Bivariate Change of Variable

i) Find $X_1 = w_1(y_1, y_2)$

$X_2 = w_2(y_1, y_2)$

ii) $J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$

iii) $g(y_1, y_2) = f(w_1(y_1, y_2), w_2(y_1, y_2)) \, |J|$

Need support for Y_1 & Y_2

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Random Variable Transformation Problems

1. Let X_1 and X_2 have the joint pdf,

$$f(x_1,x_2) = 15x_1x_2^2, 0 < x_2 < x_1 < 1$$

a. Find the joint support and pdf for $Y_1 = X_1X_2$ and $Y_2 = \frac{X_1}{X_2}$.

i) $X_1 = w_1(y_1,y_2) = \sqrt{Y_1Y_2}$

$X_2 = w_2(y_1,y_2) = \sqrt{\frac{Y_1}{Y_2}}$

ii) $J = \begin{vmatrix} \frac{1}{2}\sqrt{\frac{Y_2}{Y_1}} & \frac{1}{2}\sqrt{\frac{Y_1}{Y_2}} \\ \frac{1}{2}\sqrt{\frac{1}{Y_1Y_2}} & -\frac{1}{2}\frac{\sqrt{Y_1}}{Y_2^{3/2}} \end{vmatrix}$

$= -\frac{1}{4}\frac{1}{Y_2} - \frac{1}{4}\frac{1}{Y_2} = -\frac{1}{2}\frac{1}{Y_2}$

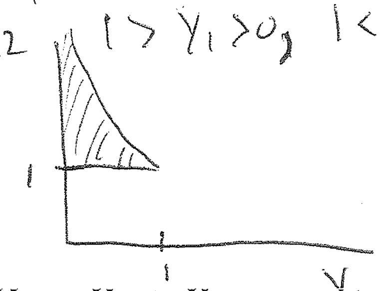
iii) $g(y_1,y_2) = 15\sqrt{y_1y_2}\left(\sqrt{\frac{y_1}{y_2}}\right)^2\left|-\frac{1}{2}\frac{1}{y_2}\right| = \frac{15}{2}\left(\frac{Y_1}{Y_2}\right)^{3/2}$, $1 > Y_1 > 0$, $1 < Y_2 < \frac{1}{Y_1}$

Need Support

a) $X_2 > 0 \Rightarrow \sqrt{\frac{Y_1}{Y_2}} > 0 \Rightarrow \boxed{Y_1 > 0}$

b) $X_1 > X_2$, $\sqrt{Y_1Y_2} > \sqrt{\frac{Y_1}{Y_2}}$, $\boxed{Y_2 > 1}$

c) $X_1 < 1 \Rightarrow Y_1Y_2 < 1 \Rightarrow \boxed{Y_2 < \frac{1}{Y_1}}$



b. Find the joint support for $Y_1 = \frac{X_2}{X_1}$ and $Y_2 = X_1 + X_2$.

i) $X_1 = \frac{Y_2}{1+Y_1}$

$X_2 = \frac{Y_1Y_2}{1+Y_1}$

ii) $J = \begin{vmatrix} \frac{-Y_2}{(1+Y_1)^2} & \frac{1}{1+Y_1} \\ \frac{Y_2(1+Y_1) - Y_1Y_2}{(1+Y_1)^2} & \frac{Y_1}{1+Y_1} \end{vmatrix} = \begin{vmatrix} -\frac{Y_2}{(1+Y_1)^2} & \frac{1}{1+Y_1} \\ \frac{Y_2}{(1+Y_1)^2} & \frac{Y_1}{1+Y_1} \end{vmatrix}$

$= -\frac{Y_1Y_2}{(1+Y_1)^3} - \frac{Y_2}{(1+Y_1)^3} = \frac{-Y_2}{(1+Y_1)^2}$

iii) $g(y_1,y_2) = \frac{Y_1^2Y_2^4\,15}{(1+Y_1)^5} = 15\left(\frac{Y_2}{1+Y_1}\right)\left(\frac{Y_1Y_2}{1+Y_1}\right)^2\left|\frac{Y_2}{(1+Y_1)^2}\right|$

Support for Y_1, Y_2 $\quad 0 < X_2 < X_1 < 1$

$X_2 > 0$

$\Rightarrow \frac{Y_1 Y_2}{1+Y_1} > 0 \Rightarrow Y_1 Y_2 > 0$ $\quad \boxed{\begin{matrix} Y_1 > 0 \\ Y_2 > 0 \end{matrix}}$

$X_1 > X_2$

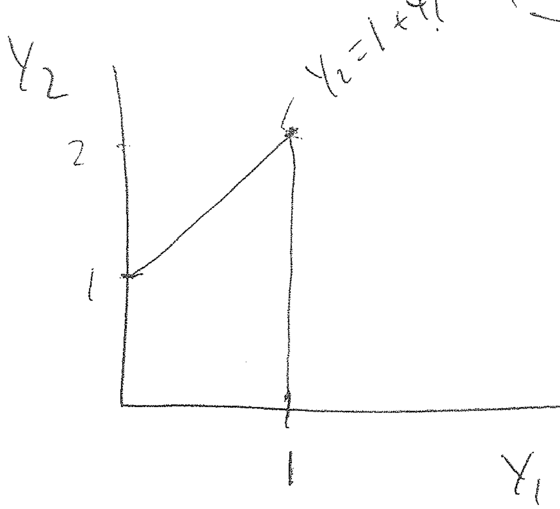
$\frac{Y_2}{1+Y_1} > \frac{Y_1 Y_2}{1+Y_1}$

$\boxed{1 > Y_1}$

$X_1 < 1$

$\frac{Y_2}{1+Y_1} < 1$

$\Rightarrow \boxed{Y_2 < 1 + Y_1}$



$$f(y_2) = \begin{cases} \int_0^1 \frac{15 y_1^2 y_2^4}{(1+y_1)^5}\, dy_1 & 0 < y_2 < 1 \\ \int_{y_2-1}^1 \frac{15 y_1^2 y_2^4}{(1+y_1)^5}\, dy_1 & 1 < y_2 < 2 \end{cases}$$

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Random Variable Transformation Problems

2. Let X and Y have the joint pdf,

$$f(x,y) = 60x^2y, x > 0, y > 0, x + y < 1$$

a. Find the joint support and pdf for $U = XY$ and $V = X$.

i) $X = V$

$Y = \frac{U}{V}$

ii) $J = \begin{vmatrix} 0 & 1 \\ \frac{1}{V} & -\frac{U}{V^2} \end{vmatrix} = -\frac{1}{V}$

iii) $g(y_1, y_2) = 60v^2 \frac{u}{v} \left|-\frac{1}{v}\right| = 60u, \quad 0 < u < v(1-v), \quad 0 < v < 1$

$X > 0 \Rightarrow \boxed{V > 0}$

$Y > 0 \Rightarrow \frac{U}{V} > 0$

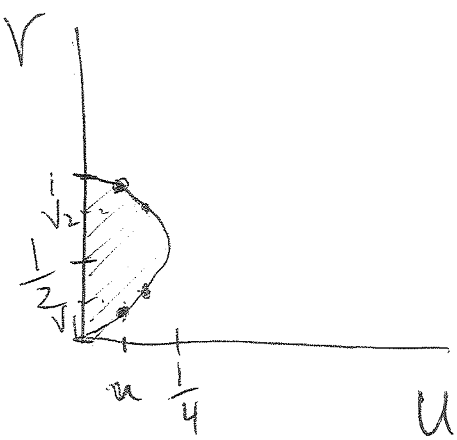
$\Rightarrow \boxed{U > 0}$

$X + Y < 1$

$V + \frac{U}{V} < 1$

$V^2 + U < V$

$\boxed{U < V(1-V)}$



~~$u = v(v-$~~

$u = v(1-v)$

$v^2 - v + u = 0$

b. Find the marginal pdf of U, $f_U(u)$.

$f_U(u) = \int_{v_1}^{v_2} 60u \, dv = 60u(v_2 - v_1), \quad 0 < u < \frac{1}{4}$

$= 60u\sqrt{1-4u}$

$v_1, v_2 = \frac{1 \pm \sqrt{1-4u}}{2}$

$v_2 = \frac{1 + \sqrt{1-4u}}{2}$

$v_1 = \frac{1 - \sqrt{1-4u}}{2}$

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Random Variable Transformation Problems

3. Let X_1, X_2, and X_3 be iid each with pdf $f(x) = e^{-x}, x > 0$, zero elsewhere. Show that

$$Y_1 = \frac{X_1}{X_1 + X_2}, Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}, Y_3 = X_1 + X_2 + X_3$$

are mutually independent.

i) We see $Y_2 Y_3 = X_1 + X_2 \Rightarrow$ $\quad X_1 = Y_1 Y_2 Y_3$

$Y_1 Y_2 Y_3 = X_1$ $\quad X_2 = (1-Y_1) Y_2 Y_3$

$\quad X_3 = (1-Y_2) Y_3$

ii) $$J = \begin{vmatrix} Y_2 Y_3 & Y_1 Y_3 & Y_1 Y_2 \\ -Y_2 Y_3 & (1-Y_1) Y_3 & (1-Y_1) Y_2 \\ 0 & -Y_3 & (1-Y_2) \end{vmatrix}$$

$$= 0 \begin{vmatrix} Y_1 Y_3 & Y_1 Y_2 \\ (1-Y_1) Y_3 & (1-Y_1) Y_2 \end{vmatrix} - (-Y_3) \begin{vmatrix} Y_2 Y_3 & Y_1 Y_2 \\ -Y_2 Y_3 & (1-Y_1) Y_2 \end{vmatrix} + (1-Y_2) \begin{vmatrix} Y_2 Y_3 & Y_1 Y_3 \\ -Y_2 Y_3 & (1-Y_1) Y_3 \end{vmatrix}$$

$$= Y_3 \left((1-Y_1) Y_2^2 Y_3 + Y_1 Y_2^2 Y_3 \right) + (1-Y_2) \left((1-Y_1) Y_2 Y_3^2 + Y_1 Y_2 Y_3^2 \right)$$

$$= Y_2^2 Y_3^2 + (1-Y_2) Y_2 Y_3^2 = Y_2 Y_3^2$$

iii) Note $f(x_1, x_2, x_3) = e^{-(x_1 + x_2 + x_3)}$, $x_1 > 0, x_2 > 0, x_3 > 0$

$g(y_1, y_2, y_3) = e^{-y_3} |J| = e^{-y_3} y_2 y_3^2$, $0 < y_1 < 1, 0 < y_2 < 1$
$0 < y_3 < \infty$

$g_1(y_1) = 1, \quad 0 < y_1 < 1$

$g_2(y_2) = 2y_2, \quad 0 < y_2 < 1$

$g_3(y_3) = \frac{1}{2} y_3^2 e^{-y_3}, \quad y_3 > 0$

Hence $g_1(y_1) g_2(y_2) g_3(y_3) = g(y_1, y_2, y_3)$ so Y_1, Y_2, Y_3 are independent.

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Random Variable Transformation Problems

4. Let X_1 and X_2 be iid with pdf $f(x) = e^{-x}, x > 0$, zero elsewhere.
   Let $Y_1 = X_1 - X_2, Y_2 = X_1 + X_2$.
   a. Find the joint pdf of Y_1 and Y_2.

i) $x_1 = \dfrac{y_1 + y_2}{2}$

$x_2 = \dfrac{y_2 - y_1}{2}$

ii) $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

iii) $g(y_1, y_2) = e^{-x_1} e^{-x_2} \left|\frac{1}{2}\right| = \cancel{\frac{1}{2} e^{-\frac{1}{2} 2 y_2}}\ \frac{1}{2} e^{-y_2}, \quad \underline{|y_1| < y_2}$

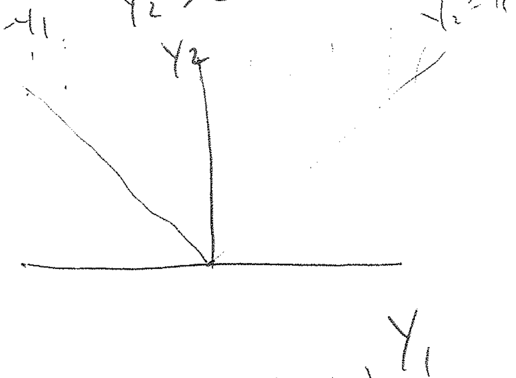
Need support for Y_1 & Y_2

$Y_2 > 0$

$X_1 > 0 \qquad X_2 > 0$

$\Rightarrow \dfrac{y_1 + y_2}{2} > 0 \qquad \Rightarrow \dfrac{y_2 - y_1}{2} > 0 \qquad \boxed{-y_2 < y_1 < y_2}$

$y_2 > -y_1 \qquad \Rightarrow \boxed{y_2 > y_1} \qquad |y_1| < y_2$



   b. Find the marginal pdfs of Y_1 and Y_2.

$f(y_1) = \displaystyle\int_{|y_1|}^{\infty} \frac{1}{2} e^{-y_2}\, dy_2 = \frac{1}{2}\left(-1 e^{-y_2}\right)\Big|_{|y_1|}^{\infty} = \frac{1}{2}\left(0 - -e^{-|y_1|}\right) = \underline{\frac{1}{2} e^{-|y_1|}}, \quad -\infty < y_1 < \infty$

$f(y_2) = \frac{1}{2} e^{-y_2} \displaystyle\int_{-y_2}^{y_2} dy_1 = \frac{1}{2} e^{-y_2}(y_2 + y_2) = \underline{y_2 e^{-y_2}}, \quad y_2 > 0$

$\Gamma(\alpha = 2, \lambda = 1)$