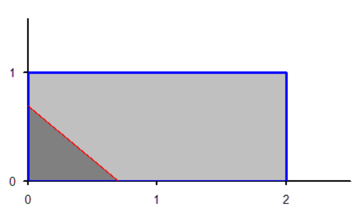
$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx = \int_{w-1}^{2} \frac{w}{3}\, dx = \frac{w}{3}(3-w) = \frac{3w - w^2}{3},$$

$$2 < w < 3.$$

OR

$$F_W(w) = P(W \le w) = P(X + Y \le w) = \ldots$$

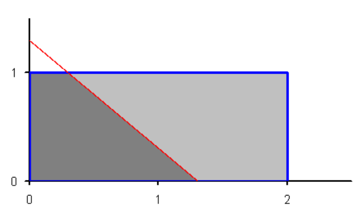
Case 1: $\quad 0 < w < 1.$



$$= \int_0^w \left(\int_0^{w-y} \frac{x+y}{3}\, dx \right) dy$$

$= \ldots$

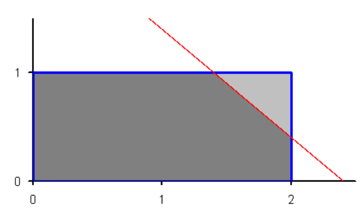
Case 2: $\quad 1 < w < 2.$



$$= \int_0^1 \left(\int_0^{w-y} \frac{x+y}{3}\, dx \right) dy$$

$= \ldots$

Case 3: $\quad 2 < w < 3.$



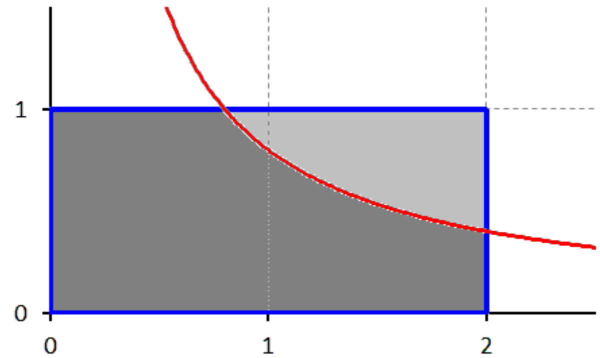
$$= 1 - \int_{w-2}^{1} \left(\int_{w-y}^{2} \frac{x+y}{3}\, dx \right) dy$$

$= \ldots$

$$f_W(w) = F_W'(w) = \ldots$$

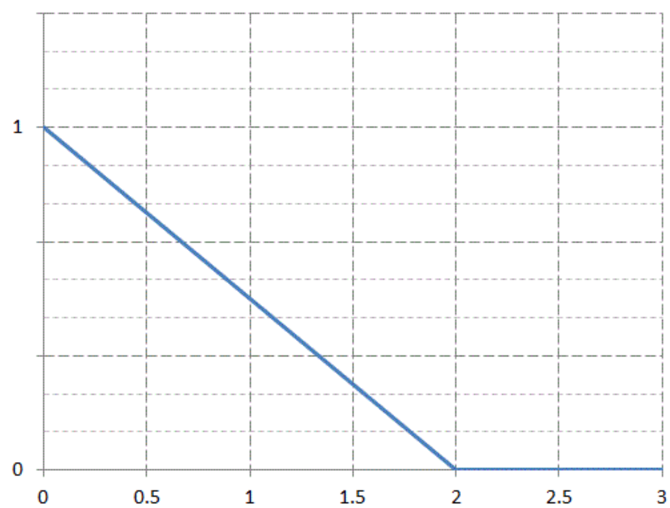
5. Find and sketch the p.d.f. of $V = X \times Y$, $f_V(v) = f_{X \times Y}(v)$.

$$F_V(v) = P(XY \le v) = \ldots$$



$$\ldots = 1 - \int_v^2 \left(\int_{v/x}^1 \frac{x+y}{3}\, dy \right) dx = \ldots = v - \frac{v^2}{4}, \qquad 0 < v < 2.$$

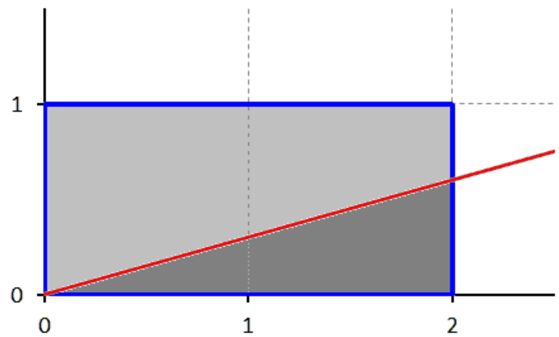
$$f_V(v) = F_V'(v) = 1 - \frac{v}{2}, \qquad 0 < v < 2.$$



6. Find and sketch the p.d.f. of $U = {}^{Y}/_{X}$, $f_U(u) = f_{Y/X}(u)$.

$$F_U(u) = P(Y \le u X) = \ldots$$

<u>Case 1</u>: $0 < u < \frac{1}{2}$.

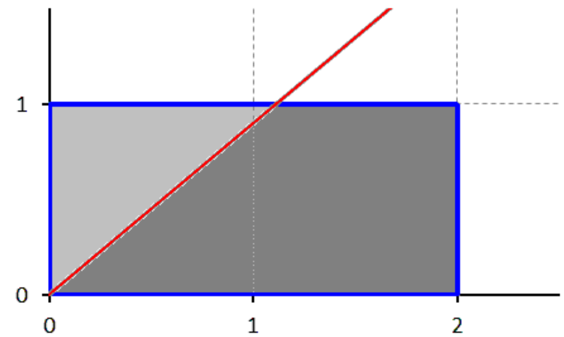


$$\ldots = \int_0^2 \left(\int_0^{ux} \frac{x+y}{3}\, dy \right) dx$$

$$= \frac{8u + 4u^2}{9}, \qquad 0 < u < \frac{1}{2}.$$

$$f_U(u) = F_U'(u) = \frac{8(1+u)}{9}, \quad 0 < u < \frac{1}{2}.$$

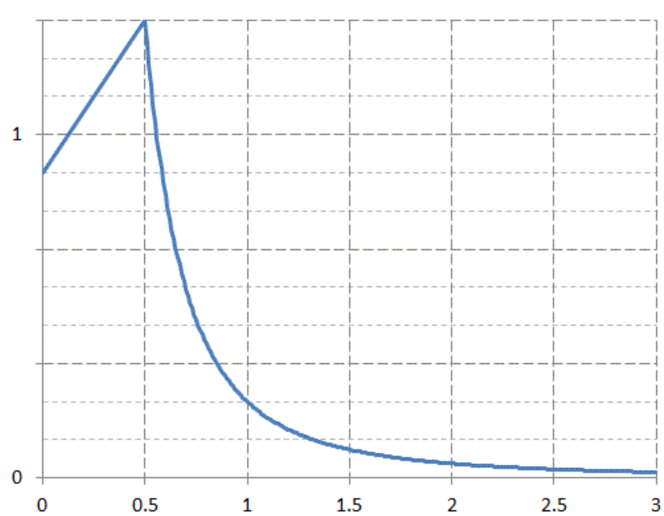
<u>Case 2</u>: $u > \frac{1}{2}$.



$$\ldots = 1 - \int_0^1 \left(\int_0^{y/u} \frac{x+y}{3}\, dx \right) dy$$

$$= 1 - \frac{1+2u}{18u^2}, \qquad u > \frac{1}{2}.$$

$$f_U(u) = F_U'(u) = \frac{(1+u)}{9u^3}, \qquad u > \frac{1}{2}.$$



7. Let $U = X + Y$ and $V = X - Y + 1$.

Find the joint probability density function of (U, V), $f_{U,V}(u, v)$.

Sketch the support of (U, V).

$$u + v = 2x + 1 \quad \Rightarrow \quad x = \frac{u + v - 1}{2}$$

$$u - v = 2y - 1 \quad \Rightarrow \quad y = \frac{u - v + 1}{2}$$

$0 < x < 2$

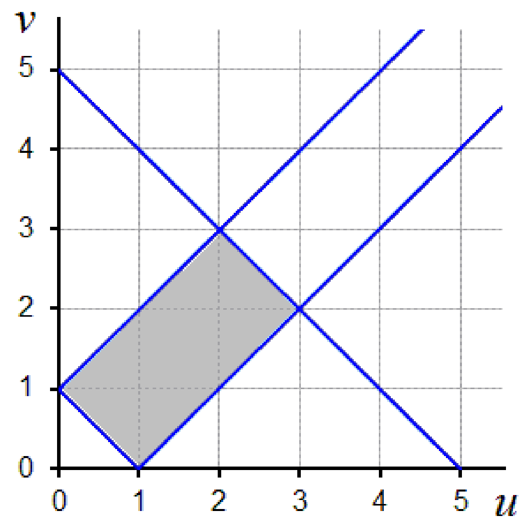
$$\Rightarrow \quad 0 < u + v - 1 < 4$$

$$\Rightarrow \quad 1 < u + v < 5$$

$0 < y < 1$

$$\Rightarrow \quad 0 < u - v + 1 < 2$$

$$\Rightarrow \quad -1 < u - v < 1$$



$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}, \qquad |J| = \frac{1}{2}.$$

$$f_{U,V}(u, v) = \frac{u}{3} \times \frac{1}{2} = \frac{u}{6}, \qquad 1 < u + v < 5, \quad -1 < u - v < 1.$$