STAT 420 Fall 2012

Homework #11

(due Friday, December 7, by 3:00 p.m.)

1. Consider the $\text{AR}(2)$ processes

$$\dot{\text{Y}}_t - 0.3\ \dot{\text{Y}}_{t-1} - 0.1\ \dot{\text{Y}}_{t-2} = e_t$$

where $\{e_t\}$ is zero-mean white noise $(\text{i.i.d. N}(0, \sigma_e^2))$, $\dot{\text{Y}}_t = \text{Y}_t - \mu$.

Based on a series of length $N = 100$, we observe $\ldots,\ y_{98} = 152,\ y_{99} = 156,$ $y_{100} = 147,\ \overline{y} = 150$. Forecast y_{101} and y_{102}.

$$\text{Y}_{N+1} = \mu + \phi_1(\text{Y}_N - \mu) + \phi_2(\text{Y}_{N-1} - \mu) + e_{N+1}$$

$$\hat{y}_{N+1} = \text{E}_N(\text{Y}_{N+1}) = \mu + \phi_1(y_N - \mu) + \phi_2(y_{N-1} - \mu)$$

$$\text{Y}_{N+2} = \mu + \phi_1(\text{Y}_{N+1} - \mu) + \phi_2(\text{Y}_N - \mu) + e_{N+2}$$

$$\hat{y}_{N+2} = \text{E}_N(\text{Y}_{N+2}) = \mu + \phi_1(\hat{y}_{N+1} - \mu) + \phi_2(y_N - \mu)$$

$$\hat{y}_{101} = \hat{\mu} + \hat{\phi}_1(y_{100} - \hat{\mu}) + \hat{\phi}_2(y_{99} - \hat{\mu})$$

$$= 150 + 0.3(147 - 150) + 0.1(156 - 150) = \mathbf{149.7}.$$

$$\hat{y}_{102} = \hat{\mu} + \hat{\phi}_1(\hat{y}_{101} - \hat{\mu}) + \hat{\phi}_2(y_{100} - \hat{\mu})$$

$$= 150 + 0.3(149.7 - 150) + 0.1(147 - 150) = \mathbf{149.61}.$$

2. Consider the AR (2) process

$$Y_t = \mu + \phi_1 (Y_{t-1} - \mu) + \phi_2 (Y_{t-2} - \mu) + e_t$$

Based on a series of length $N = 60$, we observe $\ldots,\ y_{59} = 190,\ y_{60} = 215,\ \overline{y} = 200.$

a) Suppose $r_1 = 0.40,\ r_2 = -0.26$. Use Yule-Walker equations to estimate ϕ_1 and ϕ_2.

The Yule-Walker equations for an AR (2) process are given by:

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$
$$\rho_2 = \phi_1 \rho_1 + \phi_2$$

$0.40 = \phi_1 + 0.40\ \phi_2 \qquad \times 5 \qquad 2 = 5\ \phi_1 + 2\ \phi_2$

$-0.26 = 0.40\ \phi_1 + \phi_2 \qquad \times 2 \qquad -0.52 = 0.80\ \phi_1 + 2\ \phi_2$

$\Rightarrow \quad 2.52 = 4.2\ \phi_1 \qquad \Rightarrow \quad \hat{\phi}_1 = {}^{2.52}\!/_{4.2} = \mathbf{0.60}$

$\Rightarrow \quad 0.40 = 0.60 + 0.40\ \phi_2 \qquad -0.20 = 0.40\ \phi_2$

$\Rightarrow \quad \hat{\phi}_2 = \mathbf{-0.50}.$

b) If ϕ_1 and ϕ_2 are equal to your answers to part (a), is this process stationary?

$\dot{Y}_t - 0.60\ \dot{Y}_{t-1} + 0.50\ \dot{Y}_{t-2} = e_t$

$\Phi(z) = 1 - 0.60\, z + 0.50\, z^2 = 0.$

Roots $z_{1,2} = \dfrac{0.60 \pm \sqrt{0.60^2 - 4 \cdot 1 \cdot 0.50}}{2 \cdot 0.50} = 0.60 \pm \sqrt{-1.64} = 0.60 \pm i\sqrt{1.64}$

are outside of the unit circle: $\left| z_{1,2} \right|^2 = 0.60^2 + \left(\sqrt{1.64}\right)^2 = 2 > 1.$

$\Rightarrow$ This process is stationary.

OR

An AR(2) model is stationary if

$$-1 < \phi_2 < 1, \qquad \phi_2 + \phi_1 < 1, \qquad \phi_2 - \phi_1 < 1.$$

$$-1 < -0.50 < 1, \qquad -0.50 + 0.60 < 1, \qquad -0.50 - 0.60 < 1.$$

$\Rightarrow$ This process is stationary.

c) Use your answers to part (a) to forecast y_{61}, y_{62}, and y_{63}.

$y_{59} = 190$, $y_{60} = 215$, $\overline{y} = 200$.

$$\hat{y}_{61} = \hat{\mu} + \hat{\phi}_1 (y_{60} - \hat{\mu}) + \hat{\phi}_2 (y_{59} - \hat{\mu})$$

$$= 200 + 0.60 (215 - 200) - 0.50 (190 - 200) = \mathbf{214}.$$

$$\hat{y}_{62} = \hat{\mu} + \hat{\phi}_1 (\hat{y}_{61} - \hat{\mu}) + \hat{\phi}_2 (y_{60} - \hat{\mu})$$

$$= 200 + 0.60 (214 - 200) - 0.50 (215 - 200) = \mathbf{200.9}.$$

$$\hat{y}_{63} = \hat{\mu} + \hat{\phi}_1 (\hat{y}_{62} - \hat{\mu}) + \hat{\phi}_2 (\hat{y}_{61} - \hat{\mu})$$

$$= 200 + 0.60 (200.9 - 200) - 0.50 (214 - 200) = \mathbf{193.54}.$$

3. Consider the ARMA$(1,1)$ model

$$(Y_t - 60) + 0.3\,(Y_{t-1} - 60) = e_t - 0.4\,e_{t-1}$$

which was fitted to a time series where the last 10 values are

60, 57, 52, 59, 62, 59, 63, 67, 61, 58

and the last residual is $\hat{e}_{\mathrm{N}} = -2$.

Calculate the forecasts of the next two observations, and indicate how forecasts can be calculated for lead times greater than two. Show what happens to the forecasts as the lead time becomes arbitrarily large.

$$(Y_{N+1} - 60) \;=\; -0.3\,(Y_N - 60) + e_{N+1} - 0.4\,e_N$$

$$\hat{y}_{N+1} \;=\; E_N(Y_{N+1}) \;=\; 60 - 0.3\,(y_N - 60) - 0.4\,\hat{e}_N$$

$$=\; 60 - 0.3\,(58 - 60) - 0.4\,(-2) \;=\; \mathbf{61.4}.$$

$$(Y_{N+2} - 60) \;=\; -0.3\,(Y_{N+1} - 60) + e_{N+2} - 0.4\,e_{N+1}$$

$$\hat{y}_{N+2} \;=\; E_N(Y_{N+2}) \;=\; 60 - 0.3\,(\,\hat{y}_{N+1} - 60\,)$$

$$=\; 60 - 0.3\,(61.4 - 60) \;=\; \mathbf{59.58}.$$

$$(Y_{N+l} - 60) \;=\; -0.3\,(Y_{N+l-1} - 60) + e_{N+l} - 0.4\,e_{N+l-1}$$

$l > 2$ $\qquad \hat{y}_{N+l} - 60 \;=\; -0.3\,(\,\hat{y}_{N+l-1} - 60\,)$

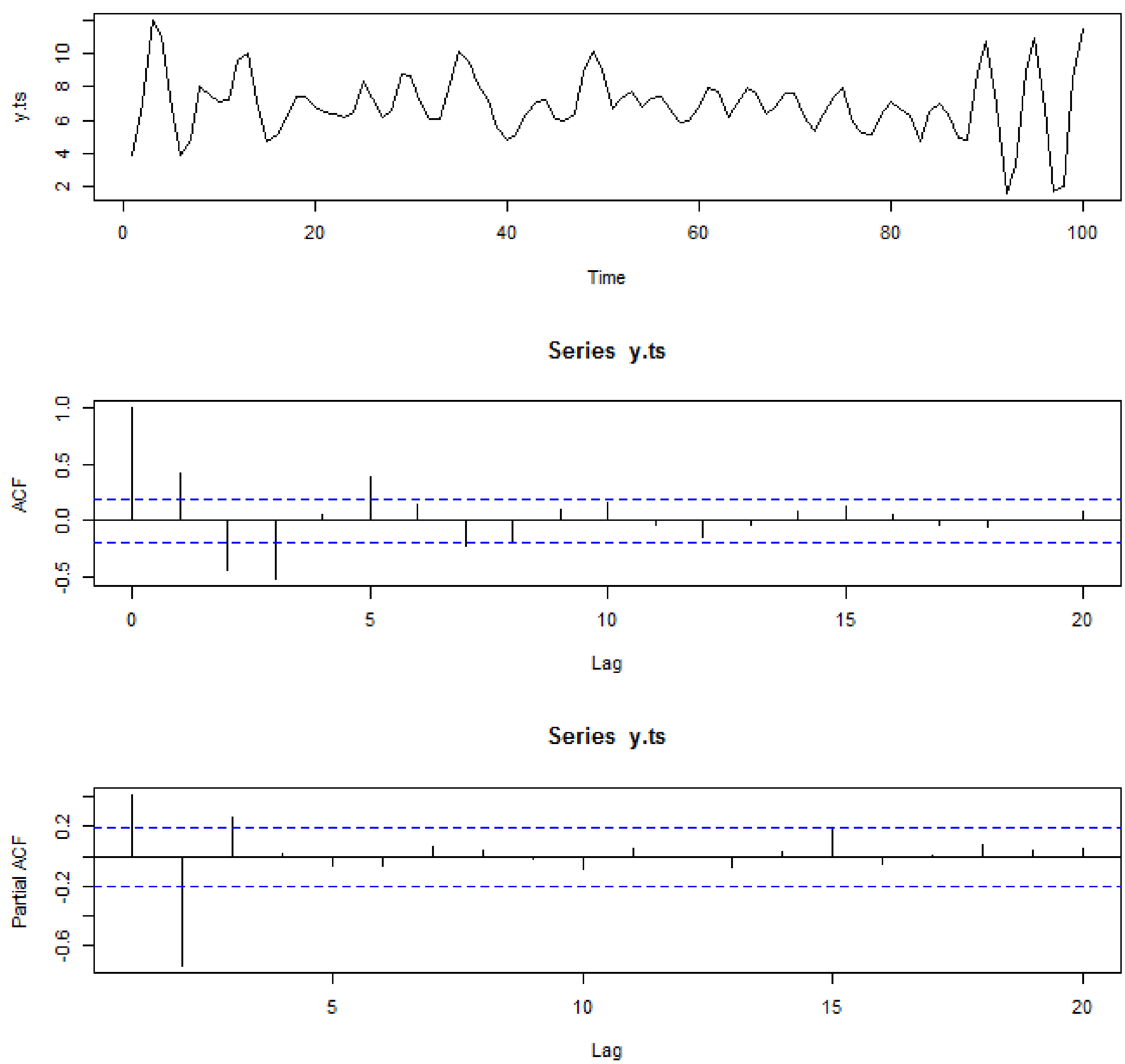
$$\Rightarrow \qquad \hat{y}_{N+l} - 60 \;=\; (-0.3)^{l-1}\,(\,\hat{y}_{N+1} - 60\,)$$

$$\Rightarrow \qquad \hat{y}_{N+l} \;=\; 60 + 1.4 \times (-0.3)^{l-1}$$

$$\Rightarrow \qquad \hat{y}_{N+l} \;\to\; 60 \quad \text{as } l \to \infty$$

4. `y.dat` contains the price of *Initech* stock at the end of the week for the last 100 weeks.

```
> y = scan(" ... /y.dat")
Read 100 items
> y.ts = ts(y)
> par(mfrow=c(3,1))
> plot(y.ts)
> acf(y.ts)
> pacf(y.ts)
```



a) Based on the sample ACF and PACF, what is an appropriate model?

That is, suggest appropriate value for p and q for ARMA(p, q).

Include the plots and justify your choice for p and q.

Three "big" spikes on PACF plot. Suggest $p = 3$.

Three "big" spikes on ACF plot (after lag 0). Suggest $q = 3$.
(The fifth spike is also "big")

ARMA$(3, 3)$.

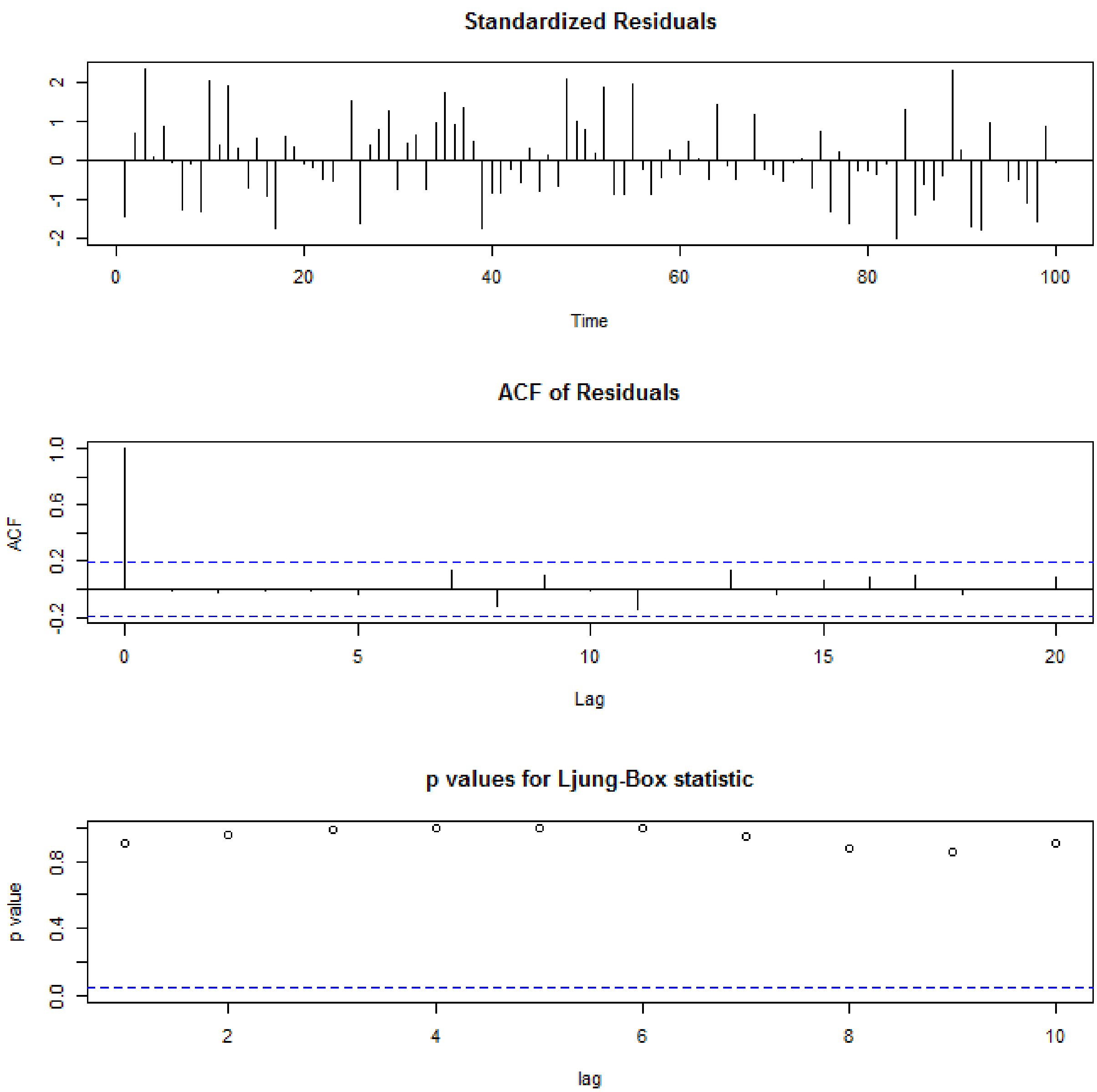
Other answers are possible too.

b) Fit the model from part (a).

```
> fit33 = arima(y.ts,order=c(3,0,3))
> fit33

Call:
arima(x = y.ts, order = c(3, 0, 3))

Coefficients:
         ar1      ar2      ar3     ma1     ma2     ma3  intercept
      0.4361  -0.8585  -0.0799  0.6479  0.6315  0.3019     6.9137
s.e.  0.5344   0.2942   0.4581  0.5243  0.3096  0.3244     0.1536

sigma^2 estimated as 0.8002:  log likelihood = -132.59,  aic = 281.18
```

```
>tsdiag(fit33)
```

Based on the diagnostics plots, is the model from part (a) an appropriate model? Justify your answer.



Residuals do not appear to have any noticeable patterns, seem to be “random”, and behave like “white noise”.

This suggests that ARMA (3, 3) is indeed an appropriate model.

c) Forecast the price of *Initech* stock at the end of the next three weeks.

```
predfit = predict(fit,n.ahead=3)
predfit
```

If we following the famous "buy low, sell high" principle, should we plan to buy or sell *Initech* stock at the end of next week? Should we plan to buy or sell *Initech* stock two weeks from now?

```
> predfit33 = predict(fit33,n.ahead=3)
> predfit33
$pred
Time Series:
Start = 101
End = 103
Frequency = 1
[1] 8.065270 3.589059 4.101339

$se
Time Series:
Start = 101
End = 103
Frequency = 1
[1] 0.8945316 1.3192266 1.3374066
```

We should be prepared to sell *Initech* stock at the end of next week.

We should be prepared to buy *Initech* stock two weeks from now.