Math 415 - Lecture 26

Orthogonal Matrices and QR Decomposition

Monday October 26th 2015

Textbook reading: Chapter 3.4

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Suggested practice exercises: 3.4: 13, 16, 17, 18. 13,

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Khan Academy video: Gram-Schmidt Example

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Strang lecture: Orthogonal Matrices and Gram-Schmidt Process.

Review

- Vectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are orthonormal if

$$\mathbf{q}_i^T \mathbf{q}_j = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

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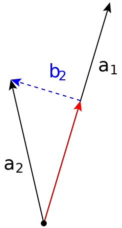

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The columns of Q are orthonormal $\iff Q^T Q = I$

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Definition

An **orthogonal matrix** is a square matrix Q with orthonormal columns.

The QR decomposition

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Theorem (QR decomposition)

Let A be an $m \times n$ matrix of rank n. There is is a orthogonal matrix $m \times n$-matrix Q and an upper triangular $n \times n$ invertible matrix R such that

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Idea. Gram-Schmidt on the columns of A to get columns of Q.

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The resulting R is indeed upper triangular, and we get:

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(Just the LU decomposition encodes the steps of Gaussian elimination, the QR decomposition encodes the steps of Gram-Schmidt.)

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$$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \langle \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \mathbf{q}_1 \rangle \mathbf{q}_1 - \langle \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \mathbf{q}_2 \rangle \mathbf{q}_2 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{q}_3$$

Solution (continued)

Hence: $Q = [\mathbf{q}_1 \ \mathbf{q}_2 \ \mathbf{q}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$

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Note R is upper triangular.

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Summarizing, we have

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Applications of $A = QR$

Using QR to solve systems of equations

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Example

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.Find the least square solution of $A\mathbf{x} = \mathbf{b}$ using QR-decomposition.

Solution

Let us first apply Gram-Schmidt to the columns of A.

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$$\mathbf{b_2} = \mathbf{a_2} - \frac{\langle \mathbf{a_2}, \mathbf{b_1} \rangle}{\langle \mathbf{b_1}, \mathbf{b_1} \rangle} \mathbf{b_1} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Solution (continued)

Normalizing we get

$$Q = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix}.$$

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Solution (continued)

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Now $A\mathbf{x} = \mathbf{b}$ is not consistent.

Solution

So we do least squares, but in this case ($A = QR$) we know the normal equations are

$$R\hat{\mathbf{x}} = Q^T\mathbf{b},$$

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So $\hat{\mathbf{x}} = \begin{bmatrix} 1/9 \\ 0 \end{bmatrix}$, and $\hat{\mathbf{b}} = A\hat{\mathbf{x}} = 1/9 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.