Math 415 - Lecture 7

LU-decomposition

Wednesday September 9th 2015

Textbook: Chapter 1.5

Suggested Practice Exercise: Chapter 1.5 Exercise 4, 5, 11, 23, 29

1 Review - Elementary matrices

- Multiply row 3 by 7:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 7g & 7h & 7i \end{bmatrix}$$

- Switch rows 2 and 3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

- $R3 \to 3R1 + R3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

- Taking the inverse of an elementary matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

2 Triangular matrices

Definition. An $n \times n$ matrix A is called upper triangular if it is of the form

$$\begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{bmatrix}.$$

An $n \times n$ matrix B is called lower triangular if it is of the form

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & \ddots & \vdots \\ * & * & * & * & * \end{bmatrix}.$$

Definition. A matrix A has LU factorization if there is a lower triangular matrix L and a upper triangular matrix U such that

$$A = LU.$$

(In practice, L will have all 1's on the main diagonal.)

When is this possible?

Theorem 1. *Let A be a $n \times n$-matrix. If A can be transformed into echelon form without the use of row exchanges, then A has LU factorization.*

Example 1. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$. Can we transform it into Echelon form without row exchanges?

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{2} & 1 & 1 \\ \boxed{4} & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}, \ell_{21} = 2$$

$$E_2 (E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{2} & 1 & 1 \\ 0 & -8 & -2 \\ \boxed{-2} & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}, \ell_{31} = -1$$

$$E_3 (E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & \boxed{-8} & -2 \\ 0 & \boxed{8} & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \ell_{32} = -1$$

We got an upper triangular matrix!

$$E_3E_2E_1A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

(Always works - if an $n \times n$ matrix is in echelon form, then it is upper triangular.)
We need to reverse these operations:

$$E_3E_2E_1A = U \implies A = E_1^{-1}E_2^{-1}E_3^{-1}U$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = L$$

This is a lower triangular matrix!

Motto
Product of lower triangulars is lower triangular.

So the LU decomposition is

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Summary:

- If A can be brought in echelon form without row exchanges we have $A = LU$,
- U - Echelon form of A
- $L = E_1^{-1}E_2^{-1}E_3^{-1}$ where E_1, E_2, E_3 were elementary matrices that put A into Echelon form. (No row exchanges!)
- $L = I+$ strictly lower triangular, and ℓ_{ij} is the factor between pivot and the entry you want to make zero in the elimination process: see the boxed numbers.

3 Row exchanges

Recall that a permutation matrix P is a square matrix obtained from the identity matrix by reordering the rows.

Theorem 2. *Let A be a $n \times n$-matrix that can be brought to echelon form. Then there is permutation matrix P such that PA has LU factorization.*

Reason: If A can be brought to echelon form with the help of row exchanges, we can do those exchange first. So there is a permutation matrix P such that PA can be brought to echelon form without row exchanges. $\square$

Example 2. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. Find PA that has a LU factorization.

Let's rearrange the rows:

- Move the 2nd row to the 1st row
- Move the 3rd row to the 2nd row
- Move the 1st row to the 3rd row

Do these moves to the identity matrix to get P:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Doing these moves to A gives the same matrix as PA:

$$PA = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can bring PA to echelon form without row exchanges (check this!) so $PA = LU$:

$$PA = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4 Applications

Theorem 3. *Let A be an $n \times n$-matrix such that $A = LU$, where L is a lower triangular matrix and U is a upper triangular matrix. Then x will be solution of*

$$Ax = b$$

if and only if x is a solution of

$$Ux = c,$$

where c satisfies $Lc = b$.

Point:$Ux = c$ and $Lc = b$ are *triangular systems*, easy to solve by substitution.

Proof. If $Lc = b$ and $Ux = c$, then

$$Ax = (LU)x = L(Ux) = Lc = b$$

On the other hand, suppose $Ax = b$. We take c to be the vector Ux. Then Lc is equal to $L(Ux) = Ax = b$, so in total we have both $Ux = c$ and $Lc = b$. $\square$

Example 3. Solve

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

We found already a LU factorization for this matrix A. So you first have to solve $Lc = b$ for c:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}.$$

Use **forward substitution:**

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 2 & 1 & 0 & -2 \\ -1 & -1 & 1 & 9 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -12 \\ 0 & -1 & 1 & 14 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

So

$$c = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

Then solve $Ux = c$ for x:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}.$$

This uses **backwards substitution.**

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 2 & 1 & 0 & 3 \\ 0 & -8 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

So

$$x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

5 Practice problems

- Is $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ upper triangular? Lower triangular?

  Yes, it is both upper and lower triangular.

- Is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ upper triangular? Lower triangular?

  No, it is neither upper nor lower triangular.

- True or false? A permutation matrix is one that is obtained by performing column exchanges on an identity matrix.

  Technically yes, but this isn't how we defined it. We defined it as row exchanges on an identity matrix.

- Why do we care about LU decomposition if we already have Gaussian elimination?

  It's faster, especially if we have to feed in lots of different values of **b**.

Example 4. Solve

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -3 \end{bmatrix}.$$

using the factorization we already have.