# Math 415 - Lecture 17

## Linear Transformations

Monday October 5th 2015

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Textbook reading: Chapter 2.6

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Suggested practice exercises: same as lecture 16

# Review

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- If $\mathbf{x_1}, \ldots, \mathbf{x_n}$ is a basis for $V$, then $T$ is determined by the values $T(\mathbf{x_1}), \ldots, T(\mathbf{x_n})$:

$$T(\mathbf{v}) = T(c_1\mathbf{x_1} + \cdots + c_n\mathbf{x_n}) = c_1 T(\mathbf{x_1}) + \cdots + c_n T(\mathbf{x_n}).$$

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# Nonstandard Bases

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**Theorem (Linear Transformation is Matrix Multiplication)**

*Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\mathcal{B} := (\mathbf{v}_1, \dots, \mathbf{v}_n)$ be a basis of $\mathbb{R}^n$ and let $\mathcal{C} := (\mathbf{w}_1, \dots, \mathbf{w}_m)$ be a basis of $\mathbb{R}^m$. Then there is a matrix $B$ such that*

$$T(\mathbf{x})_{\mathcal{C}} = B\mathbf{x}_{\mathcal{B}}, \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n.$$

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*Explicitly,*

$$B = \begin{bmatrix} T(\mathbf{v}_1)_{\mathcal{C}} & \ldots & T(\mathbf{v}_n)_{\mathcal{C}}. \end{bmatrix},$$

## Example

Let $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a + 1b \\ 1a + 3b \end{bmatrix}$. Then the matrix of $T$ is $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

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What is the coordinate matrix for $T$ with respect to $\mathcal{B} = (\mathbf{b_1}, \mathbf{b_2})$?

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$$B = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

**Summary:** The linear transformation $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a + 1b \\ 1a + 3b \end{bmatrix}$ has with respect to the standard basis the coordinate matrix $A$, but with respect to the other basis $\mathcal{B}$ the coordinate $B$:

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The linear transformation $T$ is geometrically clear in the $\mathcal{B}$ basis: $T$ is just stretching vectors by a factor 2 along $\mathbf{b_1}$ and by a factor 4 along $\mathbf{b_2}$. So using the standard basis $T$ is an obscure operation on vectors, but using the basis $\mathcal{B}$ it becomes clear. You can say that $\mathcal{B}$ is a basis adapted to $T$.

# Matrices for... Polynomials?

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**Example**

Consider the map $T : P_2 \to P_1$ given by

$$T(p(t)) = \frac{d}{dt}p(t).$$

Describe $T$ by a matrix.

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**Solution**

Wait, what?! We can't multiply a polynomial by a matrix! Use coordinate vectors instead.

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Formally,

$$D \cdot (f_{\mathcal{A}}) = T(f)_{\mathcal{B}}$$

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$$T(1)=0 \implies D\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},$$

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$$D \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The third column of $D$ is $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$. What are the remaining two columns?

$$T(1) = 0 \implies D \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$T(t) = 1 \implies D \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$T(t^2) = 2t \implies D \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

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$$D \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}.$$

Hence $D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

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On the other hand $T(f(t)) = f'(t) = -1 + 6t$, with coordinate vector $\begin{bmatrix} -1 \\ 6 \end{bmatrix}$.

# Matrices for Linear Transformations

Let's organize this.

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\begin{tikzcd}
\text{vector in } V \arrow[r, "\text{apply } T"] \arrow[d, "\text{write in coordinates}"'] & \text{vector in } W \arrow[d, "\text{write in coordinates}"] \\
\text{coordinate vector in } \mathbb{R}^n \arrow[r, "\text{multiply by } A"] & \text{coordinate vector in } \mathbb{R}^m
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\end{tikzcd}$$

In the last example this was

$$T(2 + 3t + 4t^2) = 3 + 8t$$

$$A \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

## Definition

Let $\mathcal{A} = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$ be a basis for $V$, and $\mathcal{B} = \{\mathbf{y_1}, \ldots, \mathbf{y_m}\}$ a basis for $W$.
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- has $n$ columns (one for each of the $\mathbf{x_j}$),
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$$T_{\mathcal{B}\mathcal{A}} = \begin{bmatrix} T(\mathbf{x_1})_{\mathcal{B}} & T(\mathbf{x_2})_{\mathcal{B}} & \ldots & T(\mathbf{x_n})_{\mathcal{B}} \end{bmatrix}$$

## Example

Give the matrix for $T : P_2 \to P_1$ given by

$$T(p(t)) = \frac{d}{dt}p(t).$$

in the bases $\mathcal{A} = (1, t, t^2)$ and $\mathcal{B} = (1, t)$.

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## Solution

$$T_{\mathcal{BA}} = \left[ T(1)_{\mathcal{B}} \quad T(t)_{\mathcal{B}} \quad T(t^2)_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

### Example

Recall the map $T$ given by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$. (It reflects every vector in $\mathbb{R}^2$ across the line $y = x$.)

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Recall the map $T$ given by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$. (It reflects every vector in $\mathbb{R}^2$ across the line $y = x$.)

(a) Which matrix $A$ represents $T$ with respect to the standard bases?

(b) Which matrix $B$ represents $T$ with respect to the basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$?

## Solution

(a) $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}.$

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$T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-1\end{bmatrix} = 0\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}-1\\1\end{bmatrix}$.

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Draw a picture!

# Representing Linear Maps by Matrices

## Remark

If a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is represented by the matrix $A$ with respect to the standard bases, then $T(\mathbf{x}) = A\mathbf{x}$.

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# Representing Linear Maps by Matrices

## Remark

If a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is represented by the matrix $A$ with respect to the standard bases, then $T(\mathbf{x}) = A\mathbf{x}$.
Matrix multiplication corresponds to function composition! That is, if $T_1, T_2$ are represented by $A_1, A_2$, then $T_1(T_2(\mathbf{x})) = (A_1 A_2)\mathbf{x}$.

## Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\0\\7\end{bmatrix}.$$

What is the matrix $B$ representing $T$ with respect to the following bases?

$$\mathbf{x_1} = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{x_2} = \begin{bmatrix}-1\\2\end{bmatrix} \text{ for } \mathbb{R}^2,$$

$$\mathbf{y_2} = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{y_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{y_3} = \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ for } \mathbb{R}^3.$$

## Solution

$$T(\mathbf{x_1}) =$$

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## Solution

$$T(\mathbf{x_1}) = T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) \quad = \quad T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

## Solution

$$\begin{aligned} T(\mathbf{x_1}) = T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) &= T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}4\\0\\7\end{bmatrix} = \begin{bmatrix}5\\2\\10\end{bmatrix} \end{aligned}$$

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$$
\begin{aligned}
T(\mathbf{x_1}) = T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) &= T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\\
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&= 5\begin{bmatrix}1\\1\\1\end{bmatrix} - 3\begin{bmatrix}0\\1\\0\end{bmatrix} + 5\begin{bmatrix}0\\0\\1\end{bmatrix}
\end{aligned}
$$

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\implies B &= \begin{bmatrix}5 & *\\ -3 & *\\ 5 & *\end{bmatrix}
\end{aligned}$$

$$T(\mathbf{x}_2) =$$

$$T(\mathbf{x_2}) = T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$

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$$\begin{aligned} T(\mathbf{x_2}) = T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) &= -T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= -\begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}4\\0\\7\end{bmatrix} = \begin{bmatrix}7\\-2\\11\end{bmatrix} \\ &= 7\begin{bmatrix}1\\1\\1\end{bmatrix} - 9\begin{bmatrix}0\\1\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\end{bmatrix} \end{aligned}$$

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## Remark

A matrix representing $T$ encodes in column $j$ the coefficients of $T(\mathbf{x_j})$ expressed as a linear combination of $\mathbf{y_1}, \ldots, \mathbf{y_m}$.

# Recap

**What is the Point?** Why write $T\colon V \to W$ as a matrix?

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- Replace obscure computations in $V$ and $W$ by transparent computations with matrices.

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- Replace obscure computations in $V$ and $W$ by transparent computations with matrices.
- Even if $T : \mathbb{R}^n \to \mathbb{R}^m$ (already have standard coordinates), $T$ may be simpler in a different coordinate system.

**Summary:** Given $\mathbf{v}$ in $V$, want to calculate $T(\mathbf{v})$ in $W$.

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- We know $T(\mathbf{v})$ if we know the coordinate vector $T(\mathbf{v})_{\mathcal{B}}$.
- So we know $T$ if we know the matrix $T_{\mathcal{BA}}$:

$$T(\mathbf{v})_{\mathcal{B}} = T_{\mathcal{BA}}\mathbf{v}_{\mathcal{A}}.$$

**Summary:** Given $\mathbf{v}$ in $V$, want to calculate $T(\mathbf{v})$ in $W$. Take an input basis $\mathcal{A} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ and and output basis $\mathcal{B} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_m})$.

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- We know $T(\mathbf{v})$ if we know the coordinate vector $T(\mathbf{v})_{\mathcal{B}}$.
- So we know $T$ if we know the matrix $T_{\mathcal{B}\mathcal{A}}$:

$$T(\mathbf{v})_{\mathcal{B}} = T_{\mathcal{B}\mathcal{A}} \mathbf{v}_{\mathcal{A}}.$$

The output coordinate vector equals the matrix for $T$ times the input coordinate vector.

## Example

Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let $T$ be the linear map such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\0\\7\end{bmatrix}.$$

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## Solution

The standard bases are

$$\mathbf{x_1} = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{x_2} = \begin{bmatrix}0\\1\end{bmatrix}, \quad \mathbf{y_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{y_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{y_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

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$$T(\mathbf{x_1}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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$$

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\end{aligned}
$$

$$
\begin{aligned}
T(\mathbf{x_1}) = \begin{bmatrix}1\\2\\3\end{bmatrix} &= 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix} \\
&= 1\mathbf{y_1} + 2\mathbf{y_2} + 3\mathbf{y_3} \\
T(\mathbf{x_2}) = \begin{bmatrix}4\\0\\7\end{bmatrix} &= 4\mathbf{y_1} + 0\mathbf{y_2} + 7\mathbf{y_3} \\
\implies A &= \begin{bmatrix}1 & 4\\2 & 0\\3 & 7\end{bmatrix}
\end{aligned}
$$

So $T\begin{bmatrix}2\\3\end{bmatrix} = A\begin{bmatrix}2\\3\end{bmatrix} =$

$$
\begin{aligned}
T(\mathbf{x_1}) = \begin{bmatrix}1\\2\\3\end{bmatrix} &= 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix}\\
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\end{aligned}
$$

So $T\begin{bmatrix}2\\3\end{bmatrix} = A\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}14\\4\\27\end{bmatrix}$

# Additional Problems

- Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 7 & 8 & 1 \end{bmatrix}$. Find the dimensions and a basis for all four fundamental subspaces of $A$.

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- Suppose $A$ is $5 \times 5$ and $\mathbf{v}$ is a vector in $\mathbb{R}^5$ which is not a linear combination of the columns of $A$. What can you say about the number of solutions to $A\mathbf{x} = \mathbf{0}$?

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- Suppose $A$ is $5 \times 5$ and $\mathbf{v}$ is a vector in $\mathbb{R}^5$ which is not a linear combination of the columns of $A$. What can you say about the number of solutions to $A\mathbf{x} = \mathbf{0}$?
- Let $T$ be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

What is $T\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right)$?