Math 415 - Lecture 4

Linear Combinations and Matrix operations

Monday August 31 2015

Textbook: Chapter 1.3, 1.4

Suggested Practice Exercise: Chapter 1.4 Exercise 1, 2, 10, 12, 13, 21, 30, 34, 45,

Khan Academy Video: Matrix multiplication (part I)

Review

A system such as

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

can be written in vector form as

$$x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

The right-hand side is a linear combination of the vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The row and column picture

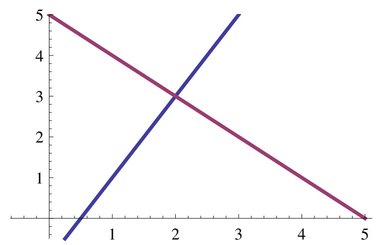
Example 1. We can think of the linear system

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

in two different geometric ways. Recall unique solution: $x = 2, y = 3$.

Row picture

- Each equation defines a line in $\mathbb{R}^2$.
- Which points lie on the intersection of these lines?
- $(2, 3)$ is the (only) intersection of the two lines $2x - y = 1$ and $x + y = 5$.

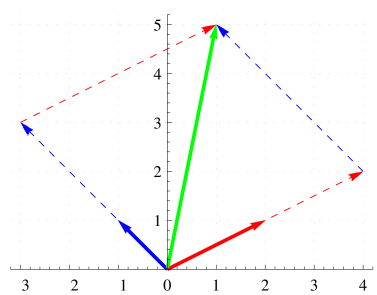


Column picture

- The system can be written as

$$x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

- Which linear combinations of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ produce $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$?
- $(2, 3)$ are the coefficients of the (only) such linear combination.



Application: Industrial Espionage.
Suppose a candy factory produces stuff , in particular:

- *Doohickeys*,
- *Nicknacks* and (of course)
- *Widgets*.

To produce these you need raw materials:

- Sugar,
- Spice and
- Everything Nice (mystery ingredient!),

in different quantities. We encode how much we need in *production vectors.* So if the Doohickeys production vector $v_D = \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix}$ then we need 10 units of Sugar, 5 units of Spice, 2 units of Everything Nice to produce one Doohickey.

Similarly we have production vectors v_N, v_W for Nicknacks and Widgets. We assume

$$v_D = \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix}, \quad v_N = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_w = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

Now let us think how much raw material the factory uses. The total consumption of raw matrial will be encoded in the *consumption vector* $B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$ of the factory. So B_1 is the amount of sugar consumed in the production, B_2 the amount of spice, etc. If the factory produces c_D Doohickeys, c_N Nicknacks and c_W widgets then the consumption vector will be

$$B = c_D v_D + c_N v_N + c_W v_W$$

More explicitly

$$B = c_D \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix} + c_N \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_W \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}.$$

So the consumption vector B is a **linear combination** of the production vectors. Now suppose that somebody (say a competitor of our factory) is interested in determining how many Doohickeys are produced. It is a trade secret, so you can not go and ask the factory. If many Doohickeys are produced the competitor might switch to the production of Whatchamacallits.

So the competitor sends a spy to the entrance of the factory and she writes down how many truck loads of raw materials enters. This determines the production vector B. Then the problem is to find the coefficients c_D, c_N, c_W, i.e., the number of Doohickeys, Nicknacks and Widgets produced.

Problem 2. *Suppose our spy observes*

$$B = \begin{bmatrix} 14 \\ 12 \\ 7 \end{bmatrix} = c_D \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix} + c_N \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_W \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}.$$

Then what is the number c_D of Doohickeys produced? How to approach the problem?

Solution 3. *Finding Linear Combinations is the same as solving Linear Systems. Write down the Augmented Matrix corresponding to the linear combination and find the Reduced Row Echelon Form*

$$\left[\begin{array}{ccc|c} 10 & 1 & 3 & 14 \\ 5 & 2 & 5 & 12 \\ 2 & 3 & 2 & 7 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

So what is c_D?

Matrix operations

Matrices are like Numbers: Matrix Algebra.

Two ways to denote $m \times n$ matrix A (m rows, n column).

- In terms of the columns of A:

$$A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array} \right]$$

- In terms of the entries of A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- a_{ij} is in the ith row and jth column
- $\mathbf{a_j}$ is j^{th} column:

$$\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

- **Main diagonal entries:** a_{11}, a_{22}, …, a_{mm} (only care about these when $m = n$)

Even more notation

- **Zero matrix:**

$$0 = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

Definition. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$ be $m \times n$-matrices and let r be a scalar. Then

- $A + B$ is defined by

$$A + B = \begin{bmatrix} \mathbf{a}_1 + \mathbf{b}_1 & \mathbf{a}_2 + \mathbf{b}_2 & \ldots & \mathbf{a}_n + \mathbf{b}_n \end{bmatrix}$$

- Moreover, rA is defined as

$$rA = \begin{bmatrix} r\mathbf{a}_1 & r\mathbf{a}_2 & \ldots & r\mathbf{a}_n \end{bmatrix}$$

Example 4. Calculate

- $$\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 8 & 3 \end{bmatrix}$$

- $$10 \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 30 \\ 30 & 10 \end{bmatrix}$$

Theorem 1. *Let A, B, and C be matrices of the same size, and let r and s be scalars.*

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + 0 = A$
- $r(A + B) = rA + rB$
- $(r + s)\, A = rA + sA$
- $r\,(sA) = (rs)\, A$

Matrices are like Numbers!

Matrix Multiplication

How to multiply matrices and vectors

Let $\mathbf{x}$ be a vector, A, B matrices.

- Multiplying B and $\mathbf{x}$ transforms $\mathbf{x}$ into the vector $B\mathbf{x}$.
- In turn, if we multiply A and $B\mathbf{x}$, we transform $B\mathbf{x}$ into $A\,(B\mathbf{x})$.
- So $A\,(B\mathbf{x})$ is the composition of two mappings.

Define the product AB so that

$$A\,(B\mathbf{x}) = (AB)\,\mathbf{x}$$

Suppose A is $m \times n$ and B is $n \times p$ where

$$B = [\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \cdots \;\; \mathbf{b}_p] \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

Define

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \ldots + x_n\mathbf{b}_n$$

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Then

$$A\,(B\mathbf{x}) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 + \ldots + x_n A\mathbf{b}_n$$

Example 5.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$B\mathbf{x} = B\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$$

Compute $A(B\mathbf{x})$ **using** $B\mathbf{x} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$:

$$A(B\mathbf{x}) = A\begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix} = (x_1 + 2x_2)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (x_2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Compute $A(B\mathbf{x})$ **using** $A\,(B\mathbf{x}) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 + \ldots + x_n A\mathbf{b}_n$:

$$A(B\mathbf{x}) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 = x_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Same answer!

Motto
Matrix Multiplication is Linear Combination!