# Math 415 - Lecture 21

Networks and linear algebra

Wednesday October 14th 2015

**Textbook reading:** Chapter 2.5.

**Suggested practice exercises:** Chapter 2.5: 1, 2, 6.

**Strang lecture:** Lecture 12: Graphs, Networks, Incidence Matrices

## 1 Review

Recall that if $V \subset \mathbb{R}^n$ is a subspace, $V^\perp$ is the *orthogonal complement* of $V$, the subspace of all vectors $\mathbf{x}$ perp to all vectors of $V$.

**Theorem 1.** ***Fundamental Theorem of Linear Algebra.***

- $\dim(V) + \dim(V^\perp) = \dim(\mathbb{R}^n) = n$.
- $\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$.
- $\mathrm{Nul}(A)^\perp = \mathrm{Col}(A^T)$.

## 2 A new perspective on $A\mathbf{x} = \mathbf{b}$

To see if $A\mathbf{x} = \mathbf{b}$ has a solution, check that

**Direct approach:** $\mathbf{b} \in Col(A)$

**Indirect approach:** $\mathbf{b} \perp Nul(A^T)$

The indirect approach means:

$$\text{if } \underbrace{\mathbf{y}^T A = \mathbf{0}}_{\mathbf{y} \in Nul(A^T)}, \text{ then } \underbrace{\mathbf{y}^T \mathbf{b} = 0}_{\mathbf{b} \perp \mathbf{y}}.$$

*Example* 2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution?

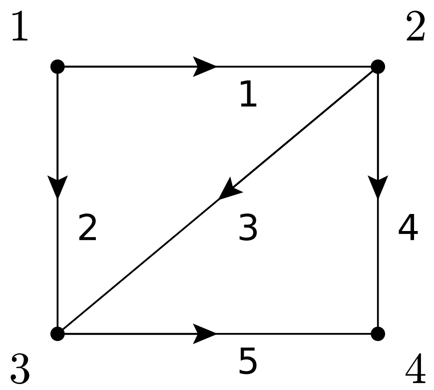
**Solution** (old)**.**

**Solution** (new)**.**

# 3 Application: Directed graphs

## 3.1 Set up

- Graphs appear in network analysis (e.g. internet) or circuit analysis.
- Arrow indicates direction of flow
- No edges from a node to itself




**Definition 3.** Let $G$ be a graph with $m$ edges and $n$ nodes. The edge-node incidence matrix of $G$ is the $m \times n$ matrix $A$ with

$$A_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j \\ +1, & \text{if edge } i \text{ enters node } j \\ 0, & \text{otherwise} \end{cases}$$

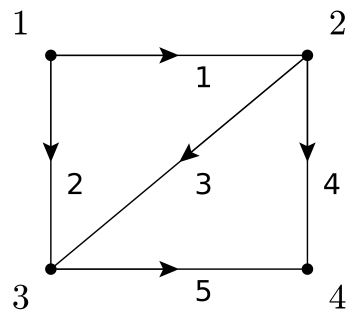
*Example* 4. Give the edge-node incidence matrix of our graph.

**Solution.**

## 3.2 Meaning of the null space

**Theorem 5.** *$dim(Nul(A))$ is the number of connected subgraphs.*

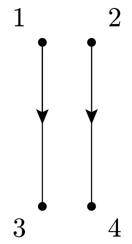
- For large graphs, disconnection may not be visually apparent
- But, we can always find out by computing $dim(Nul(A))$ using Gaussian elimination!




*Example* 6. Determine the number of connected subgraphs.

**Solution.**

$$A\mathbf{x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ -x_2 + x_3 \\ -x_2 + x_4 \\ -x_3 + x_4 \end{bmatrix}$$




*Example* 7. Give a basis for $Nul(A)$ for this graph:

**Solution.**

### 3.3 Meaning of left null space

The $\mathbf{y}$ in $\mathbf{y}^T A$ is assigning values to each edge. (Think: assigning currents to edges, so that $\mathbf{y}$ describes a *flow pattern.*)

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, A^T = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$A^T\mathbf{y} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} =$$

$$A^T\mathbf{y} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} =$$

**Idea.** So: $A^T\mathbf{y} = 0 \iff$ at each node, (directed) values assigned to edges add to zero.

When thinking of currents, this is Kirchhoff's first law: at each node, incoming and outgoing currents balance. **Flow in = Flow out.**

What is the simplest way to balance current?

Assign current in a loop! We have two loops:

$$edge_1 \to edge_3 \to -edge_2 \text{ and } edge_3 \to edge_5 \to -edge_4$$

*Example* 8. Solve $A^T\mathbf{y} = 0$ for our graph. Recall

$$A^T = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

**Solution.**

**Theorem 9.** *In general, $dim(Nul(A^T))$ is the number of (independent) loops.*

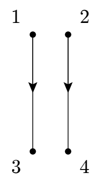
For large graphs, we now have a nice way to computationally find all loops.

# 4 Summary/Outlook

* We described a network by using a matrix $A$.
* The Null space $Nul(A)$ has as dimension the number of connected components of the network.
* The Left Null Space $Nul(A^T)$ has as dimension the number of independent loops.
* The column space $Col(A)$ and row space $Col(A^T)$ also have "geometric" meaning in terms of the network, see the book and Strang's lecture.

# 5 Practice problems

*Example* 10. Give a basis for $Nul(A^T)$ for the following graph:




**Solution.**

*Example* 11. Draw the graph with edge-node incidence matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Give a basis for $Nul(A)$ and $Nul(A^T)$.

**Solution.**