MATH 415 – Lecture 26

Review Exam 2

Thursday 16 July 2015

Fundamental Notions

Fundamental Notions

- Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are **independent** if the only relation

$$c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \mathbf{0}$$

is the one with $c_1 = c_2 \cdots c_n = 0$.

Fundamental Notions

- Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are **independent** if the only relation

$$c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \mathbf{0}$$

is the one with $c_1 = c_2 \cdots c_n = 0$.

Fundamental Notions

- Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are **independent** if the only relation

$$c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \mathbf{0}$$

is the one with $c_1 = c_2 \cdots c_n = 0$.
How to check for independence?

Fundamental Notions

- Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are **independent** if the only relation

$$c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \mathbf{0}$$

is the one with $c_1 = c_2 \cdots c_n = 0$.
How to check for independence?
The columns of a matrix A are independent $\iff$ $\mathrm{Nul}(A) = \{0\}$.

Fundamental Notions

- Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are **independent** if the only relation

$$c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \mathbf{0}$$

  is the one with $c_1 = c_2 \cdots c_n = 0$.
  How to check for independence?
  The columns of a matrix A are independent $\iff$ $\mathrm{Nul}(A) = \{0\}$.
- Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ in V are a **basis** for V if they span V, that is $V = \mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$,and they are independent.

Fundamental Notions

- Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are **independent** if the only relation

$$c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \mathbf{0}$$

  is the one with $c_1 = c_2 \cdots c_n = 0$.
  How to check for independence?
  The columns of a matrix A are independent $\iff$ $\mathrm{Nul}(A) = \{0\}$.
- Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ in V are a **basis** for V if they span V, that is $V = \mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$,and they are independent.

Fundamental Notions

- Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are **independent** if the only relation

$$c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \mathbf{0}$$

  is the one with $c_1 = c_2 \cdots c_n = 0$.
  How to check for independence?
  The columns of a matrix A are independent $\iff$ $\mathrm{Nul}(A) = \{0\}$.
- Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ in V are a **basis** for V if
  they span V, that is $V = \mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$,and
  they are independent.
  In this case, V has **dimension** n.

Fundamental Notions

- Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are **independent** if the only relation

$$c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \mathbf{0}$$

  is the one with $c_1 = c_2 \cdots c_n = 0$.
  How to check for independence?
  The columns of a matrix A are independent $\iff$ $\mathrm{Nul}(A) = \{0\}$.
- Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ in V are a **basis** for V if they span V, that is $V = \mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$,and they are independent.
  In this case, V has **dimension** n.
- Vectors $\mathbf{v}, \mathbf{m}$ in $\mathbb{R}^m$ are **orthogonal** if $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_m w_m = 0$.

Fundamental Notions, cont.

- The **length** of a vector $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

Fundamental Notions, cont.

- The **length** of a vector $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$
- The angle θ between two vectors $\mathbf{v}, \mathbf{w}$ is determined by

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos(\theta).$$

Fundamental Notions, cont.

- The **length** of a vector $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$
- The angle θ between two vectors $\mathbf{v}, \mathbf{w}$ is determined by

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos(\theta).$$

- A unit vector $\mathbf{u}$ has length 1, equivalently $\mathbf{u} \cdot \mathbf{u} = 1$.

Fundamental Notions, cont.

- The **length** of a vector $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$
- The angle θ between two vectors $\mathbf{v}, \mathbf{w}$ is determined by

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos(\theta).$$

- A unit vector $\mathbf{u}$ has length 1, equivalently $\mathbf{u} \cdot \mathbf{u} = 1$.
- If V is a subspace of $\mathbb{R}^n$, then $V^\perp$ is the subspace of all vectors perp to the vectors in V. ("Orthogonal Complement"). The dimensions satisfy

$$\dim(V) + \dim(V^\perp) = n.$$

Subspaces

Spaces of the echelon form

- From an echelon form of A, we get bases for:

Spaces of the echelon form

- From an echelon form of A, we get bases for:
  - $\mathrm{Nul}(A)$ - by solving $A\mathbf{x} = \mathbf{0}$

Spaces of the echelon form

- From an echelon form of A, we get bases for:
  - Nul(A) - by solving $A\mathbf{x} = \mathbf{0}$
  - Col(A) - by taking the pivot columns of A

Spaces of the echelon form

- From an echelon form of A, we get bases for:
  - $\text{Nul}(A)$ - by solving $A\mathbf{x} = \mathbf{0}$
  - $\text{Col}(A)$ - by taking the pivot columns of A
  - $\text{Col}(A^T)$ - by taking the nonzero rows of the echelon form

Example

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \underset{\rightsquigarrow}{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \underset{\rightsquigarrow}{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \underset{\rightsquigarrow}{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col A:

Example

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \underset{\rightsquigarrow}{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

Example

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \underset{\rightsquigarrow}{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ Basis for $\mathrm{Col}(A^T)$:

Example

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \underset{\rightsquigarrow}{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ Basis for $\text{Col}(A^T)$: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$

Example

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \underset{\rightsquigarrow}{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ Basis for Col(A^T): $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$

Basis for Nul(A):

Example

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \underset{\rightsquigarrow}{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ Basis for $\text{Col}(A^T)$: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$

Basis for $\text{Nul}(A)$: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}.$

Example

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \underset{\rightsquigarrow}{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ Basis for $\text{Col}(A^T)$: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$

Basis for $\text{Nul}(A)$: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$. Dimension of $\text{Nul}(A^T)$:

Example

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \underset{\rightsquigarrow}{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for $\operatorname{Col} A$: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ Basis for $\operatorname{Col}(A^T)$: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$

Basis for $\operatorname{Nul}(A)$: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$. Dimension of $\operatorname{Nul}(A^T)$: 2

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \mathrm{Nul}(A)$.

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \mathrm{Nul}(A)$.
- The **fundamental theorem** states that if A is an $m \times n$ matrix

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \mathrm{Nul}(A)$.
- The **fundamental theorem** states that if A is an $m \times n$ matrix
    - $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$ are orthogonal complements

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \mathrm{Nul}(A)$.
- The **fundamental theorem** states that if A is an $m \times n$ matrix
  - $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$ are orthogonal complements

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \mathrm{Nul}(A)$.
- The **fundamental theorem** states that if A is an $m \times n$ matrix
  - $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$ are orthogonal complements
    So: $\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A^T) = n$ (number of columns in A)

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \mathrm{Nul}(A)$.
- The **fundamental theorem** states that if A is an $m \times n$ matrix
  - $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$ are orthogonal complements
    So: $\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A^T) = n$ (number of columns in A)
  - $\mathrm{Nul}(A^T)$ and $\mathrm{Col}(A)$ are orthogonal complements

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \mathrm{Nul}(A)$.
- The **fundamental theorem** states that if A is an $m \times n$ matrix
  - $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$ are orthogonal complements
    So: $\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A^T) = n$ (number of columns in A)
  - $\mathrm{Nul}(A^T)$ and $\mathrm{Col}(A)$ are orthogonal complements

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \mathrm{Nul}(A)$.
- The **fundamental theorem** states that if A is an $m \times n$ matrix
  - $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$ are orthogonal complements
    So: $\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A^T) = n$ (number of columns in A)
  - $\mathrm{Nul}(A^T)$ and $\mathrm{Col}(A)$ are orthogonal complements
    So: $\dim \mathrm{Nul}(A^T) + \dim \mathrm{Col}(A) = m$ (number of rows in A)

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \text{Nul}(A)$.
- The **fundamental theorem** states that if A is an $m \times n$ matrix
  - $\text{Nul}(A)$ and $\text{Col}(A^T)$ are orthogonal complements
    So: $\dim \text{Nul}(A) + \dim \text{Col}(A^T) = n$ (number of columns in A)
  - $\text{Nul}(A^T)$ and $\text{Col}(A)$ are orthogonal complements
    So: $\dim \text{Nul}(A^T) + \dim \text{Col}(A) = m$ (number of rows in A)
  - In particular if $r = \text{rank}(A)$ (number of pivots):

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \mathrm{Nul}(A)$.
- The **fundamental theorem** states that if A is an $m \times n$ matrix
  - $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$ are orthogonal complements
    So: $\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A^T) = n$ (number of columns in A)
  - $\mathrm{Nul}(A^T)$ and $\mathrm{Col}(A)$ are orthogonal complements
    So: $\dim \mathrm{Nul}(A^T) + \dim \mathrm{Col}(A) = m$ (number of rows in A)
  - In particular if $r = \mathrm{rank}(A)$ (number of pivots):
    - $\dim \mathrm{Col}(A) = r$

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \text{Nul}(A)$.
- The **fundamental theorem** states that if A is an $m \times n$ matrix
  - $\text{Nul}(A)$ and $\text{Col}(A^T)$ are orthogonal complements
    So: $\dim \text{Nul}(A) + \dim \text{Col}(A^T) = n$ (number of columns in A)
  - $\text{Nul}(A^T)$ and $\text{Col}(A)$ are orthogonal complements
    So: $\dim \text{Nul}(A^T) + \dim \text{Col}(A) = m$ (number of rows in A)
  - In particular if $r = \text{rank}(A)$ (number of pivots):
    - $\dim \text{Col}(A) = r$
    - $\dim \text{Col}(A^T) = r$

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \text{Nul}(A)$.
- The **fundamental theorem** states that if A is an $m \times n$ matrix
  - $\text{Nul}(A)$ and $\text{Col}(A^T)$ are orthogonal complements
    So: $\dim \text{Nul}(A) + \dim \text{Col}(A^T) = n$ (number of columns in A)
  - $\text{Nul}(A^T)$ and $\text{Col}(A)$ are orthogonal complements
    So: $\dim \text{Nul}(A^T) + \dim \text{Col}(A) = m$ (number of rows in A)
  - In particular if $r = \text{rank}(A)$ (number of pivots):
    - $\dim \text{Col}(A) = r$
    - $\dim \text{Col}(A^T) = r$
    - $\dim \text{Nul}(A) = n - r$

The solution space and the fundamental theorem

- The solutions of $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x}_p + \mathrm{Nul}(A)$.
- The **fundamental theorem** states that if A is an $m \times n$ matrix
  - $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$ are orthogonal complements
    So: $\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A^T) = n$ (number of columns in A)
  - $\mathrm{Nul}(A^T)$ and $\mathrm{Col}(A)$ are orthogonal complements
    So: $\dim \mathrm{Nul}(A^T) + \dim \mathrm{Col}(A) = m$ (number of rows in A)
  - In particular if $r = \mathrm{rank}(A)$ (number of pivots):
    - $\dim \mathrm{Col}(A) = r$
    - $\dim \mathrm{Col}(A^T) = r$
    - $\dim \mathrm{Nul}(A) = n - r$
    - $\dim \mathrm{Nul}(A^T) = m - r$

Example

Consider the following subspace of $\mathbb{R}^4$:

(a) $V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a + 2b = 0, a + b + d = 0 \right\}$

(b) $V = \left\{ \begin{bmatrix} a+b-c \\ b \\ 2a+3c \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

In each case, give a basis for V and its orthogonal complement.
Try to immediately get an idea what the dimensions are going to be!

Solution:First step

Express these subspaces as one of the four subspaces of a matrix.

(a) $V =$

Solution:First step

Express these subspaces as one of the four subspaces of a matrix.

(a) $V =$

Solution:First step

Express these subspaces as one of the four subspaces of a matrix.

(a) $V = \text{Nul}\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}\right)$

(b) $V =$

Solution:First step

Express these subspaces as one of the four subspaces of a matrix.

(a) $V = \text{Nul}\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}\right)$

(b) $V =$

Solution:First step

Express these subspaces as one of the four subspaces of a matrix.

(a) $V = \text{Nul}\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}\right)$

(b) $V = \text{Col}\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}\right)$

Step II: Give a basis for each.

(a) row reductions:

$$\begin{bmatrix}1 & 2 & 0 & 0\\ 1 & 1 & 0 & 1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 2 & 0 & 0\\ 0 & -1 & 0 & 1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & -1\end{bmatrix}$$

Step II: Give a basis for each.

(a) row reductions:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Step II: Give a basis for each.

(a) row reductions:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

basis for V: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Step II: Give a basis for each.

(a) row reductions:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

basis for V: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(b) row reductions: $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

Step II: Give a basis for each.

(a) row reductions:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

basis for V: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(b) row reductions: $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

Step II: Give a basis for each.

(a) row reductions:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

basis for V: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(b) row reductions: $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

no need to continue; we already see that the colums are independent

basis for V: $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

Step III:Use the Fundamental Theorem

(a) $V^{\perp} =$

Step III:Use the Fundamental Theorem

(a) $V^{\perp} =$

Step III:Use the Fundamental Theorem

(a) $V^{\perp} = \mathrm{Col}\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^T\right)$
note the two rows are clearly independent.

Step III:Use the Fundamental Theorem

(a) $V^{\perp} = \text{Col}\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^T\right)$

note the two rows are clearly independent.

basis for $V^{\perp}$:

Step III:Use the Fundamental Theorem

(a) $V^{\perp} = \text{Col}\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^T\right)$

note the two rows are clearly independent.

basis for $V^{\perp}$: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $V^{\perp} =$

Step III:Use the Fundamental Theorem

(a) $V^{\perp} = \text{Col}\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^T\right)$

note the two rows are clearly independent.

basis for $V^{\perp}$: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $V^{\perp} =$

Step III:Use the Fundamental Theorem

(a) $V^{\perp} = \text{Col}\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^T\right)$

note the two rows are clearly independent.

basis for $V^{\perp}$: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $V^{\perp} = \text{Nul}\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}^T\right) = \text{Nul}\left(\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 3 & 1 \end{bmatrix}\right)$

Step III:Use the Fundamental Theorem

(a) $V^{\perp} = \text{Col}\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^T\right)$

note the two rows are clearly independent.

basis for $V^{\perp}$: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $V^{\perp} = \text{Nul}\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}^T\right) = \text{Nul}\left(\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 3 & 1 \end{bmatrix}\right)$

row reductions:

Step III:Use the Fundamental Theorem

(a) $V^{\perp} = \text{Col}\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^T\right)$

note the two rows are clearly independent.

basis for $V^{\perp}$: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $V^{\perp} = \text{Nul}\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}^T\right) = \text{Nul}\left(\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 3 & 1 \end{bmatrix}\right)$

row reductions:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 3 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -2/5 \\ 0 & 1 & 0 & 2/5 \\ 0 & 0 & 1 & 1/5 \end{bmatrix}$$

Step III: Use the Fundamental Theorem

basis for $V^{\perp}$: $\begin{bmatrix} 2/5 \\ -2/5 \\ -1/5 \\ 1 \end{bmatrix}$

Example

What does it mean for $A\mathbf{x} = \mathbf{b}$ if $\mathrm{Nul}(A) = \{\mathbf{0}\}$?

Example

What does it mean for $A\mathbf{x} = \mathbf{b}$ if $\mathrm{Nul}(A) = \{\mathbf{0}\}$?

Solution

It means that if there is a solution, then it is unique.
That's because all solutions to $A\mathbf{x} = \mathbf{b}$ *are given by* $\mathbf{x}_p + \mathrm{Nul}(A)$.

Linear Transformations

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map represented by the matrix

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}$$

with respect to the bases $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ of $\mathbb{R}^2$

and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ of $\mathbb{R}^3$.

(a) What is $T \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map represented by the matrix

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}$$

with respect to the bases $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ of $\mathbb{R}^2$

and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ of $\mathbb{R}^3$.

(a) What is $T \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

(b) Which matrix represents T with respect to the standard bases?

Solution

The matrix tells us that:

$$T\begin{bmatrix}0\\1\end{bmatrix} = 1\begin{bmatrix}1\\1\\0\end{bmatrix} + 2\begin{bmatrix}1\\0\\1\end{bmatrix} + 3\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}3\\4\\5\end{bmatrix}$$

Solution

The matrix tells us that:

$$T\begin{bmatrix}0\\1\end{bmatrix} = 1\begin{bmatrix}1\\1\\0\end{bmatrix} + 2\begin{bmatrix}1\\0\\1\end{bmatrix} + 3\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}3\\4\\5\end{bmatrix}$$

$$T\begin{bmatrix}1\\-1\end{bmatrix} = 0\begin{bmatrix}1\\1\\0\end{bmatrix} + 1\begin{bmatrix}1\\0\\1\end{bmatrix} + 0\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

(a) Note that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) Note that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) Note that $\begin{bmatrix}1\\1\end{bmatrix} = 2 \cdot \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) =$$

(a) Note that $\begin{bmatrix}1\\1\end{bmatrix} = 2 \cdot \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) =$$

(a) Note that $\begin{bmatrix}1\\1\end{bmatrix} = 2 \cdot \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = 2\begin{bmatrix}3\\4\\5\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}7\\8\\11\end{bmatrix}.$$

(a) Note that $\begin{bmatrix}1\\1\end{bmatrix} = 2 \cdot \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = 2\begin{bmatrix}3\\4\\5\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}7\\8\\11\end{bmatrix}.$$

(b) Note that $\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

(a) Note that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 2\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 11 \end{bmatrix}.$$

(b) Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) Note that $\begin{bmatrix}1\\1\end{bmatrix} = 2 \cdot \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = 2\begin{bmatrix}3\\4\\5\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}7\\8\\11\end{bmatrix}.$$

(b) Note that $\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) =$$

(a) Note that $\begin{bmatrix}1\\1\end{bmatrix} = 2 \cdot \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = 2\begin{bmatrix}3\\4\\5\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}7\\8\\11\end{bmatrix}.$$

(b) Note that $\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}3\\4\\5\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}4\\4\\6\end{bmatrix}.$$

(a) Note that $\begin{bmatrix}1\\1\end{bmatrix} = 2\cdot\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = 2\begin{bmatrix}3\\4\\5\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}7\\8\\11\end{bmatrix}.$$

(b) Note that $\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}3\\4\\5\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}4\\4\\6\end{bmatrix}.$$

We already know that $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\4\\5\end{bmatrix}$.

(a) Note that $\begin{bmatrix}1\\1\end{bmatrix} = 2\cdot\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = 2\begin{bmatrix}3\\4\\5\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}7\\8\\11\end{bmatrix}.$$

(b) Note that $\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$.

Hence,

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}3\\4\\5\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}4\\4\\6\end{bmatrix}.$$

We already know that $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\4\\5\end{bmatrix}$. So, T is represented by $\begin{bmatrix}4 & 3\\4 & 4\\6 & 5\end{bmatrix}$ with respect to the standard basis.

We can also use matrices to describe linear operations on polynomials.

We can also use matrices to describe linear operations on polynomials. For example, let $D\colon P_2 \to P_1$ be the differentiation linear map. We use the standard bases of polynomials $\{1, t, t^2\}$ (for the input space P_2) and $\{1, t\}$ (for the output space P_1). Then the matrix for D is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We can also use matrices to describe linear operations on polynomials. For example, let $D\colon P_2 \to P_1$ be the differentiation linear map. We use the standard bases of polynomials $\{1, t, t^2\}$ (for the input space P_2) and $\{1, t\}$ (for the output space P_1). Then the matrix for D is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Check: let $f(t) = 2.1 + 3t + 4t^2$. Then the coordinate vector of $f(t)$ is $x = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

We can also use matrices to describe linear operations on polynomials. For example, let $D\colon P_2 \to P_1$ be the differentiation linear map. We use the standard bases of polynomials $\{1, t, t^2\}$ (for the input space P_2) and $\{1, t\}$ (for the output space P_1). Then the matrix for D is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Check: let $f(t) = 2.1 + 3t + 4t^2$. Then the coordinate vector of $f(t)$ is $x = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ so that $Ax = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$.

We can also use matrices to describe linear operations on polynomials. For example, let $D\colon P_2 \to P_1$ be the differentiation linear map. We use the standard bases of polynomials $\{1, t, t^2\}$ (for the input space P_2) and $\{1, t\}$ (for the output space P_1). Then the matrix for D is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Check: let $f(t) = 2.1 + 3t + 4t^2$. Then the coordinate vector of $f(t)$ is $x = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ so that $Ax = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$. The corresponding polynomial is $3 + 8t$, which is the derivative of $f(t) = 2.1 + 3t + 4t^2$.

Check Your Understanding

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is in $\mathrm{Col}(A)$.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is in $\mathrm{Col}(A)$.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is in $\mathrm{Col}(A)$.
  That's true because $A\mathbf{x}$ are linear combinations of the columns of A.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is in $\mathrm{Col}(A)$.
  That's true because $A\mathbf{x}$ are linear combinations of the columns of A.
- A and A^T have the same rank.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is in $\mathrm{Col}(A)$.
  That's true because $A\mathbf{x}$ are linear combinations of the columns of A.
- A and A^T have the same rank.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is in $\mathrm{Col}(A)$.
  That's true because $A\mathbf{x}$ are linear combinations of the columns of A.
- A and A^T have the same rank.
  Recall that the rank of A (number of pivots of A) equals $\dim \mathrm{Col}(A)$.
  So this is another way of saying that $\dim \mathrm{Col}(A) = \dim \mathrm{Col}(A^T)$.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is in $\mathrm{Col}(A)$.
  That's true because $A\mathbf{x}$ are linear combinations of the columns of A.
- A and A^T have the same rank.
  Recall that the rank of A (number of pivots of A) equals $\dim \mathrm{Col}(A)$.
  So this is another way of saying that $\dim \mathrm{Col}(A) = \dim \mathrm{Col}(A^T)$.
- The columns of an $n \times n$ matrix are independent if and only if the rows are.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is in $\mathrm{Col}(A)$.
  That's true because $A\mathbf{x}$ are linear combinations of the columns of A.
- A and A^T have the same rank.
  Recall that the rank of A (number of pivots of A) equals $\dim \mathrm{Col}(A)$.
  So this is another way of saying that $\dim \mathrm{Col}(A) = \dim \mathrm{Col}(A^T)$.
- The columns of an $n \times n$ matrix are independent if and only if the rows are.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is in $\mathrm{Col}(A)$.
  That's true because $A\mathbf{x}$ are linear combinations of the columns of A.
- A and A^T have the same rank.
  Recall that the rank of A (number of pivots of A) equals $\dim \mathrm{Col}(A)$.
  So this is another way of saying that $\dim \mathrm{Col}(A) = \dim \mathrm{Col}(A^T)$.
- The columns of an $n \times n$ matrix are independent if and only if the rows are.
  Let r be the rank of A, and let A be $m \times n$ for now.
  The columns are independent $\iff r = n$ (so that $\dim \mathrm{Nul}(A) = 0$).
  But also: the rows are independent $\iff r = m$.
  In the case $m = n$, these two conditions are equivalent.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is orthogonal to $\mathrm{Nul}(A^T)$.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is orthogonal to $\mathrm{Nul}(A^T)$.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is orthogonal to $\mathrm{Nul}(A^T)$.
  This follows from "$A\mathbf{x} = \mathbf{b}$ has a solution x if and only if b is in $\mathrm{Col}(A)$"

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is orthogonal to Nul(A^T).
  This follows from "$A\mathbf{x} = \mathbf{b}$ has a solution x if and only if b is in Col(A)"
  together with the fundamental theorem, which says that Col(A) is the orthogonal complement of Nul(A^T).

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is orthogonal to $\text{Nul}(A^T)$.
  This follows from "$A\mathbf{x} = \mathbf{b}$ has a solution x if and only if b is in $\text{Col}(A)$"
  together with the fundamental theorem, which says that $\text{Col}(A)$ is the orthogonal complement of $\text{Nul}(A^T)$.
- The rows of A are independent if and only if $\text{Nul}(A^T) = \{\mathbf{0}\}$.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is orthogonal to $\text{Nul}(A^T)$.
  This follows from "$A\mathbf{x} = \mathbf{b}$ has a solution x if and only if b is in $\text{Col}(A)$"
  together with the fundamental theorem, which says that $\text{Col}(A)$ is the orthogonal complement of $\text{Nul}(A^T)$.
- The rows of A are independent if and only if $\text{Nul}(A^T) = \{\mathbf{0}\}$.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is orthogonal to $\mathrm{Nul}(A^T)$.
  This follows from "$A\mathbf{x} = \mathbf{b}$ has a solution x if and only if b is in $\mathrm{Col}(A)$"
  together with the fundamental theorem, which says that $\mathrm{Col}(A)$ is the orthogonal complement of $\mathrm{Nul}(A^T)$.
- The rows of A are independent if and only if $\mathrm{Nul}(A^T) = \{\mathbf{0}\}$.
  Recall that elements of $\mathrm{Nul}(A)$ correspond to linear relations between the columns of A.

Think about why each of these statements is true!

- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if $\mathbf{b}$ is orthogonal to $\text{Nul}(A^T)$.
  This follows from "$A\mathbf{x} = \mathbf{b}$ has a solution x if and only if b is in $\text{Col}(A)$"
  together with the fundamental theorem, which says that $\text{Col}(A)$ is the orthogonal complement of $\text{Nul}(A^T)$.
- The rows of A are independent if and only if $\text{Nul}(A^T) = \{\mathbf{0}\}$.
  Recall that elements of $\text{Nul}(A)$ correspond to linear relations between the columns of A.
  Likewise, elements of $\text{Nul}(A^T)$ correspond to linear relations between rows of A.

Networks

Networks

- Given a network get a matrix A with a row for each edge(arrow) and a column for each node.

Networks

- Given a network get a matrix A with a row for each edge(arrow) and a column for each node.

Networks

- Given a network get a matrix A with a row for each edge(arrow) and a column for each node. In each row we have a single -1 in the "tail of the arrow" column, a single $+1$ in "head of the arrow" column, for the rest zeroes.

Networks

- Given a network get a matrix A with a row for each edge(arrow) and a column for each node. In each row we have a single -1 in the "tail of the arrow" column, a single $+1$ in "head of the arrow" column, for the rest zeroes.
- From such a matrix you can reconstruct the network.

Networks

- Given a network get a matrix A with a row for each edge(arrow) and a column for each node. In each row we have a single -1 in the "tail of the arrow" column, a single $+1$ in "head of the arrow" column, for the rest zeroes.
- From such a matrix you can reconstruct the network.
- The null space $\mathrm{Nul}(A)$ has a basis vector for each connected component of the network.

Networks

- Given a network get a matrix A with a row for each edge(arrow) and a column for each node. In each row we have a single -1 in the "tail of the arrow" column, a single $+1$ in "head of the arrow" column, for the rest zeroes.
- From such a matrix you can reconstruct the network.
- The null space Nul(A) has a basis vector for each connected component of the network.
- The left null space Nul(A^T) has a basis vector for each small loop in the network.

That is it.

Good luck!