Math 415 - Lecture 12

Linear independence

Monday September 21st 2015

Textbook reading: Section 2.3

Suggested practice exercises: Section 2.3: 1, 2, 3, 4, 5,7, 8, 9

Khan Academy video: Introduction to Linear Independence, More on linear independence, Span and Linear Independence Example,

Strang lecture: Independence, Basis, and Dimension

1 Linear independence

Review.

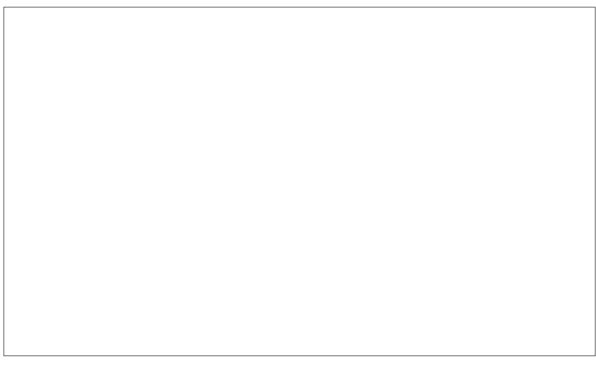
 $\bullet \ \operatorname{Span} \left\{ v_1, v_2, \ldots, v_m \right\}$ is the set of all linear combinations

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \dots + c_m\mathbf{v_m}.$$

- $\bullet \ \operatorname{Span} \left\{ v_1, v_2, \ldots, v_m \right\}$ is a vector space.
- $\operatorname{Col}(A) = \operatorname{Span}(\mathbf{a_1}, \mathbf{a_2}, \dots, \mathbf{a_n})$, if $A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \dots & \mathbf{a_n} \end{bmatrix}$. In this case $\mathbf{b} \in \operatorname{Col}(A) \iff \mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$.

Today we want to think how big the span of a bunch of vectors is. Is it a line, or a plane or

Example 1. Is Span $\left\{ \begin{bmatrix} 1\\1 \end{bmatrix}, \begin{bmatrix} 2\\2 \end{bmatrix} \right\}$ equal to \mathbb{R}^2 ?
Solution.
Example 2. Is Span $\left\{ \begin{bmatrix} 1\\1\\1 \end{bmatrix}, \begin{bmatrix} 1\\2\\3 \end{bmatrix}, \begin{bmatrix} -1\\1\\3 \end{bmatrix} \right\}$ equal to \mathbb{R}^3 ? Solution.

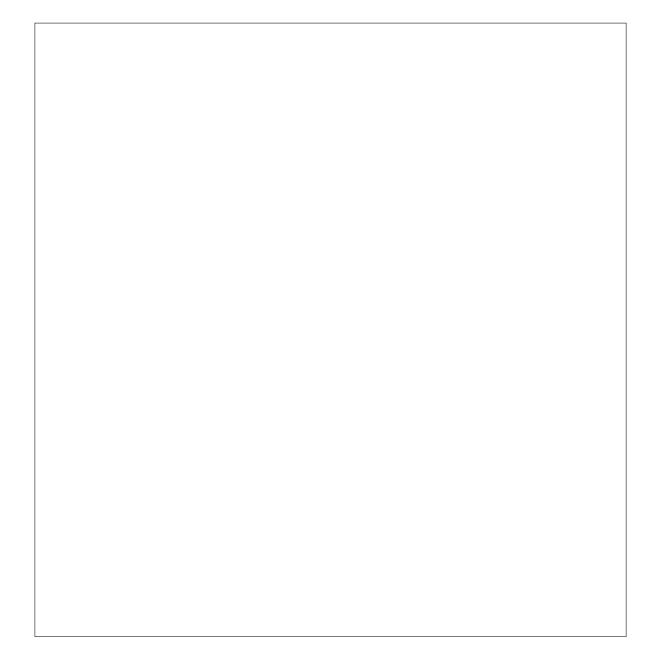


• What went wrong? Well, the three vectors that span satisfy a *relation*:

- Hence, Span $\left\{ \begin{bmatrix} 1\\1\\1 \end{bmatrix}, \begin{bmatrix} 1\\2\\3 \end{bmatrix}, \begin{bmatrix} -1\\1\\3 \end{bmatrix} \right\} = \operatorname{Span} \left\{ \begin{bmatrix} 1\\1\\1 \end{bmatrix}, \begin{bmatrix} 1\\2\\3 \end{bmatrix} \right\}.$
- We are going to say that the three vectors are **linearly dependent** because they satisfy the (non trivial) relation

$$-3\begin{bmatrix}1\\1\\1\end{bmatrix}+2\begin{bmatrix}1\\2\\3\end{bmatrix}-\begin{bmatrix}-1\\1\\3\end{bmatrix}=\mathbf{0}.$$

Definition. Vectors $\mathbf{v_1}, \dots, \mathbf{v_p}$ are said to be linearly independent if the equation
Likewise, $\mathbf{v_1}, \dots, \mathbf{v_p}$ are said to be linearly dependent
This is called a non trivial relation (when not all coefficient are zero.)
$\lceil 1 \rceil \lceil 1 \rceil \lceil -1 \rceil$
Example 3. • Are the vectors $\begin{bmatrix} 1\\1\\1 \end{bmatrix}$, $\begin{bmatrix} 1\\2\\3 \end{bmatrix}$, $\begin{bmatrix} -1\\1\\3 \end{bmatrix}$ independent?
• If possible, find a linear dependence relation among them.
Solution.



2 Linear independence of matrix columns

• Note that a linear dependence relation, such as

$$3\begin{bmatrix}1\\1\\1\end{bmatrix} - 2\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}-1\\1\\3\end{bmatrix} = \mathbf{0},$$

can be written in matrix form as

• Hence, each linear dependence relation among the columns of a matrix A corresponds to a solution to $A\mathbf{x} = 0$. The Null space determines (in)dependence!
Theorem 1. Let A be an $m \times n$ matrix. The columns of A are linearly independent. $\iff A\mathbf{x} = 0$ has only the solution $\mathbf{x} = 0$. $\iff Nul(A) = \{0\}$ $\iff A$ has n pivots. \iff there are no free variables for $A\mathbf{x} = 0$.
Example 4. Are the vectors $\begin{bmatrix} 1\\1\\1 \end{bmatrix}$, $\begin{bmatrix} 1\\2\\3 \end{bmatrix}$, $\begin{bmatrix} -1\\2\\3 \end{bmatrix}$ independent?
Solution.

Example 5. Are the vectors $\begin{bmatrix} 1\\1\\1 \end{bmatrix}$, $\begin{bmatrix} 1\\2\\3 \end{bmatrix}$, $\begin{bmatrix} -1\\1\\3 \end{bmatrix}$ independent? Solution. 3 Special cases \bullet A set of a single non-zero vector $\{\mathbf{v_1}\}$ is always linearly independent. Why? \bullet A set of two vectors $\{{\bf v_1},{\bf v_2}\}$ is linearly independent if and only if neither of the vectors is a multiple of the other.

Why?

	vectors {		j comean.					
/hy?								
	contains m				entries in	each veo	ctor, the	n the set
An	\mathbf{v}_1 set $\{\mathbf{v_1},$	$\ldots, \mathbf{v_p} \}$ (of vectors	$s in \mathbb{R}^n is$	linearly	depende	nt if $p >$	$\cdot n$.
/hy?								

Example 6. Let $A = \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$ be a two by three matrix. We want to count the free variables for $A\mathbf{x} = \mathbf{0}$. How many pivots can there be? How many free variables? Are the columns of A independent?

Solution.

4 Additional exercises

With the least amount of work possible, decide which of the following sets of vectors are linearly independent.

(a)
$$\left\{ \begin{bmatrix} 3\\2\\1 \end{bmatrix}, \begin{bmatrix} 9\\6\\4 \end{bmatrix} \right\}$$

Linearly independent, because the two vectors are not multiples of each other.

(b)
$$\left\{ \begin{bmatrix} 3\\2\\1 \end{bmatrix} \right\}$$

Linearly independent, because it is a single non-zero vector.

(c) Columns of
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \end{bmatrix}$$

Linearly dependent, because these are more than 3 (namely, 4) vectors in \mathbb{R}^3 .

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(d)
$$\left\{ \begin{bmatrix} 3\\2\\1 \end{bmatrix}, \begin{bmatrix} 9\\6\\4 \end{bmatrix}, \begin{bmatrix} 0\\0\\0 \end{bmatrix} \right\}$$

Linearly dependent, because the set includes the zero vector.