## Math 415 - Lecture 8

Inverses.

### Wednesday September 11th 2015

Textbook: Chapter 1.6

**Suggested Practice Exercise:** Chapter 1.6 Exercise 1, 2, 4, 6, 10, 11, 18, 35, 36, 37, 38, 40, 49, 50

Khan Academy Video: Inverse Matrix (part I), Inverse Matrix (part II)

### 1 Review

• Elementary matrices perform row operations:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ -2a+d & -2b+e & -2c+f \\ g & h & i \end{bmatrix}$$

• Gaussian elimination on A gives a decomposition A = LU:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

U is the echelon form, L records the reverse of the row operations we did.

• LU decomposition lets us solve  $A\mathbf{x} = \mathbf{b}$  quickly for many different  $\mathbf{b}$ .

## 1.1 Today's goal

• We know how to reverse a single row operation:

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

• Inverting a more complicated matrix is harder:

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & b & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab & -b & 1 \end{bmatrix}$$

Goal today: how to find an "inverse" to any (square!) matrix. Today A will be an  $n \times n$  matrix

## 2 The inverse of a matrix

The inverse of a real number a is denoted by  $a^{-1}$ . For example,  $7^{-1} = 1/7$  and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$
.

Remember that the identity matrix  $I_n$  is the  $n \times n$ -matrix

**Definition.** An  $n \times n$  matrix A is said to be invertible if there is an  $n \times n$  matrix C satisfying

where  $I_n$  is the  $n \times n$  identity matrix. We call C the inverse of A .

Example 1. We already know that an elementary matrix is invertible:

$$\begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In fact:

$$\begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Theorem 1.** Let A be an invertible matrix, then its inverse C is unique.

*Proof.* Assume B and C are both inverses of A. Then



- We will write  $A^{-1}$  for the inverse of A. Multiplying by  $A^{-1}$  is like "dividing by A."
- Do not write  $\frac{A}{B}$ . Why?



• Fact: if AB = I then  $A^{-1} = B$  and so BA = I. (Not so easy to show at this stage.)

**Remark.** Not all  $n \times n$  matrices are invertible. For example, the  $2 \times 2$  matrix



is not invertible. Try to find an inverse!



**Definition.** A matrix which is *not* invertible is sometimes called a singular matrix. An invertible matrix is also called nonsingular matrix.

**Theorem 2.** Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . If  $ad - bc \neq 0$ , then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If ad - bc = 0, then A is not invertible.

Proof. Calculate

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}$$

**Theorem 3.** If A is an invertible  $n \times n$  matrix, then for each **b** in  $\mathbb{R}^n$ , the equation  $A\mathbf{x} = \mathbf{b}$  has the unique solution  $\mathbf{x} = A^{-1}\mathbf{b}$ .

*Proof.* The vector  $A^{-1}\mathbf{b}$  is a solution, because

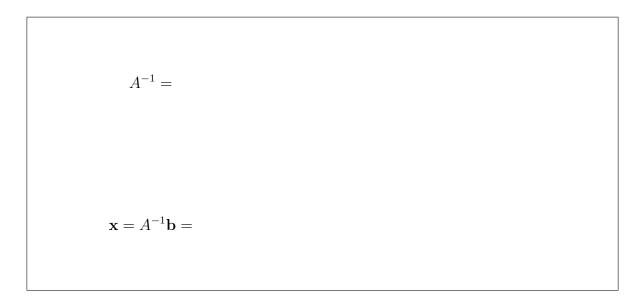


Suppose there is another solution  $\mathbf{w}$ , then

Example 2. Use the inverse of  $A=\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$  to solve  $-7x_1+3x_2=2$   $5x_1-2x_2=1$ 

**Solution.** Matrix form of the linear system:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

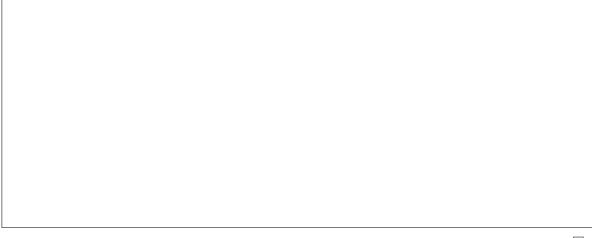


# 3 Calculational rules

**Theorem 4.** Suppose A and B are invertible. Then the following results hold:

- (a)  $A^{-1}$  is invertible and  $(A^{-1})^{-1} = A$  (i.e. A is the inverse of  $A^{-1}$ ).
- (b) AB is invertible and  $(AB)^{-1} = B^{-1}A^{-1}$
- (c)  $A^T$  is invertible and  $(A^T)^{-1} = (A^{-1})^T$

Proof.



# 4 An algorithm for computing the inverse matrix

Idea:

- To solve Ax = b we row reduce  $[A \mid b]$ .
- To solve  $AX = I_n$  we row reduce  $[A \mid I]$ .

**Theorem 5.** An  $n \times n$  matrix A is invertible if and only if A is row equivalent to  $I_n$ , and in this case, any sequence of elementary row operations that reduces A to  $I_n$  will also transform  $I_n$  to  $A^{-1}$ .

#### So here is the algorithm:

- Place A and I side-by-side to form an augmented matrix  $[A \mid I]$ . This is an  $n \times 2n$  matrix (Big Augmented Matrix), instead of  $n \times (n+1)$ , for the usual augmented matrix.
- Then perform row operations on this matrix (which will produce identical operations on A and I).
- So by Theorem 5:

$$[A \mid I]$$
 will row reduce to  $[I \mid A^{-1}]$ 

or A is not invertible.

Example 3. Find the inverse of  $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ , if it exists.

Solution:

$$[A \ I] = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

So

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0\\ 0 & 0 & 1\\ \frac{3}{2} & 1 & 0 \end{bmatrix}$$

Example 4 (Let's do the previous example step by step.).						

#### Remark. Why does this algorithm work?

• At each step, we get

$$[A \mid I] \sim [E_1A \mid E_1] \sim [E_2E_1A \mid E_2E_1] \sim \dots$$

• So each step is of the form

$$[FA \mid F], \quad F = E_r \dots E_3 E_2 E_1$$

ullet If we succeed in row reducing A to I, the final step is

$$[FA \mid F] = [I \mid F]$$

• So FA = I, which means that  $A^{-1} = F$ .