

Math 415 - Lecture 23

Projections on subspaces

Monday October 19th 2015

Textbook reading: Chapter 3.2, 3.3, 3.4

Suggested practice exercises: Chapter 3.2 Exercise 17, 18, 24, Chapter 3.4 Exercise 2, 3 and see exercise at the end of this notes

Khan Academy video: Projections onto subspaces, Visualizing a projection onto a plane, Projection is closest vector in subspace

1 Review

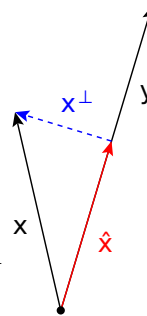
Last time

- **Orthogonal projection** of x onto y :

$$\hat{\mathbf{x}} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}} \mathbf{y}$$

“Error” $\mathbf{x}^\perp = \mathbf{x} - \hat{\mathbf{x}}$ is orthogonal to \mathbf{y} .

- If $\mathbf{y}_1, \dots, \mathbf{y}_n$ is an **orthogonal basis** of V , and \mathbf{x} is in V , then $\mathbf{x} = c_1 \mathbf{y}_1 + \dots + c_n \mathbf{y}_n$ with $c_j = \frac{\mathbf{x} \cdot \mathbf{y}_j}{\mathbf{y}_j \cdot \mathbf{y}_j}$.

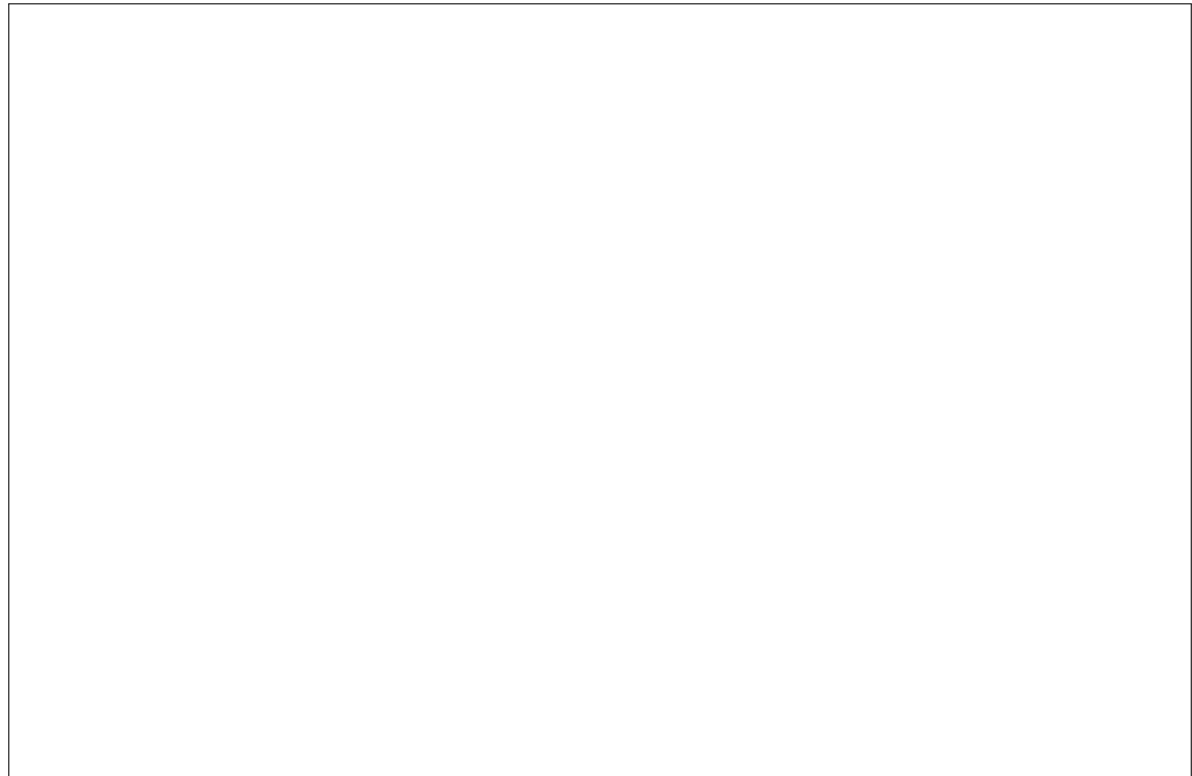


Remark. \mathbf{x} decomposes as the sum of its projections onto each vector in the orthogonal basis.

Remark. The formulas simplify when you project on *unit* vectors: all denominators are then 1.

Example 1. Express $\underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{x}}$ in terms of the basis $\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\mathbf{y}_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{y}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{y}_3}$

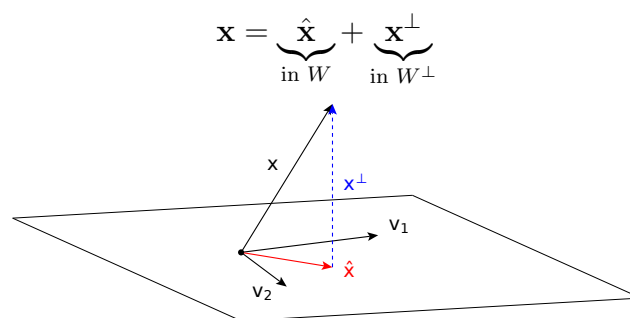
Solution.



2 Orthogonal projection on subspaces

2.1 Projecting onto a subspace

Theorem 2. *Let W be a subspace of \mathbb{R}^n . Then, each \mathbf{x} in \mathbb{R}^n can be uniquely written as*



$\hat{\mathbf{x}}$ is the **orthogonal projection** of \mathbf{x} onto W .

- $\hat{\mathbf{x}}$ is the point in W closest to \mathbf{x} . For any other \mathbf{y} in W , $\text{dist}(\mathbf{x}, \hat{\mathbf{x}}) < \text{dist}(\mathbf{x}, \mathbf{y})$.
- If $\mathbf{v}_1, \dots, \mathbf{v}_m$ is an orthogonal basis of W , then

$$\hat{\mathbf{x}} = \left(\frac{\mathbf{x} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \dots + \left(\frac{\mathbf{x} \cdot \mathbf{v}_m}{\mathbf{v}_m \cdot \mathbf{v}_m} \right) \mathbf{v}_m$$

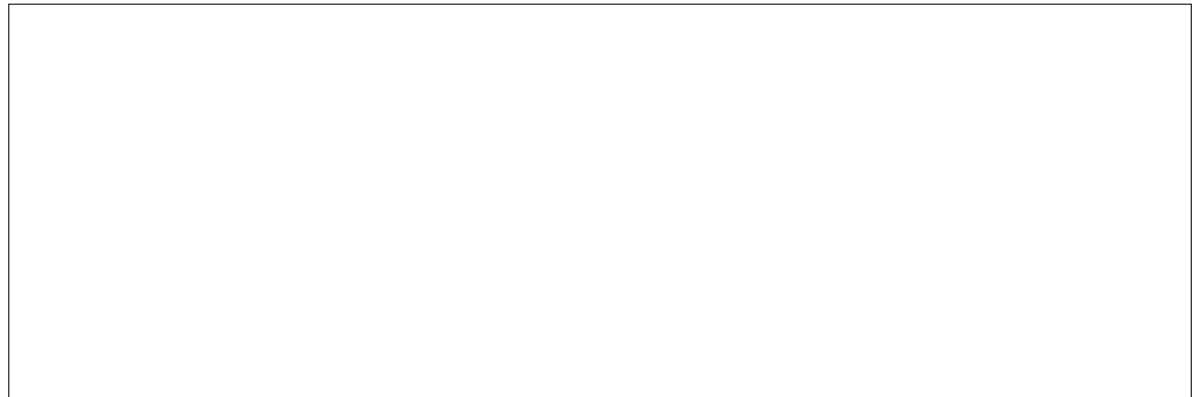
Once $\hat{\mathbf{x}}$ is determined, $\mathbf{x}^\perp = \mathbf{x} - \hat{\mathbf{x}}$.

(This is also the orthogonal projection of \mathbf{x} onto W^\perp .)

Example 3. Let $W = \text{span} \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, and $\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}$.

- Find the orthogonal projection of \mathbf{x} onto W . (Or: find the vector in W which is closest to \mathbf{x})
- Write \mathbf{x} as a vector in W plus a vector orthogonal to W .

Solution.



2.2 The matrix of a projection

Definition 4. Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be an orthogonal basis of W , a subspace of \mathbb{R}^n . Note that the projection map $\pi_W : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that sends \mathbf{x} to

$$\hat{\mathbf{x}} = \left(\frac{\mathbf{x} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \dots + \left(\frac{\mathbf{x} \cdot \mathbf{v}_m}{\mathbf{v}_m \cdot \mathbf{v}_m} \right) \mathbf{v}_m$$

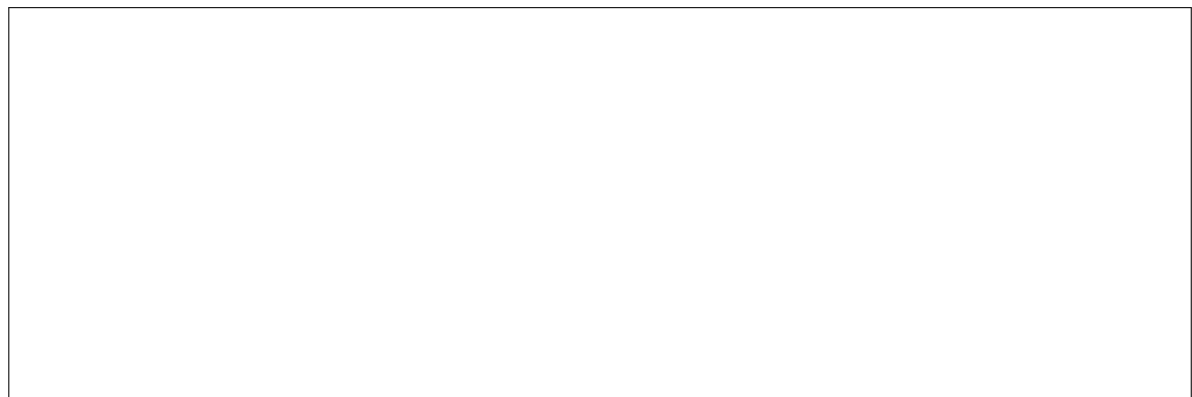
is a linear map. The matrix P representing π_W with respect to the standard basis is the **projection matrix**.

Example 5. Find the projection matrix P for the orthogonal projection onto

$$W = \text{span} \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

in \mathbb{R}^3 .

Solution.





Example 6. Let $W = \text{span} \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, and $\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}$. Find the orthogonal projection of \mathbf{x} onto W .

Solution.

Example 7. Compute P^2 when

$$P = \begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix}.$$

Solution.

3 Practice problems

3.1 Practice problems

Example 8. Find the closest point to \mathbf{x} in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ where

$$\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution.

Example 9. If P is the projection matrix for projecting on W , what is the projection matrix Q for projecting on W^\perp ?

Solution.

Example 10. Let P be the projection matrix for projecting on W , and let \mathbf{x} be some vector.

- Suppose $P\mathbf{x} = \mathbf{x}$. What can you say about \mathbf{x} ?
- Suppose $P\mathbf{x} = 0$. What can you say about \mathbf{x} ?

Solution.