Please include your name (with your last name underlined) and your netid at the top of the first page. **No credit will be given without supporting work.**

1. Let W_n denote a random variable with mean μ and variance b/n^p, where $p > 0$, μ, and $b > 0$ are constants (not functions of n). Prove that $W_n \xrightarrow{P} \mu$.
2. Suppose $X_1, \dots, X_n$ is a random sample from a Uniform$(0, \theta)$ distribution. Let $Y_n = \max(X_1, \dots, X_n)$.
   a. Show that $Y_n \xrightarrow{P} \theta$.
   b. Show that $\left(\frac{n+1}{n}\right) Y_n \xrightarrow{P} \theta$.
3. Let $X_n \sim Exponential(\lambda = n)$. Show that $X_n \xrightarrow{P} 0$.
4. Let $X_1, \dots, X_n$ be iid $U(0,1)$ and let $Y_1 = \min(X_1, \dots, X_n)$ and $Y_n = \max(X_1, \dots, X_n)$. Show that $Y_1 + Y_n \xrightarrow{P} 1$. (Hint: See Theorem 5.1.2).
5. Let Y_1 denote the minimum of a random sample of size n from a distribution that has pdf $f(x) = e^{-(x-\theta)}, \theta < x < \infty$, zero elsewhere. Let $Z_n = n(Y_1 - \theta)$. Find the cumulative distribution function (cdf) for $Z_n = n(Y_1 - \theta)$, and find the limiting cdf of Z_n as $n \to \infty$.
6. Let $\bar{X}_n$ denote the mean of a random sample of size n from a Poisson distribution with parameter $\mu = 1$.
   a. Show that the mgf of $Y_n = \sqrt{n}(\bar{X}_n - 1)$ is given by
   $$\mathrm{M}_{Y_n}(\mathrm{t}) = \exp\left[-t\sqrt{n} + n\left(e^{t/\sqrt{n}} - 1\right)\right]$$
   b. Show that the limiting moment generating function of Y_n as $n \to \infty$ is that of the standard normal distribution. Hint: Approximate $e^{t/\sqrt{n}}$ in the exponent of the mgf of Y_n using the second order Taylor series expansion
   $$e^{\delta} = 1 + \delta + \frac{1}{2}\delta^2 + o(\delta^2) \ as \ \delta \to 0.$$
7. As in the previous problem, let $\bar{X}_n$ denote the mean of a random sample of size n from a Poisson distribution with parameter $\mu = 1$. Use the Δ-method to find the limiting distribution of $\sqrt{n}\left(\sqrt{\bar{X}_n} - 1\right)$.

8. Let $X_1, X_2, \dots, X_n$ be iid random variable from a distribution with mean 0, variance σ^2 and 4th moment $E(X^4) = \mu_4 < \infty$. Let $T_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Citing any theorems that you use, find the limiting distribution of $\sqrt{n}\,(T_n - \sigma^2)$ as $n \to \infty$.
9. Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with the following pdf:

$$f(x;\theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \qquad \text{zero elsewhere},$$

where θ is an unknown fixed parameter in the range $0 < \theta < \infty$.

   a. Derive an explicit expression for the maximum likelihood estimator (MLE) $\hat{\theta}_n$ based on the sample.
   b. Derive an explicit expression for a method of moments (MoM) estimator of $\tilde{\theta}_n$ based on the sample (equate the theoretical mean and sample mean, and solve for θ).
10. Consider the same setup as in Problem 9.
   a. Show that the MoM estimator $\tilde{\theta}_n$ is a consistent estimator of θ.
   b. Show that the MLE $\hat{\theta}_n$ is a consistent estimator of θ.

================== For Graduate Students ==================================

11. Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with one of two possible pdfs:

$$f(x;\theta) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}, & -\infty < x < \infty,\ \theta = 1 \\ \dfrac{1}{\pi(1+x^2)}, & -\infty < x < \infty,\ \theta = 2 \end{cases}$$

Find the MLE $\hat{\theta}$ for $\theta \in \{1,2\}$.
12. Suppose $X_1, X_2, \dots, X_n$ are iid with the pdf $f(x;\theta) = \frac{2x}{\theta^2}, 0 < x \le \theta$, zero elsewhere.
   a. Find the MLE $\hat{\theta}$ for θ.
   b. Find a constant c so that $E(c\hat{\theta}) = \theta$.
   c. Find the MLE for the median of the distribution, i.e. the value $m(\theta)$ such that

$$F(m(\theta);\theta) = 1/2,$$

where $F(x;\theta)$ is the cumulative distribution function.