Review of Discrete and Continuous Distributions

Steven Andrew Culpepper, Ph.D.
Department of Statistics
University of Illinois at Urbana-Champaign
sculpepp@illinois.edu

STAT 410/MATH 464
Fall 2015

Random Variables

Discrete

- probability mass function (pmf)
- $p(x) = P(X = x)$
- $\forall\ x\ 0 \leq p(x) <\leq 1$
- $\sum_{\text{all } x} p(x) = 1.$
- cumulative distribution function (cdf)
- $F(x) = P(X \leq x) = \sum_{y \leq x} p(y)$

Continuous

- probability density function (pdf)
- $f(x)$
- $\forall\ x\ f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)\, dx = 1$
- cumulative distribution function
- $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\, dy$

Expected Values

Let $g(X)$ be a function of the random variable X.

Discrete

- If $\sum_{\text{all } x} |g(x)| \, p(x) < \infty$, $E[g(X)] = \sum_{\text{all } x} g(x) \, p(x)$

Continuous

- If $\int_{\infty}^{\infty} |g(x)| \, f(x) \, dx < \infty$, $E[g(X)] = \int_{-\infty}^{\infty} g(x) \, f(x) \, dx$

Common functions of $g(X)$ include:

- $g(X) = X$, expected value or μ
- $g(X) = (X - \mu)^2$, variance or σ^2
- $g(X) = X^r$, the rth moment of X
- $g(X) = e^{Xt}$, moment generating function, $\mathcal{M}(t)$

Example #1

x	$p(x)$	$F(x)$	$xp(x)$	$x^2p(x)$	$e^{xt}p(x)$
1	0.2	0.2	0.2	0.2	$0.2e^{t}$
2	0.4	0.6	0.8	1.6	$0.4e^{2t}$
3	0.3	0.9	0.9	2.7	$0.3e^{3t}$
4	0.1	1.0	0.4	1.6	$0.1e^{4t}$

- $E(X)$
- $E(X^2)$
- σ^2
- $\mathcal{M}(t)$

Example #1

x	$p(x)$	$F(x)$	$xp(x)$	$x^2p(x)$	$e^{xt}p(x)$
1	0.2	0.2	0.2	0.2	$0.2e^t$
2	0.4	0.6	0.8	1.6	$0.4e^{2t}$
3	0.3	0.9	0.9	2.7	$0.3e^{3t}$
4	0.1	1.0	0.4	1.6	$0.1e^{4t}$

- $E(X) = 0.2 + 0.8 + 0.9 + 0.4 = 2.3$
- $E(X^2)$
- σ^2
- $\mathcal{M}(t)$

Example #1

x	$p(x)$	$F(x)$	$xp(x)$	$x^2p(x)$	$e^{xt}p(x)$
1	0.2	0.2	0.2	0.2	$0.2e^{t}$
2	0.4	0.6	0.8	1.6	$0.4e^{2t}$
3	0.3	0.9	0.9	2.7	$0.3e^{3t}$
4	0.1	1.0	0.4	1.6	$0.1e^{4t}$

- $E(X) = 0.2 + 0.8 + 0.9 + 0.4 = 2.3$
- $E(X^2) = 0.2 + 1.6 + 2.7 + 1.6 = 6.1$
- σ^2
- $\mathcal{M}(t)$

Example #1

x	$p(x)$	$F(x)$	$xp(x)$	$x^2p(x)$	$e^{xt}p(x)$
1	0.2	0.2	0.2	0.2	$0.2e^{t}$
2	0.4	0.6	0.8	1.6	$0.4e^{2t}$
3	0.3	0.9	0.9	2.7	$0.3e^{3t}$
4	0.1	1.0	0.4	1.6	$0.1e^{4t}$

- $E(X) = 0.2 + 0.8 + 0.9 + 0.4 = 2.3$
- $E(X^2) = 0.2 + 1.6 + 2.7 + 1.6 = 6.1$
- $\sigma^2 = 6.1 - (2.3)^2 = 0.81$
- $\mathcal{M}(t)$

Example #1

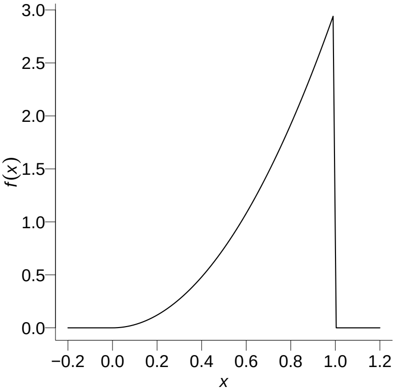
x	$p(x)$	$F(x)$	$xp(x)$	$x^2p(x)$	$e^{xt}p(x)$
1	0.2	0.2	0.2	0.2	$0.2e^{t}$
2	0.4	0.6	0.8	1.6	$0.4e^{2t}$
3	0.3	0.9	0.9	2.7	$0.3e^{3t}$
4	0.1	1.0	0.4	1.6	$0.1e^{4t}$

- $E(X) = 0.2 + 0.8 + 0.9 + 0.4 = 2.3$
- $E\left(X^2\right) = 0.2 + 1.6 + 2.7 + 1.6 = 6.1$
- $\sigma^2 = 6.1 - (2.3)^2 = 0.81$
- $\mathcal{M}(t) = 0.2e^{t} + 0.4e^{2t} + 0.3e^{3t} + 0.1e^{4t}$

Example #2

Let X be a continuous random variable with the probability density function,

$$f(x) = \begin{cases} kx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

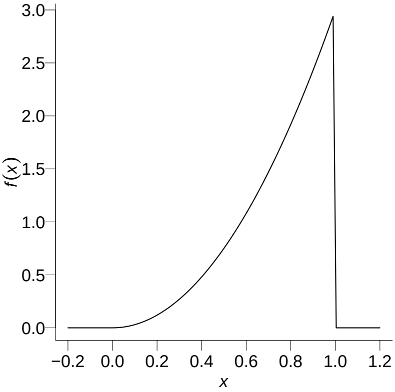


a. Find k.

Example #2

Let X be a continuous random variable with the probability density function,

$$f(x) = \begin{cases} kx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

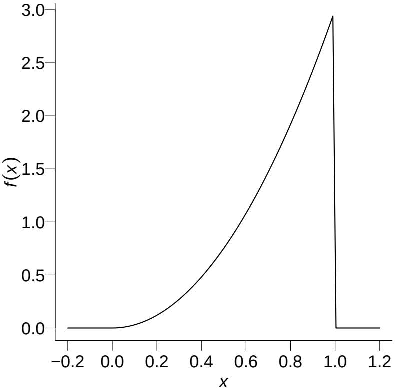


a. Find k. Need to find k so $f(x) \geq 0$ and $\int f(x)\,dx = 1$.

Example #2

Let X be a continuous random variable with the probability density function,

$$f(x) = \begin{cases} kx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



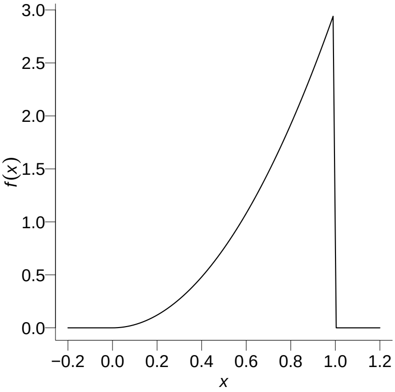
a. Find k. Need to find k so $f(x) \geq 0$ and $\int f(x)\,dx = 1$.

$$1 = k \int_0^1 x^2 dx$$

Example #2

Let X be a continuous random variable with the probability density function,

$$f(x) = \begin{cases} kx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



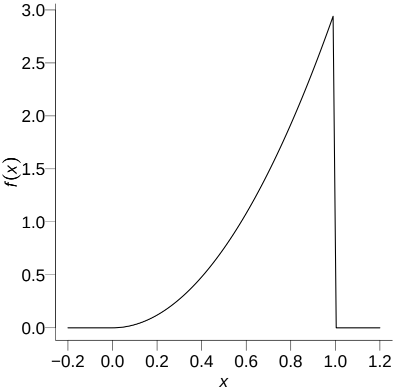
a. Find k. Need to find k so $f(x) \geq 0$ and $\int f(x)\,dx = 1$.

$$1 = k \int_0^1 x^2 dx = \frac{k}{3} x^3 \Big|_0^1$$

Example #2

Let X be a continuous random variable with the probability density function,

$$f(x) = \begin{cases} kx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



a. Find k. Need to find k so $f(x) \geq 0$ and $\int f(x)\,dx = 1$.

$$1 = k \int_0^1 x^2 dx = \frac{k}{3} x^3 \Big|_0^1 \Rightarrow k = 3$$

Example #2 (continued)

b. Find $F(x) = P(X \leq x)\ \forall$
values of x.

$$F(x) = \begin{cases} & x < 0 \\ & 0 \leq x < 1 \\ & x \geq 1 \end{cases}$$

c. Find the probability $P(0.4 \leq X \leq 0.8)$.

Example #2 (continued)

b. Find $F(x) = P(X \leq x)$ $\forall$
values of x.

$$F(x) = \begin{cases} 0 & x < 0 \\ & 0 \leq x < 1 \\ & x \geq 1 \end{cases}$$

c. Find the probability $P(0.4 \leq X \leq 0.8)$.

Example #2 (continued)

$b.$ Find $F(x) = P(X \leq x)\ \forall$
values of x.

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 3t^2 dt = x^3 & 0 \leq x < 1 \\ & x \geq 1 \end{cases}$$

$c.$ Find the probability $P(0.4 \leq X \leq 0.8)$.

Example #2 (continued)

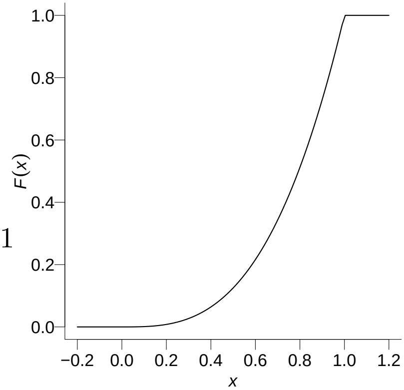
b. Find $F(x) = P(X \leq x)\ \forall$
values of x.

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 3t^2 dt = x^3 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

c. Find the probability $P(0.4 \leq X \leq 0.8)$.

Example #2 (continued)

$b.$ Find $F(x) = P(X \leq x)\ \forall$ values of x.



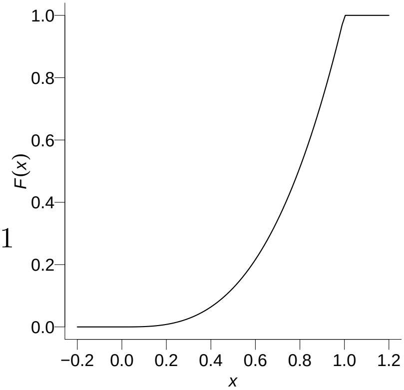
$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 3t^2 dt = x^3 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$c.$ Find the probability $P(0.4 \leq X \leq 0.8)$.

Example #2 (continued)

b. Find $F(x) = P(X \leq x)\ \forall$ values of x.

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 3t^2 dt = x^3 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$



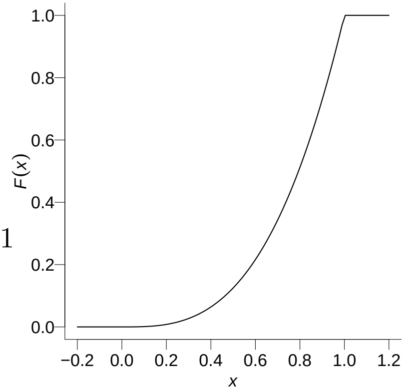
c. Find the probability $P(0.4 \leq X \leq 0.8)$.

$$P(0.4 \leq X \leq 0.8) = \int_{0.4}^{0.8} 3x^2 dx = 0.448$$

Example #2 (continued)

$b.$ Find $F(x) = P(X \leq x)\ \forall$ values of x.

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 3t^2 dt = x^3 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$



$c.$ Find the probability $P(0.4 \leq X \leq 0.8)$.

$$P(0.4 \leq X \leq 0.8) = \int_{0.4}^{0.8} 3x^2 dx = 0.448$$

$$P(0.4 \leq X \leq 0.8) = F(0.8) - F(0.4) = 0.448$$

Example #2 (continued)

d. Find the median, m, of X.

e. Find the mean, $\mu = E(X)$.

Example #2 (continued)

d. Find the median, m, of X. Recall that the median is the quantile with 50% of mass/density above and below.

e. Find the mean, $\mu = E(X)$.

Example #2 (continued)

d. Find the median, m, of X. Recall that the median is the quantile with 50% of mass/density above and below.

$$P\left(X \leq m\right) = F\left(m\right) = \int_{0}^{m} 3x^{2} dx = \frac{1}{2}$$

e. Find the mean, $\mu = E\left(X\right)$.

Example #2 (continued)

d. Find the median, m, of X. Recall that the median is the quantile with 50% of mass/density above and below.

$$P(X \leq m) = F(m) = \int_0^m 3x^2 dx = \frac{1}{2}$$
$$\Rightarrow F(m) = \frac{1}{2}$$

e. Find the mean, $\mu = E(X)$.

Example #2 (continued)

d. Find the median, m, of X. Recall that the median is the quantile with 50% of mass/density above and below.

$$P(X \leq m) = F(m) = \int_0^m 3x^2 dx = \frac{1}{2}$$
$$\Rightarrow F(m) = \frac{1}{2}$$
$$\Rightarrow m^3 = \frac{1}{2} \Rightarrow m = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

e. Find the mean, $\mu = E(X)$.

Example #2 (continued)

d. Find the median, m, of X. Recall that the median is the quantile with 50% of mass/density above and below.

$$P(X \leq m) = F(m) = \int_0^m 3x^2 dx = \frac{1}{2}$$
$$\Rightarrow F(m) = \frac{1}{2}$$
$$\Rightarrow m^3 = \frac{1}{2} \Rightarrow m = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

e. Find the mean, $\mu = E(X)$.

$$\mu = E(X) = \int_0^1 x3x^2 dx$$

Example #2 (continued)

d. Find the median, m, of X. Recall that the median is the quantile with 50% of mass/density above and below.

$$P(X \leq m) = F(m) = \int_0^m 3x^2 dx = \frac{1}{2}$$
$$\Rightarrow F(m) = \frac{1}{2}$$
$$\Rightarrow m^3 = \frac{1}{2} \Rightarrow m = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

e. Find the mean, $\mu = E(X)$.

$$\mu = E(X) = \int_0^1 x3x^2 dx = \frac{3}{4}x^4\Big|_0^1 = \frac{3}{4}$$

Example #2 (continued)

f. Find the standard deviation σ of X.

g. Find the mgf.

Example #2 (continued)

f. Find the standard deviation σ of X.

$$\sigma^2 = Var(X) = \int_0^1 x^2 3x^2 dx - \mu^2$$

g. Find the mgf.

Example #2 (continued)

f. Find the standard deviation σ of X.

$$\sigma^2 = Var\left(X\right) = \int_0^1 x^2 3x^2 dx - \mu^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$$

g. Find the mgf.

Example #2 (continued)

f. Find the standard deviation σ of X.

$$\sigma^2 = Var\left(X\right) = \int_0^1 x^2 3x^2 dx - \mu^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} \Rightarrow \sigma = \frac{1}{4}\sqrt{\frac{3}{5}}$$

g. Find the mgf.

Example #2 (continued)

f. Find the standard deviation σ of X.

$$\sigma^2 = Var\left(X\right) = \int_0^1 x^2 3x^2 dx - \mu^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} \Rightarrow \sigma = \frac{1}{4}\sqrt{\frac{3}{5}}$$

g. Find the mgf.

$$\mathcal{M}\left(t\right) = E\left(e^{Xt}\right)$$

Example #2 (continued)

f. Find the standard deviation σ of X.

$$\sigma^2 = Var\left(X\right) = \int_0^1 x^2 3x^2 dx - \mu^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} \Rightarrow \sigma = \frac{1}{4}\sqrt{\frac{3}{5}}$$

g. Find the mgf.

$$\mathcal{M}\left(t\right) = E\left(e^{Xt}\right) = \int_0^1 e^{xt} 3x^2 dx \text{ (need to i.b.p. twice)}$$

Example #2 (continued)

f. Find the standard deviation σ of X.

$$\sigma^2 = Var(X) = \int_0^1 x^2 3x^2 dx - \mu^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} \Rightarrow \sigma = \frac{1}{4}\sqrt{\frac{3}{5}}$$

g. Find the mgf.

$$\mathcal{M}(t) = E\left(e^{Xt}\right) = \int_0^1 e^{xt} 3x^2 dx \text{ (need to i.b.p. twice)}$$

Let $u = 3x^2$, $du = 6x$, $dv = e^{xt}$, $v = \frac{1}{t}e^{xt}$.

Example #2 (continued)

f. Find the standard deviation σ of X.

$$\sigma^2 = Var(X) = \int_0^1 x^2 3x^2 dx - \mu^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} \Rightarrow \sigma = \frac{1}{4}\sqrt{\frac{3}{5}}$$

g. Find the mgf.

$$\mathcal{M}(t) = E\left(e^{Xt}\right) = \int_0^1 e^{xt} 3x^2 dx \text{ (need to i.b.p. twice)}$$

Let $u = 3x^2$, $du = 6x$, $dv = e^{xt}$, $v = \frac{1}{t}e^{xt}$.

$$\mathcal{M}(t) = 3x^2 \frac{1}{t} e^{xt}\Big|_0^1 - \int_0^1 \frac{1}{t} e^{xt} 6x dx$$

Example #2 (continued)

f. Find the standard deviation σ of X.

$$\sigma^2 = Var(X) = \int_0^1 x^2 3x^2 dx - \mu^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} \Rightarrow \sigma = \frac{1}{4}\sqrt{\frac{3}{5}}$$

g. Find the mgf.

$$\mathcal{M}(t) = E\left(e^{Xt}\right) = \int_0^1 e^{xt} 3x^2 dx \text{ (need to i.b.p. twice)}$$

Let $u = 3x^2$, $du = 6x$, $dv = e^{xt}$, $v = \frac{1}{t}e^{xt}$.

$$\mathcal{M}(t) = 3x^2 \frac{1}{t} e^{xt} \Big|_0^1 - \int_0^1 \frac{1}{t} e^{xt} 6x dx = 3e^t - \int_0^1 \frac{1}{t} e^{xt} 6x dx$$

Example #2 (continued)

f. Find the standard deviation σ of X.

$$\sigma^2 = Var(X) = \int_0^1 x^2 3x^2 dx - \mu^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} \Rightarrow \sigma = \frac{1}{4}\sqrt{\frac{3}{5}}$$

g. Find the mgf.

$$\mathcal{M}(t) = E\left(e^{Xt}\right) = \int_0^1 e^{xt} 3x^2 dx \text{ (need to i.b.p. twice)}$$

Let $u = 3x^2$, $du = 6x$, $dv = e^{xt}$, $v = \frac{1}{t}e^{xt}$.

$$\mathcal{M}(t) = 3x^2 \frac{1}{t} e^{xt}\Big|_0^1 - \int_0^1 \frac{1}{t} e^{xt} 6x dx = 3e^t - \int_0^1 \frac{1}{t} e^{xt} 6x dx$$

Next, let $u = 6x$, $du = 6$, $dv = \frac{1}{t}e^{xt}$, $v = \frac{1}{t^2}e^{xt}$.

Example #2 (continued)

f. Find the standard deviation σ of X.

$$\sigma^2 = Var(X) = \int_0^1 x^2 3x^2 dx - \mu^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} \Rightarrow \sigma = \frac{1}{4}\sqrt{\frac{3}{5}}$$

g. Find the mgf.

$$\mathcal{M}(t) = E\left(e^{Xt}\right) = \int_0^1 e^{xt} 3x^2 dx \text{ (need to i.b.p. twice)}$$

Let $u = 3x^2$, $du = 6x$, $dv = e^{xt}$, $v = \frac{1}{t}e^{xt}$.

$$\mathcal{M}(t) = 3x^2 \frac{1}{t} e^{xt}\Big|_0^1 - \int_0^1 \frac{1}{t} e^{xt} 6x dx = 3e^t - \int_0^1 \frac{1}{t} e^{xt} 6x dx$$

Next, let $u = 6x$, $du = 6$, $dv = \frac{1}{t}e^{xt}$, $v = \frac{1}{t^2}e^{xt}$.

$$\mathcal{M}(t) = \frac{3}{t}e^t - 6x\frac{1}{t^2}e^{xt}\Big|_0^1 + \int_0^1 \frac{1}{t^2} e^{xt} 6dx$$

Example #2 (continued)

f. Find the standard deviation σ of X.

$$\sigma^2 = Var\left(X\right) = \int_0^1 x^2 3x^2 dx - \mu^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} \Rightarrow \sigma = \frac{1}{4}\sqrt{\frac{3}{5}}$$

g. Find the mgf.

$$\mathcal{M}\left(t\right) = E\left(e^{Xt}\right) = \int_0^1 e^{xt} 3x^2 dx \text{ (need to i.b.p. twice)}$$

Let $u = 3x^2$, $du = 6x$, $dv = e^{xt}$, $v = \frac{1}{t}e^{xt}$.

$$\mathcal{M}\left(t\right) = 3x^2 \frac{1}{t} e^{xt}\Big|_0^1 - \int_0^1 \frac{1}{t} e^{xt} 6x dx = 3e^t - \int_0^1 \frac{1}{t} e^{xt} 6x dx$$

Next, let $u = 6x$, $du = 6$, $dv = \frac{1}{t}e^{xt}$, $v = \frac{1}{t^2}e^{xt}$.

$$\mathcal{M}\left(t\right) = \frac{3}{t}e^t - 6x\frac{1}{t^2}e^{xt}\Big|_0^1 + \int_0^1 \frac{1}{t^2} e^{xt} 6 dx = \frac{3}{t}e^t - \frac{6}{t^2}e^t + \frac{6}{t^3}e^t - \frac{6}{t^3},\ t \neq 0$$

Example #2 (continued)

$h.$ Find $E\left(\sqrt{X}\right)$.

$i.$ Find $E\left[\ln\left(X\right)\right]$.

Example #2 (continued)

$h.$ Find $E\left(\sqrt{X}\right)$.

$$E\left(\sqrt{X}\right) = \int_0^1 \sqrt{x} 3x^2 dx$$

$i.$ Find $E\left[\ln(X)\right]$.

Example #2 (continued)

$h.$ Find $E\left(\sqrt{X}\right)$.

$$E\left(\sqrt{X}\right) = \int_0^1 \sqrt{x} 3x^2 dx = \int_0^1 3x^{\frac{5}{2}} dx = \frac{6}{7}$$

$i.$ Find $E\left[\ln\left(X\right)\right]$.

Example #2 (continued)

h. Find $E\left(\sqrt{X}\right)$.

$$E\left(\sqrt{X}\right) = \int_0^1 \sqrt{x} 3x^2 dx = \int_0^1 3x^{\frac{5}{2}} dx = \frac{6}{7}$$

i. Find $E\left[\ln\left(X\right)\right]$. Need to integrate by parts: $u = \ln\left(x\right)$, $du = x^{-1}$, $dv = 3x^2$, $v = x^3$.

Example #2 (continued)

h. Find $E\left(\sqrt{X}\right)$.

$$E\left(\sqrt{X}\right) = \int_0^1 \sqrt{x}3x^2dx = \int_0^1 3x^{\frac{5}{2}}dx = \frac{6}{7}$$

i. Find $E\left[\ln\left(X\right)\right]$. Need to integrate by parts: $u = \ln\left(x\right)$, $du = x^{-1}$, $dv = 3x^2$, $v = x^3$.

$$E\left[\ln\left(X\right)\right] = \int_0^1 \ln\left(x\right)3x^2dx$$

Example #2 (continued)

h. Find $E\left(\sqrt{X}\right)$.

$$E\left(\sqrt{X}\right) = \int_0^1 \sqrt{x} 3x^2 dx = \int_0^1 3x^{\frac{5}{2}} dx = \frac{6}{7}$$

i. Find $E\left[\ln\left(X\right)\right]$. Need to integrate by parts: $u = \ln\left(x\right)$, $du = x^{-1}$, $dv = 3x^2$, $v = x^3$.

$$E\left[\ln\left(X\right)\right] = \int_0^1 \ln\left(x\right) 3x^2 dx = \ln\left(x\right) x^3 \Big|_0^1 - \int_0^1 x^2 dx = -\frac{1}{3}$$

Example #2 (continued)

h. Find $E\left(\sqrt{X}\right)$.

$$E\left(\sqrt{X}\right) = \int_0^1 \sqrt{x}3x^2dx = \int_0^1 3x^{\frac{5}{2}}dx = \frac{6}{7}$$

i. Find $E\left[\ln\left(X\right)\right]$. Need to integrate by parts: $u = \ln\left(x\right)$, $du = x^{-1}$, $dv = 3x^2$, $v = x^3$.

$$E\left[\ln\left(X\right)\right] = \int_0^1 \ln\left(x\right)3x^2dx = \ln\left(x\right)x^3\Big|_0^1 - \int_0^1 x^2dx = -\frac{1}{3}$$

Note the use of L'Hopital's rule to evaluate the lower bound for the first term. That is, $\ln(1)1^3 = 0$ and

Example #2 (continued)

h. Find $E\left(\sqrt{X}\right)$.

$$E\left(\sqrt{X}\right) = \int_0^1 \sqrt{x} 3x^2 dx = \int_0^1 3x^{\frac{5}{2}} dx = \frac{6}{7}$$

i. Find $E\left[\ln\left(X\right)\right]$. Need to integrate by parts: $u = \ln\left(x\right)$, $du = x^{-1}$, $dv = 3x^2$, $v = x^3$.

$$E\left[\ln\left(X\right)\right] = \int_0^1 \ln\left(x\right) 3x^2 dx = \ln\left(x\right) x^3 \Big|_0^1 - \int_0^1 x^2 dx = -\frac{1}{3}$$

Note the use of L'Hopital's rule to evaluate the lower bound for the first term. That is, $\ln(1)1^3 = 0$ and

$$\lim_{x \to 0} \frac{\ln\left(x\right)}{x^{-3}} = \lim_{x \to 0} \frac{x^{-1}}{x^{-3}} = 0$$

Example #3: Consider $f(x) = 5x^{-6}\mathcal{I}(x > 1)$

a. Find $F(x)$.

Example #3: Consider $f(x) = 5x^{-6}\mathcal{I}(x > 1)$

a. Find $F(x)$.

$$F(x) = \int_1^x \frac{5}{t^6} dt$$

Example #3: Consider $f(x) = 5x^{-6}\mathcal{I}(x > 1)$

a. Find $F(x)$.

$$F(x) = \int_1^x \frac{5}{t^6} dt = -t^{-5}\Big|_1^x$$

Example #3: Consider $f(x) = 5x^{-6}\mathcal{I}(x > 1)$

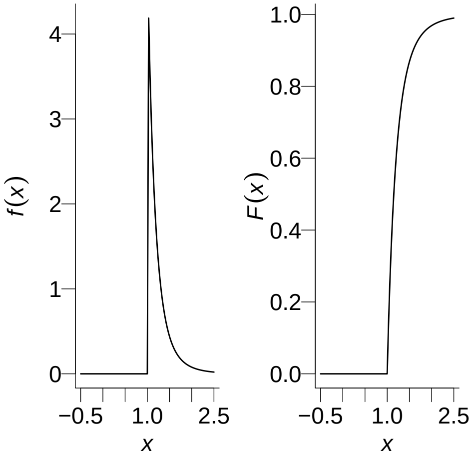
a. Find $F(x)$.

$$F(x) = \int_1^x \frac{5}{t^6} dt = -t^{-5}\Big|_1^x = -\left(x^{-5} - 1\right) = 1 - x^{-5}$$

Example #3: Consider $f(x) = 5x^{-6}\mathcal{I}(x > 1)$

a. Find $F(x)$.

$$F(x) = \int_1^x \frac{5}{t^6}dt = -t^{-5}\Big|_1^x = -\left(x^{-5} - 1\right) = 1 - x^{-5}$$



Example #3 (continued)

b. Find $E(X)$.

c. Find $Var(X)$.

d. What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

b. Find $E(X)$.

$$E(X) = \int_1^\infty x \frac{5}{x^6} dx$$

c. Find $Var(X)$.

d. What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

b. Find $E(X)$.

$$E(X) = \int_1^\infty x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^\infty$$

c. Find $Var(X)$.

d. What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

$b.$ Find $E(X)$.

$$E(X) = \int_1^{\infty} x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^{\infty} = \frac{5}{4}$$

$c.$ Find $Var(X)$.

$d.$ What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

$b.$ Find $E(X)$.

$$E(X) = \int_1^\infty x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^\infty = \frac{5}{4}$$

$c.$ Find $Var(X)$.

$$Var(X) = \int_1^\infty x^2 \frac{5}{x^6} dx - \mu^2$$

$d.$ What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

b. Find $E(X)$.

$$E(X) = \int_1^\infty x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^\infty = \frac{5}{4}$$

c. Find $Var(X)$.

$$Var(X) = \int_1^\infty x^2 \frac{5}{x^6} dx - \mu^2 = -\frac{5}{3} x^{-3} \Big|_1^\infty - \frac{25}{16}$$

d. What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

b. Find $E(X)$.

$$E(X) = \int_1^{\infty} x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^{\infty} = \frac{5}{4}$$

c. Find $Var(X)$.

$$Var(X) = \int_1^{\infty} x^2 \frac{5}{x^6} dx - \mu^2 = -\frac{5}{3} x^{-3} \Big|_1^{\infty} - \frac{25}{16} = \frac{5}{48}$$

d. What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

b. Find $E(X)$.

$$E(X) = \int_1^\infty x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^\infty = \frac{5}{4}$$

c. Find $Var(X)$.

$$Var(X) = \int_1^\infty x^2 \frac{5}{x^6} dx - \mu^2 = -\frac{5}{3} x^{-3} \Big|_1^\infty - \frac{25}{16} = \frac{5}{48}$$

d. What are the values of r where $E(X^r)$ is undefined? $E(X^r)$ is undefined for $r \geq 5$, because

Example #3 (continued)

b. Find $E(X)$.

$$E(X) = \int_1^\infty x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^\infty = \frac{5}{4}$$

c. Find $Var(X)$.

$$Var(X) = \int_1^\infty x^2 \frac{5}{x^6} dx - \mu^2 = -\frac{5}{3} x^{-3} \Big|_1^\infty - \frac{25}{16} = \frac{5}{48}$$

d. What are the values of r where $E(X^r)$ is undefined? $E(X^r)$ is undefined for $r \geq 5$, because

$$E(X^r) = \int_1^\infty x^r \frac{5}{x^6} dx \text{ diverges for } r \geq 5.$$

Example #3 (continued)

e. Find the median, m.

c. Find the 30th percentile.

Example #3 (continued)

e. Find the median, m.

$$F\left(m\right) = \frac{1}{2}$$

c. Find the 30th percentile.

Example #3 (continued)

e. Find the median, m.

$$F(m) = \frac{1}{2} \Rightarrow 1 - m^{-5} = \frac{1}{2} \Rightarrow m = \sqrt[5]{2} \approx 1.1487$$

c. Find the 30th percentile.

Example #3 (continued)

e. Find the median, m.

$$F(m) = \frac{1}{2} \Rightarrow 1 - m^{-5} = \frac{1}{2} \Rightarrow m = \sqrt[5]{2} \approx 1.1487$$

c. Find the 30th percentile.

$$F(\pi_{0.30}) = \frac{3}{10}$$

Example #3 (continued)

e. Find the median, m.

$$F(m) = \frac{1}{2} \Rightarrow 1 - m^{-5} = \frac{1}{2} \Rightarrow m = \sqrt[5]{2} \approx 1.1487$$

c. Find the 30th percentile.

$$F(\pi_{0.30}) = \frac{3}{10} \Rightarrow 1 - \pi_{0.30}^{-5} = \frac{3}{10} \Rightarrow \pi_{0.30} = \sqrt[5]{\frac{10}{7}} \approx 1.07394$$

Example #3 (continued)

b. Find $E(X)$.

c. Find $Var(X)$.

d. What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

b. Find $E(X)$.

$$E(X) = \int_1^\infty x \frac{5}{x^6} dx$$

c. Find $Var(X)$.

d. What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

b. Find $E(X)$.

$$E(X) = \int_1^{\infty} x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^{\infty}$$

c. Find $Var(X)$.

d. What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

$b.$ Find $E(X)$.

$$E(X) = \int_1^\infty x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^\infty = \frac{5}{4}$$

$c.$ Find $Var(X)$.

$d.$ What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

b. Find $E(X)$.

$$E(X) = \int_1^\infty x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^\infty = \frac{5}{4}$$

c. Find $Var(X)$.

$$Var(X) = \int_1^\infty x^2 \frac{5}{x^6} dx - \mu^2$$

d. What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

b. Find $E(X)$.

$$E(X) = \int_1^\infty x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^\infty = \frac{5}{4}$$

c. Find $Var(X)$.

$$Var(X) = \int_1^\infty x^2 \frac{5}{x^6} dx - \mu^2 = -\frac{5}{3} x^{-3} \Big|_1^\infty - \frac{25}{16}$$

d. What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

b. Find $E(X)$.

$$E(X) = \int_1^\infty x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^\infty = \frac{5}{4}$$

c. Find $Var(X)$.

$$Var(X) = \int_1^\infty x^2 \frac{5}{x^6} dx - \mu^2 = -\frac{5}{3} x^{-3} \Big|_1^\infty - \frac{25}{16} = \frac{5}{48}$$

d. What are the values of r where $E(X^r)$ is undefined?

Example #3 (continued)

b. Find $E(X)$.

$$E(X) = \int_1^\infty x \frac{5}{x^6} dx = -\frac{5}{4} x^{-4} \Big|_1^\infty = \frac{5}{4}$$

c. Find $Var(X)$.

$$Var(X) = \int_1^\infty x^2 \frac{5}{x^6} dx - \mu^2 = -\frac{5}{3} x^{-3} \Big|_1^\infty - \frac{25}{16} = \frac{5}{48}$$

d. What are the values of r where $E(X^r)$ is undefined? $E(X^r)$ is undefined for $r \geq 5$, because

$$E(X^r) = \int_1^\infty x^r \frac{5}{x^6} dx \text{ diverges for } r \geq 5.$$

Example #4: Cauchy, $f(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty$

a. Find $E(X)$.

b. Find $F(x)$.

c. Find $P(X < -1)$, $P(-1 < X < 0)$, $P(0 < X < 1)$, $P(X > 1)$.

Example #4: Cauchy, $f(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty$

a. Find $E(X)$.

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{x}{1+x^2} dx \text{ diverges}$$

b. Find $F(x)$.

c. Find $P(X < -1)$, $P(-1 < X < 0)$, $P(0 < X < 1)$, $P(X > 1)$.

Example #4: Cauchy, $f(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty$

a. Find $E(X)$.

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{x}{1+x^2} dx \text{ diverges}$$

b. Find $F(x)$.

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi} \frac{1}{1+t^2} dt$$

c. Find $P(X < -1)$, $P(-1 < X < 0)$, $P(0 < X < 1)$, $P(X > 1)$.

Example #4: Cauchy, $f(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty$

a. Find $E(X)$.

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{x}{1+x^2} dx \text{ diverges}$$

b. Find $F(x)$.

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi} \frac{1}{1+t^2} dt = \frac{1}{\pi} \arctan(x) + \frac{1}{2}, -\infty < x < \infty$$

c. Find $P(X < -1)$, $P(-1 < X < 0)$, $P(0 < X < 1)$,
$P(X > 1)$.

Example #4: Cauchy, $f(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty$

a. Find $E(X)$.

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{x}{1+x^2} dx \text{ diverges}$$

b. Find $F(x)$.

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi} \frac{1}{1+t^2} dt = \frac{1}{\pi} \arctan(x) + \frac{1}{2}, -\infty < x < \infty$$

c. Find $P(X < -1)$, $P(-1 < X < 0)$, $P(0 < X < 1)$, $P(X > 1)$. Note that $\arctan(-1) = -\frac{\pi}{4}$, $\arctan(1) = \frac{\pi}{4}$, and $arctan(0) = 0$, which implies $P(X < -1) = P(-1 < X < 0) = P(0 < X < 1) = P(X > 1) = \frac{1}{4}$.

Example #4 (continued)

$d.$ Find $M(t)$.

Example #4 (continued)

d. Find $M(t)$. We can show $M(t)$ also does not exist for the Cauchy. An integral over the support for X diverges if the integral diverges on a subset of the support for X. Note $e^x \geq \frac{x^3}{6}$ $\forall\, x > 0$ and $\frac{x^3}{1+x^2} \geq \frac{x}{2}$ $\forall\, x > 1$,

Example #4 (continued)

d. Find $M(t)$. We can show $M(t)$ also does not exist for the Cauchy. An integral over the support for X diverges if the integral diverges on a subset of the support for X. Note $e^x \geq \frac{x^3}{6}$ $\forall\, x > 0$ and $\frac{x^3}{1+x^2} \geq \frac{x}{2}$ $\forall\, x > 1$,

$$M(t) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{e^{xt}}{1+x^2} dx \geq \int_{1}^{\infty} \frac{1}{\pi} \frac{e^{xt}}{1+x^2} dx$$

Example #4 (continued)

d. Find $M(t)$. We can show $M(t)$ also does not exist for the Cauchy. An integral over the support for X diverges if the integral diverges on a subset of the support for X. Note $e^x \geq \frac{x^3}{6}$ $\forall\, x > 0$ and $\frac{x^3}{1+x^2} \geq \frac{x}{2}$ $\forall\, x > 1$,

$$
\begin{aligned}
M(t) &= \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{e^{xt}}{1+x^2} dx \geq \int_{1}^{\infty} \frac{1}{\pi} \frac{e^{xt}}{1+x^2} dx \\
&\geq \int_{1}^{\infty} \frac{1}{6\pi} \frac{x^3 t^3}{1+x^2} dx
\end{aligned}
$$

Example #4 (continued)

d. Find $M(t)$. We can show $M(t)$ also does not exist for the Cauchy. An integral over the support for X diverges if the integral diverges on a subset of the support for X. Note $e^x \geq \frac{x^3}{6}$ $\forall\, x > 0$ and $\frac{x^3}{1+x^2} \geq \frac{x}{2}$ $\forall\, x > 1$,

$$
\begin{aligned}
M(t) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{e^{xt}}{1+x^2} dx &\geq \int_{1}^{\infty} \frac{1}{\pi} \frac{e^{xt}}{1+x^2} dx \\
&\geq \int_{1}^{\infty} \frac{1}{6\pi} \frac{x^3 t^3}{1+x^2} dx \\
&\geq \frac{t^3}{12\pi} \int_{1}^{\infty} x dx = \infty
\end{aligned}
$$

MGF Theorems

- $\mathcal{M}_1(t) = \mathcal{M}_2(t)$ for some interval containing zero implies $f_1(x) = f_2(x)$.
- $\mathcal{M}^{(k)}(0) = E\left(X^k\right)$
- Let $Y = aX + b$, then $\mathcal{M}_Y(t) = e^{bt}\mathcal{M}_X(at)$

Proof.

$\square$

MGF Theorems

- $\mathcal{M}_1(t) = \mathcal{M}_2(t)$ for some interval containing zero implies $f_1(x) = f_2(x)$.
- $\mathcal{M}^{(k)}(0) = E\left(X^k\right)$
- Let $Y = aX + b$, then $\mathcal{M}_Y(t) = e^{bt}\mathcal{M}_X(at)$

Proof.

$$\mathcal{M}_Y(t) = E\left(e^{Yt}\right) = E\left(e^{(aX+b)t}\right) = e^{bt}E\left(e^{Xat}\right) = e^{bt}\mathcal{M}_X(at)$$

$\square$

Example # 5

Suppose a discrete random variable X has the following pmf,

$$P\left(X=k\right)=\begin{cases} p, & k=0 \\ \frac{1}{2^k k!}, & k=1,2,3,\ldots \end{cases}$$

a. Find p to make this a valid pmf.

b. Find $E\left(X\right)$

Example # 5

Suppose a discrete random variable X has the following pmf,

$$P(X = k) = \begin{cases} p, & k = 0 \\ \frac{1}{2^k k!}, & k = 1, 2, 3, \ldots \end{cases}$$

a. Find p to make this a valid pmf. p must be defined so $p + \sum_{k=1}^{\infty} \frac{1}{2^k k!} = 1$.

b. Find $E(X)$

Example # 5

Suppose a discrete random variable X has the following pmf,

$$P\left(X=k\right)=\begin{cases} p, & k=0 \\ \frac{1}{2^k k!}, & k=1,2,3,\ldots \end{cases}$$

a. Find p to make this a valid pmf. p must be defined so $p+\sum_{k=1}^{\infty}\frac{1}{2^k k!}=1$. Recall $\sum_{k=0}^{\infty}\frac{a^k}{k!}=e^a$ implies $\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2}\right)^k}{k!}=e^{1/2}-1 \Rightarrow p=2-e^{1/2}$.

b. Find $E\left(X\right)$

Example # 5

Suppose a discrete random variable X has the following pmf,

$$P(X = k) = \begin{cases} p, & k = 0 \\ \frac{1}{2^k k!}, & k = 1, 2, 3, \ldots \end{cases}$$

a. Find p to make this a valid pmf. p must be defined so $p + \sum_{k=1}^{\infty} \frac{1}{2^k k!} = 1$. Recall $\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$ implies $\sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)^k}{k!} = e^{1/2} - 1 \Rightarrow p = 2 - e^{1/2}$.

b. Find $E(X)$

$$E(X) = 0 \cdot p + \sum_{k=1}^{\infty} \frac{k}{2^k k!} = \sum_{k=1}^{\infty} \frac{1}{2^k (k-1)!}$$

Example # 5

Suppose a discrete random variable X has the following pmf,

$$P(X = k) = \begin{cases} p, & k = 0 \\ \frac{1}{2^k k!}, & k = 1, 2, 3, \ldots \end{cases}$$

a. Find p to make this a valid pmf. p must be defined so $p + \sum_{k=1}^{\infty} \frac{1}{2^k k!} = 1$. Recall $\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$ implies $\sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)^k}{k!} = e^{1/2} - 1 \;\Rightarrow\; p = 2 - e^{1/2}$.

b. Find $E(X)$

$$E(X) = 0 \cdot p + \sum_{k=1}^{\infty} \frac{k}{2^k k!} = \sum_{k=1}^{\infty} \frac{1}{2^k (k-1)!}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n n!} = \frac{e^{\frac{1}{2}}}{2}$$

Example # 5 (continued)

c. Find $Var\left(X\right)$

Example # 5 (continued)

c. Find $Var(X)$

$$E[X(X-1)] = \sum_{k=2}^{\infty} \frac{k(k-1)}{2^k k!} = \sum_{k=2}^{\infty} \frac{1}{2^k (k-2)!}$$

Example # 5 (continued)

c. Find $Var(X)$

$$E[X(X-1)] = \sum_{k=2}^{\infty} \frac{k(k-1)}{2^k k!} = \sum_{k=2}^{\infty} \frac{1}{2^k (k-2)!}$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{2^n n!} = \frac{e^{\frac{1}{2}}}{4}$$

Example # 5 (continued)

c. Find $Var\left(X\right)$

$$E\left[X\left(X-1\right)\right]=\sum_{k=2}^{\infty}\frac{k\left(k-1\right)}{2^{k}k!}=\sum_{k=2}^{\infty}\frac{1}{2^{k}\left(k-2\right)!}$$
$$=\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{2^{n}n!}=\frac{e^{\frac{1}{2}}}{4}$$

$$Var\left(X\right)=E\left[X\left(X-1\right)\right]+E\left(X\right)-\left[E\left(X\right)\right]^{2}$$

Example # 5 (continued)

c. Find $Var(X)$

$$E[X(X-1)] = \sum_{k=2}^{\infty} \frac{k(k-1)}{2^k k!} = \sum_{k=2}^{\infty} \frac{1}{2^k (k-2)!}$$
$$= \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{2^n n!} = \frac{e^{\frac{1}{2}}}{4}$$

$$Var(X) = E[X(X-1)] + E(X) - [E(X)]^2$$
$$= \frac{e^{\frac{1}{2}}}{4} + \frac{e^{\frac{1}{2}}}{2} - \frac{e}{4} = \frac{3}{4} e^{\frac{1}{2}} - \frac{e}{4}$$

Example # 5 (continued)

d. Find $\mathcal{M}(t)$.

e. Find $E(X)$ with $\mathcal{M}(t)$.

Example # 5 (continued)

d. Find $\mathcal{M}(t)$.

$$\mathcal{M}(t) = 1 \cdot p + \sum_{k=1}^{\infty} e^{tk} \frac{1}{2^k k!}$$

e. Find $E(X)$ with $\mathcal{M}(t)$.

Example # 5 (continued)

d. Find $\mathcal{M}(t)$.

$$\mathcal{M}(t) = 1 \cdot p + \sum_{k=1}^{\infty} e^{tk} \frac{1}{2^k k!} = 2 - e^{\frac{1}{2}} + \sum_{k=1}^{\infty} \frac{\left(\frac{e^t}{2}\right)^k}{k!}$$

e. Find $E(X)$ with $\mathcal{M}(t)$.

Example # 5 (continued)

d. Find $\mathcal{M}(t)$.

$$\mathcal{M}(t) = 1 \cdot p + \sum_{k=1}^{\infty} e^{tk} \frac{1}{2^k k!} = 2 - e^{\frac{1}{2}} + \sum_{k=1}^{\infty} \frac{\left(\frac{e^t}{2}\right)^k}{k!}$$

$$= 2 - e^{\frac{1}{2}} + e^{e^{\frac{1}{2}}} - 1 = 1 - e^{\frac{1}{2}} + e^{\frac{e^t}{2}}$$

e. Find $E(X)$ with $\mathcal{M}(t)$.

Example # 5 (continued)

d. Find $\mathcal{M}(t)$.

$$\mathcal{M}(t) = 1 \cdot p + \sum_{k=1}^{\infty} e^{tk} \frac{1}{2^k k!} = 2 - e^{\frac{1}{2}} + \sum_{k=1}^{\infty} \frac{\left(\frac{e^t}{2}\right)^k}{k!}$$

$$= 2 - e^{\frac{1}{2}} + e^{e^{\frac{1}{2}}} - 1 = 1 - e^{\frac{1}{2}} + e^{\frac{e^t}{2}}$$

e. Find $E(X)$ with $\mathcal{M}(t)$.

$$\mathcal{M}'(t) = e^{\frac{e^t}{2}} \frac{e^t}{2}$$

Example # 5 (continued)

d. Find $\mathcal{M}(t)$.

$$\mathcal{M}(t) = 1 \cdot p + \sum_{k=1}^{\infty} e^{tk} \frac{1}{2^k k!} = 2 - e^{\frac{1}{2}} + \sum_{k=1}^{\infty} \frac{\left(\frac{e^t}{2}\right)^k}{k!}$$
$$= 2 - e^{\frac{1}{2}} + e^{e^{\frac{1}{2}}} - 1 = 1 - e^{\frac{1}{2}} + e^{\frac{e^t}{2}}$$

e. Find $E(X)$ with $\mathcal{M}(t)$.

$$\mathcal{M}'(t) = e^{\frac{e^t}{2}} \frac{e^t}{2} \Rightarrow E(X) = \mathcal{M}'(0) = \frac{e^{1/2}}{2}$$

Example #6

(1.9.16) Let $\psi(t) = \ln[\mathcal{M}(t)]$ be the natural log of a mgf, which is called the cumulant generating function (cgf). Prove that $\psi'(0) = \mu$ and $\psi''(0) = \sigma^2$.

Proof.

Example #6

(1.9.16) Let $\psi(t) = \ln[\mathcal{M}(t)]$ be the natural log of a mgf, which is called the cumulant generating function (cgf). Prove that $\psi'(0) = \mu$ and $\psi''(0) = \sigma^2$.

Proof.

$$\psi'(t) = (\ln[\mathcal{M}(t)])' = \frac{\mathcal{M}'(t)}{\mathcal{M}(t)}$$

Example #6

(1.9.16) Let $\psi(t) = \ln[\mathcal{M}(t)]$ be the natural log of a mgf, which is called the cumulant generating function (cgf). Prove that $\psi'(0) = \mu$ and $\psi''(0) = \sigma^2$.

Proof.

$$\psi'(t) = (\ln[\mathcal{M}(t)])' = \frac{\mathcal{M}'(t)}{\mathcal{M}(t)}$$

$$\Rightarrow \psi'(0) = \frac{\mathcal{M}'(0)}{\mathcal{M}(0)} = \mu$$

Example #6

(1.9.16) Let $\psi(t) = \ln[\mathcal{M}(t)]$ be the natural log of a mgf, which is called the cumulant generating function (cgf). Prove that $\psi'(0) = \mu$ and $\psi''(0) = \sigma^2$.

Proof.

$$\psi'(t) = (\ln[\mathcal{M}(t)])' = \frac{\mathcal{M}'(t)}{\mathcal{M}(t)}$$

$$\Rightarrow \psi'(0) = \frac{\mathcal{M}'(0)}{\mathcal{M}(0)} = \mu$$

$$\psi''(t) = \frac{\mathcal{M}''(t)\,\mathcal{M}(t) - [\mathcal{M}'(t)]^2}{[\mathcal{M}(t)]^2}$$

Example #6

(1.9.16) Let $\psi(t) = \ln[\mathcal{M}(t)]$ be the natural log of a mgf, which is called the cumulant generating function (cgf). Prove that $\psi'(0) = \mu$ and $\psi''(0) = \sigma^2$.

Proof.

$$\psi'(t) = (\ln[\mathcal{M}(t)])' = \frac{\mathcal{M}'(t)}{\mathcal{M}(t)}$$

$$\Rightarrow \psi'(0) = \frac{\mathcal{M}'(0)}{\mathcal{M}(0)} = \mu$$

$$\psi''(t) = \frac{\mathcal{M}''(t)\mathcal{M}(t) - [\mathcal{M}'(t)]^2}{[\mathcal{M}(t)]^2}$$

$$\Rightarrow \psi''(0) = \frac{\mathcal{M}''(0)\mathcal{M}(0) - [\mathcal{M}'(0)]^2}{[\mathcal{M}(0)]^2} = \sigma^2$$

Example #7: Show cumulants beyond third order are zero for Gaussian distribution

Example #7: Show cumulants beyond third order are zero for Gaussian distribution

- The Gaussian mgf is $e^{\mu t+\frac{t^2}{2}\sigma^2}$.

Example #7: Show cumulants beyond third order are zero for Gaussian distribution

- The Gaussian mgf is $e^{\mu t + \frac{t^2}{2}\sigma^2}$.
- The Gaussian cgf is $\mu t + \frac{t^2}{2}\sigma^2$.

Example #7: Show cumulants beyond third order are zero for Gaussian distribution

- The Gaussian mgf is $e^{\mu t + \frac{t^2}{2}\sigma^2}$.
- The Gaussian cgf is $\mu t + \frac{t^2}{2}\sigma^2$.
- $\psi'(t) = \mu + t\sigma^2 \Rightarrow E(X) = \mu$.

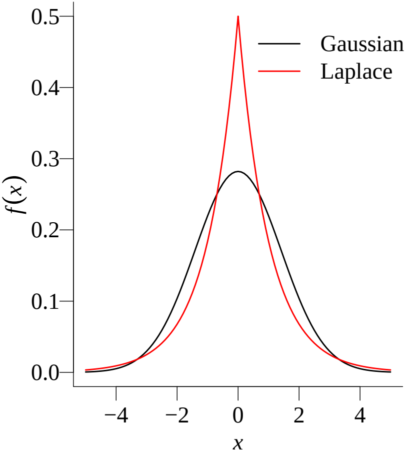
Example #7: Show cumulants beyond third order are zero for Gaussian distribution

- The Gaussian mgf is $e^{\mu t+\frac{t^2}{2}\sigma^2}$.
- The Gaussian cgf is $\mu t+\frac{t^2}{2}\sigma^2$.
- $\psi'(t)=\mu+t\sigma^2 \Rightarrow E(X)=\mu$.
- $\psi''(t)=\sigma^2 \Rightarrow Var(X)=\sigma^2$.

Example #7: Show cumulants beyond third order are zero for Gaussian distribution

- The Gaussian mgf is $e^{\mu t+\frac{t^2}{2}\sigma^2}$.
- The Gaussian cgf is $\mu t+\frac{t^2}{2}\sigma^2$.
- $\psi'(t)=\mu+t\sigma^2 \Rightarrow E(X)=\mu$.
- $\psi''(t)=\sigma^2 \Rightarrow Var(X)=\sigma^2$.
- $\psi^{(r)}(t)=0 \; r\geq 3 \Rightarrow E\left[(X-\mu)^r\right]=0$.

Example #8: Laplace distribution, $f(x) = \frac{1}{2}e^{-|x|}$



- The Gaussian density is $f(x) = \frac{1}{\sqrt{4\pi}}e^{-\frac{x^2}{4}}$.
- Recall that 68.3% of the density for the Gaussian lies $\pm\sigma$ of the mean as opposed to 86.5% for the Laplacian.

Example #8 (continued)

a. Find the mgf.

Example #8 (continued)

a. Find the mgf. Recall the definition of the absolute value,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Example #8 (continued)

a. Find the mgf. Recall the definition of the absolute value,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\mathcal{M}(t) = \int_{-\infty}^{\infty} \frac{e^{xt}}{2} e^{-|x|} dx$$

Example #8 (continued)

a. Find the mgf. Recall the definition of the absolute value,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\mathcal{M}(t) = \int_{-\infty}^{\infty} \frac{e^{xt}}{2} e^{-|x|} dx = \int_{-\infty}^{0} \frac{e^{xt}}{2} e^{-|x|} dx + \int_{0}^{\infty} \frac{e^{xt}}{2} e^{-|x|} dx$$

Example #8 (continued)

a. Find the mgf. Recall the definition of the absolute value,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\begin{aligned} \mathcal{M}(t) &= \int_{-\infty}^{\infty} \frac{e^{xt}}{2} e^{-|x|} dx = \int_{-\infty}^{0} \frac{e^{xt}}{2} e^{-|x|} dx + \int_{0}^{\infty} \frac{e^{xt}}{2} e^{-|x|} dx \\ &= \int_{-\infty}^{0} \frac{e^{xt}}{2} e^{x} dx + \int_{0}^{\infty} \frac{e^{xt}}{2} e^{-x} dx \end{aligned}$$

Example #8 (continued)

a. Find the mgf. Recall the definition of the absolute value,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\begin{aligned}
\mathcal{M}(t) &= \int_{-\infty}^{\infty} \frac{e^{xt}}{2} e^{-|x|} dx = \int_{-\infty}^{0} \frac{e^{xt}}{2} e^{-|x|} dx + \int_{0}^{\infty} \frac{e^{xt}}{2} e^{-|x|} dx \\
&= \int_{-\infty}^{0} \frac{e^{xt}}{2} e^{x} dx + \int_{0}^{\infty} \frac{e^{xt}}{2} e^{-x} dx \\
&= \int_{-\infty}^{0} \frac{e^{x(t+1)}}{2} dx + \int_{0}^{\infty} \frac{e^{x(t-1)}}{2} dx, -1 < t < 1
\end{aligned}$$

Example #8 (continued)

a. Find the mgf. Recall the definition of the absolute value,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\begin{aligned}
\mathcal{M}(t) &= \int_{-\infty}^{\infty} \frac{e^{xt}}{2} e^{-|x|} dx = \int_{-\infty}^{0} \frac{e^{xt}}{2} e^{-|x|} dx + \int_{0}^{\infty} \frac{e^{xt}}{2} e^{-|x|} dx \\
&= \int_{-\infty}^{0} \frac{e^{xt}}{2} e^{x} dx + \int_{0}^{\infty} \frac{e^{xt}}{2} e^{-x} dx \\
&= \int_{-\infty}^{0} \frac{e^{x(t+1)}}{2} dx + \int_{0}^{\infty} \frac{e^{x(t-1)}}{2} dx, -1 < t < 1 \\
&= \left. \frac{e^{x(t+1)}}{2(t+1)} \right|_{-\infty}^{0} + \left. \frac{e^{x(t-1)}}{2(t-1)} \right|_{0}^{\infty}
\end{aligned}$$

Example #8 (continued)

a. Find the mgf. Recall the definition of the absolute value,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\begin{aligned}
\mathcal{M}(t) &= \int_{-\infty}^{\infty} \frac{e^{xt}}{2} e^{-|x|} dx = \int_{-\infty}^{0} \frac{e^{xt}}{2} e^{-|x|} dx + \int_{0}^{\infty} \frac{e^{xt}}{2} e^{-|x|} dx \\
&= \int_{-\infty}^{0} \frac{e^{xt}}{2} e^{x} dx + \int_{0}^{\infty} \frac{e^{xt}}{2} e^{-x} dx \\
&= \int_{-\infty}^{0} \frac{e^{x(t+1)}}{2} dx + \int_{0}^{\infty} \frac{e^{x(t-1)}}{2} dx, -1 < t < 1 \\
&= \left. \frac{e^{x(t+1)}}{2(t+1)} \right|_{-\infty}^{0} + \left. \frac{e^{x(t-1)}}{2(t-1)} \right|_{0}^{\infty} \\
&= \frac{1}{2(t+1)} - \frac{1}{2(t-1)} = \frac{1}{1-t^2}, -1 < t < 1
\end{aligned}$$

Example #8 (continued)

b. Find $E(X)$.

c. Find $Var(X)$.

Example #8 (continued)

b. Find $E(X)$. The cgf is $\psi(t) = -\ln\left(1 - t^2\right)$ and $\psi'(t) = \frac{2t}{1-t^2} \Rightarrow E(X) = 0$.

c. Find $Var(X)$.

Example #8 (continued)

b. Find $E(X)$. The cgf is $\psi(t) = -\ln\left(1-t^2\right)$ and $\psi'(t) = \frac{2t}{1-t^2} \Rightarrow E(X) = 0$.

c. Find $Var(X)$. $\psi''(t) = \frac{2\left(1-t^2\right)+4t^2}{\left(1-t^2\right)^2} \Rightarrow Var(X) = 2$.