Math 415 - Lecture 7

LU-decomposition

Wednesday September 9th 2015

Textbook: Chapter 1.5

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Suggested Practice Exercise: Chapter 1.5 Exercise 4, 5, 11, 23, 29

Review - Elementary matrices

- Multiply row 3 by 7:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 7g & 7h & 7i \end{bmatrix}$$

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- Switch rows 2 and 3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

- $R3 \to 3R1 + R3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

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- Taking the inverse of an elementary matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^{-1} =$$

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Triangular matrices

Definition

An $n \times n$ matrix A is called upper triangular if it is of the form

$$\begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{bmatrix}.$$

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An $n \times n$ matrix B is called lower triangular if it is of the form

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & \ddots & \vdots \\ * & * & * & * & * \end{bmatrix}.$$

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$$A = LU.$$

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When is this possible?

Theorem

Let A be a $n \times n$-matrix. If A can be transformed into echelon form without the use of row exchanges, then A has LU factorization.

Example

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$.

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$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{2} & 1 & 1 \\ \boxed{4} & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}, \ell_{21} = 2$$

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$$E_2 (E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{2} & 1 & 1 \\ 0 & -8 & -2 \\ \boxed{-2} & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}, \ell_{31} = -1$$

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$$E_3 (E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & \boxed{-8} & -2 \\ 0 & \boxed{8} & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \ell_{32} = -1$$

We got an upper triangular matrix!

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$$E_3E_2E_1A = U \implies$$

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$$E_3E_2E_1A = U \implies A = E_1^{-1}E_2^{-1}E_3^{-1}U$$

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$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_3^{-1} =$$

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Product of lower triangulars is lower triangular.

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Product of lower triangulars is lower triangular.

So the LU decomposition is

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

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- If A can be brought in echelon form without row exchanges we have $A = LU$,
- U - Echelon form of A
- $L = E_1^{-1}E_2^{-1}E_3^{-1}$ where E_1, E_2, E_3 were elementary matrices that put A into Echelon form. (No row exchanges!)
- $L = I+$ strictly lower triangular, and ℓ_{ij} is the factor between pivot and the entry you want to make zero in the elimination process: see the boxed numbers.

Row exchanges

Recall that a permutation matrix P is a square matrix obtained from the identity matrix by reordering the rows.

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Theorem

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Reason:

If A can be brought to echelon form with the help of row exchanges, we can do those exchange first. So there is a permutation matrix P such that PA can be brought to echelon form without row exchanges. $\square$

Example

Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. Find PA that has a LU factorization.

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- Move the 2nd row to the 1st row
- Move the 3rd row to the 2nd row
- Move the 1st row to the 3rd row

Do these moves to the identity matrix to get P:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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$$PA = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applications

Theorem

Let A be an $n \times n$-matrix such that $A = LU$, where L is a lower triangular matrix and U is a upper triangular matrix. Then x will be solution of

$$Ax = b$$

if and only if x is a solution of

$$Ux = c,$$

where c satisfies $Lc = b$.

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Point: $Ux = c$ and $Lc = b$ are triangular systems, easy to solve by substitution.

Proof.

If $Lc = b$ and $Ux = c$, then

$$Ax = (LU)x = L(Ux) = Lc = b$$

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If $Lc = b$ and $Ux = c$, then

$$Ax = (LU)x = L(Ux) = Lc = b$$

On the other hand, suppose $Ax = b$. We take c to be the vector Ux. Then Lc is equal to $L(Ux) = Ax = b$, so in total we have both $Ux = c$ and $Lc = b$. $\square$

Example

Solve

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

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We found already a LU factorization for this matrix A. So you first have to solve $Lc = b$ for c:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}.$$

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Use **forward substitution:**

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 2 & 1 & 0 & -2 \\ -1 & -1 & 1 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 2 & 1 & 0 & -2 \\ -1 & -1 & 1 & 9 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -12 \\ 0 & -1 & 1 & 14 \end{array}\right]$$

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$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 2 & 1 & 0 & 3 \\ 0 & -8 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

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So

$$x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Practice problems

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- Is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ upper triangular? Lower triangular?
  No, it is neither upper nor lower triangular.
- True or false? A permutation matrix is one that is obtained by performing column exchanges on an identity matrix.
  Technically yes, but this isn't how we defined it. We defined it as row exchanges on an identity matrix.
- Why do we care about LU decomposition if we already have Gaussian elimination?
  It's faster, especially if we have to feed in lots of different values of $\mathbf{b}$.

Example

Solve

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -3 \end{bmatrix}.$$

using the factorization we already have.