Math 415 - Lecture 23

Projections on subspaces

Monday October 19th 2015

Textbook reading: Chapter 3.2, 3.3, 3.4

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Suggested practice exercises: Chapter 3.2 Exercise 17, 18, 24,
Chapter 3.4 Exercise 2, 3 and see exercise at the end of this notes

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Suggested practice exercises: Chapter 3.2 Exercise 17, 18, 24, Chapter 3.4 Exercise 2, 3 and see exercise at the end of this notes

Khan Academy video: Projections onto subspaces, Visualizing a projection onto a plane, Projection is closest vector in subspace

Review

- **Orthogonal projection** of x onto y:

$$\hat{\mathbf{x}} = \frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}$$

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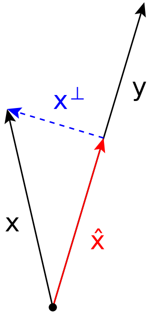

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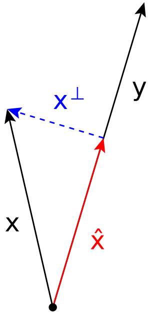

Remark

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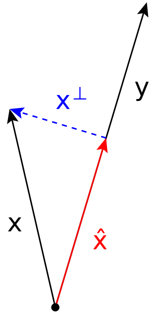

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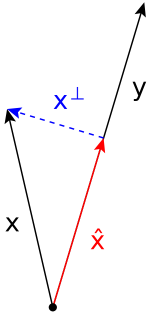

Remark

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Remark

The formulas simplify when you project on unit vectors: all denominators are then 1.

Example

Express $\underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{x}}$ in terms of the basis $\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\mathbf{y}_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{y}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{y}_3}$

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Solution.

Notice that $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ is an orthogonal basis of $\mathbb{R}^3$.

Last time

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Last time

$$\begin{bmatrix}2\\1\\1\end{bmatrix} = c_1\begin{bmatrix}1\\-1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \underbrace{\frac{\begin{bmatrix}2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\end{bmatrix}}_{\text{projection of } \mathbf{x} \text{ onto } \mathbf{y}_1} +$$

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Orthogonal projection on subspaces

Projecting onto a subspace

Theorem

Let W be a subspace of $\mathbb{R}^n$. Then, each $\mathbf{x}$ in $\mathbb{R}^n$ can be uniquely written as

$$\mathbf{x} = \underbrace{\hat{\mathbf{x}}}_{\text{in } W} + \underbrace{\mathbf{x}^{\perp}}_{\text{in } W^{\perp}}$$

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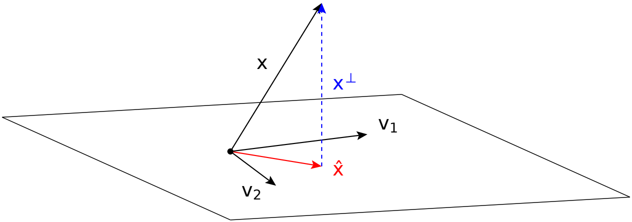

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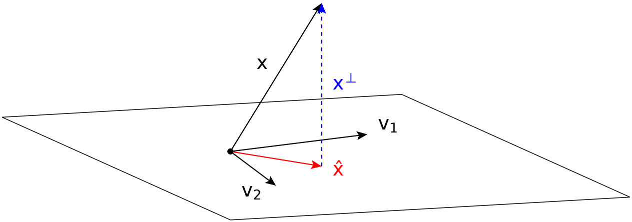
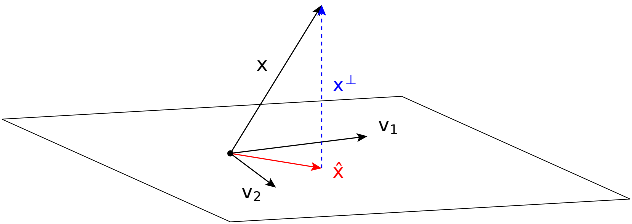
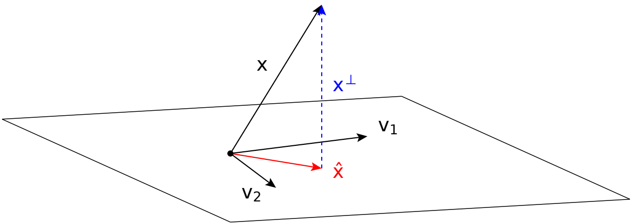


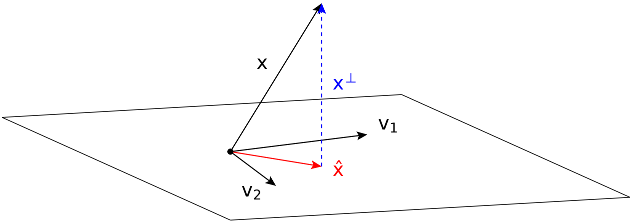
$\hat{\mathbf{x}}$ is the **orthogonal projection** of $\mathbf{x}$ onto W.

Projecting onto a subspace





- $\hat{\mathbf{x}}$ is the point in W closest to $\mathbf{x}$.



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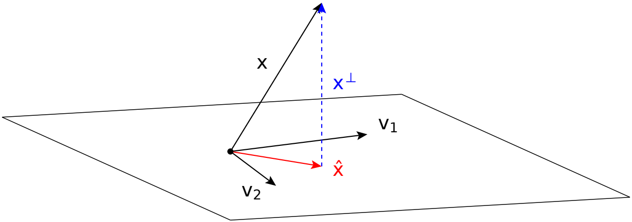
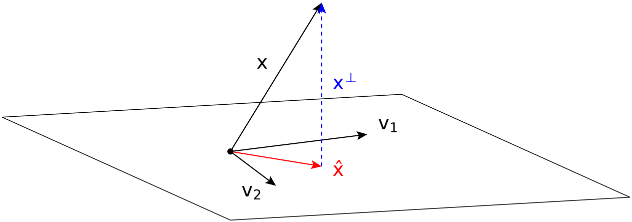


- $\hat{\mathbf{x}}$ is the point in W closest to $\mathbf{x}$. For any other $\mathbf{y}$ in W, $\text{dist}(\mathbf{x}, \hat{\mathbf{x}}) < \text{dist}(\mathbf{x}, \mathbf{y})$.



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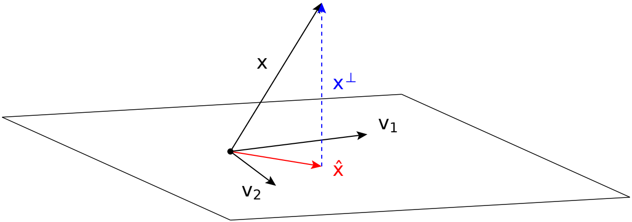

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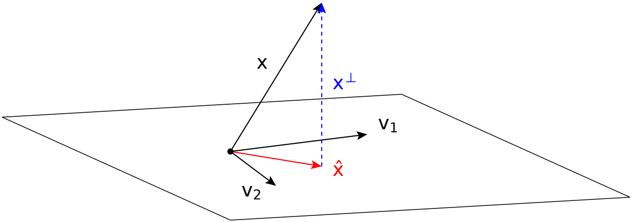

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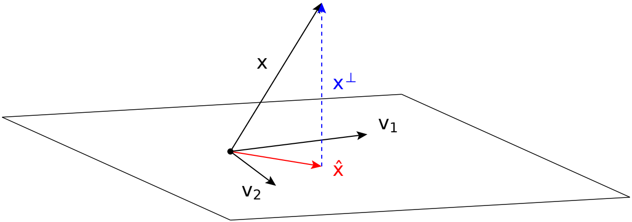

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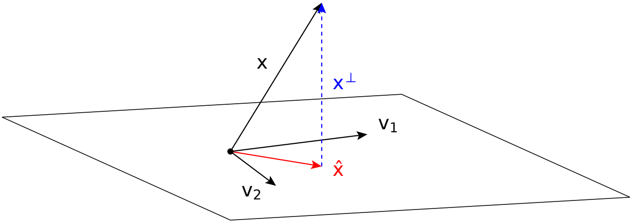

- $\hat{\mathbf{x}}$ is the point in W closest to $\mathbf{x}$. For any other $\mathbf{y}$ in W, $\text{dist}(\mathbf{x}, \hat{\mathbf{x}}) < \text{dist}(\mathbf{x}, \mathbf{y})$.
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Once $\hat{\mathbf{x}}$ is determined, $\mathbf{x}^\perp = \mathbf{x} - \hat{\mathbf{x}}$.
(This is also the orthogonal projection of $\mathbf{x}$ onto $W^\perp$.)

Example

Let $W = span\left\{\begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$, and $\mathbf{x} = \begin{bmatrix}0\\3\\10\end{bmatrix}$.

- Find the orthogonal projection of $\mathbf{x}$ onto W. (Or: find the vector in W which is closest to $\mathbf{x}$)
- Write $\mathbf{x}$ as a vector in W plus a vector orthogonal to W.

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Solution.

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Note that $\mathbf{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are an orthogonal basis for W.

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Note that $\mathbf{w}_1 = \begin{bmatrix}3\\0\\1\end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ are an orthogonal basis for W.

(We will soon learn how to construct orthogonal bases ourselves).

Hence, the orthogonal projection of $\mathbf{x}$ onto W is:

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$$\hat{\mathbf{x}} = \frac{\mathbf{x} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{x} \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}w_2$$

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$$= \frac{\begin{bmatrix}0\\3\\10\end{bmatrix}\cdot\begin{bmatrix}3\\0\\1\end{bmatrix}}{\begin{bmatrix}3\\0\\1\end{bmatrix}\cdot\begin{bmatrix}3\\0\\1\end{bmatrix}}\begin{bmatrix}3\\0\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\3\\10\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Hence, the orthogonal projection of $\mathbf{x}$ onto W is:

$$\begin{aligned}
\hat{\mathbf{x}} &= \frac{\mathbf{x}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{x}\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}w_2 \\
&= \frac{\begin{bmatrix}0\\3\\10\end{bmatrix}\cdot\begin{bmatrix}3\\0\\1\end{bmatrix}}{\begin{bmatrix}3\\0\\1\end{bmatrix}\cdot\begin{bmatrix}3\\0\\1\end{bmatrix}}\begin{bmatrix}3\\0\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\3\\10\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}\begin{bmatrix}0\\1\\0\end{bmatrix} \\
&= \frac{10}{10}\begin{bmatrix}3\\0\\1\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} =
\end{aligned}$$

Hence, the orthogonal projection of $\mathbf{x}$ onto W is:

$$\hat{\mathbf{x}} = \frac{\mathbf{x}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{x}\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}w_2$$

$$= \frac{\begin{bmatrix}0\\3\\10\end{bmatrix}\cdot\begin{bmatrix}3\\0\\1\end{bmatrix}}{\begin{bmatrix}3\\0\\1\end{bmatrix}\cdot\begin{bmatrix}3\\0\\1\end{bmatrix}}\begin{bmatrix}3\\0\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\3\\10\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

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Warning

This calculation only works for orthogonal $\mathbf{w_1}, \mathbf{w_2}$!

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Indeed, $\begin{bmatrix} -3 \\ 0 \\ 9 \end{bmatrix}$ is orthogonal to $\mathbf{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

The matrix of a projection

Definition

Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be an orthogonal basis of W, a subspace of $\mathbb{R}^n$.

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$$\hat{\mathbf{x}} = \left(\frac{\mathbf{x} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 + \ldots + \left(\frac{\mathbf{x} \cdot \mathbf{v}_m}{\mathbf{v}_m \cdot \mathbf{v}_m}\right)\mathbf{v}_m$$

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is a linear map. The matrix P representing π_W with respect to the standard basis is the **projection matrix**.

Example

Find the projection matrix P for the orthogonal projection onto

$$W = span\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

in $\mathbb{R}^3$.

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in $\mathbb{R}^3$.

Solution.

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in $\mathbb{R}^3$.

Solution. Standard basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

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Hence $P=\begin{bmatrix}\frac{9}{10} & * & *\\ 0 & * & *\\ \frac{3}{10} & * & *\end{bmatrix}$.

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Hence $P = \begin{bmatrix} \frac{9}{10} & 0 & * \\ 0 & 1 & * \\ \frac{3}{10} & 0 & * \end{bmatrix}$.

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Hence $P=\begin{bmatrix}\frac{9}{10} & 0 & \frac{3}{10}\\ 0 & 1 & 0\\ \frac{3}{10} & 0 & \frac{1}{10}\end{bmatrix}$.

Let's do the earlier example again using the matrix P.

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Example

Let $W = span\left\{\begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$, and $\mathbf{x} = \begin{bmatrix}0\\3\\10\end{bmatrix}$. Find the orthogonal projection of $\mathbf{x}$ onto W.

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Solution.

$$\hat{\mathbf{x}} = P\mathbf{x}$$

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as in the previous example.

Example

Compute P^2 when

$$P = \begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix}.$$

Explain why the answer makes sense.

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$$P^2 = P$$

Once we have projected down onto W, projecting onto W again does not change anything!

Practice problems

Practice problems

Example

Find the closest point to $\mathbf{x}$ in $span\{\mathbf{v}_1, \mathbf{v}_2\}$ where

$$\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

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Solution.

$$\hat{\mathbf{x}} = \frac{6}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{-2}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

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Solution.

$$\hat{\mathbf{x}} = \frac{6}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{-2}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix}$$

Example

If P is the projection matrix for projecting on W, what is the projection matrix Q for projecting on $W^{\perp}$?

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Solution. $Q = 1 - P$!

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Let P be the projection matrix for projecting on W, and let $\mathbf{x}$ be some vector.

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