Part 1

1. Let $X_1, \ldots, X_5$ be iid with pdf

$$f(x) = \frac{3}{(1+x)^4}, \qquad 0 < x < \infty.$$

   a. Show that the cdf is $F(x) = 1 - (1+x)^{-3}, x > 0$.

   b. Find $P(\min X_i > 0.1)$.

2. Let X and Y be independent random variables with mgfs,

$$M_X(t) = \frac{3}{4} + \frac{1}{4}e^t, \qquad t \in R$$
$$M_Y(t) = \frac{1}{2} + \frac{1}{2}e^t, \qquad t \in R$$

Find the pmf for $W = X + Y$.

3. Consider random variables X_1 and X_2 with $E(X_1) = 2$, $E(X_2) = 2$, $Var(X_1) = 2$, $Var(X_2) = 1$, and $Cov(X_1, X_2) = 1$. Let $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$. Find the correlation between Y_1 and Y_2.

4. Let the joint pdf for X and Y be,

$$f(x, y) = \frac{2}{5}, \qquad 0 < y < 1, \qquad 0 < x < 3 - y, \text{ zero otherwise}$$

Let $U = X/Y$. Find the cdf of U. (Be sure to specify the support)

5. Let the joint pdf for X and Y be,

$$f(x,y) = \frac{2}{5}, \qquad 0 < y < 1, \qquad 0 < x < 3 - y, \text{ zero otherwise}$$

Let $W = X + Y$. Find the cdf of W. (Be sure to specify the support)

6. Let the joint pdf for X and Y be,

$$f(x,y) = \frac{60}{77}x^3 y, \qquad 0 < x < 1, \qquad 0 < y < 4 - x$$

a. Show that the marginal distribution of X is,

$$f(x) = \frac{30}{77}x^3(4-x)^2, \qquad 0 < x < 1.$$

b. Find $E(Y|X)$.

7. Let the joint pdf for X and Y be,

$$f(x,y) = \frac{60}{77}x^3 y, \qquad 0 < x < 1, \qquad 0 < y < 4 - x$$

Let $U = X$ and $V = XY$. Find the joint pdf of U and V. (Be sure to specify the support)

Part 2

8. Let $X_1, \dots, X_n$ be iid with pdf

$$f(x;\theta) = \frac{1}{\theta}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{1}{2}\frac{x^2}{\theta^2}\right), \qquad 0 < x < \infty, \qquad 0 < \theta < \infty.$$

and mean and variance,

$$E(X) = \theta\sqrt{\frac{2}{\pi}}, \qquad Var(X) = \theta^2\left(1 - \frac{2}{\pi}\right).$$

a) Show that the method of moments estimator $\tilde{\theta}$, of θ is

$$\tilde{\theta} = \bar{X}_n\sqrt{\frac{\pi}{2}}.$$

b) For problem 8, is $\tilde{\theta}$ an efficient estimator of θ?

9. Let $X_1, \dots, X_n$ be iid with pdf

$$f(x;\lambda) = \frac{\lambda}{2\sqrt{x}}\exp\left(-\lambda\sqrt{x}\right), \qquad 0 < x < \infty, \qquad 0 < \lambda < \infty.$$

Note that,

$$E(X) = \frac{2}{\lambda^2}, \;\; Var(X) = \frac{20}{\lambda^4}$$

Find the limiting distribution of the method of moments estimator

$$\tilde{\lambda} = \sqrt{\frac{2}{\bar{X}_n}}.$$

10.Let $X_1, \dots, X_n$ be iid with pdf

$$f(x;\lambda) = \frac{\lambda}{2\sqrt{x}} \exp\left(-\lambda\sqrt{x}\right), \qquad 0 < x < \infty, \qquad 0 < \lambda < \infty.$$

Find $I(\lambda)$.

11. Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x;\lambda) = \frac{\lambda}{2\sqrt{x}} \exp\left(-\lambda\sqrt{x}\right), \qquad 0 < x < \infty, \qquad 0 < \lambda < \infty.$$

a) Show that $Y_i = \sqrt{X_i} \sim Exponential(\lambda)$.

b) Show that the maximum likelihood estimator

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \sqrt{X_i}},$$

is a consistent estimator of λ.

12.Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x;\theta) = \frac{1}{\theta}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{1}{2}\frac{x^2}{\theta^2}\right), \qquad 0 < x < \infty, \qquad 0 < \theta < \infty.$$

Show that,

$$Y = \sum_{i}^{n} X_i^2$$

is a sufficient statistic for θ.

13.Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x;\lambda) = \frac{\lambda}{x^{\lambda+1}}, \qquad 1 < x < \infty, \qquad 2 < \lambda < \infty.$$

a) Show that the maximum likelihood estimator is,

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \ln x_i},$$

b) For problem 13, find the likelihood ratio statistic $\Lambda(\mathbf{x})$ for, the hypothesis test: $H_0\colon \lambda = 3 \quad vs. \quad H_1\colon \lambda \neq 3.$

Part 3

14. Consider the following model for an iid sample $X_1, \dots, X_n$

$$X_i|\lambda \sim Poisson(\lambda)$$
$$\lambda \sim Gamma\left(\alpha = a, \theta = \frac{1}{b}\right)$$

so the prior distribution is proportional to,

$$h(\lambda) \propto \lambda^{a-1} e^{b\lambda}$$

a) Show that the posterior distribution of λ is $Gamma\left(\alpha = a + \sum_{i=1}^{n} x_i, \theta = \frac{1}{b+n}\right)$.

b) What is the squared loss Bayes estimator of λ?

15. Let $\lambda > 0$ and let $X_1, \ldots, X_8$ be an iid sample of size $n = 8$ with pdf,

$$f(x;\lambda) = \frac{\lambda}{(1+x)^{\lambda+1}}, \qquad 0 < x < \infty.$$

a) Show that $W = \ln(1+X) \sim Exponential(\lambda)$.

b) We wish to test $H_0: \lambda = 4$ vs. $H_1 : \lambda < 4$. Show that the uniformly most powerful rejection region with a 1% level of significance is "Reject H_0 if $\sum_{i=1}^{8} w_i \geq 4$".

Reject H_0 if ____.

c) Find the power of the test in part (a) if $\lambda = 2.25$.

Answer: ________________.

d) Suppose we observe $\sum_{i=1}^{8} \ln(1 + x_i) = 3.5$. Find the p-value of the test.

Answer: ________________.