# Math 415 - Lecture 22

Orthogonal projection

Friday October 16th 2015

**Textbook reading:** Chapter 3.2.

**Suggested practice exercises:** Chapter 3.2: 9, 10, 17, 19.

**Strang lecture:** Lecture 15: Projections onto Subspaces

## 1 Review/Outlook

- We can deal with complicated linear systems $Ax = b$ (maybe with help of a computer), but what to do when there is no exact solution?
- $Ax = b$ had no solution if $b$ is not in $Col(A)$.
- Idea: make $Ax - b$ as small as possible (when we vary $x$).
- How? *Project* $b$ on the column space $Col(A)$.
- Recall: If $v_1, v_2, \ldots, v_n$ are orthogonal (and non zero) they are independent.
- Recall: To calculate coordinates for orthogonal vectors is easy: this uses

$$v_1 \cdot (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 v_1 \cdot v_1.$$

## 2 Orthogonal Bases

**Definition 1.** A basis $v_1, v_2, \ldots, v_n$ of $\mathbb{R}^n$ is called *orthogonal* if the vectors are pairwise orthogonal, $v_i \cdot v_j = 0$ if $i \neq j$.

*Example* 2. The standard basis $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthogonal basis for $\mathbb{R}^3$. Similarly, the standard basis $e_1, e_2, \ldots, e_n$ is an orthogonal basis for $\mathbb{R}^n$.

*Example* 3. Are the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ an orthogonal basis for $\mathbb{R}^3$?

**Solution.**

**Theorem 1.** *Let* $(v_1, v_2, \ldots, v_p)$ *be an orthogonal basis of* $V \subset \mathbb{R}^n$, $w \in V$. *Then*

$$w = \frac{v_1 \cdot w}{v_1 \cdot v_1} + \cdots + \frac{v_p \cdot w}{v_p \cdot v_p}.$$

*Proof.* $\square$

*Example* 4. Express $w = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in the basis $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

**Solution.**

**Warning**
The easy formula for the coordinates only works for *orthogonal* bases.

*Example* 5. Take the basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and the vector $w = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$. Then

$$\begin{bmatrix} 4 \\ 9 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

and the coefficients are *not* the numbers you get from the easy formula. To find them you need to solve a system of equations.

*Example* 6. The standard basis $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ is orthonormal. Find the coordinates of $\begin{bmatrix}3\\7\\4\end{bmatrix}$ in the standard basis.

**Solution.**

*Example* 7. The vectors $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ form an orthogonal basis. Produce from it an *orthonormal* basis.

**Solution.**

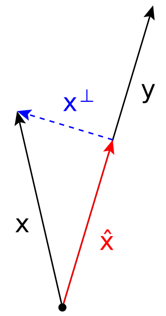
*Example* 8. Express $\begin{bmatrix}3\\7\\4\end{bmatrix}$ in the orthonormal basis $\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$.

**Solution.**

# 3 Orthogonal Projection

**Definition 9** (Orthogonal Projection)**.** The **orthogonal projection** of vector $\mathbf{x}$ on vector $\mathbf{y}$ is

The **error** is $\mathbf{x}^\perp = \mathbf{x} - \hat{\mathbf{x}}$.




- The projection $\hat{\mathbf{x}}$ is the *closest point* to $\mathbf{x}$ on the line through $\mathbf{y}$.
- The error $\mathbf{x}^\perp = \mathbf{x} - \hat{\mathbf{x}}$ is characterized by the property that it is orthogonal to $Span(\mathbf{y})$.
- We have a decomposition $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{x}^\perp$. The **projection** $\hat{\mathbf{x}}$ is in $Span(\mathbf{y})$ and $\mathbf{x}^\perp$ is orthogonal to $Span(\mathbf{y})$.

**Summary**: the projection formula is

$$\hat{\mathbf{x}} = \frac{\mathbf{y} \cdot \mathbf{x}}{\mathbf{y} \cdot \mathbf{y}} \mathbf{y}.$$

Why?

**Solution.**

*Example* 10. Find the orthogonal projection of $\mathbf{x} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ onto $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

**Solution.**

*Example* 11. What is the projection of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ onto each of the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

**Solution.**

**Theorem 2.** *If $v_1, \ldots, v_n$ is orthogonal basis of $V$ and $w \in V$ then*

$$w = c_1 v_1 + \cdots + c_n v_n, \qquad \textit{with } c_j = \frac{w \cdot v_j}{v_j \cdot v_j}.$$

So the terms in this sum are precisely the projections onto each basis vector.

# 4 Projection Matrix

If $\mathbf{y}$ is a fixed nonzero vector, we get from any vector $\mathbf{x}$ the projection $\hat{\mathbf{x}}$. There is a matrix that turns $\mathbf{x}$ into $\hat{\mathbf{x}}$. How? Rewrite the formula for $\hat{\mathbf{x}}$.

where $P = \frac{1}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}\mathbf{y}^T$. $P$ is called the *projection matrix* on the subspace $Span(\mathbf{y})$.

*Example* 12. Let $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find the projection matrix $P$ for $\mathbf{y}$ and use it to calculate the projections of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ on $\mathbf{y}$.

**Solution.**