1. Chebyshev's Inequality : $P(|X-\mu| \geq k) \leq \sigma^2/k^2$

then $P(|W_n-\mu| \geq \varepsilon) \leq \frac{b/n^p}{\varepsilon^2} = \frac{b}{n^p\varepsilon^2} \rightarrow 0$ as $n \rightarrow \infty$,

then $\lim_{n\to\infty} P(|W_n-\mu| \geq \varepsilon) = 0 \Rightarrow W_n \xrightarrow{P} \mu$.

2. a) $P(|Y_n-\theta| \geq \varepsilon) = \underbrace{P(Y_n \geq \varepsilon+\theta)}_{0} + P(Y_n \leq \theta-\varepsilon)$

$= P(Y_n \leq \theta-\varepsilon) = P(\max(X_1,\dots,X_n) \leq \theta-\varepsilon)$

$= P(X_1 \leq \theta-\varepsilon, \dots, X_n \leq \theta-\varepsilon) = P(X_1 \leq \theta-\varepsilon) \dots P(X_n \leq \theta-\varepsilon)$

$= \left(\frac{\theta-\varepsilon}{\theta}\right)^n \longrightarrow 0$ as $n \rightarrow \infty$ since $\left|\frac{\theta-\varepsilon}{\theta}\right| < 1$,

thus $Y_n \xrightarrow{P} \theta$.

b) $P(|\frac{n+1}{n}Y_n-\theta| \geq \varepsilon) = P(|\frac{n+1}{n}||Y_n-\theta| \geq \varepsilon)$

$= P(|Y_n-\theta| \geq \frac{n}{n+1}\varepsilon) = \underbrace{P(Y_n \geq \frac{n}{n+1}\varepsilon+\theta)}_{0} + P(Y_n \leq \theta - \frac{n}{n+1}\varepsilon)$

$= P(Y_n \leq \theta-\frac{n}{n+1}\varepsilon) = P(\max(X_1,\dots,X_n) \leq \theta-\frac{n}{n+1}\varepsilon)$

$= P(X_1 \leq \theta-\frac{n}{n+1}\varepsilon) \dots P(X_n \leq \theta-\frac{n}{n+1}\varepsilon)$

$= \left(\frac{\theta-\frac{n}{n+1}\varepsilon}{\theta}\right)^n = \left(1-\frac{n}{(n+1)}\frac{\varepsilon}{\theta}\right)^n \rightarrow 0$ as $n \rightarrow \infty$,

thus $\frac{n+1}{n}Y_n \xrightarrow{P} \theta$.

3. $P(|X_n-0| \geq \varepsilon) = P(X_n \geq \varepsilon) = 1-P(X_n \leq \varepsilon)$

$= 1-(1-\frac{1}{n}e^{\varepsilon/n}) = 1-1+\frac{1}{n}e^{\varepsilon/n} = 0$ as $n \rightarrow \infty$,

then $X_n \xrightarrow{P} 0$.

4. First show $Y_1 \xrightarrow{P} 0$

$$P(|Y_1 - 0| \geq \varepsilon) = P(Y_1 \geq \varepsilon) = P(\min(X_1, \dots, X_n) \geq \varepsilon)$$

$$= P(X_1 \geq \varepsilon, \dots, X_n \geq \varepsilon) = P(X_1 \geq \varepsilon) \cdots P(X_n \geq \varepsilon)$$

$$= (1-\varepsilon)^n \to 0 \text{ as } n \to \infty, \text{ then } Y_1 \xrightarrow{P} 0.$$

From 2(a), $Y_n \xrightarrow{P} 1$, then

$$P(|Y_1 + Y_n - 1| \geq \varepsilon) = P(|Y_1 + Y_n - 0 - 1| \geq \varepsilon)$$

$$\leq P(|Y_1 - 0| + |Y_n - 1| \geq \varepsilon)$$

$$\leq \underbrace{P(|Y_1 - 0| \geq \tfrac{\varepsilon}{2})}_{0} + \underbrace{P(|Y_n - 1| \geq \tfrac{\varepsilon}{2})}_{0}$$

According to definition, hence $Y_1 + Y_n \xrightarrow{P} 1$

Problem 5.

$X_i \overset{iid}{\sim} f(x) = e^{-(x-\theta)} \quad \theta < x < \infty \Rightarrow X_i - \theta \sim expo(1)$

$\therefore Y_1 - \theta = \min\{X_i - \theta, 1 \le i \le n, X_i - \theta \sim expo(1)\}$

$F_{Z_n}(z) = Pr(n(Y_1-\theta) \le z) = Pr(Y_1 - \theta \le \frac{z}{n}) = 1 - e^{-\frac{z}{n}\cdot n} = 1 - e^{-z}$

$\therefore F_{Z_n}(z) \longrightarrow F_Z(z) = 1 - e^{-z}, \quad Z \sim expo(1).$

Problem 6.

a. Recall the MGF of Poisson(μ): $\exp(\mu(e^t - 1))$

$$\begin{aligned}
M_{Y_n}(t) &= E(e^{tY_n}) \\
&= E(e^{t\sqrt{n}(\bar{X}_n - 1)}) \\
&= e^{-t\sqrt{n}} \cdot E(e^{t\sqrt{n}\cdot\frac{1}{n}\sum_{i=1}^n X_i}) \\
&= e^{-t\sqrt{n}} \cdot E(e^{\sum_{i=1}^n \frac{t}{\sqrt{n}} X_i}) \\
&= e^{-t\sqrt{n}} \cdot \prod_{i=1}^n E(e^{\frac{t}{\sqrt{n}} X_i}) \\
&= e^{-t\sqrt{n}} \cdot [\exp(e^{\frac{t}{\sqrt{n}}} - 1)]^n \\
&= \exp(-t\sqrt{n} + n(e^{\frac{t}{\sqrt{n}}} - 1)).
\end{aligned}$$

b. $e^{t/\sqrt{n}} = 1 + \frac{t}{\sqrt{n}} + \frac{1}{2}\frac{t^2}{n} + o(\frac{1}{n}) \quad$ as $n \to \infty$

$\Rightarrow n(e^{t/\sqrt{n}} - 1) = t\sqrt{n} + \frac{1}{2}(t^2) + o(1)$

$\Rightarrow -t\sqrt{n} + n(e^{t/\sqrt{n}} - 1) = \frac{1}{2}t^2 + o(1)$

$\therefore M_{Y_n}(t) = \exp(\frac{1}{2}t^2 + o(1)) \longrightarrow M_Z(t) = \exp(\frac{1}{2}t^2) \quad$ as $n \to \infty$

where $Z \sim N(0,1)$

Problem 7.

From 6 b., we conclude $Y_n = \sqrt{n}(\bar{X}_n - 1) \xrightarrow{\mathcal{D}} N(0,1)$.

Denote $g(x) = \sqrt{x} \Rightarrow g'(x) = \frac{1}{2\sqrt{x}}$, $g'(1) = \frac{1}{2} \neq 0$

By Δ-Method. $\sqrt{n}(\sqrt{\bar{X}_n} - 1) \xrightarrow{\mathcal{D}} N(0, \frac{1}{4})$.

Problem 8.

Denote $Y_i = X_i^2$. Then $\{Y_i\}$ are also random sample.

$E(Y_i) = E(X_i^2) = var(X_i) + [E(X_i)]^2 = \sigma^2 + 0 = \sigma^2$

$var(Y_i) = E(Y_i^2) - [E(Y_i)]^2 = E(X_i^4) - [E(X_i^2)]^2 = \mu_4 - \sigma^4$

By CLT. $\dfrac{\sqrt{n}(\bar{Y}_n - \sigma^2)}{\sqrt{\mu_4 - \sigma^4}} \xrightarrow{\mathcal{D}} N(0,1)$

Note that $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = T_n$

$\therefore \sqrt{n}(T_n - \sigma^2) \xrightarrow{\mathcal{D}} N(0, \mu_4 - \sigma^4)$.

9. (a) $L(\theta; x) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$

$\ell(\theta; x) = n \ln\theta + (\theta-1)\ln\left(\prod_{i=1}^{n} x_i\right)$

$= n\ln\theta + (\theta-1)\sum_{i=1}^{n} \ln x_i$

$$\frac{d\,\ell(\theta; x)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln x_i = 0 \quad \Rightarrow \hat{\theta}_n = -\frac{n}{\sum_{i=1}^{n} \ln x_i}$$

(b) $EX_i = \int_0^1 x \cdot \theta x^{\theta-1} dx = \frac{\theta}{\theta+1} x^{\theta+1} \Big|_0^1 = \frac{\theta}{\theta+1}$

$\bar{X}_n = \frac{\tilde{\theta}_n}{\tilde{\theta}_n + 1} \quad \Rightarrow \quad \tilde{\theta}_n = \frac{\bar{X}_n}{1-\bar{X}_n}$

10. (a) $\bar{X}_n \xrightarrow{P} EX_i = \frac{\theta}{\theta+1}$

$\Rightarrow \bar{X}_n \xrightarrow{D} \frac{\theta}{\theta+1}$ Let $g(x) = \frac{x}{1-x}$, $g(x)$ is cont. on $(0,1)$

$\Rightarrow g(\bar{X}_n) \xrightarrow{D} g\left(\frac{\theta}{\theta+1}\right)$ i.e. $\tilde{\theta}_n \xrightarrow{D} \theta$

$\Rightarrow \tilde{\theta}_n \xrightarrow{P} \theta$ □

(b) $\overline{-\ln X_n} \xrightarrow{P} E(-\ln X_i) = \frac{1}{\theta}$

$\Rightarrow \overline{-\ln X_n} \xrightarrow{D} \frac{1}{\theta}$ Let $g(x) = \frac{1}{x}$, $g(x)$ is cont. on $(0,1)$

$\Rightarrow g(\overline{-\ln X_n}) \xrightarrow{D} g\left(\frac{1}{\theta}\right)$ i.e. $\frac{n}{-\sum_{i=1}^{n} \ln x_i} \xrightarrow{D} \theta$

i.e. $\hat{\theta}_n \xrightarrow{D} \theta$

$\Rightarrow \hat{\theta}_n \xrightarrow{P} \theta$ □

11. $L(1; x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2}$

$L(2; x) = \prod_{i=1}^{n} \frac{1}{\pi(1+x_i^2)}$

$$\hat{\theta} = \begin{cases} 1 & L(1;x) > L(2;x) \\ 2 & L(2;x) > L(1;x) \end{cases}$$

12. (a) $L(\theta; x) = \theta^{-2n} 2^n \prod_{i=1}^{n} x_i \mathbb{1}_{(0,\theta]}(x_i)$

if $\theta < \max_i\{x_i\}$, then $L(\theta; x) = 0$

if $\theta \geq \max_i\{x_i\}$, then $L(\theta; x) > 0$ and $L(\theta; x) \downarrow$ as $\theta \uparrow$

$\Rightarrow \hat{\theta} = \max_i\{x_i\} = Y_n$

(b) $f_{X_i}(x;\theta) = \frac{2x}{\theta^2}$ $\quad F_{X_i}(x;\theta) = \frac{x^2}{\theta^2}$

$f_{Y_n}(y_n) = \binom{n}{1} \left(F_{X_i}(y_n)\right)^{n-1} f_{X_i}(y_n)$

$= 2n\,\theta^{-2n} y_n^{2n-1}$

$E Y_n = \frac{2n}{2n+1}\theta$

$\therefore c = \frac{2n+1}{2n}$ $\quad E\left(\frac{2n+1}{2n}\hat{\theta}\right) = \theta$

(c) $F(x;\theta) = \frac{x^2}{\theta^2} = \frac{1}{2} \Rightarrow m(\theta) = \sqrt{\frac{\theta^2}{2}}$

Thm 6.1.2 $\Rightarrow \widehat{m(\theta)} = m(\hat{\theta}) = \frac{1}{\sqrt{2}} \max_i\{x_i\}$