Math 415 - Lecture 12

Linear independence

Monday September 21st 2015

Textbook reading: Section 2.3

Suggested practice exercises: Section 2.3: 1, 2, 3, 4, 5,7, 8, 9

Khan Academy video: Introduction to Linear Independence, More on linear independence, Span and Linear Independence Example,

Strang lecture: Independence, Basis, and Dimension

* Exam 1 (7-8:15 pm Tuesday September 29):

* Rooms:

  - 213 Gregory Hall: AD3, ADG, ADU
  - 151 Loomis: ADC, ADD, ADL, ADM
  - 100 Gregory Hall: ADE, ADF, ADN, ADO
  - 66 Library: ADH, ADP, ADQ, ADX
  - 141 Loomis: AD1, AD2, ADS, ADT, ADW, ADZ
  - 100 MSEB: AD4, ADV, ADY, ADI, ADR
  - 150 ASL: ADA, ADB, ADJ, ADK

  MSEB is the Materials Science and Engineering Building. ASL is the Animal Science Lab.

* Conflicts: The conflict exams are at 8:00-9:20AM and 9:30-10:50AM on the same day. Email your TA with your reason for needing a conflict, and your preferred time to sign up for the conflict exam.

  The deadline for signing up for a conflict is a week before (September 22).

1 Linear independence

Review.

- Span $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_m}\}$ is the set of all linear combinations
$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_m\mathbf{v_m}.$$
- Span $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_m}\}$ is a vector space.
- $\text{Col}(A) = \text{Span}(\mathbf{a_1}, \mathbf{a_2}, \dots, \mathbf{a_n})$, if $A = \begin{bmatrix}\mathbf{a_1} & \mathbf{a_2} & \dots & \mathbf{a_n}\end{bmatrix}$. In this case
$$\mathbf{b} \in \text{Col}(A) \iff \mathbf{b} = A\mathbf{x} \text{ for some} \mathbf{x} \in \mathbb{R}^n.$$

Today we want to think how *big* the span of a bunch of vectors is. Is it a line, or a plane or ....

Example 1. Is Span $\left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix} \right\}$ equal to $\mathbb{R}^2$?

Solution. To answer the question translate to linear systems. Recall that the span is equal to
$$\{b \colon b = A\mathbf{x}\} = \left\{ \begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}\mathbf{x} : \mathbf{x} \in \mathbb{R}^2 \right\}.$$
Hence, the span is equal to $\mathbb{R}^2$ if and only if the system with augmented matrix
$$\left[\begin{array}{cc|c}1 & 2 & b_1\\1 & 2 & b_2\end{array}\right]$$
is consistent for all b_1, b_2.

To check consistency use Gaussian elimination:
$$\left[\begin{array}{cc|c}1 & 2 & b_1\\1 & 2 & b_2\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 2 & b_1\\0 & 0 & b_2 - b_1\end{array}\right]$$
When is this system consistent? The system is only consistent if $b_2 - b_1 = 0$. Hence, the span does not equal all of $\mathbb{R}^2$. The span is a line instead of a plane!

Example 2. Is Span $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix} \right\}$ equal to $\mathbb{R}^3$?

Solution. Recall that the span is equal to
$$\{b \colon b = A\mathbf{x}\} = \left\{ \begin{bmatrix}1 & 1 & -1\\1 & 2 & 1\\1 & 3 & 3\end{bmatrix}\mathbf{x} : \mathbf{x} \in \mathbb{R}^3 \right\}.$$
Hence, the span is equal to $\mathbb{R}^3$ if and only if the system with augmented matrix
$$\left[\begin{array}{ccc|c}1 & 1 & -1 & b_1\\1 & 2 & 1 & b_2\\1 & 3 & 3 & b_3\end{array}\right]$$
is consistent for all b_1, b_2, b_3.

To check consistency use Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 1 & 2 & 1 & b_2 \\ 1 & 3 & 3 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & 2 & 4 & b_3 - b_1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{array}\right]$$

When is this system consistent? The system is only consistent if $b_3 - 2b_2 + b_1 = 0$. Hence, the span does not equal all of $\mathbb{R}^3$.

- What went wrong?
  Well, the three vectors that span satisfy a *relation*:

$$\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

- Hence, $\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}.$

- We are going to say that the three vectors are **linearly dependent** because they satisfy the (non trivial) relation

$$-3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \mathbf{0}.$$

Definition. Vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are said to be **linearly independent** if the equation

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$$

has only the trivial solution (namely, $x_1 = x_2 = \cdots = x_p = 0$).

Likewise, $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are said to be **linearly dependent** if there exist coefficients $x_1, \ldots, x_p$, not all zero, such that

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_p\mathbf{v_p} = \mathbf{0}.$$

This is called a *non trivial relation* (when not all coefficient are zero.)

Example 3.

- Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ independent?
- If possible, find a linear dependence relation among them.

Solution. We need to check whether the equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a non trivial solution. The three vectors are independent if and only if there are no free variables for the system

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \mathbf{x} = \mathbf{0}.$$

To find out, we reduce the matrix to echelon form:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a column without a pivot, we do have a free variable. Hence, the three vectors are not linearly independent. To find a linear dependence relation we solve this system.

Initial steps of Gaussian elimination are as before:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free. $x_2 = -2x_3$, and $x_1 = 3x_3$. Hence, for any x_3,

$$3x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2x_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since we are only interested in one linear combination, we can set, say, $x_3 = 1:$

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

2 Linear independence of matrix columns

- Note that a linear dependence relation, such as

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \mathbf{0},$$

can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \mathbf{0}.$$

- Hence, each linear dependence relation among the columns of a matrix A corresponds to a solution to $A\mathbf{x} = \mathbf{0}$. The Null space determines (in)dependence!

Theorem 1. *Let A be an $m \times n$ matrix.*
The columns of A are linearly independent.
$\iff$ *$A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.*
$\iff$ *$Nul(A) = \{\mathbf{0}\}$*
$\iff$ *A has n pivots.* $\iff$ *there are no free variables for $A\mathbf{x} = \mathbf{0}$.*

Example 4. Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ independent?

Solution. Put the vectors in a matrix, and produce an echelon form:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

Since each column contains a pivot, there are no free variables and the three vectors are independent. These vectors span $\mathbb{R}^3$.

Example 5. Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ independent?

Solution. Put the vectors in a matrix and produce an echelon form:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the last column does not contain a pivot, there is a free variable and the three vectors are linearly dependent. They span a plane.

3 Special cases

- A set of a single non-zero vector $\{\mathbf{v_1}\}$ is always linearly independent. Why? Because $x_1\mathbf{v_1} = \mathbf{0}$ only for $x_1 = 0$.

- A set of two vectors $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent if and only if neither of the vectors is a multiple of the other.

  Why? Because if $x_1\mathbf{v_1} + x_2\mathbf{v_2} = \mathbf{0}$ with, say, $x_2 \neq 0$, then $\mathbf{v_2} = -\frac{x_1}{x_2}\mathbf{v_1}$.

- A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ containing the zero vector is linearly dependent.

  Why? Because if, say, $\mathbf{v_1} = \mathbf{0}$, then $\mathbf{v_1} + 0\mathbf{v_2} + \cdots + 0\mathbf{v_p} = \mathbf{0}$.

- If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. In other words:

  Any set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ of vectors in $\mathbb{R}^n$ is linearly dependent if $p > n$.

  Why? Let A be the matrix with columns $\mathbf{v_1}, \ldots, \mathbf{v_p}$. This is a $n \times p$ matrix.

  The columns are linearly independent if and only if each column contains a pivot.

  If $p > n$, then the matrix can have at most n pivots.

  Thus not all p columns can contain a pivot.

  In other words, the columns have to be linearly dependent.

Example 6. Let $A = \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$ be a two by three matrix. We want to count the free variables for $A\mathbf{x} = \mathbf{0}$. How many pivots can there be? How many free variables? Are the columns of A independent?

4 Additional exercises

With the least amount of work possible, decide which of the following sets of vectors are linearly independent.

(a) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$

Linearly independent, because the two vectors are not multiples of each other.

(b) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

Linearly independent, because it is a single non-zero vector.

(c) Columns of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \end{bmatrix}$

Linearly dependent, because these are more than 3 (namely, 4) vectors in $\mathbb{R}^3$.

(d) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Linearly dependent, because the set includes the zero vector.