Math 415 - Lecture 34

Discrete dynamical systems, Spectral Theorem

Wednesday November 18th 2015

Textbook reading: Chapter 5.3, Chapter 5.6 p. 297-298

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Suggested practice exercises: Chapter 5.3, 2, 3, 4, 7, 8, 9, 10, 12, 14

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Strang lecture: Lecture 25: Symmetric Matrices and Positive Definiteness

Review

Diagonalization

Suppose that A is an $n \times n$ and has independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. Then A can be **diagonalized** as $A = PDP^{-1}$.

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Calculating Powers

If $A = PDP^{-1}$ for some diagonal matrix D, then $A^n = PD^nP^{-1}$ for every n. This is helpful, because calculating powers of diagonal matrices is very easy!

Application: Discrete Dynamical Systems

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So to solve our system we need to be able to calculate high powers of the matrix A. Use eigenbasis of A for this.

Golden ratio and Fibonacci numbers

Example

'A certain man put a pair of rabbits in a place surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair from which the second month on becomes productive?' (Liber abbaci, chapter 12, p. 283-4)

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- Hence $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} \qquad \left(\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

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- But we know how to compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$ or $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$!

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- $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1^n c_1 \mathbf{v}_1 + \lambda_2^n c_2 \mathbf{v}_2$

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- Hence, $F_n = \lambda_1^n c_1 + \lambda_2^n c_2 = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$. That is **Binet's formula**.

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- but $|\lambda_2| < 1$, so $F_n \approx \lambda_1^n c_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$. In fact, $F_n = \text{round}\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$.

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Did you notice: $\frac{13}{8} = 1.625, \frac{21}{13} = 1.615, \frac{34}{21} = 1.619, \ldots$



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The **golden ratio** $\varphi = 1.618...$ Where's that from? We just showed that $F_n = \text{round}\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$. Therefore

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \left(\frac{1+\sqrt{5}}{2}\right).$$

Definition

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$\lambda_2 = 4$: We have seen $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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$$\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$\lambda_2 = 4$: We have seen $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Normalized, we get

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

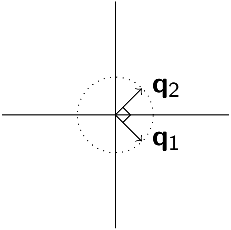
Solution (continued)

Write D: $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

Write Q: $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Get $A = QDQ^T$: $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

What does A do to the eigenvectors?



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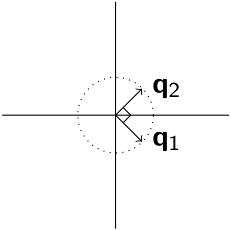

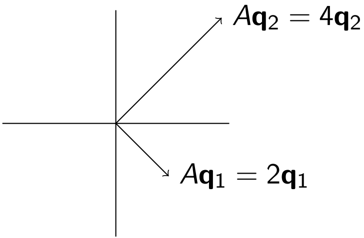

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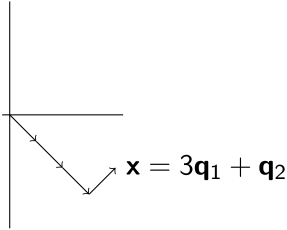

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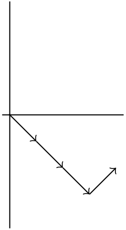

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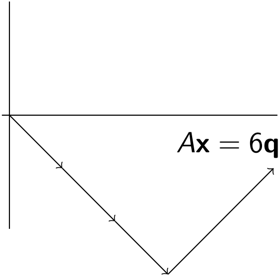

$A\mathbf{x} = 6\mathbf{q}_1 + 4\mathbf{q}_2$

Why are symmetric matrices special?

Why are symmetric matrices special? Why does spectral theorem work? If $A = A^T$, and if

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Since $\lambda_1 \neq \lambda_2$, must have $\mathbf{x} \cdot \mathbf{y} = 0$! By a similar argument you can show that the eigenvalues of a symmetric matrix **must** be real.

Example

Let $A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$.

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