Math 415 - Lecture 33

Diagonalization

Monday November 16th 2015

Textbook reading: Chapter 5.2

Suggested practice exercises: Chapter 5.2: 1, 2, 3, 4, 5, 6, 7, 15, 16, 17, 18, 19, 20, 25, 26, 29, 30, 31, 32, 33

Strang lecture: Lecture 22: Diagonalization and powers of A

1 Review

- **Eigenvector** equation: $A\mathbf{x} = \lambda\mathbf{x} \iff (A - \lambda I)\mathbf{x} = \mathbf{0}$ λ is an **eigenvalue** of $A \iff \underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial}} = 0$.
- An $n \times n$ matrix A has up to n eigenvectors for λ.
  - The **eigenspace** of λ is $\text{Nul}(A - \lambda I)$.
  - If λ has **multiplicity** m, then A has up to m (independent) eigenvectors for λ. At least one eigenvector is guaranteed (because $\det(A - \lambda I) = 0$).
  - An **Eigenbasis** for an $n \times n$ matrix A is a basis $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ of $\mathbb{R}^n$ so that each $\mathbf{v_i}$ is also an eigenvector: $A\mathbf{v_i} = \lambda_i \mathbf{v_i}$.
- Test yourself! What are the eigenvectors and eigenvalues? Is there an eigen basis?
  - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\lambda = 1, 1$ (i.e. multiplicity 2), eigenspace is $\mathbb{R}^2$. Any basis is eigen basis.
  - $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\lambda = 0, 0$, eigenspace is $\mathbb{R}^2$. Again any basis is an eigenbasis.

  These are trivial cases. Is there always an eigenbasis?

Example 1. To solve $A\mathbf{x} = \mathbf{b}$ we use row operations. If we want to find eigenvectors, $A\mathbf{x} = \lambda\mathbf{x}$, can we also use row operations? Try $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$.

- What is the echelon form U of A?
- What are the characteristic polynomials $\det(A - \lambda I)$ and $\det(U - \lambda I)$? Roots?
- Do A and U have the same eigenvalues? Eigenvectors?

Solution.

- If $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ then $U = \begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix}$.
- Then $\det(A - \lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = = \lambda^2 - 0\lambda + (-4) = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2)$, and $\det(U - \lambda I) = \lambda^2 - \mathrm{Tr}(U)\lambda + \det(U) = = \lambda^2 - 3\lambda + (-4) = (\lambda - 1)(\lambda + 4)$.
- So the eigenvalues of A and U are **DIFFERENT!**. Can check that eigenvectors are also different.

Upshot: **Don't use row operations to deal with eigenvalues and eigenvectors!** (Can use row operations to calculate determinants, though.)

Example 2. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. What is the trouble?

Solution.

- $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$ So: $\lambda = 1$ is the only eigenvalue (it has multiplicity 2).
- $\lambda = 1 : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ So the eigenspace is Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Only dimension 1!
- Trouble: We can not find an **Eigenbasis** for this matrix. This kind of problem cannot really be fixed. We have to lower our expectations and look for generalized eigenvectors. These are solutions to $(A - \lambda I)^2\mathbf{x} = \mathbf{0}, (A - \lambda I)^3\mathbf{x} = \mathbf{0}, \ldots$

2 Diagonalization

2.1 Powers of diagonal matrices

Diagonal matrices are very easy to work with.

Example 3. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. What is A^2? What is A^{100}?

Solution. $A^2 = \begin{bmatrix} 2^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & 4^2 \end{bmatrix}$ and $A^{100} = \begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 3^{100} & 0 \\ 0 & 0 & 4^{100} \end{bmatrix}$.

2.2 Powers of generic matrices

Example 4. If $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$, then $A^{100} = ?$

Solution.

- characteristic polynomial: $\begin{vmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = \cdots = (\lambda-4)(\lambda-5)$
  - $\lambda_1 = 4 : \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \mathbf{v} = \mathbf{0} \implies$ eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
  - $\lambda_2 = 5 : \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \mathbf{v} = \mathbf{0} \implies$ eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Key observation: $A^{100}\mathbf{v}_1 = \lambda_1^{100}\mathbf{v}_1$ and $A^{100}\mathbf{v}_2 = \lambda_2^{100}\mathbf{v}_2$. For A^{100}, we need $A^{100}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A^{100}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies$

$$A^{100}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = A^{100}\left(-\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = -4^{100}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\cdot 5^{100}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\implies A^{100} = \begin{bmatrix} 2\cdot 5^{100} - 4^{100} & * \\ 2\cdot 5^{100} - 2\cdot 4^{100} & * \end{bmatrix}$$

- We find the second column of A^{100} likewise. Left as exercise!

The key idea of previous example is to work with respect to an *Eigenbasis*, a basis given by eigenvectors.

- Put the eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ as columns into a matrix P.

$$A\mathbf{x}_i = \lambda\mathbf{x}_i \implies A\begin{bmatrix} | & & | \\ \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- In summary $AP = PD$. Such a diagonalization is possible if and only if A has enough eigenvectors.

So we are going to use eigenvalues and eigenvectors for A to factor A and A^{100} in a useful way. This is called *diagonalization*.

Definition. A square matrix A is said to be **diagonalizable** if there is a invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

Theorem 1. *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.*

We can express the relation between A and D in terms of change of base matrices.

$$
\begin{tikzcd}
\begin{array}{c}\text{coords for } \mathbf{x} \\ \text{in standard basis}\end{array} \arrow[rr, "A"] \arrow[d, shift right, "P^{-1}"'] & & \begin{array}{c}\text{coords for } A\mathbf{x} \\ \text{in standard basis}\end{array} \arrow[d, shift left, "P^{-1}"] \\
\begin{array}{c}\text{coords for } \mathbf{x} \\ \text{in eigen- basis}\end{array} \arrow[rr, "D"] \arrow[u, shift right, "P"'] & & \begin{array}{c}\text{coords for } A\mathbf{x} \\ \text{in eigen- basis}\end{array} \arrow[u, shift left, "P"]
\end{tikzcd}
$$

$D = P^{-1}AP$, $A = PDP^{-1}$

P changes from eigenbasis coordinates to standard coordinates, and P^{-1} goes the other way! Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^n$ and $\mathcal{B}$ the basis of eigenvectors of A, then

$$P = I_{\mathcal{E},\mathcal{B}} \text{ and } P^{-1} = I_{\mathcal{B},\mathcal{E}}.$$

3 Application: Large powers

If A has an eigenbasis, then we can raise it to large powers easily!

Theorem 5. *If $A = PDP^{-1}$, where D is a diagonal matrix, then for any m,*

$$A^m = PD^mP^{-1}$$

Proof.

$$
\begin{aligned}
A &= PDP^{-1} \\
A^m &= (PDP^{-1})^m \\
&= \underbrace{(PDP^{-1}) \cdot (PDP^{-1}) \cdots (PDP^{-1})}_{m \text{ times}} \\
&= (PD)(P^{-1} \cdot P)(DP^{-1}) \cdots (PDP^{-1}) \\
&= PD \cdot DP^{-1} \cdots PDP^{-1} \\
&= PD \cdot D \cdots D \cdot P^{-1} \\
&= PD^mP^{-1}
\end{aligned}
$$

Only the outside P and P^{-1} remain! $\square$

Finding D^m is easy!

$$D^m = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}^m = \begin{bmatrix} (\lambda_1)^m & & \\ & \ddots & \\ & & (\lambda_n)^m \end{bmatrix}$$

Why?

Example 6. Let $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$. A has eigenvectors and eigenvalues

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{with} \quad \lambda_1 = \frac{1}{2}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{with} \quad \lambda_2 = 1$$

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} \quad \text{with} \quad \lambda_3 = 2$$

Find A^{100}. Hint: Write $A = PDP^{-1}$.

Solution. Eigenvectors of A form an Eigenbasis! So we can write $A = PDP^{-1}$:

Matrix of eigenvectors $P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

Find P^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2 \rightsquigarrow R2 - R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 6 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2 \rightsquigarrow R2 - 6R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -6 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

Eigenvectors of A form a Eigenbasis! So we can write $A = PDP^{-1}$:

Matrix of eigenvalues: $D = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Finally, write $A = PDP^{-1}$**:**

$$\underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}}_{P^{-1}}$$

Take power

$$
\begin{aligned}
\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}^{100} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{100} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2^{100}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2^{100}} & 0 & 0 \\ (\frac{1}{2^{100}} - 1) & 1 & (6 \cdot 2^{100} - 6) \\ 0 & 0 & 2^{100} \end{bmatrix}
\end{aligned}
$$