Math 415 - Lecture 14

Null space and Column space basis

Friday September 25th 2015

Textbook reading: 2.4

Suggested practice exercises: Chapter 2.4 Exercise 1, 2, 3, 4, 21

Khan Academy video: Null Space and Column Space Basis, Dimension of the Null Space, Dimension of the Column Space

Strang lecture: Independence, Basis, and Dimension

* Exam 1 (7-8:15 pm Tuesday September 29):
* Rooms: look on Moodle.
* Conflicts: if you have a conflict you should have received an email about it. If not, talk to me after class.
* No Discussion Sections next week.
* No Class on Wednesday next week.
* The Exam will be part multiple choice. Bring pencils and erasers! Also bring ID.
* The material for the exam covers the lectures upto and including Lecture 12 (last Monday), and this weeks worksheet and quiz.

1 Review

- $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a **basis** of V if the vectors
  - span V, and
  - are independent.
- The **dimension** of V is the number of elements in a basis.
- The columns of A are linearly independent $\iff$ each column of A contains a pivot. $\iff$ there are no free variables. $\iff$ $\mathrm{Nul}(A) = 0$.

2 Warmup

Example 1. Find a basis and the dimension of

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}.$$

Solution. First, note that

$$W = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Is $dim\ W = 4$? No, because the third vector is the sum of the first two.

Suppose we did not notice ...

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 4 & 1 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Not a pivot in every column, hence the 4 vectors are dependent.

Remark. Not necessary here, but to get a relation, solve $A\mathbf{x} = \mathbf{0}$. Set free variable $x_3 = 1$. Then $x_4 = 0$, $x_2 = -x_3 = -1$ and $x_1 = -x_2 - 2x_3 = -1$. The relation is

$$-\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}.$$

Precisely what we "noticed" to begin with.

Hence, a basis for W is $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $dim\ W = 3$. It follows from the echelon form that these vectors are independent.

Remark. Every set of linearly independent vectors can be extended to a basis.

In other words, let $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ be linearly independent vectors in V. If V has dimension d, then we can find vectors $\mathbf{v_{p+1}}, \ldots, \mathbf{v_d}$ such that $\{\mathbf{v_1}, \ldots, \mathbf{v_d}\}$ is a basis of V.

Example 2. Consider

$$H = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

- Give a basis for H. What is the dimension of H?
- Extend the basis of H to a basis of $\mathbb{R}^3$.

Solution.

- The vectors are independent. By definition, they span H. Therefore, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for H. In particular, $dim\ H = 2$.
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is not a basis for $\mathbb{R}^3$. Why? Because a basis for $\mathbb{R}^3$ needs to contain 3 vectors. Or because, for instance, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in H. So just add this (or any other) missing vector! By construction, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is independent. Hence, this is automatically a basis of $\mathbb{R}^3$.

3 Bases for Null Spaces

To find a basis for $Nul(A)$:

- find the parametric form of the solutions to $A\mathbf{x} = \mathbf{0}$.
- express solutions $\mathbf{x}$ as a linear combination of vectors with the free variables as coefficients;
- these vectors form a basis of $Nul(A)$.

Example 3. Find a basis for $Nul(A)$ with

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 15 & 0 & 3 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 15 & 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 0 & 0 & 3 & -6 & -15 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 5 & 13 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix}$$

The solutions to $A\mathbf{x} = \mathbf{0}$ are:

$$\mathbf{x} = \begin{bmatrix} -2x_2 - 5x_4 - 13x_5 \\ x_2 \\ 2x_4 + 5x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $Nul(A) = span \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$. These vectors are independent. (Can you see why?)

Hence, $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $Nul(A)$.

Remark. If A is a matrix, $Nul(A)$ has a basis vector for each free variable. So the *dimension* of $Nul(A)$ is equal to the number of free variables!

4 Basis for Column Space

Recall that the columns of A are independent $\iff$ $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (namely, $\mathbf{x} = \mathbf{0}$) $\iff$ A has no free variables.

Theorem 1. *A basis for $Col(A)$ is given by the pivot columns of A.*

Example 4. Find a basis for $Col(A)$ with

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

Solution. $Col(A) = Span \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 22 \\ 16 \end{bmatrix} \right)$. But there could be redundant vectors among these generators. Use row operations to find the redundant vectors.

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Note that for U we have column $\mathbf{u}_2 = 2\mathbf{u_1}$ and $\mathbf{u_4} = 4\mathbf{u_1} + 5\mathbf{u_3}$. The same is true for the columns of A! Therefore $\mathbf{a_2}$ and $\mathbf{a_4}$ are redundant. The leftover columns are independent. This are the pivot columns, the first and third.

Hence, a basis for $Col(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$. This argument works in general: the pivot columns of A form a basis for $Col(A)$. So the dimension of $Col(A)$ is the number of pivots.

Remark. If A has echelon form U then any relation for the columns of U:

$$x_1\mathbf{u_1} + \cdots + x_n\mathbf{u_n} = 0$$

also holds for the columns of A:

$$x_1\mathbf{a_1} + \cdots + x_n\mathbf{a_n} = 0,$$

for the *same* scalars x_i. **Why?**

Solution. Because the relation for the columns of U is in matrix form

$$Ux = 0,$$

but this is equivalent to $Ax = 0$, which is equivalent to the relation between the columns of A.

Warning : For the basis of $Col(A)$, you have to take the columns of A, not the columns of an echelon form. Row operations do not preserve the column space.

Example 5. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Then the RREF of A is $U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$.

The second column of both A and U are redundant, so

$$Col(A) = Span(\mathbf{a_1}, \mathbf{a_2}) = Span(\mathbf{a_1}) = Span(\begin{bmatrix} 1 \\ 2 \end{bmatrix}),$$

$$Col(U) = Span(\mathbf{u_1}, \mathbf{u_2}) = Span(\mathbf{u_1}) = Span(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

So $Col(A)$ and $Col(U)$ are **NOT** equal. In contrast $Nul(A)$ and $Nul(U)$ **ARE** equal.

5 Checking Our Understanding

True or false?

1. Suppose that V has dimension n. Then any set in V containing more than n vectors must be linearly dependent. True.

2. The space P_n of polynomials of degree at most n has dimension $n+1$. True. A basis is $\left\{1, t, t^2, \ldots, t^n\right\}$.

3. The vector space of functions $f : \mathbb{R} \to \mathbb{R}$ is infinite-dimensional. True. A still-infinite-dimensional subspace are the polynomials.

4. Consider $V = \mathit{span}\left\{\mathbf{v_1}, \ldots, \mathbf{v_p}\right\}$. If one of the vectors, say $\mathbf{v_k}$, in the spanning set is a linear combination of the remaining ones, then the remaining vectors still span V. True. $\mathbf{v_k}$ is not adding anything new.