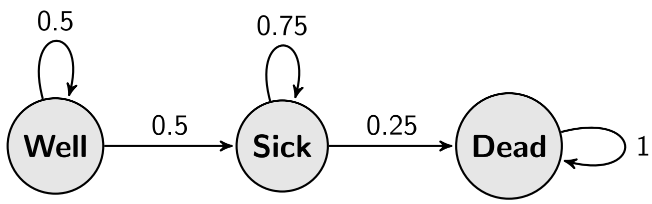
Worksheet 12 for November 17th and 19th

1. *Suppose there is an epidemic in which every month half of those who are well, become sick, and a quarter of those who are sick become dead. Set up a 3×3 transition matrix A and find the steady state $\boldsymbol{v}_\infty$.*

Solution. The transition graph is as follows:



Hence, the transition matrix is

$$A = \begin{bmatrix} .5 & 0 & 0 \\ .5 & .75 & 0 \\ 0 & .25 & 1 \end{bmatrix}.$$

Let

$$u_t = \begin{bmatrix} u_{t,W} & u_{t,S} & u_{t,D} \end{bmatrix},$$

where $u_{t,W}$ is the percentage of people that are well, $u_{t,S}$ is the percentage of people that are sick and $u_{t,D}$ the percentage of people that have died, at time t. Note that

$$u_{t+1} = Au_t.$$

The long term equlibrium is a state u_∞ such that

$$Au_\infty = u_\infty.$$

For that we need to find an eigenvector of A to the eigenvalue 1:

$$\left[\begin{array}{ccc|c} -.5 & 0 & 0 & 0 \\ .5 & -.25 & 0 & 0 \\ 0 & .25 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -.5 & 0 & 0 & 0 \\ 0 & -.25 & 0 & 0 \\ 0 & .25 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -.5 & 0 & 0 & 0 \\ 0 & -.25 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, the eigenspace of $\lambda = 1$ is spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.Since this vector is already scaled so that its entries add up to 1, we conclude

$$u_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

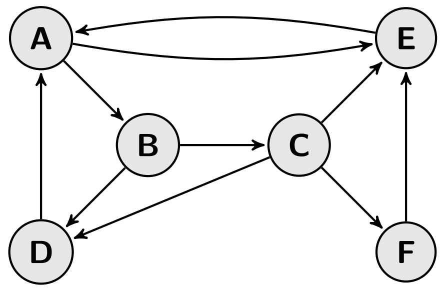
This means all people will die from the epidemic. □

2. *Find the page rank for the following system of webpages:*

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Midterm Date: November 19 7-8:15 PM, Conflict November 20, 8-9.20AM and 9:30-10:50AM, Conflict sign up deadline: November 13

Final Date: December 17 8-11AM, Conflict December 15, 8-11AM. You are allowed to take the conflict exam if you have more than two examination within 24 hours. Conflict sign up deadline: November 30



Solution. The PageRank matrix (that is, the transition matrix for the random surfer) is

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}.$$

The PageRank vector is an eigenvector of T to the eigenvalue 1. We row reduce $T-I$ as follows (do it!):

$$\left[\begin{array}{cccccc|c} -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ .5 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & .5 & -1 & 0 & 0 & 0 & 0 \\ 0 & .5 & \frac{1}{3} & -1 & 0 & 0 & 0 \\ .5 & 0 & \frac{1}{3} & 0 & -1 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -12 & 0 \\ 0 & 1 & 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Hence an eigenvector of T to the eigenvalue 1 is given by

$$\begin{bmatrix} 12 \\ 6 \\ 3 \\ 4 \\ 8 \\ 1 \end{bmatrix},$$

and the corresponding normalized PageRank vector is

$$\frac{1}{34}\begin{bmatrix} 12 \\ 6 \\ 3 \\ 4 \\ 8 \\ 1 \end{bmatrix}.$$

For the purpose of ranking this normalizing is immaterial (and it is OK if you don't normalize PageRank vectors on the exam). The ranking is A, E, B, D, C, F. $\square$

3. *Find the characteristic polynomial, eigenvalues, and a basis for the corresponding eigenspaces for the following matrix.*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Solution. We first compute the characteristic polynomial. This will be the determinant of

$$A = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{bmatrix}$$

which equals $(1-\lambda)(\lambda^2+1) = (1-\lambda)(\lambda+i)(\lambda-i)$, so our eigenvalues are 1, i, and $-i$.For $\lambda = 1$, we seek the nullspace of

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

The reduced row echelon form of this matrix is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so for $\lambda = 1$, we have $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$ as a basis for the corresponding eigenspace.For $\lambda = i$, we get

$$\begin{bmatrix} 1-i & 0 & 0 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix}$$

The reduced row echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix}$$

So we can take $\{ \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \}$ as a basis for the eigenspace corresponding to $\lambda = i$.For $\lambda = -i$, we get

$$\begin{bmatrix} 1-i & 0 & 0 \\ 0 & i & 1 \\ 0 & -1 & i \end{bmatrix}$$

The reduced row echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix}$$

So we can take $\{ \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \}$ as a basis for the eigenspace corresponding to $\lambda = -i$. □