Math 415 - Lecture 12

Linear independence

Monday September 21st 2015

Textbook reading: Section 2.3

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Suggested practice exercises: Section 2.3: 1, 2, 3, 4, 5,7, 8, 9

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Khan Academy video: Introduction to Linear Independence, More on linear independence, Span and Linear Independence Example,

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Strang lecture: Independence, Basis, and Dimension

* Exam 1 (7-8:15 pm Tuesday September 29):

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  - 213 Gregory Hall: AD3, ADG, ADU

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* Conflicts: The conflict exams are at 8:00-9:20AM and 9:30-10:50AM on the same day. Email your TA with your reason for needing a conflict, and your preferred time to sign up for the conflict exam. The deadline for signing up for a conflict is a week before (September 22).

Linear independence

Linear independence

Review.

- Span $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ is the set of all linear combinations

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_m\mathbf{v_m}.$$

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- Span $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ is a vector space.
- $\mathrm{Col}(A) = \mathrm{Span}(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n})$, if $A = \begin{bmatrix}\mathbf{a_1} & \mathbf{a_2} & \ldots & \mathbf{a_n}\end{bmatrix}$. In this case

$$\mathbf{b} \in \mathrm{Col}(A) \iff \mathbf{b} = A\mathbf{x} \text{ for some} \mathbf{x} \in \mathbb{R}^n.$$

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Today we want to think how *big* the span of a bunch of vectors is. Is it a line, or a plane or ....

Example

Is $\mathrm{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$ equal to $\mathbb{R}^2$?

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To answer the question translate to linear systems.

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To answer the question translate to linear systems. Recall that the span is equal to

$$\{b\colon\ b = A\mathbf{x}\} = \left\{\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}\mathbf{x} : \mathbf{x} \in \mathbb{R}^2\right\}.$$

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Hence, the span is equal to $\mathbb{R}^2$ if and only if the system with augmented matrix

$$\left[\begin{array}{cc|c}1 & 2 & b_1\\1 & 2 & b_2\end{array}\right]$$

is consistent for all b_1, b_2.

Linear independence

To check consistency use Gaussian elimination:

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The span is a line instead of a plane!

Example

Is $\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$ equal to $\mathbb{R}^3$?

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  Well, the three vectors that span satisfy a *relation*:

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- Hence, $\mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}.$

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- We are going to say that the three vectors are **linearly dependent** because they satisfy the (non trivial) relation
  $$-3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \mathbf{0}.$$

Linear independence

Definition

Vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are said to be **linearly independent** if the equation

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$$

has only the trivial solution (namely, $x_1 = x_2 = \cdots = x_p = 0$).

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Likewise, $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are said to be **linearly dependent** if there exist coefficients $x_1, \ldots, x_p$, not all zero, such that

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$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_p\mathbf{v_p} = \mathbf{0}.$$

This is called a *non trivial relation* (when not all coefficient are zero.)

Example

- Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix}$ independent?
- If possible, find a linear dependence relation among them.

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Solution

We need to check whether the equation

$$x_1\begin{bmatrix}1\\1\\1\end{bmatrix} + x_2\begin{bmatrix}1\\2\\3\end{bmatrix} + x_3\begin{bmatrix}-1\\1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

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has a non trivial solution. The three vectors are independent if and only if there are no free variables for the system

$$\begin{bmatrix}1 & 1 & -1\\1 & 2 & 1\\1 & 3 & 3\end{bmatrix}\mathbf{x} = \mathbf{0}.$$

Linear independence

To find out, we reduce the matrix to echelon form:

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Since there is a column without a pivot, we do have a free variable.

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Since there is a column without a pivot, we do have a free variable. Hence, the three vectors are not linearly independent. To find a linear dependence relation we solve this system.

Linear independence

Initial steps of Gaussian elimination are as before:

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x_3 is free. $x_2 = -2x_3$, and $x_1 = 3x_3$. Hence, for any x_3,

$$3x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2x_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

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Since we are only interested in one linear combination, we can set, say, $x_3 = 1$:

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Linear independence of matrix columns

Linear independence of matrix columns

- Note that a linear dependence relation, such as

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- Hence, each linear dependence relation among the columns of a matrix A corresponds to a solution to $A\mathbf{x} = \mathbf{0}$. The Null space determines (in)dependence!

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  In other words, the columns have to be linearly dependent.

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Additional exercises

With the least amount of work possible, decide which of the following sets of vectors are linearly independent.

(a) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$

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Linearly independent, because it is a single non-zero vector.

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