Math 415 - Lecture 2

Echelon Forms, General Solution.

Wednesday August 26 2015

Textbook: Chapter 1.3, Chapter 2.2 (just the pages 78 and 79)

Suggested Practice Exercise: in Chapter 1.3, Exercise 17, 23, 24, in Chapter 2.2, Exercise 2 (just reduce A, B to echelon form), 8

Khan Academy Video: Matrices: Reduced Row Echelon Form 1

1 Row Reduction and Echelon Forms

Definition. A matrix is of **Echelon form** (or **row echelon form**) if

1. All nonzero rows are above any rows of all zeros.
2. The number of *leading zeroes* in each row increase going down.

   Or: Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

A leading entry of an echelon form matrix is also called a ***PIVOT.***

Example 1. Are the following matrices in Echelon form?

(a) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Echelon form ? 1. ✓ 2. ✓ 3. ✓

(b) $\begin{bmatrix} 0 & \blacksquare & * & * & * \\ \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Not echelon form.
1. ✓ 2. Fails 3. Fails
Would be after $R1 \leftrightarrow R2$

(c) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$ Echelon form.
1. ✓ 2. ✓ 3. ✓

(d) $\begin{bmatrix} \blacksquare & 0 & 0 \\ * & \blacksquare & 0 \\ * & 0 & \blacksquare \\ * & 0 & 0 \end{bmatrix}$ Not echelon form.
1. ✓ 2. Fails 3. Fails

(e) $\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$ Echelon form.
1. ✓ 2. ✓ 3. ✓
Leading column of 0s is OK.

Why Echelon Form?
The echelon form of an augmented matrix is good if you want to know if a system is consistent, and if so if there are infinitely many solutions. If you want to find the actual solutions (if any) you need to go further:

Definition. A matrix is of the **reduced echelon form** if in addition to conditions 1, 2, and 3 above it also satisfies

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Example 2. Are the following matrices in reduced echelon form?

(a) $\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$ Reduced row echelon form.
1. ✓ 2. ✓ 3. ✓ 4. ✓ 5. ✓

(b) $\begin{bmatrix} 1 & 0 & 5 & 0 & -7 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ No:
4. Fails 5. ✓

(c) $\begin{bmatrix} 1 & 0 & -2 & 3 & 2 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ No:
4. ✓ 5. Fails

Theorem 1 (Uniqueness of The Reduced Echelon Form)**.** *Each matrix is row-equivalent to one and only one reduced echelon matrix.*

Question: Is the same statement true for Echelon from?
No:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Both are row-equivalent and in echelon form.

2 Pivots

Definition. A **pivot position** is the position of a leading entry in an echelon form of the matrix.

Definition. A **pivot** of a matrix is a (nonzero) number that appears in a pivot position.

In a Reduced Row Echelon Form matrix the pivots are 1. Pivots are used to create 0's.

Definition. A **pivot column** is a column that contains a pivot position.

Example 3. In this example, highlight the pivot positions and pivot columns.

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 7 \\ 0 & 2 & 4 & 0 & 6 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} \downarrow & \downarrow & & \downarrow & \end{array}$$

$$\begin{bmatrix} \boxed{1} & 0 & 5 & 0 & 7 \\ 0 & \boxed{2} & 4 & 0 & 6 \\ 0 & 0 & 0 & \boxed{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 4. Row reduce to echelon form and locate the pivot columns for the following matrix.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution:

$$\xrightarrow[R4\leftrightarrow R1]{} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\xrightarrow[\substack{R2\to R2+R1 \\ R3\to R3+2R1}]{} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\xrightarrow[\cdots]{} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\xrightarrow[R3\leftrightarrow R4]{} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: There is no more than one pivot in any row. There is no more than one pivot in any column.

Example 5. Row reduce to echelon form and then to reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution:

$$\xrightarrow[R1\leftrightarrow R3]{} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\xrightarrow[R2\to R2-R1]{} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\xrightarrow[R2\to R2-R1]{} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\xrightarrow[R3\to R3-\frac{3}{2}R2]{} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This is echelon form!

$$\xrightarrow[\substack{R1\to\frac{1}{3}R1 \\ R2\to\frac{1}{2}R2}]{} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow[\substack{R1\to R1-2R3 \\ R2\to R2-R1}]{} \begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow[R1\to R1+3R2]{} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This is reduced row echelon form (RREF)!

3 Solution of linear systems

Why do we care about pivots and pivot columns? Recall: each column of a coefficient matrix corresponds to one of the variables.

Definition. A **pivot variable** (or *basic variable*) is a variable that corresponds to a pivot column in the coefficient matrix of a system.

Definition. A **free variable** is variable that is *not* a pivot variable.

Example 6. Consider the following system of linear equations: $\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$

$$\begin{array}{ccccccc} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array}$$

What are the pivot columns?

1st, 3rd, and 5th columns.

What are the pivot variables? x_1, x_3, and x_5.

What are the free variables?

x_2 and x_4.

Final Step in Solving a Consistent Linear System: After the augmented matrix is in **reduced** echelon form and the system is written down as a set of equations:

Solve each equation for the pivot variable in terms of the free variables (if any) in the equation.

Example 7 (A general solution).

$$\begin{array}{ccccccc} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array} \qquad \left\{ \begin{array}{ll} x_1 & = -6x_2 - 3x_4 \\ x_2 & = \text{ free} \\ x_3 & = 8x_4 + 5 \\ x_4 & = \text{ free} \\ x_5 & = 7 \end{array} \right.$$

The **general solution** of the system provides a **parametric description of the solution set**.

- The free variables act as parameters.
- The above system has **infinitely many solutions.** Why?

Because you can pick any value of x_2 and x_4.

Warning: Use only the reduced echelon form to solve a system.

Example 8. Find the parametric description of the solution set of

$$\begin{array}{rrrrrl} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & =-5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & =9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & =15 \end{array}$$

Its augmented matrix is

$$\left[\begin{array}{rrrrr|r} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$

We determined earlier that it is reduced echelon form is

$$\left[\begin{array}{rrrrr|r} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Equation form of the RREF matrix: $\left\{\begin{array}{rrrrrl} x_1 & & -2x_3 & +3x_4 & & =-24 \\ & x_2 & -2x_3 & +2x_4 & & =-7 \\ & & & & x_5 & =4 \end{array}\right.$

Pivot variables: x_1, x_2, x_5

Free variables: x_3, x_4

General solution: $\left\{\begin{array}{ll} x_1 & = 2x_3 - 3x_4 - 24 \\ x_2 & = 2x_3 - 2x_4 - 7 \\ x_3 & = \text{ free} \\ x_4 & = \text{ free} \\ x_5 & = 4 \end{array}\right.$

4 Existence And Uniqueness

We use the *reduced* echelon form to find the complete solution of a linear system.

The question whether a system has solution and whether it is unique, is much easier to answer than to find the complete solution.

- Echelon Form $\rightarrow$ Existence & Uniqueness.
- Reduced Echelon Form $\rightarrow$ Complete Solution.

Example 9. Let us go back to the following system

$$\begin{array}{lllll} 3x_2 & -6x_3 & +6x_4 & +4x_5 & =-5 \\ 3x_1 - 7x_2 & +8x_3 & -5x_4 & +8x_5 & =9 \\ 3x_1 - 9x_2 & +12x_3 & -9x_4 & +6x_5 & =15 \end{array}$$

Is the system *Consistent*? Is the solution *Unique*? Are there free variables? To answer these question we need just an echelon form. In an earlier example, we obtained the echelon form:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & | & 15 \\ 0 & 2 & -4 & 4 & 2 & | & -6 \\ 0 & 0 & 0 & 0 & 1 & | & 4 \end{bmatrix}$$

So for the echelon form matrix

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & | & 15 \\ 0 & 2 & -4 & 4 & 2 & | & -6 \\ 0 & 0 & 0 & 0 & 1 & | & 4 \end{bmatrix}$$

1. Is the system consistent? Yes/No? YES! Why? No row $\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & b \end{array}\right]$!
2. What are the free variables? x_3, x_4.
3. How many solutions?

So we see that there are infinitely many solutions.

Theorem 2 (Existence and Uniqueness Theorem)**.** *A linear system is* ***consistent*** *if and only if an echelon form of the augmented matrix has* ***no*** *row of the form*

$$\left[\begin{array}{ccc|c} 0 & ... & 0 & b \end{array}\right],$$

*where b is nonzero.**If*** *a linear system is consistent, then the solution contains either*

- *a unique solution (when there are no free variables) or*
- *infinitely many solutions (when there is at least one free variable).*

A consistent system can have 1 or ∞ many solutions. Look at the system with augmented matrix

$$\begin{bmatrix} 3 & 4 & | & -3 \\ 2 & 5 & | & 5 \\ -2 & -3 & | & 1 \end{bmatrix}$$

How many pivot variables can this matrix have? Do you expect the system to be consistent? Well, there are at most 2 pivots, so the last row of an echelon form should be $\left[\begin{array}{cc|c} 0 & 0 & b \end{array}\right]$. We cannot predict the value of b without doing some work. We need an echelon form.

The (reduced) echelon form of

$$\begin{bmatrix} 3 & 4 & | & -3 \\ 2 & 5 & | & 5 \\ -2 & -3 & | & 1 \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 0 & | & -5 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix}$$

So what is b? Is the system consistent? So how many pivots? How many free variables? How many solutions?

Look now at the system with augmented matrix

$$\begin{bmatrix} 3 & 4 & | & -3 \\ 3 & 4 & | & -3 \\ 6 & 8 & | & -6 \end{bmatrix}$$

How many free variables can this matrix have? What is the Echelon form? $\begin{bmatrix} 3 & 4 & | & -3 \\ 0 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$. Is the system consistent? How many free variables? How many solutions?