Convergence and Large Sample Approximations – Part 1

Fall 2015
A. Stepanov
D. Simpson
S. Culpepper

Definition: Convergence in Distribution

Let $\{X_n\}$ be a sequence of random variables and let X be a random variable. Let F_{X_n} and F_X be, respectively, the cdfs of X_n and X. Let $C(F_X)$ denote the set of all points where F_X is continuous. We say that X_n **converges in distribution** to X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)\,, \forall\, x \in C(F_X)$$

We denote this convergence by

$$X_n \xrightarrow{D} X.$$

Definition: Convergence in Probability

Let $X_1, X_2, \ldots$ be an infinite sequence of random variables, and let X be another random variable. Then the sequence $\{X_n\}$ **converges in probability** to X, if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathrm{P}(|X_n - X| \geq \varepsilon) = 0, \lim_{n\to\infty} \mathrm{P}(|X_n - X| < \varepsilon) = 1$$

and write $X_n \xrightarrow{P} X$.

Theorem 1 $\qquad X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X$

Theorem 2 $\qquad X_n \xrightarrow{D} b$, b – constant $\Rightarrow X_n \xrightarrow{P} b$

Example 1 Consider a sequence of discrete random variables X_n where

$$P(X_n = 0) = \frac{1}{4} \text{ and } P\left(X_n = \frac{1}{n}\right) = \frac{3}{4}\,, \quad n = 1,2,3,\ldots$$

For each n the cdf is $F_n(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \leq x < \frac{1}{n} \\ 1, & x \geq \frac{1}{n} \end{cases}$

We see that

$$x < 0 \text{ implies } F_n(x) \to 0 \text{ as } n \to \infty$$
$$x > 0 \text{ implies } F_n(x) \to 1 \text{ as } n \to \infty$$

Therefore $F_n(x) \to F(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$ for every $x \neq 0$, i.e. for every point of continuity for F. This is the cdf of the constant 0, i.e. $P(X = 0) = 1$. Thus $X_n \xrightarrow{D} 0$.

By Theorem 2 we would conclude that $X_n \xrightarrow{P} 0$ as well. For a direct proof note that for any fixed $\epsilon > 0$ we have

$$P(|X_n| \geq \epsilon) = \begin{cases} \frac{3}{4}, & if \ \frac{1}{n} \geq \epsilon \\ 0, & if \ \frac{1}{n} < \epsilon \end{cases}$$

So the probability = 0 eventually, for all $n > \frac{1}{\epsilon}$. Since $\epsilon > 0$ was arbitrary, we conclude by definition that $X_n \xrightarrow{P} 0$.

Example 2. Let $Z_n = Z + \frac{1}{n}Y$ where Y and Z are independent $N(0,1)$ random variables. It follows that $Z_n \sim N(0, 1 + \frac{1}{n^2})$. To establish its limiting distribution consider the limiting moment generating function:

$$M_{Z_n}(t) = e^{\frac{1}{2}\left(1+\frac{1}{n^2}\right)t^2} \to e^{\frac{1}{2}t^2} = M_Z(t) \text{ as } n \to \infty$$

By a result stated below (Theorem 7) this implies that $Z_n \xrightarrow{D} Z$.

We could have also proven this directly via the cdf. Note that

$$F_{Z_n}(t) = \Phi\left(\frac{t}{\sqrt{1+\frac{1}{n^2}}}\right) \to \Phi(t) = F_Z(t)$$

for all t. Hence, by definition $Z_n \xrightarrow{D} Z$.

In fact we can show a stronger convergence. For any $\epsilon > 0$ we have

$$P(|Z_n - Z| \geq \epsilon) = P\left(\left|\frac{Y}{n}\right| \geq \epsilon\right) = P(|Y| \geq n\epsilon)$$
$$= 2\big(1 - \Phi(n\epsilon)\big) \to 0 \ \ as\, n \to \infty$$

Therefore by definition we have that $Z_n \xrightarrow{P} Z$, which also imples $Z_n \xrightarrow{D} Z$.

Example 3. Here's an example where convergence in distribution holds, but not convergence in probability: Let $Z_n = -Z$ for all *n* where $Z \sim N(0,1)$. Then $Z_n \xrightarrow{D} Z$ but $P(|Z_n - Z| \geq \epsilon) = P(|Z| \geq \epsilon) = P\left(|Z| \geq \frac{\epsilon}{2}\right) = 2\left(1 - \Phi\left(\frac{\epsilon}{2}\right)\right) > 0$ for all *n*. It follows that Z_n does *not* converge in *probability* to *Z*.

Example 4.

Let $X_1, X_2, \ldots$ be i.i.d. Uniform$(0, \theta)$. Let $Y_n = \max(X_1, X_2, \ldots, X_n)$.

First show that $Y_n \xrightarrow{P} \theta$. This follows because, given any $\epsilon > 0$ and less than θ,

$$P(|Y_n - \theta| \geq \epsilon) = P(Y_n \leq \theta - \epsilon) = \left(\frac{\theta - \epsilon}{\theta}\right)^n,$$

which converges to 0 as *n* increases, because $|(\theta - \epsilon)/\theta| < 1$.

Next find the limiting distribution of $Z_n = n(\theta - Y_n)$.

$$F_{Y_n}(x) = F_{\max X_i}(x) = \left(\frac{x}{\theta}\right)^n, 0 < x < \theta.$$

$$F_{Z_n}(z) = P[n(\theta - Y_n) \leq z] = P\left(Y_n > \theta - \frac{z}{n}\right) = 1 - \left(1 - \frac{z}{n\theta}\right)^n, 0 < z < n\theta.$$

$$\Rightarrow F_{Z_n}(z) \to 1 - e^{-\frac{z}{\theta}}, z > 0, as\ n \to \infty.$$

$Z_n \xrightarrow{D} X$, where $X \sim Exponential(\theta)$.

Example 5. Let $X_1, \dots, X_n$ be a random sample from the distribution with probability density function

$$f_X(x;\theta) = \frac{1}{\theta} x^{\frac{1-\theta}{\theta}}, 0 < x < 1, 0 < \theta < \infty$$

Let $Y_1 < Y_2 < \cdots Y_n$ denote the corresponding order statistics.

a) For which values of β does $W_n = n^\beta(1 - Y_n)$ converge in distribution? Find the limiting distribution of W_n.

$$F_{Y_n}(y) = P(Y_n \le y) = y^{\frac{n}{\theta}}, 0 < y < 1$$

$$F_{W_n}(w) = P\left[n^\beta(1 - Y_n) \le w\right] = P\left(Y_n \ge 1 - \frac{w}{n^\beta}\right)$$

$$= 1 - \left(1 - \frac{w}{n^\beta}\right)^{n/\theta}, 0 < w < n^\beta.$$

If $\beta = 1$, $\lim_{n\to\infty} F_{W_n}(w) = 1 - e^{-\frac{w}{\theta}}$, $0 < w < \infty$, Then $W_n \xrightarrow{D} X \sim Exponential(\theta)$.

If $\beta < 1$, $\lim_{n\to\infty} F_{W_n}(w) = 1$, $0 < w < \infty$, Then $W_n \xrightarrow{D} 0$ and thus $W_n \xrightarrow{P} 0$.

If $\beta > 1$, $\lim_{n\to\infty} F_{W_n}(w) = 0$, $0 < w < \infty$, Then W_n does not have a limiting distribution.

b) For which values of γ does $V_n = n^\gamma Y_1$ converge in distribution? Find the limiting distribution of V_n.

$$F_{Y_1}(y) = P(Y_1 \le y) = 1 - \left(1 - y^{\frac{1}{\theta}}\right)^n, 0 < y < 1$$

$$F_{V_n}(v) = P\left(Y_1 \le \frac{v}{n^\gamma}\right) = 1 - \left(1 - \frac{v^{\frac{1}{\theta}}}{n^{\frac{\gamma}{\theta}}}\right)^n, 0 < v < n^\gamma.$$

If $\gamma = \theta$, $\lim_{n\to\infty} F_{V_n}(v) = 1 - e^{-v^{1/\theta}}$, $0 < v < \infty$, Then $V_n \xrightarrow{D} X \sim Weibull(\theta)$.

If $\gamma < \theta$, $\lim_{n\to\infty} F_{V_n}(v) = 1\,,\ 0 < v < \infty$, Then $V_n \xrightarrow{D} 0$, and thus $V_n \xrightarrow{P} 0$.

If $\gamma > \theta$, $\lim_{n\to\infty} F_{V_n}(v) = 0\,,\ 0 < v < \infty$, Then V_n does not have a limiting distribution.

Theorem 3 $X_n \xrightarrow{D} X$, g is continuous on the support of X

$$\Rightarrow g(X_n) \xrightarrow{D} g(X)$$

Theorem 4 $X_n \xrightarrow{D} X, Y_n \xrightarrow{P} 0 \Rightarrow X_n + Y_n \xrightarrow{D} X$

Theorem 5 Slutsky's Theorem

$$X_n \xrightarrow{D} X, A_n \xrightarrow{P} a, B_n \xrightarrow{P} b$$
$$\Rightarrow A_n + B_n X_n \xrightarrow{D} a + b\,X$$

Theorem 6 $M_{X_n}(t) \to M_X(t)$ for $|t| < h \Rightarrow X_n \xrightarrow{D} X$.

Example 6. Let $X_n \sim Binomial\left(n, p = \frac{\lambda}{n}\right)$. Find the limiting distribution of X_n.

Let $X_n \sim Binomial\left(n, p = \frac{\lambda}{n}\right)$. Then

$$M_{X_n}(t) = \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^t\right)^n \to e^{\lambda(e^t - 1)} \text{ as } n \to \infty.$$

$M_X(t) = e^{\lambda(e^t-1)}$, where $X \sim Poisson(\lambda) \Rightarrow X_n \xrightarrow{D} X$ (Poisson approximation to Binomial distribution).

Example 7. Let $X_n \sim \chi^2(n)$. Recall $E(X_n) = n$ and $Var(X_n) = 2n$.

a) Let $Y_n = X_n/n$. Find the limiting distribution of Y_n.

Let $X_n \sim \chi^2(n)$ and $Y_n = X_n/n$. Then,

$$M_{Y_n}(t) = E\left[e^{\frac{X_n}{n}t}\right] = M_{X_n}\left(\frac{t}{n}\right) = \left(1 - 2\frac{t}{n}\right)^{-\frac{n}{2}} \to e^t \text{ as } n \to \infty.$$

Note $M_X(t) = e^t$, where $P(X = 1) = 1 \Rightarrow Y_n \xrightarrow{D} 1 \Rightarrow Y_n \xrightarrow{P} 1$.

b) Let $Z_n = (X_n - n)/\sqrt{2n}$. Find the limiting distribution of Z_n.

$$M_{Z_n}(t) = e^{-t\sqrt{\frac{n}{2}}} M_{X_n}\left(\frac{t}{\sqrt{2n}}\right) = e^{-t\sqrt{\frac{n}{2}}}\left(1 - 2\frac{t}{\sqrt{2n}}\right)^{-\frac{n}{2}}$$

$$= \left(e^{t\sqrt{\frac{2}{n}}} - t\sqrt{\frac{2}{n}}\,e^{t\sqrt{\frac{2}{n}}}\right)^{-\frac{n}{2}}, t < \sqrt{\frac{n}{2}}.$$

By Taylor approximation,

$$e^{t\sqrt{\frac{2}{n}}} = 1 + t\sqrt{\frac{2}{n}} + t^2\frac{1}{n} + o\left(\frac{1}{n}\right).$$

So for $t < \sqrt{\frac{n}{2}}$,

$$M_{Z_n}(t) = \left(\left(1 + t\sqrt{\frac{2}{n}} + t^2\frac{1}{n} + o\left(\frac{1}{n}\right)\right)\left(1 - t\sqrt{\frac{2}{n}}\right)\right)^{-\frac{n}{2}}$$

$$= \left(1 - \frac{t^2}{n} + o\left(\frac{1}{n}\right)\right)^{-\frac{n}{2}}$$

$$= \frac{1}{\left(\left(1 - \frac{t^2}{n} + o\left(\frac{1}{n}\right)\right)^{n}\right)^{\frac{1}{2}}}$$

$$\to \frac{1}{e^{-\frac{1}{2}t^2}} = e^{\frac{1}{2}t^2} \quad \text{as } n \to \infty$$

As $n \to \infty$, $M_{Z_n}(t) \to e^{\frac{1}{2}t^2} = M_Z(t)$, where $Z \sim N(0,1) \Rightarrow Z_n \xrightarrow{D} Z$.

Distribution-free convergence of sample averages

Weak Law of Large Numbers

$X_1, X_2, \dots, X_n$ are i.i.d. with mean μ and variance σ^2. Then

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \mu$$

Proof: For every fixed $\epsilon > 0$ we have, using Markov's inequality,

$$P(|\bar{X}_n - \mu| > \epsilon) = P((\bar{X}_n - \mu)^2 > \epsilon^2)$$

$$\leq \frac{E(\bar{X}_n - \mu)^2}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \to 0$$

as $n \to \infty$. Therefore $\bar{X}_n \xrightarrow{P} \mu$ by definition of convergence of probability.

Example 8. Let $X_1, X_2, \dots, X_n$ are i.i.d. with mean μ and variance σ^2 and finite fourth moment $\mu_4 = E(X^4)$. Then, by the weak law of large numbers,

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} E(X_1^2) = \sigma^2 + \mu^2$$

Furthermore, using our previous results we can show convergence of the sample variance:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$= \left(\frac{n}{n-1}\right)\left\{\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) - \bar{X}^2\right\} \xrightarrow{P} (1)\{(\sigma^2 + \mu^2) - \mu^2\} = \sigma^2$$

Example 9. Let $X_1, \dots, X_n$ be iid $U(0,1)$. Show the following:

a. $\bar{X}_n \xrightarrow{P} \frac{1}{2}$

b. $\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \frac{1}{2}\right)^2 \xrightarrow{P} \frac{1}{12}$

c. $\frac{1}{n}\sum_{i=1}^{n}\sqrt{X_i} \xrightarrow{P} \frac{2}{3}$

d. $\frac{1}{n}\sum_{i=1}^{n}\ln(X_i) \xrightarrow{P} -1$

e. $\frac{1}{n}\sum_{i=1}^{n}X_i^k \xrightarrow{P} \frac{1}{k+1}$

f. $\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\left(X_i > \frac{1}{2}\right) \xrightarrow{P} \frac{1}{2}$

Central Limit Theorem

$X_1, X_2, \dots, X_n$ are i.i.d. with mean μ and variance σ^2.

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\left(\sum_{i=1}^{n}X_i - n\mu\right)}{\sqrt{n}\sigma} \xrightarrow{D} Z \sim N(0,1).$$