Math 415 - Lecture 4

Linear Combinations and Matrix operations

Monday August 31 2015

Textbook: Chapter 1.3, 1.4

Suggested Practice Exercise: Chapter 1.4 Exercise 1, 2, 10, 12, 13, 21, 30, 34, 45,

Khan Academy Video: Matrix multiplication (part I)

Review

A system such as

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

can be written in vector form as

$$x \begin{bmatrix} \\ \\ \\ \end{bmatrix} + y \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

The left-hand side is a linear combination of the vectors $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$ and $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$.

The row and column picture

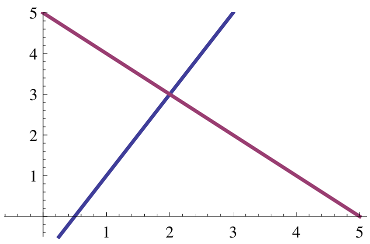
Example 1. We can think of the linear system

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

in two different geometric ways. Recall unique solution: $x = 2, y = 3$.

Row picture

- Each equation defines a line in $\mathbb{R}^2$.
- Which points lie on the intersection of these lines?
- (2, 3) is the (only) intersection of the two lines $2x - y = 1$ and $x + y = 5$.

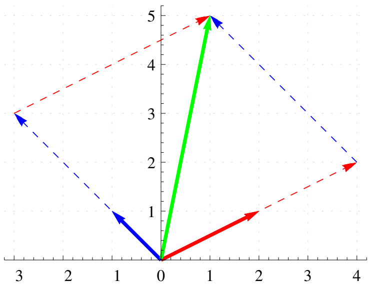


Column picture

- The system can be written as

$$x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

- Which linear combinations of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ produce $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$?
- (2, 3) are the coefficients of the (only) such linear combination.



Application: Industrial Espionage.

Suppose a candy factory produces stuff , in particular:

- *Doohickeys*,
- *Nicknacks* and (of course)
- *Widgets.*

To produce these you need raw materials:

- Sugar,
- Spice and
- Everything Nice (mystery ingredient!),

in different quantities. We need 10 units of Sugar, 5 units of Spice, 2 units of Everything Nice to produce one Doohickey. Furthermore, to produce one Nicknack one unit of Sugar, 2 units of Spice and 3 units of Everything nice are used. Finally, for one Widget 3 units of Sugars, 5 units of Spice and 2 units of Everything Nice are neccessary. Now suppose that somebody (say a competitor of our factory) is interested in determining how many Doohickeys are produced. It is a trade secret, so you can not go and ask the factory. If many Doohickeys are produced the competitor might switch to the production of Whatchamacallits. So the competitor sends a spy to the entrance of the factory and she writes down how many truck loads of raw materials enter. The spy reports back that 14 units of Sugars, 12 units of Spice and 7 units of Everything nice were delivered to the factory. Assume that everything is used for production, can you determine the number of Doohickeys produced?

Matrix operations

Matrices are like Numbers: Matrix Algebra.

Two ways to denote $m \times n$ matrix A (m rows, n column).

- In terms of the columns of A:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

- In terms of the entries of A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- a_{ij} is in the ith row and jth column
- $\mathbf{a_j}$ is j^{th} column:

$$\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

- **Main diagonal entries:** a_{11}, a_{22}, $\ldots$, a_{mm} (only care about these when $m = n$)

Even more notation

- **Zero matrix:**

$$0 = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

Definition. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$ be $m \times n$-matrices and let r be a scalar. Then

- $A + B$ is defined by

$$A + B = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

- Moreover, rA is defined as

$$rA = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Example 2. Calculate

- $$\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

- $$10 \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Theorem 1. *Let A, B, and C be matrices of the same size, and let r and s be scalars.*

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + 0 = A$
- $r(A + B) = rA + rB$
- $(r + s) A = rA + sA$
- $r(sA) = (rs) A$

Matrix Multiplication

How to multiply matrices and vectors

Let $\mathbf{x}$ be a vector, A, B matrics.

- Multiplying B and $\mathbf{x}$ transforms $\mathbf{x}$ into the vector $B\mathbf{x}$.
- In turn, if we multiply A and $B\mathbf{x}$, we transform $B\mathbf{x}$ into $A(B\mathbf{x})$.
- So $A(B\mathbf{x})$ is the composition of two mappings.

Define the product AB so that

$$A(B\mathbf{x}) = (AB)\mathbf{x}$$

Suppose A is $m \times n$ and B is $n \times p$ where

$$B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_p] \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

Define $B\mathbf{x}$

Then $A\left(B\mathbf{x}\right)$ is

$$\boxed{}$$

Example 3.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then $B\mathbf{x}$ is

$$\boxed{}$$

Compute $A(B\mathbf{x})$ **using** $B\mathbf{x} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$:

$$A(B\mathbf{x}) = A\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \boxed{}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \boxed{}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Compute $A(B\mathbf{x})$ **using** $A\left(B\mathbf{x}\right) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 + \ldots + x_n A\mathbf{b}_n$:

$$A(B\mathbf{x}) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 = x_1 \begin{bmatrix} \\ \\ \\ \end{bmatrix} + x_2 \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Same answer!