# MATH 415 – Lecture 13

18 February 2015

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    (i.e. if $\mathbf{u} \in W$ and $c \in \mathbb{R}$, then $c\mathbf{u} \in W$)
- $span\{\mathbf{v_1}, \ldots, \mathbf{v_m}\}$ is always a subspace of $V$.
  ($\mathbf{v_1}, \ldots, \mathbf{v_m}$ are vectors in $V$)

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**Example**

Is $W = \left\{ \begin{bmatrix} 2a - b & 0 \\ b & 3 \end{bmatrix} : a, b \in \mathbb{R} \right\}$ a subspace of $M_{2\times 2}$, the space of $2 \times 2$ matrices?

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**Solution**. No. $W$ does not contain the zero matrix.

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**Solution**. Write "vectors" in $W$ in the form

$$\begin{bmatrix} 2a-b & 0 \\ b & 3a \end{bmatrix} = a \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + b \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$$

to see that

$$W = span \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

Like any span, $W$ is a vector space.

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Yes. $W_2 = span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \right\}.$

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Hence, $W_2$ is a subspace of the vector space $M_{2\times 2}$ of all $2 \times 2$ matrices.

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Yes. If $p'(2) = 0$ and $q'(2) = 0$, then
$(p+q)'(2) = p'(2) + q'(2) = 0$. Likewise for scaling.

Hence, $W_5$ is a subspace of the vector space of all polynomials.

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We still have

$$W_3 = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix} + span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \right\}.$$

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Hence, $W_3$ is a subspace if and only if $\begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$ is in the span. Equivalently, we have to check whether

$$\begin{bmatrix} a+c & -2b \\ b+3c & c+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has solutions $a, b, c$. There is no solution.

What we learned before vector spaces

# Linear systems

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Sometimes, we represent the system by its augmented matrix.

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

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- infinitely many solutions.
  $\iff$ system is consistent and has at least one free variable

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- Gaussian elimination on $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$
- LU decomposition $A = LU$
- using matrix inverse, $\mathbf{x} = A^{-1}\mathbf{b}$

# Matrices and vectors

- A **linear combination** of $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ is of the form

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- *span* $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is the set of all such linear combinations
  - Spans are always vector spaces
  - For instance, a span in $\mathbb{R}^3$ can be $\{\mathbf{0}\}$, a line, a plane, or $\mathbb{R}^3$.

# Matrices and vectors

- The **transpose** $A^T$ of a matrix $A$ has rows and columns flipped.

$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \\ -1 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

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- An $m \times n$ **matrix** $A$ has $m$ rows and $n$ columns.
- The product $A\mathbf{x}$ of a matrix times a vector is

$$\begin{bmatrix} | & | & & | \\ \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n} \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n}$$

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  - row-column rule

$$(AB)_{i,j} = (row\ \ i\ \ of\ \ A) \cdot (col\ j\ of\ \ B)$$

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  - $(A^T)^{-1} = (A^{-1})^T$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - An $n \times n$ matrix $A$ is invertible
    $\iff$ $A$ has $n$ pivots
    $\iff$ $A\mathbf{x} = \mathbf{b}$ has a unique solution (if true for one $\mathbf{b}$, then true for all $\mathbf{b}$)

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  - **Scaling** : Multiply all entries in a row by a non-zero constant
- Each elementary row operation can be encoded as multiplication with an **elementary matrix**.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e-a & f-b & g-c & h-d \\ i & j & k & l \end{bmatrix}$$

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- **Gaussian elimination** can bring any matrix into an **echelon form**. It proceeds by **elementary row operations** :
  - **Replacement** : Add one row to a multiple of another row
  - **Interchange** : Interchange two rows
  - **Scaling** : Multiply all entries in a row by a non-zero constant
- Each elementary row operation can be encoded as multiplication with an **elementary matrix**.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e-a & f-b & g-c & h-d \\ i & j & k & l \end{bmatrix}$$

- We can continue row reduction to obtain the (unique) RREF.

# Using Gaussian elimination

Gaussian elimination and row reductions allow us to:

- solve systems of linear equations

$$\left[\begin{array}{cccc|c} 0 & 3 & -6 & 4 & -5 \\ 3 & -7 & 8 & 8 & 9 \\ 3 & -9 & 12 & 6 & 15 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -24 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$x_1 = -24 + 2x_3, \quad x_2 = -7 + 2x_3, \quad x_3 \textit{ free}, \quad x_4 = 4$$

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- compute the LU decomposition $A = LU$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ & -8 & -2 \\ & & 1 \end{bmatrix}$$

## Using Gaussian elimination

- compute the inverse of a matrix

to find $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$, we use Gauss–Jordan:

$$\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

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(Each solution $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ gives a linear combination $\begin{bmatrix}1\\2\\3\end{bmatrix} = x_1\begin{bmatrix}1\\1\\1\end{bmatrix} + x_2\begin{bmatrix}1\\2\\0\end{bmatrix}$).

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- So the solution is *unique* if and only if $Nul(A) = 0$.

# Midterm

7:00PM-8:15PM, Thursday, February 19th

Students last name A-G: 114 David Kinley Hall
Students last name H-Ra: 100 Noyes Lab
Students last name Re-Zu: 112 Greg Hall

Bring university ID. No books, notes, or electronic devices.

Good luck!