# Math 415 - Lecture 29

Determinants

Wednesday November 4th 2015

**Textbook reading:** Chapters 4.2, 4.3

**Suggested practice exercises:** Chapter 4.2, # 1, 2, 4, 5, 10, 14, 15, 17, 18, 19, 20, 22, 23

**Khan Academy video:** $3\times 3$ Determinant, $n\times n$ Determinant, Determinants along other rows/ columns,

**Strang lecture:** Lecture 18: Properties of determinants, Lecture 19: Determinant formulas and cofactors

## 1 Determinants

**Definition.** The determinant is characterized by:

- the normalization $\det I_{n\times n} = 1$,
- and how it is affected by elementary row operations:
  - **(Replacement)** Add a multiple of one row to another row. *Does not change* the determinant.
  - **(Interchange)** Interchange two rows. *Reverses the sign* of the determinant.
  - **(Scaling)** Multiply all entries in a row by $s$. *Multiplies* the determinant by $s$.

**Important Fact**
The determinant of a triangular matrix is the product of the diagonal entries.

$$\det \begin{bmatrix} 2 & 3 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 6 \end{bmatrix} = 2 \cdot 4 \cdot 6.$$

*Example* 1 (Generic matrix). Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

**Solution.**

*Example* 2 (Reality check). Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

**Solution.**

*Example* 3 (Larger matrix). Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

**Solution.**

**Important properties**

- $\det(A) = 0 \iff A$ is not invertible. Why?

- $\det(AB) = \det(A)\det(B)$ **Challenge**: Figure out why! (Matrix multiplication can be seen as linear combinations of rows)

- $\det(A^{-1}) = \frac{1}{\det(A)}$ Why? Because $AA^{-1} = I$.

- $\det(A^T) = \det(A)$. (Think about why this works at home.)

**Remark.** $\det(A^T) = \det(A)$ means that everything you know about determinants in terms of *rows* of $A$ is also true for the columns.For instance:

- If you exchange two *columns* in a determinant you get a minus sign.
- You can add a multiple of a *column* to another column without changing the determinant.
- If your matrix has equal *column* the determinant is zero.
- If your matrix has a zero *column* the determinant is zero.

*Example* 4. Recall that $AB = \mathbf{0}$, then it does not follow that $A = \mathbf{0}$ or $B = \mathbf{0}$. However, show that $\det(A) = 0$ or $\det(B) = 0$.

**Solution.**

## 2 A "bad" way to compute determinants, Cofactor expansion

**Fact 5.**

$$\det \begin{bmatrix} a & b & c \\ * & * & * \\ * & * & * \end{bmatrix} = \det \begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} + \det \begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & c \\ * & * & * \\ * & * & * \end{bmatrix}$$

We can use this idea to calculate an $n \times n$ determinant in terms of $n$ determinants of smaller matrices.

*Example* 6. What is the determinant $\begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$? What about $\begin{bmatrix} 0 & b & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$?

**Solution.**

$$\det \begin{bmatrix} a & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ * & B \end{bmatrix} = a \det \begin{bmatrix} B \end{bmatrix},$$

where $B$ is the $2 \times 2$ right lower block. Same way, with a twist:

$$\det \begin{bmatrix} 0 & b & 0 \\ v_1 & v_2 & v_3 \end{bmatrix} = \boxed{-1} b \begin{bmatrix} 1 & 0 & 0 \\ v_2 & v_1 & v_3 \end{bmatrix} = -b \det \begin{bmatrix} v_1 & v_3 \end{bmatrix}.$$

We can use this idea to calculate an $n \times n$ determinant in terms of $n$ determinants of $(n-1) \times (n-1)$ matrices. Then repeat ....

*Example* 7. Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

**Solution.**

Each term in the cofactor expansion is $\pm 1$ times an entry times a smaller determinant (row and column of entry deleted). The $\pm$ is assigned to each entry according to

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}.$$

There is nothing special about the first row. We can use any other row or column. For example, let's use the second column:

**Solution.**

Let use the third column:

**Solution.**

## Why not cofactor expansion

Why is the method of cofactor expansion not practical (except when there are lots of zeroes in your matrix.)? Because to compute a large $n \times n$ matrix,

- one reduces to $n$ determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,
- and so on.

In the end, we have $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ many numbers to add.WAY TOO MUCH WORK! Already

$$25! = 15511210043330985984000000 \approx 1.55 \cdot 10^{25}.$$

Context: today's fastest computer, Tianhe-2, runs at 34 pflops ($3.4 \cdot 10^{16}$ operations per second).By the way: "fastest" is measured by computing LU decompositions!

# 3 Practice Problems

## 3.1

*Example* 8. Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$. Use your favorite method (or a mix of methods!)

**Solution.**

- What's **wrong**?!

$$\det(A^{-1}) = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}(da - (-b)(-c) = 1$$

The correct calculation is:

*Example* 9. Suppose $A$ is a $3 \times 3$ matrix with $\det(A) = 5$. What is $\det(2A)$?

**Solution.**

## Imaginary unit and Fibonacci numbers

*Example* 10. First off, say hello to our new friend: $i$, the **imaginary unit**. It is infamous for $\boxed{i^2 = -1}$. Let us calculate some determinants.

$$\begin{aligned}
|1| &= 1 \\
\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} &= 1 - i^2 = 2 \\
\begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} &= 1\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} - i\begin{vmatrix} i & 0 \\ i & 1 \end{vmatrix} = 2 - i^2 = 3 \\
\begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} &= 1\begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} - i\begin{vmatrix} i & 0 & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 3 - i^2\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} = 5
\end{aligned}$$

*Example* 11 (continued).

$$\begin{vmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & i & \\ & & i & 1 & i \\ & & & i & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} - i^2 \begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 5 + 3 = 8$$

The Fibonacci numbers!

Do you know about the connection of Fibonacci numbers and rabbits? If not, Google is your friend.