Worksheet 10 for November 3rd and 5th

1. *Let* $\mathbf{v}_1 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ *and* $\mathbf{v}_3 = \begin{bmatrix}2\\1\\0\\-1\end{bmatrix}$. *Using Gram-Schmidt, find an orthonormal basis for* $W = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, *using* $\mathbf{v}_1, \mathbf{v}_2,$ *and* $\mathbf{v}_3$.

Solution. Set

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}}{\left\|\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\right\|} = \begin{bmatrix}\frac{1}{\sqrt{3}}\\0\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}$$

Then

$$\mathbf{u}_2 = \frac{\mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1}{\|\mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1\|} = \frac{\begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \left(\begin{bmatrix}\frac{1}{\sqrt{3}}\\0\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{3}}\\0\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}}{\left\|\begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \left(\begin{bmatrix}\frac{1}{\sqrt{3}}\\0\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{3}}\\0\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}\right\|} = \frac{\begin{bmatrix}\frac{2}{3}\\0\\-\frac{1}{3}\\-\frac{1}{3}\end{bmatrix}}{\left\|\begin{bmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{bmatrix}\right\|} = \sqrt{\frac{3}{2}}\begin{bmatrix}\frac{2}{3}\\0\\-\frac{1}{3}\\-\frac{1}{3}\end{bmatrix} = \begin{bmatrix}\frac{2}{\sqrt{6}}\\0\\-\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\end{bmatrix}$$

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM
Midterm Date: November 19 7-8:15 PM, Conflict November 20, 8-9.20AM and 9:30-10:50AM, Conflict sign up deadline: November 13
Final Date: December 17 8-11AM, Conflict December 15, 8-11AM. You are allowed to take the conflict exam if you have more than two examination within 24 hours. Conflict sign up deadline: November 30

Finally,

$$\mathbf{u}_3 = \frac{\mathbf{v}_3 - (\mathbf{u}_1 \cdot \mathbf{v}_3)\,\mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{v}_3)\mathbf{u}_2}{\|\mathbf{v}_3 - (\mathbf{u}_1 \cdot \mathbf{v}_3)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{v}_3)\mathbf{u}_2\|}$$

$$= \frac{\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} - \left(\begin{bmatrix} \frac{2}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right) \begin{bmatrix} \frac{2}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}}{\left\| \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} - \left(\begin{bmatrix} \frac{2}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right) \begin{bmatrix} \frac{2}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \right\|}$$

$$= \sqrt{\frac{2}{3}} \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Now $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of W. □

2. *Let* $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(i) *Calculate $A^T A$. What does this tell you aboutthe columns of A?*

(ii) *Find an orthonormal basis $\{q_1, q_2\}$ for* Col(A) *(startingwith the columns of A!). Put $Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$. What is Q^{-1}?*

Solution. (i) We have:

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Since only entries on the main diagonal are nonzero, columns of A are orthogonal to each other.

(ii) Since we already know that columns of A are orthogonal to each other, to find an orthonormal basis for Col(A) it is enough to divide each column by its length. Hence: (note that for non-zero vectors, orthogonality implies linear independence)

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Q is an orthogonal matrix, so:

$$Q^{-1} = Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

□

3. *Let*

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

the matrix for rotation by the angle θ (counterclockwise).

(i) *Calculate $Q_\theta^T Q_\theta$. What does this tell you aboutthe columns of Q_θ?*

(ii) *What is Q_θ^{-1}? Express Q_θ^{-1}in terms of another rotation matrix Q_ϕ.*

(iii) *Show that if $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ then the vector $\mathbf{x}$ and the rotated vector$Q_\theta \mathbf{x}$ have the same length.*

Solution. (i) We have:

$$Q_\theta^T Q_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This shows that the columns of Q_θ form an orthonormal basis for $\mathbb{R}^2$.

(ii) By the first part, we have $Q_\theta^{-1} = Q_\theta^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. It is easy to see that the inverse of the rotation by θ is the rotation by $-\theta$, therefore:

$$Q_\theta^{-1} = Q_{-\theta}$$

(iii) We have:

$$Q_\theta \mathbf{x} = Q_\theta \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{bmatrix}$$

Thus,

$$\begin{aligned} \text{length}(Q_\theta \mathbf{x}) &= \sqrt{(a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2} = \sqrt{a^2(\cos^2\theta + \sin^2\theta) + b^2(\cos^2\theta + \sin^2\theta)} \\ &= \sqrt{a^2 + b^2} = \text{length}(\mathbf{x}) \end{aligned}$$

□

4. *Let P be the matrix*

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(i) *Compute the dot products between every two columns of P.*

(ii) *What is P^{-1}?*

Now let P be an arbitrary $n \times n$ permutation matrix, so each row and each column hasa single non zero entry 1. Write $P = \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix}$.

(iii) *What is the dot product between the columns of$P$, i.e., what is $P_i^T P_j$?*

(iv) *What is P^{-1}?*

Solution. (i) See (iii) below.

(ii) See (iv) below. The inverse of P is:

$$P^{-1} = P^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(iii) We have $P_i^T P_j = 0$ if $i \neq j$ and $P_i^T P_i = 1$. This is because the columns of P are the standard basis vectors of $\mathbb{R}^n$ in a different order.

(iv) From the first part, we know that columns of P form an orthonormal basis, i.e., P is orthogonal. Hence, we have:

$$P^{-1} = P^T$$

□

5. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

a. *Find the QR decomposition of A: write $A = QR$ where Q is a matrix with orthonormal columns and R is an upper triangular matrix.*

b. *Let* $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. *Use the QR decomposition of A to find the least squares solution of $A\hat{\mathbf{x}} = \mathbf{b}$ (by solving $R\hat{\mathbf{x}} = Q^T\mathbf{b}$).*

Solution. **a.** We start with columns of $A(= [\mathbf{v}_1\ \mathbf{v}_2])$ and we use Gram-Schmidt to find columns of $Q(= [\mathbf{q}_1\ \mathbf{q}_2])$:

$$q_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

and,

$$\mathbf{q}_2 = \frac{\mathbf{v}_2 - (\mathbf{q}_1 \cdot \mathbf{v}_2)\mathbf{q}_1}{\|\mathbf{v}_2 - (\mathbf{q}_1 \cdot \mathbf{v}_2)\mathbf{q}_1\|} = \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}}{\left\| \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\|} = \frac{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}}{\left\| \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} \right\|} = \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Hence,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

We have:

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix}$$

b. We have to solve $R\hat{\mathbf{x}} = Q^T\mathbf{b}$:

$$\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Therefore, $\hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$. □

6. **a.** *Recall that the orthogonal projection onto Col(A) has projection matrix $A(A^TA)^{-1}A^T$. How does this formula simplify in the case when A has orthonormal columns?*

b. *Let* $Q = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{5} \\ 0 & -\frac{4}{5} \end{bmatrix}$. *What is the projection matrix corresponding to the orthogonal projection onto Col(Q)?*

c. *Let* $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$. *What is the projection matrix corresponding to the orthogonal projection onto Col(Q)? Explain why your answer is not surprising.*

Solution. **a.** If A ha orthonormal columns then $A^TA = I$. So the projection matrix is:

$$A(A^TA)^{-1}A^T = AA^T$$

b. Since Q has orthonormal columns, th projection matrix is:

$$QQ^T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{5} \\ 0 & -\frac{4}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{9}{25} & -\frac{12}{25} \\ 0 & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}$$

c. Q has orthonormal columns, so the projection matrix is QQ^T which is equal to I(since Q is orthogonal).Since columns of Q are linearly independent and Q has 3 columns, columns of Q form a basis for $\mathbb{R}^3$. Hence, Col(Q) $= \mathbb{R}^3$ and projection of each vector in $\mathbb{R}^3$ onto Col(Q) is itself, i.e., the projection matrix is I. □

7. *Quarterly economic data is subject to seasonal fluctuations. A curve that approximates the gross domestic product (GDP) of a country might be of the form*

$$y = \beta_0 + \beta_1 x + \beta_2 \sin(2\pi x/4),$$

where is x is the time in quarters of a year. The term $\beta_0 + \beta_1 x$ gives the basic GDP growth trend of the economy, while the sine term reflects the seasonal changes. Assume the GDP data are $(x_1, y_1), \ldots, (x_n, y_n)$.

(i) *Give the design matrix that leads to a least-square fit to the equation above.*

(ii) *(Highly Optional) GDP data for US economy is available at http://www.bea.gov/. Using the above, can you find the GDP growth trend of the US economy.*

Solution. (i) Plugging in our data points we get that our design matrix is

$$\begin{bmatrix} 1 & x_1 & \sin(2\pi x_1/4) \\ 1 & x_2 & \sin(2\pi x_2/4) \\ \vdots & \vdots & \vdots \\ 1 & x_n & \sin(2\pi x_n/4) \end{bmatrix}$$

(ii) This is left as an exercise.

□

8. *According to Kepler's first law, a comet should have an elliptic, parabolic or hyperbolic orbit. In suitable polar coordinates, the position (r, θ) of a comet satisfies an equation*

$$r = \beta + e(r \cdot \cos(\theta)),$$

where β is a constant and e is the eccentricity of the the orbit, with $0 \leq e < 1$ for an ellipse, $e = 1$ for a parabola and $e > 1$ for a hyperbola. Suppose observations of a newly discovered comet provide the data below.

θ	*.88*	*1.10*	*1.42*	*1.77*	*2.14*
r	*3.00*	*2.30*	*1.65*	*1.25*	*1.01*

Use least square methods to find the type of the orbit, and predict where the comet will be when $\theta = 4.6$ *(radians).*

Solution. We first set up the design matrix. Plugging in our values for r and θ, we get the following overdetermined system.

$$B = \begin{bmatrix} 1 & 3.00\cos(.88) \\ 1 & 2.30\cos(1.10) \\ 1 & 1.65\cos(1.42) \\ 1 & 1.25\cos(1.77) \\ 1 & 1.01\cos(2.14) \end{bmatrix} \begin{bmatrix} \beta \\ e \end{bmatrix} = \begin{bmatrix} 3.00 \\ 2.30 \\ 1.65 \\ 1.25 \\ 1.01 \end{bmatrix}$$

Multiplying both sides of this by B^T and truncating at 3 significant figures, we get

$$\begin{bmatrix} 5 & 2.41 \\ 2.41 & 5.16 \end{bmatrix} \begin{bmatrix} \beta \\ e \end{bmatrix} = \begin{bmatrix} 9.21 \\ 7.68 \end{bmatrix}$$

Solving this numerically, we get that $\beta \approx 1.45$, and $e \approx .810$. Thus this comet has an elliptical orbit. $\square$