Math 415 - Lecture 20

Fundamental Theorem of Linear algebra, orthogonal complement of

fundamental subspaces of a matrix

Monday October 12th 2015

Textbook reading: Chapter 3.1

Suggested practice exercises: Chapter 2.6, 5,6,7,36,37

Khan Academy video: Orthogonal complements

Strang lecture: Lecture 14: Orthogonal vectors and subspaces

1 Review

1.1 Orthogonality and FTLA

- $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are **orthogonal** iff $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w} = v_1w_1 \cdots + v_nw_n = 0$.
  - Non-zero orthogonal vectors are independent.
- If V is a subspace of $\mathbb{R}^n$ then the *orthogonal complement* of V is

$$V^\perp = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{x} = 0, \text{for all } \mathbf{v} \in V\}$$

- If $W = V^\perp$ then $W^\perp = V$.
- In other words $(V^\perp)^\perp = V$.
- $\dim(V) + \dim(V^\perp) = \dim(\mathbb{R}^n) = n$

Example 1. Let V be the horizontal x-y-plane in $\mathbb{R}^3$ and W the vertical y-z-plane.

- Is W the orthogonal complement of V?
- Is it true that W is orthogonal to V?
- What is the orthogonal complement of V?

Example 2. Given

$$\text{Nul}\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right),$$

get

$$\text{Col}\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}^T\right) = span\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

Why?

Theorem 1 (Fundamental Theorem of Linear Algebra)**.** *Let A be a $m \times n$-matrix. Then*

- *$Nul(A)$ is the orthogonal complement of $Col(A^T)$ (in $\mathbb{R}^n$). Also, dim $Nul(A)+$ dim $Col(A^T) = (n-r)+r = n$.*
- *$Col(A)$ is the orthogonal complement of $Nul(A^T)$ (in $\mathbb{R}^m$).*

Why? Suppose $\mathbf{x} \in Nul(A)$. That is,

$$A\mathbf{x} = \mathbf{0}$$

What does this mean? (Think row-column rule).

- It means that the inner product of every row of A(transposed!) with $\mathbf{x}$ is zero. But that implies that $\mathbf{x}$ is **orthogonal to the row space.**

1.2 FLTA in action

Example 3. Find all vectors orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Solution. This means: Find the orthogonal complement of $Col\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$.

Use the Fundamental Theorem: This is $Nul\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}^T\right) = Nul\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right)$

Compute Nul space: $Nul\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = Nul\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right)$

Basis for Nul: $\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$

Final answer: the set of vectors orthogonal to $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\end{bmatrix}$ is $span\left\{\begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$

Alternative solution. The FLTA is not magic! You can do this the old-fashioned way!

Looking for all $\mathbf{x}$ so that

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \mathbf{x} = 0 \text{ and } \begin{bmatrix}0\\1\\1\end{bmatrix} \cdot \mathbf{x} = 0$$

Matrix form:

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\end{bmatrix}\mathbf{x} = \begin{bmatrix}0\\0\end{bmatrix}$$

Get Null space:

$$\mathbf{x} \in Nul\left(\begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\end{bmatrix}\right)$$

This is the same null space we obtained from the FTLA.

Example 4. Let $V = \left\{\begin{bmatrix}a\\b\\c\end{bmatrix} : a+b=2c\right\}$. Find a basis for the orthogonal complement of V.

Solution. Write as Null space: $V = Nul\left(\begin{bmatrix}1 & 1 & -2\end{bmatrix}\right)$.

By FTLA: the orthogonal complement is $Col\left(\begin{bmatrix}1 & 1 & -2\end{bmatrix}^T\right)$.

Basis for the orthogonal complement: $\begin{bmatrix}1\\1\\-2\end{bmatrix}$

Geometrically this makes sense: V is a plane with normal vector $\begin{bmatrix}1\\1\\-2\end{bmatrix}$

Alternative solution.

Notice that $a+b=2c \iff \begin{bmatrix}a\\b\\c\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-2\end{bmatrix} = 0.$

Interpret the above: V is actually defined as the orthogonal complement of

$$span\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix}\right\}.$$

Example 5. Let $V = \left\{ \begin{bmatrix} 2a+b \\ -b \\ a+b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Find the orthogonal complement of V.

Solution. Write as Column space: $V = span\left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$, so $V = Col\left(\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \right)$

By FTLA the orthogonal complement is $Nul\left(\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \right)$

Get RREF to compute Null space: $\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$

Basis for the Null space: $\left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$

So the orthogonal complement to V is: $span\left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$

Directions and Equations. Let V be a subspace of $\mathbb{R}^n$. Then there are *two* ways of describing V.

By directions: If $V = Col(A)$ then you know that any vector $\mathbf{v}$ in V is a linear combination of the columns of A, so you know in which directions $\mathbf{v}$ can point.

By equations: If $V = Nul(B)$ then you know that any $\mathbf{v}$ in V satisfies the equations $\mathbf{R_i^T} \cdot \mathbf{v} = 0$, for all rows $\mathbf{R_i}$ of B.

Both descriptions are useful, and we will often switch between them, to answer any particular question we want to answer.

2 A new perspective on $A\mathbf{x} = \mathbf{b}$

To see if $A\mathbf{x} = \mathbf{b}$ has a solution, check that

Direct approach: $\mathbf{b} \in Col(A)$

Indirect approach: $\mathbf{b} \perp Nul(A^T)$

The indirect approach means:

$$\text{if } \underbrace{\mathbf{y}^T A = \mathbf{0}}_{\mathbf{y} \in Nul(A^T)}, \text{ then } \underbrace{\mathbf{y}^T \mathbf{b} = 0}_{\mathbf{b} \perp \mathbf{y}}.$$

Example 6. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution?

Solution (old)**. Write augmented matrix, get Echelon form:**

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 1 & b_2 \\ 0 & 5 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -5 & -3b_1 + b_2 \\ 0 & 0 & -3b_1 + b_2 + b_3 \end{array}\right]$$

When is this consistent? Whenever $-3b_1 + b_2 + b_3 = 0$.

Solution (new)**. Indirect approach says:** $A\mathbf{x} = \mathbf{b}$ solvable $\iff$ $\mathbf{b} \perp Nul(A^T)$.

Find basis for $Nul(A^T)$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 1 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

so $Nul(A^T)$ has basis $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

Need $\mathbf{b} \perp Nul(A^T)$: $A\mathbf{x} = \mathbf{b}$ is solvable $\iff$ $\mathbf{b} \cdot \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = 0$

This is the same condition as before!

3 Motivation

3.1 How to find almost-solutions

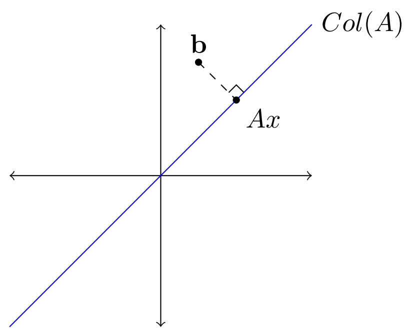
Why do we care about orthogonality? Not all linear systems have solutions. In fact, for many applications, data needs to be fitted and there is **no hope** for a perfect match. For example, $A\mathbf{x} = \mathbf{b}$ with

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

has no solution: $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is not in $Col(A) = span \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Idea. Instead of giving up, we want the $\mathbf{x}$ which makes $A\mathbf{x}$ and $\mathbf{b}$ as close as possible.

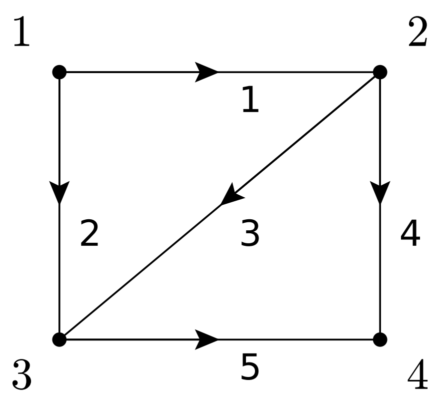
Such $\mathbf{x}$ is characterized by $A\mathbf{x}$ being **orthogonal** to the error $\mathbf{b} - A\mathbf{x}$.



4 Application: Directed graphs

4.1 Set up

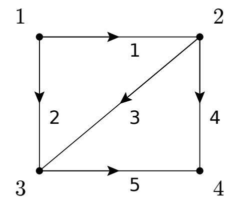
- Graphs appear in network analysis (e.g. internet) or circuit analysis.
- Arrow indicates direction of flow
- No edges from a node to itself



Definition 7. Let G be a graph with m edges and n nodes. The edge-node incidence matrix of G is the $m \times n$ matrix A with

$$A_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j \\ +1, & \text{if edge } i \text{ enters node } j \\ 0, & \text{otherwise} \end{cases}$$

Example 8. Give the edge-node incidence matrix of our graph.



Solution.

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- Each column represents a node
- Each row represents an edge

We are going to use linear algebra to study networks!