# Worksheet 5 (September 22nd and 24th)

**1.** *Determine which of the following sets are subspaces of the indicated vector spaces and* **give reasons**. *For any sets that are subspaces, find a matrix $A$ such that $W_i = Null(A)$ or $W_i = Col(A)$.*

(a) $W_1 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 2b = c, 4a + 2c = 0 \right\} \subseteq \mathbb{R}^3$,

(b) $W_2 = \left\{ \begin{bmatrix} a-b \\ c \\ a+c \\ a-2b-c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \subseteq \mathbb{R}^4$,

(c) $W_3 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a \cdot b \geq 0 \right\} \subseteq \mathbb{R}^2$.

(d) $W_4 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a^2 + b^2 \leq 1 \right\} \subseteq \mathbb{R}^2$.

*Also sketch the sets $W_3$ and $W_4$ and give geometric reasons why $W_3$ and $W_4$ either are or are not subspaces of $\mathbb{R}^2$.*

**2.** *Is $H = \left\{ \begin{bmatrix} a+1 \\ a \end{bmatrix} : a \text{ in } \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$? Why or why not?*

*Is $K = \left\{ \begin{bmatrix} a+1 \\ b \end{bmatrix} : a \text{ and } b \text{ in } \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$? Why or why not?*

**3.** *Are the following subspaces of $M_{2\times 2}$, the set of all $2 \times 2$ matrices?*

(a) *$S$, the set of all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that $ad - bc = 0$*

(b) *$V$, the set of all $2 \times 2$ matrices such that $B^T = B$*

**4.** *For the $3 \times 5$ matrix*

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

*determine the nullspace, $\mathrm{Nul}(A)$, of $A$. Write your answer as the span of a set of vectors.*

**5.** *Consider the $2 \times 2$ matrix*

$$A = \begin{bmatrix} 2 & -6 \\ -5 & 15 \end{bmatrix}.$$

*Determine which of the following vectors*

$$\begin{bmatrix} -6 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 15 \end{bmatrix}, \begin{bmatrix} 4 \\ -10 \end{bmatrix}$$

*belong to the column space, $\mathrm{Col}(A)$, of $A$. Also, find $B$ such that $\mathrm{Col}(A) = \mathrm{Nul}(B)$.*

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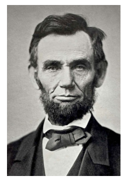
**Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM**

**Midterm Dates: September 29th, October 22nd, November 19th (All Midterms 7-8:15 PM, see** `learn.illinois.edu` **for locations)**

**6.** *Let us consider the vector space $M_{m\times n}$ of $m \times n$-matrices. We can think of a* ***grayscale picture*** *consisting of $m \times n$ many pixels as a matrix $(a_{ij})_{1\le i\le m, 1\le j\le n}$ in this vector space where all the $a_{ij}$'s are between 0 and 1. Here the entry $a_{ij}$ of a grayscale picture represents the grayscale value of the pixel at position $i, j$ in this picture. So the value 1 means the pixel is white and 0 means the pixel is black. Many operations on this vector space correspond to functions your favorite image manipulation software can carry out. For example, let $P_1$ and $P_2$ be two grayscale pictures. Then taking the linear combination $\frac{1}{2}P_1 + \frac{1}{2}P_2$ is the same as blending the two pictures together. For example,*

$0.5\cdot$  $+0.5\cdot$  $=$  .

*In this exercise we will look at a few other operations.*

(a) *Which vector space operation of $M_{m\times n}$ corresponds to changing the brightness of a grayscale picture?*

(b) *Let $P$ be a grayscale picture and let $B$ be the $m \times n$-matrix all whose entries are 1. Calculating $B - P$ correspond to which function of your favorite image manipulation software?*

(c) *Let suppose that $m = n$. Let $P$ be a grayscale picture and let $C$ be the $m \times m$-matrix of the form*

$$\begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & 0 & 1 & 0 \\ 0 & \dots & 0 & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}.$$

*What happens to the picture $P$ if you multiply it with $C$ from left (ie. calculate $CP$)? What happens to $P$ if you multiply it with $C$ from right?*

## The following may be useful in the above problems:

**Definition.** A **subspace** of a vector space $V$ is a subset $H$ of $V$ that has three properties:

(1) The zero vector $V$ is in $H$.
(2) For each $\mathbf{u}$ and $\mathbf{v}$ in $H$, $\mathbf{u}+\mathbf{v}$ is in $H$. (In this case, we say $H$ is **closed under vector addition**.)
(3) For each $\mathbf{u}$ in $H$ and each scalar $c \in \mathbb{R}$, $c\mathbf{u}$ is in $H$. (In this case, we say $H$ is **closed under scalar multiplication**.)

**Theorem.** If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space $V$, the the **subset** $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of $V$, is also a **subspace** of $V$.

**Theorem** (Zero Test)**.** If $H$ is a **subset** of the vector space $V$, and the zero vector $\mathbf{0}$ is **not** in $H$, then $H$ is **not** a subspace of $V$. (Caution: The converse of the zero test is not always true!)

**Definition.** The **nullspace** of an $m \times n$ matrix $A$, written $\text{Nul}(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In other words,

$$\text{Nul}(A) = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

**Definition.** The **column space** of an $m \times n$ matrix $A$, written $\text{Col}(A)$, is the set of all linear columns of $A$. In other words, if $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$, then

$$\text{Col}(A) = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}.$$