Math 415 - Lecture 20

Fundamental Theorem of Linear algebra, orthogonal complement of fundamental subspaces of a matrix

Monday October 12th 2015

Textbook reading: Chapter 3.1

Suggested practice exercises: Chapter 2.6, 5,6,7,36,37

Khan Academy video: Orthogonal complements

Strang lecture: Lecture 14: Orthogonal vectors and subspaces

1 Review

1.1 Orthogonality and FTLA

- $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are **orthogonal** iff $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w} = v_1w_1 \cdots + v_nw_n = 0$.
  - This simple criterion is equivalent to Pythagoras' theorem.
  - Non-zero orthogonal vectors are independent.
- If V is a subspace of $\mathbb{R}^n$ then the *orthogonal complement* of V is

$$V^\perp = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{x} = 0, \text{for all } \mathbf{v} \in V\}$$

- If $W = V^\perp$ then $W^\perp = V$.
- In other words $(V^\perp)^\perp = V$.
- $\dim(V) + \dim(V^\perp) = \dim(\mathbb{R}^n) = n$

Example 1. Let V be the horizontal x-y-plane in $\mathbb{R}^3$ and W the vertical y-z-plane.

- Is it true that W is orthogonal to V?
- Is W the orthogonal complement of V?
- What is the orthogonal complement of V?

Solution.

Theorem 1 (Fundamental Theorem of Linear Algebra). *Let A be a $m \times n$-matrix. Then*

- *$Nul(A)$ is the orthogonal complement of $Col(A^T)$ (in $\mathbb{R}^n$). Also, dim $Nul(A)$ + dim $Col(A^T) = (n-r)+r = n$.*
- *$Col(A)$ is the orthogonal complement of $Nul(A^T)$ (in $\mathbb{R}^m$).*

Solution.

Example 2. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. What is the orthogonal complement of $\mathrm{Nul}(A)$?

Solution.

Example 3. Find all vectors orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Solution.

Alternative solution. The FLTA is not magic! You can do this the old-fashioned way!

Example 4. Let $V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b = 2c \right\}$. Find a basis for the orthogonal complement of V.

Solution.

Example 5. Let $V = \left\{ \begin{bmatrix} 2a+b \\ -b \\ a+b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Find the orthogonal complement of V.

Solution.

2 A new perspective on $A\mathbf{x} = \mathbf{b}$

To see if $A\mathbf{x} = \mathbf{b}$ has a solution, check that

Direct approach: $\mathbf{b} \in Col(A)$

Indirect approach: $\mathbf{b} \perp Nul(A^T)$

The indirect approach means:

Example 6. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution?

Solution (old).

Solution (new).

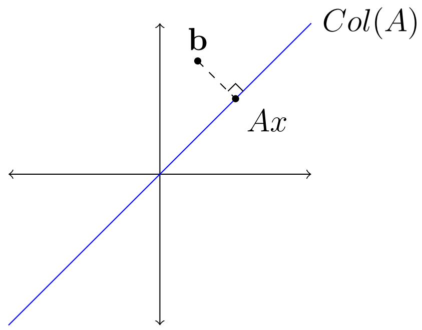
Why do we care about orthogonality?Not all linear systems have solutions. For example, $A\mathbf{x} = \mathbf{b}$ with

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

has no solution: $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is not in $Col(A) = span \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

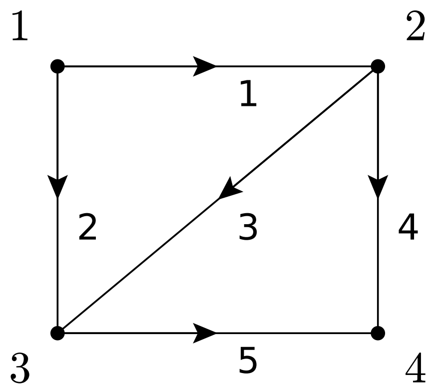
Idea. Instead of giving up, we want the $\mathbf{x}$ which makes $A\mathbf{x}$ and $\mathbf{b}$ as close as possible.

Such $\mathbf{x}$ is characterized by $A\mathbf{x}$ being **orthogonal** to the error $\mathbf{b} - A\mathbf{x}$.



3 Application: Directed graphs

- Graphs appear in network analysis (e.g. internet) or circuit analysis.
- Arrow indicates direction of flow
- No edges from a node to itself



Definition 7. Let G be a graph with m edges and n nodes. The edge-node incidence matrix of G is the $m \times n$ matrix A with

$$A_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j \\ +1, & \text{if edge } i \text{ enters node } j \\ 0, & \text{otherwise} \end{cases}$$

Example 8. Give the edge-node incidence matrix of our graph.

Solution.