Math 415 - Lecture 11

Column space, Solution to $A\mathbf{x} = b$

Friday September 18th 2015

Textbook: Chapter 2.1, 2.2.

Suggested practice exercises: Chapter 2.1: 3, 21, 28. Chapter 2.2: 33 and additional exercises at the end of this lecture.

Khan Academy videos: Introduction to the Null Space of a Matrix, Calculating the Null Space of a Matrix, Column Space of a Matrix

1 Review

Definition. The **nullspace** of an $m \times n$ matrix A, written as $Nul(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$Nul(A) = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

Theorem 1. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions $\mathbf{x}$ to the system $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.*

For example

$$\text{Nul}\left(\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}\right) = \text{Nul}\left(\begin{bmatrix} 1 & 2 & 0 & 13 & 33 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix}\right).$$

This corresponds to the solution:

$$\begin{aligned} x_1 &= -2x_2 - 13x_4 - 33x_5 \\ x_3 &= 6x_4 + 15x_5. \end{aligned}$$

Write this as a linear combination:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}.$$

This means that

$$\mathrm{Nul}(A) = span \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Remark. If $\mathrm{Nul}(A) \neq \{\mathbf{0}\}$, then the number of vectors in the spanning set for $\mathrm{Nul}(A)$ equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.
In this example, we had 3 free variables (x_2, x_4, and x_5) so there were 3 vectors in the spanning set for $\mathrm{Nul}(A)$. More about this later!

2 Column Spaces

Definition. The **column space**, written as $Col(A)$, of an $m \times n$ matrix A is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n} \end{bmatrix}$,

then $Col(A) = \mathrm{Span}\{\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}\}$.

Example 1.

- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $Col(A) = \mathrm{Span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix})$. This is all of $\mathbb{R}^2$!
- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $Col(A) = \mathrm{Span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \mathrm{Span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$. This is the x_1 axis in $\mathbb{R}^2$!
- If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $Col(A) = \mathrm{Span}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix})$. This is the zero subspace of $\mathbb{R}^2$!

Theorem 2. *The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.*

Why is it a subspace? Because it is a Span!

Remark. If A is $m \times n$ (m rows, n columns) then

- $Col(A)$ is a subspace of the output space $\mathbb{R}^m$.
- $Nul(A)$ is a subspace of the input space $\mathbb{R}^n$.

Theorem 3. *Let A be an $m \times n$ matrix.* $\mathbf{b}$ *is in $Col(A)$ iff there is an* $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ *in $\mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$.*

Proof. Suppose $A\mathbf{x} = \mathbf{b}$. Then

$$\mathbf{b} = A\mathbf{x} = \underbrace{x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n}}_{(\textit{lin. comb. of } \mathbf{a_1}, \ldots, \mathbf{a_n})}.$$

$\square$

Example 2. Find a matrix A such that $W = Col(A)$ where

$$W = \left\{ \begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

Solution.

$$\begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}.$$

So

$$W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\} = Col \left(\begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 1 & 1 \end{bmatrix} \right).$$

Therefore

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 1 & 1 \end{bmatrix}.$$

3 $Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Theorem 4. *Let A be an $m \times n$ matrix, let $\mathbf{b} \in \mathbb{R}^m$, and let $\mathbf{x_p} \in \mathbb{R}^n$ such that*

$$A\mathbf{x_p} = \mathbf{b}.$$

Then the set of solutions $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$ is exactly

$$\mathbf{x_p} + Nul(A).$$

So every solution of $A\mathbf{x} = \mathbf{b}$ is of the form

$$\mathbf{x_p} + \mathbf{x_n}$$

where $\mathbf{x_n}$ is some vector in $Nul(A)$.

Proof. Let $\mathbf{x_p} \in \mathbb{R}^n$ such that $A\mathbf{x_p} = \mathbf{b}$. Suppose $\mathbf{x}$ is also in $\mathbb{R}^n$ with $A\mathbf{x} = \mathbf{b}$. Then

$$A(\mathbf{x} - \mathbf{x_p}) = A\mathbf{x} - A\mathbf{x_p} = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Therefore, $\mathbf{x} - \mathbf{x_p} = \mathbf{x_n}$ is in $Nul(A)$, so $\mathbf{x} = \mathbf{x_p} + \mathbf{x_n}$. $\square$

Remark. We often call $\mathbf{x_p}$ a *particular solution* of $A\mathbf{x} = \mathbf{b}$. The theorem then says that every solution to $A\mathbf{x} = \mathbf{b}$ is the sum of one particular solution $\mathbf{x_p}$ and all the solutions to $A\mathbf{x} = \mathbf{0}$ (the null space).

Example 3. Let $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$. Solve $A\mathbf{x} = \mathbf{b}$.

Step 1 : Reduce $A\mathbf{x} = \mathbf{b}$ to $U\mathbf{x} = \mathbf{c}$.

$$\begin{bmatrix} A \mid \mathbf{b} \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 7 & 5 \\ -1 & -3 & 3 & 4 & 5 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 6 & 6 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$$U\mathbf{x} = \mathbf{c}.$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

Step 2 : Find a particular solution to $U\mathbf{x} = \mathbf{c}$.

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Could pick any value for the free variables (x_2 and x_4). Trick: Set them both to 0. Then

$$3x_3 = 3 \Rightarrow x_3 = 1.$$
$$x_1 + 3x_3 = 1 \Rightarrow x_1 = -2.$$

$$\text{So } \mathbf{x_p} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ is a particular solution to } A\mathbf{x} = \mathbf{b}.$$

Step 3 : Find **all** the solutions to $A\mathbf{x} = \mathbf{0}$ to find $Nul(A)$.

$$\begin{bmatrix} U \mid \mathbf{0} \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Every vector in $Nul(A)$ is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Step 4 : To find all the solutions to $A\mathbf{x} = \mathbf{b}$, add a particular solution $\mathbf{x_p}$ to the null space of A. So the set of solutions is

$$\mathbf{x_p} + Nul(A).$$

$$\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \text{Span} \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

and each solution $\mathbf{x}$ is of the form

$$\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Remark.

- If A is a matrix with echelon form U, then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$, which is the same as the space of solutions of $Ux = 0$ (That is the point of echelon form matrices!) which is $Nul(U)$.
- Not true that $Col(A) = Col(U)$! Why?

Example 4. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $U = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

$$Col(A) = \text{Span}(\begin{bmatrix} 1 \\ 2 \end{bmatrix}), \quad Col(U) = \text{Span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

Additional Exercises

1. True or false?
   - (i) The solutions to $A\mathbf{x} = \mathbf{0}$ form a vector space. True. This is the null space $Nul(A)$.
   - (ii) The solutions to $A\mathbf{x} = \mathbf{b}$ form a vector space. False, unless $\mathbf{b} = 0$.
2. Find an explicit description for $Nul(A)$ where

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}.$$

3. Show that the given set W is a subspace (by showing that W is the column space or null space of some matrix A) or find a specific example that shows that W is not a subspace.

   (i) $W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 5x - 1 = y + 2z \right\}.$

   (ii) $W_2 = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a = 2b + c,\ 2a = c - 3d \right\}.$

4. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Find a smallest spanning set for $W = Col(A)$. Find a matrix B such that $W = Nul(B)$.

5. Let $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find a smallest spanning set for $W = Nul(A)$. Find a matrix B such that $W = Col(B)$