Math 415 - Lecture 23

Projections on subspaces

Monday October 19th 2015

Textbook reading: Chapter 3.2, 3.3, 3.4

Suggested practice exercises: Chapter 3.2 Exercise 17, 18, 24, Chapter 3.4 Exercise 2, 3 and see exercise at the end of this notes

Khan Academy video: Projections onto subspaces, Visualizing a projection onto a plane, Projection is closest vector in subspace

1 Review

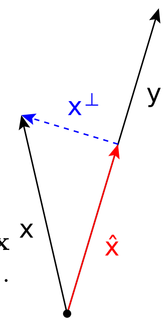
Last time

- **Orthogonal projection** of x onto y:

$$\hat{\mathbf{x}} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}} \mathbf{y}$$

  "Error" $\mathbf{x}^{\perp} = \mathbf{x} - \hat{\mathbf{x}}$ is orthogonal to $\mathbf{y}$.

- If $\mathbf{y}_1, \ldots, \mathbf{y}_n$ is an **orthogonal basis** of V, and $\mathbf{x}$ is in V, then $\mathbf{x} = c_1\mathbf{y}_1 + \ldots + c_n\mathbf{y}_n$ with $c_j = \frac{\mathbf{x} \cdot \mathbf{y}_j}{\mathbf{y}_j \cdot \mathbf{y}_j}$.



Remark. $\mathbf{x}$ decomposes as the sum of its projections onto each vector in the orthogonal basis.

Remark. The formulas simplify when you project on *unit* vectors: all denominators are then 1.

Example 1. Express $\underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{x}}$ in terms of the basis $\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\mathbf{y}_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{y}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{y}_3}$

Solution.
Notice that $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ is an orthogonal basis of $\mathbb{R}^3$.

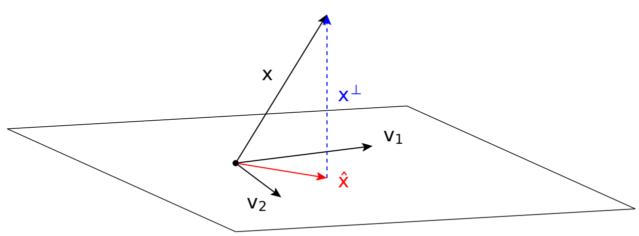
$$\begin{aligned}
\begin{bmatrix}2\\1\\1\end{bmatrix} &= c_1 \begin{bmatrix}1\\-1\\0\end{bmatrix} + c_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + c_3 \begin{bmatrix}0\\0\\1\end{bmatrix} \\
&= \underbrace{\frac{\begin{bmatrix}2\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\0\end{bmatrix}} \begin{bmatrix}1\\-1\\0\end{bmatrix}}_{\text{projection of } \mathbf{x} \text{ onto } \mathbf{y}_1} + \underbrace{\frac{\begin{bmatrix}2\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix}} \begin{bmatrix}1\\1\\0\end{bmatrix}}_{\text{projection of } \mathbf{x} \text{ onto } \mathbf{y}_2} + \underbrace{\frac{\begin{bmatrix}2\\1\\1\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix}}{\begin{bmatrix}0\\0\\1\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix}} \begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{projection of } \mathbf{x} \text{ onto } \mathbf{y}_3} \\
&= \frac{1}{2} \begin{bmatrix}2\\1\\1\end{bmatrix} + \frac{3}{2} \begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}
\end{aligned}$$

2 Orthogonal projection on subspaces

2.1 Projecting onto a subspace

Theorem 2. *Let W be a subspace of $\mathbb{R}^n$. Then, each $\mathbf{x}$ in $\mathbb{R}^n$ can be uniquely written as*

$$\mathbf{x} = \underbrace{\hat{\mathbf{x}}}_{\text{in } W} + \underbrace{\mathbf{x}^\perp}_{\text{in } W^\perp}$$



$\hat{\mathbf{x}}$ is the **orthogonal projection** of $\mathbf{x}$ onto W.

- $\hat{\mathbf{x}}$ is the point in W closest to $\mathbf{x}$. For any other $\mathbf{y}$ in W, $\text{dist}(\mathbf{x}, \hat{\mathbf{x}}) < \text{dist}(\mathbf{x}, \mathbf{y})$.
- If $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is an orthogonal basis of W, then

  $$\hat{\mathbf{x}} = \left(\frac{\mathbf{x} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 + \ldots + \left(\frac{\mathbf{x} \cdot \mathbf{v}_m}{\mathbf{v}_m \cdot \mathbf{v}_m}\right) \mathbf{v}_m$$

  Once $\hat{\mathbf{x}}$ is determined, $\mathbf{x}^\perp = \mathbf{x} - \hat{\mathbf{x}}$.

  (This is also the orthogonal projection of $\mathbf{x}$ onto $W^\perp$.)

Example 3. Let $W = span\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, and $\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}$.

- Find the orthogonal projection of $\mathbf{x}$ onto W. (Or: find the vector in W which is closest to $\mathbf{x}$)
- Write $\mathbf{x}$ as a vector in W plus a vector orthogonal to W.

Solution.

Note that $\mathbf{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are an orthogonal basis for W.
(We will soon learn how to construct orthogonal bases ourselves).
Hence, the orthogonal projection of $\mathbf{x}$ onto W is:

$$\begin{aligned}
\hat{\mathbf{x}} &= \frac{\mathbf{x} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{x} \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} w_2 \\
&= \frac{\begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \frac{10}{10} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}
\end{aligned}$$

Warning
This calculation only works for *orthogonal* $\mathbf{w_1}, \mathbf{w_2}$!

$\hat{\mathbf{x}}$ is the vector in W which best approximates $\mathbf{x}$.
Orthogonal projection of $\mathbf{x}$ onto the orthogonal complement of W:

$$\mathbf{x}^{\perp} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 9 \end{bmatrix}$$

Hence,

$$\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}}_{\text{in } W} + \underbrace{\begin{bmatrix} -3 \\ 0 \\ 9 \end{bmatrix}}_{\text{in } W^{\perp}}$$

Indeed, $\begin{bmatrix} -3 \\ 0 \\ 9 \end{bmatrix}$ is orthogonal to $\mathbf{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

2.2 The matrix of a projection

Definition 4. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be an orthogonal basis of W, a subspace of $\mathbb{R}^n$. Note that the projection map $\pi_W : \mathbb{R}^n \to \mathbb{R}^n$ that sends $\mathbf{x}$ to

$$\hat{\mathbf{x}} = \left(\frac{\mathbf{x} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 + \ldots + \left(\frac{\mathbf{x} \cdot \mathbf{v}_m}{\mathbf{v}_m \cdot \mathbf{v}_m}\right)\mathbf{v}_m$$

is a linear map. The matrix P representing π_W with respect to the standard basis is the **projection matrix**.

Example 5. Find the projection matrix P for the orthogonal projection onto

$$W = span\left\{\begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

in $\mathbb{R}^3$.

Solution. Standard basis: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$.

The first column of P encodes the projection of $\begin{bmatrix}1\\0\\0\end{bmatrix}$:

$$\frac{\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}3\\0\\1\end{bmatrix}}{\begin{bmatrix}3\\0\\1\end{bmatrix} \cdot \begin{bmatrix}3\\0\\1\end{bmatrix}}\begin{bmatrix}3\\0\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\1\\0\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix}}\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{3}{10}\begin{bmatrix}3\\0\\1\end{bmatrix}.$$

Hence $P = \begin{bmatrix}\frac{9}{10} & * & * \\ 0 & * & * \\ \frac{3}{10} & * & *\end{bmatrix}$.

The second column of P encodes the projection of $\begin{bmatrix}0\\1\\0\end{bmatrix}$:

$$\frac{\begin{bmatrix}0\\1\\0\end{bmatrix} \cdot \begin{bmatrix}3\\0\\1\end{bmatrix}}{\begin{bmatrix}3\\0\\1\end{bmatrix} \cdot \begin{bmatrix}3\\0\\1\end{bmatrix}}\begin{bmatrix}3\\0\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\1\\0\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\1\\0\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix}}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Hence $P = \begin{bmatrix} \frac{9}{10} & 0 & * \\ 0 & 1 & * \\ \frac{3}{10} & 0 & * \end{bmatrix}$.

The third column of P encodes the projection of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$:

$$\frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $P = \begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix}$.

Let's do the earlier example again using the matrix P.

Example 6. Let $W = span \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, and $\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}$. Find the orthogonal projection of $\mathbf{x}$ onto W.

Solution.

$$\hat{\mathbf{x}} = P\mathbf{x} = \begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix},$$

as in the previous example.

Example 7. Compute P^2 when

$$P = \begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix}.$$

Explain why the answer makes sense.

Solution.

$$\begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix} = \begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix}$$

$$P^2 = P$$

Once we have projected down onto W, projecting onto W again does not change anything!

3 Practice problems

3.1 Practice problems

Example 8. Find the closest point to $\mathbf{x}$ in $span\{\mathbf{v}_1, \mathbf{v}_2\}$ where

$$\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution.

$$\hat{\mathbf{x}} = \frac{6}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{-2}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix}$$

Example 9. If P is the projection matrix for projecting on W, what is the projection matrix Q for projecting on $W^{\perp}$?

Solution. $Q = 1 - P$!

Example 10. Let P be the projection matrix for projecting on W, and let $\mathbf{x}$ be some vector.

- Suppose $P\mathbf{x} = \mathbf{x}$. What can you say about $\mathbf{x}$?
- Suppose $P\mathbf{x} = 0$. What can you say about $\mathbf{x}$?