Worksheet 6 for October 6th and 8th

1. *Determine a basis for each of the following subspaces:*

(i) $H = \left\{ \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} : s, t \in \mathbb{R} \right\}$,

(ii) $K = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a - 3b + c = 0 \right\}$,

(iii) $\mathrm{Col}\left(\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right)$,

(iv) $\mathrm{Nul}\left(\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right)$.

Solution. (i): Every vector in H is of the form

$$\begin{bmatrix} 4s \\ -3s \\ t \end{bmatrix} = s \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where s, t range freely over $\mathbb{R}$. Thus

$$H = \mathrm{span} \left\{ \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Since these two vectors are linearly independent (they are not multiples of each other), the set

$$\left\{ \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis of H.
(ii) We observe that $K = \mathrm{Nul}(A)$ where

$$A = \begin{bmatrix} 1 & -3 & 1 & 0 \end{bmatrix}.$$

Since A is already in (reduced) row echelon form, we see that a is the pivot variable and b, c, d are the free variables. Thus we get as the general solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$,

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3b - c \\ b \\ c \\ d \end{bmatrix} = b \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where b, c, d range freely over $\mathbb{R}$. Thus,

$$K = \mathrm{Nul}(A) = \mathrm{span}\left(\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right).$$

It is clear that these three vectors are linearly independent (this follows from a routine argument involving the bottom 3 rows only). It follows that

$$\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

forms a basis for K.
(iii) The matrix

$$A := \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

is already of echelon form. Hence its pivot columns form a basis of $\mathrm{Col}(A)$. Hence

$$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

is basis of $\mathrm{Col}(A)$.
Alternatively, we could observe that the span of the 1st, 4th and 5th columns of A is all of $\mathbb{R}^3$. Thus $\mathrm{Col}(A) = \mathbb{R}^3$ and so *any* basis of $\mathbb{R}^3$ would also be a basis of $\mathrm{Col}(A)$ in this case.
(iv) As above in (iii), let

$$A := \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We bring A to reduced echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - 3R2} \begin{bmatrix} 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence the free variables of $A\mathbf{x} = 0$ are x_2, x_5. We have $A\mathbf{x} = 0$ iff

$$\begin{aligned} x_1 &= -2x_2 + 3x_5 \\ x_3 &= -x_5 \\ x_4 &= 0. \end{aligned}$$

Hence every vector $\mathbf{v} \in \mathrm{Nul}(A)$ is of the form

$$\begin{bmatrix} -2x_2 + 3x_5 \\ x_2 \\ -x_5 \\ 0 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence

$$\mathrm{Nul}(A) = \left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

and so

$$\left\{ \begin{bmatrix} -2 \\ \mathbf{1} \\ 0 \\ 0 \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} -3 \\ \mathbf{0} \\ -1 \\ 0 \\ \mathbf{1} \end{bmatrix} \right\}$$

is a basis of $\mathrm{Nul}(A)$.[Note: if you transform A into reduced echelon form and write $\mathrm{Nul}(A)$ as span of vectors whose coefficients are the free variables (as we did above) then those vectors always will be linearly independent (Why? in the set above, since the free variables were x_2 and x_5, you can make an argument that the vectors are linearly independence using the 2nd and 5th entries of the column vectors (in **bold** above)). So here there is no need to check linear independence, since this particular method guarantees that they are!] $\square$

2. *Determine the rank of A and the dimension of* $\mathrm{Nul}(A)$, $\mathrm{Col}(A)$, $\mathrm{Nul}(A^T)$, *and* $\mathrm{Col}(A^T)$ *where*

$$A := \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}.$$

Solution. In order to determine the dimension of the two subspaces, we just have to determine the number of pivot columns and free variables of A. So we bring A to echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix} \xrightarrow{R2\to R2-R1, R3\to R3-2R1, R4\to R4-3R1} \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 0 & 0 & -3 & 6 & 0 \\ 0 & 0 & -9 & 18 & -7 \\ 0 & 0 & -9 & 18 & -15 \end{bmatrix}$$

$$\xrightarrow{R3\to R3-3R2, R4\to R4-3R2} \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 0 & 0 & -3 & 6 & 0 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & -15 \end{bmatrix}$$

$$\xrightarrow{R4\to R4-(15/7)R2} \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 0 & 0 & -3 & 6 & 0 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon form of A has three pivots columns and two non-pivot columns. Hence the rank of A is 3 and $\dim \mathrm{Col}(A)$ = number of pivot columns = 3 and $\dim \mathrm{Nul}(A)$ = number of free variables = 2. $\square$

3. *Let A, B be two 4×3 matrices. Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ be the columns of A and let $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ be the columns of B.*

(i) *Suppose that $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are linearly independent. Find a basis for* $\mathrm{Col}(A)$ *and a basis for* $\mathrm{Nul}(A)$. *What are the dimensions of* $\mathrm{Col}(A)$ *and* $\mathrm{Nul}(A)$*?*

(ii) *Suppose that $\boldsymbol{b}_1, \boldsymbol{b}_2$ are linearly independent and $\boldsymbol{b}_3 = 2\boldsymbol{b}_1 + 7\boldsymbol{b}_2$. Find a basis for* $\mathrm{Col}(B)$ *and a basis for* $\mathrm{Nul}(B)$.

Solution. (i) $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a basis of $\mathrm{Col}(A)$, because it spans $\mathrm{Col}(A)$ (they are the columns of A!) and they are linearly independent (by assumption). Since the basis of $\mathrm{Col}(A)$ has size 3, the dimension of $\mathrm{Col}(A)$ is 3. Since the rank of A is 3 which is the total number of columns of A, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (since there are no free variables). Hence

$$\mathrm{Nul}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \mathrm{span}(),$$

i.e., it is the span of the set of no vectors. Technically speaking in this case we say that the empty set $\emptyset$ is a basis for $\mathrm{Nul}(A)$, and the dimension of $\mathrm{Nul}(A)$ is zero since it's basis consists of zero vectors.(ii) $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis of $\mathrm{Col}(B)$: by assumption it is linearly independent and $\mathbf{b}_3 = 2\mathbf{b}_1 + 7\mathbf{b}_2$, so we can show each column of B as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$. Hence, we have $\mathrm{span}(\mathbf{b}_1, \mathbf{b}_2) = \mathrm{span}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = \mathrm{Col}(B)$. Also $\dim \mathrm{Col}(B) = 2$ since the basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ contains 2 vectors. Now let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be in $\mathrm{Nul}(B)$. Then

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3.$$

Since $\mathbf{b}_3 = 2\mathbf{b}_1 + 7\mathbf{b}_2$, this happens iff

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3(2\mathbf{b}_1 + 7\mathbf{b}_2) = (x_1 + 2x_3)\mathbf{b}_1 + (x_2 + 7x_3)\mathbf{b}_2.$$

Since $\mathbf{b}_1, \mathbf{b}_2$ are linearly independent, the equation $y_1\mathbf{b}_1 + y_2\mathbf{b}_2 = 0$ has only the solution $y_1 = y_2 = 0$. Hence the above equation gives

$$0 = x_1 + 2x_3 = x_2 + 7x_3.$$

Hence every vector in $\mathrm{Nul}(B)$ is of the form

$$\begin{bmatrix} -2x_3 \\ -7x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -7 \\ 1 \end{bmatrix},$$

and so $\left\{ \begin{bmatrix} -2 \\ -7 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathrm{Nul}(B)$ and the dimension of $\mathrm{Nul}(B)$ is 1. $\square$

4. *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *and let* $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

(i) *Let* $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$*.Express* $\mathbf{v}$ *in terms of the basis* $\mathcal{B}$ *(i.e., realize* $\mathbf{v}$ *as a linear combination of the vectors from* $\mathcal{B}$*). What is the coordinate vector* $\mathbf{v}_\mathcal{B}$ *of* $\mathbf{v}$ *in terms of the basis* $\mathcal{B}$*.*

(ii) *Let* $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$*.Express* $\mathbf{w}$ *in terms of the basis* $\mathcal{B}$*. What is the coordinate vector* $\mathbf{w}_\mathcal{B}$ *of* $\mathbf{w}$ *in terms of the basis* $\mathcal{B}$*.*

Solution. (i) We need to solve

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 3 \end{array}\right] \xrightarrow{R2 \to R2 - R1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & 1 \end{array}\right]$$
$$\xrightarrow{R2 \to R2/(-2)} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} \end{array}\right]$$
$$\xrightarrow{R1 \to R1 - R2} \left[\begin{array}{cc|c} 1 & 0 & \frac{5}{2} \\ 0 & 1 & -\frac{1}{2} \end{array}\right].$$

Hence

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence the coordinate vector $\mathbf{v}_\mathcal{B}$ is $\begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \end{bmatrix}$.

(ii) As in (i) we can show

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence the coordinate vector $\mathbf{w}_\mathcal{B}$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\square$

5. *Consider the vector space* $\mathbb{P}_2$ *of polynomials of degree at most* 2*. Let* $\mathcal{B} = \{t^2+t+1, t+1, 1\}$*.*

(a) *Check that* $\mathcal{B}$ *is a basis for* $\mathbb{P}_2$*.*

(a) *Suppose* $p(t) \in \mathbb{P}_2$ *has coordinate vector* $p(t)_\mathcal{B} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$*. What is* $p(t)$*?*

Solution. (a) To check that $\mathcal{B}$ is a basis for $\mathbb{P}_2$, we need to check two things.**First**: We need to check that $\mathcal{B}$ spans all of $\mathbb{P}_2$. This means that given an arbitrary element $at^2 + bt + c \in \mathbb{P}_2$, we must show that it can be written as a linear combination of the elements of $\mathcal{B}$. This can be done because we can write t^2, t and 1 as linear combinations of elements of $\mathcal{B}$, i.e.,

$$\begin{aligned} at^2 + bt + c &= a((t^2+t+1) - (t+1)) + b((t+1) - 1) + c \cdot 1 \\ &= a(t^2+t+1) + (b-a)(t+1) + (c-b) \cdot 1 \end{aligned}$$

where we use the color blue to indicate elements of $\mathcal{B}$.**Second**: Next we check that $t^2 + t + 1, t + 1$ and 1 are indeed linearly independent. Suppose we have a linear combination

$$a(t^2+t+1) + b(t+1) + c = 0.$$

Our job is to show that $a = b = c = 0$ is the only solution. First its clear that $a = 0$ since the right hand side of the above expression is 0 so it has degree less than 2 as a polynomial. Thus the right hand side must have degree less than 2 also, so $a = 0$, for otherwise the right hand side would have degree 2 ($t^2 + t + 1$ is the only basis element that has a t^2 in it). Thus we have reduced to

$$b(t+1) + c = 0.$$

By a similar argument, it must also be the case that $b = 0$. But then we are left with $c = 0$.

(b) Since $p(t)_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, we have that

$$p(t) = 2 \cdot (t^2 + t + 1) + 1 \cdot (t + 1) + 3 \cdot 1 = 2t^2 + 3t + 6.$$

□

6. *Which of the following mappings T are linear? Justify your answer!*

(a) *$T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 x_2$.*

(b) *$T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ z \end{bmatrix}$.*

(c) *$T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = e^x$.*

(d) *$T : \mathbb{P}_3 \to \mathbb{P}_3$ defined by $T(f(t)) = \frac{d}{dt} f(t)$, where $\mathbb{P}_3$ is the space of all polynomials of degree at most 3.*

(e) *$T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\mathbf{v}) = \mathbf{v} + \mathbf{v}_0$, for some fixed $\mathbf{v}_0 \neq \mathbf{0} \in \mathbb{R}^n$.*

Solution. (a) No, T is not linear. Counterexample:

$$1 = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 0 + 0 = 0.$$

(b) Yes, T is linear. For arbitrary $\alpha, \beta \in \mathbb{R}$ we have:

$$T\left(\alpha \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \beta \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \\ \alpha z_1 + \beta z_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \alpha z_1 + \beta z_2 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ z_1 \end{bmatrix} + \beta \begin{bmatrix} x_2 \\ z_2 \end{bmatrix}$$

$$= \alpha T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + \beta T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$$

(c) No, T is not linear since it does not send the zero vector (which in this case is the same thing as the real number $0 \in \mathbb{R}$) to the zero vector. Specifically, $T(0) = e^0 = 1 \neq 0$.

(d) Yes, T is linear. For any $f(t), g(t) \in \mathbb{P}_3$ and scalars $\alpha, \beta \in \mathbb{R}$, we have

$$T(\alpha f(t) + \beta g(t)) = \frac{d}{dt}(\alpha f(t) + \beta g(t)) = \alpha \frac{d}{dt} f(t) + \beta \frac{d}{dt} g(t) = \alpha T(f(t)) + \beta T(g(t)).$$

(e) No, T is not linear since it send the zero vector to the vector $\mathbf{v}_0$ (i.e., $T(\mathbf{0}) = \mathbf{v}_0$).

□

7. *Let* $L\colon \mathbb{R}^2 \to \mathbb{R}^3$ *be a linear transformation such that*

$$L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\8\\4\end{bmatrix}, \quad L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\0\\1\end{bmatrix}.$$

What is $L\left(\begin{bmatrix}2\\1\end{bmatrix}\right)$*?*

Solution. Since L is linear, we get

$$L\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = L\left(2\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = 2L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 2\begin{bmatrix}2\\8\\4\end{bmatrix} + \begin{bmatrix}3\\0\\1\end{bmatrix} = \begin{bmatrix}7\\16\\9\end{bmatrix}. \qquad \square$$

8. *Consider the following eight vectors in $\mathbb{R}^8$:*

$$\mathcal{B} = \left(\begin{bmatrix}1\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\-1\\1\end{bmatrix} \right), \ \boldsymbol{w} = \begin{bmatrix}100\\200\\44\\50\\20\\20\\4\\2\end{bmatrix}.$$

(1) *Show that $\mathcal{B}$ is a basis of $\mathbb{R}^8$.*
(2) *Calculate $\boldsymbol{w}_{\mathcal{B}}$.*
(3) *Suppose Alice needs to inform Bob about $\boldsymbol{w}$. However, Alice can transmit only four non-zero numbers (and not eight) to Bob; that is Alice can only send Bob a vector in $\mathbb{R}^8$ which has at most four non-zero entries. Luckily Bob does not need to know the exact values in $\boldsymbol{w}$, a good approximation suffices. Using your result in (2), which vector should Alice communicate to Bob?*

Solution. (1) One solution would be to count the number of pivots in the reduced row echelon form of the matrix whose columns are the members of $\mathcal{B}$. There will be pivots in each of the columns of this matrix, so $\mathcal{B}$ is linearly independent. As $\mathcal{B}$ has eight elements, this means it is a basis for $\mathbb{R}^8$.We will show this is independent by inspection. Suppose we have $\mathcal{B} = \{w_1, \ldots, w_8\}$, and $a_1w_1 + \ldots + a_8w_8 = 0$. Note first that w_1 and w_5 the only ones to be nonzero in the first and second entry. Looking at these entries, we must have that $a_1 = a_2 = 0$. Looking at the third/forth entry and so on, we can show that all the a_i are 0, so $\mathcal{B}$ is a set of 8 linearly independent vectors in $\mathbb{R}^8$ and thus a basis.

(2) We have to solve:

$$\begin{bmatrix}1&0&0&0&-1&0&0&0\\1&0&0&0&1&0&0&0\\0&1&0&0&0&-1&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&-1&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&-1\\0&0&0&1&0&0&0&1\end{bmatrix} \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\\x_8\end{bmatrix} = \begin{bmatrix}100\\200\\44\\50\\20\\20\\4\\2\end{bmatrix}.$$

Solving this system, we see that

$$\boldsymbol{w}_{\mathcal{B}} = \begin{bmatrix}150\\47\\20\\3\\50\\3\\0\\-1\end{bmatrix}.$$

(3) Alice should communicate only the values in $\boldsymbol{w}_{\mathcal{B}}$ that are far away from 0. Thus Alice should send

$$\boldsymbol{v}_{\mathcal{B}} := \begin{bmatrix} 150 \\ 47 \\ 20 \\ 0 \\ 50 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Bob would use $\boldsymbol{v}_{\mathcal{B}}$ to reconstruct a vector $\boldsymbol{v}$ whose coordinate vector with respect to $\mathcal{B}$ is $\boldsymbol{v}_{\mathcal{B}}$. So

$$\boldsymbol{v} = 150\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 47\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 20\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 50\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 47 \\ 47 \\ 20 \\ 20 \\ 0 \\ 0 \end{bmatrix}.$$

That is not exactly $\boldsymbol{w}$, but it is close enough for Bob's purposes.
This is the idea behind many data compression methods, in particular JPEG. Note that the choice of the basis $\mathcal{B}$ is crucial and we used the fact that always two entries in $\boldsymbol{w}$ that are next to each other, are similar in value. If you use JPEG to compress an image, then JPEG will be more efficient if the image is rather uniform (ie big parts of it are of similar colour).

$\square$

The following may be useful in the above problems:

Definition. The **nullspace** of an $m \times n$ matrix A, written $\mathrm{Nul}(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In other words,$\mathrm{Nul}(A) = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$.

Definition. The **column space** of an $m \times n$ matrix A, written $\mathrm{Col}(A)$, is the set of all linear combinations of columns of A. In other words, if $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$, then$\mathrm{Col}(A) = \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Definition. A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V is a **basis** if

(1) $V = \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$, and
(2) the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly independent.

Definition. A vector space has **dimension** d if it has a basis consisting of d vectors.

Definition. Let V and W be vector spaces. A map $T : V \to W$ is a **linear transformation** if

$$T(c\mathbf{x} + d\mathbf{y}) = cT(\mathbf{x}) + dT(\mathbf{y})$$

for all $c, d \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$.