HW 2 Solutions

1. $m_x(t) = E(e^{tx}) = \sum_x e^{tx} p(x)$; then

$$f_x(x) = p(X=x) = \begin{cases} \frac{1}{2} & x=-1 \\ \frac{1}{4} & x=0 \\ \frac{1}{5} & x=1 \\ \frac{1}{20} & x=5 \end{cases}$$

2. a) $M(t) = \int_0^1 e^{xt}\,dx = \frac{e^{xt}}{t}\Big|_0^1 = \frac{e^t - 1}{t}$ for $t \neq 0$

If $t=0$, then $\int_0^1 e^{xt}\,dx = \int_0^1 1\,dx = 1$,

$$m(t) = \begin{cases} e^t - 1/t & t \neq 0 \\ 1 & t = 0 \end{cases}$$

b) $m(t) = \int_0^\infty e^{x(t-1)} = \frac{e^{x(t-1)}}{t-1}\Big|_0^\infty = \frac{1}{1-t}$ for $t<1$

Integral only converges for $t<1$

c) $m_x(t) = E(e^{tx}) = \int e^{tx} f_x(x)\,dx$

a) $f(x) = 1$, $x \in (0,1)$

b) $f(x) = e^{-x}$, $x \in (0,\infty)$

3. a) $P(X=Y) = P(0,0) + P(1,1) = 0.1 + 0.2 = 0.3$

b) $P(Y>0 \mid X>0) = P(Y=1 \mid X=1) + P(Y=2 \mid X=1)$

$$= \frac{P(1,1)}{P(X=1)} + \frac{P(1,2)}{P(X=1)} = \frac{0.2}{0.7} + \frac{0.4}{0.7}$$

$$= \frac{0.6}{0.7} = \frac{6}{7}$$

c) $E(X+Y) = \sum (x+y)\, p(x+y)$

$= 0\,(P(0,0)) + 1\,(P(0,1)+P(1,0)) + 2\,(P(0,2)+P(1,1))$
$+ 3\,(P(1,2))$

$= 0 \cdot 0.1 + 1 \cdot 0.25 + 2 \cdot 0.25 + 3 \cdot 0.4 = 1.95$

d) $E(Y|X=1) = 0 \cdot P(Y=0|X=1) + 1 \cdot P(Y=1|X=1)$
$+ 2 \cdot P(Y=2|X=1)$

$= \dfrac{P(1,1)}{P(X=1)} + \dfrac{2P(1,2)}{P(X=1)} = \dfrac{0.2}{0.7} + \dfrac{0.8}{0.7} = \dfrac{10}{7}$

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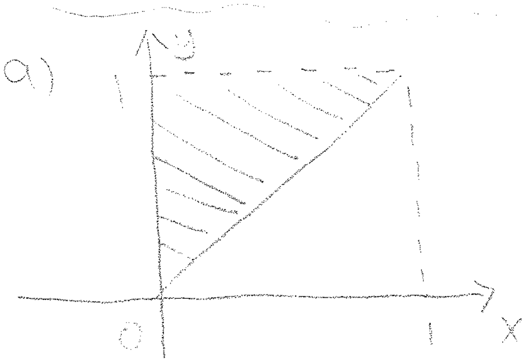
4. a)



b) $1 = \int_0^1 \int_0^y f_{XY}(x,y)\, dx\, dy$

$= \int_0^1 \int_0^y c\, dx\, dy$

$= \int_0^1 cx \Big|_0^y dy = \int_0^1 cy\, dy$

$= \left(\dfrac{cy^2}{2}\right)\Big|_0^1 = \dfrac{c}{2} \Rightarrow c = 2$

(c) $P(X > \frac{Y}{2}) = \int_0^1 \int_{y/2}^{y} f_{XY}(x,y)\, dx\, dy = \int_0^1 \int_{y/2}^{y} 2\, dx\, dy$

$= \int_0^1 y\, dy = \dfrac{1}{2}$

5. a) $P(Z \le 0) = 0$

Because $X > 0,\ y > 0 \Rightarrow Z = X + y > 0$.

b) For $z > 0$

$F_Z(z) = P(Z \le z) = P(X+Y \le z) = \int_0^z \int_0^{z-x} e^{-x-y}\, dy\, dx$

$= (-z-1)e^{-z} + 1$

$f_Z(z) = F_Z'(z) = ze^{-z} \Rightarrow f_Z(z) = \begin{cases} 0 & z \le 0 \\ ze^{-z} & z > 0 \end{cases}$

(c) $M_{X,Y}(t_x, t_y) = E(e^{t_x X} e^{t_y Y})$

$= \int_0^\infty \int_0^\infty e^{x t_x + y t_y - x - y} \, dx \, dy = \int_0^\infty \int_0^\infty e^{x(t_x - 1) + y(t_y - 1)} \, dx \, dy$

let $u = x(t_x - 1) + y(t_y - 1)$

$du = t_x - 1 \, dx$ $\quad \boxed{t_x < 1}$

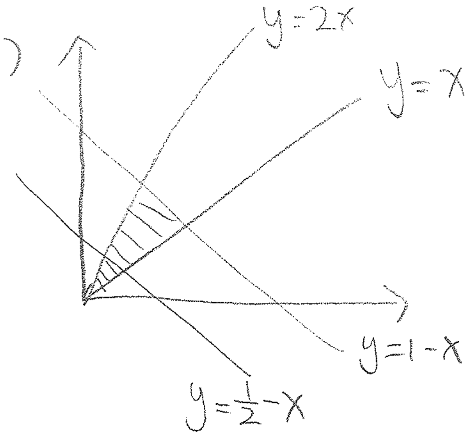
$\Rightarrow \int_0^\infty \frac{1}{t_x - 1} \int_{y(t_y-1)}^{-\infty} e^u \, du = \int_0^\infty \frac{1}{t_x - 1} e^u \Big|_{y(t_y-1)}^{-\infty} dy$

$= -\frac{1}{t_x - 1} \int_0^\infty e^{y(t_y - 1)} dy = \frac{1}{1 - t_x} \int_0^\infty e^{y(t_y-1)} dy$

let $s = y(t_y - 1)$, $ds = t_y - 1 \, dy$, $\boxed{t_y < 1}$

$= \frac{1}{(1-t_x)(t_y - 1)} \int_0^{-\infty} e^s \, ds = \frac{1}{(1-t_x)(1-t_y)} \quad t_x, t_y < 1$

6. (a)



(b) $c\left(\int_0^{1/3} \int_x^{2x} dy\,dx + \int_{1/3}^{1/2} \int_x^{1-x} dy\,dx\right)$

$= c\left(\int_0^{1/3} y \Big|_x^{2x} dx + \int_{1/3}^{1/2} y \Big|_x^{1-x} dx\right)$

$= c\left(\int_0^{1/3} x\,dx + \int_{1/3}^{1/2} 1 - 2x \, dx\right.$

$= c\left(\frac{x^2}{2} \Big|_0^{1/3} + (x - x^2) \Big|_{1/3}^{1/2}\right)$

$= c(1/18 + 1/36) = c\left(\frac{1}{12}\right) = 1 \Rightarrow c = 12$

(c) $\int_{X+Y>1/2} f \, dx\,dy = 1 - \int_0^{1/6} \int_x^{2x} 12 \, dy\,dx - \int_{1/6}^{1/4} \int_x^{1/2 - x} 12 \, dy\,dx$

$= 1 - 6x^2 \Big|_0^{1/6} - 6x - 12x^2 \Big|_{1/6}^{1/4} = 1 - \frac{1}{6} - \frac{1}{12} = \frac{3}{4}$

7. a. $f_X(x) = \begin{cases} \int_x^{2x} 12\,dy \\ \int_x^{1-x} 12\,dy \end{cases}$

$= \begin{cases} 12x & 0 < x < \frac{1}{3} \\ 12(1-2x) & \frac{1}{3} \le x < \frac{1}{2} \end{cases}$

b. $f_Y(y) = \begin{cases} \int_{\frac{y}{2}}^{y} 12\,dx \\ \int_{\frac{y}{2}}^{1-y} 12\,dx \end{cases}$

$= \begin{cases} 6y & 0 < y < \frac{1}{2} \\ 6(2-3y) & \frac{1}{2} \le y < \frac{2}{3} \end{cases}$

c. $f_{X,Y} \neq f_X \cdot f_Y \Rightarrow$ Not independent

8. a.

X \ Y	-1	0	1	
0	0	0.5	0	0.5
1	0.125	0.25	0.125	0.5
	0.125	0.75	0.125	

$EX = \frac{1}{2}$

$EY = 0$

$E(Y|X=0) = 0$

$E(Y|X=1) = 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{4} = 0$

b. $\rho_{X,Y} = \dfrac{E[(X-EX)(Y-EY)]}{\sigma_X \sigma_Y} = \dfrac{0 \times 0.5 + \frac{1}{2} \times 0.125 + 0 \times 0.25 + (-\frac{1}{2}) \times 0.125}{\sigma_X \sigma_Y} = \dfrac{0}{\sigma_X \sigma_Y}$

$= 0$

c. $P(X=0) = \frac{1}{2}$ $\quad P(Y=-1) = \frac{1}{8}$

$P(X=0, Y=-1) = 0 \neq P(X=0)P(Y=-1) = \frac{1}{16}$ $\Rightarrow$ Not independent

9. a. $Cov(2X+3Y, X-Y) = Cov(2X, X) - Cov(2X, Y) + Cov(3Y, X) - Cov(3Y, Y)$

$= -1$

b. $E(X+Y|X) = E(X|X) + E(Y|X) = X$

c. $E(E(X+Y|X)) = \int_x \int_{x+y} x+y\, f_{X+Y|X}(x+y|x)\, d(x+y) \cdot f_X(x)\,dx = \int_{x+y} f_{X+Y}(x+y) \cdot (x+y)\, d(x+y)$

$= E(X+Y)$

10. $\mu = \int_0^\infty x f_X(x)\,dx \geq \int_{2\mu}^\infty x f_X(x)\,dx \geq 2\mu \int_{2\mu}^\infty f_X(x)\,dx = 2\mu \cdot P(X \geq 2\mu)$

$\overset{\mu > 0}{\Longrightarrow} \quad P(X \geq 2\mu) \leq \frac{1}{2}$

11. take $u(x) = \frac{(x-\mu)^4}{\sigma^4}$, $c = d^4$ ($u(x)$ is nonnegative and $c>0$)

$P(u(X) \geq c) \leq \frac{E[u(X)]}{c} \quad \Rightarrow P\left(\frac{(X-\mu)^4}{\sigma^4} \geq d^4\right) \leq \frac{K}{d^4}$

12. a. $\phi[E(X)] \leq E[\phi(X)]$ ϕ convex

Let $\phi = x^4$ $[E(X)]^4 \leq E(X^4)$

b. Let $\phi = x^2$ $X = X^2$ $[EX^2]^2 \leq E[(X^2)^2] = E(X^4)$