Math 415 - Lecture 2

Echelon Forms, General Solution.

Wednesday August 26 2015

Textbook: Chapter 1.3, Chapter 2.2 (just the pages 78 and 79)

Suggested Practice Exercise: in Chapter 1.3, Exercise 17, 23, 24, in Chapter 2.2, Exercise 2 (just reduce A, B to echelon form), 8

Khan Academy Video: Matrices: Reduced Row Echelon Form 1

1 Row Reduction and Echelon Forms

Definition. A matrix is of **Echelon form** (or **row echelon form**) if

1. All nonzero rows are above any rows of all zeros.
2. The number of *leading zeroes* in each row increase going down.
3. All entries in a column below a leading entry are zero.

A leading entry of an echelon form matrix is also called a ***PIVOT.***

Example 1. Are the following matrices in Echelon form?

(a) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Echelon form? 1. $\boxed{}$ 2. $\boxed{}$ 3. $\boxed{}$

(b) $\begin{bmatrix} 0 & \blacksquare & * & * & * \\ \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Echelon form? 1. $\boxed{}$ 2. $\boxed{}$ 3. $\boxed{}$

(c) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$ Echelon form? 1. $\boxed{}$ 2. $\boxed{}$ 3. $\boxed{}$

(d) $\begin{bmatrix} \blacksquare & 0 & 0 \\ * & \blacksquare & 0 \\ * & 0 & \blacksquare \\ * & 0 & 0 \end{bmatrix}$ Echelon form? 1. $\boxed{}$ 2. $\boxed{}$ 3. $\boxed{}$

(e) $\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$ 1. $\boxed{}$ 2. $\boxed{}$ 3. $\boxed{}$

Definition. A matrix is of the **reduced echelon form** if in addition to conditions 1, 2, and 3 above it also satisfies

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Example 2. Are the following matrices in reduced echelon form?

(a) $\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$ $\boxed{}$

(b) $\begin{bmatrix} 1 & 0 & 5 & 0 & -7 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\boxed{}$

(c) $\begin{bmatrix} 1 & 0 & -2 & 3 & 2 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ $\boxed{}$

Theorem 1 (Uniqueness of The Reduced Echelon Form)**.** *Each matrix is row-equivalent to one and only one reduced echelon matrix.*

Question: Is the same statement true for Echelon from?

2 Pivots

Definition. A **pivot position** is the position of a leading entry in an echelon form of the matrix.

Definition. A **pivot** of a matrix is a (nonzero) number that appears in a pivot position.

In a Reduced Row Echelon Form matrix the pivots are 1. Pivots are used to create 0's.

Definition. A **pivot column** is a column that contains a pivot position.

Example 3. In this example, highlight the pivot positions and pivot columns.

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 7 \\ 0 & 2 & 4 & 0 & 6 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 4. Row reduce to echelon form and locate the pivot columns for the following matrix.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution:

Note: There is no more than one pivot in any row. There is no more than one pivot in any column.

Example 5. Row reduce to echelon form and then to reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution:

3 Solution of linear systems

Definition. A **pivot variable** (or *basic variable*) is a variable that corresponds to a pivot column in the coefficient matrix of a system.

Definition. A **free variable** is variable that is *not* a pivot variable.

Example 6. Consider the following system of linear equations: $\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$

$$\begin{array}{ccccccc} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array}$$

What are the pivot columns?

What are the pivot variables?

What are the free variables?

Final Step in Solving a Consistent Linear System: After the augmented matrix is in **reduced** echelon form and the system is written down as a set of equations:

Solve each equation for the pivot variable in terms of the free variables (if any) in the equation.

Example 7 (A general solution).

$$\begin{array}{rrrrl} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array} \qquad \left\{ \begin{array}{rl} x_1 & = \boxed{} \\ x_2 & = \boxed{} \\ x_3 & = \boxed{} \\ x_4 & = \boxed{} \\ x_5 & = \boxed{} \end{array} \right.$$

The **general solution** of the system provides a **parametric description of the solution set**.

- The free variables act as parameters.
- The above system has **infinitely many solutions.** Why?

Warning: Use only the reduced echelon form to solve a system.

Example 8. Find the parametric description of the solution set of

$$\begin{array}{rrrrrl} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 15 \end{array}$$

Its augmented matrix is

$$\left[\begin{array}{rrrrr|r} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$

We determined earlier that it is reduced echelon form is

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Equation form of the RREF matrix: $\begin{cases} x_1 \quad\quad -2x_3 + 3x_4 \quad\quad = -24 \\ \quad\quad x_2 - 2x_3 + 2x_4 \quad\quad = -7 \\ \quad\quad\quad\quad\quad\quad\quad\quad x_5 = 4 \end{cases}$

Pivot variables:

$\boxed{}$

Free variables:

$\boxed{}$

General solution: $\begin{cases} x_1 = \boxed{} \\ x_2 = \boxed{} \\ x_3 = \boxed{} \\ x_4 = \boxed{} \\ x_5 = \boxed{} \end{cases}$

4 Existence And Uniqueness

Example 9. Let us go back to the following system

$$\begin{aligned}
3x_2 \quad & - 6x_3 & + 6x_4 & + 4x_5 & = -5 \\
3x_1 - 7x_2 \quad & + 8x_3 & - 5x_4 & + 8x_5 & = 9 \\
3x_1 - 9x_2 \quad & + 12x_3 & - 9x_4 & + 6x_5 & = 15
\end{aligned}$$

In an earlier example, we obtained the echelon form:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & | & 15 \\ 0 & 2 & -4 & 4 & 2 & | & -6 \\ 0 & 0 & 0 & 0 & 1 & | & 4 \end{bmatrix}$$

So for the echelon form matrix

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & | & 15 \\ 0 & 2 & -4 & 4 & 2 & | & -6 \\ 0 & 0 & 0 & 0 & 1 & | & 4 \end{bmatrix}$$

1. Is the system consistent? Yes/No? Why?
2. What are the free variables?
3. How many solutions?

Theorem 2 (Existence and Uniqueness Theorem)**.** *A linear system is **consistent** if and only if an echelon form of the augmented matrix has **no** row of the form*

$$\left[\begin{array}{ccc|c} 0 & \dots & 0 & b \end{array}\right],$$

*where b is nonzero.**If** a linear system is consistent, then the solution contains either*

- *a unique solution (when there are no free variables) or*
- *infinitely many solutions (when there is at least one free variable).*

Example 10. The (reduced) echelon form of

$$\begin{bmatrix} 3 & 4 & | & -3 \\ 2 & 5 & | & 5 \\ -2 & -3 & | & 1 \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 0 & | & -5 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix}$$

Is the system consistent? How many pivots? How many free variables? How many solutions?

Example 11. The echelon form of

$$\begin{bmatrix} 3 & 4 & | & -3 \\ 3 & 4 & | & -3 \\ 6 & 8 & | & -6 \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 3 & 4 & | & -3 \\ 0 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

What can you say about the number of solutions?