Quiz 15

1. Suppose $L_1 \subseteq \Sigma^*$ and $h : \Sigma^* \to \Delta^*$ is a homomorphism. Let $L_2 = h(L_1)$. Which of the following statements is necessarily true, given that context-free languages are closed under homomorphic images?

   (A) L_1 and L_2 are both context-free languages.

   (B) Either L_1 and L_2 are both context-free languages or neither one of them is.

   (C) If L_2 is a context-free language then L_1 is a context-free language.

   (D) If L_2 is not a context-free language then L_1 is not a context-free language.

   Correct answer is (D).

2. Consider the following "proof" that context-free languages are closed under inverse homomorphic images. Suppose L_1 is a context-free language, h is a homomorphism, and $L_2 = h^{-1}(L_1)$. Suppose, for contradiction, that L_2 is not context-free. Then $h(L_2) = L_1$ is not context-free because context-free languages are closed under homomorphic images. This contradicts our assumption that L_1 is context-free. Pick the strongest statement that describes why this proof is incorrect.

   (A) The proof is incorrect because $h(L_2)$ need not be equal to L_1.

   (B) The proof is incorrect because closure under homomorphic images does not mean that if L_2 is not context-free then $h(L_2)$ is not context-free.

   (C) The proof is incorrect because non context free languages may not be closed under homomorphic images.

   (D) All of the options are correct.

   Correct answer is (D).

3. Consider the following proof that context-free languages are closed under homomorphic images. Observe that every regular language is also context-free since an NFA is a special PDA that never pushes or pops from its stack. We proved that regular languages are closed under homomorphic images. Hence, context-free languages must also be closed under homomorphic images. Which of the following statements is true?

   (A) The proof is correct.

   (B) The proof is incorrect because closure of regular languages under homomorphic images does not imply closure of context-free languages under homomorphic images.

   (C) The proof is incorrect because there are regular languages that are not context-free.

   (D) The proof is incorrect because regular languages are not closed under homomorphic images.

   Correct answer is (B).

4. Context-free languages are not closed under intersection. This means that

   (A) If L_1, L_2 are non-context-free languages then $L_1 \cap L_2$ is not context-free.

   (B) If $L_1 \cap L_2$ is context-free then L_1 and L_2 are context-free languages.

   (C) There are context-free languages L_1 and L_2 such that $L_1 \cap L_2$ is not context-free.

   (D) If $L_1 \cap L_2$ is not context-free then either L_1 or L_2 is not context-free.

Correct answer is (C).

5. Let $G_1 = (\{S_1, A\}, \{a, b\}, \{S_1 \rightarrow aA,\ A \rightarrow bA \mid \epsilon\}, S_1)$ and $G_2 = (\{S_2, A\}, \{a, b\}, \{S_2 \rightarrow bA,\ A \rightarrow aA \mid \epsilon\}, S_2)$. Construct grammar G to describe $\mathbf{L}(G_1) \cup \mathbf{L}(G_2)$ by adding a new start symbol and rules from S that go to either S_1 or S_2. More precisely, $G = (\{S, S_1, S_2, A\}, \{a, b\}, \{S \rightarrow S_1 \mid S_2,\ S_1 \rightarrow bA,\ S_2 \rightarrow aA,\ A \rightarrow bA \mid aA \mid \epsilon\}, S)$. Which of the following is the *strongest* statement that is true?

   (A) $\mathbf{L}(G) = \mathbf{L}(G_1) \cup \mathbf{L}(G_2)$.

   (B) $\mathbf{L}(G) \neq \mathbf{L}(G_1) \cup \mathbf{L}(G_2)$ and this shows that context-free languages are not closed under $\cup$.

   (C) $\mathbf{L}(G) \neq \mathbf{L}(G_1) \cup \mathbf{L}(G_2)$ and this shows that our proof establishing the closure of context-free languages under $\cup$ is incorrect.

   (D) $\mathbf{L}(G) \neq \mathbf{L}(G_1) \cup \mathbf{L}(G_2)$.

   Correct answer is (D).