Practice Problems 3

1. The number of patients in the emergency room of a local hospital during four shifts is recorded.

Morning (6 – 12)	Afternoon (12 – 6)	Evening (6 – 12)	Night (12 – 6)
3	5	5	3
2	8	8	5
5	9	10	5
7	6	8	2
3	3	6	8
1	5	11	4
$\bar{y}_1 = 3.5$	$\bar{y}_2 = 6.0$	$\bar{y}_3 = 8.0$	$\bar{y}_4 = 4.5$
$s_1^2 = 4.7$	$s_2^2 = 4.8$	$s_3^2 = 5.2$	$s_4^2 = 4.3$

Consider the model $Y_{ij} = \mu_j + \varepsilon_{ij}$, where ε_{ij}'s are i.i.d. $N(0, \sigma^2)$.

a) At $\alpha = 0.01$, can one conclude that there is a difference in the average number of patients for the four shifts?

b) Use a 99% confidence level and Tukey's pairwise comparison procedure to compare the average number of patients in the emergency room for Evening with that for Morning.

c) Use a 99% confidence level and Scheffé's multiple comparison procedure to compare the average number of patients in the emergency room for Evening with that for Morning.

d) Use a 99% confidence level and Scheffé's multiple comparison procedure to compare the average number of patients in the emergency room for Evening with that for Morning and Night.

e) Use a 99% confidence level and Scheffé's multiple comparison procedure to compare the average number of patients in the emergency room for Afternoon and Evening with that for Morning and Night.

2. a) The American Car Company is interested in the average number of cars coming off three assembly lines with obvious defects (determined by visual inspection). The number of defects per week for each of the assembly lines is recorded for several weeks. The data follow.

Assembly Line

1	2	3
49	50	61
61	68	44
55	57	53
67	64	47
48	73	58
	66	49
	56	

Given the observations above, test for differences in the mean number of visually defective cars coming off the assembly lines using the ANOVA F test. Use $\alpha = 0.10$.

b) Construct three 90% (individual) confidence intervals for the differences $\mu_1 - \mu_2$, $\mu_1 - \mu_3$, $\mu_2 - \mu_3$.

c) Use Bonferroni method to construct three simultaneous 90% confidence intervals for the pairwise differences of means.

d) Use Tukey's method to construct three simultaneous 90% confidence intervals for the pairwise differences of means.

e) Use a 90% confidence level and Scheffé's multiple comparison procedure to construct confidence interval for the following contrasts:

i) $\mu_2 - \mu_3$; ii) $\mu_2 - \dfrac{\mu_1 + \mu_3}{2}$.

f) Test for differences in the mean number of visually defective cars coming off the assembly lines using the Kruskal-Wallis test. Use $\alpha = 0.10$.

3. Following is a partial ANOVA table.

Source	Sum of squares	df	Mean square	F
Between		2		
Within			20	
Total	500	11		

Complete the table, and answer the following questions. Use the 0.05 significance level.

a) How many groups (treatments) are there?

b) What was the total sample size?

c) What is the critical value of F?

d) Write out the null and alternative hypotheses.

e) What is your conclusion regarding the null hypothesis?

4. Following is a partial ANOVA table.

Source	Sum of squares	df	Mean square	F
Between		2		
Within			20	
Total	500	11		

Complete the table, and answer the following questions. Use the 0.05 significance level.

a) How many groups (treatments) are there?

b) What was the total sample size?

c) What is the critical value of F?

d) Write out the null and alternative hypotheses.

e) What is your conclusion regarding the null hypothesis?

5. A consumer agency is investigating claims that some bars routinely underfill their drinks. Six pints selected at random from each of the four bars under investigation contain the following amounts of beer, in ounces:

							$\overline{y}_j$	s_j^2
Bar 1	15.97	15.99	16.13	16.09	15.99	16.01	16.03	0.00416
Bar 2	16.12	16.05	16.13	16.07	16.03	16.14	16.09	0.00212
Bar 3	16.07	15.97	15.99	15.93	15.97	16.01	15.99	0.00224
Bar 4	16.09	15.95	16.09	15.93	16.01	15.99	16.01	0.00464

Assume the populations of beer amounts in pints are normal with equal variances. We want to determine whether there is significant evidence of a difference in the average beer amounts in pints from the four bars.

$$H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$$

a) Perform the appropriate test at a 5% level of significance.

b) Use a 95% confidence level and Scheffé's multiple comparison procedure to compare the average beer amount in pints for Bar 2 with the average beer amounts for Bars 1, 3, and 4.

6. A consumer agency is investigating claims that some bars water down their light beers on tap. The alcohol percentage of three popular brands of light beer on tap was measured at five bars.

	Bar					
Beer Brand	1	2	3	4	5	$\bar{y}_{i\bullet}$
Boors Light	4.15	4.17	4.16	4.08	4.19	4.15
Ciller Lite	4.06	4.18	4.10	4.05	4.16	4.11
Mud Light	4.06	4.13	4.13	4.08	4.10	4.10
$\bar{y}_{\bullet j}$	4.09	4.16	4.13	4.07	4.15	$\bar{y}_{\bullet\bullet} = 4.12$

Hint: $\sum_i \sum_j \left(y_{ij} - \bar{y}_{\bullet\bullet}\right)^2 = 0.0314.$

a) Is there a significant difference in alcohol percentage between beer brands? Perform the appropriate test at a 10% level of significance.

b) Is there a significant difference in alcohol percentage between bars? Perform the appropriate test at a 5% level of significance.

7. Each of three cars is driven with each of four different brands of gasoline. The number of miles per gallon driven for each of the $IJ = (3)(4) = 12$ different combinations is recorded in the table below.

Car	Gasoline 1	2	3	4	$\overline{Y}_{i\bullet}$
1	31	32	23	26	28
2	36	38	28	34	34
3	23	29	27	21	25
$\overline{Y}_{\bullet j}$	30	33	26	27	29

$Y_{ij} = \mu + \mathrm{Car}_i + \mathrm{Gas}_j + \varepsilon_{ij}$, $\quad i = 1, 2, 3, \quad j = 1, 2, 3, 4.$

ε_{ij} are independent $N(0, \sigma^2)$ random variables,

$\mathrm{Car}_1 + \mathrm{Car}_2 + \mathrm{Car}_3 = 0,$ $\quad \mathrm{Gas}_1 + \mathrm{Gas}_2 + \mathrm{Gas}_3 + \mathrm{Gas}_4 = 0.$

a) Construct the ANOVA table.

Hint: $\displaystyle\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij} - \overline{y}_{\bullet\bullet}\right)^2 = 318.$

b) Is there a significant interaction effect between cars and gasoline? Use a 5% level of significance.

c) Test for differences in cars. Use a 5% level of significance.

H_0: $\mathrm{Car}_1 = \mathrm{Car}_2 = \mathrm{Car}_3 = 0$

d) Test for differences in brands of gasoline. Use a 5% level of significance.

H_0: $\mathrm{Gas}_1 = \mathrm{Gas}_2 = \mathrm{Gas}_3 = \mathrm{Gas}_4 = 0$

8. A two-factor analysis of variance experiment was performed with $I = 4$, $J = 3$, and $K = 3$ (a 4×3 factorial experiment with 3 replicates).

	Factor B			
Factor A	1	2	3	$\bar{y}_{i\bullet\bullet}$
1	17 13 18	15 19 17	19 23 21	18
2	20 18 16	18 14 13	17 21 16	17
3	16 16 13	17 14 17	16 13 13	15
4	22 17 18	16 18 14	16 20 21	18
$\bar{y}_{\bullet j\bullet}$	17	16	18	$\bar{y}_{\bullet\bullet\bullet} = 17$

a) Complete the ANOVA table.

ANOVA table:

Source	**SS**	**DF**	**MS**	**F**
Row	______	______	______	______
Column	______	______	______	______
Interaction	______	______	______	______
Residuals	120	______	______	
Total	258	______		

b) Conduct the tests of factor interaction and main effects, if appropriate, each at a 5% level of significance.

9. A two-factor analysis of variance experiment was performed with $I = 2$, $J = 3$, and $K = 4$ (a 2×3 factorial experiment with 4 replicates).

ANOVA table:

Source	**SS**	**DF**	**MS**	**F**
Factor A	4	______	______	______
Factor B	18	______	______	______
Interaction	12	______	______	______
Residuals	______	______	______	
Total	70	______		

a) Complete the ANOVA table.

b) Conduct the tests of factor interaction and main effects, if appropriate, each at a 5% level of significance.

Answers:

1. $Y_{ij} = \mu_j + \varepsilon_{ij}$, where ε_{ij}'s are i.i.d. $N(0, \sigma^2)$.

a) $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$

$J = 4$. $N = 24$.

$$\bar{y} = \frac{6 \cdot 3.5 + 6 \cdot 6.0 + 6 \cdot 8.0 + 6 \cdot 4.5}{24} = 5.5.$$

$$\text{SSB} = 6 \cdot (3.5 - 5.5)^2 + 6 \cdot (6.0 - 5.5)^2 + 6 \cdot (8.0 - 5.5)^2 + 6 \cdot (4.5 - 5.5)^2 = 69.$$

$$\text{MSB} = \frac{\text{SSB}}{J-1} = \frac{69}{3} = 23.$$

$$\text{SSW} = 5 \cdot 4.7 + 5 \cdot 4.8 + 5 \cdot 5.2 + 5 \cdot 4.3 = 95.$$

$$\text{MSW} = \frac{\text{SSW}}{N-J} = \frac{95}{20} = 4.75.$$

$$\text{SSTot} = \text{SSB} + \text{SSW} = 69 + 95 = 164.$$

$$F = \frac{\text{MSB}}{\text{MSW}} = \frac{23}{4.75} = \mathbf{4.8421}.$$

ANOVA table:

Source	SS	DF	MS	F
Between	69	3	23	4.8421
Within	95	20	4.75	
Total	164	23		

Critical Value(s): $F_{0.01}(3, 20) = \mathbf{4.94}$.

Decision: **Do NOT Reject H_0 at $\alpha = 0.01$**.

b) $\left(\overline{Y}_i - \overline{Y}_j\right) \pm \frac{q_{\gamma,J,N-J}}{\sqrt{2}} \cdot s_{pooled} \cdot \sqrt{\frac{1}{n_i} + \frac{1}{n_j}}$

$q_{0.01,4,20} = 5.02.$

$(8.0 - 3.5) \pm \frac{5.02}{\sqrt{2}} \cdot \sqrt{4.75} \cdot \sqrt{\frac{1}{6} + \frac{1}{6}}$ **4.5 ± 4.4666**

c) $\sum_{j=1}^{J} c_j \overline{y}_j \pm \sqrt{F_\alpha(J-1, N-J)} \cdot \sqrt{\text{MSW}} \cdot \sqrt{(J-1) \cdot \sum_{j=1}^{J} \frac{c_j^2}{n_j}}$

$c_M = -1, \quad c_A = 0, \quad c_E = 1, \quad c_N = 0.$

$F_{0.01}(3, 20) = 4.94.$

$(8.0 - 3.5) \pm \sqrt{4.94} \cdot \sqrt{4.75} \cdot \sqrt{3 \cdot \left(\frac{1}{6} + \frac{1}{6}\right)}$ **4.5 ± 4.844**

d) $\sum_{j=1}^{J} c_j \overline{y}_j \pm \sqrt{F_\alpha(J-1, N-J)} \cdot \sqrt{\text{MSW}} \cdot \sqrt{(J-1) \cdot \sum_{j=1}^{J} \frac{c_j^2}{n_j}}$

$c_M = -\frac{1}{2}, \quad c_A = 0, \quad c_E = 1, \quad c_N = -\frac{1}{2}.$

$F_{0.01}(3, 20) = 4.94.$

$\left(8.0 - \frac{3.5 + 4.5}{2}\right) \pm \sqrt{4.94} \cdot \sqrt{4.75} \cdot \sqrt{3 \cdot \left(\frac{1}{24} + \frac{1}{6} + \frac{1}{24}\right)}$ **4 ± 4.195**

e) $\sum_{j=1}^{J} c_j \overline{y}_j \pm \sqrt{F_\alpha(J-1, N-J)} \cdot \sqrt{\text{MSW}} \cdot \sqrt{(J-1) \cdot \sum_{j=1}^{J} \frac{c_j^2}{n_j}}$

$c_M = -\frac{1}{2}, \quad c_A = \frac{1}{2}, \quad c_E = \frac{1}{2}, \quad c_N = -\frac{1}{2}.$

$F_{0.01}(3, 20) = 4.94.$

$\left(\frac{6.0 + 8.0}{2} - \frac{3.5 + 4.5}{2}\right) \pm \sqrt{4.94} \cdot \sqrt{4.75} \cdot \sqrt{3 \cdot \left(\frac{1}{24} + \frac{1}{24} + \frac{1}{24} + \frac{1}{24}\right)}$ **3 ± 3.425**

2. a)

```
> Y = c(49,61,55,67,48, 50,68,57,64,73,66,56, 61,44,53,47,58,49)
> A = c( 1, 1, 1, 1, 1,  2, 2, 2, 2, 2, 2, 2,  3, 3, 3, 3, 3, 3)
> summary(aov(glm(Y ~ factor(A))))
            Df Sum Sq Mean Sq F value Pr(>F)
factor(A)    2  330.0   165.0  2.8846 0.0871 .
Residuals   15  858.0    57.2
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

p-value = 0.0871 < 0.10. **Reject $H_0: \mu_1 = \mu_2 = \mu_3$ at $\alpha = 0.10$.**

OR

$n_1 = 5$, $\bar{y}_1 = 56$, $s_1^2 = 65$,

$n_2 = 7$, $\bar{y}_2 = 62$, $s_2^2 = 63.66667$,

$n_3 = 6$, $\bar{y}_3 = 52$, $s_3^2 = 43.2$,

$J = 3$. $N = n_1 + n_2 + n_3 = 5 + 7 + 6 = 18$.

$$\bar{y} = \frac{n_1 \cdot \bar{y}_1 + n_2 \cdot \bar{y}_2 + \ldots + n_J \cdot \bar{y}_J}{N} = \frac{5 \cdot 56 + 7 \cdot 62 + 6 \cdot 52}{18} = \frac{1026}{18} = 57.$$

$$\text{SSB} = n_1 \cdot \left(\bar{y}_1 - \bar{y}\right)^2 + n_2 \cdot \left(\bar{y}_2 - \bar{y}\right)^2 + \ldots + n_J \cdot \left(\bar{y}_J - \bar{y}\right)^2$$
$$= 5 \cdot (56 - 57)^2 + 7 \cdot (62 - 57)^2 + 6 \cdot (52 - 57)^2 = 5 + 175 + 150 = 330.$$

$$\text{MSB} = \frac{\text{SSB}}{J-1} = \frac{330}{2} = 165.$$

$$\text{SSW} = \left(n_1 - 1\right) \cdot s_1^2 + \left(n_2 - 1\right) \cdot s_2^2 + \ldots + \left(n_J - 1\right) \cdot s_J^2$$
$$= 4 \cdot 65 + 6 \cdot 63.66667 + 5 \cdot 43.2 = 260 + 382 + 216 = 858.$$

$$\text{MSW} = \frac{\text{SSW}}{N-J} = \frac{858}{15} = 57.2.$$

$$\text{SSTot} = \text{SSB} + \text{SSW} = 330 + 858 = 1188.$$

$F = \frac{\text{MSB}}{\text{MSW}} = \frac{165}{57.2} \approx 2.8846.$

ANOVA table:

Source	SS	DF	MS	F
Between	330	2	165	2.8846
Within	858	15	57.2	
Total	1188	17		

$F > F_{0.10}(2, 15) = 2.70.$ **Reject** $\mathbf{H_0} : \mu_1 = \mu_2 = \mu_3$ **at** $\alpha = \mathbf{0.10}$.

p-value = [=FDIST(2.8846,2,15)] = 0.0871.

b) $\overline{Y}_i - \overline{Y}_j \pm t_{\gamma/2}(N - J \text{ d.f.}) \cdot \sqrt{\text{MSW}} \cdot \sqrt{\frac{1}{n_i} + \frac{1}{n_j}}$

$t_{0.10/2}(15) = t_{0.05}(15) = 1.753.$

$\mu_A - \mu_B$ $\quad 56 - 62 \pm 1.753 \sqrt{57.2} \sqrt{\frac{1}{5} + \frac{1}{7}}$ $\quad \mathbf{-6 \pm 7.763}$

$\mu_A - \mu_C$ $\quad 56 - 52 \pm 1.753 \sqrt{57.2} \sqrt{\frac{1}{5} + \frac{1}{6}}$ $\quad \mathbf{4 \pm 8.028}$

$\mu_B - \mu_C$ $\quad 62 - 52 \pm 1.753 \sqrt{57.2} \sqrt{\frac{1}{7} + \frac{1}{6}}$ $\quad \mathbf{10 \pm 7.376}$ ☺

c) $\overline{Y}_i - \overline{Y}_j \pm t_{(\gamma/m)/2}(N - J \text{ d.f.}) \cdot \sqrt{\text{MSW}} \cdot \sqrt{\frac{1}{n_i} + \frac{1}{n_j}}$ if m intervals

$t_{(0.10/3)/2}(15) = t_{0.10/6}(15) = 2.342925.$

```
> qt(1-.10/6,15)
[1] 2.342925
```

OR =TINV(0.10/3,15)

$\mu_A - \mu_B \qquad 56 - 62 \ \pm\ 2.342959\ \sqrt{57.2}\ \sqrt{\frac{1}{5}+\frac{1}{7}}$ **– 6 ± 10.376**

$\mu_A - \mu_C \qquad 56 - 52 \ \pm\ 2.342959\ \sqrt{57.2}\ \sqrt{\frac{1}{5}+\frac{1}{6}}$ **4 ± 10.730**

$\mu_B - \mu_C \qquad 62 - 52 \ \pm\ 2.342959\ \sqrt{57.2}\ \sqrt{\frac{1}{7}+\frac{1}{6}}$ **10 ± 9.858** ☺

d) $\left(\overline{Y}_i - \overline{Y}_j\right) \pm \frac{q_{\gamma, J, N-J}}{\sqrt{2}} \cdot \sqrt{\text{MSW}} \cdot \sqrt{\frac{1}{n_i}+\frac{1}{n_j}}$

```
> qtukey(0.90,3,15)
[1] 3.139694
```

$\mu_A - \mu_B \qquad 56 - 62 \ \pm\ \frac{3.139694}{\sqrt{2}}\ \sqrt{57.2}\ \sqrt{\frac{1}{5}+\frac{1}{7}}$ **– 6 ± 9.832**

$\mu_A - \mu_C \qquad 56 - 52 \ \pm\ \frac{3.139694}{\sqrt{2}}\ \sqrt{57.2}\ \sqrt{\frac{1}{5}+\frac{1}{6}}$ **4 ± 10.167**

$\mu_B - \mu_C \qquad 62 - 52 \ \pm\ \frac{3.139694}{\sqrt{2}}\ \sqrt{57.2}\ \sqrt{\frac{1}{7}+\frac{1}{6}}$ **10 ± 9.342** ☺

OR

```
> TukeyHSD(aov(Y ~ factor(A)),conf.level=0.90)
  Tukey multiple comparisons of means
    90% family-wise confidence level

Fit: aov(formula = Y ~ factor(A))

$`factor(A)`
      Diff          lwr         upr       p adj
2-1      6    -3.831663  15.8316634   0.3882972
3-1     -4   -14.167311   6.1673112   0.6646181
3-2    -10   -19.341518  -0.6584825   0.0754136
```
☺

e) $$\sum_{j=1}^{J} c_j \, \overline{\mathrm{Y}}_j \pm \sqrt{F_\gamma(J-1, N-J)} \cdot \sqrt{\mathrm{MSW}} \cdot \sqrt{(J-1) \cdot \sum_{j=1}^{J} \frac{c_j^2}{n_j}}$$

$\mathrm{F}_{0.10}(2, 15) = 2.70.$

i) $c_{\mathrm{A}} = 0, \qquad c_{\mathrm{B}} = 1, \qquad c_{\mathrm{C}} = -1.$

$$(62 - 52) \pm \sqrt{2.70} \cdot \sqrt{57.2} \cdot \sqrt{2 \cdot \left(\frac{1}{7} + \frac{1}{6}\right)} \qquad \mathbf{10 \pm 9.778} \qquad ☺$$

ii) $c_{\mathrm{A}} = -\frac{1}{2}, \qquad c_{\mathrm{B}} = 1, \qquad c_{\mathrm{C}} = -\frac{1}{2}.$

$$\left(62 - \frac{56+52}{2}\right) \pm \sqrt{2.70} \cdot \sqrt{57.2} \cdot \sqrt{2 \cdot \left(\frac{0.25}{5} + \frac{1}{7} + \frac{0.25}{6}\right)} \qquad \mathbf{8 \pm 8.511}$$

f)

3	3	1	1	3	2	3	1	2
44	47	48	49	49	50	53	55	56
1	2	3	4.5	4.5	6	7	8	9

2	3	1	3	2	2	1	2	2
57	58	61	61	64	66	67	68	73
10	11	12.5	12.5	14	15	16	17	18

$\bar{r}_1 = \frac{44}{5} = 8.8$ $\qquad \bar{r}_2 = \frac{89}{7} \approx 12.7143$ $\qquad \bar{r}_3 = \frac{38}{6} \approx 6.3333$

$\bar{r} = 9.5$ $\qquad \text{K} = \frac{12}{N(N+1)} \sum_{j=1}^{J} n_j \left(\bar{r}_j - \bar{r}\right)^2 \approx \mathbf{4.73467}.$

$\chi^2_\alpha(J-1) = \chi^2_{0.10}(2) = 4.605.$ $\qquad \text{K} > 4.605.$ $\qquad$ **Reject $\mathbf{H_0}$** at $\alpha = 0.10.$

```
> 1-pchisq(4.73467,2)
[1] 0.09373018
```

```
=CHIDIST(4.73467,2)
```

p-value $\approx 0.09373 < 0.10.$ $\qquad$ **Reject $\mathbf{H_0}$** at $\alpha = 0.10.$

OR

```
> Y = c(49,61,55,67,48,  50,68,57,64,73,66,56,  61,44,53,47,58,49)
> A = c( 1, 1, 1, 1, 1,   2, 2, 2, 2, 2, 2, 2,   3, 3, 3, 3, 3, 3)
>
> kruskal.test(Y ~ factor(A))

        Kruskal-Wallis rank sum test

data:  Y by factor(A)
Kruskal-Wallis chi-squared = 4.7445, df = 2, p-value = 0.09327
```

p-value $= 0.09327 < 0.10.$ $\qquad$ **Reject $\mathbf{H_0}$** at $\alpha = 0.10.$

3.

Source	**Sum of squares**	**df**	**Mean square**	**F**
Between	**320**	2	**160**	**8.00**
Within	**180**	**9**	20	
Total	500	11		

a) $J = \mathbf{3}$.

b) $N = \mathbf{12}$.

c) $F_{0.05}(2, 9) = \mathbf{4.26}$.

d) $H_0: \mu_1 = \mu_2 = \mu_3$ vs. H_1: at least two of μ_i are different.

e) **Reject H_0.**

4. df Between + df Within = df Total. $\Rightarrow$ df Within = 9.

$\dfrac{\text{SSWithin}}{\text{df Within}}$ = MSWithin. $\Rightarrow$ SSWithin = 180.

SSBetween + SSWithin = SSTotal. $\Rightarrow$ SSBetween = 320.

$\dfrac{\text{SSBetween}}{\text{df Between}}$ = MSBetween. $\Rightarrow$ MSBetween = 160.

$\text{F} = \dfrac{\text{MSBetween}}{\text{MSWithin}}$. $\Rightarrow$ F = 8.

Source	**Sum of squares**	**df**	**Mean square**	**F**
Between	**320**	2	**160**	**8.00**
Within	**180**	**9**	20	
Total	500	11		

a) $J = \mathbf{3}$.   b) $N = \mathbf{12}$.   c) $\text{F}_{0.05}(2, 9) = \mathbf{4.26}$.

d) $\text{H}_0 : \mu_1 = \mu_2 = \mu_3$ vs. H_1 : at least two of μ_i are different.

e) **Reject $\mathbf{H_0}$**.

5.

a) $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$

$J = 4.$ $N = n_1 + n_2 + \dots + n_J = 24.$

$$\bar{y} = \frac{n_1 \cdot \bar{y}_1 + n_2 \cdot \bar{y}_2 + \dots + n_J \cdot \bar{y}_J}{N} = \frac{16.03 + 16.09 + 15.99 + 16.01}{4} = 16.03.$$

$$\text{SSB} = n_1 \cdot \left(\bar{y}_1 - \bar{y}\right)^2 + n_2 \cdot \left(\bar{y}_2 - \bar{y}\right)^2 + \dots + n_J \cdot \left(\bar{y}_J - \bar{y}\right)^2$$
$$= 6 \cdot (0.00)^2 + 6 \cdot (0.06)^2 + 6 \cdot (-0.04)^2 + 6 \cdot (-0.02)^2 = 0.0336.$$

$$\text{MSB} = \frac{\text{SSB}}{J-1} = \frac{0.0336}{3} = 0.0112.$$

$$\text{SSW} = \left(n_1 - 1\right) \cdot s_1^2 + \left(n_2 - 1\right) \cdot s_2^2 + \dots + \left(n_J - 1\right) \cdot s_J^2$$
$$= 5 \cdot 0.00416 + 5 \cdot 0.00212 + 5 \cdot 0.00224 + 5 \cdot 0.00464 = 0.0658.$$

$$\text{MSW} = \frac{\text{SSW}}{N-J} = \frac{0.0658}{20} = 0.00329.$$

$$\text{SSTot} = \text{SSB} + \text{SSW} = 0.0336 + 0.0658 = 0.0994.$$

$$\text{F} = \frac{\text{MSB}}{\text{MSW}} = \frac{0.0112}{0.00329} = \mathbf{3.4042553}.$$

Source	*SS*	*df*	*MS*	*F*
Between	0.0336	3	0.0112	**3.404**
Within	0.0658	20	0.00329	
Total	0.0994	23		

Critical Value(s): $F_{0.05}(3, 20) = \mathbf{3.10}$.

Decision: **Reject H_0 at $\alpha = 0.05$**.

b) $$\sum_{j=1}^{J} c_j \, \overline{y}_j \pm \sqrt{F_\alpha(J-1, N-J)} \cdot \sqrt{\mathrm{MSW}} \cdot \sqrt{(J-1) \cdot \sum_{j=1}^{J} \frac{c_j^2}{n_j}}$$

$$c_1 = -{}^1\!/_3, \quad c_2 = 1, \quad c_3 = -{}^1\!/_3, \quad c_4 = -{}^1\!/_3. \qquad \mathrm{F}_{0.05}(3, 20) = 3.10.$$

$$\left(16.09 - \frac{16.03 + 15.99 + 16.01}{3}\right) \pm \sqrt{3.10} \cdot \sqrt{0.00329} \cdot \sqrt{3 \cdot \left(\frac{1}{54} + \frac{1}{6} + \frac{1}{54} + \frac{1}{54}\right)}$$

$\mathbf{0.8 \pm 0.082458}$ $\mathbf{(-0.002458, 0.162458)}$

6.

a) $\text{SSA} = J\sum_{i=1}^{I}\left(\overline{y}_{i\bullet} - \overline{y}_{\bullet\bullet}\right)^2 = 5\cdot[\,(4.15-4.12)^2 + (4.11-4.12)^2 + (4.10-4.12)^2\,]$

$= 0.0070.$

$\text{SSB} = I\sum_{j=1}^{J}\left(\overline{y}_{\bullet j} - \overline{y}_{\bullet\bullet}\right)^2 = 3\cdot[\,(4.09-4.12)^2 + (4.16-4.12)^2 + (4.13-4.12)^2$

$+ (4.07-4.12)^2 + (4.15-4.12)^2\,] = 0.0180.$

SSResiduals = SSTotal – SSA – SSB = 0.0314 – 0.0070 – 0.0180 = 0.0064.

ANOVA table:

Source	SS	DF	MS	F
Row (A)	**0.0070**	$I-1=$ **2**	**0.0035**	**4.375**
Column (B)	**0.0180**	$J-1=$ **4**	**0.0045**	**5.625**
Residuals	**0.0064**	$(I-1)(J-1)=$ **8**	**0.0008**	
Total	0.0314	$IJ-1=$ **14**		

a) Critical Value: $F_{0.10}(2, 8) = \mathbf{3.11}$.

F = 4.375 > 3.11. Decision: **Reject $\mathbf{H_0}$** at $\alpha = 0.10$.

b) Critical Value: $F_{0.05}(4, 8) = \mathbf{3.84}$.

F = 5.625 > 3.84. Decision: **Reject $\mathbf{H_0}$** at $\alpha = 0.05$.

7.

a) $\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}-\overline{y}_{\bullet\bullet}\right)^2 = 318 \quad = \text{SSTotal.}$

$\text{SSA} = J\sum_{i=1}^{I}\left(\overline{y}_{i\bullet}-\overline{y}_{\bullet\bullet}\right)^2 = 4\cdot[(28-29)^2+(34-29)^2+(25-29)^2] = 168.$

$\text{SSB} = I\sum_{j=1}^{J}\left(\overline{y}_{\bullet j}-\overline{y}_{\bullet\bullet}\right)^2 = 3\cdot[(30-29)^2+(33-29)^2+(26-29)^2+(27-29)^2] = 90.$

SSResid = SSTotal – SSA – SSB = 318 – 168 – 90 = 60.

ANOVA table:

Source	SS	DF	MS	F
Row (Car)	**168**	$I-1=$ **2**	**84**	**8.4**
Column (Gas)	**90**	$J-1=$ **3**	**30**	**3**
Residuals	**60**	$(I-1)(J-1)=$ **6**	**10**	
Total	318	$IJ-1=$ **11**		

b) We cannot estimate interaction effect with only one observation per sell.
Therefore, we cannot tell if it is significant or not.

c) Critical Value: $F_{0.05}(2, 6) =$ **5.14**.

$F = 8.4 > 5.14.$ Decision: **Reject H_0**.

d) Critical Value: $F_{0.05}(3, 6) =$ **4.76**.

$F = 3 < 4.76.$ Decision: **Do NOT Reject H_0**.

8.

$$\text{SSA} = JK\sum_{i=1}^{I}\left(\overline{y}_{i\bullet}-\overline{y}_{\bullet\bullet}\right)^2 = 3\times3\times[\,(18-17)^2+(17-17)^2 + (15-17)^2+(18-17)^2\,] = 54.$$

$$\text{SSB} = IK\sum_{j=1}^{J}\left(\overline{y}_{\bullet j}-\overline{y}_{\bullet\bullet}\right)^2 = 4\times3\times[\,(17-17)^2+(16-17)^2+(18-17)^2\,] = 24.$$

SSAB = SSTot – SSA – SSB – SSRes = 258 – 54 – 24 – 120 = 60.

ANOVA table:

Source	**SS**	**DF**	**MS**	**F**
Row	**54**	**3**	**18**	**3.6**
Column	**24**	**2**	**12**	**2.4**
Interaction	**60**	**6**	**10**	**2**
Residuals	120	**24**	**5**	
Total	258	**35**		

b) Critical Value:

Interaction	$F_{0.05}(6, 24) = \mathbf{2.51}$.	Not significant
Factor A	$F_{0.05}(3, 24) = \mathbf{3.01}$.	Significant
Factor B	$F_{0.05}(2, 24) = \mathbf{3.40}$.	Not significant

9.

a) ANOVA table:

Source	**SS**	**DF**	**MS**	**F**
Factor A	4	$I-1=\mathbf{1}$	**4**	**2**
Factor B	18	$J-1=\mathbf{2}$	**9**	**4.5**
Interaction	12	$(I-1)(J-1)=\mathbf{2}$	**6**	**3**
Residuals	**36**	$IJ(K-1)=\mathbf{18}$	**2**	
Total	70	$IJK-1=\mathbf{23}$		

b)

	Critical Value:	
Interaction	$F_{0.05}(2, 18)=\mathbf{3.55}$.	Not significant
Factor A	$F_{0.05}(1, 18)=\mathbf{4.41}$.	Not significant
Factor B	$F_{0.05}(2, 18)=\mathbf{3.55}$.	Significant