Worksheet 11 for November 10th and 12th

1. **a.** *Compare* $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and the "row flipped" determinant* $\det \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

b. *If* $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, *what is* $\det(A)$*?*

c. *If* $A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{bmatrix}$, *what is* $\det(A)$*?*

d. *If* $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix}$, *what is* $\det(A)$*?*

e. *If* $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$, *find* $\det(A)$ *byexpanding along the last column.*

Solution. **a.** We have:

$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 1 \cdot 4 - 2 \cdot 3 = -2$$

and,

$$\det\left(\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}\right) = 3 \cdot 2 - 4 \cdot 1 = 2$$

So, $\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = -\det\left(\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}\right)$.

b. We transform A into an upper triangular matrix using row operations:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R5, R2 \leftrightarrow R4} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since we swap rows twice, we have:

$$\det(A) = -(-\det(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix})) = 1$$

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Midterm Date: November 19 7-8:15 PM, Conflict November 20, 8-9.20AM and 9:30-10:50AM, Conflict sign up deadline: November 13

Final Date: December 17 8-11AM, Conflict December 15, 8-11AM. You are allowed to take the conflict exam if you have more than two examination within 24 hours. Conflict sign up deadline: November 30

c. We transform A into an upper triangular matrix using row operations:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1, R3 \to R3 - 3R1} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & -6 \end{bmatrix}$$

Since the row operations that we used do not change the value of the determinant, we have:

$$\det(A) = \det(\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & -6 \end{bmatrix}) = 1 \cdot 0 \cdot (-6) = 0$$

d. We transform A into an upper triangular matrix using row operations:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1, R3 \to R3 - 3R1} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R2} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the row operations that we used do not change the value of the determinant, we have:

$$\det(A) = \det\left(\begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\right) = 0$$

e. We have:

$$\det(A) = 3\det\left(\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\right) - 1\det\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\right) + 3\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\right) = 3 \cdot 1 - 1 \cdot (-1) + 3 \cdot (-4) = -8$$

□

2. *True or False? Justify your answers!*

a. *Let Q be a 3×3 orthogonal matrix. Then det$(Q) = 1$.*

b. *If det$(A) =$det$(B) = 0$ then det$(A + B) = 0$.*

c. *Let A be a 3×3 matrix so that det$(A) = 0$. Then $A\mathbf{x} = \mathbf{b}$ has exactly one solution for each vector $\mathbf{b}$.*

d. *Let A be a 3×3 matrix so that det$(A) = 9$. Then det$(2A) = 18$.*

e. *Let R be a 2×3 matrix. Then det$(R^T R) = 0$.*

f. *Let R be a 2×3 matrix. Then det$(RR^T) = 0$.*

Solution. **a.** False, we have $QQ^T = I$ so $\det(Q)\det(Q^T) = \det(Q)^2 = \det(I) = 1$. Hence, $\det(Q) = 1$ or -1 but it is not necessarily equal to 1 or necessarily equal to -1. Consider the following examples:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

b. False, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

c. False, we have that A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has exactly one solution for each vector $\mathbf{b}$.

d. False, $\det(2A) = 2^3 \det(A) = 72$.

e. Let $R = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$. Write $\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.Then

$$R^T R = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} (a_1\boldsymbol{a} + b_1\boldsymbol{b}) & (a_2\boldsymbol{a} + b_2\boldsymbol{b}) & (a_3\boldsymbol{a} + b_3\boldsymbol{b}) \end{bmatrix}.$$

Therefore the columns of $R^T R$ are in span$(\boldsymbol{a}, \boldsymbol{b})$. Since $R^T R$ has three columns, they have to be linearly dependent. Hence $R^T R$ is not invertible and $\det(R^T R) = 0$.

f. False, consider $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then $RR^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and therefore $\det(RR^T) = 1$. Therefore the statement is not true. □

3. *True or False? Justify your answers!*

a. *We say A and B ($n \times n$ matrices) are similar if $A = DBD^{-1}$ for an invertible matrix D. Let A and B be similar matrices, then det(A) =det(B).*

b. *Let A and B be 3×3 matrices. If det(A) =det(B) then A and B are similar. [Note: number of pivots in DBD^{-1} is equal to the number of pivots in B. (Why?) Use this fact to find a counter example.]*

c. *Someone tells you that the zero vector is an eigenvector of a 2×2 matrix A. Is this possible?*

d. *An $n \times n$ matrix A always has n distinct eigenvalues.*

Solution. **a.** True, we have:

$$\det(A) = \det(DBD^{-1}) = \det(D)\det(B)\det(D^{-1}) = \det(D)\det(B)\frac{1}{\det(D)} = \det(B)$$

b. False, consider $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then the number of pivots in DBD^{-1} is 1 but the number of pivots in A is equal to 2. Thus, it is not possible to find D so that $A = DBD^{-1}$.

c. This is false. By convention, the zero vector is **never** an eigenvector.

d. False, the $n \times n$ identity matrix I_n (where $n \geq 2$) has only one eigenvalue $\lambda = 1$. This eigenvalue occurs with multiplicity n.

□

4. *For each of the following matrices, determine the characteristic polynomial $p(\lambda)$ of the matrix, determine the eigenvalues of the matrix and for each eigenvalue, determine (a basis for) the eigenspace that is associated to that eigenvalue.*

a. $\begin{bmatrix} 4 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$,

b. $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$,

c. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution. **a.** We have:

$$p(\lambda) = \det \begin{bmatrix} 4-\lambda & 0 & -2 \\ 1 & 1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda)(4-\lambda)$$

Hence, the eigenvalues of A are 2,4, and 1. For $\lambda = 2$:

$$\begin{bmatrix} 2 & 0 & -2 \\ 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2\to R2-1/2R1, R1\to 1/2R1, R2\to -R2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 4$:

$$\begin{bmatrix} 0 & 0 & -2 \\ 1 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R2\to R2+R3, R1\to R1-R3, R3\to -1/2R2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 1$:

$$\begin{bmatrix} 3 & 0 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2\to R2-2R3, R1\to R1+2R3, R1\to R1-3R2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

b. We have:

$$p(\lambda) = \det \begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix} = (3-\lambda)(-3-\lambda) - 16 = \lambda^2 - 25 = (\lambda-5)(\lambda+5)$$

Hence, the eigenvalues of A are 5 and -5. For $\lambda = 5$:

$$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \xrightarrow{R2\to R2+2R1, R1\to -1/2R1} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = -5$:

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \xrightarrow{R2\to R2-1/2R1, R1\to 1/8R1} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

c. We have:

$$p(\lambda) =$$

$$\det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda)((1-\lambda)(1-\lambda)-1) - (-\lambda) + (1-(1-\lambda))$$

$$= (1-\lambda)(-\lambda)(2-\lambda) + 2\lambda = -\lambda((1-\lambda)(2-\lambda)-2) = \lambda^2(3-\lambda)$$

Hence, the eigenvalues of A are 0 and 3. For $\lambda = 0$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R2\to R2-R1, R3\to R3-R1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is $\operatorname{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 3$:

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{R1\leftrightarrow R3, R2\to R2-R1, R3\to R3+2R1, R3\to R3+R2} \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2\to -1/3R2, R1\to R1-R2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. □

5. *Let A be an $n \times n$-matrix with eigenvalue λ. Which of the following statements are true:*

a. *λ^2 is an eigenvalue of A^2,*
b. *λ^{-1} is an eigenvalue of A^{-1},*
c. *$\lambda + 1$ is an eigenvalue of $A + I$.*

Solution. All three statements are correct. For **a.**, let $\mathbf{v}$ be an eigenvector of A to the eigenvalue λ. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda\lambda\mathbf{v} = \lambda^2\mathbf{v}.$$

So $\mathbf{v}$ is an eigenvector of A^2 to eigenvalue λ^2. So λ^2 is an eigenvalue of A^2.
For **b.**, let $\mathbf{v}$ be an eigenvector of A to the eigenvalue λ. Then

$$A^{-1}(\lambda\mathbf{v}) = A^{-1}(A\mathbf{v}) = (A^{-1}A)\mathbf{v} = \mathbf{v} = \lambda^{-1}(\lambda\mathbf{v}).$$

Hence $\lambda\mathbf{v}$ is an eigenvector of A^{-1} to eigenvalue λ^{-1}. So λ^{-1} is an eigenvalue of A^{-1}.
For **c.**, let $\mathbf{v}$ be an eigenvector of A to the eigenvalue λ. Then

$$(A+I)(\mathbf{v}) = A\mathbf{v} + \mathbf{v} = \lambda\mathbf{v} + \mathbf{v} = (\lambda+1)\mathbf{v}.$$

Hence $(\lambda+1)\mathbf{v}$ is an eigenvector of $A+I$ to eigenvalue $\lambda+1$. So $\lambda+1$ is an eigenvalue of $A+I$. □

6. *Let A, B be two $n \times n$-matrices such that $AB = BA$.*

a. *Suppose $\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ. Is $B\boldsymbol{v}$ an eigenvector of A? If so, what is the eigenvalue of that eigenvector?*

b. *Suppose A has eigenvectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ with distinct eigenvalues $\lambda_1 \neq \ldots \neq \lambda_n$. Is each $\boldsymbol{v_i}$ also an eigenvector of B? (This question is a bit tricker. Hint: Note that each of the eigenspaces of A has dimension 1 and then use your answer to* ***a.****).*

Solution. **a.** We first must consider the case that $B\boldsymbol{v} = 0$, in which case $B\boldsymbol{v}$ cannot be an eigenvector. In the other case, consider $A(B\boldsymbol{v}) = (AB)\boldsymbol{v}$, and since $AB = BA$, this is the same as $(BA)\boldsymbol{v} = B(A\boldsymbol{v}) = B(\lambda\boldsymbol{v}) = \lambda(B\boldsymbol{v})$ (as $\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ), so since $A(B\boldsymbol{v}) = \lambda B\boldsymbol{v}$, $B\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ. So overall either $B\boldsymbol{v} = 0$ or $B\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ_i.

b. Take the i-th eigenvalue $\boldsymbol{v_i}$ of A. Since there are n distinct eigenvalue, $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ form a basis of $\mathbb{R}^n$. Thus the eigenspace of A for eigenvalue λ_i is $\text{span}(\boldsymbol{v_i})$. Since $\boldsymbol{v_i}$ is an eigenvector of A with eigenvalue λ_i, we get$A\boldsymbol{v_i} = \lambda_i \boldsymbol{v_i}$. Then by **a.** either $B\boldsymbol{v_i} = 0$ or $B\boldsymbol{v_i}$ is an eigenvector of A to the eigenvalue λ_i. If $B\boldsymbol{v_i} = 0$, then $\boldsymbol{v_i}$ is eigenvector of B with eigenvalue 0. So it is left to consider the case that $B\boldsymbol{v_i}$ is an eigenvector of A to the eigenvalue λ_i. Then $B\boldsymbol{v_i}$ is in the eigenspace of A of the eigenvalue λ_i. Then $B\boldsymbol{v_i}$ is in the span of $\boldsymbol{v_i}$. Therefore $B\boldsymbol{v_i}$ is a multiple of $\boldsymbol{v}_i$. Hence $\boldsymbol{v_i}$ is an eigenvector of B. $\square$

7. *Let $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$, $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto A \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathcal{B} = \{ \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix} \}$.*

a. *If $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, what is $\boldsymbol{v}_{\mathcal{B}}$?*

b. *If $\boldsymbol{v}_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, what is $\boldsymbol{v}$?*

c. *What is $T_{\mathcal{B},\mathcal{B}}$?*

Solution. Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^2$. Note that $T_{\mathcal{E},\mathcal{E}} = A$. The bases change matrix $I_{\mathcal{B},\mathcal{E}}$ is $I_{\mathcal{E},\mathcal{B}}^{-1}$. We know that

$$I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}.$$

This matrix is orthogonal, therefore

$$I_{\mathcal{B},\mathcal{E}} = I_{\mathcal{E},\mathcal{B}}^{-1} = I_{\mathcal{E},\mathcal{B}}^{T} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}.$$

a. Using the base change matrix, we get

$$\boldsymbol{v}_{\mathcal{B}} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{8}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}.$$

b. Here

$$\boldsymbol{v} = I_{\mathcal{E},\mathcal{B}} \boldsymbol{v}_{\mathcal{B}} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{\sqrt{5}} \\ \frac{4}{\sqrt{5}} \end{bmatrix}.$$

c. We calculate

$$T_{\mathcal{B},\mathcal{B}} = I_{\mathcal{B},\mathcal{E}} T_{\mathcal{E},\mathcal{E}} I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}.$$

$\square$

8. *Let* $\mathcal{B} := \{\boldsymbol{b_1}, \boldsymbol{b_2}\}$ *and* $\mathcal{C} := \{\boldsymbol{c_1}, \boldsymbol{c_2}\}$ *be two bases of* $\mathbb{R}^2$ *such that*

$$\boldsymbol{b_1} = 6\boldsymbol{c_1} - 2\boldsymbol{c_2} \textit{ and } \boldsymbol{b_2} = 9\boldsymbol{c_1} - 4\boldsymbol{c_2}.$$

Determine $I_{\mathcal{C},\mathcal{B}}$ *and* $I_{\mathcal{B},\mathcal{C}}$*!*

Solution. $I_{\mathcal{C},\mathcal{B}}$ is the matrix representing the identity transformation I with input basis $\mathcal{B}$ and output basis $\mathcal{C}$. Since

$$\boldsymbol{b_1} = 6\boldsymbol{c_1} - 2\boldsymbol{c_2} \text{ and } \boldsymbol{b_2} = 9\boldsymbol{c_1} - 4\boldsymbol{c_2},$$

we get that $I(b_1)_{\mathcal{C}} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ and $I(b_2)_{\mathcal{C}} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$.Therefore we have that

$$I_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}.$$

Now $I_{\mathcal{B},\mathcal{C}}$ will be the inverse of this matrix, that is it represents the identity transformation with input basis $\mathcal{B}$ and output basis $\mathcal{C}$. Thus we invert the previous matrix to get $\begin{bmatrix} \frac{2}{3} & \frac{3}{2} \\ \frac{-1}{3} & -1 \end{bmatrix}$. □

9. *Let* A *be a* $n \times n$*-matrix and let* $T : \mathbb{R}^n \to \mathbb{R}^n$ *be the linear transformation such that* $T(\boldsymbol{v}) = A\boldsymbol{v}$ *for all* $\boldsymbol{v}$ *in* $\mathbb{R}^n$*. Let* $\mathcal{E}$ *be the standard basis of* $\mathbb{R}^n$*. True or false?*

- **a.** *Let* $\mathcal{B} := \{\boldsymbol{b_1}, \ldots, \boldsymbol{b_n}\}$ *be a basis of* $\mathbb{R}^n$*. All* $\boldsymbol{b_i}$*'s are eigenvectors of* A *if and only if* $T_{\mathcal{B},\mathcal{B}}$ *is diagonal.*
- **b.** *The matrix* A *is invertible if and only if there is a basis* $\mathcal{C} := \{\boldsymbol{c_1}, \ldots, \boldsymbol{c_n}\}$ *of* $\mathbb{R}^n$ *such that* $T_{\mathcal{C},\mathcal{E}} = I_{n\times n}$.

Solution.

- **a.** This is true. Suppose all the $\boldsymbol{b_i}'s$ are eigenvectors of A. Then $T(\boldsymbol{b_i}) = \lambda_i \boldsymbol{b_i}$, where λ_i is the eigenvalue corresponding to $\boldsymbol{b_i}$. Thus the matrix $T_{\mathcal{B},\mathcal{B}}$ is diagonal with entries λ_i down the diagonal.Suppose $T_{\mathcal{B},\mathcal{B}}$ is diagonal, with entry λ_i in column i. Then let $\boldsymbol{e_i}$ be the vector with 1 in the i-th row and 0 elsewhere. Then $T_{\mathcal{B},\mathcal{B}}\boldsymbol{e_i} = \lambda_i \boldsymbol{e_i}$. Recalling the definition of T, this means that $A\boldsymbol{b_i} = \lambda_i \boldsymbol{b_i}$, and thus $\boldsymbol{b_i}$ is an eigenvector of A with eigenvalue λ_i.
- **b.** Suppose the matrix A is invertible. Then set $\mathcal{C} = \{T\boldsymbol{e_1}, \ldots, T\boldsymbol{e_n}\}$. This then will have $T_{\mathcal{C},\mathcal{E}} = I_{n\times n}$.Suppose that there is a basis $\mathcal{C} = \{\boldsymbol{c_1}, \ldots \boldsymbol{c_n}\}$ such that $T_{\mathcal{C},\mathcal{E}} = I_{n\times n}$. Then $A\boldsymbol{e_i} = \boldsymbol{c_i}$ for each i. However $A\boldsymbol{e_i}$ is the i-th column of A. Thus the columns of A are linearly independent, so since A is also square, A is invertible.

□