Math 415 - Lecture 6

Elementary Matrices, LU Decomposition

Friday September 4th 2015

Textbook: Chapter 1.4, 1.5

Suggested Practice Exercise: Chapter 1.4 Exercise 22, 27, Chapter 1.5: 4, 5, 11, 23, 29

Khan Academy Video: Matrix multiplication (part I), Matrix multiplication (part II), Defined and undefined matrix operations

Review of matrix multiplication

- **Matrix multiplication is linear combination**: $A\mathbf{x}$ is a linear combination of the columns of A with weights given by the entries of $\mathbf{x}$.

  Example 1.

  $$Ax = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

- **Linear Combination is Linear System**

  Example 2.

  $$x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \leftrightarrow \begin{aligned} x_1 + 2x_2 + 3x_3 &= -2 \\ 4x_1 + (-1)x_2 + 0x_3 &= 4 \end{aligned}$$

- $A\mathbf{x} = \mathbf{b}$ is the matrix form of the linear system with augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$.

  Example 3.

  $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \leftrightarrow \begin{aligned} x_1 + 2x_2 + 3x_3 &= -2 \\ 4X_1 + (-1)x_2 + 0x_3 &= 4 \end{aligned}$$

  $$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 4 & -1 & 0 & 4 \end{array}\right]$$

- Each column of AB is a linear combination of the columns of A with weights given by the corresponding column of B: $AB = A\begin{bmatrix}\mathbf{b}_1 & \dots & \mathbf{b_p}\end{bmatrix} = \begin{bmatrix}A\mathbf{b_1} & \dots & A\mathbf{b_p}\end{bmatrix}$

  Example 4. If $A = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$, $B = \begin{bmatrix}3 & 4\\0 & 2\end{bmatrix}$, then

$$AB = \begin{bmatrix}A\begin{bmatrix}3\\0\end{bmatrix} & A\begin{bmatrix}4\\2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}3\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}2\\1\end{bmatrix} & 4\begin{bmatrix}1\\2\end{bmatrix}+2\begin{bmatrix}2\\1\end{bmatrix}\end{bmatrix}$$
$$= \begin{bmatrix}3 & 8\\6 & 10\end{bmatrix}$$

- Row-column rule: The ij-th entry of AB is $a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$.

  Example 5. If $A = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$, $B = \begin{bmatrix}3 & 4\\0 & 2\end{bmatrix}$, then the 22 entry of AB is

$$AB_{22} = \begin{bmatrix}\\2 & 1\end{bmatrix}\begin{bmatrix} & 4\\ & 2\end{bmatrix} = 2\times 4 + 1\times 2 = 10$$

- Matrix multiplication is not commutative: usually, $AB \neq BA$.

Transpose

Definition. If A is $m \times n$, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A. In terms of matrix elements $(A^T)_{ij} = A_{ji}$.

Example 6. Let $A = \begin{bmatrix}1 & 2 & 3 & 4 & 5\\6 & 7 & 8 & 9 & 8\\7 & 6 & 5 & 4 & 3\end{bmatrix}$. The $A^T =$

Example 7. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute AB, $(AB)^T$, $A^T B^T$ and $B^T A^T$.

Solution.

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$A^T B^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Conclusion. The transpose of a product is the product of transposes **IN OPPOSITE ORDER**:

$$(AB)^T = B^T A^T$$

Definition. A is **symmetric** if $A = A^T$.

Example 8. Which of these is symmetric?

$$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}^T =$$

$$\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}^T =$$

Theorem 1. *Let A and B denote matrices whose sizes are appropriate for the following sums and products.*

(a) $\left(A^T\right)^T = A$,

(b) $(A + B)^T = A^T + B^T$

(c) For any scalar r, $(rA)^T = rA^T$

(d) $(AB)^T = B^T A^T$

Example 9. Prove that $(ABC)^T = C^T B^T A^T$.

Solution.

Elementary matrices

Definition. The $n \times n$ **identity matrix** I_n has all entries 0, except on the main diagonal where the entries are 1. For example

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition. An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix. A **permutation matrix** is one that is obtained by performing row exchanges on an identity matrix.

Example 10. Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$.

E_1, E_2, and E_3 are elementary matrices. Why?

Solution.

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

Theorem 2. *If an elementary row operation is performed on an $m \times n$- matrix A, the resulting matrix can be written as EA, where the $m \times m$-matrix E is created by performing the same row operations on I_m.*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} =$$

We write this as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Remark. Elementary matrices are *invertible* because row operations are *reversible.* To determine the inverse of an elementary matrix E, determine the elementary row operation needed to transform E back into I and apply this operation to I to find the inverse.

Example 11. Compute the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Solution.

Example 12. Compute the following products using the row or column interpretation of matrix multiplication. (Don't just use the row-column rule.)

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} =$$