Stat 410 Homework 7 Fall 2015

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Bert and Ernie noticed that the following are satisfied when Cookie Monster eats cookies:

(a) The number of cookies eaten during non-overlapping time intervals are independent;

(b) The probability of exactly one cookie eaten in a sufficiently short interval of length h is approximately λh;

(c) The probability of two or more cookies eaten in a sufficiently short interval is essentially zero.

Therefore, X_t, the number of cookies eaten by Cookie Monster by time t, is a Poisson process, and for any $t > 0$, the distribution of X_t is Poisson(λt).

However, Bert and Ernie could not agree on the value of λ, the average number of cookies that Cookie Monster eats per minute. Bert claimed that it equals 1.5, but Ernie insisted that it has been less than 1.5 ever since Cookie Monster was forced to eat broccoli and carrots. Thus, the two friends decided to test

$$H_0: \lambda = 1.5 \quad \text{versus} \quad H_1: \lambda < 1.5$$

1. Bert decided to count the number of cookies Cookie Monster would eat in 7 minutes, X_7, and then Reject H_0 if X_7 is too small.

   a) Help Bert to find the best (uniformly most powerful) Rejection Region with the significance level α closest to 0.05. (Hint: $X_7 \le c$)

   b) Find the power of the test from part (a) if $\lambda = 1$.

2. Bert decided to Reject H_0 if Cookie Monster eats at most 6 cookies in 7 minutes.

   a) Find the significance level α for this rejection region.

   b) Suppose Cookie Monster ate 4 cookies in 7 minutes. Find the p-value of the test.

3. Ernie, who was less patient than Bert, decided to measure how much time Cookie Monster needs to eat the first 4 cookies, T_4, and then Reject H_0 if T_4 is too large. Help Ernie to find the best (uniformly most powerful) rejection region with the significance level $\alpha = 0.05$. (Hint: $T_4 \geq c$. If T has a Gamma (α , $\theta = 1/\lambda$) distribution, where α is an integer, then $2T/\theta$ has a χ^2 (2α) distribution (a chi-square distribution with 2α degrees of freedom).

4. Ernie decided to Reject H_0 if it takes Cookie Monster longer than 5 minutes to eat the first 4 cookies.
   a) Find the significance level α for this rejection region. Hint: If T has a Gamma(α, $\theta = 1/\lambda$) distribution, where α is an integer, then
   $$F_T(t) = P(T \leq t) = P(Y \geq \alpha) \text{ and } P(T > t) = P(Y \leq \alpha - 1),$$
   where Y has a Poisson (λt) distribution.
   b) Find the power of the test from part (a) if $\lambda = 1$.
   c) It took Cookie Monster 6 minutes to eat the first 4 cookies. Find the p-value of the test.

That's all for Bert and Ernie!

5. Suppose a pollster is interested in understanding whether candidate A or candidate B would win an election. Let X denote the number of respondents to a telephone poll that indicate a preference for candidate A out of a total of n respondents. Consider the following Bayesian formulation,
   $$X|\theta \sim Binomial(n, \theta)$$
   $$\theta \sim Beta(a, b)$$
   a. Suppose the pollster expects a close race with θ near 0.5. Set $a = b = 135$ and find the posterior distribution for θ.
   b. Find the posterior mean for θ and express it as a weighted average of the maximum likelihood estimate for θ and the mean of the prior pdf.

6. Under the same scenario as in the previous problem, use the posterior distribution to test the hypothesis H_0: Candidate A wins vs. H_1: Candidate B wins if $x = 255$ out of $n = 500$ respondents phoned supported candidate A (Hint: Use the pbeta R function).

7. Under the same scenario as in the previous two problems compute a 95% Bayesian credible interval for θ (Hint: Use the pbeta R function).

8. Let Y_n be the nth order statistic of a random sample of size n from a distribution with pdf $f(x|\theta) = \frac{1}{\theta},\ 0 < x < \theta$, zero elsewhere. Take the loss function to be $\mathcal{L}[\theta, \delta(\mathbf{x})] = [\theta - \delta(y_n)]^2$. Let θ be an observed value of the random variable Θ, which has the prior pdf $h(\theta) = \beta\alpha^\beta/\theta^{\beta+1}$, $\alpha < \theta < \infty$, zero elsewhere, with $\alpha > 0, \beta > 0$.
    a. Find the posterior distribution for Θ.
    b. Find the Bayes solution $\delta(y_n)$ for a point estimate of θ.

9. Under the same scenario as in Problem 8, find a 95% Bayesian credible interval for θ.

10. Suppose that $Y \sim Gamma(\alpha = r, \lambda = \theta)$ and $\lambda \sim Gamma(\alpha = s, \lambda = \mu)$. Show that the posterior pdf is $\Theta | Y = y \sim Gamma(\alpha = r + s, \lambda = y + \mu)$.

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For Graduate students:

11. Reconsider the model in problem 5. Show that the marginal distribution of X is

$$g_1(x) = \frac{\binom{x+a-1}{x}\binom{n-x+b-1}{n-x}}{\binom{n+a+b-1}{n}}$$

12. Suppose $X_1, X_2, \ldots, X_n$ are conditionally independent given θ and distributed as $N(\theta, \sigma^2)$, and suppose that $\theta \sim N(\mu, \tau^2)$. Show that the posterior distribution for θ given $X_1, X_2, \ldots, X_n$ is $N(\tilde{X}, v)$ where

$$\tilde{X} = w\bar{X} + (1-w)\mu, \qquad v = w\frac{\sigma^2}{n}, \quad \text{and} \quad w = \frac{1}{1 + \frac{\sigma^2}{n\tau^2}}$$