Math 415 - Lecture 8

Inverses.

Wednesday September 11th 2015

Textbook: Chapter 1.6

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Suggested Practice Exercise: Chapter 1.6 Exercise 1, 2, 4, 6, 10, 11, 18, 35, 36, 37, 38, 40, 49, 50

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Khan Academy Video: Inverse Matrix (part I), Inverse Matrix (part II)

Review

- Elementary matrices perform row operations:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ -2a+d & -2b+e & -2c+f \\ g & h & i \end{bmatrix}$$

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- Gaussian elimination on A gives a decomposition $A = LU$:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

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- LU decomposition lets us solve $A\mathbf{x} = \mathbf{b}$ quickly for many different $\mathbf{b}$.

Today's goal

- We know how to reverse a single row operation:

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Goal today: how to find an "inverse" to any (square!) matrix.
Today A will be an $n \times n$ matrix

The inverse of a matrix

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Remember that the identity matrix I_n is the $n \times n$-matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

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$$CA = AC = I_n$$

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Example

We already know that an elementary matrix is invertible:

$$\begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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(Check this at home!) So the definition works!

Theorem

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- Fact: if $AB = I$ then $A^{-1} = B$ and so $BA = I$. (Not so easy to show at this stage.)

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Definition

A matrix which is *not* invertible is sometimes called a singular matrix. An invertible matrix is also called nonsingular matrix.

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc \neq 0$, *then* A *is invertible and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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$$\begin{aligned} A\mathbf{w} &= \mathbf{b} \\ A^{-1}A\mathbf{w} &= A^{-1}\mathbf{b} \\ I_n\mathbf{w} &= A^{-1}\mathbf{b} \\ \mathbf{w} &= A^{-1}\mathbf{b} \end{aligned}$$

$\square$

Example

Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

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$$\begin{aligned} A^{-1} &= \frac{1}{14-15} \begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \\ \mathbf{x} &= A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix} \end{aligned}$$

Check this works!

Calculational rules

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$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I \quad \checkmark$$

An algorithm for computing the inverse matrix

Idea:

- To solve $Ax = b$ we row reduce $[\ A \mid b\ ]$.

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Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1}.

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- So by Theorem 5:

$$[\ A \mid I\ ] \text{ will row reduce to } \left[\ I \mid A^{-1}\ \right]$$

or A is not invertible.

Example

Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

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Check at home that $AA^{-1} = I_3$.

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