Math 415 - Lecture 10

Span is a subspace, Null Space

Wednesday September 16th 2015

Textbook: Chapter 2.1, 2.2.

Suggested practice exercises: Chapter 2.1: 3, 21, 28. Chapter 2.2: 33 and additional exercises in this lecture note.

Khan Academy videos: Linear Subspaces, Introduction to the Null Space of a Matrix, Calculating the Null Space of a Matrix

1 Review of vector space and subspace

- A vector space is a set of vectors which can be added and scaled (without leaving the space!); subject to the "usual" rules.
- The set of all polynomials of degree up to 2 is a vector space. Why?

  Note how it "works" just like $\mathbb{R}^3$.

- The set of all polynomials of degree exactly 2 is **not** a vector space. Why?

- Easy test: Is the zero vector in the set? (If not, then it's **not** a vector space.)

Example 1. Let V be the set of all function $f : \mathbb{R} \to \mathbb{R}$. Is V a vector space?

Solution.

Definition. A **subspace** of a vector space V is a subset H of V that has three properties:

1. The zero vector of V is in H.
2. For each $\mathbf{u}$ and $\mathbf{v}$ are in H, $\mathbf{u} + \mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)
3. For each $\mathbf{u}$ in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

Problem 2. *Find as many subspaces in* $\mathbb{R}^2$ *as you can.*

2 A Shortcut for Determining Subspaces

Definition. Recall that *span* $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$ is the collection of all vectors that can be written as

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_p\mathbf{v_p},$$

where $x_1, x_2, \ldots, x_p$ are scalars.

Theorem 1. *If* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$ *are in a vector space* V*, then span* $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$ *is a subspace of* V*.*

Example 3. Is $V = \left\{ \begin{bmatrix} a+2b \\ 2a-3b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$? Why or why not?

Solution.

Example 4. Is $H = \left\{ \begin{bmatrix} a+2b \\ a+1 \\ a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^3$? Why or why not?

Solution.

Example 5. Is the set H of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of M_{2x2}?

Solution.

Problem 6. *Determine which of the following sets are subspaces and give reasons:*

1. $W_1 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 2b = c, 4a + 2c = 1 \right\}.$

2. $W_2 = \left\{ \begin{bmatrix} a-b \\ c \\ a+c \\ a-2b-c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$

3. $W_3 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a \cdot b \geq 0 \right\}.$

3 Null Spaces

Definition. The **nullspace** of an $m \times n$ matrix A, written as $Nul(A)$, is

Theorem 2. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.Equivalently, the set of all solutions to the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.*

Proof: $Nul(A)$ is a subset of $\mathbb{R}^n$ since A has n columns. We have to verify properties (a), (b), and (c) of the definition of a subspace.

Property (a): Show that $\mathbf{0}$ is in $Nul(A)$.

Property (b): If $\mathbf{u}$ and $\mathbf{v}$ are in $Nul(A)$, show that $\mathbf{u} + \mathbf{v}$ is also in $Nul(A)$.

Property (c): If $\mathbf{u}$ is in $Nul(A)$ and c is a scalar, show that $c\mathbf{u}$ is also in $Nul(A)$.

Remark.

- Since properties (a), (b), and (c) hold, $Nul(A)$ is a subspace of $\mathbb{R}^n$.
- Since $Nul(A)$ is a subspace, it is closed under linear combinations. You can add solutions of $A\mathbf{x} = 0$ and get a new solution! This is very important. Not true for $A\mathbf{x} = \mathbf{b}$ for $b \neq 0$. Here you cannot add solutions!
- Solving $A\mathbf{x} = \mathbf{0}$ yields an explicit description of $Nul(A)$.

Example 7. Find and explicit description of $Nul(A)$ where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

Solution.

Remark. If $Nul(A) \neq \{\mathbf{0}\}$, then the number of vectors in the spanning set for $Nul(A)$ equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.

In this example, we had 3 free variables (x_2, x_4, and x_5) so there were 3 vectors in the spanning set for $Nul(A)$. More about this later!