Math 415 - Lecture 35

Quadratic forms

Monday November 30th 2015

Textbook reading: Chapter 6.2

Suggested practice exercises: Chapter 6.2, # 1, 2, 4, 5

Strang lecture: Lecture 27: Positive definite matrices and minima

1 Review

Spectral theorem:

- A is a symmetric matrix if $A^T = A$. e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 5 \end{bmatrix}$
- Any $n \times n$ symmetric matrix A has n real eigenvalues and an orthonormal eigenbasis $\{\mathbf{q}_1, \dots, \mathbf{q}_n\}$.
- So, we can write
$$A = QDQ^T$$
where
$$D = \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\text{matrix of eigenvalues}} \text{ and } Q = \underbrace{\begin{bmatrix} | & & | \\ \mathbf{q}_1 & \cdots & \mathbf{q}_n \\ | & & | \end{bmatrix}}_{\text{matrix of eigenvectors}}$$

2 Quadratic forms

2.1 Quadratic forms

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function with critical point at $\mathbf{0}$. This means that all partial derivatives at $\mathbf{0}$ vanish. Is $\mathbf{0}$ a max, min, or neither? How to tell?

- Look at the **quadratic** part of f!

Definition. A **quadratic form** $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial (in n variables) with every term degree two.

e.g., for $n = 2$

$$f(x, y) = 3x^2 + 4xy - 5y^2$$

Example 1. Let

$$f(x, y) = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Expand $f(x, y)$ as a polynomial in x and y. The dot denotes the dot product!

Solution. Expanding we get

$$\begin{aligned} f(x, y) &= \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 3x + 2y \\ 2x - 5y \end{bmatrix} \\ &= 3x^2 + 4xy - 5y^2 \end{aligned}$$

This is the quadratic function from before!

Theorem 1. *Any quadratic form $f(x_1, x_2, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ can be written*

$$f(\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x} = \mathbf{x}^T A\mathbf{x}$$

for a symmetric matrix A.

We see symmetric matrices show up "in the wild!"

Example 2. Write $f(x, y, z) = 5x^2 + 7y^2 + 3z^2 + 2xy - 2yz$ as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where A is symmetric.

Solution. A 3×3 symmetric matrix has the form $A = \begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix}$. In general,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax^2 + by^2 + cz^2 + (2d)xy + (2e)yz + (2f)xz$$

So, $A = \begin{bmatrix} 5 & 1 & 0 \\ 1 & 7 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

2.2 Principal axes for a quadratic form

Intermezzo: From Eigenbasis to Standard Basis and back.

- A symmetric, so $A = QDQ^T$.

- If $x \in \mathbb{R}^n$ and $x_Q = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is the coordinate vector of x in the Q basis, then $x = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n = Q x_Q$.
- This means that to find the Q coordinate vector for x, multiply by $Q^{-1} = Q^T$:

$$\boxed{x_Q = Q^T x}$$

There is always a "nicest possible" coordinate system for each quadratic form. Just use an eigenbasis of A.

Theorem 2. *Let A be a symmetric matrix, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ an orthonormal basis of eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_n$. Write*

$$\mathbf{x} = c_1 \mathbf{v}_n + \cdots + c_n \mathbf{v}_n \quad \text{\tiny \textit{How?}}$$

Then,

$$q(\mathbf{x}) := \mathbf{x}^T A \mathbf{x} = \lambda_1 (c_1)^2 + \cdots + \lambda_n (c_n)^2$$

Proof. A is symmetric. So write $A = QDQ^T$. Let's find $\mathbf{x}^T A \mathbf{x} = \mathbf{x}^T QDQ^T \mathbf{x}$. We know $Q^T \mathbf{x}$ writes $\mathbf{x}$ in $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ coordinates. So

$$Q^T \mathbf{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

D is the matrix of eigenvalues. So,

$$D \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \lambda_1 c_1 \\ \vdots \\ \lambda_n c_n \end{bmatrix}$$

Since $\mathbf{x}^T Q = (Q^T x)^T$, we have $x^T Q = \begin{bmatrix} c_1 & \ldots & c_n \end{bmatrix}$. Thus,

$$\mathbf{x}^T A \mathbf{x} = \begin{bmatrix} c_1 & \ldots & c_n \end{bmatrix} \begin{bmatrix} \lambda_1 c_1 \\ \ldots \\ \lambda_n c_n \end{bmatrix} = \lambda_1 (c_1)^2 + \cdots + \lambda_n (c_n)^2$$

□

Example 3. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Let $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$.

- Find the eigenvalues λ_1, λ_2 and **orthonormal** eigenbasis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for A.
- Compute $q(\mathbf{x})$ using the formula $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$.
- Compute $q(\mathbf{x})$ using the theorem ($q(\mathbf{x}) = \lambda_1 (c_1)^2 + \lambda_2 (c_2)^2$.)

Are the answers the same? **This is a silly Example. To calculate $q(x)$ you never would go through the eigenvalues.**

Solution. Eigenvalues: Sum $\lambda_1 + \lambda_2 = \text{Tr}(A) = 2$ Product $\lambda_1\lambda_2 = \det(A) = -3$. So, $\lambda_1 = 3$, $\lambda_2 = -1$.

Eigenbasis: $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

Compute using formula:

$$\begin{aligned}
q(\mathbf{x}) &= \mathbf{x}^T A\mathbf{x} \\
&= \begin{bmatrix}2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix} \\
&= \begin{bmatrix}2 & 0\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix} \\
&= 4
\end{aligned}$$

Using theorem: $\mathbf{x} = \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} = \sqrt{2}\mathbf{v}_1 + \sqrt{2}\mathbf{v}_2$. So,

$$\begin{aligned}
q(\mathbf{x}) &= \lambda_1(c_1)^2 + \lambda_2(c_2)^2 \\
&= 3(\sqrt{2})^2 + (-1)(\sqrt{2})^2 \\
&= 4
\end{aligned}$$

Get same answer!

We have $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ and

$$q(\mathbf{x}) = \lambda_1(c_1)^2 + \cdots + \lambda_n(c_n)^2$$

- So up to coordinate change, $\mathbf{q}$ is a weighted sum of squares.
- The eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are called principal axes

Definition 4. Let A be a symmetric $n \times n$. We say A is **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for all non zero $\mathbf{x} \in \mathbb{R}^n$.

Theorem 3. *Let A be a symmetric $n \times n$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Then*

1. *If all $\lambda_i > 0$, then A is positive definite,*
2. *If all $\lambda_i < 0$, then $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$*
3. *If some $\lambda_i > 0$, some $\lambda_j < 0$, $\mathbf{x}^T A\mathbf{x}$ will have both positive and negative values.*

2.3 Completing the squares

Basic Question. Let A be a symmetric matrix, and $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. Is $q(\mathbf{x})$ always ≥ 0? Or always ≤ 0? How to decide? Write $q(\mathbf{x})$ as a sum of squares!

Example 5. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, so that $q(\mathbf{x}) = x^2 + 4xy + y^2$. Write $q(\mathbf{x})$ as a sum of squares. Is $q(\mathbf{x})$ always positive?

Solution. * $q(\mathbf{x}) = x^2 + 4xy + y^2 = (x + 2y)^2 - 3y^2$.

* Sometimes you get something positive, sometimes something negative.

There are many ways of writing $q(\mathbf{x})$ as a sum of squares. Today we are using eigenvalues to do this.