Quiz 2

1. Consider the sequence defined inductively as follows: $a_0 = 0$, and $a_n = a_{\lceil n/2 \rceil} + a_{\lfloor n/2 \rfloor}$. We claim that $a_n = n$ for all n. We prove this by induction. For the base case observe that $a_0 = 0$ by definition. Assume that for all $n < k$, we have $a_n = n$. Now $a_k = a_{\lceil k/2 \rceil} + a_{\lfloor k/2 \rfloor}$ by definition. From the induction hypothesis, we have $a_{\lceil k/2 \rceil} = \lceil k/2 \rceil$ and $a_{\lfloor k/2 \rfloor} = \lfloor k/2 \rfloor$. Thus, $a_k = a_{\lceil k/2 \rceil} + a_{\lfloor k/2 \rfloor} = \lceil k/2 \rceil + \lfloor k/2 \rfloor = k$. Thus, the claim is established by induction.

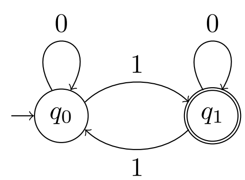
   (A) The proof is correct.

   (B) For some values of k, the induction hypothesis does not allow us to conclude $a_{\lceil k/2 \rceil} = \lceil k/2 \rceil$

   (C) For some values of k, the induction hypothesis does not allow us to conclude $a_{\lfloor k/2 \rfloor} = \lfloor k/2 \rfloor$.

   (D) For some values of k, $\lceil k/2 \rceil + \lfloor k/2 \rfloor \neq k$.

   The correct answer is (B).



2. Consider the automaton M_1 shown above. We will show that $w \in \mathbf{L}(M_1)$ iff w has an odd number of 1s. We will prove this by induction on the length of w. For the base case, observe that when $w = \epsilon$, w has an even number of 1s. Further, $q_0 \xrightarrow{\epsilon}_{M_1} q_0$, and so M_1 does not accept w. For the induction step, consider two cases. When $w = 0u$, w has an odd number of 1s iff u has an odd number of 1s, iff (by ind. hyp.) $q_0 \xrightarrow{u}_{M_1} q_1$ iff $q_0 \xrightarrow{w=0u}_{M_1} q_1$ (since $\delta(q_0, 0) = q_0$). When $w = 1u$, w has an odd number of 1s iff u has an even number of 1s iff (by ind. hyp.) $q_1 \xrightarrow{u}_{M_1} q_1$ iff $q_0 \xrightarrow{w=1u}_{M_1} q_1$ (since $\delta(q_0, 1) = q_1$). Thus, the claim is established by induction.

   (A) The proof is correct.

   (B) The proof is incorrect because not all possible strings w have been considered in the induction step.

   (C) The proof is incorrect because there is no basis for concluding that $q_1 \xrightarrow{u}_{M_1} q_1$ by induction hypothesis in the case when $w = 1u$.

   (D) The proof is incorrect because the base case has not been proved.

   Correct answer is (C).

3. Recall the language $L_2 \subseteq \{0,1\}^*$ defined as

$$L_2 = \{u10 \mid u \in \{0,1\}^*\} \cup \{u11 \mid u \in \{0,1\}^*\}$$

   That is, L_2 contains all strings that have a 1 as the second last symbol. Consider the following proof that any DFA recognizing L_2 must have at least 4 states: Let M be a DFA with initial state q_0 recognizing L_2 having less than 4 states. Let $q_0 \xrightarrow{00}_M A$, $q_0 \xrightarrow{01}_M B$, $10 \xrightarrow{C}_{q_0}$, $q_0 \xrightarrow{11}_M D$, where

A, B, C, D are some states of M. Now, if M has less than 4 states, then two out of A, B, C, D must be the same state. If $A = B$ then M either accepts both 000 and 010 or neither; but only 010 should be accepted. If $A = C$ then M either accepts both 00 and 10 or neither; again only 10 must be accepted. If $A = D$ then M either accepts both 00 and 11 or neither; however, only 11 must be accepted. If $B = C$ then M either accepts both 01 and 10 or neither; again only 10 should be accepted. If $B = D$ then M either accepts both 01 and 11 or neither; only 11 should be accepted. Finally, if $C = D$ then M either accepts both 100 and 110 or neither; however, only 110 should be accepted. Thus, A, B, C, D must all be different, and M cannot have less than 4 states.

(A) The proof is correct.

(B) The proof is incorrect because there is a DFA with 3 states that recognizes L_2.

(C) The proof does not show that there is no DFA with less than 4 states accepting L_2. It only shows that a specific DFA M that has 5 states q_0, A, B, C, and D cannot accept L_2.

(D) The proof is incorrect because there is no basis for assuming that two out of A, B, C, D must be the same.

Correct answer is (A).

4. Recall the language $L_2 \subseteq \{0,1\}^*$ defined as

$$L_2 = \{u10 \mid u \in \{0,1\}^*\} \cup \{u11 \mid u \in \{0,1\}^*\}$$

That is, L_2 contains all strings that have a 1 as the second last symbol. Consider the following proof that any DFA recognizing L_2 must have at least 4 states: Any DFA recognizing L_2 has to remember the last two symbols of the input string. Since there are 4 strings of length 2, it must have 4 states.

(A) The proof is correct.

(B) The proof is incorrect because there is a DFA with 3 states that recognizes L_2.

(C) The proof is incorrect because there is no basis for assuming that a DFA recognizing L_2 has to remember that last two symbols of the input string.

(D) The proof is incorrect because this statement has to be proved by induction.

Correct answer is (C).