Worksheet 8 for October 20th and 22ndth

1. *Let*

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}.$$

(a) *Draw a directed graph with numbered edges and nodes, whose edge-node incidence matrix is A.*

(b) *Find a basis for the solutions to $A\mathbf{x} = 0$ in two ways: by using the matrix A, and then by using a property of the graph.*

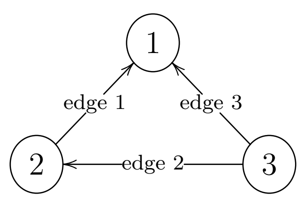
(c) *Find a basis for the solutions to $A^T\mathbf{y} = 0$ in two ways: by using the matrix A, and then by using a property of the graph.*

(d)* *Conclude from the fundamental theorem that a vector $\mathbf{b}$ is in the column space of A if and only if it satisfies $b_1 + b_2 - b_3 = 0$. What does this condition mean when the b's are potential differences?*

(e)* *Conclude from the fundamental theorem that a vector $\mathbf{f}$ is in the row space of A if and only if satisfies $f_1 + f_2 + f_3 = 0$. What does that mean when the f's are net currents into the nodes?*

($*$): *Questions (d) and (e) will not be on the Midterm.*

Solution. (a)



(b) From matrix: we transform A into the row reduced echelon form:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{R3 \to R3 - R1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R3 \to R3 - R2, R1 \to R1 + R2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So,

$$\mathrm{Nul}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = x_3, x_2 = x_3 \right\} = \left\{ \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \mathrm{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Thus, a basis for $\mathrm{Nul}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Using graph, since the graph is connected a basis for $\mathrm{Nul}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Midterm Dates: September 29th, October 22nd, November 19th (All Midterms 7-8:15 PM, see `learn.illinois.edu` **for locations)**

(c) From matrix: we transform A^T into the row reduced echelon form:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{R2 \to R2+R1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{R3 \to R3+R2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So,

$$\text{Nul}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = -x_3, x_2 = -x_3 \right\} = \left\{ \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Thus, a basis for $\text{Nul}(A)$ is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Using graph, there is only one loop: edge_1,$-\text{edge}_3$,edge_2.

So, a basis for $\text{Nul}(A^T)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

(d) $\mathbf{b}$ is in $\text{Col}(A)$ if and only if $\mathbf{b}$ is orthogonal to $\text{Nul}(A^T)$,i.e. :

$$\mathbf{b} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = b_1 + b_2 - b_3 = 0$$

This means sum of potential differences around a loop is zero, Kirchhoff's voltage law.

(e) $\mathbf{f}$ is in $\text{Col}(A^T)$ if and only if $\mathbf{f}$ is orthogonal to $\text{Nul}(A)$,i.e. :

$$\mathbf{f} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = f_1 + f_2 + f_3 = 0$$

This means sum of net currents over connect subgraphs is zero. □

2. *Consider the matrix:*

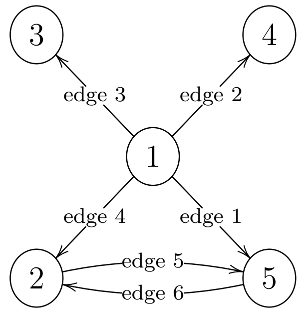
$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

(a) *Draw a directed graph with numbered edges and nodes, whose edge-node incidence matrix is A.*

(b) *Use a property of the graph to find a basis for* $\text{Nul}(A)$.

(c) *Use a property of the graph to find a basis for* $\text{Nul}(A^T)$.

Solution. (a)



(b) Since the graph is connected a basis for Nul(A) is $\left\{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right\}$.

(c) There are two (independent) loops: edge_1,edge_6,$-\text{edge}_4$ and edge_5,edge_6.

So a basis for Nul(A^T) is $\left\{\begin{bmatrix}1\\0\\0\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\1\end{bmatrix}\right\}$.

$\square$

3. *Let* $\mathbf{v} = \begin{bmatrix}1\\1\\1\end{bmatrix}$. *Find the projections* $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$ *of the vectors*

$$\mathbf{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}2\\-1\\-1\end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix}2\\2\\2\end{bmatrix}$$

onto the $\mathbf{v}$. *Interpret your results geometrically.*

Solution. We have,

$$\hat{\mathbf{a}} = \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix},$$

$$\hat{\mathbf{b}} = \frac{\mathbf{b}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\begin{bmatrix}2\\-1\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{0}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

$$\hat{\mathbf{c}} = \frac{\mathbf{c} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}. \qquad \square$$