Math 415 - Lecture 9

Vector spaces and subspaces

Monday September 14th 2015

Textbook: Chapter 2.1.

Suggested practice exercises: Chapter 2.1: 1, 2, 10, 11, 17, 18.

Khan Academy video: Linear Subspaces

We know how to find the inverse of a 2×2 matrix. What about 3×3,....,$n\times n$? We use:

Theorem 1. *An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1}.*

Note that this tells us a lot about $Ax = b$ if A is invertible.

- $Ax = b$ has how many pivots in A?
- How many free variables?
- Can $Ax = b$ be inconsistent?

Here is the algorithm to find the inverse of a matrix A, called the *Gauss-Jordan Method*

- Place A and I side-by-side to form an augmented matrix $[\ A \mid I\ ]$. This is an $n \times 2n$ matrix (*B*ig *A*ugmented *M*atrix), instead of $n \times (n + 1)$, for the usual augmented matrix.
- Then perform row operations on this matrix (which will produce identical operations on A and I).
- So by the Theorem:

$$[\ A \mid I\ ] \text{ will row reduce to } \left[\ I \mid A^{-1}\ \right]$$

or A is not invertible.

Example 1. Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$[A \mid I\ ] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$$

Example 2 (Let's do the previous example step by step.).

$$[\ A \mid I\ ] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R1 \to \frac{1}{2}R1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R2 \to R2 + 3R1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R2 \leftrightarrow R3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

Check at home that $AA^{-1} = I_3$.

Remark. Why does this algorithm work?

- At each step, we get

$$[\ A \mid I\ ] \rightsquigarrow [\ E_1A \mid E_1\ ] \rightsquigarrow [\ E_2E_1A \mid E_2E_1\ ] \rightsquigarrow \ldots$$

- So each step is of the form

$$[\ FA \mid F\ ], \quad F = E_r \ldots E_3E_2E_1$$

- If we succeed in row reducing A to I, the final step is

$$[\ FA \mid F\ ] = [\ I \mid F\ ]$$

- So $FA = I$, which means that $A^{-1} = F$.

Example 3. Use the Gauss Jordan method to compute the inverse of

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Solution.

$$[\ A \mid I\ ] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R3\to R1+R3}{\to} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\underset{R3\to R2+R3}{\to} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

Failure: the reduced row echelon form of A will not be I, so A has no inverse!

Practice Problems. Find the inverse of A:

- $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.
- $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$. Hint: What is $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?
- $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
- $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 8 & 0 \\ 9 & 0 & 1 & 0 \end{bmatrix}$.
- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

1 Vector Spaces and Subspaces

- The most important property of column vectors in $\mathbb{R}^n$ is that you can take *linear combinations* of them.
- There are many mathematical objects $X, Y, \ldots$ for which a linear combination $cX + dY$ make sense, and have the usual properties of linear combination in $\mathbb{R}^n$

- We are going to define a *vector space* in general as a collection of objects for which linear combinations make sense. The objects of such a set are called vectors.

Definition. A **vector space** is a non-empty set V of objects, called *vectors*, for which linear combinations make sense. More precisely: on V there are defined two operations, called *addition* and *multiplication* by scalars (real numbers), subject to the ten axioms below. The axioms must hold for all u, v, and w in V and for all scalars c and d.

1. $\mathbf{u} + \mathbf{v}$ is in V. (V is "closed under addition".)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

Definition Continued

6. $c\mathbf{u}$ is in V. (V is "closed under scalar multiplication".)
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $(cd)\mathbf{u} = c(d\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$.

2 Vector Space Examples

Example 4. Let $M_{2x2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$. This is a vector space.

We need to say what the two operations are. Addition:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}.$$

Scalar Multiplication:

$$e \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea & eb \\ ec & ed \end{bmatrix}.$$

Next we need to say what the zero vector is. Question: What is the matrix $\mathbf{0}$ such that $\mathbf{0} + A = A$ for any (2×2) matrix A? Answer: We see that the $\mathbf{0}$ vector is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then we need to check all the 10 axioms. They follow from the corresponding properties of ordinary numbers.

Remarks

- We can take instead of matrices of size 2×2 matrices of any shape: you can check that the set $M_{m\times n}$ of $m \times n$ matrices is also a vector space, in the same way as we indicated above.
- Confusing: in the vector space $M_{2\times 2}$ the vectors are in fact 2×2 matrices!
- In the definition of the vector space $M_{2\times 2}$ the multiplication of matrices plays no role; matrix multiplication will show up when we study the connections *between* vector spaces.
- a "vector" $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ behaves very much like a column vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. A fancy person would say that the vector spaces $M_{2\times 2}$ and $\mathbb{R}^4$ are *isomorphic*.

Example 5. Let $n \geq 0$ be an integer and let

$$\mathbf{P}_n = \text{the set of all polynomials of degree at most } n.$$

This is a vector space.

Members of $\mathbf{P}_n$ have the form

$$\mathbf{p}(t) = a_0 + a_1 t + \cdots + a_n t^n$$

where $a_0, a_1, \ldots, a_n$ are real numbers and t is a variable. We will just verify 3 out of the 10 axioms here.

Vector Space Examples

Let $\mathbf{p}(t) = a_0 + a_1 t + \cdots + a_n t^n$ and $\mathbf{q}(t) = b_0 + b_1 t + \cdots + b_n t^n$ and let c be a scalar. The polynomial $\mathbf{p} + \mathbf{q}$ is defined as follows:

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t).$$

Therefore,

$$\begin{aligned} (\mathbf{p} + \mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t) \\ &= (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n. \end{aligned}$$

which is also a polynomial of degree at most n. So $\mathbf{p} + \mathbf{q}$ is in $\mathbf{P}_n$ (i.e. $\mathbf{P}_n$ is closed under addition). This verifies Axiom 1.

Vector Space Examples

Next we need to find a zero vector. **Question:** What this is the polynomial $\mathbf{0}(t)$ such that $\mathbf{0}(t) + p(t) = p(t)$? **Answer:** Take $\mathbf{0}(t) = 0 + 0t + \cdots + 0t^n$ (zero vector in $\mathbf{P}_n$) Then

$$\begin{aligned} (\mathbf{p} + \mathbf{0})(t) &= (a_0 + a_1 t + \cdots + a_n t^n) + (0 + 0t + \cdots + 0t^n) \\ &= (a_0 + 0) + (a_1 + 0)t + \cdots + (a_n + 0)t^n \\ &= a_0 + a_1 t + \cdots + a_n t^n \end{aligned}$$

and so $\mathbf{p} + \mathbf{0} = \mathbf{p}$. This verifies Axiom 4. Next we define scalar multiplication. Remember $\mathbf{p}(t) = a_0 + a_1 t + \cdots + a_n t^n$. We define

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = (ca_0) + (ca_1)t + \cdots + (ca_n)t^n$$

which is in $\mathbf{P}_n$. so that Axiom 6 holds. The other 7 axioms also hold, so $\mathbf{P}_n$ is a vector space.

3 Subspaces

New vector spaces may be formed from subsets of other vector spaces. These are called **subspaces**.

Definition. A *subspace* of a vector space V is a subset H of V that satisfies 3 properties:

- The zero vector (of V) belongs to H.
- If $\mathbf{u}, \mathbf{v}$ both belong to H also the sum $\mathbf{u} + \mathbf{v}$ belongs to H. (H is *closed* under vector addition).
- If $\mathbf{u}$ is in H and c is any scalar also $c\mathbf{u}$ belongs to H. (H is closed under scalar multiplication.)

Note that if the subset H satisfies these three properties, then H itself is a vector space.

Example 6. $Z = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^2$. Why?

Check:

- $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in Z.
- $\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+0 \end{bmatrix}$ is in Z.
- $c \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c0 \\ c0 \end{bmatrix}$ is in Z.

Z is called the zero subspace of $\mathbb{R}^2$. Every vectorspace has a zero subspace consisting just of the zero vector.

Example 7. $H = \mathit{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^2$. Why?

Check:

- $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in H.

- $\begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix}$ is in H.
- $c \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} ca \\ ca \end{bmatrix}$ is in H.

Example 8. Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Show that H is a subspace of $\mathbb{R}^3$.

Verify properties $1, 2$, and 3 of the definition of a subspace.

- The zero vector of $\mathbb{R}^3$ is in H.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H, \quad (a = b = 0)$$

Subspaces

- Adding two vectors in H always produces another vector whose second entry is 0 and therefore the sum of two sectors in H is also in H. (H is closed under addition.)

$$\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} c \\ 0 \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ 0 \\ b+d \end{bmatrix}.$$

- Multiplying a vector in H by a scalar produces another vector in H. (H is closed under scalar multiplication.)

$$c \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} ca \\ 0 \\ cb \end{bmatrix}.$$

Since those three properties hold, H is a subspace of $\mathbb{R}^3$.

Remark. Vectors $(a, 0, b)$ look and act like the points (a, b) in $\mathbb{R}^2$. But they are **not** the same!

Example 9. Is $H = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$? (i.e. does H satisfy the properties of a subspace?)

H does not contain the zero vector (property 1).

$$\begin{bmatrix} x \\ x+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

cannot be true for any value of x. Therefore, H is **not** a subspace!

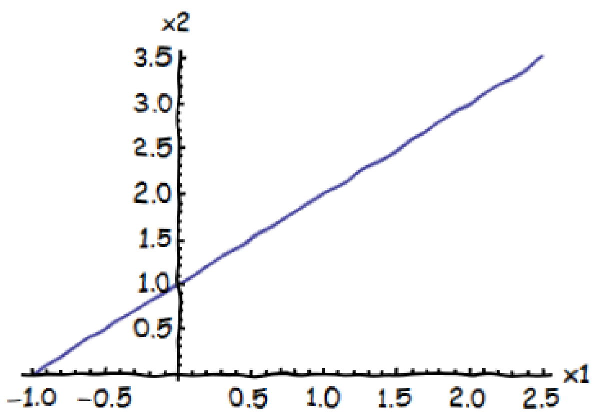
Example 10. Is $H = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$? (i.e. does H satisfy the properties of a subspace?)

Another way to show that H is not a subspace of $\mathbb{R}^2$ is to check whether H is closed under addition (property 2).

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in H$$

but

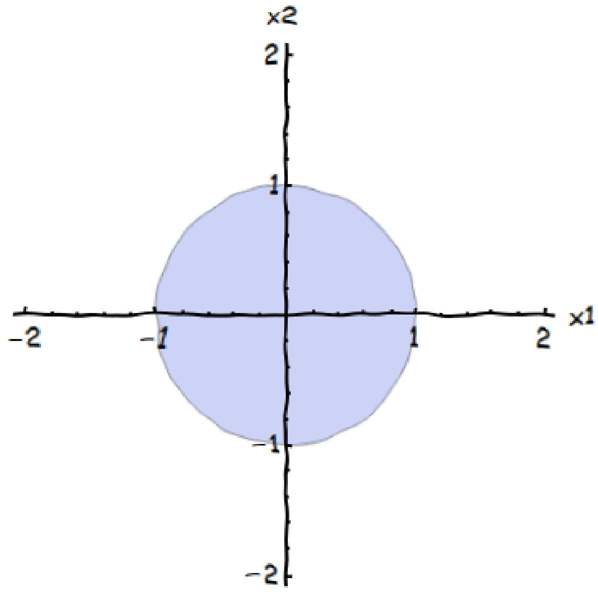
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \notin H.$$

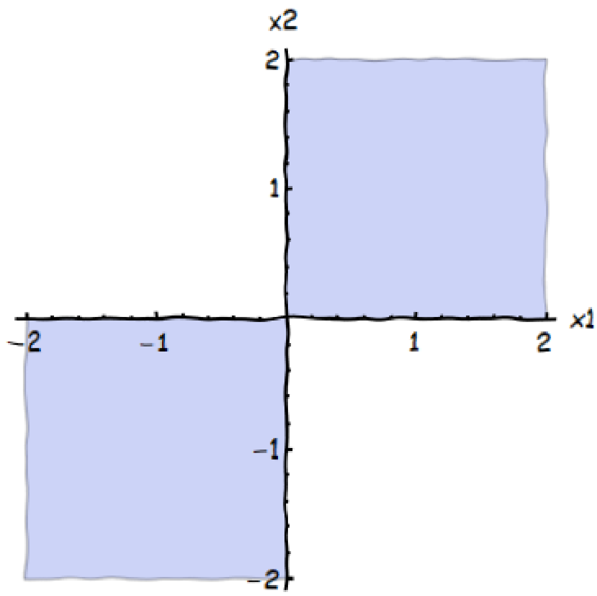


Problem 11. *Find as many subspaces in $\mathbb{R}^2$ as you can.*

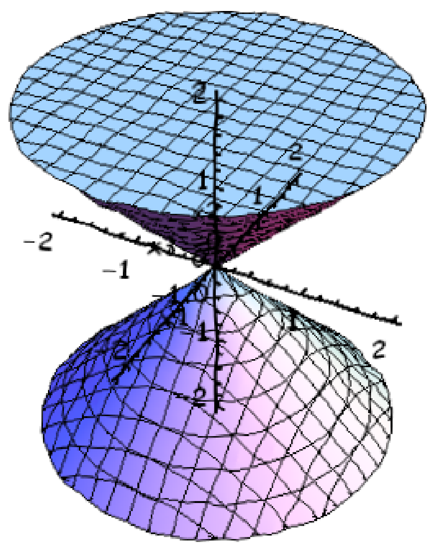
Think of this at home.

Example 12. Is one of the following a subspace of $\mathbb{R}^2$?





Example 13. Is this a subspace of $\mathbb{R}^3$?



2
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2
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-2