Math 415 - Lecture 6

Elementary Matrices, LU Decomposition

Friday September 4th 2015

Textbook: Chapter 1.4, 1.5

Suggested Practice Exercise: Chapter 1.4 Exercise 22, 27, Chapter 1.5: 4, 5, 11, 23, 29

Khan Academy Video: Matrix multiplication (part I), Matrix multiplication (part II), Defined and undefined matrix operations

Review of matrix multiplication

- **Matrix multiplication is linear combination**: $A\mathbf{x}$ is a linear combination of the columns of A with weights given by the entries of $\mathbf{x}$.

  Example 1.

$$Ax = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

- **Linear Combination is Linear System**

  Example 2.

$$x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \leftrightarrow \begin{aligned} x_1 + 2x_2 + 3x_3 &= -2 \\ 4x_1 + (-1)x_2 + 0x_3 &= 4 \end{aligned}$$

- $A\mathbf{x} = \mathbf{b}$ is the matrix form of the linear system with augmented matrix $\begin{bmatrix} A & | & \mathbf{b} \end{bmatrix}$.

  Example 3.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \leftrightarrow \begin{aligned} x_1 + 2x_2 + 3x_3 &= -2 \\ 4x_1 + (-1)x_2 + 0x_3 &= 4 \end{aligned}$$

$$\leftrightarrow \begin{bmatrix} 1 & 2 & 3 & | & -2 \\ 4 & -1 & 0 & | & 4 \end{bmatrix}$$

- Each column of AB is a linear combination of the columns of A with weights given by the corresponding column of B: $AB = A \begin{bmatrix} \mathbf{b_1} & \ldots & \mathbf{b_p} \end{bmatrix} = \begin{bmatrix} A\mathbf{b_1} & \ldots & A\mathbf{b_p} \end{bmatrix}$

  Example 4. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$, then

$$AB = \begin{bmatrix} A \begin{bmatrix} 3 \\ 0 \end{bmatrix} & A \begin{bmatrix} 4 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} & 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 8 \\ 6 & 10 \end{bmatrix}$$

- Row-column rule: The ij-th entry of AB is $a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$.

  Example 5. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$, then the 22 entry of AB is

$$AB_{22} = \begin{bmatrix} & \\ 2 & 1 \end{bmatrix} \begin{bmatrix} & 4 \\ & 2 \end{bmatrix} = 2 \times 4 + 1 \times 2 = 10$$

- Matrix multiplication is not commutative: usually, $AB \neq BA$.

Powers of A

Powers of A

We write: $A^k = A \cdots A$, k-times.

For which matrices A does this make sense? If A is $m \times n$ what can m, n be?

Example 6.

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 21 & 8 \end{bmatrix}$$

Calculating high powers of large matrix is hard. We will later learn a clever way to do this efficiently.

Transpose

Definition. If A is $m \times n$, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A. In terms of matrix elements $(A^T)_{ij} = A_{ji}$.

Example 7. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \end{bmatrix}$.

Then $A^T = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 7 & 6 \\ 3 & 8 & 5 \\ 4 & 9 & 4 \\ 5 & 8 & 3 \end{bmatrix}$

Example 8. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute AB, $(AB)^T$, $A^T B^T$ and $B^T A^T$.

Solution.

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 10 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 1 & 1 \\ 4 & 10 \end{bmatrix}$$

$$A^T B^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 10 \end{bmatrix}$$

Conclusion

The transpose of a product is the product of transposes **IN OPPOSITE ORDER**:

$$(AB)^T = B^T A^T$$

Definition. A is **symmetric** if $A = A^T$.

Example 9. Which of these is symmetric?

$$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}^T =$$

$$\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}^T =$$

Theorem 1. *Let A and B denote matrices whose sizes are appropriate for the following sums and products.*

(a) $\left(A^T\right)^T = A$,

(b) $(A+B)^T = A^T + B^T$

(c) For any scalar r, $(rA)^T = rA^T$

(d) $(AB)^T = B^T A^T$

Example 10. Prove that $(ABC)^T = C^T B^T A^T$.

Solution. By part d of the Theorem,$(ABC)^T = (A(BC))^T = (BC)^T A^T = C^T B^T A^T$.

Elementary matrices

Definition. The $n \times n$ **identity matrix** I_n has all entries 0, except on the main diagonal where the entries are 1. For example

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition. An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix. A **permutation matrix** is one that is obtained by performing row exchanges on an identity matrix.

Example 11. Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$.

E_1, E_2, and E_3 are elementary matrices. Why? Are there any permutation matrices?

Solution. Observe the following products and describe how these products can be obtained by elementary row operations on A.

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g+3a & h+3b & i+3c \end{bmatrix}$$

Theorem 2. *If an elementary row operation is performed on an* $m \times n$*- matrix* A*, the resulting matrix can be written as* EA*, where the* $m \times m$*-matrix* E *is created by performing the same row operations on* I_m*.*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We write this as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

More on inverses soon.

Remark. Elementary matrices are *invertible* because row operations are *reversible*. To determine the inverse of an elementary matrix E, determine the elementary row operation needed to transform E back into I and apply this operation to I to find the inverse.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Example 12. Compute the following products using the row or column interpretation of matrix multiplication. (Don't just use the row-column rule.)

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ -2 & 1 & -1 \end{bmatrix}$$