Math 415 - Lecture 18

Inner Product and Orthogonality

Wednesday October 7th 2015

Textbook reading: Ch 3.1

Suggested practice exercises: 1, 2, 4, 5, 14, 16

Khan Academy video: Vector Dot Product and Vector Length

Strang lectures: Lecture 30: Linear Transformations / Lecture 14: Orthogonality

Applications: Information retrieval

1 Review

- A **linear map** $T : V \to W$ satisfies

$$T(c\mathbf{x} + d\mathbf{y}) = cT(\mathbf{x}) + dT(\mathbf{y}).$$

- $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is linear. A is the matrix representing T in the standard bases. For example, $T(\mathbf{e_1}) = A\mathbf{e_1}$ = first column of A.

$$\left(\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)$$

- Any $T \colon V \to W$ can be represented by a matrix.

What is the Point? Why write $T \colon V \to W$ as a matrix?

- Replace obscure computations in V and W by transparent computations with matrices.

- Even if $T : \mathbb{R}^n \to \mathbb{R}^m$ (already have standard coordinates), T may be simpler in a different coordinate system.

Summary: Given $\mathbf{v}$ in V, want to calculate $T(\mathbf{v})$ in W. Take an input basis $\mathcal{A} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ and and output basis $\mathcal{B} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_m})$.

- The abstract input vector $\mathbf{v}$ and the coordinate vector $\mathbf{v}_{\mathcal{A}}$ determine each other.
- The abstract output vector $T(\mathbf{v})$ and the coordinate vector $T(\mathbf{v})_{\mathcal{B}}$ determine each other.
- So we know T if we know the matrix $T_{\mathcal{B}\mathcal{A}}$:

$$T(\mathbf{v})_{\mathcal{B}} = T_{\mathcal{B}\mathcal{A}} \mathbf{v}_{\mathcal{A}}.$$

The output coordinate vector equals the matrix for T times the input coordinate vector.

Formula For the Coordinate matrix. To write $T\colon V \to W$ as a matrix, take an input basis $\mathcal{A} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ and and output basis $\mathcal{B} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_m})$. Then

$$T_{\mathcal{B}\mathcal{A}} = \begin{bmatrix} T(\mathbf{x_1})_{\mathcal{B}} & T(\mathbf{x_2})_{\mathcal{B}} & \ldots & T(\mathbf{x_n})_{\mathcal{B}} \end{bmatrix}$$

Example 1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be reflection across the x-y plane, $(x, y, z) \mapsto (x, y, -z)$. Determine the matrix representing T in the basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Solution.

Example 2. Let $T : \mathbb{P}_3 \to \mathbb{P}_2$ be the linear map given by

$$T(p(t)) = \frac{d}{dt}p(t).$$

What's the matrix A representing T in the standard bases?

Solution.

What is $Col(A)$ and $Nul(A)$ for $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$?

Solution.

Let's try differentiating $7t^3 - t + 3$ using the matrix A.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 7 \end{bmatrix} = \boxed{}$$

2 Inner Product and Distances

Definition. The **inner product** (or **dot product**) of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ is

Example 3.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} =$$

Theorem 1. *Let* $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$, *and let* c *be any scalar. Then*

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(c) $(c\mathbf{u}) \cdot \mathbf{v} = c\,(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

(d) $\mathbf{u} \cdot \mathbf{u} \geq \mathbf{0}$, *and* $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ *if and only if* $\mathbf{u} = \mathbf{0}$.

Definition.

- The **norm** (or **length**) of a vector $\mathbf{v} \in \mathbb{R}^n$ is

- The **distance** between points $v, w \in \mathbb{R}^n$ is

Example 4. (a)

$$\left\| \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\| = \boxed{}$$

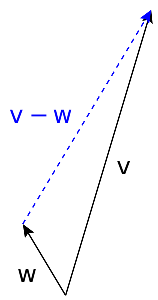
(b)

$$dist\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) = \boxed{}$$

3 Inner Product and Angles

We can use the dot product to compute angles too.

Theorem 2. *If* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *are linearly independent, they form an angle* θ*, and*



Example 5. What is the angle formed in $\mathbb{R}^3$ between the vectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}?$$

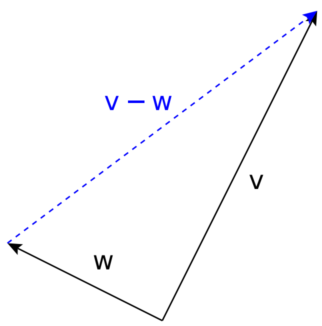
(A base jumper runs at a cliff at a 45° angle, then jumps straight away from the cliff and 45° downwards; what angle does he turn as he jumps?)

Solution.

4 Orthogonal vectors

Definition. $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ are **orthogonal** if

Remark. We write $\mathbf{v} \perp \mathbf{w}$ when $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. Nonzero vectors $\mathbf{v}$, $\mathbf{w}$ are orthogonal if and only if they are perpendicular.



Example 6. Are the following vectors orthogonal?

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Example 7. Let $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Is the set of vectors orthogonal to $\mathbf{v}$ a subspace of $\mathbb{R}^3$?

$$V = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{v} \cdot \mathbf{x} = 0\}$$

Solution.

Definition. If V is a subspace of $\mathbb{R}^n$, a vector $\mathbf{x}$ is **orthogonal to** V if it is orthogonal to every vector in V.

Example 8. Let $V = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$. Is $\mathbf{x} = \begin{bmatrix}-1\\1\end{bmatrix}$ orthogonal to V?

Solution.