Math 415 - Lecture 18

Inner Product and Orthogonality

Wednesday October 7th 2015

Textbook reading: Ch 3.1

Suggested practice exercises: 1, 2, 4, 5, 14, 16

Khan Academy video: Vector Dot Product and Vector Length

Strang lectures: Lecture 30: Linear Transformations / Lecture 14: Orthogonality

Applications: Information retrieval

1 Review

- A **linear map** $T : V \to W$ satisfies

$$T(c\mathbf{x} + d\mathbf{y}) = cT(\mathbf{x}) + dT(\mathbf{y}).$$

- $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is linear. A is the matrix representing T in the standard bases. For example, $T(\mathbf{e_1}) = A\mathbf{e_1}$ = first column of A.

$$\left(\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)$$

- Any $T \colon V \to W$ can be represented by a matrix.

What is the Point? Why write $T \colon V \to W$ as a matrix?

- Replace obscure computations in V and W by transparent computations with matrices.
- Even if $T : \mathbb{R}^n \to \mathbb{R}^m$ (already have standard coordinates), T may be simpler in a different coordinate system.

Summary: Given $\mathbf{v}$ in V, want to calculate $T(\mathbf{v})$ in W. Take an input basis $\mathcal{A} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ and and output basis $\mathcal{B} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_m})$.

- The abstract input vector $\mathbf{v}$ and the coordinate vector $\mathbf{v}_{\mathcal{A}}$ determine each other.
- The abstract output vector $T(\mathbf{v})$ and the coordinate vector $T(\mathbf{v})_{\mathcal{B}}$ determine each other.
- So we know T if we know the matrix $T_{\mathcal{BA}}$:

$$T(\mathbf{v})_{\mathcal{B}} = T_{\mathcal{BA}}\mathbf{v}_{\mathcal{A}}.$$

The output coordinate vector equals the matrix for T times the input coordinate vector.

Formula For the Coordinate matrix. To write $T\colon V \to W$ as a matrix, take an input basis $\mathcal{A} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ and and output basis $\mathcal{B} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_m})$. Then

$$T_{\mathcal{BA}} = \begin{bmatrix} T(\mathbf{x_1})_{\mathcal{B}} & T(\mathbf{x_2})_{\mathcal{B}} & \ldots & T(\mathbf{x_n})_{\mathcal{B}} \end{bmatrix}$$

Example 1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be reflection across the x-y plane, $(x, y, z) \mapsto (x, y, -z)$. Determine the matrix representing T in the basis $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$.

$T\colon (x, y, z) \mapsto (x, y, -z)$. So calculate

$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} - 2\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = (-1)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Hence

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix}.$$

Example 2. Let $T : \mathbb{P}_3 \to \mathbb{P}_2$ be the linear map given by

$$T(p(t)) = \frac{d}{dt}p(t).$$

What's the matrix A representing T in the standard bases?

Solution. The standard bases are $\{1, t, t^2, t^3\}$ for $\mathbb{P}_2$ and $\{1, t, t^2\}$ for $\mathbb{P}_2$. The matrix A has 4 columns and 3 rows.

- $T(1) = 0$, so the first column is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

- $T(t) = 1$, so the second column is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

- $T(t^2) = 2t$, so the third column is $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

- $T(t^3) = 3t^2$, so the last (fourth) column is $\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$.

So the matrix A representing T in the standard bases is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

What is $Col(A)$ and $Nul(A)$ for $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$?

Solution. $Col(A) = \mathbb{R}^3$. Every quadratic polynomial is the derivative of some cubic polynomial.

$$Nul(A) = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The corresponding polynomial is $p(t) = 1$. That makes sense because differentiation kills constant polynomials.

Let's try differentiating $7t^3 - t + 3$ using the matrix A.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 21 \end{bmatrix}$$

We get $-1 + 0t + 21t^2$, which is indeed the derivative of $7t^3 - t + 3$.

2 Inner Product and Distances

Definition. The **inner product** (or **dot product**) of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ is

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = v_1 w_1 + \cdots + v_n w_n.$$

Example 3.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = 1 - 2 - 6 = -7$$

Theorem 1. *Let* $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$*, and let* c *be any scalar. Then*

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

(d) $\mathbf{u} \cdot \mathbf{u} \geq \mathbf{0}$*, and* $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ *if and only if* $\mathbf{u} = \mathbf{0}$.

Definition.

- The **norm** (or **length**) of a vector $\mathbf{v} \in \mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

- The **distance** between points $v, w \in \mathbb{R}^n$ is

$$dist\,(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$

Example 4. (a)

$$\left\| \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11}$$

(b)

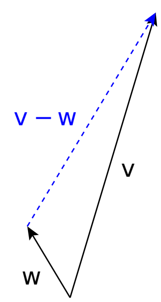
$$dist\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) = \| \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix} \| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

3 Inner Product and Angles

We can use the dot product to compute angles too.

Theorem 2. *If* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *are linearly independent, they form an angle* θ*, and*

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta$$



Example 5. What is the angle formed in $\mathbb{R}^3$ between the vectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}?$$

(A base jumper runs at a cliff at a 45° angle, then jumps straight away from the cliff and 45° downwards; what angle does he turn as he jumps?)

Solution.

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$\|v\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

$$\|w\| = \sqrt{0^2 + (-1)^2 + (-1)^2} = \sqrt{2}$$

$$v \cdot w = -1$$

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$$

$$-1 = 2\cos\theta \quad \Rightarrow \quad \cos\theta = -\frac{1}{2}$$

$$\theta = 120^\circ$$

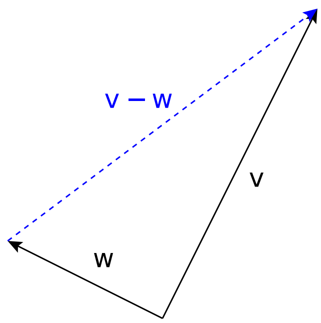
4 Orthogonal vectors

Definition. $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ are **orthogonal** if

$$\mathbf{v} \cdot \mathbf{w} = 0.$$

Remark. We write $\mathbf{v} \perp \mathbf{w}$ when $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. Nonzero vectors $\mathbf{v}$, $\mathbf{w}$ are orthogonal if and only if they are perpendicular.

Nonzero vectors $\mathbf{v}$, $\mathbf{w}$ are orthogonal if and only if they are perpendicular. We can derive this from Pythagoras' theorem. $\mathbf{v} \perp \mathbf{w} \iff \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 = \|\mathbf{v} - \mathbf{w}\|^2 \iff \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} = (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) \iff \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{v} - 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} \iff \mathbf{v} \cdot \mathbf{w} = 0$



Example 6. Are the following vectors orthogonal?

(a) $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}$

$$\begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}-2\\1\end{bmatrix} = 1 \cdot (-2) + 2 \cdot 1 = 0$$

So yes, they're orthogonal.

(b) $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\1\end{bmatrix}$

$$\begin{bmatrix}1\\2\\1\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\1\end{bmatrix} = 1 \cdot (-2) + 2 \cdot 1 + 1 \cdot 1 = 1$$

So no, they're not orthogonal.

Example 7. Let $\mathbf{v} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$. Is the set of vectors orthogonal to $\mathbf{v}$ a subspace of $\mathbb{R}^3$?

$$V = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{v} \cdot \mathbf{x} = 0\}$$

Solution.

$$\begin{aligned} &\mathbf{v} \cdot \mathbf{x} = 0 \\ \Leftrightarrow &\begin{bmatrix}2 & -1 & 1\end{bmatrix} \mathbf{x} = 0 \end{aligned}$$

V is just the null space of the matrix $\mathbf{v}^T = \begin{bmatrix}2 & -1 & 1\end{bmatrix}$. So yes, it is a subspace.

Definition. If V is a subspace of $\mathbb{R}^n$, a vector $\mathbf{x}$ is **orthogonal to** V if it is orthogonal to every vector in V.

Example 8. Let $V = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$. Is $\mathbf{x} = \begin{bmatrix}-1\\1\end{bmatrix}$ orthogonal to V?

Solution.

$$\begin{bmatrix}-1\\1\end{bmatrix} \cdot \begin{bmatrix}a\\a\end{bmatrix} = -a + a = 0$$

So yes.