Math 415 - Lecture 14

Null sapce and Column space basis

Friday September 25th 2015

Textbook reading: 2.4

Suggested practice exercises: Chapter 2.4 Exercise 1, 2, 3, 4, 21

Khan Academy video: Null Space and Column Space Basis, Dimension of the Null Space, Dimension of the Column Space

Strang lecture: Independence, Basis, and Dimension

1 Review

- $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a **basis** of V if the vectors
  - span V, and
  - are independent.
- The **dimension** of V is the number of elements in a basis.
- The columns of A are linearly independent $\iff$ each column of A contains a pivot. $\iff$ there are no free variables.

2 Warmup

Example 1. Find a basis and the dimension of

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Solution.

Remark. Every set of linearly independent vectors can be extended to a basis.

In other words, let $\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ be linearly independent vectors in V. If V has dimension d, then we can find vectors $\mathbf{v_{p+1}}, \dots, \mathbf{v_d}$ such that $\{\mathbf{v_1}, \dots, \mathbf{v_d}\}$ is a basis of V.

Example 2. Consider

$$H = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

- Give a basis for H. What is the dimension of H?
- Extend the basis of H to a basis of $\mathbb{R}^3$.

Solution.

3 Bases for Null Spaces

To find a basis for $Nul(A)$:

- find the parametric form of the solutions to $A\mathbf{x} = \mathbf{0}$.
- express solutions $\mathbf{x}$ as a linear combination of vectors with the free variables as coefficients;
- these vectors form a basis of $Nul(A)$.

Example 3. Find a basis for $Nul(A)$ with

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 15 & 0 & 3 \end{bmatrix}.$$

Solution.

Remark. If A is a matrix, $Nul(A)$ has a basis vector for each free variable. So the *dimension* of $Nul(A)$ is equal to the number of free variables!

4 Basis for Column Space

Recall that the columns of A are independent $\iff$ $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (namely, $\mathbf{x} = \mathbf{0}$) $\iff$ A has no free variables.

Theorem 1. *A basis for $Col(A)$ is given by the pivot columns of A.*

Example 4. Find a basis for $Col(A)$ with

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

Solution.

Remark. If A has echelon form U then any relation between the columns of U:

$$x_1\mathbf{u_1} + \cdots + x_n\mathbf{u_n} = 0$$

also holds for the columns of A:

$$x_1\mathbf{a_1} + \cdots + x_n\mathbf{a_n} = 0,$$

for the *same* scalars x_i.

Why?

Solution.

Warning : For the basis of $Col(A)$, you have to take the columns of A, not the columns of an echelon form. Row operations do not preserve the column space.

Example 5. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Then the RREF of A is $U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$.

5 Checking Our Understanding

True or false?

1. Suppose that V has dimension n. Then any set in V containing more than n vectors must be linearly dependent.

2. The space P_n of polynomials of degree at most n has dimension $n + 1$.

3. The vector space of functions $f : \mathbb{R} \to \mathbb{R}$ is infinite-dimensional.

4. Consider $V = \textit{span}\,\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$. If one of the vectors, say $\mathbf{v_k}$, in the spanning set is a linear combination of the remaining ones, then the remaining vectors still span V.