Math 415 - Lecture 26

Orthogonal Matrices and QR Decomposition

Wednesday October 28th 2015

Textbook reading: Chapter 3.4

Suggested practice exercises: 3.4: 13, 16, 17, 18. 13,

Khan Academy video: Gram-Schmidt Example

Strang lecture: Orthogonal Matrices and Gram-Schmidt Process.

1 Review

- Vectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are orthonormal if

$$\mathbf{q}_i^T \mathbf{q}_j = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

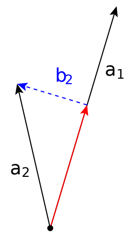
- **Gram-Schmidt** orthonormalization: input: basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$ for V. output: orthonormal basis $\mathbf{q}_1, \ldots, \mathbf{q}_n$ for V.

$$\begin{aligned} \mathbf{b}_1 = \mathbf{a}_1, &\quad \mathbf{q}_1 = \tfrac{\mathbf{b}_1}{\|\mathbf{b}_1\|} \\ \mathbf{b}_2 = \mathbf{a}_2 - \langle \mathbf{a}_2, \mathbf{q}_1 \rangle \mathbf{q}_1, &\quad \mathbf{q}_2 = \tfrac{\mathbf{b}_2}{\|\mathbf{b}_2\|} \\ \mathbf{b}_3 = \mathbf{a}_3 - \langle \mathbf{a}_3, \mathbf{q}_1 \rangle \mathbf{q}_1 - \langle \mathbf{a}_3, \mathbf{q}_2 \rangle \mathbf{q}_2, &\quad \mathbf{q}_3 = \tfrac{\mathbf{b}_3}{\|\mathbf{b}_3\|} \\ \cdots &\quad \cdots \end{aligned}$$

[-1cm]

Fact 1. *if A is any matrix A^TA is the matrix of dot products of the columns of A: Write $A = \begin{bmatrix} \mathbf{a_1}, \ldots \mathbf{a_n} \end{bmatrix}$ then*

$$A^TA = \begin{bmatrix} \mathbf{a_1} \cdot \mathbf{a_1} & \mathbf{a_1} \cdot \mathbf{a_2} & \mathbf{a_1} \cdot \mathbf{a_3} & \ldots \\ \mathbf{a_2} \cdot \mathbf{a_1} & \mathbf{a_2} \cdot \mathbf{a_2} & \mathbf{a_2} \cdot \mathbf{a_3} & \ldots \\ \mathbf{a_3} \cdot \mathbf{a_1} & \mathbf{a_3} \cdot \mathbf{a_2} & \mathbf{a_3} \cdot \mathbf{a_3} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$



Theorem 1. *The columns of Q are orthonormal* $\iff$ $Q^T Q = I$

Definition. An **orthogonal matrix** is a square matrix Q with orthonormal columns.

2 The QR decomposition

In linear algebra "everything" is a matrix factorization.

- Gaussian elimination in terms of matrices: $A = LU$
- Gram-Schmidt in terms of matrices $A = QR$

Theorem 2 (QR decomposition)**.** *Let A be an $m \times n$ matrix of rank n. There is is a orthogonal matrix $m \times n$-matrix Q and an upper triangular $n \times n$ invertible matrix R such that*

$$A = QR.$$

Recipe

In general, to obtain $A = QR$:

- Gram-Schmidt on (columns of) A, to get (columns of) Q.
- Then $R = Q^T A$.

The resulting R is indeed upper triangular, and we get:

$$\begin{bmatrix} | & | & \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots \\ | & | & \end{bmatrix} = \begin{bmatrix} | & | & \\ \mathbf{q}_1 & \mathbf{q}_2 & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T\mathbf{a}_1 & \mathbf{q}_1^T\mathbf{a}_2 & \mathbf{q}_1^T\mathbf{a}_3 & \cdots \\ & \mathbf{q}_2^T\mathbf{a}_2 & \mathbf{q}_2^T\mathbf{a}_3 & \\ & & \mathbf{q}_3^T\mathbf{a}_3 & \\ & & & \ddots \end{bmatrix}$$

Example 3. Find the QR decomposition of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$.

Solution.

Example 4. Find the QR decomposition of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

Solution.

2.1 Using QR to solve systems of equations

QR decomposition can be used to solve systems of linear equations.

$$\begin{aligned} A\mathbf{x} = \mathbf{b} \quad &\iff \quad QR\mathbf{x} = \mathbf{b} \\ &\iff \quad R\mathbf{x} = Q^T\mathbf{b} \end{aligned}$$

$R\mathbf{x} = Q^T\mathbf{b}$ is triangular, so solve it by back substitution. QR is a little slower than LU, but makes up in numerical stability.

Theorem 2. *Let A be matrix with linear independent columns. Suppose $A\mathbf{x} = \mathbf{b}$ has no solution. Then the solution of $R\mathbf{x} = Q^T\mathbf{b}$ is the least square solution of $A\mathbf{x} = \mathbf{b}$.*

Proof.

$\square$

Remark. $R\mathbf{x} = Q^T\mathbf{b}$ always gives the best possible solution to $A\mathbf{x} = \mathbf{b}$.

Example 5. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.Find the least square solution of $A\mathbf{x} = \mathbf{b}$ using QR-decomposition.

Solution.