Math 415 - Lecture 7

LU-decomposition continued

Wednesday September 9th 2015

Textbook: Chapter 1.5

Suggested Practice Exercise: Chapter 1.5 Exercise 4, 5, 11, 23, 29

1 Review - Elementary matrices

- Multiply row 3 by 7:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 7g & 7h & 7i \end{bmatrix}$$

- Switch rows 2 and 3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

- $R3 \to 3R1 + R3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

- Taking the inverse of an elementary matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

2 Triangular matrices

Definition. An $n \times n$ matrix A is called upper triangular if it is of the form

$$\begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{bmatrix}.$$

An $n \times n$ matrix B is called lower triangular if it is of the form

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & \ddots & \vdots \\ * & * & * & * & * \end{bmatrix}.$$

Example 1. Give a few examples!

Definition. A matrix A has LU factorization if there is a lower triangular matrix L and a upper triangular matrix U such that

$$A = LU.$$

(In practice, L will have all 1's on the main diagonal.)

When is this possible?

Theorem 1. *Let A be a $n \times n$-matrix. If A can be transformed into echelon form without the use of row exchanges, then A has LU factorization.*

Example 2. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$.

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{2} & 1 & 1 \\ \boxed{4} & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} =$$

$$E_2 (E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{2} & 1 & 1 \\ 0 & -8 & -2 \\ \boxed{-2} & 7 & 2 \end{bmatrix} =$$

$$E_3 (E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & \boxed{-8} & -2 \\ 0 & \boxed{8} & 3 \end{bmatrix} =$$

$$E_3 E_2 E_1 A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

So,

$$A =$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = L$$

So the LU decomposition is

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

An easy way to compute L:

3 Row exchanges

Recall that a permutation matrix P is a square matrix obtained from the identity matrix by reordering the rows.

Theorem 2. *Let A be a $n \times n$-matrix that can be brought to echelon form. Then there is permutation matrix P such that PA has LU factorization.*

Why?

Example 3. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. Find PA that has a LU factorization.

Solution.

4 Applications

Theorem 3. *Let A be an $n \times n$-matrix such that $A = LU$, where L is a lower triangular matrix and U is a upper triangular matrix. Then x will be solution of*

$$Ax = b$$

if and only if x is a solution of

$$Ux = c,$$

where c satisfies $Lc = b$.

Why?

Example 4. Solve

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

We found already a LU factorization for this matrix A. So you first have to solve $Lc = b$ for c:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}.$$

Use **forward substitution:**

Then solve $Ux = c$ for x:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}.$$

This uses **backwards substitution.**

Practical question. Why do we care about LU decomposition if we already have Gaussian elimination?

Solution.