Worksheet 3 for September 8th and 10th

1. (1) *True or False: If A and x are real numbers such that $Ax = 0$, then either $A = 0$ or $x = 0$*

(2) *Find a nonzero matrix A and vector x such that $Ax = 0$ but x is nonzero.*

(3) *Show that if A is a 2×2 matrix with pivots in each column, then $Ax = 0$ implies that $x = 0$.*

2. *Let θ be a real number. Then consider the following matrix:*

$$A_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

(1) *What does A_θ do to $\mathbb{R}^2$ geometrically? For a hint: consider what A_θ does to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.*

(2) *Using the geometric intuition from the previous part, show that $A_{\theta_1+\theta_2} = A_{\theta_1}A_{\theta_2}$*

(3) *Using the previous, we see that $A_{2\theta} = A_\theta^2$. Considering both sides of this equality, what trigonometric identity do you discover?*

3. *Let*

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

(1) *Is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ of the form Ax for some x? Set up a system and solve.*

(2) *Now do the same as in (1), but by thinking of vectors of the form Ax as linear combinations of the columns of A. What form do such linear combinations take?*

4. *Let*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

(1) *What is A^{100}?*

(2) *Can you calculate B^{100} by hand?*

5. *The processors of a supercomputer are inspected weekly in order to determine their condition. The condition of a processor can either be perfect, good or bad. A perfect processor is still perfect after one week with probability 0.7, with probability 0.2 the state is good and with probability 0.1 it is bad. A processor in good condition is still good after one week with probability 0.6 and bad with probability 0.4. A bad processor stays bad.*

(a) *What is the probability that a processor in perfect condition is in bad condition after two weeks?*

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Midterm Dates: September 29th, October 22nd, November 19th (All Midterms 7-8:15 PM, see `learn.illinois.edu` **for locations)**

(b) *Let us see how this connects to matrix multiplication. Write down the matrix*

$$T = \begin{bmatrix} T_{p,p} & T_{g,p} & T_{b,p} \\ T_{p,g} & T_{g,g} & T_{b,g} \\ T_{p,b} & T_{g,b} & T_{b,b} \end{bmatrix},$$

where the entries of T correspond to the probability that the condition of a processor changes from one week to the next. So for example, $T_{p,g}$ is the probability that a processor in perfect condition is in good condition after one week. Calculate T^2! The entry in the third row and the first column of T^2 should be equal to your result in (a). Why is that?

(c) *How can you use matrix multiplication to determine the probability that a processor in perfect condition is in bad condition after n weeks?*

6. (1) *Find a matrix E such that:*

$$E \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 - 2R_1 \\ R_3 \end{bmatrix}$$

Which matrix E' undoes the row operation implemented by E? What is $E'E$? Is E invertible, and if so, what is E^{-1}?

(2) *Find a matrix F such that:*

$$F \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_2 \\ R_1 \\ R_3 \end{bmatrix}$$

Which matrix F' undoes the row operation implemented by F? What is $F'F$? Is F invertible, and if so, what is F^{-1}?

(3) *Find a matrix G such that:*

$$G \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ 3R_2 \\ R_3 \end{bmatrix}$$

Which matrix G' undoes the row operation implemented by G? What is $G'G$? Is G invertible, and if so, what is G^{-1}?

The following may be useful in the above problems:

Definition. An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A.