

Math 415 - Lecture 1

Introduction

Monday August 24 2015

- Textbook: Chapter 1.3, Chapter 2.2 (just the pages 78 and 79)
- Suggested Practice Exercise: in Chapter 1.3, Exercise 1, 3, 5, 6, 11
- Khan Academy Video: Matrices: Reduced Row Echelon Form 1

1 Systems of Linear Equations

Definition. A **linear equation** is an equation of the form

where a_1, \dots, a_n, b are numbers and x_1, \dots, x_n are variables.

Example 1. Which of the following equations are linear equations (or can be rearranged to become linear equations)?

| | | |
|---------------------------------|--|-------------------|
| $4x_1 - 5x_2 + 2 = x_1$ | | Linear/Nonlinear? |
| $x_2 = 2(\sqrt{6} - x_1) + x_3$ | | Linear/Nonlinear? |
| $4x_1 - 6x_2 = x_1x_2$ | | Linear/Nonlinear? |
| $x_2 = 2\sqrt{x_1} - 7$ | | Linear/Nonlinear? |

Definition. A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same set of variables, say, x_1, x_2, \dots, x_n .

Definition. A **solution** of a linear system is a list (s_1, s_2, \dots, s_n) of numbers that makes each equation in the system true when the values s_1, s_2, \dots, s_n are substituted for x_1, x_2, \dots, x_n , respectively.

Definition. The **solution set** of a system of linear equations is the set of all possible solutions of a linear system.

Example 2. Two equations in two variables:

$$\begin{aligned}x_1 + x_2 &= 1 \\ -x_1 + x_2 &= 0.\end{aligned}$$

What is a solution for this system of linear equations?

Example 3. Does every system of linear equation have a solution?

$$\begin{aligned}x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8.\end{aligned}$$

Example 4. How many solutions are there to the following system?

$$\begin{aligned}x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6\end{aligned}$$

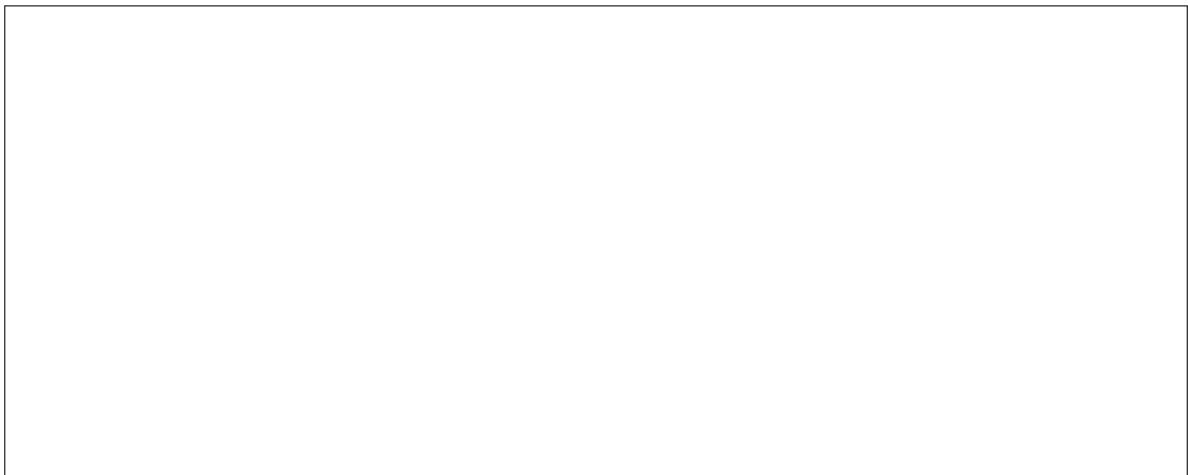


Theorem 1. *This is all there is: A linear system has either*

one unique solution or no solution or infinitely many solutions.

Can you draw the set of solutions of the above equations?

$$\begin{aligned}x_1 + x_2 &= 1 \\ -x_1 + x_2 &= 0.\end{aligned}$$



$$\begin{aligned}x_1 - 2x_2 &= -3 \\2x_1 - 4x_2 &= 8.\end{aligned}$$

$$\begin{aligned}x_1 + x_2 &= 3 \\-2x_1 - 2x_2 &= -6\end{aligned}$$

1.1 Strategies for solving systems of linear equations

Definition. Two systems are **equivalent** if they have the same solution set.

The general strategy is to replace one system with an equivalent system that is easier to solve.

Example 5. Consider

$$\begin{array}{rclcl}x_1 & - & 2x_2 & = & -1 \\-x_1 & + & 3x_2 & = & 3\end{array}$$



1.2 Matrix Notation

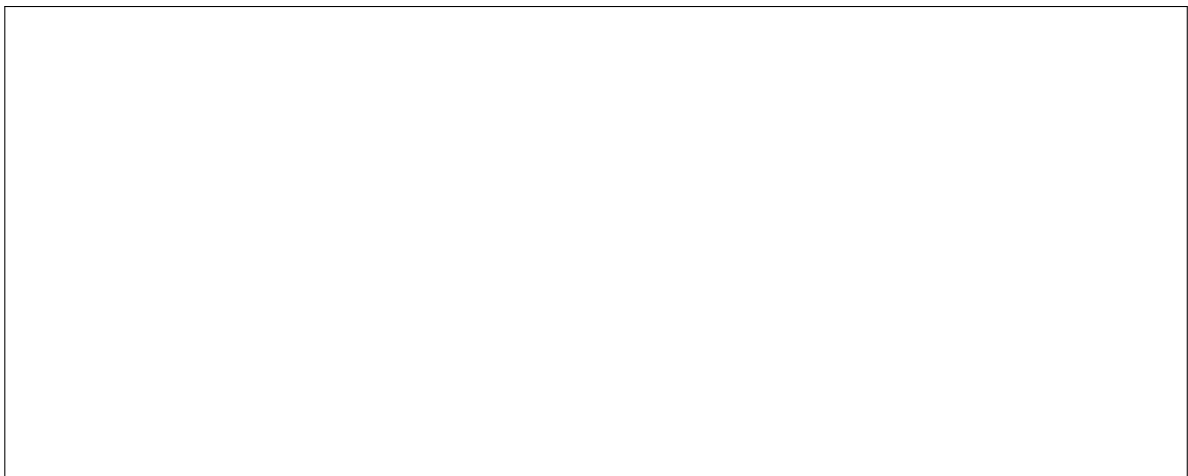
From a system of equations, we can get:

Coefficient Matrix

$$\begin{array}{rclcl} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

Augmented matrix

$$\begin{array}{rclcl} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array} \right]$$



Definition. An **elementary row operation** is one of the following

(Replacement) Add a multiple of one row to another row,

(Interchange) Interchange two rows, or

(Scaling) Multiply all entries in a row by a nonzero constant.

Definition. Two matrices are **row equivalent**, if one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Theorem 2. *If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.*

Example 6. Solve the following system (or show there is no solution):

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\2x_2 - 8x_3 &= 8 \\-4x_1 + 5x_2 + 9x_3 &= -9\end{aligned}$$

$$\begin{array}{lcl}x_1 - 2x_2 + x_3 = 0 & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \\2x_2 - 8x_3 = 8 \\-4x_1 + 5x_2 + 9x_3 = -9 \\ \hline x_1 - 2x_2 + x_3 = 0 & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right] \\2x_2 - 8x_3 = 8 \\-3x_2 + 13x_3 = -9 \\ \hline x_1 - 2x_2 + x_3 = 0 & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right] \\x_2 - 4x_3 = 4 \\-3x_2 + 13x_3 = -9 \\ \hline x_1 - 2x_2 + x_3 = 0 & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right] \\x_2 - 4x_3 = 4 \\x_3 = 3 \\ \hline x_1 - 2x_2 = -3 & \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right] \\x_2 = 16 \\x_3 = 3 \\ \hline x_1 = 29 & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right] \\x_2 = 16 \\x_3 = 3\end{array}$$

Solution: (29, 16, 3)

Check: Is (29, 16, 3) a solution of the *original* system?

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\2x_2 - 8x_3 &= 8 \\-4x_1 + 5x_2 + 9x_3 &= -9 \\ \\29 - 32 + 3 &= 0 \quad \checkmark \\32 - 24 &= 8 \quad \checkmark \\-116 + 80 + 27 &= -9 \quad \checkmark\end{aligned}$$

Example 9. For what values of h will the following system be consistent?

$$\begin{array}{rclcl} 3x_1 & - & 9x_2 & = & 4 \\ -2x_1 & + & 6x_2 & = & h \end{array}$$

Solution:

Reduce to triangular form.

$$\left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array} \right] \rightarrow \left[\begin{array}{cc|c} & & \end{array} \right] \rightarrow \left[\begin{array}{cc|c} & & \end{array} \right]$$

System is consistent if and only if h is