Worksheet 11 for November 10th and 12th

1. **a.** *Compare* $\det\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ *and the "row flipped" determinant* $\det\begin{bmatrix}3 & 4\\1 & 2\end{bmatrix}$.

b. *If* $A = \begin{bmatrix}0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & 0\\1 & 0 & 0 & 0 & 0\end{bmatrix}$, *what is* $\det(A)$*?*

c. *If* $A = \begin{bmatrix}1 & 1 & 4\\2 & 2 & 5\\3 & 3 & 6\end{bmatrix}$, *what is* $\det(A)$*?*

d. *If* $A = \begin{bmatrix}1 & 4 & 5\\2 & 5 & 7\\3 & 6 & 9\end{bmatrix}$, *what is* $\det(A)$*?*

e. *If* $A = \begin{bmatrix}1 & 2 & 3\\3 & 2 & 1\\1 & 1 & 3\end{bmatrix}$, *find* $\det(A)$ *by expanding along the last column.*

2. *True or False? Justify your answers!*

a. *Let Q be a 3×3 orthogonal matrix. Then det$(Q) = 1$.*
b. *If det(A) =det$(B) = 0$ then det$(A + B) = 0$.*
c. *Let A be a 3×3 matrix so that det$(A) = 0$. Then $A\mathbf{x} = \mathbf{b}$ has exactly one solution for each vector $\mathbf{b}$.*
d. *Let A be a 3×3 matrix so that det$(A) = 9$. Then det$(2A) = 18$.*
e. *Let R be a 2×3 matrix. Then det$(R^T R) = 0$.*
f. *Let R be a 2×3 matrix. Then det$(RR^T) = 0$.*

3. *True or False? Justify your answers!*

a. *We say A and B ($n \times n$ matrices) are similar if $A = DBD^{-1}$ for an invertible matrix D. Let A and B be similar matrices, then det(A) =det(B).*
b. *Let A and B be 3×3 matrices. If det(A) =det(B) then A and B are similar. [Note: number of pivots in DBD^{-1} is equal to the number of pivots in B. (Why?) Use this fact to find a counter example.]*
c. *Someone tells you that the zero vector is an eigenvector of a 2×2 matrix A. Is this possible?*
d. *An $n \times n$ matrix A always has n distinct eigenvalues.*

4. *For each of the following matrices, determine the characteristic polynomial $p(\lambda)$ of the matrix, determine the eigenvalues of the matrix and for each eigenvalue, determine (a basis for) the eigenspace that is associated to that eigenvalue.*

Tutoring Room (443 Altgeld Hall): Mon 4-6 PM, Tue 5-7 PM, Wed 6-8 PM

Midterm Date: November 19 7-8:15 PM, Conflict November 20, 8-9.20AM and 9:30-10:50AM, Conflict sign up deadline: November 13

Final Date: December 17 8-11AM, Conflict December 15, 8-11AM. You are allowed to take the conflict exam if you have more than two examination within 24 hours. Conflict sign up deadline: November 30

a. $\begin{bmatrix} 4 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$,

b. $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$,

c. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

5. *Let A be an $n \times n$-matrix with eigenvalue λ. Which of the following statements are true:*

a. λ^2 *is an eigenvalue of* A^2,

b. λ^{-1} *is an eigenvalue of* A^{-1},

c. $\lambda + 1$ *is an eigenvalue of* $A + I$.

6. *Let A, B be two $n \times n$-matrices such that $AB = BA$.*

a. *Suppose $\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ. Is $B\boldsymbol{v}$ an eigenvector of A? If so, what is the eigenvalue of that eigenvector?*

b. *Suppose A has eigenvectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ with distinct eigenvalues $\lambda_1 \neq \ldots \neq \lambda_n$. Is each $\boldsymbol{v_i}$ also an eigenvector of B? (This question is a bit tricker. Hint: Note that each of the eigenspaces of A has dimension 1 and then use your answer to* ***a****.).*

7. *Let* $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$, $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by* $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto A \begin{bmatrix} x \\ y \end{bmatrix}$, *and* $\mathcal{B} = \{ \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix} \}$.

a. *If* $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, *what is* $\boldsymbol{v}_{\mathcal{B}}$?

b. *If* $\boldsymbol{v}_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, *what is* $\boldsymbol{v}$?

c. *What is* $T_{\mathcal{B},\mathcal{B}}$?

8. *Let $\mathcal{B} := \{\boldsymbol{b_1}, \boldsymbol{b_2}\}$ and $\mathcal{C} := \{\boldsymbol{c_1}, \boldsymbol{c_2}\}$ be two bases of $\mathbb{R}^2$ such that*

$$\boldsymbol{b_1} = 6\boldsymbol{c_1} - 2\boldsymbol{c_2} \text{ and } \boldsymbol{b_2} = 9\boldsymbol{c_1} - 4\boldsymbol{c_2}.$$

Determine $I_{\mathcal{C},\mathcal{B}}$ and $I_{\mathcal{B},\mathcal{C}}$!

9. *Let A be a $n \times n$-matrix and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation such that $T(\boldsymbol{v}) = A\boldsymbol{v}$ for all $\boldsymbol{v}$ in $\mathbb{R}^n$. Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^n$. True or false?*

a. *Let $\mathcal{B} := \{\boldsymbol{b_1}, \ldots, \boldsymbol{b_n}\}$ be a basis of $\mathbb{R}^n$. All $\boldsymbol{b_i}$'s are eigenvectors of A if and only if $T_{\mathcal{B},\mathcal{B}}$ is diagonal.*

b. *The matrix A is invertible if and only if there is a basis $\mathcal{C} := \{\boldsymbol{c_1}, \ldots, \boldsymbol{c_n}\}$ of $\mathbb{R}^n$ such that $T_{\mathcal{C},\mathcal{E}} = I_{n \times n}$.*