Math 415 - Lecture 29

Determinants

Wednesday November 4th 2015

Textbook reading: Chapters 4.2, 4.3

Suggested practice exercises: Chapter 4.2, # 1, 2, 4, 5, 10, 14, 15, 17, 18, 19, 20, 22, 23

Khan Academy video: 3×3 Determinant, $n \times n$ Determinant, Determinants along other rows/ columns,

Strang lecture: Lecture 18: Properties of determinants, Lecture 19: Determinant formulas and cofactors

1 Determinants

For the next few lectures, all matrices are square! Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The determinant of

- a 2×2 matrix is $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$,
- a 1×1 matrix is $\det([a]) = a$.
- What is the determinant of an $n \times n$ matrix?

Goal (Point of the determinant)
A is invertible $\iff \det(A) \neq 0$

Notation: We will write both $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ for the determinant.

We want to define for every $n \times n$ matrix A a number $\det(A)$.

Definition. The determinant is characterized by:

- the normalization $\det I_{n\times n} = 1$,
- and how it is affected by elementary row operations:
  - **(Replacement)** Add a multiple of one row to another row. Does not change the determinant.
  - **(Interchange)** Interchange two rows. Reverses the sign of the determinant.
  - **(Scaling)** Multiply all entries in a row by s. Multiplies the determinant by s.

This allows us to compute the determinant using just row operations!

Important Fact
The determinant of a triangular matrix is the product of the diagonal entries.

Example

$$\det \begin{bmatrix} 2 & 3 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 6 \end{bmatrix} = 2 \cdot 4 \cdot 6.$$

Why? Take out the diagonal entries, and then use row operations to get the identity matrix.

Example 1 (Generic matrix). Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

Solution.

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} &\overset{\substack{R2\to R2-3R1 \\ R3\to R3-2R1}}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & -4 & 1 \end{vmatrix} \\ &\overset{R3\to R3-\frac{4}{7}R2}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & -\frac{1}{7} \end{vmatrix} \\ &= 1 \cdot (-7) \cdot (-\frac{1}{7}) = 1 \end{aligned}$$

Example 2 (Reality check). Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solution.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{R2\to R2-\frac{c}{a}R1}{=} \begin{vmatrix} a & b \\ 0 & d-\frac{c}{a}b \end{vmatrix} = a\left(d - \frac{c}{a}b\right) = ad - bc$$

NB: this only works if $a \neq 0$. What do you do if $a = 0$?

Example 3 (Larger matrix). Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

Solution. The matrix looks complicated, but we can do this in two steps!

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix} \overset{R4 \to R4 - \frac{3}{2}R3}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & \frac{7}{2} \end{vmatrix} = 1 \cdot 2 \cdot 2 \cdot \frac{7}{2} = 14$$

The following important properties follow from behavior under row operations.

Important properties

- $\det(A) = 0 \iff A$ is not invertible. Why? Because $\det(A) = 0$, if and only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).
- $\det(AB) = \det(A)\det(B)$ **Challenge**: Figure out why! (Matrix multiplication can be seen as linear combinations of rows)
- $\det(A^{-1}) = \frac{1}{\det(A)}$ Why? Because $AA^{-1} = I$. Since $\det(AA^{-1}) = \det(I) = 1$ and $\det(AA^{-1}) = \det(A)\det(A^{-1})$, we have $\det(A)\det(A^{-1}) = 1$.
- $\det(A^T) = \det(A)$. (Think about why this works at home.)

Remark. $\det(A^T) = \det(A)$ means that everything you know about determinants in terms of *rows* of A is also true for the columns.For instance:

- If you exchange two *columns* in a determinant you get a minus sign.
- You can add a multiple of a *column* to another column without changing the determinant.
- If your matrix has equal *columns* the determinant is zero.
- If your matrix has a zero *column* the determinant is zero.

Example 4. Recall that $AB = \mathbf{0}$, then it does not follow that $A = \mathbf{0}$ or $B = \mathbf{0}$. However, show that $\det(A) = 0$ or $\det(B) = 0$.

Solution. If $AB = \mathbf{0}$, then $\det(AB) = \det(\mathbf{0}) = 0$. Follows from $\det(AB) = \det(A)\det(B)$.

2 A "bad" way to compute determinants, Cofactor expansion

Idea. The determinant is linear in each row or column.

(In some texts this is one of the basic assumptions about det.)

Fact 5.

$$\det\begin{bmatrix} a & b & c \\ * & * & * \\ * & * & * \end{bmatrix} = \det\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} + \det\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & c \\ * & * & * \\ * & * & * \end{bmatrix}$$

We can use this idea to calculate an $n \times n$ determinant in terms of n determinants of smaller matrices.

Example 6. What is the determinant $\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$? What about $\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$?

Solution.

$$\det\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a\det\begin{bmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a\det\begin{bmatrix} 1 & 0 \\ * & B \end{bmatrix} = a\det\begin{bmatrix} B \end{bmatrix},$$

where B is the 2×2 right lower block. Same way, with a twist:

$$\det\begin{bmatrix} 0 & b & 0 \\ v_1 & v_2 & v_3 \end{bmatrix} = \boxed{-1}b\det\left(\begin{bmatrix} 1 & 0 & 0 \\ v_2 & v_1 & v_3 \end{bmatrix}\right) = -b\det\begin{bmatrix} v_1 & v_3 \end{bmatrix}.$$

We can use this idea to calculate an $n \times n$ determinant in terms of n determinants of $(n-1) \times (n-1)$ matrices. Then repeat ....

Example 7. Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution. We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix}$$

$$\overset{\text{i.e.}}{=} 1 \cdot \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 1 \times (-1) - 2 \cdot (-1) + 0 = 1$$

Each term in the cofactor expansion is ± 1 times an entry times a smaller determinant (row and column of entry deleted). The $\pm$ is assigned to each entry according to

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}.$$

There is nothing special about the first row. We can use any other row or column.

For example, let's use the second column:

Solution.

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & & 0 \\ & + & \\ 2 & & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & & 0 \\ 3 & & 2 \\ & - & \end{vmatrix}$$

$$= -2 \cdot (-1) + (-1) \cdot 1 - 0 = 1$$

Same answer!

Let use the third column:

Solution.

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 & \\ & & - \\ 2 & 0 & \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 & \\ 3 & -1 & \\ & & + \end{vmatrix}$$

$$= 0 - 2 \cdot (-4) + 1 \cdot (-7) = 1$$

Same answer!

Why not cofactor expansion

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)? Because to compute a large $n \times n$ matrix,

- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,
- and so on.

In the end, we have $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ many numbers to add.WAY TOO MUCH WORK! Already

$$25! = 15511210043330985984000000 \approx 1.55 \cdot 10^{25}.$$

Context: today's fastest computer, Tianhe-2, runs at 34 pflops ($3.4 \cdot 10^{16}$ operations per second).By the way: "fastest" is measured by computing LU decompositions!

3 Practice Problems

3.1

Example 8. Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$. Use your favorite method (or a mix of methods!)

Solution. The final answer should be -10.

- What's **wrong**?!

$$\det(A^{-1}) = \det \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}(da - (-b)(-c) = 1$$

The correct calculation is:

$$\det \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(ad-bc)^2} \det \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}$$

Example 9. Suppose A is a 3×3 matrix with $\det(A) = 5$. What is $\det(2A)$?

Solution. A has three rows. Multiplying all 3 of them produces $2A$. Hence, $\det(2A) = 2^3 \det(A) = 40$.

Imaginary unit and Fibonacci numbers

Example 10. First off, say hello to our new friend: i, the **imaginary unit**. It is infamous for $\boxed{i^2 = -1}$. Let us calculate some determinants.

$$\begin{aligned}
|1| &= 1 \\
\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} &= 1 - i^2 = 2 \\
\begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} &= 1\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} - i\begin{vmatrix} i & 0 \\ i & 1 \end{vmatrix} = 2 - i^2 = 3 \\
\begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} &= 1\begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} - i\begin{vmatrix} i & 0 & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 3 - i^2\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} = 5
\end{aligned}$$

Example 11 (continued).

$$\begin{vmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & i & \\ & & i & 1 & i \\ & & & i & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} - i^2\begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 5 + 3 = 8$$

The Fibonacci numbers!

Do you know about the connection of Fibonacci numbers and rabbits? If

not, Google is your friend.