Math 415 - Lecture 35

Quadratic forms

Monday November 30th 2015

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Textbook reading: Chapter 6.2

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Suggested practice exercises: Chapter 6.2, # 1, 2, 4, 5

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Strang lecture: Lecture 27: Positive definite matrices and minima

Review

Spectral theorem:

- A is a symmetric matrix if $A^T = A$. e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 5 \end{bmatrix}$

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- Any $n \times n$ symmetric matrix A has n real eigenvalues and an orthonormal eigenbasis $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$.
- So, we can write

$$A = QDQ^T$$

where

$$D = \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\text{matrix of eigenvalues}} \quad \text{and } Q = \underbrace{\begin{bmatrix} | & & | \\ \mathbf{q}_1 & \cdots & \mathbf{q}_n \\ | & & | \end{bmatrix}}_{\text{matrix of eigenvectors}}$$

Quadratic forms

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- Look at the **quadratic** part of f!

Definition

A **quadratic form** $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial (in n variables) with every term degree two.

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e.g., for $n = 2$

$$f(x, y) = 3x^2 + 4xy - 5y^2$$

Example

Let

$$f(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Expand $f(x,y)$ as a polynomial in x and y.

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Solution

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Expanding we get

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This is the quadratic function from before!

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Theorem

Any quadratic form $f(x_1, x_2, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ *can be written*

$$f(\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x}$$

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for a symmetric matrix A.

We see symmetric matrices show up "in the wild!"

Example

Write $f(x, y, z) = 5x^2 + 7y^2 + 3z^2 + 2xy - 2yz$ as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where A is symmetric.

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A 3×3 symmetric matrix has the form $A = \begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix}$.

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So, $A = \begin{bmatrix} 5 & 1 & 0 \\ 1 & 7 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Principal axes for a quadratic form

Intermezzo: From Eigenbasis to Standard Basis and back.

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- If $x \in \mathbb{R}^n$ and $x_Q = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is the coordinate vector of x in the Q basis, then
  $x = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n = Q x_Q$.

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  $x = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n = Q x_Q$.
- This means that to find the Q coordinate vector for x, multiply by $Q^{-1} = Q^T$:

$$\boxed{x_Q = Q^T x}$$

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Then,

$$q(\mathbf{x}) := \mathbf{x}^T A \mathbf{x} = \lambda_1 (c_1)^2 + \cdots + \lambda_n (c_n)^2$$

Proof.

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D is the matrix of eigenvalues. So,

$$D\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \lambda_1 c_1 \\ \vdots \\ \lambda_n c_n \end{bmatrix}$$

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$$\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} c_1 & \ldots & c_n \end{bmatrix} \begin{bmatrix} \lambda_1 c_1 \\ \ldots \\ \lambda_n c_n \end{bmatrix} = \lambda_1 (c_1)^2 + \cdots + \lambda_n (c_n)^2$$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Let $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$.

- Find the eigenvalues λ_1, λ_2 and **orthonormal** eigenbasis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for A.
- Compute $q(\mathbf{x})$ using the formula $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$.
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Are the answers the same? **This is a silly Example. To calculate $q(x)$ you never would go through the eigenvalues.**

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Product $\lambda_1\lambda_2 = \det(A)$

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So, $\lambda_1 = 3$, $\lambda_2 = -1$.

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Eigenbasis: $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Compute using formula:

$$\begin{aligned} q(\mathbf{x}) &= \mathbf{x}^T A \mathbf{x} \\ &= \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \\ &= 4 \end{aligned}$$

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Using theorem: $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \sqrt{2}\mathbf{v}_1 + \sqrt{2}\mathbf{v}_2$. So,

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Using theorem: $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \sqrt{2}\mathbf{v}_1 + \sqrt{2}\mathbf{v}_2$. So,

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Get same answer!

We have $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ and

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Principal axes for a quadratic form

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3. *If some $\lambda_i > 0$, some $\lambda_j < 0$, $\mathbf{x}^T A\mathbf{x}$ will have both positive and negative values.*

Completing the squares

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