Math 415 - Lecture 32

Complex numbers and eigenvectors

Wednesday November 11th 2015

Textbook reading: first part of Chapter 5.5

Suggested practice exercises: 5.5 1, 2, 3,

Khan Academy video: Complex Numbers (part 1)

Strang lecture: Lecture 21: Eigenvalues and eigenvectors

1 Review

- If $A\mathbf{x} = \lambda\mathbf{x}$ then $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ. All eigenvectors (plus $\mathbf{0}$) with eigenvalue λ form **eigenspace** of λ.
- λ is an eigenvalue of $A \iff \underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial}} = 0$. Why? Because $A\mathbf{x} = \lambda\mathbf{x} \iff (A - \lambda I)\mathbf{x} = \mathbf{0}$. By the way: this means that the eigenspace of λ is just $\mathrm{Nul}(A - \lambda I)$.
- E.g. if $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$ then $\det(A - \lambda I) = (3-\lambda)(6-\lambda)(2-\lambda)$.
- Eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ of A corresponding to different eigenvalues are independent.
- By the way:
  - product of eigenvalues = determinant
  - sum of eigenvalues = "trace" (sum of diagonal entries)

2 Eigenbasis?

An $n \times n$ matrix A has up to n different eigenvalues. Namely, the roots of degree n characteristic polynomial $\det(A - \lambda I)$.

- For each eigenvalue λ, A has at least one eigenvector. That is because $\mathrm{Nul}(A - \lambda I)$ has dimension at least one.
- If λ has multiplicity m, then A has up to m (independent) eigenvectors for λ.

Ideally, we would like to find a total of n (independent) eigenvectors for A. This would give an **EIGENBASIS**. Why can there be no more than n independent eigenvectors?!

Two sources of trouble: eigenvalues can be

- complex numbers (that is, not enough real roots), or
- repeated roots of the characteristic polynomial.

Example 1. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Geometrically, what is the trouble?

Solution.

3 Complex numbers review

Definition. $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

- $i = \sqrt{-1}$, or $i^2 = -1$.
- Any point in $\mathbb{R}^2$ can be viewed as a complex number:
  $\binom{x}{y} \leftrightarrow x + iy$

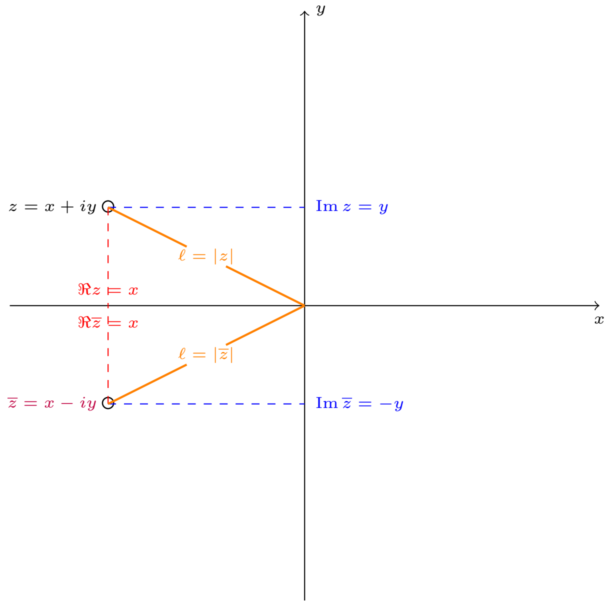
  Let $z = x + iy$ be a complex number

Real part The real part of z, denoted $\Re(z)$ is defined by $\Re(z) = x$.

Imaginary part The imaginary part of z, denoted $\mathrm{Im}(z)$ is defined by $\mathrm{Im}(z) = y$.

Complex conjugate The complex conjugate of z, denoted $\overline{z}$, is defined by $\overline{z} = x - iy$.

Absolute value The absolute value, or magnitude of z, denoted $|z|$ or $\|z\|$, is given by $|z| = \sqrt{x^2 + y^2}$.



Adding complex numbers

Definition. Given $z = x + iy$, $w = u + iv$, we define

Remark. This corresponds exactly to addition of vectors in $\mathbb{R}^2$.

Multiplying complex numbers

Definition. Given $z = x + iy$, $w = u + iv$, we define

Absolute value and complex conjugate

Remark.

- $\overline{\overline{z}} = z$
- $|z|^2 = z\overline{z}$
- $|z| = |\overline{z}|$

Proof. $\square$

3.1 Complex Linear Algebra

Until now we took as our scalars the real numbers. In particular we used the vector space $\mathbb{R}^n$ of column vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

If c is a real number (a scalar) we defined

$$c\mathbf{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}.$$

Now we want to use **COMPLEX** scalars. We need a new context to make sense of this.

Definition. $\mathbb{C}^n$ is the (complex) vector space of *complex* column vectors $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$, where $z_1, z_2, \ldots z_n$ are complex numbers.

- Now multiplication by a complex scalar makes sense.
- We can define subspaces, Span, independence, basis, dimension for $\mathbb{C}^n$ in the usual way.
- We can multiply complex vectors by complex matrices. Column space and Null space still make sense.
- The only difference is the dot product, you need to use the complex conjugate to get a good notion of length. (Later more.)

4 Back to eigenvectors

Example 2. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution (continued)**.**

Theorem 1. *If A is a matrix with real entries and λ is a* ***complex eigenvalue****, then $\bar{\lambda}$ is also a complex eigenvalue. Furthermore, if $\mathbf{x}$ is an eigenvector with eigenvalue λ, then $\bar{\mathbf{x}}$ is an eigenvector with eigenvalue $\bar{\lambda}$.*

Proof.

$\square$

Remark. Note that we are using vectors in $\mathbb{C}^2$, instead of vectors in $\mathbb{R}^2$. Works pretty much the same!

Example 3. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution.

- Trouble: We can not find an **Eigenbasis** for this matrix. This kind of problem cannot really be fixed. We have to lower our expectations and look for generalized eigenvectors. These are solutions to $(A - \lambda I)^2\mathbf{x} = \mathbf{0}, (A - \lambda I)^3\mathbf{x} = \mathbf{0}, \ldots$

5 Practice problems

Example 4. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.

Solution.