Math 415 - Lecture 17

Linear Transformations

Monday October 5th 2015

Textbook reading: Chapter 2.6

Suggested practice exercises: same as lecture 16

1 Review

- A map $T : V \to W$ between vector spaces is **linear** if
  - $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
  - $T(c\mathbf{x}) = cT(\mathbf{x})$
- If $\mathbf{x_1}, \dots, \mathbf{x_n}$ is a basis for V, then T is determined by the values $T(\mathbf{x_1}), \dots, T(\mathbf{x_n})$:
$$T(\mathbf{v}) = T(c_1\mathbf{x_1} + \cdots + c_n\mathbf{x_n}) = c_1T(\mathbf{x_1}) + \cdots + c_nT(\mathbf{x_n}).$$
- Let A be an $m \times n$ matrix.
  - $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is linear.
  - *Every* linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is of the form $T(\mathbf{x}) = A\mathbf{x}$.

2 Nonstandard Bases

Until now we have used the standard bases to describe $T \colon \mathbb{R}^n \to \mathbb{R}^m$. Often it is useful to use other bases.

Theorem 1 (Linear Transformation is Matrix Multiplication)**.** *Let $T \colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\mathcal{B} := (\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ be a basis of $\mathbb{R}^n$ and let $\mathcal{C} := (\boldsymbol{w}_1, \dots, \boldsymbol{w}_m)$ be a basis of $\mathbb{R}^m$. Then there is a matrix B such that*
$$T(\mathbf{x})_{\mathcal{C}} = B\mathbf{x}_{\mathcal{B}}, \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n.$$
Explicitly,
$$B = \begin{bmatrix} T(\boldsymbol{v}_1)_{\mathcal{C}} & \dots & T(\boldsymbol{v}_n)_{\mathcal{C}}. \end{bmatrix},$$

Example 1. Let $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a + 1b \\ 1a + 3b \end{bmatrix}$. Then the matrix of T is $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. But let us use, instead of the standard basis, another basis adapted to T. Put

$$\mathbf{b_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{b_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

What is the coordinate matrix for T with respect to $\mathcal{B} = (\mathbf{b_1}, \mathbf{b_2})$?

Solution.

3 Matrices for... Polynomials?

Let P_n be the vector space of polynomials of degree at most n.
Example 2. Consider the map $T : P_2 \to P_1$ given by

$$T(p(t)) = \frac{d}{dt}p(t).$$

Describe T by a matrix.

Solution.

4 Matrices for Linear Transformations

Let $T : V \to W$ be a linear transformation, $\mathcal{A} = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$ be an *input basis* for V, and $\mathcal{B} = \{\mathbf{y_1}, \ldots, \mathbf{y_m}\}$ an *output basis* for W. Each vector in V has a coordinate vector in $\mathbb{R}^n$, each vector in W has a coordinate vector in $\mathbb{R}^m$. T now corresponds to a matrix from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$\begin{tikzcd}
\text{vector in } V \arrow[r, "\text{apply } T"] \arrow[d, "\text{write in coordinates}"'] & \text{vector in } W \arrow[d, "\text{write in coordinates}"] \\
\text{coordinate vector in } \mathbb{R}^n \arrow[r, "\text{multiply by } A"] & \text{coordinate vector in } \mathbb{R}^m
\end{tikzcd}$$

In the last example this was

Definition. Let $\mathcal{A} = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$ be a basis for V, and $\mathcal{B} = \{\mathbf{y_1}, \ldots, \mathbf{y_m}\}$ a basis for W. The matrix $T_{\mathcal{BA}}$ representing T with respect to these bases

- has n columns (one for each of the $\mathbf{x_j}$),
- the j-th column is the coordinate vector of $T(\mathbf{x_j})$ in the basis $\mathcal{B}$.

$$T_{\mathcal{BA}} = \begin{bmatrix} T(\mathbf{x_1})_{\mathcal{B}} & T(\mathbf{x_2})_{\mathcal{B}} & \ldots & T(\mathbf{x_n})_{\mathcal{B}} \end{bmatrix}$$

Example 3. Give the matrix for $T : P_2 \to P_1$ given by

$$T(p(t)) = \frac{d}{dt} p(t).$$

in the bases $\mathcal{A} = (1, t, t^2)$ and $\mathcal{B} = (1, t)$.

Solution.

Example 4. Recall the map T given by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$. (It reflects every vector in $\mathbb{R}^2$ across the line $y = x$.)

(a) Which matrix A represents T with respect to the standard bases?

(b) Which matrix B represents T with respect to the basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$?

Solution.

Remark. If a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is represented by the matrix A with respect to the standard bases, then $T(\mathbf{x}) = A\mathbf{x}$. Matrix multiplication corresponds to function composition! That is, if T_1, T_2 are represented by A_1, A_2, then $T_1(T_2(\mathbf{x})) = (A_1 A_2)\mathbf{x}$.

Example 5. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\0\\7\end{bmatrix}.$$

What is the matrix B representing T with respect to the following bases?

$$\mathbf{x_1} = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{x_2} = \begin{bmatrix}-1\\2\end{bmatrix} \text{ for } \mathbb{R}^2, \ \mathbf{y_2} = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{y_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{y_3} = \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ for } \mathbb{R}^3.$$

Solution.

Remark. A matrix representing T encodes in column j the coefficients of $T(\mathbf{x_j})$ expressed as a linear combination of $\mathbf{y_1}, \ldots, \mathbf{y_m}$.

5 Recap

What is the Point? Why write $T\colon V \to W$ as a matrix?

- Replace obscure computations in V and W by transparent computations with matrices.
- Even if $T : \mathbb{R}^n \to \mathbb{R}^m$ (already have standard coordinates), T may be simpler in a different coordinate system.

Summary: Given $\mathbf{v}$ in V, want to calculate $T(\mathbf{v})$ in W. Take an input basis $\mathcal{A} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ and and output basis $\mathcal{B} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_m})$.

- We know $\mathbf{v}$ if we know the coordinate vector $\mathbf{v}_{\mathcal{A}}$.
- We know $T(\mathbf{v})$ if we know the coordinate vector $T(\mathbf{v})_{\mathcal{B}}$.
- So we know T if we know the matrix $T_{\mathcal{B}\mathcal{A}}$:

$$T(\mathbf{v})_{\mathcal{B}} = T_{\mathcal{B}\mathcal{A}} \mathbf{v}_{\mathcal{A}}.$$

The output coordinate vector equals the matrix for T times the input coordinate vector.

Example 6. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let T be the linear map such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\0\\7\end{bmatrix}.$$

What is the matrix A representing T with respect to the standard bases? Use that to calculate $T\begin{bmatrix}2\\3\end{bmatrix}$.

Solution.

6 Additional Problems

- Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 7 & 8 & 1 \end{bmatrix}$. Find the dimensions and a basis for all four fundamental subspaces of A.

- Suppose A is 5×5 and v is a vector in $\mathbb{R}^5$ which is not a linear combination of the columns of A. What can you say about the number of solutions to $A\mathbf{x} = \mathbf{0}$?

- Let T be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

What is $T\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right)$?