STAT 410 HW #4 Solution

1. $E(X^k) = \int_0^\infty x^k \cdot \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}\, dx$

$= \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{k+\alpha-1} e^{-x}\, dx$

$= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$

$= \frac{(\alpha+k-1)\Gamma(\alpha+k-1)}{\Gamma(\alpha)}$

$= \cdots$

$= \frac{(\alpha+k-1)(\alpha+k-2)\cdots(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)}$

$= (\alpha+k-1)(\alpha+k-2)\cdots(\alpha+1)\alpha$

2. $X \sim \text{Gamma}(2,2) \sim \chi^2(4)$

$P(X > 7.779) = 1 - P(X \le 7.779) \doteq 0.1$

3. a. $P(\text{Gamma}(7, \frac{1}{3}) > 3) \doteq 0.207$

b. $P(1 < \text{Gamma}(7, \frac{1}{3}) < 2) = P(\text{Gamma}(7, \frac{1}{3}) < 2) - P(\text{Gamma}(7, \frac{1}{3}) < 1) \doteq 0.360$

4. $Y = (1, -1, 2)\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

a. $EY = (1,-1,2)\vec{\mu} = (1,-1,2)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5$ $\quad Var(Y) = (1,-1,2)\Sigma\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 5$

b. $P(X_1 > X_2 + X_3 - 4) = P(-X_1 + X_2 + X_3 < 4)$

$-X_1 + X_2 + X_3 \sim N(4, 2)$

$P(N(4,2) < 4) = P(N(0,1) < 0) = 0.5$

5. $P(Y=1) = P(X > 0)$

$= 1 - P(X \le 0)$

$= 1 - P\left(\frac{X-\eta}{1} \le -\eta\right)$

$= 1 - \Phi(-\eta)$

$= \Phi(\eta)$

6. a. $M_A(t) = E(e^{At})$

$$= E(e^{SZt})$$

$$= E\left(E(e^{SZt}|S)\right)$$

$$= \frac{1}{2}E(e^{Zt}) + \frac{1}{2}E(e^{-Zt})$$

$$= \frac{1}{2}e^{\frac{1}{2}t^2} + \frac{1}{2}e^{\frac{1}{2}t^2}$$

$$= e^{\frac{1}{2}t^2}$$

$\Rightarrow \quad A = SZ \sim N(0,1)$

b. $P(Z+A \le x) = P(Z+A\le x, S=1) + P(Z+A\le x, S=-1)$

$$= \frac{1}{2}P(2Z\le x) + \frac{1}{2}P(0\le x)$$

$$= \begin{cases} \frac{1}{2}\Phi(\frac{x}{2}) & x<0 \\ \frac{1}{2}\Phi(\frac{x}{2}) + \frac{1}{2} & x\ge 0 \end{cases}$$

the CDF has a jump at $x=0$ $\Rightarrow$ $Z+A$ is not normally distributed

7. $E(e^{Yt}) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} = M_Y(t)$

$E(X) = E(e^Y) = M_Y(t)\Big|_{t=1} = e^{\mu + \frac{1}{2}\sigma^2}$

$E(X^2) = E(e^{2Y}) = M_Y(t)\Big|_{t=2} = e^{2\mu+2\sigma^2}$

$Var(X) = E(X^2) - [E(X)]^2 = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$

8. $E(\bar{X}) = E(\vec{a}'\underset{\sim}{X}/n) = \frac{1}{n}E(\vec{a}'\underset{\sim}{X}) = \frac{1}{n}\vec{a}'\vec{\mu} = \frac{1}{n}(1,1,\cdots,1)\begin{pmatrix}\mu_1\\ \mu_2\\ \vdots \\ \mu_n\end{pmatrix} = \frac{1}{n}\sum_{i=1}^{n}\mu_i$

$Var(\bar{X}) = \frac{1}{n^2}\vec{a}'I_n\vec{a} = \frac{1}{n}$

$\bar{X} \sim N(\frac{1}{n}\sum_{i=1}^{n}\mu_i, \frac{1}{n})$

9. $$\Sigma = \begin{pmatrix} 1-\rho & & 0 \\ & 1-\rho & \\ 0 & \ddots & 1-\rho \end{pmatrix}_{n\times n} + \begin{pmatrix} \rho & \rho & \cdots & \rho \\ \rho & \rho & & \vdots \\ \vdots & & \ddots & \vdots \\ \rho & \cdots & & \rho \end{pmatrix}_{n\times n} = \begin{pmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & & & \rho \\ \rho & & \ddots & & \vdots \\ \vdots & & & & \rho \\ \rho & \rho & \cdots & \rho & 1 \end{pmatrix}_{n\times n}$$

a. $Var(X_i) = (\Sigma)_{ii} = 1 \qquad i=1,2,\cdots,n$

b. $Cov(X_i, X_j) = (\Sigma)_{ij} = \rho \qquad 1\le i<j\le n$

$Corr(X_i, X_j) = \frac{\rho}{1\cdot 1} = \rho \qquad 1\le i<j\le n$

10. $E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} \mu_i$

$$Var(\bar{X}) = \frac{1}{n^2}(1, 1, \cdots, 1)\,\Sigma \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{1}{n^2}\cdot n\cdot(1+(n-1)\rho) = \frac{1}{n} + \frac{n-1}{n}\rho$$

$$\bar{X} \sim N\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i,\ \frac{1}{n} + \frac{n-1}{n}\rho\right)$$

11. $$E(\Phi(a-bX)) = \int_{-\infty}^{+\infty} \Phi(a-bx) f_X(x)\,dx$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{a-bx} f_Z(z)\,dz\, f_X(x)\,dx \qquad Z, X \sim N(0,1)$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{a-bx} f_{ZX}(z,x)\,dz\,dx$$

$$= Pr(Z + bX \le a) \qquad Z + bX \sim N(0, 1+b^2)$$

$$= Pr\left(\frac{Z+bX}{\sqrt{1+b^2}} \le \frac{a}{\sqrt{1+b^2}}\right)$$

$$= \Phi\left(\frac{a}{\sqrt{1+b^2}}\right)$$

12. $$f(x) = C\cdot e^{-\ln(2)x^2} \sim N\left(0, \frac{1}{2\ln 2}\right)$$

$$C = \frac{1}{\sqrt{2\pi}\cdot\sigma} = \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2\ln 2}}} = \sqrt{\frac{\ln 2}{\pi}}$$

a. $$P(0.4 < X < 1.3) = P\left(\frac{0.4-0}{\sqrt{\frac{1}{2\ln 2}}} < \frac{X-0}{\sqrt{\frac{1}{2\ln 2}}} < \frac{1.3-0}{\sqrt{\frac{1}{2\ln 2}}}\right)$$

$$= P\left(0.4\sqrt{2\ln 2} < N(0,1) < 1.3\sqrt{2\ln 2}\right)$$

$$\approx 0.256$$

b. $$\int_{0.5}^{1.8} 2^{-x^2}dx = \frac{\sqrt{\pi}}{\sqrt{\ln 2}}\int_{0.5}^{1.8} f(x)\,dx$$

$$= \frac{\sqrt{\pi}}{\sqrt{\ln 2}} P(0.5 < X < 1.8)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\ln 2}} P\left(0.5\sqrt{2\ln 2} < N(0,1) < 1.8\sqrt{2\ln 2}\right) \approx 0.556$$