Math 415 - Lecture 32

Complex numbers and eigenvectors

Wednesday November 11th 2015

Textbook reading: first part of Chapter 5.5

Suggested practice exercises: 5.5 1, 2, 3

Khan Academy video: Complex Numbers (part 1)

Strang lecture: Lecture 21: Eigenvalues and eigenvectors

SOME DATES.

* Friday November 13th: No class.
* Next week there will be discussion sections, but no quiz. Prepare for the midterm!
* Thursday November 19th, 7-8:15PM: Midterm 3.
* Friday November 20th: No class.
* November 23-27th Thanksgiving break: no class.
* Wednesday December 9th: last day of class
* Thursday December 17th: Final Exam.

1 Review

1.1 Properties of eigenvectors and eigenvalues

- If $A\mathbf{x} = \lambda\mathbf{x}$ then $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ.
- λ is an eigenvalue of $A \iff \underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial}} = 0.$

Definition 1. An $n \times n$ matrix A is a **Markov matrix** if has non negative entries, and the entries in each column add to 1.

Theorem 1. *Let A be a Markov matrix. Then*

(i) 1 is an eigenvalue of A and any other eigenvalue λ satisfies $|\lambda| \leq 1$.

(ii) If A has only positive entries, then any other eigenvalue satisfies $|\lambda| < 1$.

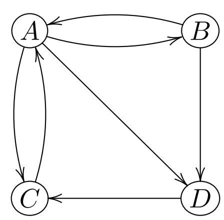
Theorem 2. *Let A be an $n \times n$-Markov matrix with only positive entries and let $\boldsymbol{v} \in \mathbb{R}^n$. Then*

$$\boldsymbol{v}_\infty := \lim_{k\to\infty} A^k \boldsymbol{v} \text{ exists,}$$

*and $A\boldsymbol{v}_\infty = \boldsymbol{v}_\infty$. In this case $\boldsymbol{v}_\infty$ is often called the **steady state**.*

2 Page rank (or: the 25000000000 $ eigenvector)

Google's success is based on an algorithm to rank webpages, the **Page rank**, named after Google founder Larry Page. The idea is to determine how likely it is that a web user randomly gets to a given webpage. The webpages are ranked by these probabilities.
Suppose the internet consisted of the only four webpages A, B, C, D linked as in the following graph.



Imagine a surfer following these links at random. For the probability $PR_n(A)$ that she is at A (after n steps), we need to think about how she could have reached A. We add:

- the probability that she was at B (at exactly one step before), and left for A,(that's $PR_{n-1}(B) \cdot \frac{1}{2}$)
- the probability that she was at C, and left for A,
- the probability that she was at D, and left for A.

Hence: $PR_n(A) = PR_{n-1}(B) \cdot \frac{1}{2} + PR_{n-1}(C) \cdot \frac{1}{1} + PR_{n-1}(D) \cdot \frac{0}{1}$.

Encode the probabilties at step n in a state vector with four entries. $\begin{bmatrix} PR_n(A) \\ PR_n(B) \\ PR_n(C) \\ PR_n(D) \end{bmatrix} =$

$$\begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} PR_{n-1}(A) \\ PR_{n-1}(B) \\ PR_{n-1}(C) \\ PR_{n-1}(D) \end{bmatrix}$$

Definition 2. The **PageRank vector** is the long-term equilibrium. It is an eigenvector of the Markov matrix with eigenvalue 1.

Let's call the Markov matrix with the probabilities T:

- $T - 1I = \begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

  $\implies$ eigenspace of $\lambda = 1$ is spanned by $\begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}$.

- Now we need to make the entries add up to 1.

$$\begin{bmatrix} PR(A) \\ PR(B) \\ PR(C) \\ PR(D) \end{bmatrix} = \frac{3}{16} \begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix}.$$

  This is the PageRank vector.

- The corresponding ranking of the webpages is A, C, D, B.

Remark. In practical situations the system might be too large for finding the eigenvalues by row operations.

- Google reports having met 60 trillion webpages. Google's search index is over $100,000,000$ gigabytes. Number of Google's servers is secret: about $2,500,000$ More than 1,000,000,000 websites (i.e. hostnames; about 75% not active)

- Thus we have a gigantic but very sparse matrix.

An alternative to row operations is the **power method** (see Theorem 2):

Power method
If T is an (acyclic and irreducible) Markov matrix, then for any $\mathbf{v}_0$ the vectors $T^n \mathbf{v}_0$ converge to an eigenvector with eigenvalue 1

Here:

$$T = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Start with an arbitrary state vector, hit it with powers of T.

$$\left(\begin{bmatrix} PR(A) \\ PR(B) \\ PR(C) \\ PR(D) \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix} \right), T \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.083 \\ 0.333 \\ 0.208 \end{bmatrix}$$

. Note that the ranking of the webpages is already A, C, D, B if we stop here.

$$T \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.083 \\ 0.333 \\ 0.208 \end{bmatrix}, \quad T^2 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.333 \\ 0.167 \end{bmatrix}, \quad T^3 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.396 \\ 0.125 \\ 0.292 \\ 0.188 \end{bmatrix}$$

Remark.

- If all entries of T are positive (no zero entries!), then the power method is guaranteed to work.
- In the context of PageRank, we can make sure that this is the case by replacing T with

$$(1-p) \cdot \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} + p \cdot \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

  Just to make sure: still a Markov matrix, now with positive entries Google used to use $p = 0.15$.

3 Eigenbasis?

3.1 Number of (independent) eigenvectors

An $n \times n$ matrix A has up to n different eigenvalues. Namely, the roots of degree n characteristic polynomial $\det(A - \lambda I)$.

- For each eigenvalue λ, A has at least one eigenvector. That is because $\mathrm{Nul}(A - \lambda I)$ has dimension at least one.
- If λ has multiplicity m, then A has up to m (independent) eigenvectors for λ.

Ideally, we would like to find a total of n (independent) eigenvectors for A. This would give an **EIGENBASIS**. Why can there be no more than n independent eigenvectors?!

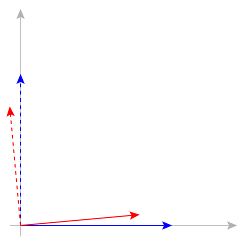
Two sources of trouble: eigenvalues can be

- complex numbers (that is, not enough real roots), or
- repeated roots of the characteristic polynomial.

3.2 Trouble I: complex eigenvalues

Example 3. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Geometrically, what is the trouble?

Solution.



$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$

i.e. multiplication by A is a rotation by $90°$ (counter-clockwise).

Which vector is parallel after rotation by $90°$? Trouble.

4 Complex numbers review

Definition. $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

- $i = \sqrt{-1}$, or $i^2 = -1$.
- Any point in $\mathbb{R}^2$ can be viewed as a complex number:
  $\binom{x}{y} \leftrightarrow x + iy$

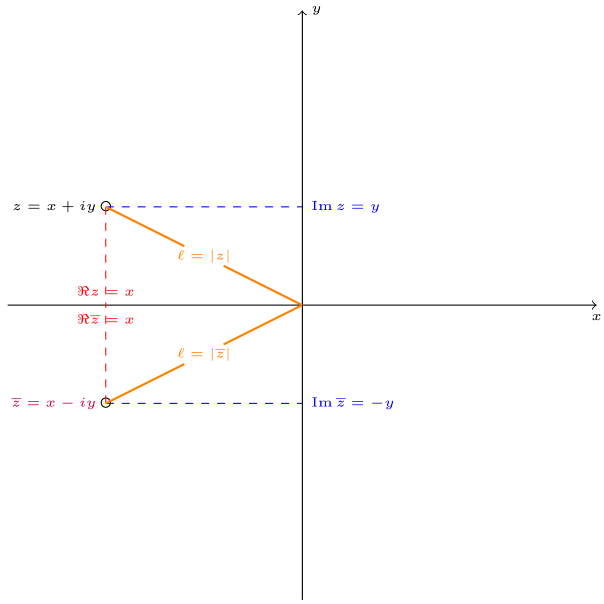
Definition. Let $z = x + iy$ be a complex number

Real part The real part of z, denoted $\Re(z)$ is defined by $\Re(z) = x$.

Imaginary part The imaginary part of z, denoted $\operatorname{Im}(z)$ is defined by $\operatorname{Im}(z) = y$.

Complex conjugate The complex conjugate of z, denoted $\overline{z}$, is defined by $\overline{z} = x - iy$.

Absolute value The absolute value, or magnitude of z, denoted $|z|$ or $\|z\|$, is given by $|z| = \sqrt{x^2 + y^2}$.



Adding complex numbers

Definition. Given $z = x + iy$, $w = u + iv$, we define

$$z + w = (x + u) + i(y + v)$$

Remark. This corresponds exactly to addition of vectors in $\mathbb{R}^2$.

Multiplying complex numbers

Definition. Given $z = x + iy$, $w = u + iv$, we define

$$\begin{aligned} zw &= (x + iy)(u + iv) \\ &= xu + x(iv) + (iy)u + (iy)(iv) \\ &= (xu - yv) + i(xv + yu) \end{aligned}$$

Absolute value and complex conjugate

Remark.

- $\overline{\overline{z}} = z$
- $|z|^2 = z\overline{z}$
- $|z| = |\overline{z}|$

Proof.

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - x(iy) + (iy)x - (iy)(iy) \\ &= x^2 + y^2 \end{aligned}$$

□

4.1 Complex Linear Algebra

Until now we took as our scalars the real numbers. In particular we used the vector space $\mathbb{R}^n$ of column vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

If c is a real number (a scalar) we defined

$$c\mathbf{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}.$$

Now we want to use **COMPLEX** scalars. We need a new context to make sense of this.

Definition. $\mathbb{C}^n$ is the (complex) vector space of *complex* column vectors $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$, where $z_1, z_2, \ldots . z_n$ are complex numbers.

- Now multiplication by a complex scalar makes sense.
- We can define subspaces, Span, independence, basis, dimension for $\mathbb{C}^n$ in the usual way.
- We can multiply complex vectors by complex matrices. Column space and Null space still make sense.
- The only difference is the dot product, you need to use the complex conjugate to get a good notion of length. (Later more.)

5 Back to eigenvectors

Example 4. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Now, we can use complex numbers!

Solution (continued).

- $\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$ So the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$.

- $\lambda_1 = i$: $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \mathbf{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ Let us check $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix}$

- $\lambda_2 = -i$: $\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \mathbf{x}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Summary: We had $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

- Eigenvalues: $i, -i$ These are conjugates!
- Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ These are also conjugates!

Theorem 3. *If A is a matrix with real entries and λ is a **complex eigenvalue**, then $\bar{\lambda}$ is also a complex eigenvalue. Furthermore, if $\mathbf{x}$ is an eigenvector with eigenvalue λ, then $\bar{\mathbf{x}}$ is an eigenvector with eigenvalue $\bar{\lambda}$.*

Remark. Note that we are using vectors in $\mathbb{C}^2$, instead of vectors in $\mathbb{R}^2$. Works pretty much the same!

5.1 Trouble II: generalized eigenvectors

Example 5. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. What is the trouble?

Solution.

- $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$ So: $\lambda = 1$ is the only eigenvalue (it has multiplicity 2).
- $\lambda = 1$: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ So the eigenspace is Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Only dimension 1!
- Trouble: We can not find an **Eigenbasis** for this matrix. This kind of problem cannot really be fixed. We have to lower our expectations and look for generalized eigenvectors. These are solutions to $(A - \lambda I)^2 \mathbf{x} = \mathbf{0}, (A - \lambda I)^3 \mathbf{x} = \mathbf{0}, \ldots$

6 Practice problems

Example 6. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.