Math 415 - Lecture 12

Linear independence

Monday September 21st 2015

Textbook reading: Section 2.3

Suggested practice exercises: Section 2.3: 1, 2, 3, 4, 5,7, 8, 9

Khan Academy video: Introduction to Linear Independence, More on linear independence, Span and Linear Independence Example,

Strang lecture: Independence, Basis, and Dimension

1 Linear independence

Review.

- Span $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_m}\}$ is the set of all linear combinations

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_m\mathbf{v_m}.$$

- Span $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_m}\}$ is a vector space.
- $\mathrm{Col}(A) = \mathrm{Span}(\mathbf{a_1}, \mathbf{a_2}, \dots, \mathbf{a_n})$, if $A = \begin{bmatrix}\mathbf{a_1} & \mathbf{a_2} & \dots & \mathbf{a_n}\end{bmatrix}$. In this case $\mathbf{b} \in \mathrm{Col}(A) \iff \mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$.

Today we want to think how *big* the span of a bunch of vectors is. Is it a line, or a plane or ....

Example 1. Is $\operatorname{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$ equal to $\mathbb{R}^2$?

Solution.

Example 2. Is $\operatorname{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix}\right\}$ equal to $\mathbb{R}^3$?

Solution.

- What went wrong? Well, the three vectors that span satisfy a *relation*:

- Hence, $\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

- We are going to say that the three vectors are **linearly dependent** because they satisfy the (non trivial) relation

$$-3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \mathbf{0}.$$

Definition. Vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are said to be **linearly independent** if the equation

Likewise, $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are said to be **linearly dependent**

This is called a *non trivial relation* (when not all coefficient are zero.)

Example 3.

- Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ independent?
- If possible, find a linear dependence relation among them.

Solution.

2 Linear independence of matrix columns

- Note that a linear dependence relation, such as

$$3\begin{bmatrix}1\\1\\1\end{bmatrix} - 2\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}-1\\1\\3\end{bmatrix} = \mathbf{0},$$

can be written in matrix form as

- Hence, each linear dependence relation among the columns of a matrix A corresponds to a solution to $A\mathbf{x} = \mathbf{0}$. The Null space determines (in)dependence!

Theorem 1. *Let A be an $m \times n$ matrix.*
The columns of A are linearly independent.
$\iff$ $A\mathbf{x} = \mathbf{0}$ *has only the solution* $\mathbf{x} = \mathbf{0}$.
$\iff$ $Nul(A) = \{\mathbf{0}\}$
$\iff$ *A has n pivots.* $\iff$ *there are no free variables for* $A\mathbf{x} = \mathbf{0}$.

Example 4. Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ independent?

Solution.

Example 5. Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ independent?

Solution.

3 Special cases

- A set of a single non-zero vector $\{\mathbf{v_1}\}$ is always linearly independent.

  Why?

- A set of two vectors $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent if and only if neither of the vectors is a multiple of the other.

  Why?

- A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ containing the zero vector is linearly dependent.

Why?

- If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. In other words:

Any set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ of vectors in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Why?

Example 6. Let $A = \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$ be a two by three matrix. We want to count the free variables for $A\mathbf{x} = \mathbf{0}$. How many pivots can there be? How many free variables? Are the columns of A independent?

Solution.

4 Additional exercises

With the least amount of work possible, decide which of the following sets of vectors are linearly independent.

(a) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$

Linearly independent, because the two vectors are not multiples of each other.

(b) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

Linearly independent, because it is a single non-zero vector.

(c) Columns of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \end{bmatrix}$

Linearly dependent, because these are more than 3 (namely, 4) vectors in $\mathbb{R}^3$.

(d) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Linearly dependent, because the set includes the zero vector.