

Math 415 - Lecture 11

Column space, Solution to $Ax = b$

Friday September 18th 2015

Textbook: Chapter 2.1, 2.2.

Textbook: Chapter 2.1, 2.2.

Suggested practice exercises: Chapter 2.1: 3, 21, 28. Chapter 2.2:
33 and additional exercises at the end of this lecture.

Textbook: Chapter 2.1, 2.2.

Suggested practice exercises: Chapter 2.1: 3, 21, 28. Chapter 2.2: 33 and additional exercises at the end of this lecture.

Khan Academy videos: Introduction to the Null Space of a Matrix, Calculating the Null Space of a Matrix, Column Space of a Matrix

Review

Definition

The **nullspace** of an $m \times n$ matrix A , written as $Nul(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$Nul(A) = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

Definition

The **nullspace** of an $m \times n$ matrix A , written as $Nul(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$Nul(A) = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

Theorem

The null space of an $m \times n$ matrix A is a subspace of \mathbb{R}^n . Equivalently, the set of all solutions \mathbf{x} to the system $A\mathbf{x} = \mathbf{0}$ is a subspace of \mathbb{R}^n .

For example

$$\text{Nul} \left(\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \right) = \text{Nul} \left(\begin{bmatrix} 1 & 2 & 0 & 13 & 33 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \right).$$

This corresponds to the solution:

For example

$$\text{Nul} \left(\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \right) = \text{Nul} \left(\begin{bmatrix} 1 & 2 & 0 & 13 & 33 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \right).$$

This corresponds to the solution:

$$x_1 = -2x_2 - 13x_4 - 33x_5$$

$$x_3 = 6x_4 + 15x_5.$$

For example

$$\text{Nul} \left(\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \right) = \text{Nul} \left(\begin{bmatrix} 1 & 2 & 0 & 13 & 33 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \right).$$

This corresponds to the solution:

$$x_1 = -2x_2 - 13x_4 - 33x_5$$

$$x_3 = 6x_4 + 15x_5.$$

Write this as a linear combination:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} =$$

For example

$$\text{Nul} \left(\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \right) = \text{Nul} \left(\begin{bmatrix} 1 & 2 & 0 & 13 & 33 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \right).$$

This corresponds to the solution:

$$x_1 = -2x_2 - 13x_4 - 33x_5$$

$$x_3 = 6x_4 + 15x_5.$$

Write this as a linear combination:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} =$$

For example

$$\text{Nul} \left(\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \right) = \text{Nul} \left(\begin{bmatrix} 1 & 2 & 0 & 13 & 33 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \right).$$

This corresponds to the solution:

$$x_1 = -2x_2 - 13x_4 - 33x_5$$

$$x_3 = 6x_4 + 15x_5.$$

Write this as a linear combination:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}.$$

This means that

$$\text{Nul}(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

This means that

$$\text{Nul}(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Remark

If $\text{Nul}(A) \neq \{\mathbf{0}\}$, then the number of vectors in the spanning set for $\text{Nul}(A)$ equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.

This means that

$$\text{Nul}(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Remark

If $\text{Nul}(A) \neq \{\mathbf{0}\}$, then the number of vectors in the spanning set for $\text{Nul}(A)$ equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.

This means that

$$\text{Nul}(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Remark

If $\text{Nul}(A) \neq \{\mathbf{0}\}$, then the number of vectors in the spanning set for $\text{Nul}(A)$ equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.

In this example, we had **3 free variables** (x_2 , x_4 , and x_5) so there were **3 vectors** in the spanning set for $\text{Nul}(A)$. More about this later!

Column Spaces

Column Spaces

Definition

The **column space**, written as $Col(A)$, of an $m \times n$ matrix A is the set of all linear combinations of the columns of A .

Column Spaces

Definition

The **column space**, written as $Col(A)$, of an $m \times n$ matrix A is the set of all linear combinations of the columns of A .

If $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$, then $Col(A) = \text{Span} \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$.

Example

- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\text{Col}(A) =$

Example

- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $Col(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

Example

- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $Col(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. This is all of \mathbb{R}^2 !

Example

- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\text{Col}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. This is all of \mathbb{R}^2 !
- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $\text{Col}(A) =$

Example

- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\text{Col}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. This is all of \mathbb{R}^2 !
- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $\text{Col}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) =$

Example

- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\text{Col}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. This is all of \mathbb{R}^2 !
- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $\text{Col}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

Example

- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $Col(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. This is all of \mathbb{R}^2 !
- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $Col(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.
This is the x_1 axis in \mathbb{R}^2 !

Example

- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $Col(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. This is all of \mathbb{R}^2 !
- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $Col(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.
This is the x_1 axis in \mathbb{R}^2 !
- If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $Col(A) =$

Example

- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $Col(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. This is all of \mathbb{R}^2 !
- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $Col(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.
This is the x_1 axis in \mathbb{R}^2 !
- If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $Col(A) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$.

Example

- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\text{Col}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. This is all of \mathbb{R}^2 !
- If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $\text{Col}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.
This is the x_1 axis in \mathbb{R}^2 !
- If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\text{Col}(A) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$. This is the zero subspace of \mathbb{R}^2 !

Column Spaces

Theorem

The column space of an $m \times n$ matrix A is a subspace of \mathbb{R}^m .

Column Spaces

Theorem

The column space of an $m \times n$ matrix A is a subspace of \mathbb{R}^m .

Why is it a subspace?

Column Spaces

Theorem

The column space of an $m \times n$ matrix A is a subspace of \mathbb{R}^m .

Why is it a subspace? Because it is a Span!

Column Spaces

Theorem

The column space of an $m \times n$ matrix A is a subspace of \mathbb{R}^m .

Why is it a subspace? Because it is a Span!

Remark

If A is $m \times n$ (m rows, n columns) then

Column Spaces

Theorem

The column space of an $m \times n$ matrix A is a subspace of \mathbb{R}^m .

Why is it a subspace? Because it is a Span!

Remark

If A is $m \times n$ (m rows, n columns) then

- $Col(A)$ is a subspace of the output space

Column Spaces

Theorem

The column space of an $m \times n$ matrix A is a subspace of \mathbb{R}^m .

Why is it a subspace? Because it is a Span!

Remark

If A is $m \times n$ (m rows, n columns) then

- $Col(A)$ is a subspace of the output space \mathbb{R}^m .

Column Spaces

Theorem

The column space of an $m \times n$ matrix A is a subspace of \mathbb{R}^m .

Why is it a subspace? Because it is a Span!

Remark

If A is $m \times n$ (m rows, n columns) then

- $Col(A)$ is a subspace of the output space \mathbb{R}^m .
- $Nul(A)$ is a subspace of the input space

Column Spaces

Theorem

The column space of an $m \times n$ matrix A is a subspace of \mathbb{R}^m .

Why is it a subspace? Because it is a Span!

Remark

If A is $m \times n$ (m rows, n columns) then

- $Col(A)$ is a subspace of the output space \mathbb{R}^m .
- $Nul(A)$ is a subspace of the input space \mathbb{R}^n .

Column Spaces

Theorem

The column space of an $m \times n$ matrix A is a subspace of \mathbb{R}^m .

Why is it a subspace? Because it is a Span!

Remark

If A is $m \times n$ (m rows, n columns) then

- $Col(A)$ is a subspace of the output space \mathbb{R}^m .
- $Nul(A)$ is a subspace of the input space \mathbb{R}^n .

Column Spaces

Theorem

Let A be an $m \times n$ matrix.

Column Spaces

Theorem

Let A be an $m \times n$ matrix.

\mathbf{b} is in $\text{Col}(A)$ iff there is an $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in \mathbb{R}^n such that $A\mathbf{x} = \mathbf{b}$.

Column Spaces

Theorem

Let A be an $m \times n$ matrix.

\mathbf{b} is in $\text{Col}(A)$ iff there is an $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in \mathbb{R}^n such that $A\mathbf{x} = \mathbf{b}$.

Proof.

Suppose $A\mathbf{x} = \mathbf{b}$. Then

Column Spaces

Theorem

Let A be an $m \times n$ matrix.

\mathbf{b} is in $\text{Col}(A)$ iff there is an $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in \mathbb{R}^n such that $A\mathbf{x} = \mathbf{b}$.

Proof.

Suppose $A\mathbf{x} = \mathbf{b}$. Then

$$\mathbf{b} = A\mathbf{x} = \underbrace{x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n}_{(\text{lin. comb. of } \mathbf{a}_1, \dots, \mathbf{a}_n)}.$$



Column Spaces

Example

Find a matrix A such that $W = \text{Col}(A)$ where

$$W = \left\{ \begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

Column Spaces

Solution

$$\begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}.$$

Column Spaces

Solution

$$\begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}.$$

So

$$W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}$$

Column Spaces

Solution

$$\begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}.$$

So

$$W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\} = \text{Col} \left(\begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 1 & 1 \end{bmatrix} \right).$$

Column Spaces

Solution

$$\begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}.$$

So

$$W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\} = \text{Col} \left(\begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 1 & 1 \end{bmatrix} \right).$$

Therefore

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 1 & 1 \end{bmatrix}.$$

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Theorem

Let A be an $m \times n$ matrix, let $\mathbf{b} \in \mathbb{R}^m$, and let $\mathbf{x}_p \in \mathbb{R}^n$ such that

$$A\mathbf{x}_p = \mathbf{b}.$$

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Theorem

Let A be an $m \times n$ matrix, let $\mathbf{b} \in \mathbb{R}^m$, and let $\mathbf{x}_p \in \mathbb{R}^n$ such that

$$A\mathbf{x}_p = \mathbf{b}.$$

Then the set of solutions $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$ is exactly

$$\mathbf{x}_p + Nul(A).$$

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Theorem

Let A be an $m \times n$ matrix, let $\mathbf{b} \in \mathbb{R}^m$, and let $\mathbf{x}_p \in \mathbb{R}^n$ such that

$$A\mathbf{x}_p = \mathbf{b}.$$

Then the set of solutions $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$ is exactly

$$\mathbf{x}_p + Nul(A).$$

So every solution of $A\mathbf{x} = \mathbf{b}$ is of the form

$$\mathbf{x}_p + \mathbf{x}_n$$

where \mathbf{x}_n is some vector in $Nul(A)$.

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Proof.

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Proof.

Let $\mathbf{x}_p \in \mathbb{R}^n$ such that $A\mathbf{x}_p = \mathbf{b}$.

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Proof.

Let $\mathbf{x}_p \in \mathbb{R}^n$ such that $A\mathbf{x}_p = \mathbf{b}$. Suppose \mathbf{x} is also in \mathbb{R}^n with $A\mathbf{x} = \mathbf{b}$.

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Proof.

Let $\mathbf{x}_p \in \mathbb{R}^n$ such that $A\mathbf{x}_p = \mathbf{b}$. Suppose \mathbf{x} is also in \mathbb{R}^n with $A\mathbf{x} = \mathbf{b}$. Then

$$A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Proof.

Let $\mathbf{x}_p \in \mathbb{R}^n$ such that $A\mathbf{x}_p = \mathbf{b}$. Suppose \mathbf{x} is also in \mathbb{R}^n with $A\mathbf{x} = \mathbf{b}$. Then

$$A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Therefore, $\mathbf{x} - \mathbf{x}_p = \mathbf{x}_n$ is in $Nul(A)$, so $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$.

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Proof.

Let $\mathbf{x}_p \in \mathbb{R}^n$ such that $A\mathbf{x}_p = \mathbf{b}$. Suppose \mathbf{x} is also in \mathbb{R}^n with $A\mathbf{x} = \mathbf{b}$. Then

$$A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Therefore, $\mathbf{x} - \mathbf{x}_p = \mathbf{x}_n$ is in $Nul(A)$, so $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$. □

Remark

We often call \mathbf{x}_p a **particular solution** of $A\mathbf{x} = \mathbf{b}$. The theorem then says that every solution to $A\mathbf{x} = \mathbf{b}$ is the sum of one particular solution \mathbf{x}_p and all the solutions to $A\mathbf{x} = \mathbf{0}$ (the null space).

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Example

Let $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$. Solve $A\mathbf{x} = \mathbf{b}$.

Step 1 : Reduce $A\mathbf{x} = \mathbf{b}$ to $U\mathbf{x} = \mathbf{c}$.

Step 1 : Reduce $A\mathbf{x} = \mathbf{b}$ to $U\mathbf{x} = \mathbf{c}$.

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 7 & 5 \\ -1 & -3 & 3 & 4 & 5 \end{array} \right]$$

Step 1 : Reduce $A\mathbf{x} = \mathbf{b}$ to $U\mathbf{x} = \mathbf{c}$.

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 7 & 5 \\ -1 & -3 & 3 & 4 & 5 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 6 & 6 & 6 \end{array} \right]$$

Step 1 : Reduce $A\mathbf{x} = \mathbf{b}$ to $U\mathbf{x} = \mathbf{c}$.

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 7 & 5 \\ -1 & -3 & 3 & 4 & 5 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 6 & 6 & 6 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Step 1 : Reduce $A\mathbf{x} = \mathbf{b}$ to $U\mathbf{x} = \mathbf{c}$.

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 7 & 5 \\ -1 & -3 & 3 & 4 & 5 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 6 & 6 & 6 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

$$U\mathbf{x} = \mathbf{c}.$$

$$\left[\begin{array}{cccc} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Step 2 : Find a particular solution to $U\mathbf{x} = \mathbf{c}$.

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Step 2 : Find a particular solution to $U\mathbf{x} = \mathbf{c}$.

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Could pick any value for the free variables (x_2 and x_4).

Nul(A) and solutions to $A\mathbf{x} = \mathbf{b}$

Step 2 : Find a particular solution to $U\mathbf{x} = \mathbf{c}$.

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Could pick any value for the free variables (x_2 and x_4).

Trick: Set them both to 0.

Nul(A) and solutions to $A\mathbf{x} = \mathbf{b}$

Step 2 : Find a particular solution to $U\mathbf{x} = \mathbf{c}$.

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Could pick any value for the free variables (x_2 and x_4).

Trick: Set them both to 0. Then

$$3x_3 = 3 \Rightarrow x_3 = 1.$$

Nul(A) and solutions to $A\mathbf{x} = \mathbf{b}$

Step 2 : Find a particular solution to $U\mathbf{x} = \mathbf{c}$.

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Could pick any value for the free variables (x_2 and x_4).

Trick: Set them both to 0. Then

$$\begin{aligned} 3x_3 &= 3 \Rightarrow x_3 = 1. \\ x_1 + 3x_3 &= 1 \Rightarrow x_1 = -2. \end{aligned}$$

Nul(A) and solutions to $A\mathbf{x} = \mathbf{b}$

Step 2 : Find a particular solution to $U\mathbf{x} = \mathbf{c}$.

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Could pick any value for the free variables (x_2 and x_4).

Trick: Set them both to 0. Then

$$\begin{aligned} 3x_3 &= 3 \Rightarrow x_3 = 1. \\ x_1 + 3x_3 &= 1 \Rightarrow x_1 = -2. \end{aligned}$$

$$\text{So } \mathbf{x}_p = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ is a particular solution to } A\mathbf{x} = \mathbf{b}.$$

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Step 3 : Find **all** the solutions to $A\mathbf{x} = \mathbf{0}$ to find $Nul(A)$.

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Step 3 : Find **all** the solutions to $A\mathbf{x} = \mathbf{0}$ to find $Nul(A)$.

$$[U \mid \mathbf{0}] = \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Step 3 : Find **all** the solutions to $A\mathbf{x} = \mathbf{0}$ to find $Nul(A)$.

$$[U \mid \mathbf{0}] = \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$Nul(A)$ and solutions to $Ax = b$

Step 3 : Find **all** the solutions to $Ax = \mathbf{0}$ to find $Nul(A)$.

$$\begin{aligned} [U \mid \mathbf{0}] &= \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \\ &\left[\begin{array}{cccc|c} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]. \end{aligned}$$

$Nul(A)$ and solutions to $Ax = b$

Step 3 : Find **all** the solutions to $Ax = \mathbf{0}$ to find $Nul(A)$.

$$\begin{aligned} [U \mid \mathbf{0}] &= \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \\ &\left[\begin{array}{cccc|c} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]. \end{aligned}$$

Every vector in $Nul(A)$ is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}$$

Nul(A) and solutions to $A\mathbf{x} = \mathbf{b}$

Step 3 : Find **all** the solutions to $A\mathbf{x} = \mathbf{0}$ to find $Nul(A)$.

$$\begin{aligned} [U \mid \mathbf{0}] &= \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \\ &\left[\begin{array}{cccc|c} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]. \end{aligned}$$

Every vector in $Nul(A)$ is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Step 4 : To find all the solutions to $A\mathbf{x} = \mathbf{b}$, add a particular solution \mathbf{x}_p to the null space of A .

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Step 4 : To find all the solutions to $A\mathbf{x} = \mathbf{b}$, add a particular solution \mathbf{x}_p to the null space of A . So the set of solutions is

$$\mathbf{x}_p + Nul(A).$$

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Step 4 : To find all the solutions to $A\mathbf{x} = \mathbf{b}$, add a particular solution \mathbf{x}_p to the null space of A . So the set of solutions is

$$\mathbf{x}_p + Nul(A).$$

$$\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \text{Span} \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$Nul(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Step 4 : To find all the solutions to $A\mathbf{x} = \mathbf{b}$, add a particular solution \mathbf{x}_p to the null space of A . So the set of solutions is

$$\mathbf{x}_p + Nul(A).$$

$$\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \text{Span} \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

and each solution \mathbf{x} is of the form

$$\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$.

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why?

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$,

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$, which is the same as the space of solutions of $Ux = 0$

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$, which is the same as the space of solutions of $Ux = 0$ (That is the point of echelon form matrices!)

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$, which is the same as the space of solutions of $Ux = 0$ (That is the point of echelon form matrices!) which is $Nul(U)$.

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$, which is the same as the space of solutions of $Ux = 0$ (That is the point of echelon form matrices!) which is $Nul(U)$.
- Not true that $Col(A) = Col(U)$! Why?

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$, which is the same as the space of solutions of $Ux = 0$ (That is the point of echelon form matrices!) which is $Nul(U)$.
- Not true that $Col(A) = Col(U)$! Why?

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$, which is the same as the space of solutions of $Ux = 0$ (That is the point of echelon form matrices!) which is $Nul(U)$.
- Not true that $Col(A) = Col(U)$! Why?

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $U =$

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$, which is the same as the space of solutions of $Ux = 0$ (That is the point of echelon form matrices!) which is $Nul(U)$.
- Not true that $Col(A) = Col(U)$! Why?

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $U = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$, which is the same as the space of solutions of $Ux = 0$ (That is the point of echelon form matrices!) which is $Nul(U)$.
- Not true that $Col(A) = Col(U)$! Why?

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $U = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

$$Col(A) =$$

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$, which is the same as the space of solutions of $Ux = 0$ (That is the point of echelon form matrices!) which is $Nul(U)$.
- Not true that $Col(A) = Col(U)$! Why?

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $U = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

$$Col(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right),$$

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$, which is the same as the space of solutions of $Ux = 0$ (That is the point of echelon form matrices!) which is $Nul(U)$.
- Not true that $Col(A) = Col(U)$! Why?

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $U = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

$$Col(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right), \quad Col(U) =$$

Remark

- If A is a matrix with echelon form U , then $Nul(A) = Nul(U)$. Why? Because $Nul(A)$ is the set of solutions of $Ax = 0$, which is the same as the space of solutions of $Ux = 0$ (That is the point of echelon form matrices!) which is $Nul(U)$.
- Not true that $Col(A) = Col(U)$! Why?

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $U = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

$$Col(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right), \quad Col(U) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

Additional Exercises

1. True or false?

Additional Exercises

1. True or false?

(i) The solutions to $A\mathbf{x} = \mathbf{0}$ form a vector space.

Additional Exercises

1. True or false?

(i) The solutions to $A\mathbf{x} = \mathbf{0}$ form a vector space.

Additional Exercises

1. True or false?

- (i) The solutions to $A\mathbf{x} = \mathbf{0}$ form a vector space. True. This is the null space $Nul(A)$.

Additional Exercises

1. True or false?

- (i) The solutions to $A\mathbf{x} = \mathbf{0}$ form a vector space. True. This is the null space $Nul(A)$.
- (ii) The solutions to $A\mathbf{x} = \mathbf{b}$ form a vector space.

Additional Exercises

1. True or false?

- (i) The solutions to $A\mathbf{x} = \mathbf{0}$ form a vector space. True. This is the null space $Nul(A)$.
- (ii) The solutions to $A\mathbf{x} = \mathbf{b}$ form a vector space.

Additional Exercises

1. True or false?

- (i) The solutions to $A\mathbf{x} = \mathbf{0}$ form a vector space. True. This is the null space $Nul(A)$.
- (ii) The solutions to $A\mathbf{x} = \mathbf{b}$ form a vector space. False, unless $\mathbf{b} = \mathbf{0}$.

Additional Exercises

1. True or false?

- (i) The solutions to $A\mathbf{x} = \mathbf{0}$ form a vector space. True. This is the null space $Nul(A)$.
- (ii) The solutions to $A\mathbf{x} = \mathbf{b}$ form a vector space. False, unless $\mathbf{b} = \mathbf{0}$.

2. Find an explicit description for $Nul(A)$ where

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}.$$

Additional Exercises

3. Show that the given set W is a subspace (by showing that W is the column space or null space of some matrix A) or find a specific example that shows that W is not a subspace.

Additional Exercises

3. Show that the given set W is a subspace (by showing that W is the column space or null space of some matrix A) or find a specific example that shows that W is not a subspace.

(i) $W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 5x - 1 = y + 2z \right\}.$

Additional Exercises

3. Show that the given set W is a subspace (by showing that W is the column space or null space of some matrix A) or find a specific example that shows that W is not a subspace.

$$(i) \quad W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 5x - 1 = y + 2z \right\}.$$

$$(ii) \quad W_2 = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a = 2b + c, 2a = c - 3d \right\}.$$

Additional Exercises

4. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Find a smallest spanning set for $W = \text{Col}(A)$. Find a matrix B such that $W = \text{Nul}(B)$.

Additional Exercises

4. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Find a smallest spanning set for $W = Col(A)$. Find a matrix B such that $W = Nul(B)$.
5. Let $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find a smallest spanning set for $W = Nul(B)$. Find a matrix A such that $W = Col(A)$.