Math 415 - Lecture 30

Eigenvectors and Eigenvalues

Friday November 6th 2015

Textbook reading: Chapter 5.1

Suggested practice exercises: 12, 20, 21, 22, 36

Khan Academy video: Introduction to Eigenvalues and Eigenvectors, Proof of formula for determining Eigenvalues, Finding Eigenvectors and Eigenspaces example

Strang lecture: Lecture 21: Eigenvalues and eigenvectors

1 Review

Definition. The determinant is characterized by:

- the normalization $\det I_{n\times n} = 1$,
- and how it is affected by elementary row operations:
  - **(Replacement)** Add a multiple of one row to another row. Does not change the determinant.
  - **(Interchange)** Interchange two rows. Reverses the sign of the determinant.
  - **(Scaling)** Multiply all entries in a row by s. Multiplies the determinant by s.
- For triangular A the determinant is just product of the diagonal entries.

This allows us to compute the determinant using just row operations!. Bring A into echelon form= triangular form, keeping track how the determinant changes under the row operations you are using.

- What's **wrong**?!

$$\det(A^{-1}) = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}(da - (-b)(-c) = 1$$

The correct calculation is:

$$\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(ad-bc)^2}\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}$$

Example 1. Suppose A is a 3×3 matrix with $\det(A) = 5$. What is $\det(2A)$?

Solution 2. *A has three rows. Multiplying all 3 of them produces $2A$. Hence, $\det(2A) = 2^3 \det(A) = 40$.*

2 Eigenvectors and eigenvalues

2.1 Definition

Throughout, A will be an $n \times n$ matrix.

Definition. An **eigenvector** of A is a nonzero $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x} \text{ for some scalar } \lambda.$$

The scalar λ is the corresponding **eigenvalue**.

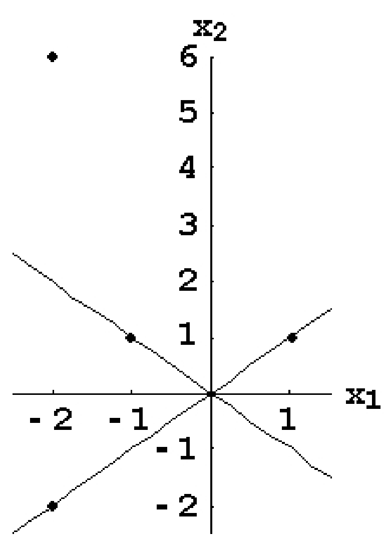
In words, eigenvectors are those $\mathbf{x}$, for which $A\mathbf{x}$ is parallel to $\mathbf{x}$.

Example 3. Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$. Is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an eigenvector?

Solution.

$$A\mathbf{x} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2\mathbf{x}$$

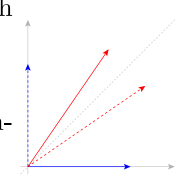
Hence, $\mathbf{x}$ is an eigenvector of A with eigenvalue -2.



2.2 Geometric interpretation

Example 4. Use your geometric understanding to find the eigenvectors and the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution. $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$, i.e. multiplication with A is reflection through the line $y = x$.



- $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 1$.

- $A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ So $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = -1$.

Example 5. Use your geometric understanding to find the eigenvectors and the eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution. $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, i.e. multiplication with A is projection on the x-axis.

- $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 1$.

- $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ So $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 0$.

Summary

* Eigenvectors $\mathbf{x}$ get stretched by eigenvalue λ under multiplication by A:

$$A\mathbf{x} = \lambda\mathbf{x}.$$

* Eigenvectors $\mathbf{x}$ **CANNOT** be zero. Why? $A\mathbf{0} = \lambda\mathbf{0}$ for any λ. Not useful!

* Eigenvalues λ **CAN** be zero. See the projection example.

Problems

* How to find possible eigenvalues for A? This uses determinants.

* How to find eigenvectors? This uses null spaces.

3 Eigenspaces

Definition. The **eigenspace** of A corresponding to λ is the set of all $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$. It consists of all the eigenvectors of A with eigenvalue λ, and also the zero vector.

Example 6. We saw the projection matrix P of the projection onto a subspace V has two eigenvalues $\lambda = 0, 1$.

- The eigenspace of $\lambda = 1$ is V.
- The eigenspace of $\lambda = 0$ is $V^\perp$.

4 How to solve $A\mathbf{x} = \lambda\mathbf{x}$

Key observation: $\mathbf{x} \neq 0$ is an eigenvector means:

$$\begin{aligned} &A\mathbf{x} = \lambda\mathbf{x} \\ \iff &A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0} \\ \iff &(A - \lambda I)\mathbf{x} = \mathbf{0} \end{aligned}$$

This $\mathbf{x}$ is a non trivial solution! This can happen $\iff$ the square matrix $A - \lambda I$ is not invertible $\iff$ $\det(A - \lambda I) = 0$

Recipe
To find the eigenvectors and eigenvalues of A:

- First, find the eigenvalues using λ is an eigenvalue $\iff$ $\det(A - \lambda I) = 0$
- Then, for each eigenvalue λ, find the corresponding eigenvectors by solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$. So you need to find the null space $\mathrm{Nul}(A - \lambda I)$.

4.1 The characteristic polynomial

Example 7. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Solution.

- $A - \lambda I = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$
- $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$

  $$= \lambda^2 - 6\lambda + 8 = 0 \implies \lambda_1 = 2, \lambda_2 = 4$$

  This is the **characteristic polynomial** of A. Its roots are the eigenvalues of A.

- Next, find the eigenvectors with eigenvalue $\lambda_1 = 2$:

$$A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \left(A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \right)$$

  Solutions to $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$ have basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So: $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 2$. All other eigenvectors with eigenvalue $\lambda = 2$ are multiples of $\mathbf{x}_1$. Span $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is the **eigenspace** for the eigenvalue $\lambda = 2$.

- Find the eigenvectors with eigenvalue $\lambda_2 = 4$:

$$A - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \left(A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \right)$$

  Solutions to $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$ have basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So: $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = 4$. The eigenspace of $\lambda = 4$ is Span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

4.2 Triangular matrices

Example 8. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution.

- The characteristic polynomial is:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & 3 \\ 0 & 6-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda)(2-\lambda)$$

- A has eigenvalues $2, 3, 6$.

  $\boxed{\text{The eigenvalues of a triangular matrix are its diagonal entries.}}$

- $\lambda_1 = 2$:

$$(A - \lambda_1 I)\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 10 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \mathbf{x}_1 = \begin{bmatrix} 2 \\ -5/2 \\ 1 \end{bmatrix}$$

- $\lambda_2 = 3$:

$$(A - \lambda_2 I)\mathbf{x} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 10 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- $\lambda_3 = 6$:

$$(A - \lambda_3 I)\mathbf{x} = \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 10 \\ 0 & 0 & -4 \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \mathbf{x}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

- Each of those matrices had a one-dimensional null space. So our eigenvectors are not unique. They are unique up to scaling.

- In summary, $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$ has eigenvalues $2, 3, 6$ with corresponding eigenvectors

$$\begin{bmatrix} 2 \\ -5/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}.$$

  These three vectors are independent. By the next result, this is always so.

4.3 Independent eigenvectors

Theorem 1. *If $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are eigenvectors of A corresponding to different eigenvalues, then they are independent.*

Proof. Suppose, for contradiction, that $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are dependent. By kicking out some vectors, we may assume that there is (up to multiples) only one linear relation: $c_1\mathbf{x}_1 + \ldots + c_m\mathbf{x}_m = \mathbf{0}$. In other words, the matrix with columns $\mathbf{x}_1, \ldots, \mathbf{x}_m$ has one-dimensional null space. Now multiply this relation with A:

$$A(c_1\mathbf{x}_1 + \ldots + c_m\mathbf{x}_m) = c_1\lambda_1\mathbf{x}_1 + \ldots + c_m\lambda_m\mathbf{x}_m = \mathbf{0}$$

This is a second independent relation! In other words, two independent vectors living in a one-dimensional vector space. Contradiction. $\square$

5 Relations between eigenvalues

5.1 Product of Eigenvalues

If A is $n \times n$ get in principle n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. How are these eigenvalues related?

Theorem 2. *The product of eigenvalues $\lambda_1\lambda_2 \ldots \lambda_n$ is equal to the determinant of A.*

Proof. The characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ has constant term $\det(A)$. On the other hand $p(\lambda)$ factors, because the roots are the eigenvalues we get $p(\lambda) = (\pm 1)(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$, which has constant term $\lambda_1\lambda_2 \ldots \lambda_n$. $\square$

Example 9. Let $A = \begin{bmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{bmatrix}$. Then the eigenvalues are λ_1, λ_2 and $\det(A) = \lambda_1\lambda_2$.

5.2 Sum of Eigenvalues

What other relations are there between the eigenvalues?

Definition 10. Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ be $n \times n$. Then the **TRACE** of A is the sum of the diagonal entries: $\mathrm{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

Theorem 3. *Let A be $n \times n$. Then the trace of A is the* ***sum*** *of eigenvalues:*

$$\mathrm{Tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Example 11. Let $A = \begin{bmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{bmatrix}$. What are the eigenvalues and what is $\mathrm{Tr}(A)$?

Solution. The eigenvalues are λ_1, λ_2 and $\mathrm{Tr}(A) = \lambda_1 + \lambda_2$.

5.3 The Characteristic Polynomial for 2×2

2×2 matrices are easy.

Theorem 4. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the characteristic polynomial is*

$$p(\lambda) = \lambda^2 - (a + d)\lambda + ad - bc = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A).$$

Example 12. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. What are the eigenvalues and what is the characteristic polynomial?

Solution. $\mathrm{Tr}(A) = 6$, $\det(A) = 8$, so $p(\lambda) = \lambda^2 - 6\lambda + 8$. Also in terms of eigenvalues $\mathrm{Tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1\lambda_2$. So $\lambda_1 = 2, \lambda_2 = 4$

6 Practice problems

Example 13. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

Example 14. What are the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 4 \end{bmatrix}$. No calculations!