Math 415 - Lecture 18

Inner Product and Orthogonality

Wednesday October 7th 2015

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Textbook reading: Ch 3.1

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Suggested practice exercises: 1, 2, 4, 5, 14, 16

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Khan Academy video: Vector Dot Product and Vector Length

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Strang lectures: Lecture 30: Linear Transformations / Lecture 14: Orthogonality

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Applications: Information retrieval

Review

- A **linear map** $T : V \to W$ satisfies

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- $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is linear. A is the matrix representing T in the standard bases.
  For example, $T(\mathbf{e_1}) = A\mathbf{e_1} =$ first column of A.

$$\left(\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)$$

- Any $T \colon V \to W$ can be represented by a matrix.

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- Replace obscure computations in V and W by transparent computations with matrices.
- Even if $T : \mathbb{R}^n \to \mathbb{R}^m$ (already have standard coordinates), T may be simpler in a different coordinate system.

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- So we know T if we know the matrix $T_{\mathcal{BA}}$:

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The output coordinate vector equals the matrix for T times the input coordinate vector.

Formula For the Coordinate matrix. To write $T\colon V \to W$ as a matrix, take an input basis $\mathcal{A} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ and and output basis $\mathcal{B} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_m})$. Then

$$T_{\mathcal{B}\mathcal{A}} = \begin{bmatrix} T(\mathbf{x_1})_{\mathcal{B}} & T(\mathbf{x_2})_{\mathcal{B}} & \ldots & T(\mathbf{x_n})_{\mathcal{B}} \end{bmatrix}$$

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$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = (-1)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Hence

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-2 & 0 & -1\end{bmatrix}.$$

Example

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What's the matrix A representing T in the standard bases?

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Solution

The standard bases are $\left\{1, t, t^2, t^3\right\}$ for $\mathbb{P}_2$ and $\left\{1, t, t^2\right\}$ for $\mathbb{P}_2$.

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The matrix A has 4 columns and 3 rows.

- $T(1) = 0$, so the first column is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
- $T(t) = 1$, so the second column is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.
- $T(t^2) = 2t$, so the third column is $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.
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So the matrix A representing T in the standard bases is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

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The corresponding polynomial is $p(t) = 1$. That makes sense because differentiation kills constant polynomials.

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$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 21 \end{bmatrix}$$

We get $-1 + 0t + 21t^2$, which is indeed the derivative of $7t^3 - t + 3$.

Inner Product and Distances

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$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = v_1 w_1 + \cdots + v_n w_n.$$

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Theorem

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(c) $(c\mathbf{u}) \cdot \mathbf{v} = c\,(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

(d) $\mathbf{u} \cdot \mathbf{u} \geq \mathbf{0}$, *and* $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ *if and only if* $\mathbf{u} = \mathbf{0}$.

Definition

- The **norm** (or **length**) of a vector $\mathbf{v} \in \mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

- The **distance** between points $v, w \in \mathbb{R}^n$ is

$$dist\left(\mathbf{v}, \mathbf{w}\right) = \|\mathbf{v} - \mathbf{w}\|.$$

Example

(a)

$$\left\| \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\|$$

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Inner Product and Angles

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$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$

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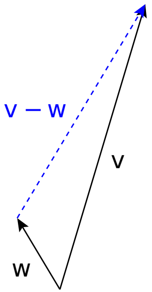

Example

What is the angle formed in $\mathbb{R}^3$ between the vectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}?$$

(A base jumper runs at a cliff at a 45° angle, then jumps straight away from the cliff and 45° downwards; what angle does he turn as he jumps?)

Solution

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Orthogonal vectors

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Remark

We write $\mathbf{v} \perp \mathbf{w}$ when $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. Nonzero vectors $\mathbf{v}$, $\mathbf{w}$ are orthogonal if and only if they are perpendicular.

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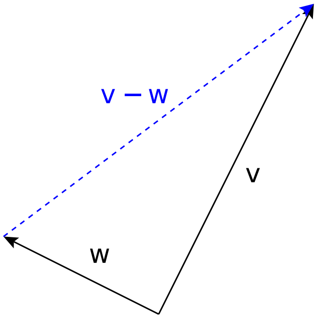
Nonzero vectors **v**, **w** are orthogonal if and only if they are perpendicular. We can derive this from Pythagoras' theorem.



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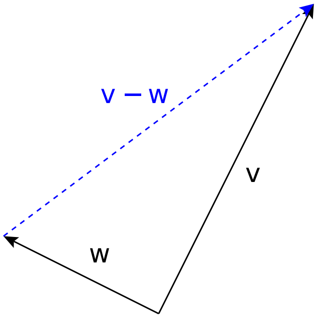

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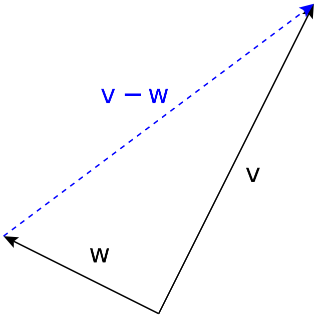

$\mathbf{v} \perp \mathbf{w}$

$\iff \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 = \|\mathbf{v} - \mathbf{w}\|^2$

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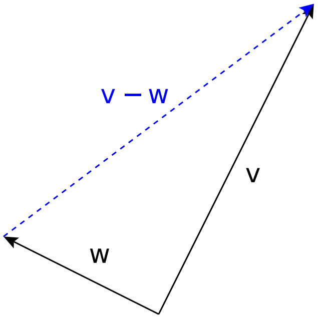

$$\begin{aligned} &\mathbf{v} \perp \mathbf{w} \\ \iff\ & \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 = \|\mathbf{v} - \mathbf{w}\|^2 \\ \iff\ & \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} = (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) \end{aligned}$$

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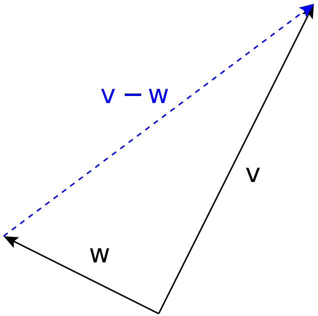

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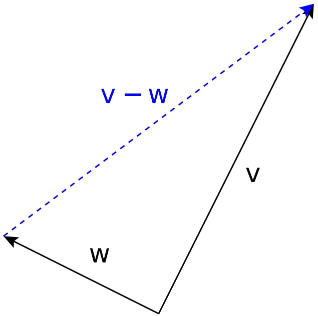

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&\iff \mathbf{v} \cdot \mathbf{w} = 0
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So no, they're not orthogonal.

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