Problem 1. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) *[3 points]* Find the eigenvalues of A.

(b) *[5 points]* For each eigenvalue of A, determine a basis of the corresponding eigenspace.

(c) *[7 points]* Find a solution to the initial value problem

$$\frac{d}{dt}u = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} u, \quad u(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Simplify your solution as far as possible.

a) $\det(A - \lambda I) = (2-\lambda)\left((1-\lambda)^2 - 1\right) = (2-\lambda)(1 - 2\lambda + \lambda^2 - 1)$

$= (2-\lambda)\lambda(-2+\lambda)$

So $\lambda_1 = 0 \qquad \lambda_2 = 2$

b) for $\lambda_1 = 0$:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{basis of eigenspace } \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

for $\lambda_2 = 2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{basis of eigenspace } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Problem 2. Consider a country that is divided in three regions A, B and C. A study finds that each year 5% of the residents of region A move to region B, and 5% move to region C. Of the residents of region B, 15% move to region A, and 10% move to region C. And of the residents of region C, 10% leave region C for region A, and 5% leave for region B.

(a) *[3 points]* Set up the 3×3 transition matrix.
(b) *[5 points]* Find the steady state.
(c) *[2 points]* In the steady state, what is the percentage of the population of the country living in region A?

a)

$$M = \begin{bmatrix} 0.9 & 0.15 & 0.1 \\ 0.05 & 0.75 & 0.05 \\ 0.05 & 0.10 & 0.85 \end{bmatrix}$$

b)

$$M - I = \begin{bmatrix} -0.1 & 0.15 & 0.1 \\ 0.05 & -0.25 & 0.05 \\ 0.05 & 0.10 & -0.15 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -\frac{13}{7} \\ 0 & 1 & -\frac{4}{7} \\ 0 & 0 & 0 \end{bmatrix}$$

Basis of $\mathrm{Nul}(M-I)$: $\begin{bmatrix} \frac{13}{7} \\ \frac{4}{7} \\ 1 \end{bmatrix}$

$\Rightarrow$ Steady state : $\dfrac{1}{\frac{13}{7}+\frac{4}{7}+1}\begin{bmatrix} \frac{13}{7} \\ \frac{4}{7} \\ 1 \end{bmatrix} = \dfrac{7}{24}\begin{bmatrix} \frac{13}{7} \\ \frac{4}{7} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{13}{24} \\ \frac{4}{24} \\ \frac{7}{24} \end{bmatrix}$

c) $\dfrac{13}{24} \approx 54\%$

Problem 3. *[10 points]* Find the least squares line for the four data points (0, 1), (1, 0), (2, −2), (−2, 3). Make sure to show all your work!

Solve:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}}_{\beta} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}}_{b}$$

$$A \quad \beta = b$$

Find least squares sol.!

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$

$\Rightarrow$ Solve $\begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$

$\Rightarrow \beta_1 = \frac{4}{5}$ and $\beta_2 = -\frac{6}{5}$

$\Rightarrow$ the least squares line is

$$y = \frac{4}{5} - \frac{6}{5} x.$$

Problem 4. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix}$.

(a) *[5 points]* Determine the LU decomposition of A.

(b) *[5 points]* Determine A^{-1}.

(If you have time, check your answer for both questions.)

a) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix} \underset{R2 \to R2 - 2R1}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 0 \end{bmatrix}$

$\underset{R3 \to R3 - 2R2}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

b) $A^{-1} = \begin{bmatrix} \frac{5}{3} & -\frac{1}{3} & -\frac{1}{12} \\ 0 & 0 & \frac{1}{4} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$

Problem 5. Let V be the set of all solutions $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ to the system of equations

$$x_1 - 2x_2 + 3x_3 + 2x_4 = 0.$$

(a) *[4 points]* Determine a basis for V.
(b) *[2 points]* Find a matrix A such that $V = \mathrm{Nul}(A)$, and find a matrix B such that $V = \mathrm{Col(B)}$.
(c) *[4 points]* Determine a basis for the orthogonal complement of V.

a) Note that $V = \mathrm{Nul}\left(\begin{bmatrix} 1 & -2 & 3 & 2 \end{bmatrix}\right)$
Basis of $\mathrm{Nul}\left(\begin{bmatrix} 1 & -2 & 3 & 2 \end{bmatrix}\right)$ is

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

b) $V = \mathrm{Nul}\left(\begin{bmatrix} 1 & -2 & 3 & 2 \end{bmatrix}\right)$

$= \mathrm{Col}\left(\begin{bmatrix} 2 & -3 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$

c) $V = \mathrm{Nul}(\begin{bmatrix} 1 & -2 & 3 & 2 \end{bmatrix}$

$\Rightarrow V^{\perp} = \mathrm{Col}\left(\begin{bmatrix} 1 & -2 & 3 & 2 \end{bmatrix}^T\right)$

$= \mathrm{span}\left\{\begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix}\right\}$

$\Rightarrow \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix}$ basis of $V^{\perp}$.

Problem 6. Let $A = \begin{bmatrix} 4 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$.

(a) *[4 points]* Apply Gram–Schmidt to the columns of A to obtain an orthonormal basis for Col(A).

(b) *[3 points]* Determine the orthogonal projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto Col(A).

(c) *[3 points]* Determine the QR decomposition of A.

a) $q_1 = \frac{1}{4}\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$b_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$q_2 = \frac{b_2}{\|b_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ $\Rightarrow$ orthonormal basis for Col(A).

$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right)$

b)

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{Col(A)} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

c) $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} \end{bmatrix}$, $R = Q^TA = \begin{bmatrix} 4 & 1 \\ 0 & \sqrt{2} \end{bmatrix}$

MULTIPLE CHOICE

(28 questions, 2 points each)

Instructions for multiple choice questions

- No reason needs to be given. There is always exactly one correct answer.
- Enter your answer on the scantron sheet that is included with your exam. In addition, on your exam paper, circle the choices you made on the scantron sheet.
- Use a **number 2 pencil** to shade the bubbles completely and darkly.
- Do **NOT** cross out your mistakes, but rather erase them thoroughly before entering another answer.
- Before beginning, please code in your name, UIN, and netid in the appropriate places. In the **'Section' field on the scantron**, please enter

  000 if Armin Straub is your instructor,

  001 if Philipp Hieronymi is your instructor.

MC 1. If A is a $m \times n$ matrix and B is a $n \times \ell$ matrix, then what is $(AB)^T$?

(a) $A^T B^T$,
(b) $B^T A^T$,
(c) AB^T,
(d) BA^T.

MC 2. Let A be an $n \times n$ matrix. Are the following two statements always correct?

(U1) The matrix A^T has the same eigenvectors as A.
(U2) The matrix A^T has the same eigenvalues as A.

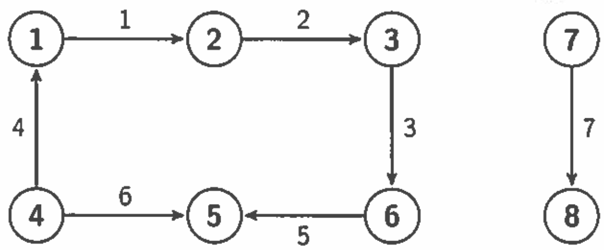
Then:

(a) Statement U1 and Statement U2 are correct.
(b) Only Statement U1 is correct.
(c) Only Statement U2 is correct.
(d) Neither Statement U1 nor Statement U2 is correct.

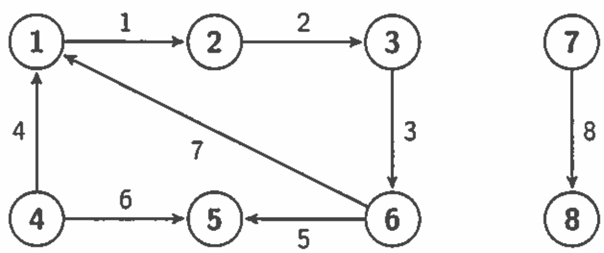
MC 3. Let A be an $m \times n$ matrix. Then $A^T A$ is a

(a) $m \times n$ matrix,
(b) $m \times m$ matrix,
(c) $n \times m$ matrix,
(d) $n \times n$ matrix.

MC 4. Let A be the edge-node incidence matrix of the directed graph below.



Let B be the edge-node incidence matrix of the directed graph below.



Then:

(a) $\text{Nul}(A) = \text{Nul}(B)$
(b) $\text{Nul}(A) \neq \text{Nul}(B)$, but $\dim \text{Nul}(A) = \dim \text{Nul}(B)$
(c) $\dim \text{Nul}(A) \neq \dim \text{Nul}(B)$, but $\text{Nul}(A) = \text{Nul}(B)$
(d) $\dim \text{Nul}(A) \neq \dim \text{Nul}(B)$ and $\text{Nul}(A) \neq \text{Nul}(B)$

MC 5. Let A, B be the edge-node incidence matrices of the directed graphs from the previous problem. Then:

(a) $\text{Nul}(A^T) = \text{Nul}(B^T)$
(b) $\text{Nul}(A^T) \neq \text{Nul}(B^T)$, but $\dim \text{Nul}(A^T) = \dim \text{Nul}(B^T)$
(c) $\dim \text{Nul}(A^T) \neq \dim \text{Nul}(B^T)$, but $\text{Nul}(A^T) = \text{Nul}(B^T)$
(d) $\dim \text{Nul}(A^T) \neq \dim \text{Nul}(B^T)$ and $\text{Nul}(A^T) \neq \text{Nul}(B^T)$

MC 6. Let A be an $n \times n$ matrix, let D be a diagonal matrix and let P be an invertible matrix such that $A = PDP^{-1}$. Which of the following statements is always true?

(a) $\text{Nul}(A) = \text{Nul}(D)$
(b) $\dim \text{Nul}(A) = \dim \text{Nul}(D)$
(c) $\dim \text{Nul}(A) \neq \dim \text{Nul}(D)$
(d) none of the above

MC 7. Let A be an $n \times n$ matrix and P be an invertible $n \times n$ matrix. Consider the following two statements:

(S1) A and PAP^{-1} have the same determinant.

(S2) A is invertible if and only if PAP^{-1} is invertible.

Are these statements always correct?

(a) Statement S1 and Statement S2 are correct.

(b) Only Statement S1 is correct.

(c) Only Statement S2 is correct.

(d) Neither Statement S1 nor Statement S2 is correct.

$\det(A) = \det(PAP^{-1})$
$= \det(P)\det(A)\det(P)^{-1}$
$= \det(A)$

MC 8. Let A be an $n \times n$ matrix, and let $\boldsymbol{v}_1, \boldsymbol{v}_2$ in $\mathbb{R}^n$ be such that $\boldsymbol{v}_1$ is an eigenvector of A to eigenvalue λ_1 and $\boldsymbol{v}_2$ is an eigenvector of A to eigenvalue λ_2.
Consider the following two statements:

(T1) If $\lambda_1 \neq \lambda_2$, then $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly independent.

(T2) If $\lambda_1 \neq \lambda_2$, then $\boldsymbol{v}_1, \boldsymbol{v}_2$ are orthogonal to each other.

Are these statements always correct?

(a) Statement T1 and Statement T2 are correct.

(b) Only Statement T1 is correct.

(c) Only Statement T2 is correct.

(d) Neither Statement T1 nor Statement T2 is correct.

MC 9. Let $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Let $\boldsymbol{w}_1$ be the orthogonal projection of $\boldsymbol{v}_1$ onto W, and let $\boldsymbol{w}_2$ be the orthogonal projection of $\boldsymbol{v}_2$ onto W. Then:

(a) $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(b) $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(c) $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(e) $\boldsymbol{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

MC 10. Let A be an $m \times n$ matrix. Which of the following is always correct?

(a) $\dim \mathrm{Nul}(A) = \dim \mathrm{Nul}(A^T)$
(b) $\dim \mathrm{Nul}(A) = n - m + \dim \mathrm{Nul}(A^T)$
(c) $\dim \mathrm{Nul}(A) = m - n + \dim \mathrm{Nul}(A^T)$
(d) $\dim \mathrm{Nul}(A) = m - \dim \mathrm{Nul}(A^T)$
(e) none of the above.

$$\dim \mathrm{Nul}(A) = n - \dim \mathrm{Col}\, A$$
$$= n - \dim \mathrm{Col}\, A^T$$
$$= n - (m - \mathrm{Nul}\, A^T)$$

MC 11. What is the dimension of span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$?

(a) 1
(b) 2
(c) 3
(d) 4
(e) none of the above

MC 12. Let B be a 3×3 matrix with eigenvalues $0, 1, 2$. What does that tell you about the rank of B?

(a) The rank of B is 0.
(b) The rank of B is 1.
(c) The rank of B is 2.
(d) The rank of B is 3.
(e) Nothing. (That is, the rank of B can be either $0, 1, 2$ or 3.)

MC 13. For what values of α is the system

$$\begin{aligned} x_1 + \alpha x_2 &= 0 \\ x_1 - x_3 &= 1 \\ x_1 + 2\alpha x_2 + x_3 &= 2 \end{aligned}$$

consistent?

(a) It is consistent for $\alpha = 0$.
(b) It is consistent for $\alpha = 1$.
(c) It is consistent for $\alpha = 2$.
(d) It is consistent for all values of α.
(e) It is always inconsistent.

$$\left[\begin{array}{ccc|c} 1 & \alpha & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 2\alpha & 1 & 2 \end{array}\right]$$
$$\downarrow$$
$$\left[\begin{array}{ccc|c} 1 & \alpha & 0 & 0 \\ 0 & -\alpha & -1 & 1 \\ 0 & \alpha & 1 & 2 \end{array}\right]$$
$$\downarrow$$
$$\left[\begin{array}{ccc|c} 1 & \alpha & 0 & 0 \\ 0 & -\alpha & -1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

MC 14. Let A be an $n \times n$ matrix. Which of the following is always correct?

(a) $\det(-A) = -\det(A)$,
(b) $\det(-A) = \det(A)$,
(c) $\det(-A) = (-1)^n \det(A)$,
(d) $\det(-A) = 0$,
(e) none of the above.

MC 15. Let $\mathbb{P}_n$ be the vector space of polynomials of degree up to n. Consider the following bases:

$$\overset{-\mathcal{B}}{1, 1+t, 1+t+t^2} \text{ of } \mathbb{P}_2, \text{ and } \overset{\mathcal{C}}{1, 1-t} \text{ of } \mathbb{P}_1.$$

With respect to these bases the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_1$ is represented by the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \quad T(t)_{\mathcal{C}} = A\, t_{\mathcal{B}} = A\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

What is $T(t)$?

$$\Rightarrow T(t) = 1-t$$

(a) $1+t$
(b) $1-t$
(c) t
(d) 1
(e) none of the above

MC 16. Let $T : \mathbb{P}_2 \to \mathbb{P}_1$ be the linear transformation from the previous problem. What is the matrix representing T with respect to the standard bases of $\mathbb{P}_2$ and $\mathbb{P}_1$?

(a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

(e) none of the above

$T(t)_{\mathcal{C}} = 1-t$
$T(1) = 1$

$T(t^2) = T(1+t+t^2) - T(1+t) = -T(1+t)$
$= -(1 + (1-t)) = -2+t$

MC 17. Consider the following two matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Which of the following is correct?

(a) $\mathrm{Col}(A) = \mathrm{Col}(B)$ and $\mathrm{Col}(A^T) = \mathrm{Col}(B^T)$
(b) $\mathrm{Col}(A) = \mathrm{Col}(B)$ and $\mathrm{Col}(A^T) \neq \mathrm{Col}(B^T)$
(c) $\mathrm{Col}(A) \neq \mathrm{Col}(B)$ and $\mathrm{Col}(A^T) = \mathrm{Col}(B^T)$
(d) $\mathrm{Col}(A) \neq \mathrm{Col}(B)$ and $\mathrm{Col}(A^T) \neq \mathrm{Col}(B^T)$

MC 18. Let W be the vector space of all polynomials $p(t)$ of degree up to 3 with the property that $p(7) = 0$. What is the dimension of W?

(a) 1
(b) 2
(c) 3
(d) 4
(e) W is not a vector space.

MC 19. Are the following two statements correct?

(S1) The set of polynomials $p(t)$ satisfying $p(0) = 0$ is a vector space.
(S1) The set of polynomials $p(t)$ satisfying $p'(0) = 0$ is a vector space.

Then:

(a) Statement S1 and Statement S2 are correct.
(b) Only Statement S1 is correct.
(c) Only Statement S2 is correct.
(d) Neither Statement S1 nor Statement S2 is correct.

MC 20. Consider the vector space of continuous functions $\mathbb{R} \to \mathbb{R}$, which are periodic with period 2π, together with the inner product

$$\langle f, g\rangle = \int_0^{2\pi} f(t)g(t)\mathrm{d}t.$$

Then the orthogonal projection of $f(t)$ onto $\operatorname{span}\{\sin(t), \sin(2t)\}$ is:

(a) $\dfrac{\int_0^{2\pi} f(t)\sin(t)\mathrm{d}t}{\int_0^{2\pi}\sin(t)^2\mathrm{d}t}\sin(t) + \dfrac{\int_0^{2\pi} f(t)\sin(2t)\mathrm{d}t}{\int_0^{2\pi}\sin(2t)^2\mathrm{d}t}\sin(2t)$

(b) $\dfrac{\int_0^{2\pi} f(t)\sin(t)\mathrm{d}t}{\int_0^{2\pi} f(t)^2\mathrm{d}t}\sin(t) + \dfrac{\int_0^{2\pi} f(t)\sin(2t)\mathrm{d}t}{\int_0^{2\pi} f(t)^2\mathrm{d}t}\sin(2t)$

(c) $\dfrac{\int_0^{2\pi} f(t)\sin(t)\mathrm{d}t}{\int_0^{2\pi} f(t)^2\mathrm{d}t}f(t) + \dfrac{\int_0^{2\pi} f(t)\sin(2t)\mathrm{d}t}{\int_0^{2\pi} f(t)^2\mathrm{d}t}f(t)$

(d) 0
(e) none of the above

MC 21. Consider the following two subsets of $\mathbb{R}^2$:

$$V_0 = \left\{\begin{bmatrix} a \\ b\end{bmatrix} : ab = 0\right\}, \quad V_1 = \left\{\begin{bmatrix} a \\ b\end{bmatrix} : ab = 1\right\}$$

Then:

(a) V_0 and V_1 are vector spaces.
(b) Only V_0 is a vector space.
(c) Only V_1 is a vector space.
(d) Neither V_0 nor V_1 is a vector space.

MC 22. Suppose that $\hat{\boldsymbol{x}}$ is a least squares solution of the linear system $A\boldsymbol{x} = \boldsymbol{b}$. Which of the following is always true?

(a) $A\hat{\boldsymbol{x}} = \boldsymbol{b}$
(b) $A^T A\hat{\boldsymbol{x}} = \boldsymbol{b}$
(c) $A\hat{\boldsymbol{x}} - \boldsymbol{b}$ is orthogonal to Col(A)
(d) $A\hat{\boldsymbol{x}} - \boldsymbol{b}$ is orthogonal to $\boldsymbol{b}$
(e) none of the above

MC 23. Which of the following sets of polynomials is linearly dependent?

(a) $\{1, t, t^2\}$
(b) $\{1 + t, t, t^2\}$
(c) $\{1 + t, 1 + t^2, t^2\}$
(d) $\{1 + t, 1 + t + t^2, t + t^2\}$
(e) none of the above

MC 24. Consider the following two matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Are the following two statements correct?

(S1) A is the projection matrix corresponding to orthogonal projection onto Col(A).
(S2) B is the projection matrix corresponding to orthogonal projection onto Col(B).

Then:

(a) Statement S1 and Statement S2 are correct.
(b) Only Statement S1 is correct.
(c) Only Statement S2 is correct.
(d) Neither Statement S1 nor Statement S2 is correct.

MC 25. Consider $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Is the vector $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the row spaces of these matrices?

(a) $\boldsymbol{v}$ is in Col(A^T) but not in Col(B^T).
(b) $\boldsymbol{v}$ is in Col(B^T) but not in Col(A^T).
(c) $\boldsymbol{v}$ is in Col(A^T) and in Col(B^T).
(d) $\boldsymbol{v}$ is neither in Col(A^T) nor in Col(B^T).

MC 26. Let A be a 3×7 matrix, and $\boldsymbol{x}$ a vector in $\mathbb{R}^7$ such that $A\boldsymbol{x} = \mathbf{0}$. Then the vector $\boldsymbol{x}$ is always orthogonal to:

(a) $\mathrm{Col}(A)$
(b) $\mathrm{Col}(A^T)$
(c) $\mathrm{Nul}(A)$
(d) $\mathrm{Nul}(A^T)$

MC 27. If A is a 64×17 matrix with rank 11, then what is the largest number of linearly independent vectors $\boldsymbol{x}$ satisfying $A\boldsymbol{x} = \mathbf{0}$ that one can find?

(a) 6
(b) 11
(c) 17
(d) 53
(e) none of the above

64

12

rank = 11 $\Rightarrow$ 11 pivot cols.
$\Rightarrow$ 6 non-pivot cols.
$\Rightarrow$ dim Nul A = 6.

MC 28. Consider the following two subsets of the space of 2×2 matrices.

$$V_0 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = 0 \right\}, \quad V_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 0 \text{ is an eigenvalue of } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}$$

Then:

(a) V_0 and V_1 are vector spaces.
(b) Only V_0 is a vector space.
(c) Only V_1 is a vector space.
(d) Neither V_0 nor V_1 is a vector space.

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{in } V_1} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{not in } V_1}$$