Math 415 - Lecture 5

Matrices and Linear Systems

Wednesday September 2nd 2015

Textbook: Chapter 1.4

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Suggested Practice Exercise: Chapter 1.4 Exercise 1, 2, 10, 12, 13, 21, 30, 34, 45,

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Khan Academy Video: Matrix multiplication (part I), Matrix multiplication (part II), Defined and undefined matrix operations

Review Matrix Multiplication

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Motto 1

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$$Ax = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n, \quad \text{if } A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a_n} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Problem

Consider the linear combination

$$3\begin{bmatrix}1\\2\end{bmatrix} - 1\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} = b.$$

Write the linear combination b as a matrix multiplication $b = Ax$.

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$A = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$ and $x = \begin{bmatrix}3\\-1\end{bmatrix}$.

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From now on we will write $Ax = b$ for the system of equations with augmented matrix $\left[\begin{array}{c|c} A & b \end{array}\right]$.

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Let A be a matrix, $\mathbf{x}, \mathbf{y}$ vectors and c, d scalars. If the input vector is a linear combination then also the output vector is a linear combination:

$$A(c\mathbf{x} + d\mathbf{y}) = cA\mathbf{x} + dA\mathbf{y}.$$

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So we see that we get infinitely many new solutions $x + cz$, if we have found just two solutions.

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- Let B be $n \times p$: input $x \in \mathbb{R}^p$, output $c = Bx \in \mathbb{R}^n$.
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Definition

The **machine** AB takes as input $x \in \mathbb{R}^p$ and produces as output $A(Bx) \in \mathbb{R}^n$.

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The **machine** AB takes as input $x \in \mathbb{R}^p$ and produces as output $A(Bx) \in \mathbb{R}^n$.

So given two matrices A and B (of the right size) we defined a machine that we call AB.

Theorem

The machine AB is in fact a matrix of size $m \times p$ given explicitly by

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}.$$

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$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p] \,.$$

Example

Previous example, again

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

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Previous example, again

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$$AB = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

Example

Compute AB where

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$$

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$$\begin{aligned} A\mathbf{b}_1 &= 2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 6\begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 6 \\ 0 \end{bmatrix} + \begin{bmatrix} -12 \\ -30 \\ 6 \end{bmatrix} \end{aligned}$$

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Note that $A\mathbf{b}_1$ is a linear combination of the columns of A and $A\mathbf{b}_2$ is a linear combination of the columns of A.

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Note that $A\mathbf{b}_1$ is a linear combination of the columns of A and $A\mathbf{b}_2$ is a linear combination of the columns of A. Each column of AB is a linear combination of the columns of A using weights from the corresponding columns of B.

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- AB is 4×2,
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If AB is defined, let $(AB)_{ij}$ denote the entry in the ith row and jth column of AB. Then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

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If you know about dot products you see that every entry in the product AB is the dot product of a row vector (of A) and a column vector (of B).

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$A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined.

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Row-Column Rule for Computing AB

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Let A be $m \times n$ and B and C have sizes for which the indicated sums and products are defined.

(a) $A(BC) = (AB)C$ *(associative law of multiplication)*

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(b) $A(B + C) = AB + AC$, $(B + C)A = BA + CA$ *(distributive laws)*

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(d) $r(AB) = (rA)B = A(rB)$ *for any scalar r*

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(e) $I_m A = A = A I_n$ *(identity for matrix multiplication)*

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Matrices are like numbers.

Theorem

Let A be $m \times n$ and B and C have sizes for which the indicated sums and products are defined.

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Here $I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$ is the identity matrix af size n.

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&= \begin{bmatrix} 1 & 0 \\ 9 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 21 & 8 \end{bmatrix}
\end{aligned}$$

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