Solutions for Problem Set 1
CS 373: Theory of Computation

Assigned: January 17, 2013 Due on: January 24, 2013

Problem 1. [Category: Comprehension+Proof]

1. Let $A = \{1,2,3\}$, $B = \{\emptyset, \{1\}, \{2\}\}$, and $C = \{1, 2, \{1,2\}\}$. Compute $A \cup B$, $A \cap B$, $B \cap C$, $A \cap C$, $A \times B$, $A \times C$, $C \setminus A$, $C \setminus B$, $A \times B \times C$, and 2^B. Recall that 2^A denotes the *power set* of A, and $A \setminus B$ denotes A *set difference* B. **[5 points]**

2. Prove for any two sets A and B, $A \times B = B \times A$ if and only if $A = B$ or $A = \emptyset$ or $B = \emptyset$. **[5 points]**

Solution:

1. The answers are as follows.

$$
\begin{aligned}
&A \cup B = \{1,2,3,\emptyset,\{1\},\{2\}\} \\
&A \cap B = \emptyset \\
&B \cap C = \emptyset \\
&A \cap C = \{1,2\} \\
&A \times B = \{(1,\emptyset),(1,\{1\}),(1,\{2\}),(2,\emptyset),(2,\{1\}),(2,\{2\}),(3,\emptyset),(3,\{1\}),(3,\{2\})\} \\
&A \times C = \{(1,1),(1,2),(1,\{1,2\}),(2,1),(2,2),(2,\{1,2\}),(3,1),(3,2),(3,\{1,2\})\} \\
&C \setminus A = \{\{1,2\}\} \\
&C \setminus B = \{1,2,\{1,2\}\} = C \\
&A \times B \times C = \{(1,\emptyset,1),(1,\emptyset,2),(1,\emptyset,\{1,2\}),(1,\{1\},1),(1,\{1\},2),(1,\{1\},\{1,2\}),(1,\{2\},1),(1,\{2\},2), \\
&\qquad (1,\{2\},\{1,2\}),(2,\emptyset,1),(2,\emptyset,2),(2,\emptyset,\{1,2\}),(2,\{1\},1),(2,\{1\},2),(2,\{1\},\{1,2\}), \\
&\qquad (2,\{2\},1),(2,\{2\},2),(2,\{2\},\{1,2\}),(3,\emptyset,1),(3,\emptyset,2),(3,\emptyset,\{1,2\}),(3,\{1\},1),(3,\{1\},2), \\
&\qquad (3,\{1\},\{1,2\}),(3,\{2\},1),(3,\{2\},2),(3,\{2\},\{1,2\})\} \\
&2^B = \{\emptyset,\{\emptyset\},\{\{1\}\},\{\{2\}\},\{\emptyset,\{1\}\},\{\emptyset,\{2\}\},\{\{1\},\{2\}\},\{\emptyset,\{1\},\{2\}\}\}
\end{aligned}
$$

2. Observe that $\emptyset \times B = \emptyset = A \times \emptyset$ for any sets A and B. Also note that if $A = B$ then clearly $A \times B = B \times A$. The main thing to prove is the converse of the statement. Let A, B be such that $A \times B = B \times A$, and $A \neq \emptyset$ and $B \neq \emptyset$. We need to show that then $A = B$. Let a be any element of A. Since $B \neq \emptyset$ there is $b \in B$. Thus, $(a,b) \in A \times B$. Since $A \times B = B \times A$, $(a,b) \in B \times A$. From the definition of $B \times A$, it follows that $a \in B$. This shows that $A \subseteq B$. A similar argument shows that $B \subseteq A$, and thus $A = B$.

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Problem 2. [Category: Comprehension] Consider the following DFA M_0 over the alphabet $\{0,1\}$.

1. Describe formally what the following are for automaton M_0: set of states, initial state, final states, and transition function. **[4 points]**

2. What are $\hat{\delta}_{M_0}(A,\epsilon)$, $\hat{\delta}_{M_0}(A,1011)$, $\hat{\delta}_{M_0}(B,101)$, and $\hat{\delta}_{M_0}(C,10110)$? **[4 points]**

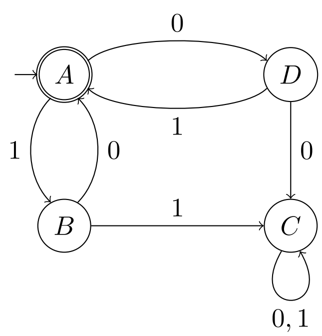

Figure 1: DFA M_0 for Problem 2

3. What is $\mathbf{L}(M_0)$? **[1 points]**

4. What is the language recognized if we change the initial state to B? What is the language recognized if we change the set of final states to be $\{B\}$ (with initial state A)? **[1 points]**

Solution:

1. States: $\{A, B, C, D\}$; Initial state: A; Final states: $\{A\}$; and transitions given by the following matrix

	0	1
A	D	B
B	A	C
C	C	C
D	C	D

2. $\hat{\delta}_{M_0}(A, \epsilon) = \{A\}$; $\hat{\delta}_{M_0}(A, 1011) = \{C\}$; $\hat{\delta}_{M_0}(B, 101) = \{C\}$; $\hat{\delta}_{M_0}(C, 10110) = \{C\}$.

3. Let us call a string $w \in \{0,1\}^*$ to be *proper* if in every prefix u of w has at most one more 0 than 1 and at most one more 1 than 0. Then

$$\mathbf{L}(M_0) = \{w \in \{0,1\}^* \mid w \text{ is proper and has equal number of 0s and 1s}\}$$

Other ways to express this language include $\mathbf{L}(M_0) = \{10, 01\}^*$, or

$$\mathbf{L}(M_0) = \{w \in \{0,1\}^* \mid \; |w| \text{ is even and for every odd position } i, \text{ symbol at position } i \text{ is different than the symbol at position } i+1\}.$$

You were not required to prove that your answer is correct but we do it here to give you some practice. We will establish by induction on $|w|$ the following statements

(a) $A \in \hat{\delta}_{M_0}(A, w)$ iff $w \in \mathbf{L}(M_0)$

(b) $A \in \hat{\delta}_{M_0}(B, w)$ iff $w = 0u$ where $u \in \mathbf{L}(M_0)$

(c) $A \in \hat{\delta}_{M_0}(C, w)$ iff $w \in \emptyset$

(d) $A \in \hat{\delta}_{M_0}(D, w)$ iff $w = 1u$ where $u \in \mathbf{L}(M_0)$

Base Case: Since $|w| = 0$, we know that $w = \epsilon$. Observe that $\epsilon \in \mathbf{L}(M_0)$ and $\hat{\delta}_{M_0}(q, \epsilon) = \{q\}$ for any $q \in \{A, B, C, D\}$. Thus, $A \in \hat{\delta}_{M_0}(q, \epsilon)$ iff $q = A$, establishing all the four statements.

Induction Hypothesis: Assume that (a),(b),(c),(d) hold for strings w of length i.

Induction Step: Consider w of length $i+1$. Without loss of generality, we may assume that $w = av$, where $a \in \{0,1\}$ and v is of length i. We have a few subcases to consider.

Subcase 1: Observe that $\hat{\delta}_{M_0}(A, 0v) = \hat{\delta}_{M_0}(\delta(A,0), v)$ (by the proposition that we proved in class). Thus, we have

$$\begin{aligned} A \in \hat{\delta}_{M_0}(A, 0v) &\text{iff } A \in \hat{\delta}_{M_0}(D, v) && (\delta(A,0) = D) \\ &\text{iff } v = 1u \text{ where } u \in \mathbf{L}(M_0) && (\text{ind. hyp.}) \\ &\text{iff } w = 0v \in \mathbf{L}(M_0) \end{aligned}$$

The other subcases are similar.

Subcase 2: Again, $A \in \hat{\delta}_{M_0}(A, 1v)$ iff $A \in \hat{\delta}_{M_0}(B, v)$ iff $v = 0u$ where $u \in \mathbf{L}(M_0)$ iff $w = 0v \in \mathbf{L}(M_0)$.

Subcase 3: $A \in \hat{\delta}_{M_0}(B, 0v)$ iff $A \in \hat{\delta}_{M_0}(A, v)$ iff $v \in \mathbf{L}(M_0)$.

Subcase 4: $A \in \hat{\delta}_{M_0}(B, 1v)$ iff $A \in \hat{\delta}_{M_0}(D, v)$ iff $v \in \emptyset$.

Subcase 5: For any $a \in \{0,1\}$, $A \in \hat{\delta}_{M_0}(C, av)$ iff $A \in \hat{\delta}_{M_0}(C, v)$ iff $v \in \emptyset$ iff $w = av \in \emptyset$.

Subcase 6: $A \in \hat{\delta}_{M_0}(D, 0v)$ iff $A \in \hat{\delta}_{M_0}(C, v)$ iff $v \in \emptyset$.

Subcase 7: $A \in \hat{\delta}_{M_0}(D, 1v)$ iff $A \in \hat{\delta}_{M_0}(A, v)$ iff $v \in \mathbf{L}(M_0)$.

4. When the initial state is changed to B the language is

$$\{w \in \{0,1\} \mid w = 1u \text{ where } u \in \mathbf{L}(M_0)\}$$

Here $\mathbf{L}(M_0)$ refers to the set defined in the previous part. When the set of final states is changed to $\{B\}$, the language is

$$\{w \in \{0,1\}^* \mid w = u1 \text{ where } u \in \mathbf{L}(M_0)\}$$

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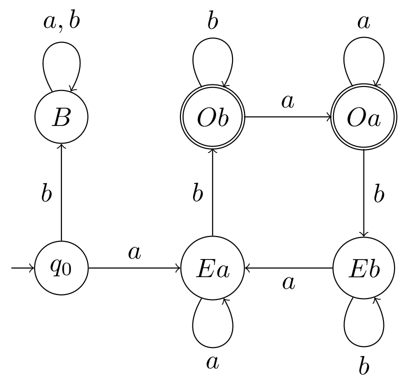
Problem 3. [Category: Design] Design DFAs to recognize the following languages; the alphabet in each case is $\{a,b\}$. You need not prove that your construction is correct, but your construction must be clear and understandable.

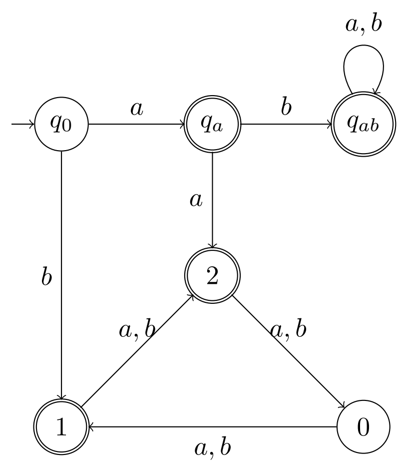
1. $L_1 = \{w \in \{a,b\}^* \mid w \text{ starts with } a \text{ and has an odd number of } ab \text{ substrings}\}$. For example, $aaab \in L_1$ (has only 1 ab), $bab \notin L_1$ (does not begin with a), and $abbab \notin L_1$ (has 2 ab substrings). **[5 points]**
2. $L_2 = \{w \in \{a,b\}^* \mid w \text{ starts with } ab \text{ or } |w| \text{ is not divisible by } 3\}$. For example, $abb \in L_2$ (because it begins with ab), $baba \in L_2$ (because $|baba| = 4$ is not divisible by 3), and $bab \notin L_2$. **[5 points]**

Solution:

1. After checking that the first symbol is an 'a', the automaton will keep track of the parity of the number of ab substrings it has seen so far. To do that it will have an initial state q_0; a state B the remembers that the input began with a b; and states Pc that record that the string so far begins with an a and has parity (odd/even) P number of abs and ends with symbol c (which is either a or b). This can be drawn as below.



2. The automaton will check if the first two symbols are ab; if so it will "ignore" what it sees afterwards and accept no matter what it sees. On the other hand if the first two symbols are not ab then it will keep a count (modulo 3) of how many symbols of w it has read. The automaton's transition diagram looks as follows: The state q_0 is the initial state, q_a remembers that the first symbol was an a, q_{ab}



remembers that the first two symbols are ab, state i (for $i \in \{0, 1, 2\}$) remembers that the number of symbols read so far modulo 3 is i.

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