Math 415 - Lecture 1

Introduction

Monday August 24 2015

- Textbook: Chapter 1.3, Chapter 2.2 (just the pages 78 and 79)

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- Suggested Practice Exercise: in Chapter 1.3, Exercise 1,3, 5, 6, 11

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- Khan Academy Video: Matrices: Reduced Row Echelon Form 1

Systems of Linear Equations

Definition

A **linear equation** is a equation of the form

$$a_1 x_1 + \ldots + a_n x_n = b$$

where $a_1, ..., a_n, b$ are numbers and $x_1, ..., x_n$ are variables.

Example

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\end{array}
$$

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$4x_1 - 5x_2 + 2 = x_1$	$3x_1 - 5x_2 = -2$	*Linear.*
$x_2 = 2(\sqrt{6} - x_1) + x_3$	$2x_1 + x_2 - x_3 = 2\sqrt{6}$	*Linear.*
$4x_1 - 6x_2 = x_1 x_2$	$4x_1 - 6x_2 = \underline{x_1 x_2}$	*Not linear.*
$x_2 = 2\sqrt{x_1} - 7$		

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This course will focus on linear equations.

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A **solution** of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

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Definition

The **solution set** of a system of linear equations is the set of all possible solutions of a linear system.

Example

Two equations in two variables:

$$\begin{aligned} x_1 + x_2 &= 1 \\ -x_1 + x_2 &= 0. \end{aligned}$$

What is a solution for this system of linear equations?

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Plug into first equation. $x_1 + .5 = 1 \quad \Rightarrow \quad x_1 = .5$

$(x_1, x_2) = (.5, .5)$ is the only solution.

Example

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Subtract from second equation. $0 = 14$

The equation $0 = 14$ is always false, so no solutions exist.

Example

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Any value of x_1 works. $x_2 = 3 - x_1$. Infinitely many solutions.

Theorem

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Can you draw the set of solutions of the above equations?

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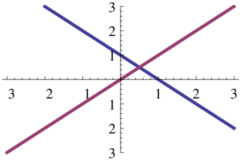

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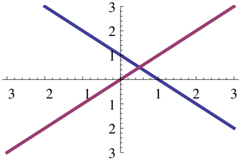

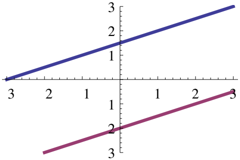

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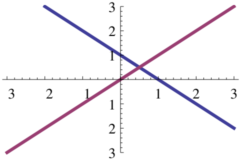

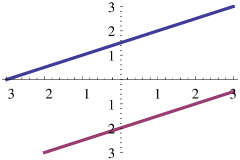

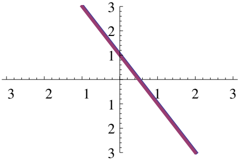

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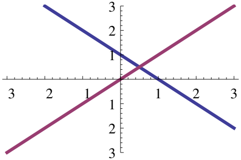

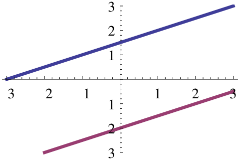

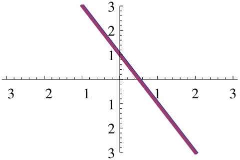

(The numbers in the graphs are not quite right.)

Take away: Whenever you have a linear system with n equations, then the set of solutions of this system is precisely the intersection of the sets of solutions of each of the n equations on its own.

Strategies for solving systems of linear equations

Definition

Two systems are **equivalent** if they have the same solution set.

The general strategy is to replace one system with an equivalent system that is easier to solve.

Example

Consider

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array}$$

The general strategy is to replace one system with an equivalent system that is easier to solve.

Example

Consider

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array}$$

$$R2 \to R2 + R1 \quad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ 0 & + & x_2 & = & 2 \end{array}$$

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$x_2 = 2$, so $x_1 = 3$.

Matrix Notation

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From a system of equations, we can get:

Coefficient Matrix

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From a system of equations, we can get:

Coefficient Matrix

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

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Solution: $x_1 = 3, \ x_2 = 2$

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Theorem

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Example

Solve the following system (or show there is no solution):

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

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$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 &= 29 \\ {}_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Solution: $(29, 16, 3)$

Check: Is $(29, 16, 3)$ a solution of the **original** system?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

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$$\begin{array}{rcrcrcrl} 29 & - & 32 & + & 3 & = & 0 & \checkmark \\ & & 32 & - & 24 & = & 8 & \checkmark \\ -116 & + & 80 & + & 27 & = & -9 & \checkmark \end{array}$$

Two Fundamental Questions (Existence and Uniqueness)

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(1) Is the system consistent? (I.e. does a solution **exist**?)

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(1) Is the system consistent? (I.e. does a solution **exist**?)

(2) If a solution exists, is it **unique**? (I.e. is there one only one solution?)

Example

Is this system consistent? If so, is the solution unique?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

In the last example, this system was reduced to the triangular form:

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This is sufficient to see that the system is consistent and unique. Why?

In the last example, this system was reduced to the triangular form:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

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- The last row determines x_3 uniquely.
- Knowing x_3, the second row determines x_2 uniquely.
- Knowing x_2 and x_3, the first row determines x_1 uniquely.
- So, exactly one possible solution (x_1, x_2, x_3).

Example

Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \left[\begin{array}{rrr|r} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right]$$

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Equation notation of triangular form:

$$\begin{array}{rrrcr} x_1 & -2x_2 & +3x_3 & = & -1 \\ & 3x_2 & -6x_3 & = & 8 \\ & & 0 & = & -3 \end{array}$$

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The original system is inconsistent!

Example

For what values of h will the following system be consistent?

$$\begin{array}{rcrcr} 3x_1 & - & 9x_2 & = & 4 \\ -2x_1 & + & 6x_2 & = & h \end{array}$$

Solution:

$$\left[\begin{array}{rr|r} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right]$$

Solution:

$$\left[\begin{array}{rr|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \underset{R1\to\frac{1}{3}R1}{\longrightarrow} \left[\begin{array}{rr|c} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array}\right]$$

Solution:

$$\left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \xrightarrow[R1 \to \frac{1}{3}R1]{} \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array}\right] \xrightarrow[R2 \to R2 + 2R1]{} \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h + \frac{8}{3} \end{array}\right]$$

Solution:

$$\begin{aligned} & \left[\begin{array}{rr|r} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \\ \underset{R1\to\frac{1}{3}R1}{\longrightarrow} & \left[\begin{array}{rr|r} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array}\right] \\ \underset{R2\to R2+2R1}{\longrightarrow} & \left[\begin{array}{rr|c} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h+\frac{8}{3} \end{array}\right] \end{aligned}$$

System is consistent if and only if $h = -\frac{8}{3}$.