Math 415 - Lecture 4

Linear Combinations and Matrix operations

Monday August 31 2015

Textbook: Chapter 1.3, 1.4

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Suggested Practice Exercise: Chapter 1.4 Exercise 1, 2, 10, 12, 13, 21, 30, 34, 45,

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Khan Academy Video: Matrix multiplication (part I)

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The right-hand side is a linear combination of the vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Example

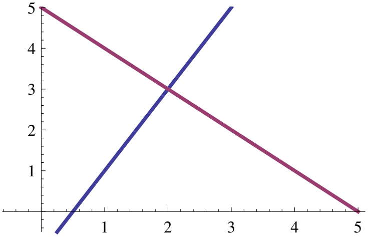
We can think of the linear system

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

in two different geometric ways. Recall unique solution: $x = 2, y = 3$.

Row picture

- Each equation defines a line in $\mathbb{R}^2$.



Row picture

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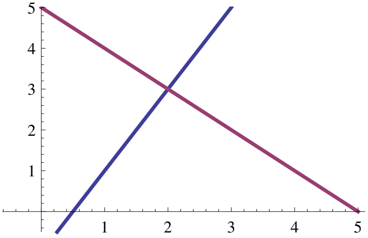
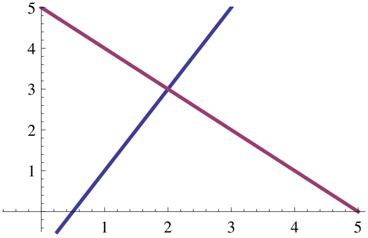
- Which points lie on the intersection of these lines?



Row picture

- Each equation defines a line in $\mathbb{R}^2$.
- Which points lie on the intersection of these lines?
- $(2, 3)$ is the (only) intersection of the two lines $2x - y = 1$ and $x + y = 5$.



Column picture

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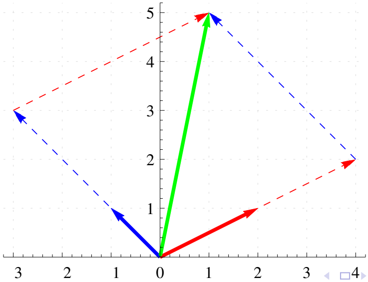

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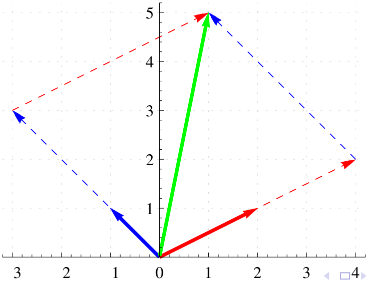

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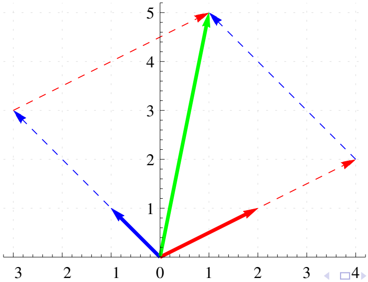
- Which linear combinations of $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$ produce $\begin{bmatrix}1\\5\end{bmatrix}$?
- (2, 3) are the coefficients of the (only) such linear combination.



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in different quantities. We encode how much we need in *production vectors*. So if the Doohickeys production vector

$v_D = \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix}$ then we need 10 units of Sugar, 5 units of Spice, 2 units of Everything Nice to produce one Doohickey.

Application: Industrial Espionage, Continued.

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$$B = c_D v_D + c_N v_N + c_W v_W$$

Application: Industrial Espionage, Continued.

More explicitly

$$B = c_D \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix} + c_N \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_W \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}.$$

Application: Industrial Espionage, Continued.

More explicitly

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So the consumption vector B is a **linear combination** of the production vectors.

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So the competitor sends a spy to the entrance of the factory and she writes down how many truck loads of raw materials enters. This determines the production vector B. Then the problem is to find the coefficients c_D, c_N, c_W, i.e., the number of Doohickeys, Nicknacks and Widgets produced.

Application: Industrial Espionage, Continued.

Problem

Suppose our spy observes

$$B = \begin{bmatrix} 14 \\ 12 \\ 7 \end{bmatrix} =$$

Application: Industrial Espionage, Continued.

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Suppose our spy observes

$$B = \begin{bmatrix} 14 \\ 12 \\ 7 \end{bmatrix} = c_D \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix} + c_N \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_W \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}.$$

Then what is the number c_D of Doohickeys produced? How to approach the problem?

Solution

Finding Linear Combinations is the same as solving Linear Systems.

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So what is c_D*?*

Matrices are like Numbers: Matrix Algebra.

Two ways to denote $m \times n$ matrix A (m rows, n column).

- In terms of the columns of A:

$$A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array} \right]$$

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- In terms of the entries of A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

More notation

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- **Main diagonal entries:** $a_{11}, a_{22}, \ldots, a_{mm}$ (only care about these when $m = n$)

Even more notation

- **Zero matrix:**

$$0 = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

Definition

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$ be $m \times n$-matrices and let r be a scalar.

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- $A + B$ is defined by

$$A + B = \begin{bmatrix} \mathbf{a}_1 + \mathbf{b}_1 & \mathbf{a}_2 + \mathbf{b}_2 & \ldots & \mathbf{a}_n + \mathbf{b}_n \end{bmatrix}$$

- Moreover, rA is defined as

$$rA = \begin{bmatrix} r\mathbf{a}_1 & r\mathbf{a}_2 & \ldots & r\mathbf{a}_n \end{bmatrix}$$

Example

Calculate

- $$\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} =$$

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- $$\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 8 & 3 \end{bmatrix}$$

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- $$10 \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} =$$

Example

Calculate

- $$\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 8 & 3 \end{bmatrix}$$
- $$10 \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 30 \\ 30 & 10 \end{bmatrix}$$

Theorem

Let A, B, and C be matrices of the same size, and let r and s be scalars.

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + 0 = A$
- $r(A + B) = rA + rB$
- $(r + s)\, A = rA + sA$
- $r\,(sA) = (rs)\, A$

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Define the product AB so that

$$A(B\mathbf{x}) = (AB)\mathbf{x}$$

Suppose A is $m \times n$ and B is $n \times p$ where

$$B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_p] \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

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$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \ldots + x_n\mathbf{b}_n$$

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$$A\left(B\mathbf{x}\right) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 + \ldots + x_n A\mathbf{b}_n$$

Example

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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$$A(B\mathbf{x}) = A \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix} = (x_1 + 2x_2) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (x_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$B\mathbf{x} = B\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$$

Compute $A(B\mathbf{x})$ using $B\mathbf{x} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$:

$$A(B\mathbf{x}) = A\begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix} = (x_1 + 2x_2)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (x_2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Compute $A(B\mathbf{x})$ using $A\left(B\mathbf{x}\right) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 + \ldots + x_n A\mathbf{b}_n$:

Example

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$B\mathbf{x} = B\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$$

Compute $A(B\mathbf{x})$ using $B\mathbf{x} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$:

$$A(B\mathbf{x}) = A\begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix} = (x_1 + 2x_2)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (x_2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Compute $A(B\mathbf{x})$ using $A\,(B\mathbf{x}) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 + \ldots + x_n A\mathbf{b}_n$:

$$A(B\mathbf{x}) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2$$

Example

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$B\mathbf{x} = B \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$$

Compute $A(B\mathbf{x})$ using $B\mathbf{x} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$:

$$A(B\mathbf{x}) = A \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix} = (x_1 + 2x_2) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (x_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Compute $A(B\mathbf{x})$ using $A\,(B\mathbf{x}) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 + \ldots + x_n A\mathbf{b}_n$:

$$A(B\mathbf{x}) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 = x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$B\mathbf{x} = B\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$$

Compute $A(B\mathbf{x})$ using $B\mathbf{x} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$:

$$A(B\mathbf{x}) = A\begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix} = (x_1 + 2x_2)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (x_2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Compute $A(B\mathbf{x})$ using $A\left(B\mathbf{x}\right) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 + \ldots + x_n A\mathbf{b}_n$:

$$A(B\mathbf{x}) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 = x_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Same answer!

Matrix Multiplication

Motto

Matrix Multiplication is Linear Combination!