STAT 410 Summer 2014

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Examples for 06/18/2014

Functions of One Random Variable

Example 1:

x	$p_X(x)$
1	0.2
2	0.4
3	0.3
4	0.1

$\text{Y} = \text{X}^2$

$y = x^2$	$p_Y(y) = p_X(\sqrt{y})$
1	0.2
4	0.4
9	0.3
16	0.1

Need to keep track of changes in support

Example 2:

x	$p_X(x)$
-2	0.2
0	0.4
2	0.3
3	0.1

$\text{Y} = \text{X}^2$

y	$p_Y(y)$
0	$p_X(0) = 0.4$
4	$p_X(-2) + p_X(2) = 0.5$
9	$p_X(3) = 0.1$

Example 3:

X ~ Poisson(λ): $$p_X(x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}, \quad x = 0, 1, 2, 3, 4, \ldots .$$

$\text{Y} = \text{X}^2$ $\Rightarrow$ $$p_Y(y) = \frac{\lambda^{\sqrt{y}} \cdot e^{-\lambda}}{(\sqrt{y})!}, \quad y = 0, 1, 4, 9, 16, \ldots .$$

Let X be a continuous random variable.

Let $Y = g(X)$. What is the probability distribution of Y?

1. Cumulative Distribution Function approach:

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = \int_{\{x:\, g(x) \le y\}} f_X(x)\,dx = \ldots$$

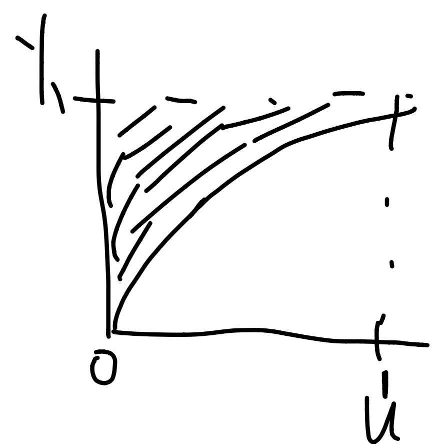
2. Moment-Generating Function approach:

$$M_Y(t) = E(e^{Y \cdot t}) = E(e^{g(X)\cdot t}) = \int_{-\infty}^{\infty} e^{g(x)\cdot t} f_X(x)\,dx = \ldots$$

3. Change of Variables Approach

1. Let U be a Uniform(0, 1) random variable:

$$f_U(u) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases} \qquad F_U(u) = \begin{cases} 0 & u < 0 \\ u & 0 \le u < 1 \\ 1 & u \ge 1 \end{cases}$$



Consider $Y = U^2$. What is the probability distribution of Y?

$$F_Y(y) = P(Y \le y) = P(U^2 \le y)$$

$$F(y) = \begin{cases} y < 0 & P(U^2 \le y) = 0 & F_Y(y) = 0. \\ 0 \le y < 1 & P(U^2 \le y) = P(U \le \sqrt{y}) = \sqrt{y} & F_Y(y) = \sqrt{y}. \\ y \ge 1 & P(U^2 \le y) = 1 & F_Y(y) = 1. \end{cases}$$

$$f_Y(y) = F'_Y(y) = \begin{cases} 1/(2\sqrt{y}) & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Theorem: Change of Variables

$$P(X \leq x) = F_X(x)$$

The pdf is $F_X'(x) = f_X(x)$

Suppose we transform:

$y = G(X)$ where G is one-to-one

$\Rightarrow X = G^{-1}(Y)$

Case #1: Monotone increasing $\frac{d\,G^{-1}(y)}{dy} \geq 0$

$$P(Y \leq y) = P(G(X) \leq y) = P(X \leq G^{-1}(y))$$

$$= F_X(G^{-1}(Y))$$

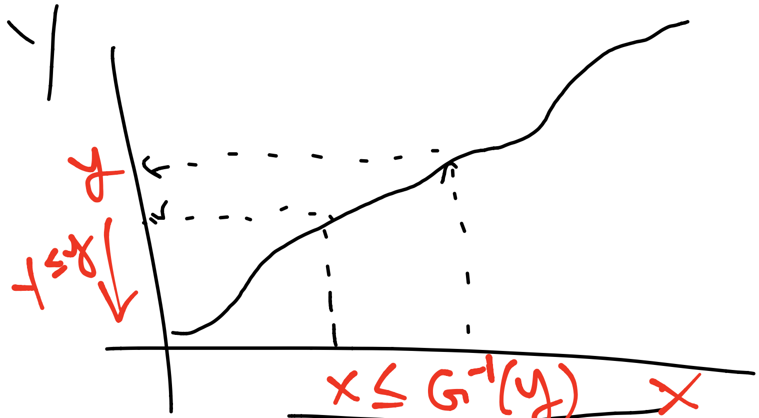
The pdf. is

$$f_Y(y) = \frac{d}{dy} F_X(G^{-1}(y)) = F_X'(G^{-1}(y)) \frac{d}{dy} G^{-1}(y)$$

$$= f_X(G^{-1}(y)) \left| \frac{d}{dy} G^{-1}(y) \right|$$

* Used chain rule!!

(a) $Y = G(X)$



$Y < y$

$\iff X \leq G^{-1}(y)$

b/c large X ⟺ larger

Case # 2 (i.e. $\frac{dG^{-1}(y)}{dy} < 0$)

(b) Here

$\uparrow X \Rightarrow \downarrow Y$

$= P(Y \leq y) = P(X \geq G^{-1}(y))$

$F_Y(y) =$

$$P(Y < y) = P(X \geq G^{-1}(y)) = 1 - F_X(G^{-1}(y))$$

$$f_Y(y) = -F_X'(G^{-1}(y)) \frac{dG^{-1}(y)}{dy}$$

but $-\frac{dG^{-1}(y)}{dy} > 0$, so,

$$f_y(y) = f_x(G^{-1}(y))\left|\frac{d\,G^{-1}(y)}{dy}\right| \text{ for case 2}$$

This holds for both cases $\blacksquare$

Theorem X – continuous r.v. with p.d.f. $f_X(x)$.

$Y = g(X)$ $g(x)$ – one-to-one, differentiable

$dx/dy = d[g^{-1}(y)]/dy$

$$\Rightarrow \qquad f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

Approach #3

$g(u) = u^2$ $\qquad g^{-1}(y) = \sqrt{y} = y^{1/2}$ $\qquad du/dy = \frac{1}{2} y^{-1/2}$

$$f_Y(y) = f_U(g^{-1}(y)) \left| \frac{du}{dy} \right| = (1) \left| \frac{1}{2} y^{-1/2} \right| = \frac{1}{2} y^{-1/2} \qquad 0 < y < 1$$

2. Consider a continuous random variable X with p.d.f.

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

a) Find the probability distribution of $Y = \sqrt{X}$.

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Approach: 1

CDF approach

$y < 0$ $\qquad F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = 0.$

$y \ge 0$ $\qquad F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2) = F_X(y^2).$

$0 \le y < 1$ $\qquad F_Y(y) = F_X(y^2) = y^4.$

$y \ge 1$ $\qquad F_Y(y) = F_X(y^2) = 1.$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y^4 & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases} \qquad f_Y(y) = F'_Y(y) = \begin{cases} 4y^3 & 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

Change of Var. Approach

OR

2

$$g(x) = \sqrt{x} \qquad g^{-1}(y) = y^2 \qquad dx/dy = 2y$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (2y^2)(2y) = 4y^3, \qquad 0 < y < 1.$$

b) Find the probability distribution of $W = \dfrac{1}{X+1}$.

Note: $0 < x < 1 \quad \Rightarrow \quad \left.\frac{1}{2} < w < 1 \right\} *$

$$F_W(w) = P(W \le w) = P\left(\frac{1}{X+1} \le w\right) = P\left(X \ge \frac{1}{w} - 1\right) = 1 - F_X\left(\frac{1}{w} - 1\right)$$

$$= 1 - \left(\frac{1}{w} - 1\right)^2 = \frac{2}{w} - \frac{1}{w^2}, \qquad \frac{1}{2} < w < 1.$$

$$= 1 - \frac{(1-w)^2}{w^2} = 1 - \frac{1}{w^2} + \frac{2}{w} - 1$$

$$f_W(w) = F'_W(w) = -\frac{2}{w^2} + \frac{2}{w^3} = \frac{2-2w}{w^3}, \qquad \frac{1}{2} < w < 1.$$

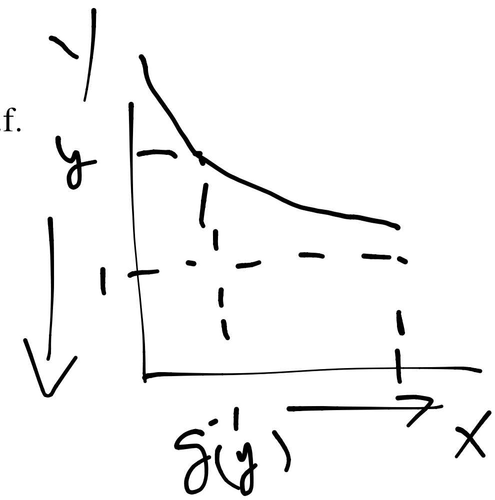
$$= \frac{2(1-w)}{w^3}$$

OR

$$g(x) = \frac{1}{x+1} \qquad g^{-1}(w) = \frac{1}{w} - 1 \qquad dx/dw = -\frac{1}{w^2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \left[2\left(\frac{1}{w} - 1\right)\right]\left(\frac{1}{w^2}\right) = \frac{2-2w}{w^3}, \qquad \frac{1}{2} < w < 1.$$

3. Consider a continuous random variable X with p.d.f.



$$f_X(x) = \begin{cases} 6x^5 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the probability distribution of $Y = 1/X^2$.

1

Support of $X = \{ 0 < x < 1 \}$

$Y = 1/X^2 \quad \Rightarrow \quad$ Support of $Y = \{ y > 1 \}$

2

$g(x) = 1/x^2 \qquad X = g^{-1}(y) = 1/\sqrt{y} = y^{-1/2}$

$$dx/dy = -\frac{1}{2} y^{-3/2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (6y^{-5/2})(\frac{1}{2} y^{-3/2}) = 3y^{-4} \qquad y > 1.$$

OR

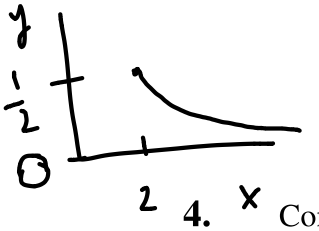
3

$$f_X(x) = \begin{cases} 6x^5 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < 0 \\ x^6 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$F_Y(y) = P(Y \le y) = P(1/X^2 \le y) = P(X \ge 1/\sqrt{y}) = 1 - F_X(1/\sqrt{y})$$

$$= 1 - y^{-3}, \qquad y > 1.$$

$$f_Y(y) = F_Y'(y) = 3y^{-4}, \qquad y > 1.$$



4. Consider a continuous random variable X with the p.d.f. $f_X(x) = \dfrac{24}{x^4}$, $x > 2$.

a) Let $Y = 1/X$. Find the p.d.f. of Y, $f_Y(y)$.

$F_x = 24\int_2^x t^{-4}\,dt = \frac{24}{-3}\, t^{-3}\Big|_2^x$
$= -8\left(x^{-3} - \frac{1}{8}\right)$
$= 1 - \frac{8}{x^3}$

1

Support of X $= \{\, x > 2 \,\}$

$Y = 1/X \quad \Rightarrow \quad$ Support of Y $= \{\, 0 < y < 1/2 \,\}$

2a

$g(x) = 1/x \qquad g^{-1}(y) = 1/y \qquad dx/dy = -1/y^2$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| = (24\,y^4)(y^{-2}) = 24\,y^2, \quad 0 < y < 1/2.$$

OR

2b

$F_X(x) = 1 - \dfrac{8}{x^3}, \qquad x > 2.$

$F_Y(y) = P(Y \le y) = P(1/X \le y) = P(X \ge 1/y) = 1 - F_X(1/y) = 8\,y^3,$

$0 < y < 1/2.$

$f_Y(y) = 24\,y^2, \;\; 0 < y < 1/2.$

b) Let $Y = 1/X^2$. Find the p.d.f. of Y, $f_Y(y)$.

Support of X $= \{\, x > 2 \,\}$

$Y = 1/X^2 \quad \Rightarrow \quad$ Support of Y $= \{\, 0 < y < 1/4 \,\}$

$g(x) = 1/x^2 \qquad g^{-1}(y) = 1/\sqrt{y} = y^{-1/2} \qquad dx/dy = -\frac{1}{2}\, y^{-3/2}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (24y^2)(\tfrac{1}{2}y^{-3/2}) = 12y^{1/2} = 12\sqrt{y},$$

$$0 < y < {}^1\!/_4.$$

OR

$$F_X(x) = 1 - \frac{8}{x^3}, \qquad x > 2.$$

$$F_Y(y) = P(Y \le y) = P({}^1\!/_{X^2} \le y) = P(X \ge {}^1\!/_{\sqrt{y}})$$

$$= 1 - F_X({}^1\!/_{\sqrt{y}}) = 8y^{3/2}, \qquad 0 < y < {}^1\!/_4.$$

$$f_Y(y) = 12y^{1/2} = 12\sqrt{y}, \quad 0 < y < {}^1\!/_4.$$

$-3<x<-1$

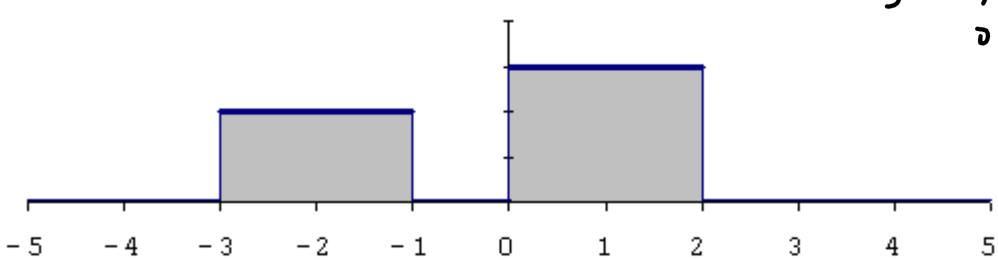
5. Consider a continuous random variable X with p.d.f.

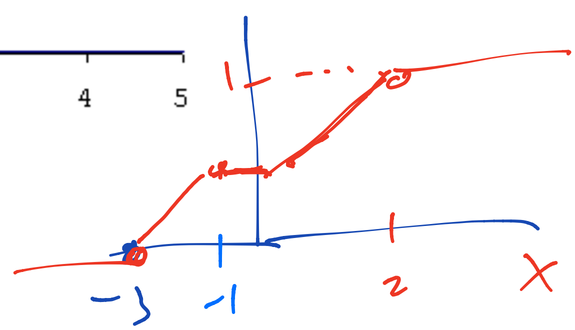
$$\int_{-3}^{x} \frac{1}{5}\,dt = \frac{1}{5}(x+3)$$

$$f_X(x) = \begin{cases} 0.2 & -3 < x < -1 \\ 0.3 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases} \Rightarrow F_x = $$

$$= \frac{2}{5} + \int_0^x \frac{3}{10}\,dt = \frac{2}{5} + \frac{3}{10}x$$

$0<x<2$





Find the probability distribution of $Y = X^2$.

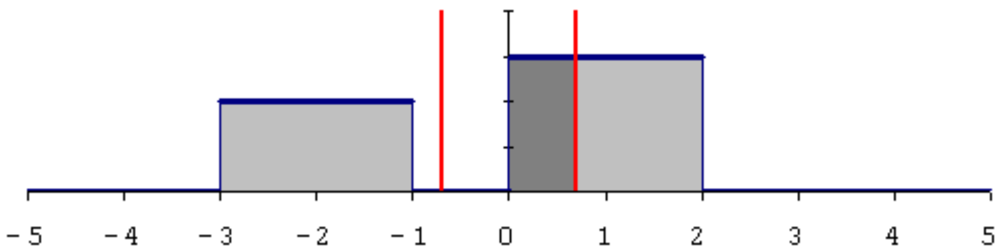
$y < 0$ $\qquad P(X^2 \le y) = 0$ $\qquad F_Y(y) = 0.$

$y \ge 0$ $\qquad F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$

$F_Y = P(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$= .3\sqrt{y}$

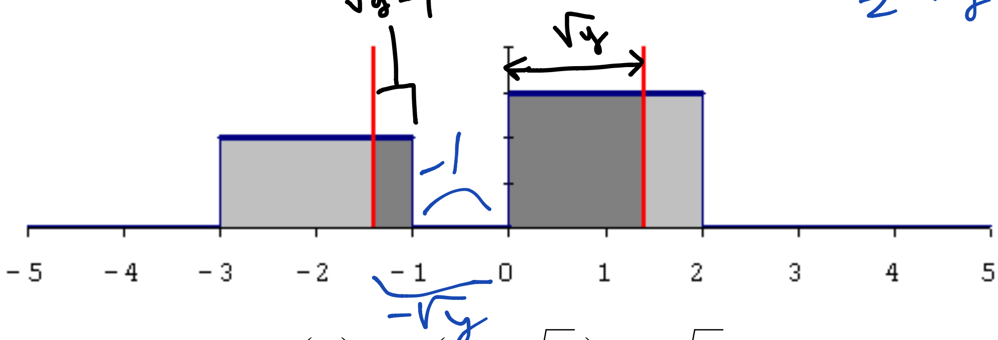
Case 1: $0 \le y < 1$ $\Rightarrow$ $0 \le \sqrt{y} < 1$



$$F_Y(y) = 0.3\sqrt{y}.$$

$F_Y = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{3}{10}\sqrt{y} + \frac{2}{5} - \frac{1}{5}(-\sqrt{y}+3)$

Case 2: $1 \le y < 4$ $\Rightarrow$ $1 \le \sqrt{y} < 2$ $= \frac{1}{2}\sqrt{y} - \frac{1}{5}$



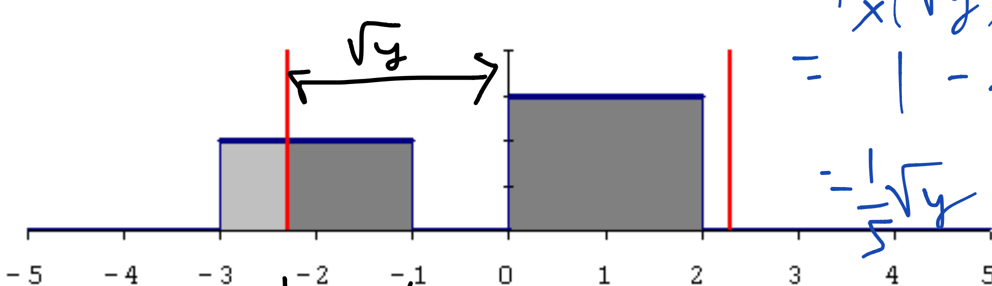
$$F_Y(y) = 0.2(-1 + \sqrt{y}) + 0.3\sqrt{y}.$$

Case 3: $4 \le y < 9$ $\Rightarrow$ $2 \le \sqrt{y} < 3$

$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$= 1 - \frac{1}{5}(-\sqrt{y}+3)$

$= \frac{1}{5}\sqrt{y} - \frac{2}{5}$



$$F_Y(y) = \underbrace{0.2(-1 + \sqrt{y})}_{\sqrt{y}-1} + 0.6. \quad = \frac{1}{5}\sqrt{y} + .4$$

Case 4: $y \ge 9$ $F_Y(y) = 1.$

6. Let $\lambda > 0$ and let X be a random variable with the probability density function

$$f(x) = \frac{\lambda}{x^{\lambda+1}}, \qquad x > 1, \qquad \text{zero otherwise.}$$

Let $\mathrm{W} = \ln \mathrm{X}$. What is the probability distribution of W?

a) Determine the probability distribution of W by finding the c.d.f. of W

$$\mathrm{F_W}(w) = \mathrm{P}(\mathrm{W} \le w) = \mathrm{P}(\ln \mathrm{X} \le w).$$

"Hint": Find $\mathrm{F_X}(x)$ first.

$$\mathrm{F_X}(x) = \int_1^x \frac{\lambda}{y^{\lambda+1}}\,dy = 1 - \frac{1}{x^{\lambda}}., \qquad x > 1.$$

$$\mathrm{F_Y}(y) = \mathrm{P}(\mathrm{W} \le w) = \mathrm{P}(\mathrm{X} \le e^{w}) = \mathrm{F_X}(e^{w}) = 1 - e^{-\lambda w}, \quad w > 0.$$

$$f_{\mathrm{W}}(w) = \lambda e^{-\lambda w}, \quad w > 0.$$

$\Rightarrow$ W has Exponential distribution with mean $1/\lambda$.

b) Determine the probability distribution of W by finding the m.g.f. of W

$$\mathrm{M_W}(t) = \mathrm{E}(e^{\mathrm{W}\cdot t}) = \mathrm{E}(e^{\ln \mathrm{X}\cdot t}).$$

$$\mathrm{M_W}(t) = \mathrm{E}(e^{\mathrm{W}\cdot t}) = \mathrm{E}(e^{\ln \mathrm{X}\cdot t}) = \mathrm{E}(\mathrm{X}^t) = \int_1^{\infty} \left(x^t \cdot \frac{\lambda}{x^{\lambda+1}} \right) dx$$

$$= \int_1^{\infty} \lambda\, x^{t-\lambda-1}\,dx = \frac{\lambda}{\lambda - t} = \frac{1}{1 - \frac{1}{\lambda}t}, \qquad t < \lambda.$$

$\Rightarrow$ W has Exponential distribution with mean $1/\lambda$.

c) Determine the probability distribution of W by finding the p.d.f. of W, $f_W(w)$, using the change-of-variable technique.

$$w = \ln(x) \qquad x = g^{-1}(w) = e^{w} \qquad \frac{dx}{dw} = e^{w}$$

$$x > 1 \quad \Rightarrow \quad w > 0$$

$$f_W(w) = f_X(g^{-1}(w)) \left| \frac{dx}{dw} \right| = \frac{\lambda}{\left(e^{w}\right)^{\lambda+1}} \cdot \left| e^{w} \right| = \lambda e^{-\lambda w}, \quad w > 0.$$

$\Rightarrow$ W has Exponential distribution with mean $1/\lambda$.

Consider a continuous random variable X, with p.d.f. f and c.d.f. F, where F is strictly increasing on some interval I, F = 0 to the left of I, and F = 1 to the right of I. I may be a bounded interval or an unbounded interval such as the whole real line. $F^{-1}(u)$ is then well defined for $0 < u < 1$.

Fact 1:

Let U ~ Uniform(0, 1), and let $X = F^{-1}(U)$. Then the c.d.f. of X is F.

Proof: $P(X \le x) = P(F^{-1}(U) \le x) = P(U \le F(x)) = F(x)$.

OR CDF of U is $P(U \le u) = u$

Fact 2:

Let $U = F(X)$; then U has a Uniform(0, 1) distribution.

Proof: $P(U \le u) = P(F(X) \le u) = P(X \le F^{-1}(u)) = F(F^{-1}(u)) = u$. ← CDF of $U \sim U(0,1)$

$\bar{F}(x) = \int_{-\infty}^{x} f(t)\,dt$

$= \int_{-\infty}^{x} \frac{e^{-t}}{(1+e^{-t})^2}\,dt$ $\quad u = 1+e^{-t}$, $\quad du = -e^{-t}\,dt$

$= -\int u^{-2}\,du = \frac{-1}{-1}u^{-1}\Big|$

$= \frac{1}{1+e^{-t}}\Big|_{-\infty}^{x} = \frac{1}{1+e^{-x}}$

HW **7.** Let X have a **logistic distribution** with p.d.f.

$$f(x) = \frac{e^{-x}}{\left(1+e^{-x}\right)^2}, \qquad -\infty < x < \infty.$$

Show that $\quad \mathrm{Y} = \dfrac{1}{1+e^{-\mathrm{X}}} \quad$ has a $\mathrm{U}(0,1)$ distribution.

$$\mathrm{F_X}(x) = \frac{1}{1+e^{-x}}, \qquad -\infty < x < \infty.$$

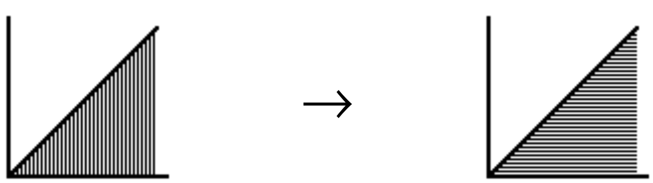
$\Rightarrow \qquad \mathrm{Y} = \mathrm{F_X}(\mathrm{X})$ has a Uniform(0, 1) distribution by Fact 2.

~ **1.9.20** (7th edition) **1.9.19** (6th edition)

Let X be a nonnegative continuous random variable with p.d.f. $f(x)$ and c.d.f. $F(x)$. Show that

$$\mathrm{E}(\mathrm{X}) = \int_0^{\infty} (1-F(x))\,dx.$$

$$\mathrm{E}(\mathrm{X}) = \int_0^{\infty} x\,f(x)\,dx = \int_0^{\infty}\left(\int_0^{x} dy\right) f(x)\,dx = \int_0^{\infty}\left(\int_0^{x} f(x)\,dy\right) dx$$



$$\int_0^{\infty}\left(\int_0^{x} f(x)\,dy\right) dx = \int_0^{\infty}\left(\int_y^{\infty} f(x)\,dx\right) dy$$

$$\Rightarrow \qquad \mathrm{E}(\mathrm{X}) = \int_0^{\infty}\left(\int_y^{\infty} f(x)\,dx\right) dy = \int_0^{\infty} \mathrm{P}(\mathrm{X} > y)\,dy = \int_0^{\infty} (1-F(y))\,dy.$$

Example: Find the expected value of an Exponential (θ) distribution.

For Exponential (θ), $1 - \mathrm{F}(x) = \mathrm{P}(\mathrm{X} > x) = e^{-x/\theta}, \quad t > 0.$

$$\mathrm{E}(\mathrm{X}) = \int_0^\infty (1 - F(x))\,dx = \int_0^\infty e^{-x/\theta}\,dx = \theta.$$

1.9.21 (7th edition) **1.9.20** (6th edition)

Let X be a random variable of the discrete type with pmf $p(x)$ that is positive on the nonnegative integers and is equal to zero elsewhere. Show that

$$\mathrm{E}(\mathrm{X}) = \sum_{x=0}^{\infty} [1 - \mathrm{F}(x)],$$

where $\mathrm{F}(x)$ is the cdf of X.

$$1 - \mathrm{F}(x) = \mathrm{P}(\mathrm{X} > x) = p(x+1) + p(x+2) + p(x+3) + p(x+4) + \ldots$$

$$\begin{array}{ll}
1 - \mathrm{F}(0) & p(1) + p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + \ldots \\
1 - \mathrm{F}(1) & \phantom{p(1) + {}} p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + \ldots \\
1 - \mathrm{F}(2) & \phantom{p(1) + p(2) + {}} p(3) + p(4) + p(5) + p(6) + p(7) + \ldots \\
1 - \mathrm{F}(3) & \phantom{p(1) + p(2) + p(3) + {}} p(4) + p(5) + p(6) + p(7) + \ldots \\
1 - \mathrm{F}(4) & \phantom{p(1) + p(2) + p(3) + p(4) + {}} p(5) + p(6) + p(7) + \ldots \\
\ldots & \qquad\qquad\qquad \ldots
\end{array}$$

$$\Rightarrow \quad \sum_{x=0}^{\infty} [1 - \mathrm{F}(x)] = 1 \times p(1) + 2 \times p(2) + 3 \times p(3) + 4 \times p(4) + \ldots = \mathrm{E}(\mathrm{X}).$$

Example: Find the expected value of a Geometric (p) distribution.

For Geometric (p), $1 - \mathrm{F}(x) = \mathrm{P}(\mathrm{X} > x) = (1 - p)^x, \quad x = 0, 1, 2, \ldots .$

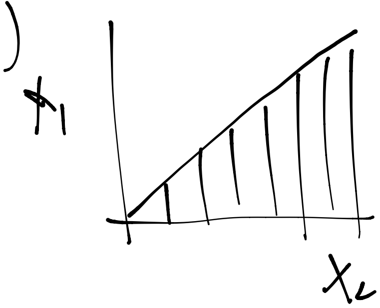
$$\mathrm{E}(\mathrm{X}) = \sum_{x=0}^{\infty} [1 - \mathrm{F}(x)] = \sum_{x=0}^{\infty} [1 - p]^x = \frac{1}{1 - [1 - p]} = \frac{1}{p}.$$

$X_1 \sim \exp(\lambda_1)$, $X_2 \sim \exp(\lambda_2)\, I(X_2 > 1)$

$Y = \frac{X_1}{X_2}$ $\quad F_1(x_1) = 1 - e^{-\lambda_1 x_1}$

$$f_2(x_2) = \frac{\lambda_2 e^{-\lambda_2 x_2}}{e^{\lambda_2}} = \lambda_2 e^{-\lambda_2 (x_2 + 1)}$$

$$F_2(x_2) = 1 - e^{-\lambda_2 (x_2 + 1)}$$



$$F_Y(y) = P(Y \le y) = P(X_1 \le y X_2)$$

$$= \int_1^\infty F_1(y X_2)\, f_2(X_2)\, dx_2$$

$$= \int_1^\infty \left(1 - e^{-\lambda_1 y X_2}\right) \lambda_2 e^{-\lambda_2 (x_2 + 1)}\, dX_2$$

$$= 1 - \lambda_2 \int_1^\infty e^{-(\lambda_1 y + \lambda_2) x_2 - \lambda_2}\, dx_2$$

$$= 1 - \frac{\lambda_2}{e^{\lambda_2}} \left(\frac{-1}{\lambda_1 y + \lambda_2} e^{-(\lambda_1 y + \lambda_2) x_2} \right) \Big|_1^\infty$$

$$= 1 - \frac{\lambda_2}{\lambda_1 y + \lambda_2} \frac{e^{-\lambda_1 y}}{e^{2\lambda_2}} \qquad y > 0$$

$$\Rightarrow f_Y(y) = F_Y'(y)$$

$$= \frac{\lambda_2}{e^{2\lambda_2}} \left\{ \frac{-\lambda_1 e^{-\lambda_1 y}(\lambda_1 y + \lambda_2) - e^{-\lambda_1 y}\lambda_1}{(\lambda_1 y + \lambda_2)^2} \right\}$$

$$= \frac{\lambda_1 \lambda_2}{e^{2\lambda_2}} e^{-\lambda_1 y} \left\{ \frac{\lambda_1 y + \lambda_2 - 1}{(\lambda_1 y + \lambda_2)^2} \right\}$$

$$\lambda_1 = \lambda_2 = 1$$

$$\Rightarrow e^{-(y+2)} \frac{y}{(y+1)^2}, \quad y > 0$$