1. a) $\hat{\theta} = \prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x_i!} \Rightarrow l(\hat{\theta}) = \sum_{i=1}^{n} \log\left(\frac{\theta^{x_i} e^{-\theta}}{x_i!}\right)$

$= \sum_{i=1}^{n} \log(\theta^{x_i}) - \log(x_i!) - \theta n \Rightarrow l'(\theta) = \frac{n\bar{x}}{\theta} - n \overset{set}{=} 0$

$\Rightarrow \hat{\theta} = \bar{X}$

b) According to CLT, $\sqrt{n}(\hat{\theta} - \theta) \sim N(0, \theta)$
since the variance of a Poisson distribution is θ.

2. $\hat{\theta} \xrightarrow{P} \theta$, then when n is large, $\sqrt{n}(\hat{\theta} - \theta) \sim N(0, \hat{\theta})$.
(We don't know what θ is, but we do know $\hat{\theta}$, so we are use $\hat{\theta}$ to find a C.I. for θ),
then $C.I. = \hat{\theta} \pm Z_{97.5} \sqrt{\frac{\hat{\theta}}{n}} = \bar{X} \pm 1.96 \sqrt{\frac{\bar{X}}{n}}$

3. $f(x_i;\theta) = \prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x_i!} = \left(\prod_{i=1}^{n} \theta^{x_i} e^{-\theta}\right)\left(\prod_{i=1}^{n} \frac{1}{x_i!}\right) = \underbrace{\left(e^{-n\theta} \theta^{\sum X_i}\right)}_{\phi(u(x_i);\theta)} \underbrace{\left(\prod_{i=1}^{n} \frac{1}{X_i!}\right)}_{h(x_i)}$

So $Y = \sum X_i$ is sufficient by factorization theorem.

4. a) $E(Y) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i^2) = \frac{1}{n}\sum_{i=1}^{n}\left(Var(X_i) + E(X_i)^2\right)$

$= \frac{1}{n}\sum_{i=1}^{n}(\theta + 0^2) = \theta$

b) $Var(Y) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i^2) = \frac{1}{n^2}\sum_{i=1}^{n} E(X_i^4) - E(X_i^2)^2$

$E(X_i^2) = Var(X_i) + E(X_i)^2 = \theta$

$E(X_i^4) = E[(X_i - \mu)^4] = 3\sigma^4 = 3\theta^2$ since $\mu = 0$

So $Var(Y) = \frac{1}{n^2}\sum_{i=1}^{n} E(X_i^4) - E(X_i^2)^2 = \frac{1}{n^2}\sum_{i=1}^{n}(3\theta^2 - \theta^2) = \frac{2\theta^2}{n}$

5. a) $I(\theta) = -E\left[\frac{\partial^2 \log(f(X;\theta))}{\partial\theta^2}\right] = -E\left[\frac{\partial^2}{\partial\theta^2}\log\left(\frac{1}{\sqrt{2\pi\theta}}e^{-x^2/2\theta}\right)\right]$

$= -E\left[\frac{\partial^2}{\partial\theta^2}\left[\log(1) - \log(\sqrt{2\pi\theta}) - \frac{x^2}{\theta}\right]\right] = -E\left[\frac{\partial^2}{\partial\theta^2}\left[-\frac{1}{2}\log(2\pi\theta) - \frac{x^2}{2\theta}\right]\right]$

$= -E\left[\frac{\partial}{\partial\theta}\left(-\frac{1}{2}\left(\frac{2\pi}{2\pi\theta}\right) + \frac{x^2}{2\theta^2}\right)\right] = -E\left[\frac{\partial}{\partial\theta}\left(-\frac{1}{2\theta} + \frac{x^2}{2\theta^2}\right)\right]$

$= -E\left[\frac{1}{2\theta^2} - \frac{x^2}{\theta^3}\right] = -\frac{1}{2\theta^2} + \frac{1}{\theta^3}E(X^2) = -\frac{1}{2\theta^2} + \frac{1}{\theta^2} = \frac{1}{2\theta^2}$

b) Rao-Cramer Lower Bound

$\frac{1}{nI(\theta)} = 2\theta^2/n$ MLE: $L(\theta;x) = \left(\frac{1}{\sqrt{2\pi\theta}}\right)^n e^{-\sum x_i^2/2\theta}$

$l(\theta,x) = -\frac{n}{2}\ln(2\pi\theta) - \sum x_i^2/2\theta \Rightarrow \frac{\partial l(\theta,x)}{\partial\theta} = -\frac{n}{2\theta} + \frac{\sum x_i^2}{2\theta^2} \overset{set}{=} 0$

$\Rightarrow \hat{\theta} = \frac{1}{n}\sum x_i^2$.

From 4, $E(\hat{\theta}) = \theta$ and $Var(\hat{\theta}) = 2\theta^2/n$, hence efficient

c) $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{D} N(0, 1/I(\theta)) = N(0, 2\theta^2)$

6. a) $f_X(x_i;\theta) = \prod_{i=1}^{n}(\theta+1)(1-x_i)^\theta = (\theta+1)^n\left[\prod_{i=1}^{n}(1-x_i)^\theta\right]$

$= (\theta+1)^n\left[\prod_{i=1}^{n}(1-x_i)\right]^\theta$

$\therefore \prod_{i=1}^{n}(1-x_i)$ is sufficient.

b) $I(\theta) = -E\left(-\frac{1}{(\theta+1)^2}\right) = \frac{1}{(\theta+1)^2}$

7. $EY = 2EX_i = 2\int_0^\infty x \frac{3\theta^3}{(x+\theta)^4}dx$

$= 2\theta^3 \left\{ x\left[-(x+\theta)^{-3}\right]\Big|_0^\infty - \int_0^\infty -(x+\theta)^{-3}dx \right\}$

$= \theta$

$\therefore Y = 2\bar{X}$ is unbiased for θ

$\frac{\partial^2}{\partial\theta^2}\left[\log f(x;\theta)\right] = -\frac{3}{\theta^2} + \frac{4}{(x+\theta)^2}$

$I(\theta) = \frac{3}{\theta^2} - E\left(\frac{4}{(x+\theta)^2}\right)$

$= \frac{3}{\theta^2} - \int_0^\infty \frac{4}{(x+\theta)^2}\frac{3\theta^3}{(x+\theta)^4}dx$

$= \frac{3}{\theta^2} - \frac{12}{5}\frac{1}{\theta^2}$

$= \frac{3}{5}\frac{1}{\theta^2}$

$Var(Y) = 4Var(\bar{X}) = \frac{4}{n}Var(X_i) = \frac{4}{n}\left(\theta^2 - \frac{\theta^2}{4}\right) = \frac{3}{n}\theta^2$

efficiency $= \dfrac{\frac{1}{nI(\theta)}}{Var(Y)} = \frac{5}{9}$

8. $L(\theta;x) = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}$ $\quad \ell(\theta;x) = \Sigma x_i \cdot \log\theta + (n-\Sigma x_i)\log(1-\theta)$

$\frac{\partial \ell(\theta;x)}{\partial\theta} = \frac{1}{\theta}\Sigma x_i - \frac{1}{1-\theta}(n-\Sigma x_i) = 0 \quad \Rightarrow \hat{\theta} = \frac{1}{n}\Sigma x_i = \bar{X}$

$\Lambda = \frac{L(\frac{1}{3})}{L(\hat{\theta})} = \left(\frac{\frac{1}{3}}{\bar{X}}\right)^{\Sigma x_i}\left(\frac{\frac{2}{3}}{1-\bar{X}}\right)^{n-\Sigma x_i}$

$-2\ln\Lambda = -2\left[\Sigma x_i \cdot \log\frac{1}{3\bar{X}} + (n-\Sigma x_i)\log\frac{2}{3(1-\bar{X})}\right]$

9. (a) $\chi^2_W = nI(\theta_0)(\hat{\theta}_n - \theta_0)^2$

$I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2}\log p(x;\theta)\right) = \frac{1}{\theta(1-\theta)}$

$\chi^2_W = \frac{9}{2}n\left(\bar{X}_n - \frac{1}{3}\right)^2$

(b) $\chi^2_R = \frac{[\ell'(\theta_0;x)]^2}{nI(\theta_0)}$

$\ell'(\theta_0;x) = \frac{9}{2}\Sigma x_i - \frac{3}{2}n$

$\chi^2_R = \frac{2}{9}\cdot\frac{1}{n}\cdot\left(\frac{9}{2}\Sigma x_i - \frac{3}{2}n\right)^2$

10. $L(\theta) = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$ $\ell(\theta; x) = n\log\theta + (\theta-1)\sum_{i=1}^{n} \log x_i$

$\frac{\partial}{\partial\theta}(\ell(\theta;x)) = \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i = 0 \Rightarrow \hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log x_i}$

$\Lambda = \frac{L(1)}{L(\hat{\theta})} = \frac{1}{\hat{\theta}^n \left(\prod_{i=1}^{n} x_i\right)^{\hat{\theta}-1}} = \left(-\frac{\sum \log x_i}{n}\right)^n \left(\prod x_i\right)^{\frac{n}{\sum \log x_i}+1}$

$-2\ln\Lambda = -2\left[n\log\left(-\frac{\sum\log x_i}{n}\right) + n + \sum \log x_i\right]$

11. $\bar{X}_n \sim N(\theta, \frac{\sigma^2}{n})$

$E(\bar{X}_n^2) = Var(\bar{X}_n) + (E(\bar{X}_n))^2 = \frac{\sigma^2}{n} + \theta^2$

$E(\bar{X}_n^2 - \sigma^2/n) = \theta^2 \Rightarrow \bar{X}_n^2 - \sigma^2/n$ is unbiased for θ^2

$I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2}(\log f(x;\theta))\right) = -E\left(-\frac{1}{\sigma^2}\right) = \frac{1}{\sigma^2}$

lower bound $= \frac{(k'(\theta))^2}{n I(\theta)} = \frac{(2\theta)^2}{n \cdot \frac{1}{\sigma^2}} = \frac{4\theta^2\sigma^2}{n}$

$Var\left(\bar{X}_n^2 - \frac{\sigma^2}{n}\right) = Var(\bar{X}_n^2) = \frac{4\theta^2\sigma^2}{n} + \frac{2\sigma^4}{n^2}$

$\therefore$ efficiency $= \frac{4\theta^2\sigma^2/n}{4\theta^2\sigma^2/n + 2\sigma^4/n^2} = \frac{1}{1 + \sigma^2/2n\theta^2}$

12. when $\frac{\pi}{4} < x < \frac{\pi}{2}$, $\Lambda = \frac{\sin x}{\sin x} = 1$ cannot reject

when $0 < x < \frac{\pi}{4}$, $\Lambda = \frac{\sin x}{\cos x} = \tan x \leq c$

$P(\tan x \leq c \mid H_0) = 0.1 \iff P(x \leq \arctan c \mid H_0) = 0.1$

$\int_0^{\arctan c} \sin x \, dx = 0.1 \Rightarrow c = \tan(\arccos 0.9) \approx 0.484$

$\therefore$ reject H_0 when $\Lambda \leq 0.484$

Power $= P(\tan x \leq 0.484 \mid H_a) = \int_0^{\arctan 0.484} \cos x \, dx \approx 0.436$