Math 415 - Lecture 21

Networks and linear algebra

Wednesday October 14th 2015

Textbook reading: Chapter 2.5.

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Suggested practice exercises: Chapter 2.5: 1, 2, 6.

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Strang lecture: Lecture 12: Graphs, Networks, Incidence Matrices

Review

Recall that if $V \subset \mathbb{R}^n$ is a subspace, $V^\perp$ is the orthogonal complement of V, the subspace of all vectors $\mathbf{x}$ perp to all vectors of V.

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Fundamental Theorem of Linear Algebra.

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Both descriptions are useful, and we will often switch between them, to answer any particular question we want to answer.

A new perspective on $A\mathbf{x} = \mathbf{b}$

To see if $A\mathbf{x} = \mathbf{b}$ has a solution, check that

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The indirect approach means:

$$\text{if } \underbrace{\mathbf{y}^T A = \mathbf{0}}_{\mathbf{y} \in Nul(A^T)}, \text{ then } \underbrace{\mathbf{y}^T \mathbf{b} = 0}_{\mathbf{b} \perp \mathbf{y}}.$$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution?

Solution (old)

Write augmented matrix, get Echelon form:

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 1 & b_2 \\ 0 & 5 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -5 & -3b_1 + b_2 \\ 0 & 0 & -3b_1 + b_2 + b_3 \end{array}\right]$$

When is this consistent?

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When is this consistent? Whenever $-3b_1 + b_2 + b_3 = 0$.

Solution (new)

Indirect approach says: $A\mathbf{x} = \mathbf{b}$ solvable $\iff$ $\mathbf{b} \perp Nul(A^T)$.

Find basis for $Nul(A^T)$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 1 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

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This is the same condition as before!

Application: Directed graphs

Set up

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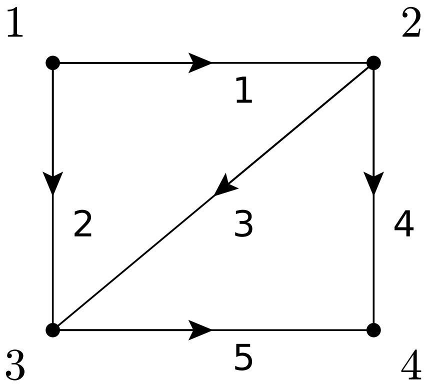
- Graphs appear in network analysis (e.g. internet) or circuit analysis.



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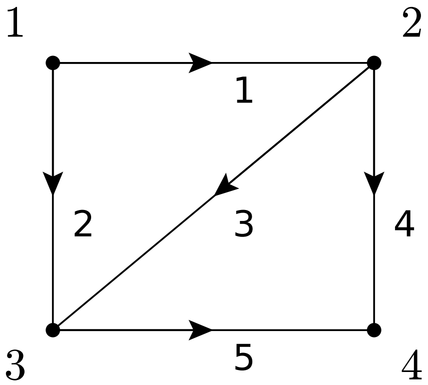

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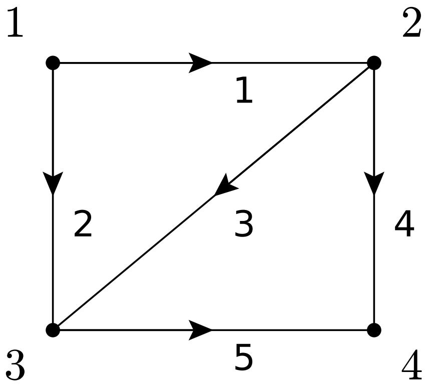
- Arrow indicates direction of flow
- No edges from a node to itself



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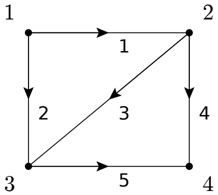
So each row (describing an edge=arrow) contains a single -1 (the tail of the arrow), a single $+1$ (the head of the arrow), and for the rest zeroes.

Example

Give the edge-node incidence matrix of our graph.



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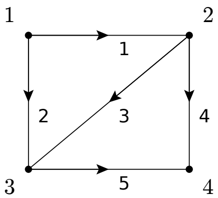

Solution

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Meaning of the null space

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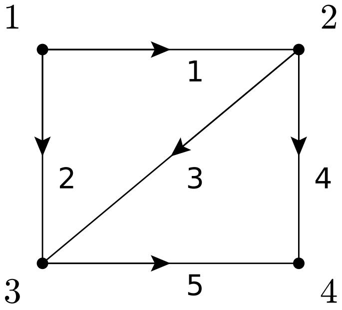
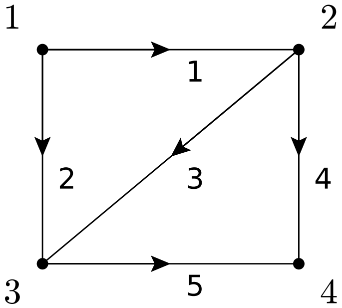
- For large graphs, disconnection may not be visually apparent
- But, we can always find out by computing $dim(Nul(A))$ using Gaussian elimination!

Meaning of the null space



The $\mathbf{x}$ in $A\mathbf{x}$ assigns values to each node.



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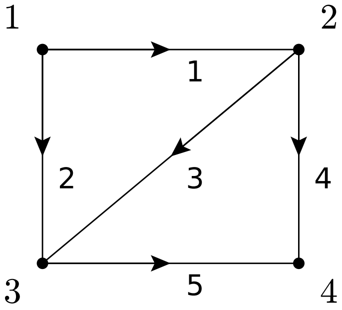

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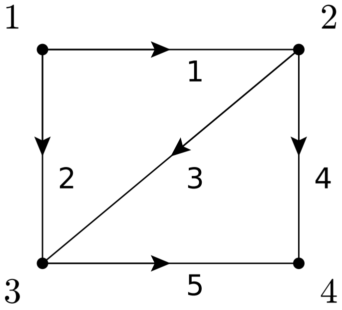

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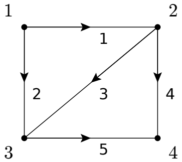

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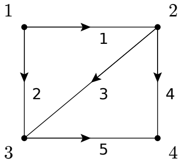

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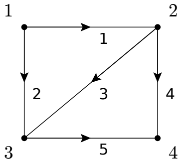

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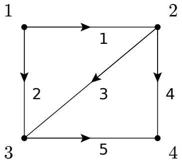

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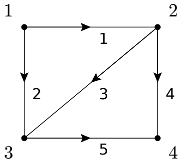

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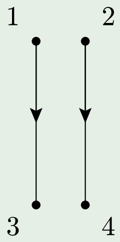
This always happens as long as G is connected.

Example

Give a basis for $Nul(A)$ for this graph:



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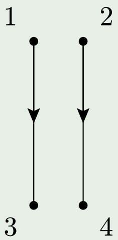
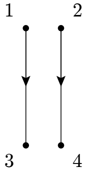


Meaning of the null space



If $A\mathbf{x} = 0$, then

$$\underbrace{x_1 = x_3}_{\text{connected by an edge}} \quad \text{and} \quad \underbrace{x_2 = x_4}_{\text{connected by an edge}}$$

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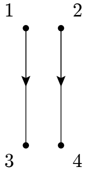

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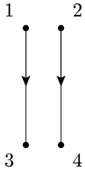

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So, $Nul(A)$ has basis: $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$

Just to make sure, the edge-node incidence matrix is:

$$A = \begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}$$

Meaning of left null space

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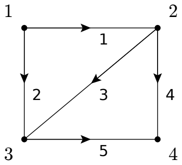

The $\mathbf{y}$ in $\mathbf{y}^T A$ is assigning values to each edge.

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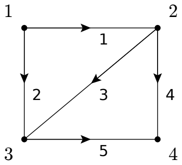
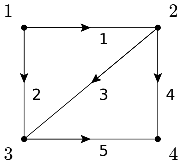


The $\mathbf{y}$ in $\mathbf{y}^T A$ is assigning values to each edge. (Think: assigning currents to edges, so that $\mathbf{y}$ describes a *flow pattern*.)



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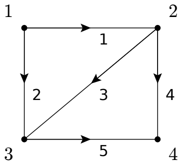

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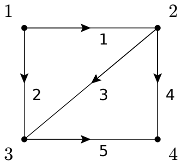

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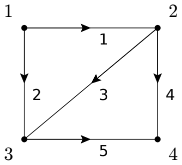

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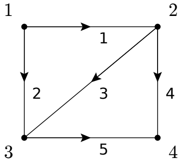
Meaning of left null space



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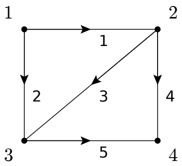

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Idea

So: $A^T\mathbf{y} = 0 \iff$ at each node, (directed) values assigned to edges add to zero.

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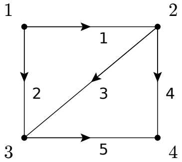

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Idea

So: $A^T\mathbf{y} = 0 \iff$ at each node, (directed) values assigned to edges add to zero.

When thinking of currents, this is Kirchhoff's first law: at each node, incoming and outgoing currents balance. **Flow in = Flow out.**

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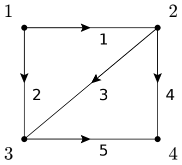
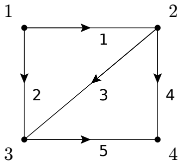


What is the simplest way to balance current?

Meaning of left null space



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Assign current in a loop!

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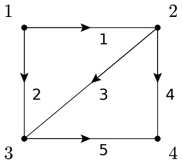

What is the simplest way to balance current?
Assign current in a loop! We have two loops:

$$edge_1 \to edge_3 \to -edge_2 \text{ and } edge_3 \to edge_5 \to -edge_4$$

Example

Solve $A^T\mathbf{y} = 0$ for our graph. Recall

$$A^T = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

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Solution

Get RREF:

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \overset{RREF}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Meaning of left null space

The parametric solution is: $\begin{bmatrix} y_3 - y_5 \\ -y_3 + y_5 \\ y_3 \\ -y_5 \\ y_5 \end{bmatrix}$

So a basis for $Nul(A^T)$ is: $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

Observation: These two basis vectors correspond to loops.

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Note: get the "simpler" loop $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ as $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

Theorem

In general, $\dim(\mathrm{Nul}(A^T))$ is the number of (independent) loops.

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For large graphs, we now have a nice way to computationally find all loops.

Summary/Outlook

* We described a network by using a matrix A.

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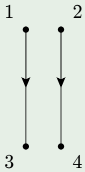
* The Left Null Space $Nul(A^T)$ has as dimension the number of independent loops.
* The column space $Col(A)$ and row space $Col(A^T)$ also have "geometric" meaning in terms of the network, see the book and Strang's lecture.

Practice problems

Problem 1

Example

Give a basis for $Nul(A^T)$ for the following graph:



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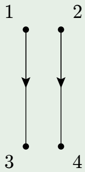

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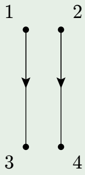

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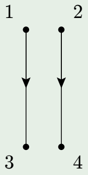

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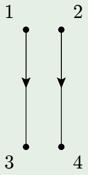

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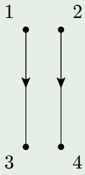

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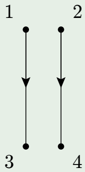

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Indeed, $Nul(A^T) = \{0\}$.

Problem 2

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Draw the graph with edge-node incidence matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

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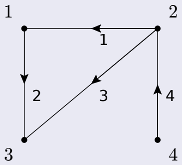
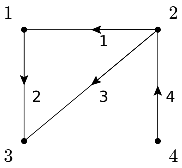


Figure : The graph

Problem 2



If $A\mathbf{x} = 0$, then

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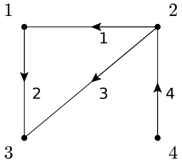

If $A\mathbf{x} = 0$, then $x_1 = x_2 = x_3 = x_4$ (all connected by edges.)

So, $Nul(A)$ has basis

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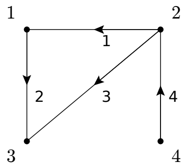

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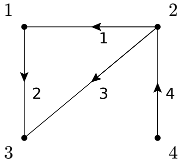

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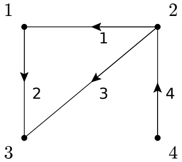

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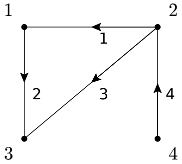

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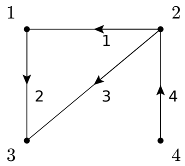

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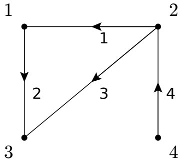

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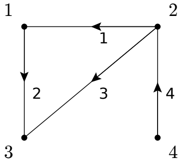

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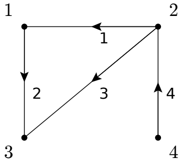

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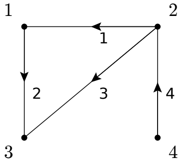

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