Math 415 - Lecture 6

Elementary Matrices, LU Decomposition

Friday September 4th 2015

Textbook: Chapter 1.4, 1.5

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Suggested Practice Exercise: Chapter 1.4 Exercise 22, 27, Chapter 1.5: 4, 5, 11, 23, 29

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Khan Academy Video: Matrix multiplication (part I), Matrix multiplication (part II), Defined and undefined matrix operations

Review of Matrix Multiplication

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- **Linear Combination is Linear System**

Example

$$x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \leftrightarrow$$

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$$= \begin{bmatrix}3 & 8\\ 6 & 10\end{bmatrix}$$

- Row-column rule: The ij-th entry of AB is $a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$.

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$$AB_{22} =$$

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$$AB_{22} = \begin{bmatrix} & \\ 2 & 1 \end{bmatrix} \begin{bmatrix} & 4 \\ & 2 \end{bmatrix} = 2 \times 4 + 1 \times 2 = 10$$

- Matrix multiplication is not commutative: usually, $AB \neq BA$.

Powers of A

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Calculating high powers of large matrix is hard. We will later learn a clever way to do this efficiently.

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Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \end{bmatrix}$.

Then $A^T = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 7 & 6 \\ 3 & 8 & 5 \\ 4 & 9 & 4 \\ 5 & 8 & 3 \end{bmatrix}$

Example

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute AB, $(AB)^T$, $A^T B^T$ and $B^T A^T$.

Solution

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Transpose

Conclusion

The transpose of a product is the product of transposes **IN OPPOSITE ORDER**:

$$(AB)^T = B^T A^T$$

Definition

A is **symmetric** if $A = A^T$.

Example

Which of these is symmetric?

$$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}^T =$$

$$\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}^T =$$

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(d) $(AB)^T = B^T A^T$

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Definition

The $n \times n$ **identity matrix** I_n has all entries 0, except on the main diagonal where the entries are 1. For example

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Definition

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.
A **permutation matrix** is one that is obtained by performing row exchanges on an identity matrix.

Example

Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$.
E_1, E_2, and E_3 are elementary matrices. Why? Are there any permutation matrices?

Solution

Observe the following products and describe how these products can be obtained by elementary row operations on A.

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

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$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g+3a & h+3b & i+3c \end{bmatrix}$$

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If an elementary row operation is performed on an $m \times n$- matrix A, the resulting matrix can be written as EA, where the $m \times m$-matrix E is created by performing the same row operations on I_m.

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We write this as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

More on inverses soon.

Remark

Elementary matrices are *invertible* because row operations are *reversible*. To determine the inverse of an elementary matrix E, determine the elementary row operation needed to transform E back into I and apply this operation to I to find the inverse.

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Example

Compute the following products using the row or column interpretation of matrix multiplication. (Don't just use the row-column rule.)

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} =$$

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