Worksheet 10 for November 3rd and 5th

1. *Let* $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ *and* $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. *Using Gram-Schmidt, find an orthonormal basis for* $W = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, *using* $\mathbf{v}_1, \mathbf{v}_2,$ *and* $\mathbf{v}_3$.

2. *Let* $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(i) *Calculate* $A^T A$. *What does this tell you about the columns of* A*?*

(ii) *Find an orthonormal basis* $\{q_1, q_2\}$ *for* $\operatorname{Col}(A)$ *(starting with the columns of* A*!). Put* $Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$. *What is* Q^{-1}*?*

3. *Let*

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

the matrix for rotation by the angle θ *(counterclockwise).*

(i) *Calculate* $Q_\theta^T Q_\theta$. *What does this tell you about the columns of* Q_θ*?*

(ii) *What is* Q_θ^{-1}*? Express* Q_θ^{-1} *in terms of another rotation matrix* Q_ϕ.

(iii) *Show that if* $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ *then the vector* $\mathbf{x}$ *and the rotated vector* $Q_\theta \mathbf{x}$ *have the same length.*

4. *Let* P *be the matrix*

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(i) *Compute the dot products between every two columns of* P.

(ii) *What is* P^{-1}*?*

Now let P *be an arbitrary* $n \times n$ *permutation matrix, so each row and each column has a single non zero entry 1. Write* $P = \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix}$.

(iii) *What is the dot product between the columns of* P, *i.e., what is* $P_i^T P_j$*?*

(iv) *What is* P^{-1}*?*

5. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

a. *Find the QR decomposition of* A*: write* $A = QR$ *where* Q *is a matrix with orthonormal columns and* R *is an upper triangular matrix.*

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Midterm Date: November 19 7-8:15 PM, Conflict November 20, 8-9.20AM and 9:30-10:50AM, Conflict sign up deadline: November 13

Final Date: December 17 8-11AM, Conflict December 15, 8-11AM. You are allowed to take the conflict exam if you have more than two examination within 24 hours. Conflict sign up deadline: November 30

b. *Let* $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. *Use the QR decomposition of A to find the least squares solution of* $A\hat{\mathbf{x}} = \mathbf{b}$ *(by solving* $R\hat{\mathbf{x}} = Q^T\mathbf{b}$*).*

6. **a.** *Recall that the orthogonal projection onto Col*(A) *has projection matrix* $A(A^TA)^{-1}A^T$. *How does this formula simplify in the case when A has orthonormal columns?*

b. *Let* $Q = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{5} \\ 0 & -\frac{4}{5} \end{bmatrix}$. *What is the projection matrix corresponding to the orthogonal projection onto Col*(Q)*?*

c. *Let* $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$. *What is the projection matrix corresponding to the orthogonal projection onto Col*(Q)*? Explain why your answer is not surprising.*

7. *Quarterly economic data is subject to seasonal fluctuations. A curve that approximates the gross domestic product (GDP) of a country might be of the form*

$$y = \beta_0 + \beta_1 x + \beta_2 \sin(2\pi x/4),$$

where is x *is the time in quarters of a year. The term* $\beta_0 + \beta_1 x$ *gives the basic GDP growth trend of the economy, while the sine term reflects the seasonal changes. Assume the GDP data are* $(x_1, y_1), \ldots, (x_n, y_n)$.

(i) *Give the design matrix that leads to a least-square fit to the equation above.*

(ii) *(Highly Optional) GDP data for US economy is available at http://www.bea.gov/. Using the above, can you find the GDP growth trend of the US economy.*

8. *According to Kepler's first law, a comet should have an elliptic, parabolic or hyperbolic orbit. In suitable polar coordinates, the position* (r, θ) *of a comet satisfies an equation*

$$r = \beta + e(r \cdot \cos(\theta)),$$

where β *is a constant and* e *is the eccentricity of the the orbit, with* $0 \le e < 1$ *for an ellipse,* $e = 1$ *for a parabola and* $e > 1$ *for a hyperbola. Suppose observations of a newly discovered comet provide the data below.*

θ	.88	1.10	1.42	1.77	2.14
r	3.00	2.30	1.65	1.25	1.01

Use least square methods to find the type of the orbit, and predict where the comet will be when $\theta = 4.6$ *(radians).*