MATH 415 – Lecture 37

Review for Exam 3

Thursday 30 July 2015

Big topics for midterm 3

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- Orthogonal projection

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- Orthogonal projection
- Least Squares

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- Gram-Schmidt

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- Orthogonal projection
- Least Squares
- Gram-Schmidt
- Determinants

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- Orthogonal projection
- Least Squares
- Gram-Schmidt
- Determinants
- Eigenvalues and eigenvectors

Big topics for midterm 3

- Orthogonal projection
- Least Squares
- Gram-Schmidt
- Determinants
- Eigenvalues and eigenvectors
- Diagonalization

Orthogonal Projection

- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an **orthogonal basis** for V, and $\mathbf{x}$ is in V, then

$$\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n \qquad \text{with } c_j =$$

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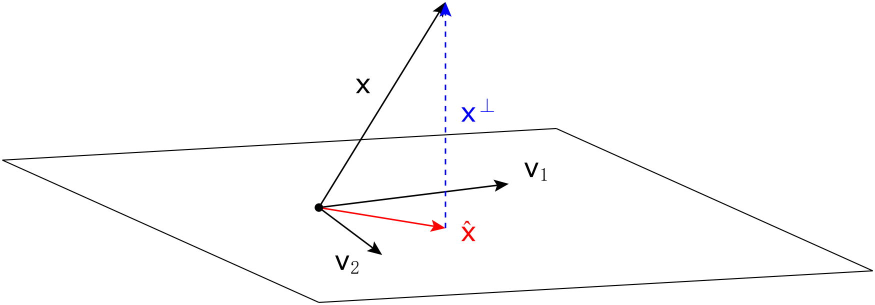

$\mathbf{x}$
$\mathbf{x}^{\perp}$
$\mathbf{v}_1$
$\hat{\mathbf{x}}$
$\mathbf{v}_2$

Example

What is the orthogonal projection of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ onto Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$?

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Solution

The projection is $\begin{bmatrix}1\\1\\0\end{bmatrix}$.

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What is the orthogonal projection of $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ onto $\text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$?

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Solution (First try:)

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Is the projection $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ *?No!*

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Is the projection $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ *?No! Wrong approach!!* *(This is because the basis is not orthogonal.)*

Solution (Corrected:)

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Then compute:

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Answer: $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$

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What would Gram-Schmidt do? $\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \implies \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}$.

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The projection is $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$.

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- The functions

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  are orthogonal basis for this space.
- Expanding a function $f(x)$ in this basis produces its **Fourier series**

$$f(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \ldots$$

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You don't need to know how to compute this integral. But, you should be able to find the norm (length) of a function! Lengths of other functions work the same way!

Example

If $f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \dots$
how can we compute b_2?

Example

If $f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \dots$ how can we compute b_2?

Solution

$b_2 \sin(2x)$ *is the orthogonal projection of* f *onto the span of* $\sin(2x)$.

Example

If $f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \ldots$ how can we compute b_2?

Solution

$b_2 \sin(2x)$ is the orthogonal projection of f onto the span of $\sin(2x)$.

Example

If $f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \ldots$ how can we compute b_2?

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Projections onto the span of an other function work the same way! Again, you should be able to project functions onto the span of a function!

Least Squares

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Solution (continued)

Here, we need to find a least squares solution to

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Line of best fit: $y = 2/7 + 5/14x$.

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Gram-Schmidt

Recipe. (Gram-Schmidt orthonormalization)

Given a basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$, produce an orthonormal basis $\mathbf{q}_1, \ldots, \mathbf{q}_n$.

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- An **orthogonal matrix** is a square matrix Q with orthonormal columns. Equivalently, $Q^T Q = I$ (also true for non-square matrices).
- Apply Gram-Schmidt to the (independent) columns of A to obtain the **QR decomposition** $A = QR$.
  - Q has orthonormal columns (the output vectors of Gram-Schmidt).
  - $R = Q^T A$ is upper triangular.

Example

Find the QR decomposition of $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$.

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Solution (continued)

Hence: $$Q = [\mathbf{q}_1, \mathbf{q}_2] = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} \\ 0 & \frac{5}{\sqrt{45}} \end{bmatrix}$$

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Determinants

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$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & & 0 \\ & + & \\ 2 & & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & & 0 \\ 3 & & 2 \\ & - & \end{vmatrix}$$

$$= -2 \cdot (-1) + (-1) \cdot 1 - 0 = 1$$

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Solution

The determinant is 0 *because the matrix is not invertible (second and third columns are the same).*

Eigenvalues and eigenvectors

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Diagonalization

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- Then it is easy to calculate the action of powers of A on $\mathbf{x}$:

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n.$$

GOOD LUCK!