Math 415 - Lecture 16

Linear Transformations

Friday October 2nd 2015

Textbook reading: Chapter 2.6

Suggested practice exercises: Chapter 2.6: 5, 6, 7, 36, 37

Khan Academy videos: Linear Transformations / Linear Transformations as Matrix Vector Products / Linear Transformation Examples: Rotations in $\mathbb{R}^2$

Strang lecture: Lecture 30: Linear Transformations

1 Review

If $\mathcal{B} = (\mathbf{b_1}, \dots, b_p)$ is a basis for a vector space V then the coordinate vector of a vector $\mathbf{w} \in V$ is the column vector

$$\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}, \quad \text{if } \mathbf{w} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \cdots + c_p\mathbf{b_p}$$

Example 1. Let $V = \mathbb{R}^2$, $\mathcal{B} = (\mathbf{b_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Solution.

Example 2. Still $V = \mathbb{R}^2$, $\mathcal{B} = (\mathbf{b_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ a basis for V. Suppose $\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ is a coordinate vector with respect to the basis $\mathcal{B}$. What is the vector $\mathbf{w}$, with respect to the standard basis?

Solution.

Remark. Translating to the standard basis is always easy. To go from the standard basis to a new basis requires solving a system of equations, so is generally harder.

2 Linear Transformations

Let V and W be vector spaces.

Definition. A map $T : V \to W$ is a linear transformation if

for all $\mathbf{x}, \mathbf{y} \in V$ and all $c, d \in \mathbb{R}$. In other words, a linear transformation respects addition and scaling.

- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- $T(c\mathbf{x}) = cT(\mathbf{x})$

It also sends the zero vector in V to the zero vector in W:

- $T(\mathbf{0}) = \mathbf{0}$ (because $T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}$)

Example 3. Let $V = \mathbb{R}, W = \mathbb{R}$. Then the map $f(x) = 3x$ is linear. Why?

Solution.

Example 4. Let A be an $m \times n$ matrix. Then the map $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. Why?

Solution.

Example 5. Let P_n be the vector space of all polynomials of degree at most n. Consider the map $T : P_n \to P_{n-1}$ given by

$$T(p(t)) = \frac{d}{dt}p(t).$$

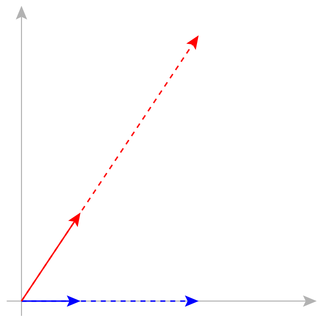
This map is linear! Why?

Solution.

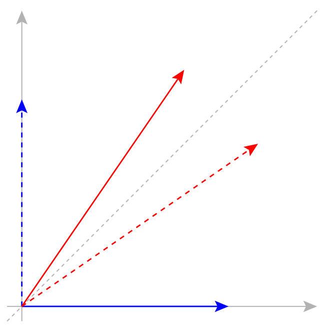
3 Important Geometric Examples

Let's consider some linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ which are defined by matrix multiplication ($\mathbf{x} \mapsto A\mathbf{x}$). In fact, it turns out that all linear maps $\mathbb{R}^n \to \mathbb{R}^m$ are given by $\mathbf{x} \mapsto A\mathbf{x}$ for some $m \times n$ matrix A.

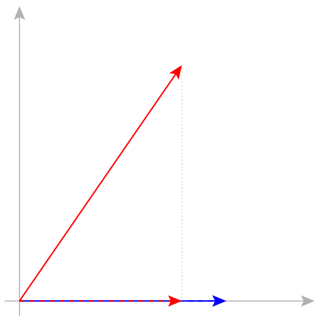
Example 6 (Stretching). The matrix $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ gives the map $x \mapsto c\mathbf{x}$, It stretches every vector in $\mathbb{R}^2$ by a factor c.



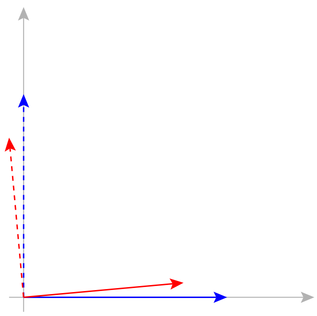
Example 7 (Reflection). The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$. It reflects every vector in $\mathbb{R}^2$ across the line $y = x$.



Example 8 (Projection.). The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$. It projects every vector in $\mathbb{R}^2$ onto the x-axis.



Example 9 (Rotation by 90°.). The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$. It rotates every vector in $\mathbb{R}^2$ counter-clockwise by 90 degrees.



4 Representing linear maps by matrices

Motto
If you know T on a basis, you know T everywhere.

- Let $\mathbf{x_1}, \ldots, \mathbf{x_n}$ be an input basis, a basis for V. A linear map $T : V \to W$ is determined by the values $T(\mathbf{x_1}), \ldots, T(\mathbf{x_n})$.

- Why?

Solution.

Example 10. Suppose $T : R^2 \to \mathbb{R}^3$ is a linear map so that

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

What is

$$T \begin{bmatrix} 1 \\ 2 \end{bmatrix}?$$

Solution.

Summary: The linear transformation

$$T\colon \mathbb{R}^2 \to \mathbb{R}^3, \quad T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\-3\end{bmatrix}$$

is the same as multiplying by the matrix

$$A = \begin{bmatrix} T\begin{bmatrix}1\\0\end{bmatrix} & T\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} & \begin{bmatrix}0\\0\\-3\end{bmatrix}\end{bmatrix}$$

We say that the linear transformation T is represented by the matrix A, or that A is the *coordinate matrix* of the linear transformation T, (with respect to the standard bases).

Example 11. Let $T_\alpha\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the "rotation over α radians (counterclockwise)" map. So $T_\alpha(\mathbf{x})$ is the vector obtained by rotating $\mathbf{x}$ over angle α. Can you find a matrix so that $T_\alpha(\mathbf{x}) = A_\alpha \mathbf{x}$?

Solution.

Theorem 1 (Linear Transformation is Matrix Multiplication, Standard basis)**.** *Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a matrix A such that*

$$T(\mathbf{x}) = A\mathbf{x}, \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n.$$

Explicitly,

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix},$$

where $e_1, e_2, \dots, e_n$ is the standard basis of $\mathbb{R}^n$.

Proof.

$\square$

Example 12. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+3b-c \\ -a+b+2c \end{bmatrix}$. What is the matrix representing T (with respect to the standard bases)?

Solution.

5 Nonstandard Bases

Untill now we have used the standard bases to describe $T\colon \mathbb{R}^n \to \mathbb{R}^m$. Often it is useful to use other bases.

Example 13. Let $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a+1b \\ 1a+3b \end{bmatrix}$. Then the matrix of T is $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. But let us use, instead of the standard basis, another basis adapted to T. Put

$$\mathbf{b_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{b_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

What is the coordinate matrix for T with respect to $\mathcal{B} = (\mathbf{b_1}, \mathbf{b_2})$?

Solution.

6 Additional Problems

- Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 7 & 8 & 1 \end{bmatrix}$. Find the dimensions and a basis for all four fundamental subspaces of A.

- Suppose A is 5×5 and v is a vector in $\mathbb{R}^5$ which is not a linear combination of the columns of A. What can you say about the number of solutions to $A\mathbf{x} = \mathbf{0}$?

- Let T be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

  What is $T\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right)$?