## Math 415 - Lecture 14

Null sapce and Column space basis

#### Friday September 25th 2015

Textbook reading: 2.4

Suggested practice exercises: Chapter 2.4 Exercise 1, 2, 3, 4, 21

**Khan Academy video:** Null Space and Column Space Basis, Dimension of the Null Space, Dimension of the Column Space

Strang lecture: Independence, Basis, and Dimension

### 1 Review

- $\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$  is a **basis** of V if the vectors
  - span V, and
  - are independent.
- $\bullet$  The **dimension** of V is the number of elements in a basis.
- The columns of A are linearly independent  $\iff$  each column of A contains a pivot.  $\iff$  there are no free variables.

## 2 Warmup

Example 1. Find a basis and the dimension of

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}.$$

Solution.		

Remark. Every set of linearly independent vectors can be extended to a basis.

In other words, let  $\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$  be linearly independent vectors in V. If V has dimension d, then we can find vectors  $\mathbf{v_{p+1}}, \dots, \mathbf{v_d}$  such that  $\{\mathbf{v_1}, \dots, \mathbf{v_d}\}$  is a basis of V.

Example 2. Consider

$$H = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

- Give a basis for H. What is the dimension of H?
- Extend the basis of H to a basis of  $\mathbb{R}^3$ .

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# 3 Bases for Null Spaces

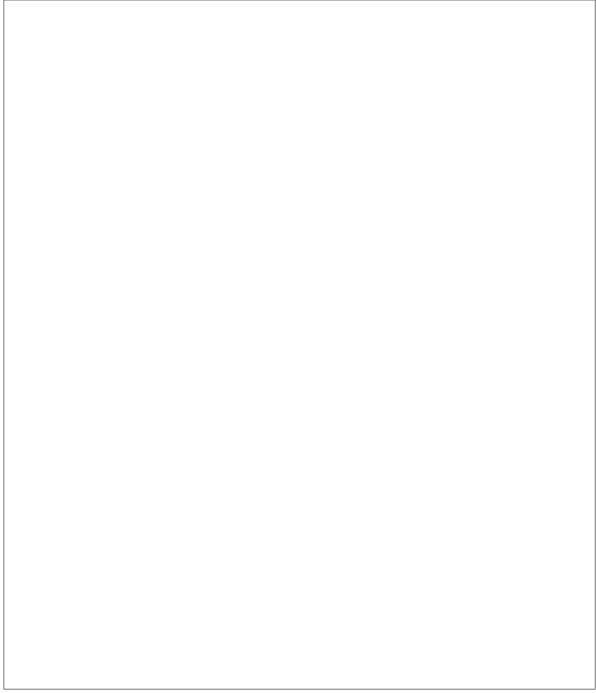
To find a basis for Nul(A):

- find the parametric form of the solutions to  $A\mathbf{x} = \mathbf{0}$ .
- $\bullet$  express solutions **x** as a linear combination of vectors with the free variables as coefficients;
- these vectors form a basis of Nul(A).

Example 3. Find a basis for Nul(A) with

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 15 & 0 & 3 \end{bmatrix}.$$

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**Remark.** If A is a matrix, Nul(A) has a basis vector for each free variable. So the dimension of Nul(A) is equal to the number of free variables!

## 4 Basis for Column Space

Recall that the columns of A are independent  $\iff A\mathbf{x} = \mathbf{0}$  has only the trivial solution (namely,  $\mathbf{x} = \mathbf{0}$ )  $\iff A$  has no free variables.

**Theorem 1.** A basis for Col(A) is given by the pivot columns of A.

Example 4. Find a basis for Col(A) with

Solution.

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

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**Remark.** If A has echelon form U then any relation between the columns of U:

$$x_1\mathbf{u_1} + \dots + x_n\mathbf{u_n} = 0$$

also holds for the columns of A:

$$x_1\mathbf{a_1} + \dots + x_n\mathbf{a_n} = 0,$$

for the same scalars  $x_i$ .

Why?

Solution.

**Warning**: For the basis of Col(A), you have to take the columns of A, not the columns of an echelon form. Row operations do not preserve the column space.

Example 5. Let  $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ . Then the RREF of A is  $U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ .

# 5 Checking Our Understanding

True or false?

1.	Suppose that $V$ has dimension $n$ . Then any set in $V$ containing more than $n$
	vectors must be linearly dependent.
2.	The space $P_n$ of polynomials of degree at most $n$ has dimension $n+1$ .
3.	The vector space of functions $f: \mathbb{R} \to \mathbb{R}$ is infinite-dimensional.
4.	Consider $V = span \{ \mathbf{v_1}, \dots, \mathbf{v_p} \}$ . If one of the vectors, say $\mathbf{v_k}$ , in the spanning set is a linear combination of the remaining ones, then the remaining vectors still span $V$ .