Math 415 - Lecture 9

Vector spaces and subspaces

Monday September 14th 2015

Textbook: Chapter 2.1.

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Suggested practice exercises: Chapter 2.1: 1, 2, 10, 11, 17, 18.

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Khan Academy video: Linear Subspaces

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- Then perform row operations on this matrix (which will produce identical operations on A and I).
- So by the Theorem:

$$[\ A \mid I\ ] \text{ will row reduce to } \left[\ I \mid A^{-1}\ \right]$$

or A is not invertible.

Example

Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

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Check at home that $AA^{-1} = I_3$.

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One more example

Example

Use the Gauss Jordan method to compute the inverse of

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

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Failure: the reduced row echelon form of A will not be I, so A has no inverse!

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Vector Spaces and Subspaces

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- The most important property of column vectors in $\mathbb{R}^n$ is that you can take *linear combinations* of them.
- There are many mathematical objects $X, Y, \ldots$ for which a linear combination $cX + dY$ make sense, and have the usual properties of linear combination in $\mathbb{R}^n$
- We are going to define a *vector space* in general as a collection of objects for which linear combinations make sense. The objects of such a set are called vectors.

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Addition:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} =$$

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Let $M_{2x2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$. This is a vector space.

We need to say what the two operations are.

Addition:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}.$$

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Then we need to check all the 10 axioms. They follow from the corresponding properties of ordinary numbers.

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- a "vector" $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ behaves very much like a column vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. A fancy person would say that the vector spaces $M_{2\times 2}$ and $\mathbb{R}^4$ are *isomorphic*.

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Let $\mathbf{p}(t) = a_0 + a_1 t + \cdots + a_n t^n$ and $\mathbf{q}(t) = b_0 + b_1 t + \cdots + b_n t^n$ and let c be a scalar.

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Therefore,

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which is also a polynomial of degree at most n. So $\mathbf{p} + \mathbf{q}$ is in $\mathbf{P}_n$ (i.e. $\mathbf{P}_n$ is closed under addition). This verifies Axiom 1.

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which is in $\mathbf{P}_n$. so that Axiom 6 holds.

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The other 7 axioms also hold, so $\mathbf{P}_n$ is a vector space.

Subspaces

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Note that if the subset H satisfies these three properties, then H itself is a vector space.

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Example

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- $\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+0 \end{bmatrix}$ is in Z.
- $c \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c0 \\ c0 \end{bmatrix}$ is in Z.

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Z is called the zero subspace of $\mathbb{R}^2$.

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Z is called the zero subspace of $\mathbb{R}^2$. Every vectorspace has a zero subspace consisting just of the zero vector.

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Check:

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- $\begin{bmatrix}a\\a\end{bmatrix} + \begin{bmatrix}b\\b\end{bmatrix} = \begin{bmatrix}a+b\\a+b\end{bmatrix}$ is in H.
- $c\begin{bmatrix}a\\a\end{bmatrix} = \begin{bmatrix}ca\\ca\end{bmatrix}$ is in H.

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Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Show that H is a subspace of $\mathbb{R}^3$.

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Verify properties $1, 2$, and 3 of the definition of a subspace.

- The zero vector of $\mathbb{R}^3$ is in H.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H, \quad (a = b = 0)$$

Subspaces

- Adding two vectors in H always produces another vector whose second entry is 0 and therefore the sum of two sectors in H is also in H. (H is closed under addition.)

$$\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} c \\ 0 \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ 0 \\ b+d \end{bmatrix}.$$

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$$\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} c \\ 0 \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ 0 \\ b+d \end{bmatrix}.$$

- Multiplying a vector in H by a scalar produces another vector in H. (H is closed under scalar multiplication.)

$$c \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} ca \\ 0 \\ cb \end{bmatrix}.$$

Subspaces

Since those three properties hold, H is a subspace of $\mathbb{R}^3$.

Remark

Vectors $(a, 0, b)$ look and act like the points (a, b) in $\mathbb{R}^2$.

But they are **not** the same!

Subspaces

Example

Is $H = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$?

(i.e. does H satisfy the properties of a subspace?)

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H does not contain the zero vector (property 1).

$$\begin{bmatrix} x \\ x+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

cannot be true for any value of x.

Therefore, H is **not** a subspace!

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Another way to show that H is not a subspace of $\mathbb{R}^2$ is to check whether H is closed under addition (property 2).

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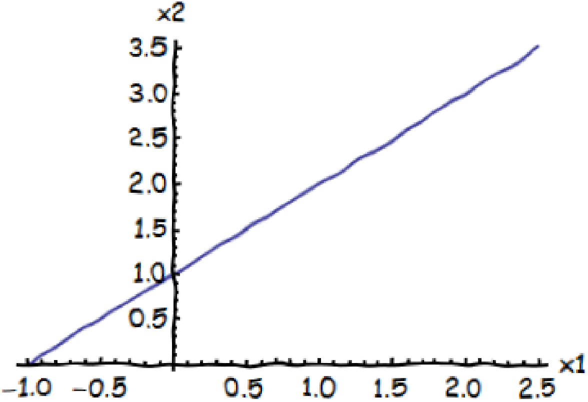
Another way to show that H is not a subspace of $\mathbb{R}^2$ is to check whether H is closed under addition (property 2).

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in H$$

but

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \notin H.$$


x2
3.5
3.0
2.5
2.0
1.5
1.0
0.5
x1
−1.0
−0.5
0.5
1.0
1.5
2.0
2.5

Problem

Find as many subspaces in $\mathbb{R}^2$ as you can.

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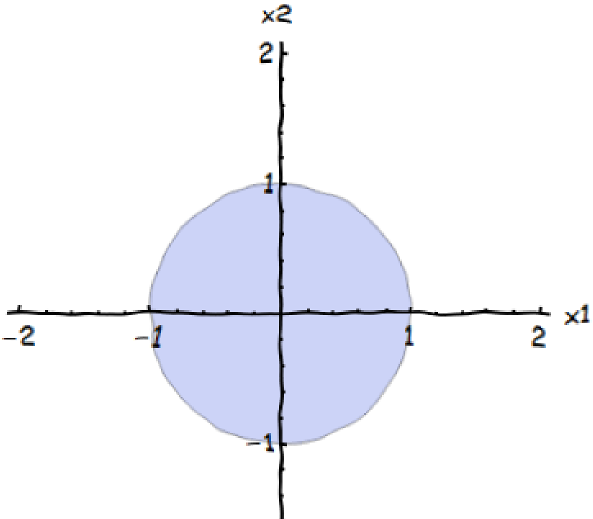
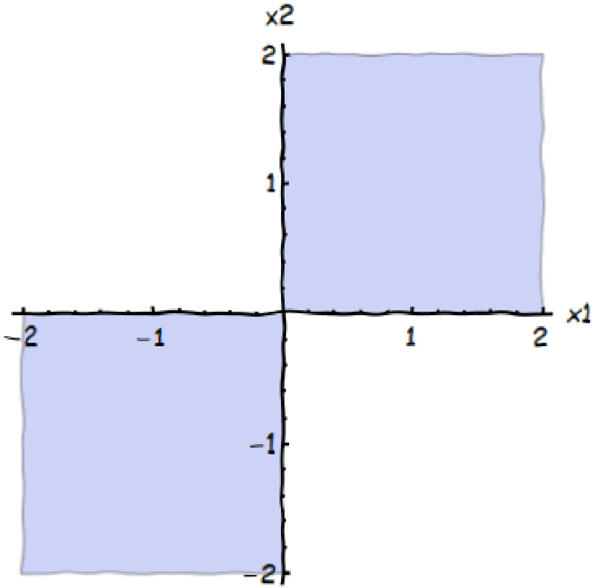
Find as many subspaces in $\mathbb{R}^2$ *as you can.*

Think of this at home.

Example

Is one of the following a subspace of $\mathbb{R}^2$?




x2
2
1
-1
-2
x1
-2
-1
1
2

Example

Is this a subspace of $\mathbb{R}^3$?