Math 415 - Lecture 11

Column space, Solution to $A\mathbf{x} = b$

Friday September 18th 2015

Textbook: Chapter 2.1, 2.2.

Suggested practice exercises: Chapter 2.1: 3, 21, 28. Chapter 2.2: 33 and additional exercises at the end of this lecture.

Khan Academy videos: Introduction to the Null Space of a Matrix, Calculating the Null Space of a Matrix, Column Space of a Matrix

1 Review

Definition. The **nullspace** of an $m \times n$ matrix A, written as $Nul(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$Nul(A) = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

Theorem 1. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions $\mathbf{x}$ to the system $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.*

2 Column Spaces

Definition. The **column space**, written as $Col(A)$, of an $m \times n$ matrix A is

Example 1. • If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

• If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,

• If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

Theorem 2. *The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.*

Why is it a subspace?

Remark. If A is $m \times n$ (m rows, n columns) then

- $Col(A)$ is a subspace of $\boxed{}$
- $Nul(A)$ is a subspace of $\boxed{}$

Why?

Theorem 3. *Let A be an $m \times n$ matrix.* $\mathbf{b}$ *is in* $Col(A)$ *iff there is an* $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ *in* $\mathbb{R}^n$ *such that* $A\mathbf{x} = \mathbf{b}$.

Proof.

$\square$

Example 2. Find a matrix A such that $W = Col(A)$ where

$$W = \left\{ \begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

Solution.

3 $Nul(A)$, $Col(A)$ and solutions to $A\mathbf{x} = \mathbf{b}$

Theorem 4. *Let A be an $m \times n$ matrix, let $\mathbf{b} \in \mathbb{R}^m$, and let $\mathbf{x_p} \in \mathbb{R}^n$ such that*

$$A\mathbf{x_p} = \mathbf{b}.$$

Then the set of solutions $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$ is exactly

$$\mathbf{x_p} + Nul(A).$$

So every solution of $A\mathbf{x} = \mathbf{b}$ is of the form

$$\mathbf{x_p} + \mathbf{x_n}$$

where $\mathbf{x_n}$ is some vector in $Nul(A)$.

Proof.

□

Remark. We often call $\mathbf{x_p}$ a *particular solution* of $A\mathbf{x} = \mathbf{b}$. The theorem then says that every solution to $A\mathbf{x} = \mathbf{b}$ is the sum of one particular solution $\mathbf{x_p}$ and all the solutions to $A\mathbf{x} = \mathbf{0}$ (the null space).

Example 3. Let $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$. Solve $A\mathbf{x} = \mathbf{b}$.

Step 1 : Reduce $A\mathbf{x} = \mathbf{b}$ to $U\mathbf{x} = \mathbf{c}$.

Step 2 : Find a particular solution to $U\mathbf{x} = \mathbf{c}$.

Step 3 : Find **all** the solutions to $A\mathbf{x} = \mathbf{0}$ to find $Nul(A)$.

Step 4 : To find all the solutions to $A\mathbf{x} = \mathbf{b}$, add a particular solution $\mathbf{x_p}$ to the null space of A.

Remark.

- If A is a matrix with echelon form U, then $Nul(A) = Nul(U)$. Why?

- Not true that $Col(A) = Col(U)$! Why?

Additional Exercises

1. True or false?

   (i) The solutions to $A\mathbf{x} = \mathbf{0}$ form a vector space. True. This is the null space $Nul(A)$.

   (ii) The solutions to $A\mathbf{x} = \mathbf{b}$ form a vector space. False, unless $\mathbf{b} = 0$.

2. Find an explicit description for $Nul(A)$ where

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}.$$

3. Show that the given set W is a subspace (by showing that W is the column space or null space of some matrix A) or find a specific example that shows that W is not a subspace.

   (i) $W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 5x - 1 = y + 2z \right\}.$

(ii) $W_2 = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a = 2b + c,\ 2a = c - 3d \right\}$.

4. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Find a smallest spanning set for $W = Col(A)$. Find a matrix B such that $W = Nul(B)$.

5. Let $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find a smallest spanning set for $W = Nul(A)$. Find a matrix B such that $W = Col(B)$