- Finish Order stats.
- Discuss Exam 1.

~~Suppose~~ Fish in a lake

Suppose the size of fish in a lake is independent and uniform on $(0,8)$. A fisherperson catches 5 fish. X_1, X_2, X_3, X_4, X_5 denote weights

$Y_k =$ k^{th} smallest fish.

$$X \sim U(0,8) \Rightarrow F(x) = \frac{x}{8}, \quad 0<x<8$$

a) $$P(Y_1 \le 2) = 1 - P(Y_1 > 2) = 1 - P(X_1>2, X_2>2, X_3>2, X_4>2, X_5>2)$$

$$= 1 - \prod_{i=1}^{n} P(X_i > 2)$$ independence

CDF defn

$$= 1 - \prod_{i=1}^{n} (1 - F_{X_i}(2))$$ identical dist.

$$= 1 - \prod_{i=1}^{n} (1 - F(2)) = 1 - (1-F(2))^5$$

$$= 1 - \left(1 - \frac{2}{8}\right)^5 = 1 - \left(\frac{3}{4}\right)^5 \approx .763$$

OR

$$f_{Y_1}(x) = n(1-F(x))^{n-1} f(x) = 5\left(1 - \frac{x}{8}\right)^4 \frac{1}{8}, \quad 0 < x < 8$$

$$P(Y_1 \le 2) = \frac{5}{8}\int_0^2 \left(1 - \frac{x}{8}\right)^4 dx =$$

b) Find $P(Y_5 > 7)$. the prob. the largest fish weighs more than 7 pounds.

$$= 1 - P(Y_5 \le 7) = 1 - P(X_1 \le 7, X_2 \le 7, X_3 \le 7, X_4 \le 7, X_5 \le 7)$$

$$= 1 - \prod_{i=1}^{5} P(X_i \le 7) \quad \text{(indep)}$$

$$= 1 - [F(7)]^5 = 1 - \left[\frac{7}{8}\right]^5 \approx .4871$$

OR

$$f_{\max x_i}(x) = n[F(x)]^{n-1} f(x)$$

$$= 5\left(\frac{x}{8}\right)^4 \frac{1}{8} = \frac{5}{8^5} x^4, \quad 0 < x < 8$$

$$P(Y_5 > 7) = \frac{5}{8^5}\int_7^8 x^4\,dx = \ldots$$

c) $P(6 < Y_5 < 7)$ Prob. the largest fish caught is b/w 6 & 7 pounds.

$$= P(Y_5 < 7) - P(Y_5 < 6) = \left(\frac{7}{8}\right)^5 - \left(\frac{6}{8}\right)^5 \approx .2756$$

OR

$$P(6 < Y_5 < 7) = \frac{5}{8^5}\int_6^7 x^4\,dx = \ldots$$

d) What is the prob. the 2^{nd} largest fish weighs b/w 4 and 6 lbs. $Y_k = k^{th}$ smallest fish

$k = ?$, $k = 4$

$$P(4 < Y_4 < 6) = P(Y_4 < 6) - P(Y_4 < 4)$$

Recall $P(Y_k < x) = \sum_{i=k}^{n} \binom{n}{i} [F(x)]^i [1 - F(x)]^{n-i}$

$$\Rightarrow \sum_{i=4}^{5} \binom{5}{i} [F(6)]^i [1 - F(6)]^{5-i} - \sum_{i=4}^{5} \binom{5}{i} [F(4)]^i [1 - F(4)]^{5-i}$$

$$= 5 \cdot \left(\frac{6}{8}\right)^4 \left(1 - \frac{6}{8}\right) + 1 \cdot \left(\frac{6}{8}\right)^5 - 5 \cdot \left(\frac{4}{8}\right)^4 \left(\frac{4}{8}\right) - 1 \cdot \left(\frac{4}{8}\right)^5 = .4453$$

$=$

OR

$$f_{Y_4}(x) = \frac{5!}{(4-1)!\,(5-4)!} \left(\frac{x}{8}\right)^{4-1} \left(1 - \frac{x}{8}\right)^{5-4} \underbrace{\frac{1}{8}}_{f(x)}$$

$$P(4 < Y_4 < 6) = \int_4^6 f_{Y_4}(x)\, dx = \ldots$$

Suppose $X_1, \dots, X_n$ are independent r.v.'s where

$X_i \sim \text{Geometric}(p_i)$ for $i = 1, \dots, n$. Find $\underline{P(Y_1 = y)}$

Survival analysis

Recall the pmf is $P(X=k) = (1-p)^{k-1} p$

$$P(X > y) = \sum_{k=y+1}^{\infty} p(1-p)^{k-1} = p(1-p)^y \underbrace{\sum_{k=0}^{\infty} (1-p)^k}_{\text{geometric series}}$$

$\frac{1}{p}$

$$\boxed{P(X>y) = (1-p)^y}$$

$$P(Y_1 = y) = P(Y_1 > y-1) - P(\underline{Y_1 > y})$$

need $P(Y_1 > y) = P(X_1 > y) P(X_2 > y) \dots P(X_n > y)$

$$= (1-p_1)^y (1-p_2)^y \cdots (1-p_n)^y = \left[\prod_{i=1}^n (1-p_i)\right]^y$$

$$P(Y_1 = y) = \left[\prod_{i=1}^n (1-p_i)\right]^{y-1} - \left[\prod_{i=1}^n (1-p_i)\right]^y$$

$$= \left[\prod_{i=1}^n (1-p_i)\right]^{y-1} \left(1 - \prod_{i=1}^n (1-p_i)\right)$$

$$Y_1 \sim \text{Geometric}\left(1 - \prod_{i=1}^n (1-p_i)\right)$$

Suppose X_1, X_2, X_3, X_4 are iid and cont. w/ pdf $f(x)$. Find the joint pdf of Y_1, Y_2, Y_3, Y_4 if $X_i \sim Exp(1)$.

$$\Rightarrow f(x) = e^{-x}$$

Use Theorem 4.4.1 $g(y_1, \dots, y_n) = n!\, f(y_1) \cdots f(y_n)$, $a < y_1 < y_2 < \cdots < y_n < b$

a) $g(y_1, y_2, y_3, y_4) = 4!\, e^{-y_1} e^{-y_2} e^{-y_3} e^{-y_4}$

$= 24\, e^{-\sum_{i=1}^{4} y_i}$, $0 < y_1 < y_2 < y_3 < y_4 < \infty$

b) Find $g(y_1, y_2, y_4) = \int_{y_2}^{y_4} g(y_1, y_2, y_3, y_4)\, dy_3$

c) $g(y_3 \mid y_1, y_2, y_4) = \dfrac{g(y_1, y_2, y_3, y_4)}{g(y_1, y_2, y_4)}$