STAT 420

Time Series: y_t, $t = 1, 2, \ldots, N$.

Stationary process ≈ a random process where all of its statistical properties do not vary with time.

$E(Y_t) = \mu$. $\mathrm{Var}(Y_t) = \sigma_Y^2$.

$$\gamma(k) = \mathrm{Cov}(Y_t, Y_{t+k}) = E[(Y_t - \mu)(Y_{t+k} - \mu)]$$
$$= \mathrm{Cov}(Y_t, Y_{t-k}) = E[(Y_t - \mu)(Y_{t-k} - \mu)].$$

$$\gamma(0) = \mathrm{Var}(Y_t) = \sigma_Y^2.$$

$$\rho_k = \mathrm{Corr}(Y_t, Y_{t+k}) = \frac{\mathrm{Cov}(Y_t, Y_{t+k})}{\sqrt{\mathrm{Var}(Y_t)\mathrm{Var}(Y_{t+k})}} = \frac{E(Y_t - \mu)(Y_{t+k} - \mu)}{\sigma_Y^2},$$
$$k = \pm 1, \pm 2, \ldots .$$

$\rho_k = \gamma(k) / \gamma(0)$. $\rho_0 = 1$.

Sample autocorrelation coefficient: $$r_k = \frac{\sum_{t=1}^{N-k}(y_t - \bar{y})(y_{t+k} - \bar{y})}{\sum_{t=1}^{N}(y_t - \bar{y})^2}$$

1. Calculate r_1 and r_2 for the time series 16 22 19 25 18

(Note: In practice reliable autocorrelation estimates are only obtained from series consisting of approximately 50 observations or more.)

$$r_k = \frac{\sum_{t=1}^{N-k}\left(y_t - \bar{y}\right)\left(y_{t+k} - \bar{y}\right)}{\sum_{t=1}^{N}\left(y_t - \bar{y}\right)^2}$$

$$\bar{y} = \frac{16+22+19+25+18}{5} = 20.$$

y_t	$y_t - \bar{y}$	$\left(y_t - \bar{y}\right)^2$	$\left(y_t - \bar{y}\right)\left(y_{t+1} - \bar{y}\right)$	$\left(y_t - \bar{y}\right)\left(y_{t+2} - \bar{y}\right)$
16	-4	16	-8	4
22	2	4	-2	10
19	-1	1	-5	2
25	5	25	-10	---
18	-2	4	---	---
		50	-25	16

$$r_1 = \frac{\sum_{t=1}^{N-1}\left(y_t - \bar{y}\right)\left(y_{t+1} - \bar{y}\right)}{\sum_{t=1}^{N}\left(y_t - \bar{y}\right)^2} = \frac{-25}{50} = \mathbf{-0.5}.$$

$$r_2 = \frac{\sum_{t=1}^{N-2}\left(y_t - \bar{y}\right)\left(y_{t+2} - \bar{y}\right)}{\sum_{t=1}^{N}\left(y_t - \bar{y}\right)^2} = \frac{16}{50} = \mathbf{0.32}.$$

Consider the following "regression" (autoregressive) model: AR(1)

$$(Y_t - \quad) = \phi \cdot (Y_{t-1} - \quad) + e_t$$

$$E(e_t) = 0, \qquad Var(e_t) = \sigma_e^2 \qquad \text{for all } t$$

$$E(e_t e_s) = 0, \qquad \text{for } t \neq s$$

$$E(e_t Y_s) = 0, \qquad \text{for } s < t$$

Then

$$\begin{aligned}
\gamma(0) &= Var(Y_t) = E[(Y_t - \quad)^2] \\
&= E[(\phi(Y_{t-1} - \quad) + e_t)^2] \\
&= \phi^2 E[(Y_{t-1} - \quad)^2] + 2\phi E[(Y_{t-1} - \quad)e_t] + E[e_t^2] \\
&= \phi^2 Var(Y_{t-1}) + Var(e_t) \\
&= \phi^2 \gamma(0) + \sigma_e^2
\end{aligned}$$

Therefore,

$$\gamma(0) = Var(Y_t) = \frac{\sigma_e^2}{1-\phi^2}, \qquad \Rightarrow \qquad \text{need } |\phi| < 1.$$

$$\begin{aligned}
\gamma(k) &= Cov(Y_t, Y_{t-k}) = E[(Y_t - \quad)(Y_{t-k} - \quad)] \\
&= E[(\phi(Y_{t-1} - \quad) + e_t)(Y_{t-k} - \quad)] \\
&= \phi E[(Y_{t-1} - \quad)(Y_{t-k} - \quad)] + E[e_t(Y_{t-k} - \quad)] \\
&= \phi\gamma(k-1), \qquad k \geq 1.
\end{aligned}$$

Then $\rho_k = \phi \rho_{k-1}, \qquad k \geq 1.$

Therefore, $\rho_k = \rho_{-k} = \phi^k, \qquad k \geq 1.$