Math 415 - Lecture 22

Orthogonal projection

Friday October 16th 2015

Textbook reading: Chapter 3.2.

Suggested practice exercises: Chapter 3.2: 9, 10, 17, 19.

Strang lecture: Lecture 15: Projections onto Subspaces

1 Review/Outlook

- We can deal with complicated linear systems $Ax = b$ (maybe with help of a computer), but what to do when there is no exact solution?
- $Ax = b$ had no solution if b is not in $Col(A)$.
- Idea: make $Ax - b$ as small as possible (when we vary x).
- How? *Project* b on the column space $Col(A)$.
- Recall: If $v_1, v_2, \dots, v_n$ are orthogonal (and non zero) they are independent.
- Recall: To calculate coordinates for orthogonal vectors is easy: this uses

$$v_1 \cdot (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 v_1 \cdot v_1.$$

2 Orthogonal Bases

2.1 Orthogonal Basis

Definition 1. A basis $v_1, v_2, \dots, v_n$ of $\mathbb{R}^n$ is called *orthogonal* if the vectors are pairwise orthogonal, $v_i \cdot v_j = 0$ if $i \neq j$.

Example 2. The standard basis $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthogonal basis for $\mathbb{R}^3$. Similarly, the standard basis $e_1, e_2, \dots, e_n$ is an orthogonal basis for $\mathbb{R}^n$.

Example 3. Are the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ an orthogonal basis for $\mathbb{R}^3$?

Solution. Just check:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = ??, \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = ??, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = ??$$

So this *is* an orthogonal basis. Note that we don't have to check it is a basis: orthogonality implies independence, and 3 independent vectors form a basis in $\mathbb{R}^3$.

Example 4. Suppose $v_1, v_2, \ldots, v_n$ form an orthogonal basis of $\mathbb{R}^n$, $w \in \mathbb{R}^n$. Find the coordinates of w. That is, find constants $c_1, c_2, \ldots, c_n$ so that

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

Solution. Take dot product with v_1 on both sides:

$$v_1 \cdot w = v_1 \cdot (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 v_1 \cdot v_1.$$

Hence $c_1 = \frac{v_1 \cdot w}{v_1 \cdot v_1}$ and more generally $c_i = \frac{v_i \cdot w}{v_i \cdot v_i}$.

Easy (and Important) Formula
If $v_1, v_2, \ldots, v_p$ form an orthogonal basis of $V \subset \mathbb{R}^n$, $w \in V$, then $w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$, with

$$c_i = \frac{v_i \cdot w}{v_i \cdot v_i}.$$

Special Case
If $v_1, v_2, \ldots, v_p$ is orthonormal then

$$c_i = v_i \cdot w.$$

Example 5. Express $w = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in the basis $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =$$

We use the formula for the coordinates:

$$c_1 = ?? c_2 = ?? c_3 = ??,$$

Warning
The easy formula for the coordinates only works for *orthogonal* bases.

Example 6. Take the basis $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}$ and the vector $w = \begin{bmatrix}4\\9\end{bmatrix}$. Then

$$\begin{bmatrix}4\\9\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\3\end{bmatrix},$$

and the coefficients are *not* the numbers you get from the easy formula. To find them you need to solve a system of equations.

Example 7. The standard basis $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ is orthonormal. Find the coordinates of $\begin{bmatrix}3\\7\\4\end{bmatrix}$ in the standard basis.

Solution. This is trivial of course,

$$\begin{bmatrix}3\\7\\4\end{bmatrix} = ?? \qquad \begin{bmatrix}1\\0\\0\end{bmatrix} + ?? \qquad \begin{bmatrix}0\\1\\0\end{bmatrix} + ?? \qquad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

But ....

Solution (continued)**.** note that the coordinates are dot products with orthonormal vectors:

$$\begin{bmatrix}3\\7\\4\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} = ?? \qquad , \quad \begin{bmatrix}3\\7\\4\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} = \qquad , \quad \begin{bmatrix}3\\7\\4\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} = ?? \qquad .$$

Example 8. The vectors $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ form an orthogonal basis. Produce from it an *orthonormal* basis.

Solution. We just divide by the lengths of these vectors (this will keep them orthogonal).

$$\begin{bmatrix}1\\-1\\0\end{bmatrix} \text{ has length } ?? \qquad = ?? \qquad , \text{ normalized: } ?? \qquad \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

Solution (continued)**.**

$$\begin{bmatrix}1\\1\\0\end{bmatrix} \text{ has length } ?? \qquad = ?? \qquad , \text{ normalized: } ?? \qquad \begin{bmatrix}1\\1\\0\end{bmatrix}$$

and

$$\begin{bmatrix}0\\0\\1\end{bmatrix}$$

is already normalized. So we get as orthonormal basis

$$?? \quad \begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad ?? \quad \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad ?? \quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Example 9. Express $\begin{bmatrix}3\\7\\4\end{bmatrix}$ in the orthonormal basis $\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$.

Solution. Just calculate dot products:

$$c_1 = \quad = ?? \qquad , \quad c_2 = ?? \qquad = ?? \qquad ,$$

Solution (continued)**.**

$$c_3 = ?? \qquad = ??$$

so that

$$\begin{bmatrix}3\\7\\4\end{bmatrix} = ?? \qquad \begin{bmatrix}1\\-1\\0\end{bmatrix} + ?? \qquad \begin{bmatrix}1\\1\\0\end{bmatrix} + ?? \qquad \begin{bmatrix}0\\0\\1\end{bmatrix} =$$

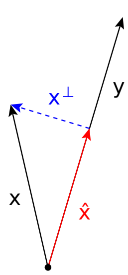
$$??$$

3 Orthogonal Projection

Definition 10 (Orthogonal Projection)**.** The **orthogonal projection** of vector $\mathbf{x}$ on vector $\mathbf{y}$ is

$$\hat{\mathbf{x}} = \frac{\mathbf{y}\cdot\mathbf{x}}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}.$$

The **error** is $\mathbf{x}^{\perp} = \mathbf{x} - \hat{\mathbf{x}}$.



- The projection $\hat{\mathbf{x}}$ is the *closest point* to $\mathbf{x}$ on the line through $\mathbf{y}$.
- The error $\mathbf{x}^{\perp} = \mathbf{x} - \hat{\mathbf{x}}$ is characterized by the property that it is orthogonal to $Span(\mathbf{y})$.
- We have a decomposition $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{x}^{\perp}$. The **projection** $\hat{\mathbf{x}}$ is in $Span(\mathbf{y})$ and $\mathbf{x}^{\perp}$ is orthogonal to $Span(\mathbf{y})$.

Summary: the projection formula is

$$\hat{\mathbf{x}} = \frac{\mathbf{y}\cdot\mathbf{x}}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}.$$

Why?

Solution.

- We know $\hat{\mathbf{x}}$ is in the direction of $\mathbf{y}$, so $\hat{\mathbf{x}} = c\mathbf{y}$ for some constant c.
- The error $\mathbf{x} - \hat{\mathbf{x}}$ orthogonal to $\mathbf{y}$.
- So $0 = \mathbf{y} \cdot (\mathbf{x} - \hat{\mathbf{x}}) = \mathbf{y} \cdot (\mathbf{x} - c\mathbf{y}) = \mathbf{y} \cdot \mathbf{x} - c\mathbf{y} \cdot \mathbf{y}$.
- Solving for c gives $c = \frac{\mathbf{y}\cdot\mathbf{x}}{\mathbf{y}\cdot\mathbf{y}}$.

Example 11. Find the orthogonal projection of $\mathbf{x} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ onto $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Solution.

$$\hat{\mathbf{x}} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}\mathbf{y} = \frac{-8.3 + 4.1}{3^2 + 1^2} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}.$$

The error is

$$\mathbf{x}^{\perp} = \mathbf{x} - \hat{\mathbf{x}} = \begin{bmatrix} -8 \\ 4 \end{bmatrix} - \begin{bmatrix} -6 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

Note that vector $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and error $\mathbf{x}^{\perp} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ are orthogonal.

Example 12. What is the projection of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ onto each of the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

Solution.

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ onto } \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} : \frac{2.1 + 1.(-1) + 1.0}{1^2 + (-1)^2 + 0^2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ onto } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : \frac{2.0 + 1.0 + 1.1}{0^2 + 0^2 + 1^2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note that these sum up to $\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \mathbf{x}$. Why? because ...

Theorem 1. *If $v_1, \ldots, v_n$ is orthogonal basis of V and $w \in V$ then*

$$w = c_1 v_1 + \cdots + c_n v_n, \qquad \textit{with } c_j = \frac{w \cdot v_j}{v_j \cdot v_j}.$$

So the terms in this sum are precisely the projections onto each basis vector.

4 Projection Matrix

If $\mathbf{y}$ is a fixed nonzero vector, we get from any vector $\mathbf{x}$ the projection $\hat{\mathbf{x}}$. There is a matrix that turns $\mathbf{x}$ into $\hat{\mathbf{x}}$. How? Rewrite the formula for $\hat{\mathbf{x}}$.

$$\hat{\mathbf{x}} = \frac{\mathbf{y}\cdot\mathbf{x}}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y} = \frac{1}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}(\mathbf{y}^T\mathbf{x}) = \left(\frac{1}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}\mathbf{y}^T\right)\mathbf{x} = P\mathbf{x},$$

where $P = \frac{1}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}\mathbf{y}^T$. P is called the *projection matrix* on the subspace $Span(\mathbf{y})$.

Example 13. Let $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find the projection matrix P for $\mathbf{y}$ and use it to calculate the projections of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ on $\mathbf{y}$.

Solution.

$$P = \frac{1}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}\mathbf{y}^T = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

- If $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then $\hat{\mathbf{x}} = P\mathbf{x} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- If $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then $\hat{\mathbf{x}} = P\mathbf{x} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{x}$! Why?
- If $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then $\hat{\mathbf{x}} = P\mathbf{x} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$! Why?