Math 415 - Lecture 28

Change of base, Image Compression

Monday November 2nd 2015

Textbook reading: Notes by Strang

Suggested practice exercises:

Khan Academy video:

Strang lecture: Change of basis; image compression

1 Review

* If $\mathcal{B}, \mathcal{C}$ are bases of $\mathbb{R}^n$, get *coordinate vectors* $x_\mathcal{B}, x_\mathcal{C}$ for any $x \in \mathbb{R}$.

* *Change of basis matrices*: $I_{\mathcal{C},\mathcal{B}}, I_{\mathcal{B},\mathcal{C}}$ such that

$$x_\mathcal{B} = I_{\mathcal{B},\mathcal{C}} x_\mathcal{C}, \quad x_\mathcal{C} = I_{\mathcal{C},\mathcal{B}}.$$

* Inverses: $I_{\mathcal{C},\mathcal{B}}^{-1} = I_{\mathcal{B},\mathcal{C}}$.

* Easy case: If $\mathcal{E}$ is the standard basis: then

$$I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}, \quad I_{\mathcal{B},\mathcal{E}} = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}^{-1}.$$

Theorem 1. *Let $\mathcal{U} := (\mathbf{u_1}, \dots, \mathbf{u_n})$ be an orthonormal basis of $\mathbb{R}^n$ and $U = \begin{bmatrix} \mathbf{u_1} & \dots & \mathbf{u_n} \end{bmatrix}$. Then for every $\mathbf{v} \in \mathbb{R}^n$*

$$v_\mathcal{U} = U^T v.$$

Why? $I_{\mathcal{E},\mathcal{U}} = \begin{bmatrix} \mathbf{u_1} & \dots & \mathbf{u_n} \end{bmatrix} = U$. But U has orthonormal columns, so $I_{\mathcal{U},\mathcal{E}} = U^{-1} = U^T$.

Example 1. Let $\mathcal{U} := (\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix})$. Determine $\begin{bmatrix} 2 \\ 4 \end{bmatrix}_\mathcal{U}$.

Solution. We have $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. This is the change of basis matrix from the $\mathcal{U}$ basis to the standard basis. So to go the other direction take the inverse. In this case inverse is transpose, so

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix}_{\mathcal{U}} = U^T \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$$

Check:

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{6}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Example 2. Let $\mathcal{B} := (\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix})$. Let $A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} \end{bmatrix}$. How can you easily compute $A_{\mathcal{B}} := \begin{bmatrix} \mathbf{a_1}_{\mathcal{B}} & \mathbf{a_2}_{\mathcal{B}} \end{bmatrix}$, ie the matrix whose are $\mathcal{B}$-coordinates of the columns of A?

Solution. To get the $\mathcal{B}$ coordinate vectors, multiply each column of A by U^T, where $U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$. So the wanted matrix is $A_{\mathcal{B}} = U^T A$.

Theorem 2. *Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^n$, let $\mathcal{B} := (\mathbf{u_1}, \dots, \mathbf{u_n})$ be an orthonormal basis of $\mathbb{R}^n$ and $U = \begin{bmatrix} \mathbf{u_1} & \dots & \mathbf{u_n} \end{bmatrix}$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then*

$$T_{\mathcal{B},\mathcal{B}} = U^T T_{\mathcal{E},\mathcal{E}} U,$$

or equivalently,

$$T_{\mathcal{E},\mathcal{E}} = U T_{\mathcal{B},\mathcal{B}} U^T.$$

Example 3. Let $\mathcal{B} := (\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix})$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformation given by

$$T(\mathbf{v}) = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix} \mathbf{v}.$$

Determine $T_{\mathcal{B},\mathcal{B}}$!

Solution. $T_{\mathcal{B},\mathcal{B}} = I_{\mathcal{B}\mathcal{E}} A I_{\mathcal{E}\mathcal{B}}$, where $A = T_{\mathcal{E}\mathcal{E}}$ is the matrix of T with respect to the standard basis. Now $I_{\mathcal{E}\mathcal{B}} = \begin{bmatrix} b_1 & b_1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = U$ So

$$T_{\mathcal{B},\mathcal{B}} = U^T A U = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

Use $T_{\mathcal{B},\mathcal{B}}$ to calculate $T(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix})$.

Solution.

$$T(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix})_{\mathcal{B}} = T_{\mathcal{B}\mathcal{B}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

This means that $T(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = 2 \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We will call such vectors eigenvectors and the number 2 will be called an eigenvalue. More about this soon!

2 Data compression

Let consider the following basis $\mathcal{H}$ of $\mathbb{R}^8$:

$$\left(\begin{bmatrix} \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right)$$

(i) Is $\mathcal{H}$ orthogonal?

(ii) Is $\mathcal{H}$ orthonormal?

This basis $\mathcal{H}$ is called **Haar Wavelet basis**. We will see in the following that $\mathcal{B}$ is much more effective than the standard basis (at least for certain applications).

Example 4. Find the coordinate vector of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 88 \\ 90 \\ 92 \\ 93 \\ 92 \\ 92 \\ 94 \\ 95 \end{bmatrix}$ with respect to $\mathcal{H}$?

Solution.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{H}} = \begin{bmatrix} \sqrt{8} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 88 \\ 90 \\ 92 \\ 93 \\ 92 \\ 92 \\ 94 \\ 95 \end{bmatrix}_{\mathcal{H}} = \begin{bmatrix} 260.2 \\ -3.5 \\ -3.5 \\ -2.5 \\ -1.4 \\ -0.7 \\ 0 \\ -0.7 \end{bmatrix}.$$

How could one use that for (lossy) data compression?

- Pick $\epsilon > 0$, and set all entries of the vectors with absolute value at most ϵ to 0.
- The $\mathcal{H}$-coordinate vector has more of these small entries, so more values become 0.

Let's do it with pictures!

2.1 Image compression

Consider 8×8-matrix, i.e., a 8×8-grayscale picture:

$$A = \begin{bmatrix} 88 & 88 & 89 & 90 & 92 & 94 & 96 & 97 \\ 90 & 90 & 91 & 92 & 93 & 95 & 97 & 97 \\ 92 & 92 & 93 & 94 & 95 & 96 & 97 & 97 \\ 93 & 93 & 94 & 95 & 96 & 96 & 96 & 96 \\ 92 & 93 & 95 & 96 & 96 & 96 & 96 & 95 \\ 92 & 94 & 96 & 98 & 99 & 99 & 98 & 97 \\ 94 & 96 & 99 & 101 & 103 & 103 & 102 & 101 \\ 95 & 97 & 101 & 104 & 106 & 106 & 105 & 105 \end{bmatrix}$$

Let suppose we want to replace each column of A by its $\mathcal{H}$-coordinate. By Theorem, we have to calculate $H^T A$, where

$$H = \begin{bmatrix} \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

We get

$$H^T A = \begin{bmatrix} 260.22 & 263.4 & 267.29 & 268.35 & 268.35 & 273.3 & 282.49 & 289.56 \\ -3.54 & -6.72 & -4.95 & -3.18 & -2.47 & -4.6 & -6.72 & -8.84 \\ -3.5 & -1.5 & -1.5 & -1.5 & -3. & -4. & -5. & -6.5 \\ -2.5 & -3. & -1.5 & 0. & 0.5 & 1.5 & 1.5 & 1. \\ -1.41 & 0. & 0. & 0. & -0.71 & -1.41 & -1.41 & -1.41 \\ -0.71 & -0.71 & -0.71 & -0.71 & -0.71 & -1.41 & -1.41 & -2.12 \\ 0. & -1.41 & -0.71 & 0. & 0. & 0. & 0. & 0. \\ -0.71 & 0. & 0. & 0. & 0.71 & 0.71 & 0.71 & 0. \end{bmatrix}$$

How could one use that for (lossy) image compression?

- Pick $\epsilon > 0$, and set all entries of $H^T A$ with absolute value at most ϵ to 0.
- $H^T A$ has more of these small entries, so more values become 0.

Already good, but we can do even better! Replace the rows of $H^T A$ by their $\mathcal{H}$-coordinates. For that we just need to calculate $H^T AH$! Why? We calculate

$$H^T AH = \begin{bmatrix} 768.25 & -19.25 & -6.01 & -15.2 & -2.25 & -0.75 & -3.5 & -5. \\ -14.5 & 1.5 & -1.06 & 4.24 & 2.25 & -1.25 & 1.5 & 1.5 \\ -9.37 & 3.71 & -1. & 2.25 & -1.41 & 0. & 0.71 & 1.06 \\ -0.88 & -4.07 & -2. & -0.25 & 0.35 & -1.06 & -0.71 & 0.35 \\ -2.25 & 1.25 & -0.71 & 0.35 & -1. & 0. & 0.5 & 0. \\ -3. & 1. & 0. & 0.71 & 0. & 0. & 0.5 & 0.5 \\ -0.75 & -0.75 & -0.35 & 0. & 1. & -0.5 & 0. & 0. \\ 0.5 & -1. & -0.35 & 0.35 & -0.5 & 0. & 0. & 0.5 \end{bmatrix}.$$

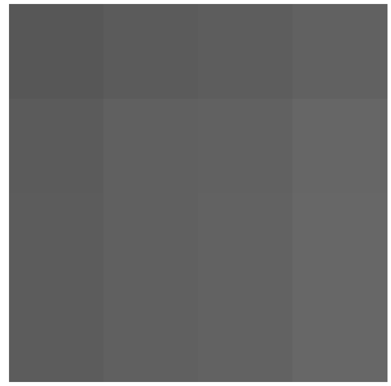
This just a change of base: $H^T AH = A_{\mathcal{H},\mathcal{H}}$. Image compression algorithms usually then divide every entry by an integers (this process is called quantization) and then rounds each entries to the nearest integer. Say we divide here by 12, then our matrix becomes the matrix B:

$$B = \begin{bmatrix} 64 & -2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To recover an image, we have to reverse the process. How do you do that? So let's calculate $H(12B)H^T$:

$$\begin{bmatrix} 87.3 & 87.3 & 91.5 & 91.5 & 93.3 & 93.3 & 97.5 & 97.5 \\ 87.3 & 87.3 & 91.5 & 91.5 & 93.3 & 93.3 & 97.5 & 97.5 \\ 91.5 & 91.5 & 95.7 & 95.7 & 97.5 & 97.5 & 101.7 & 101.7 \\ 91.5 & 91.5 & 95.7 & 95.7 & 97.5 & 97.5 & 101.7 & 101.7 \\ 92.4 & 92.4 & 96.6 & 96.6 & 98.4 & 98.4 & 102.6 & 102.6 \\ 92.4 & 92.4 & 96.6 & 96.6 & 98.4 & 98.4 & 102.6 & 102.6 \\ 92.4 & 92.4 & 96.6 & 96.6 & 98.4 & 98.4 & 102.6 & 102.6 \\ 92.4 & 92.4 & 96.6 & 96.6 & 98.4 & 98.4 & 102.6 & 102.6 \end{bmatrix}$$

Let's compare the images. The original is on the left, the compressed image on the right:

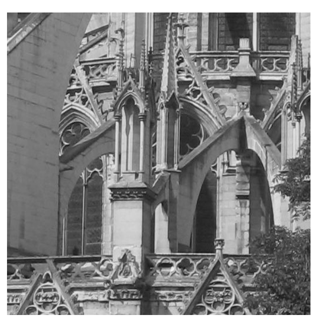
The compression ratio of an image is the ratio of the non-zero elements in the original matrix to the non-zero elements in the matrix representing the compressed image. The matrix

$$\begin{bmatrix} 64 & -2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has only 6 non-zero entry, while matrix A has 64. So the compression ratio is 64/6. That's pretty high!

2.2 JPEG

So does JPEG works? Given an image, let's say a 512×512 pixel grayscale image of the flying buttresses of the Notre Dame Cathedral in Paris:



This picture is split into blocks of 8×8-pixels. The block in top left corner is given by our matrix A. As the next step the JPEG algorithm does precisely what we did above.