Math 415 - Lecture 13

Basis and Dimension

Wednesday September 23rd 2015

Textbook reading: Chapter 2.3

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Suggested practice exercises: Chapter 2.3 Exercise 1, 2, 3, 5, 6, 9, 11, 16, 19, 20, 22, 27.

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Strang lecture: Independence, Basis, and Dimension

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Review

- Vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are *linearly* **Dependent** if

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_p\mathbf{v_p} = \mathbf{0},$$

and not all the coefficients are zero.

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- Any set of 11 vectors in $\mathbb{R}^{10}$ is linearly dependent. Why?

Redundant Vectors.

Definition

In a list of vectors $(\mathbf{v_1}, \ldots, \mathbf{v_p})$ in a vector space V we call $\mathbf{v_k}$ *redundant* if v_k is a linear combination of the previous vectors. In this case $\text{Span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{k-1}}, \mathbf{v_k}) = \text{Span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{k-1}})$, i.e., you can delete the redundant vector and get the same span.

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Example

Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}.$

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Solution

Since $\mathbf{v_3} = \mathbf{v_1} + \mathbf{v_2}$, $\mathbf{v_3}$ is redundant and $\text{Span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \text{Span}(\mathbf{v_1}, \mathbf{v_2})$.

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Today we are going to study sets of vectors without redundant elements.

A Basis of a Vector Space

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Definition

A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in V is a **basis** of V if

- $V = \text{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$, and
- the vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are linearly independent.

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Fact: $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in V is a basis of V if and only if every vector $\mathbf{w}$ in V can be uniquely expressed as $\mathbf{w} = c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p}$.

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Fact: A basis is a *minimal spanning set*: the elements of the basis span V but you cannot delete any of these elements and still get all of V. There are no redundant vectors.

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Let $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Show that $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is a basis of $\mathbb{R}^3$. (It is called the **standard basis**.)

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Solution

- Clearly, $\text{Span}\,\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\} = \mathbb{R}^3$.
- $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ are independent, because

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- $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ are independent, because $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has a pivot in each column, no free variables.

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V is said to have **dimension** p if it has a basis consisting of p vectors.

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A basis of $\mathbb{R}^3$ cannot have less than 3 vectors,because 2 vectors span at most a plane.

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A basis of $\mathbb{R}^3$ cannot have less than 3 vectors,because 2 vectors span at most a plane. (Challenge: can you think of an argument that is more "rigorous"?)

A Basis of a Vector Space

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$\mathbb{R}^3$ has dimension 3.

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Indeed, the standard basis

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Likewise, $\mathbb{R}^n$ has dimension n.

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Its standard basis is $1, t, t^2, t^3, \ldots$ Why?

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Its standard basis is $1, t, t^2, t^3, \ldots$ Why?

Solution

This is indeed a basis, because any polynomial can be written as a unique linear combination:

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

for some n.

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Recall that vectors in V form a **basis** of V if

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These are two conditions. If we know the dimension of V, we only need to check one of these two conditions:

Theorem

Suppose that V has dimension d.

- *A set of d vectors in V are a basis if they span V.*
- *A set of d vectors in V are a basis if they are linearly independent.*

Why?

Solution

Theorem

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A Basis of a Vector Space

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Are the following sets a basis for $\mathbb{R}^3$?

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Since each column contains a pivot, the three vectors are independent. Hence, this is a basis for $\mathbb{R}^3$.

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Let P_2 be the space of polynomials of degree at most 2.

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  This is indeed a basis because every polynomial

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  Hence, $\{t, 1-t, 1+t-t^2\}$ is a basis of P_2.

Shrinking and Exanding Sets of Vectors

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We can find a basis for $V = \text{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ by discarding, if necessary, some of the vectors in the spanning set.

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Example

Produce a basis of $\mathbb{R}^2$ from the vectors

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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The remaining vectors $\{\mathbf{v_1}, \mathbf{v_3}\}$ are a basis for $\mathbb{R}^2$, because the two vectors are clearly linearly independent.

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$\mathbf{v_1}$ is independent.

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- The only 3-dimensional subspace is $\mathbb{R}^3$ itself.

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   True. $\mathbf{v_k}$ is not adding anything new.