Portfolio

MATH 476

Ahad Jiva

June 8, 2025

Exercise 1

Forward Contract Payoff

1. The payoff from a long position (buying the asset) in a forward contract is $S_T - K$.
2. The payoff from a short position (selling the asset) in a forward contract is $K - S_T$.

Exercise 2

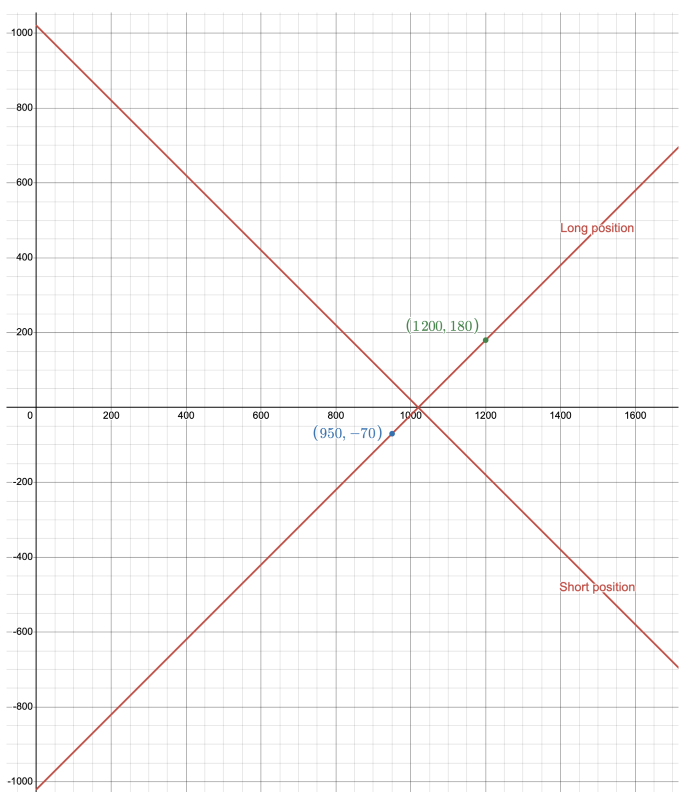
Forward Contract on Stock Index

We know the current price is $1000 and the 6-month forward price is $1020.

1. If the price is $950 in 6 months, the long position will lose $70 (950 - 1020).
2. If the price is $1200 in 6 months, the long position will gain $180 (1200 - 1020).

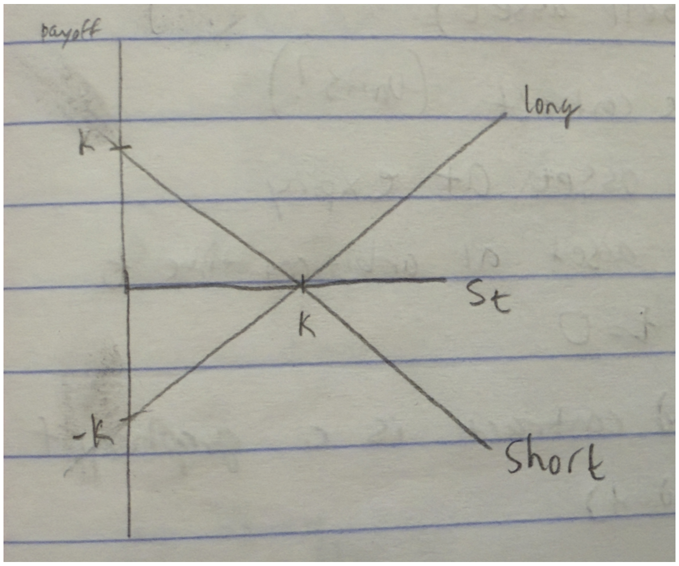
The forward contract allows for a profit if the value of the asset increases after 6 months, without having to actually own the asset.

Payoff Diagram:



Exercise 3

Payoff Diagrams for Forward Contract



Exercise 4

Forward Contract on Foreign Exchange
The bank agrees to a 6-month forward contract to purchase 1 million GBP in 6 months.

1. If the spot price is 1.3000 in 6 months, the bank will make $(1.3000 - 1.2230) \cdot 1000000 = \$77000$.

2. If the spot price is 1.2000 in 6 months, the bank will lose $(1.2000 - 1.2230) \cdot 1000000 = \$23000$.

Exercise 5

Forward Contract on Foreign Exchange
An investor enters into a short forward contract to sell 100,000 GBP for USD at 1.3000 USD per pound.

1. If the spot price is 1.2900 at the end of the contract, the short position gains $(1.3000 - 1.2900) \cdot 100000 = \$1000$.

2. If the spot price is 1.3200 at the end of the contract, the short position loses $(1.3000 - 1.3200) \cdot 100000 = \$2000$.

Exercise 6

Forward Contract on Foreign Exchange
A trader enters into a short forward contract to sell 100 million yen at \$0.0090 per yen.

1. If the spot price is 0.0084 at the end of the contract, the short position gains $(0.0090 - 0.0084) \cdot 100000000 = \$60000$.

2. If the spot price is 0.0101 at the end of the contract, the short position loses $(0.0090 - 0.0101) \cdot 100000000 = \$110000$.

Exercise 7

ECO with T = 10 days, K = \$250

1. If $S_T = \$270$, then the holder of the ECO will exercise the option and the payoff will be $270 - 250 = \$20$
2. If $S_T = \$230$, then the holder of the ECO will let the option expire worthless. The payoff is zero and the holder of the option only loses the option premium.

Exercise 8

Expected Value $= \frac{1}{2} \cdot \$20 + \frac{1}{2} \cdot \$0 = \$10$

Exercise 9

Suppose that an investor did indeed pay c = 10 dollars for an ECO.

1. If $S_T = \$270$, then the payoff is \$20. The net profit is $20 - 10 = \$10$. In this case the net profit is 100% of the initial cost.
2. If $S_T = \$230$, then the payoff is \$0. The net profit is $0 - 10 = -\$10$. In this case, the loss is 100% of the initial cost.

Exercise 10

Suppose that the investor purchases the stock for \$250 outright instead of buying an option.

1. If $S_T = \$270$, then the profit is \$20, which is 8% of the initial cost.
2. If $S_T = \$230$, then the profit is -\$20, which is also 8% of the initial cost.

Compared to buying a call option, purchasing the stock outright has less risk in terms of potential percentage gained or lost. However, the initial cost is much higher.

Exercise 11

EPO with 100 shares of underlying stock, $K = \$70$, current price is \$65. If $S_T = \$55$, then the holder will exercise the option. The payoff will be $100 \cdot (70 - 55) = \$1500$.

Exercise 12

ECO with $K = \$100$, 100 underlying shares, $c = \$500$, $S_T = \%102$.

1. Option 1: Exercise option. The per-share gain is \$2, so the total gain is \$200. Subtracting the \$500 initial cost, the investor would lose \$300.
2. Option 2: Let option expire worthless. The total gain is \$0 and the option cost \$500, so the total loss is \$500.

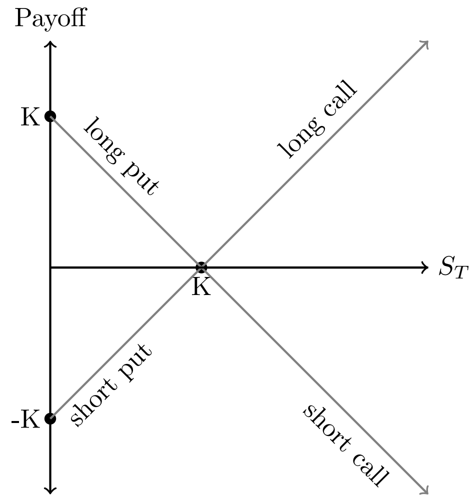
In this case, exercising the option would let the investor reduce their losses.

Exercise 13

1. Long position in ECO: If $S_T \leq K$, the long position lets the option expire worthless. The payoff is 0. If $S_T > K$, the long position exercises the option. The payoff is $S_T - K$. Thus, the payoff is $\max(S_T - K, 0)$.
2. Short position in ECO: If $S_T \leq K$, then the long will let the option expire and the payoff to the short is 0. If $S_T > K$, the long will exercise and the payoff to the short is $K - S_T$. Thus, the payoff is $\min(K - S_T, 0)$.

3. Long position in EPO: If $S_T \leq K$, then the payoff will be $K - S_T$. If $S_T > K$, then the payoff will be 0.
4. Short position in EPO: If $S_T \leq K$, then the payoff is $S_T - K$. If $S_T > K$, then the payoff will be 0. So the payoff is $\min(S_T - K, 0)$.

Exercise 14

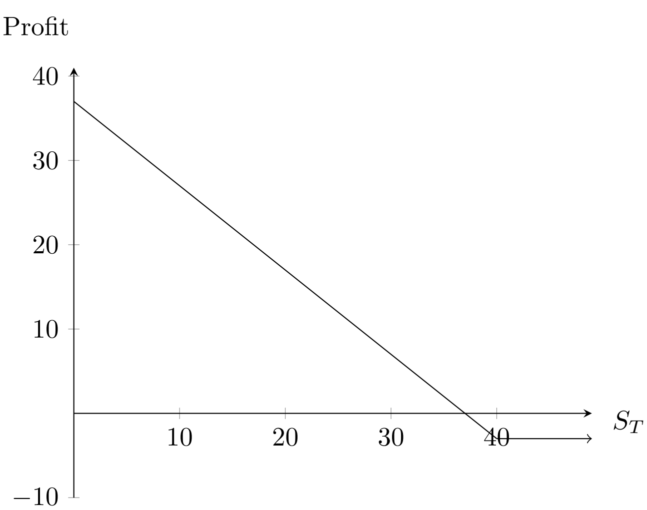


Exercise 15

Investor buys EPO for \$3, current price is \$42, and $K = \$40$. The profit is calculated as

$$\begin{cases} 40 - S_T - 3 & S_T < 40 \\ -3 & S_T \geq 40 \end{cases}$$

Since we want the trade to be profitable, we want $40 - S_T - 3 > 0$, or $S_T < 37$. The option will be exercised if $S_T \leq 40$, since that means the payoff will be positive (but not necessarily the profit). The profit diagram is as follows:

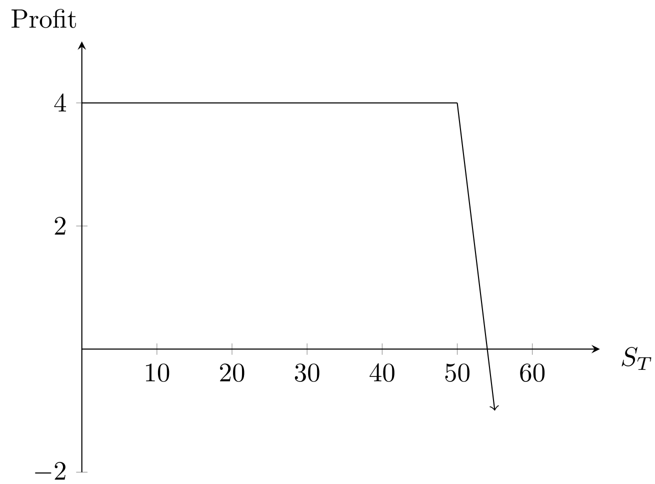


Exercise 16

Investor sells ECO for \$4, $K = \$50$, current price is \$47. The profit is calculated as

$$\begin{cases} 0 + 4 & S_T \leq 0 \\ 50 - S_T + 4 & S_T > 50 \end{cases}$$

Since we want the trade to be profitable, we want $S_T \leq 54$. The option will be exercised when $S_T < 50$, since this is when the payoff is acceptable. The profit diagram is as follows:



Exercise 17

The investor has a short position on an ECO and a long position on an EPO. There are two cases to consider:

1. The long position will exercise if $S_T > K$. Thus, the investor will have to sell to the long position for K. At time $t = T$, the payoff is

   $$\begin{cases} 0 & S_T \leq K \\ K - S_T & S_T > K \end{cases}$$

   or $-\max\{S_T - K, 0\}$.

2. The investor will exercise their long position on the EPO if $S_T < K$. At time $t = T$ the payoff is

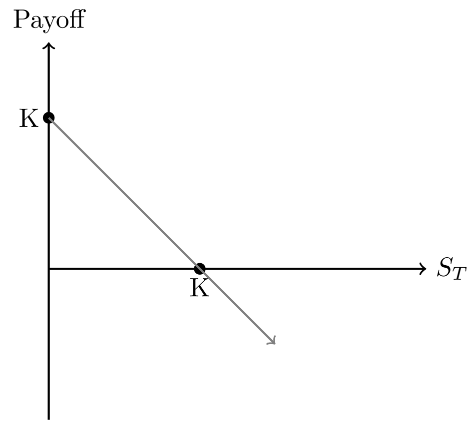
   $$\begin{cases} K - S_T & S_T < K \\ 0 & S_T \geq K \end{cases}$$

   or $-\max\{K - S_T, 0\}$.

Then the overall payoff at expiry will be

$$-\max\{S_T - K, 0\} + \max\{K - S_T, 0\} = \begin{cases} 0 + K - S_T & S_T < K \\ K - S_T + 0 & S_T \geq K \end{cases} = K - S_T.$$

The payoff diagram looks like the following:



Exercise 18

1. Long Position on an ECO: $K = \$45$, $c = \$3$, expiry $t = T$. The profit is represented as

$$\begin{cases} S_T - 45 - 3 & S_T \geq 45 \\ -3 & S_T < 45 \end{cases}$$

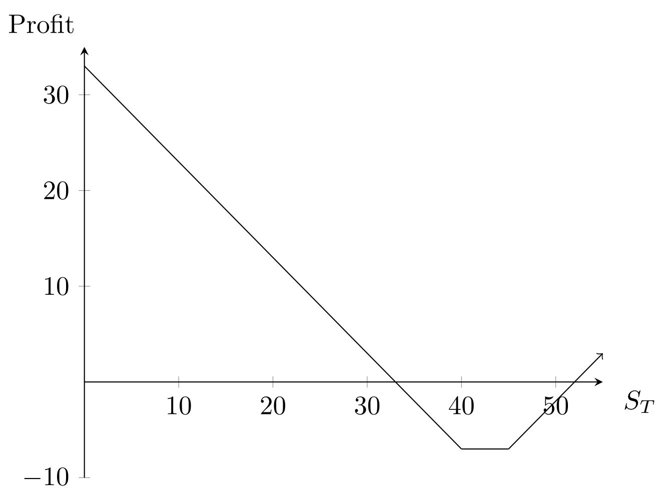
2. Long Position on an EPO: $K = \$40$, $c = \$4$, expiry $t = T$. The profit is represented as

$$\begin{cases} 40 - 4 - S_T & S_T < 40 \\ -4 & S_T \geq 40 \end{cases}$$

Then the net profit is

$$\begin{cases} 36 - S_T - 3 & S_T < 40 \\ -3 - 4 & 40 \leq S_T \leq 45 \\ S_T - 48 - 4 & S_T > 45 \end{cases}$$

The profit diagram is as follows:



Exercise 19

An American option will always be worth as much as a European option on the same asset with the same strike price and exercise date because if the holder of the American option doesn't exercise until the expiry date, the option is no different from a European option. Having the right to exercise the option before the expiry date is an additional right that the American option has, and as such it must be at least as valuable as a similar European option.

Exercise 20

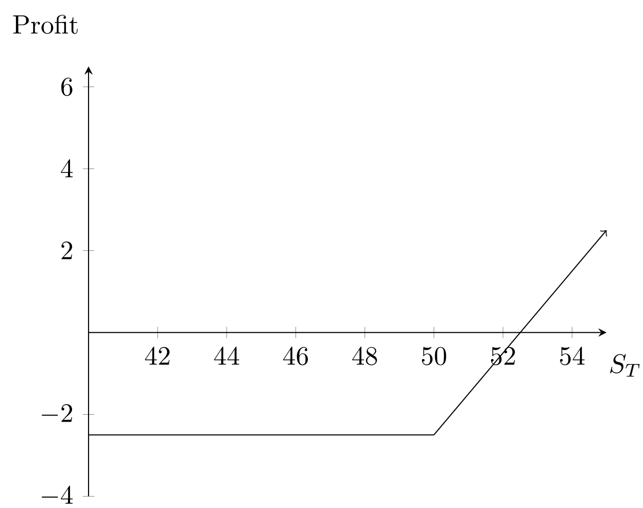
Variable	*European call*	*European put*	*American call*	*American put*
Current stock price	+	−	+	−
Strike price	−	+	−	+
Time to expiration	?	?	+	+
Volatility	+	+	+	+
Risk-free interest rate	+	−	+	−

Exercise 21

The trader will make a profit when the long position in the contract exercises, and the profit will be $S_T - K - c$, which in this case is $S_T - 34$. So the investor will make a profit as long as $S_T \geq 34$.

Exercise 22

The holder of the option will make a profit if the price of the stock at maturity is greater than or equal to \$52.5. The option will be exercised if the price at maturity is greater than or equal to \$50, which means that the condition for exercising does not necessarily result in a profit.



Exercise 23

$K = 20$, $S_0 = 18$, $c = 2$. Suppose that $S_T = 25$. If the option is held until September, at which the stock price is \$25, then the long position in the call will exercise the option. Then the investor will have to buy a share for \$25 and sell it to the long position for \$20, taking a loss of \$5. Since the trader wrote the option and sold it for \$2, then the total cash flow is a loss of \$3.

Exercise 24

1. Trader A: forward contract, $K = 1000$, so profit is $S_T - 1000$.
2. Trader B: call option, $c = 100$, $K = 1000$, so profit is

$$\begin{cases} -100 & S_T \leq K \\ S_T - 1100 & S_T > K \end{cases}$$

So if $S_T > K$, then Trader A does better, and if $S_T \leq K$, then Trader A loses less money until $S_T = 900$, below which Trader A will lose more money than Trader B.

Exercise 25

1. Purchasing shares outright: $S_0 = 316.50$, so upfront cost is \$31,650. If $S_T = 400$, then the per share profit is \$83.50, so the total profit is \$8,350. If $S_T = 300$, then the per share loss is \$16.50, so the total loss is \$1,650.

2. Purchasing call options: $c = 21.70$, so the upfront cost is \$2,170. If $S_T = 400$, then the profit is $-2170 + (400 - 320) \cdot 100 = 5,830$. If $S_T = 300$, we can let the options expire worthless. So the loss is \$2,170.

Exercise 26

On May 21, 2020, an investor owns 100 Apple shares. The investor is comparing two alternatives to limit risk. The first involves buying one December put option contract with a strike price of \$290. The second involves instructing a broker to sell the 100 shares as soon as Apple's price reaches \$290. Discuss the advantages and disadvantages of the two strategies.

Solution: Let's look at the two possible choices separately:

1. Buying a put option: $K = \$290, c = \$21.30$. Then the profit is

$$\begin{cases} -(21.30)(1000) & S_T > K \\ K - S_T - 2130 & S_T \leq K \end{cases}$$

2. Use a stop-loss to sell AAPL at \$290. This option is less risky, but it could lose out on profits if the market is volatile.

Using a stop-loss is a less risky and simpler strategy that will ensure an investor will receive an exact amount for their shares. On the other hand, using a put option could allow the investor to make a profit in the future if the stock price is predicted to decline. Given that the investor holds shares at this instant, using a put option allows them to spend a portion of the value of the shares as "insurance", bounding the maximum possible loss. Also, if the market is volatile, the stop-loss could be triggered before a breakout rally, causing the investor to lose out on potential profits.

Exercise 27

Find the payoff from a bull spread.

1. Case 1: $S_T > K_2$. Then the payoff is $(S_T - K_1) + (K_2 - S_T) = K_2 - K_1$.

2. Case 2: $K_2 > S_T > K_1$. Then the payoff is $(0 + S_T - K_1) = S_T - K_1$.

3. Case 3: $S_T < K_1$. Then the payoff is zero.

So a trader that uses a bull spread is hoping that the stock price will increase.

Exercise 28

An investor buys for \$3 a 3-month European call with a strike price of \$30 and sells for \$1 a 3-month European call with a strike price of \$35. Find the profit from this bull spread in each of the following cases:

1. $S_T = \$25$. We know that $K_1 = \$30$ since it is the long position and that $K_2 = \$35$ since it is the short position. Then $S_T < K_1 < K_2$. Using the payoff form a bull spread from exercise 27, we know the payoff is 0. Then the profit is $-\$2$.

2. $S_T = \$34$. Then $K_1 < S_T < K_2$. So we know that the payoff is $S_T - K_1$. So the payoff is \$4. Then the profit is \$2.

3. $S_T = \$40$. Then $S_T > K_2$. So the payoff is $K_2 - K_1$, or \$5. Then the profit is \$3.

Exercise 29

Find the payoff from a bear spread.

1. Case 1: $S_T < K_2 < K_1$. Then the payoff is $(K_1 - S_T) + (S_T - K_2) = K_1 - K_2$.
2. Case 2: $K_2 < S_T < K_1$. Then the payoff is $K_1 - S_T$.
3. Case 3: $S_T > K_1$. Then the payoff is zero.

So a trader using a bear spread is hoping that the stock price will decrease.

Exercise 30

Long EPO, $K_1 = 35$, $c = 3$. Short EPO, $K_2 = 30$, $c = 1$

1. $S_T = 25$. Then the payoff is $K_1 - K_2 = 35 - 30 = 5$. The profit is $5 - 3 + 1 = \$3$.
2. $S_T = 34$. Then the payoff is $K_1 - S_T = 35 - 34 = 1$. The profit is $1 - 2 = -\$1$.
3. $S_T = 40$. Then the payoff is 0, and the profit is -\$2.

Exercise 31

Find the payoff from a straddle (long on an ECO and long on an EPO, both same strike price and expiry). We know the payoff from the long ECO is

$$\begin{cases} S_T - K & S_T > K \\ 0 & S_T \leq K \end{cases}$$

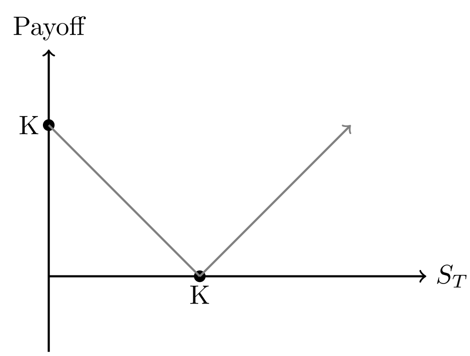
and the payoff from the long EPO is

$$\begin{cases} K - S_T & S_T < K \\ 0 & S_T \geq K \end{cases}$$

Then the total payoff is

$$\begin{cases} S_T - K & S_T > K \\ K - S_T & S_T \leq K \end{cases}$$

This can be represented with the following payoff diagram



Exercise 32

Long ECO with $K = 60$, $c = 6$, and long EPO with $K = 60$, $c = 4$

1. The profit from the ECO is $\begin{cases} -6 & S_T \leq 60 \\ S_T - 66 & S_T > 60 \end{cases}$
2. The profit from the EPO is $\begin{cases} 56 - S_T & S_T \leq 60 \\ -4 & S_T > 60 \end{cases}$

Then the total profit is

$$\begin{cases} 50 - S_T & S_T \leq 60 \\ S_T - 70 & S_T > 60 \end{cases}$$

Thus there will be a loss when $50 < S_T < 70$.

Exercise 33

Initial = 150, $r = 0.08$, find value after 20 days (365 days in a year). We use the usual formula: $V(t) = (1 + tr)P$. Plugging in the given numbers, we get \$150.66.

Exercise 34

We need \$1000 after 3 months with annual interest at 8%. Using the usual formula: $V(t) = (1 + tr)P$. We plug in the given values and isolate P to get \$980.39 of initial capital.

Exercise 35

Initial deposit of \$100, annual interest of 10%, find value after two years when compounded

1. Annually: Then we have $V(2) = (1 + \frac{10}{1})^{2\cdot 1} \cdot 100 = \$121$
2. Monthly: Then we have $V(2) = (1 + \frac{10}{12})^{2\cdot 12} \cdot 100 = \$122.03$

Exercise 36

Show if $m < k$, then

$$(1 + \tfrac{r}{m})^m < (1 + \tfrac{r}{k})^k$$

Proof: Let $f(x) = (1 + \frac{r}{x})^x$ for $x > 0$. It is enough to show that $f(x)$ is increasing. Take the natural log of both sides and use log rules to get:

$$\ln(f(x)) = \ln((1 + \tfrac{r}{x})^x) = x \ln(1 + \tfrac{r}{x})$$

Now take the derivative of both sides and simplify:

$$\begin{gathered} \tfrac{1}{f(x)} f'(x) = \ln(1 + \tfrac{r}{x}) + x(\tfrac{1}{1+\frac{r}{x}})(-\tfrac{r}{x^2}) \\ f'(x) = f(x)\big(\ln(1 + \tfrac{r}{x}) - \tfrac{r}{x+r}\big) \end{gathered}$$

Observe that $f(x) > 0$, so we just need to show that the other term is positive.

$$\begin{gathered} \ln(1 + \tfrac{r}{x}) - \tfrac{r}{x+r} > 0 \\ (x + r) \cdot \ln(1 + \tfrac{r}{x}) > r \\ e^{x+r} + (1 + \tfrac{r}{x}) > e^{x+r} \end{gathered}$$

We know this must be true, therefore $f'(x)$ is indeed greater than zero. So $f(x)$ is indeed increasing and we are done. $\square$

Exercise 37

Solve $\frac{dV}{dt} = r \cdot V$

We can use separation of variables to solve this differential equation.

$$\frac{dV}{V} = r \cdot dt$$

Integrate both sides:

$$\int \frac{dV}{V} = \int r \cdot dt$$
$$\ln(|v|) = rt + C$$
$$V = e^{rt+C} = Ce^{rt}$$

Now we can use our initial condition to find a specific solution.

$$V(0) = 0$$
$$V(0) = Ce^0$$
$$P = C$$

Thus $V = Pe^{rt}$.

Exercise 38

Continous compounding is given by $V(t) = \lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^{tm} P$.

1. Show that $e = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$. To start, we can take the natural log of both sides:

$$\ln(e) = \ln(\lim_{x\to\infty}(1 + \frac{1}{x})^x)$$
$$1 = \lim_{x\to\infty}(\ln(1 + \frac{1}{x})^x)$$
$$1 = \lim_{x\to\infty}(x\ln(1 + \frac{1}{x}))$$
$$1 = \lim_{x\to\infty}\left(\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}\right)$$

   Now we can use L'Hopital's Rule to evaluate this limit:

$$1 = \lim_{x\to\infty}\left(\frac{\frac{1}{1+\frac{1}{x}}(\frac{-1}{x^2})}{(\frac{-1}{x^2})}\right)$$
$$1 = \lim_{x\to\infty}(\frac{1}{1+\frac{1}{x}}) = \frac{1}{1} = 1$$

   Thus we achieve the desired result.

2. Now we want to obtain a closed form expression for $V(t)$. By using the proof above, we can assert that

$$e^r = \lim_{x\to\infty}(1 + \frac{r}{x})^x$$

   We know that $V(t) = \lim\limits_{m\to\infty}\left(1 + \dfrac{r}{m}\right)^{tm} P$. We can rewrite this as

$$\left(\lim_{m\to\infty}(1 + \frac{r}{m})^m\right)^t \cdot P =$$
$$e^{rt} \cdot P = Pe^{rt}$$

   Thus the closed form for $V(t)$ is $V(t) = Pe^{rt}$, and this matches our result from solving the differential equation.

Exercise 39

Prove that $c \leq S_0$. Suppose not, for the sake of contradiction. Then $S_0 < c$. Consider the portfolio where we are short one ECO and long one share of stock. Then the initial cash flow is $c - S_0$, which is positive. Consider the two cases:

1. $S_T \leq K$. Then the long position in the ECO will not exercise. The payoff to the investor is c, which is positive.

2. $S_T > K$. Then the long position will exercise the ECO. Then the payoff to the investor is $K - S_T + S_T = K$, which is positive.

Thus we have an arbitrage opportunity, which is a contradition. Thus $c \leq S_0$. $\square$

Exercise 40

Prove that $p \leq Ke^{-rT}$. Suppose not, for the sake of contradiction. Then $p > Ke^{-rT}$. Consider the portfolio where we are short one EPO and have Ke^{-rT} in cash. Then the initial cash flow is positive.

1. Case 1: $S_T < K$. Then the payoff is $p - Ke^{-rT}$, which is positive.
2. Case 2: $S_T \geq K$. Then the payoff is $(p - Ke^{-rT}) + K$, which is also positive.

Thus we have an arbitrage opportunity, which is a contradiction. Thus $p \leq Ke^{-rT}$. $\square$

Exercise 41

Prove that $c \geq S_0 - Ke^{-rT}$. Consider the following portfolios:

1. Portfolio A: one ECO and cash equal to Ke^{-rT}
2. Portfolio B: one share of the underlying stock

At expiry, the value of each portfolio is

1. Portfolio A: $\max(S_T - K, 0) + Ke^{-rT}e^{rT}$
2. Portfolio B: S_T.

Note that in all cases, the value of Portfolio A is greater than Portfolio B. So by the No-Arbitrage property, $V_0(A) \geq V_0(B)$. Then $c + Ke^{-rT} \geq S_0$, so $c \geq S_0 - Ke^{-rT}$. $\square$

Exercise 42

Prove that $p \geq Ke^{-rT} - S_0$. Consider the portfolio where we are long one EPO, long one share, and borrow Ke^{-rT} in cash. The initial cash flow is therefore positive. Suppose, for the sake of contradiction, that $p < Ke^{-rT} - S_0$. Then at expiry, the payoff will be $\max(K - S_T, 0) + S_T - Ke^{-rT}e^{rT}$, which can be simplified to $\max(0, S_T - K)$, which we know is postive. Thus we have a contradiction to the no arbitrage principle. Thus $p \geq Ke^{-rT} - S_0$. $\square$

Exercise 43

Find the lower bound of a 2-month EPO with $S_0 = 58$, $K = 65$, and $r = 0.05$. We use the formula from above for a put option: $p \geq Ke^{-rT} - S_0$. Plugging in the given values, we find that the lower bound for this put is $6.46.

Exercise 44

Show that $c + Ke{-}rT = p + S_0$, i.e. show put-call parity.

Consider the following portfolios:

1. Portfolio A: Long one ECO and cash equal to Ke^{-rT}
2. Portfolio B: Long one EPO and one share of stock

At expiry, the portfolios will be worth:

1. Portfolio A: $\max(S_T - K, 0) + Ke^{-rT}e^{-rT}$, which can be simplified to $\max(S_T, K)$.
2. Portfolio B: $\max(K - S_T, 0) + S_T$, which can be simplified to $\max(S_T, K)$.

Thus at expiry, both the portfolios are worth the same. By no-arbitrage, they must be worth the same at $t = 0$, thus $c + Ke{-}rT = p + S_0$. $\square$

Exercise 45

$S_0 = 19$, $c = 1$, $K = 20$, $r = 0.04$, expiry in 3 months means $T = 1/4$. We can use put-call parity to calculate the price of a put option with same strike price and expiry.

$$\begin{gathered} c + Ke^{-rT} = p + S_0 \\ 1 + 20e^{-0.04 \cdot 0.25} = p + 19 \\ p = 1 + 20e^{-0.04 * 0.25} - 19 \\ p = 1.80 \end{gathered}$$

Exercise 46

$S_0 = 130$, expiry in one year means $T = 1$, $c = 20$, $p = 5$, $K = 120$. We can use put-call parity to calculate the risk-free interest rate.

$$\begin{gathered} c + Ke^{-rT} = p + S_0 \\ 20 + 120e^{-r \cdot 1} = 5 + 130 \\ e^{-r} = \frac{5+130-20}{120} \\ e^{r} = \frac{120}{115} \\ r = \ln(\frac{120}{115}) = 0.0426 \end{gathered}$$

Thus the risk-free interest rate is 4.26%.

Exercise 47

$S_0 = 31$, $c = 3$, $p = 2.25$, $K = 30$, $T = 0.25$, $r = 0.1$. Note that put-call parity does not hold here:

$$\begin{gathered} c + Ke^{-rT} = p + S_0 \\ 3 + 30e^{-0.1 \cdot 0.25} = 2.25 + 31 \\ 32.26 \neq 33.25 \end{gathered}$$

Thus we should be able to construct an arbitrage opportunity. Consider a portfolio where we buy the call option and short-sell the put option and the stock. At $t = 0$, the cash flow is

$$-c + p + S_0 = \$30.25$$

Thus we have positive cash flow at $t = 0$. We can then invest this at the risk-free interest rate. At expiry, this will be worth \$31.02. At expiry, we have two cases.

1. $S_T \leq 30$. Then we let the ECO expire, the EPO will be exercised, and we return the stock we shorted. The payoff will be $0 - 30 = -30$, and since we started with \$31.02, we have \$1.02 profit.

2. $S_T > 30$. Then we exercise the ECO, the EPO expires worthless, and we return the stock we shorted. The payoff is $-K + S_T - S_T = -30$. Since we started with \$31.02 we have \$1.02 profit.

Thus in all cases, we make a profit with positive initial cash flow, and this was an arbitrage opportunity. $\square$

Exercise 48

$p = 2.50$, $S_0 = 47$, $r = 0.06$, $K = 50$. What opportunities are there for an arbitrageur?

We start with the put-call parity equation: $c + Ke^{-rT} = p + S_0$. Plugging in the given values, we reach $c + 50e^{\frac{-0.06}{12}} = 49.5$, which means that $c \geq 0$. Thus an arbitrageur should go long in $c + Ke^{-rT}$ and short in $p + S_0$. Specifically, they should borrow \$49.50 at 6% interest for a month, go long in one share of the stock, and go long in a put option. In all cases, there will be positive initial cash flow and positive profit at expiry.

Exercise 49

Put-call parity will not hold for American options because they can be exercised any time before the expiry date. This means that with enough volatility, we could violate the put-call parity equation without violating the no arbitrage principle.

Exercise 50

Prove that it is never optimal to exercise an American call option early.

It suffices to show that $c > C$, since we have already argued previously that $C \geq c$ (see Ex. 19). Suppose not, for the sake of contradiction. Then $C > c$ (here C is the price of an ACO, c is the price of an ECO). Conside the portfolio where we are short one ACO, long one ECO, both with same expiry and strike price. Then our initial cash flow is positive. Consider the following cases:

1. Case 1: ACO expires worthless. Then the ECO also expires worthless, and our payoff is zero. Our profit is positive.

2. Case 2: ACO is exercised. Then the ECO is also exercised. So the payoff is zero, and our profit is positive.

In all cases, we make a positive profit with positive initial cash flow. This is a contradiction to the no-arbitrage principle. Thus $c = C$.

Exercise 51

Show that it can be optimal to exercise an APO before expiry. Suppose that S_0 is 100, $K = 90$. Consider the scenario where the stock price drops to \$89 right after initiation. If we exercise the APO, we get a payoff of $K - S_T$, which is \$1. We can then invest this amount in the risk-free interest rate, which if high enough, will net us more money than waiting for option expiry.

Exercise 52

As the risk free interest rate increases and volatility decreases, exercising an APO early becomes more attractive because the higher interest rate means we can use the money from the exercised option to invest in a risk free asset and get a good return. Also, the low volatility means that there is little chance of a large price movement that would make exercising early attractive.

Exercise 53

Show that $S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$. We split this into two inequalities and prove them separately.

1. Prove that $S_0 - K \leq C - P$. Suppose not, for the sake of contradiction, i.e. $S_0 - K > C - P$. Consider the portfolio where we are long one ACO, short one APO and one share of stock. Note that the initial cash flow is positive. If $S_T > K$, then the payoff at expiry is $Ke^{rT} - K$, which is positive. If $S_T \leq K$, then the payoff is also $Ke^{rT} - K$, which is positive. This is a contradiction to the no arbitrage principle, thus $S_0 - K \leq C - P$.

2. Prove that $C - P \leq S_0 - Ke^{-rT}$. We know that $C = c$ and $P \geq p$, because it is never optimal to exercise an American option early. If we make this substitution, we get $c - p = S_0 - Ke^{-rT}$, which is put-call parity (we know this is true). Since $P \geq p$, we conclude that $C - P \leq S_0 - Ke^{-rT}$.

Thus we have proved both inequalities. $\square$

Exercise 54

We have three call options with prices $c(K_1), c(K_2), c(K_3)$ with $K_1 < K_2 < K_3$. We also have three put options with prices $p(K_1), p(K_2), p(K_3)$. All of these options have the same expiry.

1. Show $c(K_1) \geq c(K_2)$. Suppose not for the sake of contradiction, ie $c(K_1) < c(K_2)$. Consider the portfolio where we are short in $c(K_2)$ and long in $c(K_1)$. The initial cash flow is $c(K_2) - c(K_1) > 0$, and let CF_0 represent this amount.

   (a) Case 1: $S_T < K_1 < K_2$. Then none of the call options will be exercised. The payoff is zero and the profit is $0 + CF_0 \cdot e^{rT}$, which is positive.

(b) Case 2: $K_1 < S_T < K_2$. Then the $c(K_1)$ option will be exercised, which is our long position. So the payoff is $S_T - K_1$ and the profit is $S_T - K_1 + CF_0 \cdot e^{rT}$, which is positive.

(c) Case 3: $K_1 < K_2 < S_T$. Then both options will be exercised. The payoff will be $S_T - K_1 - S_T + K_2 = K_2 - K_1$, which is positive, and thus the profit is $K_2 - K_1 + CF_0 \cdot e^{rT}$, which is positive.

Thus in all cases, we have a risk free profit with no initial investment, ie an arbitrage opportunity. Contradition. Thus $c(K_1) \geq c(K_2)$. $\square$

2. Show $p(K_2) \geq p(K_1)$. Suppose not for the sake of contradiction, ie $p(K_2) < p(K_1)$ or $p(K_1) - p(K_2) > 0$. Consider the portfolio where we are short in $p(K_1)$ and long in $p(K_2)$. Then the initial cash flow is $p(K_1) - p(K_2) > 0$, and let CF_0 represent this amount.

   (a) Case 1: $S_T < K_1 < K_2$. Then both put options will be exercised. The payoff is $S_T - K_1 + K_2 - S_T = K_2 - K_1$. So the profit is $K_2 - K_1 + CF_0 \cdot e^{rT}$, which is positive.

   (b) Case 2: $K_1 < S_T < K_2$. Then the $p(K_2)$ option will be exercised. The payoff is $K_2 - S_T$ and the profit is $K_2 - S_T + CF_0 \cdot e^{rT}$, which is positive.

   (c) Case 3: $K_1 < K_2 < S_T$. Then none of the put options will be exercised and the payoff is zero. Then the profit is $CF_0 \cdot e^{rT}$, which is positive.

   Thus in all cases, we have a risk free profit with no initial investment, ie an arbitrage opportunity. Contradiction. Thus $p(K_2) \geq p(K_1)$. $\square$

3. Show $c(K_1) - c(K_2) \leq K_2 - K_1$. Suppose not for the sake of contradiction, ie $c(K_1) - c(K_2) > K_2 - K_1$. Consider the portfolio where we are short in $c(K_1)$ and long in $c(K_2)$, and also we hold cash equivalent to $(K_1 - K_2) \cdot e^{-rT}$. Then the initial cash flow is $c(K_1) - c(K_2) + (K_1 - K_2) \cdot e^{-rT}$, which we denote as CF_0, which is positive.

   (a) Case 1: $S_T < K_1 < K_2$. Then neither call option is exercised. The payoff is zero, and the profit is $CF_0 \cdot e^{rT}$.

   (b) Case 2: $K_1 < S_T < K_2$. Then $c(K_1)$ is exercised. The payoff is $K_1 - S_T$, and thus the profit is $K_1 - S_T + CF_0 \cdot e^{rT}$, which is positive.

   (c) Case 3: $K_1 < K_2 < S_T$. Then both call options are exercised. The payoff is $K_2 - K_1$, which when added to $CF_0 \cdot e^{rT}$ to get profit is positive.

   Thus in all cases, we have a risk free profit with no initial investment, ie an arbitrage opportunity. Contradition. Then $c(K_1) - c(K_2) \leq K_2 - K_1$. $\square$

4. Show $p(K_2) - p(K_1) \leq K_2 - K_1$. Suppose not for the sake of contradiction, ie $p(K_2) - p(K_1) > K_2 - K_1$. Consider the portfolio where we are short in $p(K_1)$ and long in $p(K_2)$, and also we hold cash equivalent to $(K_1 - K_2) \cdot e^{-rT}$. Then the initial cash flow is $p(K_1) - p(K_2) + (K_1 - K_2) \cdot e^{-rT}$, denoted as CF_0, which is positive.

   (a) Case 1: $S_T < K_1 < K_2$. Then both put options will be exercised. The payoff will be $K_2 - K_1$, which when added to $CF_0 \cdot e^{rT}$ to get profit, gives us a positive value.

   (b) Case 2: $K_1 < S_T < K_2$. Then $p(K_2)$ is exercised. The payoff will be $K_2 - S_T$, which is positive. When added to $CF_0 \cdot e^{rT}$, we get a positive profit.

   (c) Case 3: $K_1 < K_2 < S_T$. Then neither put option will be exercised. The payoff is zero and the total profit is $CF_0 \cdot e^{rT}$ which is positive.

   Thus in all cases we get a risk free profit will no initial investment, ie an arbitrage opportunity. Contradiction. Then $p(K_2) - p(K_1) \leq K_2 - K_1$. $\square$

Exercise 55

Convexity: $c(K_1), c(K_2), c(K_3)$ ECO prices, $K_1 < K_2 < K_3$.

1. Show that $K_2 = \lambda K_1 + (1-\lambda)K_3$, ie $\exists \lambda \in \mathbb{R}$ such that $0 < \lambda < 1$. We can use algebra to obtain $\lambda = \frac{K_3 - K_2}{K_3 - K_1}$ Since we know $K_3 \neq K_1$ and $K_3 \neq K_2$, we know λ is not zero and we don't have to worry about zero denominators. Since $K_1 < K_2$, $K_3 - K_1 > K_3 - K_2$. Then $\lambda < 1$. Thus $0 < \lambda < 1$ as desired. $\square$

2. Show with λ from part a that $c(K_2) \leq \lambda c(K_1) + (1-\lambda)c(K_3)$. Suppose not, for the sake of contradiction. Then $c(K_2) > \lambda c(K_1) + (1-\lambda)c(K_3)$. Consider the portfolio where we short-sell $c(K_2)$, long in $\lambda c(K_1) + (1-\lambda)c(K_3)$. The initial cash flow is $c(K_2) - \lambda c(K_1) - (1-\lambda)c(K_3) > 0$, and we denote this value as CF_0.

   (a) Case 1: $S_T \leq K_1 < K_2 < K_3$. None of the options are exercised. Then the profit is $0 + CF_0 e^{rT}$, which is positive.

   (b) Case 2: $K_1 < S_T < K_2 < K_3$. Then $\lambda c(K_1)$ is exercised. The profit is $\lambda(S_T - K_1) + CF_0 e^{rT}$, which is positive.

   (c) Case 3: $K_1 < K_2 < S_T < K_3$. Then $\lambda c(K_1)$ and $c(K_2)$ are exercised. The profit is $K_2 - \lambda K_1 - (1 - \lambda)S_T + CF_0 e^{rT}$, which is positive.

   (d) Case 4: $K_1 < K_2 < K_3 < S_T$. Then all options are exercised. The profit is $0 + CF_0 e^{rT}$, which is positive.

   Thus in all cases we get a positive profit from a positive initial cash flow, contradicting the no arbitrage principle. Thus $c(K_2) \leq \lambda c(K_1) + (1-\lambda)c(K_3)$. □

Exercise 56

$c(K_1), c(K_2), c(K_3)$ ECO prices, $K_1 < K_2 < K_3$. Also, $K_3 - K_2 = K_2 - K_1$. All options have the same maturity. Show that $c(K_2) \leq \frac{c(K_1)+c(K_3)}{2}$.

Consider the portfolio where we are long in $c(K_1), c(K_3)$ and short in 2 contracts of $c(K_2)$. The initial cash flow is $-c(K_1) - c(K_3) + 2c(K_2)$, which is positive. We will denote this value as CF_0.

1. Case 1: $S_T < K_1 < K_2 < K_3$. None of the call options will be exercised. The payoff is zero and the profit is $CF_0 e^{rT}$, which is positive.

2. Case 2: $K_1 < S_T < K_2 < K_3$. Then $c(K_1)$ will be exercised. The profit will be $S_T - K_1 + CF_0 e^{rT}$, which is positive.

3. Case 3: $K_1 < K_2 < S_T < K_3$. Then $c(K_1)$ and $c(K_2)$ will be exercised. The profit will be $(K_2 - K_1) - (S_T - K_2) + CF_0 e^{rT}$, which is positive.

4. Case 4: $K_1 < K_2 < K_3 < S_T$. Then all options will be exercised and the profit will be $0 + CF_0 e^{rT}$, which is positive.

Thus in all cases we get positive profit with a positive initial cash flow, which means the no arbitrage principle is violated. Thus $c(K_2) \leq \frac{c(K_1)+c(K_3)}{2}$ as desired. □

Exercise 57

Same as Ex. 56 but for put options, ie show that $p(K_2) \leq \frac{p(K_1)+p(K_3)}{2}$. Consider the portfolio where we are long in $p(K_1), p(K_3)$ and short in 2 contracts of $p(K_2)$. The initial cash flow is $-p(K_1) - p(K_3) + 2p(K_2)$, which is positive. We will denote this value as CF_0.

1. Case 1: $S_T < K_1 < K_2 < K_3$. Then all of the options will be exercised. The payoff will be zero and the profit is $0 + CF_0 e^{rT}$.

2. Case 2: $K_1 < S_T < K_2 < K_3$. Then $p(K_2)$ and $p(K_3)$ will be exercised. The profit will be $(K_3 - S_T) + (K_2 - K_1) + CF_0 e^{rT}$, which is positive.

3. Case 3: $K_1 < K_2 < S_T < K_3$. Then $p(K_3)$ will be exercised. The profit will be $(K_3 - S_T) + CF_0 e^{rT}$, which is positive.

4. Case 4: $K_1 < K_2 < K_3 < S_T$. Then none of the put options will be exercised. The payoff is zero and the profit is $0 + CF_0 e^{rT}$.

Thus in all cases we have positive profit from a positive initial cash flow, which contradicts the no-arbitrage principle. Thus $p(K_2) \leq \frac{p(K_1)+p(K_3)}{2}$ as desired. □

Exercise 58

Show at time T, the value of the option is either \$1 or \$0.
If $S_T = 22$, then the payoff is \$1, so the value of the option will be \$1. If $S_T = 18$, then the payoff is zero and the option will be worth zero.

Exercise 59

Consider the portfolio where we are long in Δ shares and short 1 ECO. If $S_T = 22$, then the payoff will be $22\Delta - 1$.

Exercise 60

Consider the same portfolio as in exercise 59. If $S_T = 18$, then we have Δ shares worth 18 each and the ECO is worthless. So we have 18Δ.

Exercise 61

To find a value of Δ that makes this portfolio riskless, we simply set the two values equal to each other and find Δ.

$$22\Delta - 1 = 18\Delta$$
$$\Delta = 0.25$$

Then the value of the portfolio is always \$4.5 at time T.

Exercise 62

$r = 12\%$, $T = 3/12 = 0.25$. $V(0) = 4.5 \cdot e^{-0.12 \cdot 0.25} = 4.367$.

Exercise 63

Let f denote the price of an ECO at $t = 0$. Consider the portfolio where we are long 1/n shares and short one ECO. Then at $t = 0$, the value of the portfolio is $20\Delta - f = 4.367$, so then $f = 0.633$.

Exercise 64

If $S_T = S_0 u$, the portfolio is equal to $\Delta S_0 u - f_u$.
If $S_T = S_0 d$, the portfolio is equal to $\Delta S_0 d - f_d$.
The portfolio is riskless if $\Delta S_0 d - f_d = \Delta S_0 u - f_u$, i.e. $\Delta = \frac{f_u - f_d}{S_0(u-d)}$.

Exercise 65

At $t = 0$, the portfolio is worth $\Delta S_0 - f$. At expiry, the portfolio is worth $\Delta S_0 d - f_d = \Delta S_0 u - f_u$, where $\Delta = \frac{f_u - f_d}{S_0(u-d)}$. In the absence of arbitrage, $\Delta S_0 - f = (\Delta S_0 u - f_u)e^{-rT}$. Now isolate f.

$$\Delta S_0 - \Delta S_0 u e^{-rT} + f_u e^{-rT} = f$$
$$f = \Delta S_0(1 - ue^{-rT}) + f_u e^{-rT}. \text{ Now sub in our expression for } \Delta.$$
$$f = S_0 \left(\frac{f_u - f_d}{S_0(u-d)}\right)(1 - ue^{-rT}) + f_u e^{-rT}$$
$$f = \frac{f_u(1 - de^{-rT}) + f_d(ue^{-rT}) - 1}{u-d}, \text{ where } f_u = \max(S_0 u - K, 0) \text{ and } f_d = \max(S_0 d - K, 0)$$

Now, let's make the algebraic substitution $p = \frac{e^{rT} - d}{u-d}$. So $1 - p = \frac{u - e^{rT}}{u-d}$. Then

$$f = \frac{f_u(1 - de^{-rT}) + f_d(ue^{-rT} - 1)}{u-d}$$
$$f = e^{-rT}(pf_u + (1-p)f_d)$$

as desired. □

Exercise 66

From Exercise 62 we have $r = 12\%$, $T = 3/12 = 0.25$. We will use the formula from the previous exercise.

$$f = e^{-rT}(pf_u + (1-p)f_d)$$
$$f = e^{-0.12\cdot 0.25}(p(1) + (1-p)(0))$$

We can calculate p using the formula from Exercise 65 and plug in the given values

$$p = \frac{e^{rT}-d}{u-d}$$
$$p = \frac{e^{0.12\cdot 0.25}-0.9}{1.1-0.9}$$
$$f = \$0.633$$

as desired. □

Exercise 67

$S_0 = 100$, in 6 months expected to increase or decrease by 10%, $r = 8\%$ compounded continously.
Using the formula for p from before, we get $p = 0.887$. Then using the formula for f from before

$$f = e^{-rT}(pf_u + (1-p)f_d)$$

Substituting in our known values, we get $f = 9.61$.

Exercise 68

Same situation as the previous exercise, but now we are pricing a put option. We can use put-call parity in this case.

$$c + Ke^{-rT} = p + S_0$$
$$9.61 + 100e^{-0.08\cdot 0.25} = p + 100$$
$$p = 1.92$$

Exercise 69

$S_0 = 25$, after 2 months the stock price will be either 23 or 27. $r = 10\%$, continuous compounding. What is the value of a derivative that pays S_T^2 at this time?
Using the formula for p, we get $p = 1.157$. Then using the formula for f we have
$f = e^{-0.1}(1.157(27)^2 + (1-1.157)(23)^2) = 639.3$

Exercise 70

Show that $f = e^{-2r\Delta t}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}]$.

Using the one-step binomial tree, we know that $f = e^{-rT}(pf_u + (1-p)f_d)$. We can then use this formula as f_u and f_d in the second step of the two-step binomial tree.

$$f = e^{-rT}(pe^{-rT}(pf_u + (1-p)f_d) + (1-p)e^{-rT}(pf_u + (1-p)f_d))$$
$$f = e^{-2r\Delta t}(p(pf_{uu}) + p(1-p)f_{ud} + p(1-p)f_{du} + (1-p)(1-p)f_{dd})$$
$$f = e^{-2r\Delta t}(p^2 f_{uu} + 2(1-p)f_{ud} + (1-p)^2 f_{dd})$$

as desired. □

Exercise 71

$S_0 = 50$, over each of the next two 3-month periods it is expected to up by 6% or down by 5%. The risk free interest rate is 5% with continuous compounding. What is the value of a 6-month ECO with a strike price of \$51?

We can use the formula from the previous exercise and plug in the appropriate values. First let's calculate p

$$p = \frac{e^{0.05 \cdot 0.25} - 0.95}{1.06 - 0.95}$$
$$p = 0.5689$$

Now we can plug this into our formula for f to get

$$f = e^{-2r\Delta t}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}]$$
$$f = e^{-2 \cdot 0.05 \cdot 0.25}(0.5689^2 \cdot 5.18 + 2 \cdot 0.5689 \cdot (1 - 0.5689) \cdot (-0.65) + (1 - 0.5689)^2 \cdot 0)$$
$$f = 1.635$$

Exercise 72

Show that $\mathbb{E}[S_n] = np$.
First note that we can break up S_n Bernoulli trials into n S_1 trials, ie the number of successes in n Bernoulli trials is the same as the number of successes in one Bernoulli trial n times, ie $S_n = S_1 + S_1 + ... + S_1$. By the linearity of expected value, we can write

$$\mathbb{E}[S_n] = \mathbb{E}[S_1] + \mathbb{E}[S_1] + \mathbb{E}[S_1] + ... + \mathbb{E}[S_1].$$

Let's assign a value of 1 to a success in a Bernoulli trial and a value of 0 to a failure. We know the expected value for S_1 is

$$\sum_{x \in \Omega} x \cdot p(x)$$

In this case, there are only two possible outcomes, with a success having probability p and failure $1 - p$. So the expected value is $(1 \cdot p) + (0 \cdot (1 - p)) = p$. Then we have

$$\mathbb{E}[S_n] = \mathbb{E}[S_1] + \mathbb{E}[S_1] + \mathbb{E}[S_1] + ... + \mathbb{E}[S_1] = p + p + p + ... + p = np$$

as desired. □

Exercise 73

Show that $V(X) = \mathbb{E}[X^2] - \mu^2$.

$V(X) = \mathbb{E}[X^2 - 2\mu X + \mu^2]$
$V(X) = \mathbb{E}[X^2] - \mathbb{E}[2\mu X] + \mathbb{E}[\mu^2]$
$V(X) = \mathbb{E}[X^2] + \mu^2 - 2\mu\mathbb{E}[X]$
$V(X) = \mathbb{E}[X^2] + \mu^2 - 2\mu^2 = \mathbb{E}[X^2] - \mu^2$ as desired. □

Exercise 74

Show that $\mathbb{V}[S_n] = \sigma^2 = npq$.
First note $\mathbb{E}[X^2] = ((1^2)p + (0^2)(1 - p))$. Observe that for a Bernoulli trial, $\mu = p$, or $\mu^2 = p^2$. So then $\text{Var}(X) = p - p^2$ since variance $= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$. Then we have

$$\text{Var}(S_n) = \text{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{Var}(X_i) = n(p - p^2) = np(1 - p) = npq$$

as desired. □

Exercise 75

Show that $\int_{-\infty}^{\infty} \phi(x)dx = 1$, where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.
Suppose that I is equal to this integral. We will switch to polar coordinates to prove the identity.

$$I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dxdy$$

Now we can make the substitution $x^2 + y^2 = r^2$ and change the bounds of integration accordingly.

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-\frac{1}{2}r^2} r dr d\theta$$

Since the integrand is entirely in r, we can separate the integrals and cancel the $d\theta$ integral from 0 to 2π with the $\frac{1}{2\pi}$ outside.

$$\int_0^{\infty} e^{-\frac{1}{2}r^2} r dr$$

Now we can use u-substitution to finish the improper integral.

$$u = \frac{1}{2}r^2, du = rdr \rightarrow \int_0^{\infty} e^{-u} du$$

$$-e^{-u}|_0^{\infty} = 0 + 1 = 1$$

as desired. $\square$

Exercise 76

For a standard normal distribution, show that $\mu = 0$ and $\sigma = 1$

First let's calculate the mean.

$$\int_{-\infty}^{\infty} x\phi(x)dx = \int_{-\infty}^{0} x\phi(x)dx + \int_0^{\infty} x\phi(x)dx$$

Note that $f(x) = \phi(x)$ and $f(-x) = -f(x)$, where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

So $\int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx = \int_0^{\infty} f(-x)dx + \int_0^{\infty} f(x)dx = -\int_0^{\infty} f(x)dx + \int_0^{\infty} f(x)dx = 0$

as desired. Now we can calculate the standard deviation as follows:

$$\int_{-\infty}^{\infty} (x-\mu)^2 \phi(x)dx = \int_{-infty}^{\infty} x^2\phi(x)dx.$$

Note that $x^2\phi(x)$ is an even function, so

$$2\int_0^{\infty} x^2\phi(x)dx = 2\int_0^{\infty} x^2(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2})$$

Recall that $\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}dt$ (This is the gamma function). Consider the substitution $t = \frac{1}{2}x^2$, $dt = xdx$. Then

$$2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t}\frac{1}{2}tdt = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t}(t^{3/2-1})(\frac{1}{\sqrt{1/2}})dt =$$

$$\sqrt{2}\Gamma(3/2) = \frac{\sqrt{2\pi}}{2} \frac{2}{\sqrt{2\pi}} = 1$$

as desired. $\square$

Exercise 77

Show that for any x, $1 - N(x) = N(-x)$.

Note that $1 - N(x) = 1 - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. Also, note that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = 1$. Then

$$1 - N(x) = 1 - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} =$$

$$\int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = N(x)$$

as desired. $\square$

Exercise 78

X a normal distribution with $\mu = 70$ and $\sigma = 10$

1. $P(X > 50) = 1 - P(X \leq 50) = 1 - P(z \leq \frac{50-70}{10}) = 1 - P(z \leq -2) = 0.9773$
2. $P(X < 60) = P(z < \frac{60-70}{10}) = P(z < -1) = N(-1) = 1 - N(1) = 0.1587$
3. $P(X > 90) = 1 - P(X \leq 90) = 1 - P(z \leq \frac{90-70}{10}) = 1 - P(z \leq 2) = 1 - 0.9973 = 0.0027$
4. $P(60 < X < 80)$
   (a) $P(X < 80) = P(z < \frac{80-70}{10}) = N(1) = 0.8413$
   (b) $P(60 < X) = 1 - P(X \leq 60) = N(-1) = 1 - N(1) = 0.1587$

   Thus $P(60 < X < 80) = P(X < 80) - P(X \leq 60) = 0.6826$

Exercise 79

Show that $c = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(S_0 u^j d^{m-j} - K, 0)$.

Using the binomial tree model, if there have been j up movements and $n - j$ down movements in an n-step tree, then $S_T = S_0 u^j d^{n-j}$. The value of the ECO at $t = T$ is $\max(S_0 u^j d^{n-j} - K, 0)$. Consider each movement of the stock (each step in the binomial tree) as a Bernoulli trial with probability of an up movement as p and the probability of a down movement as $1 - p$. Then the probability of exactly j ups and $n - j$ downs is $\binom{n}{j} p^j (1-p)^{n-j}$. Thus the expected value of the payoff of the ECO at expiry is

$$\mathbb{E}[\text{payoff}] = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \cdot \max(S_0 u^j d^{n-j} - K, 0).$$

By no arbitrage, $c = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(S_0 u^j d^{m-j} - K, 0)$, as desired. $\square$

Exercise 80

Show that the terms of $c = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0)$ are nonzero if and only if $j > a = \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$.
When are the terms of this sum nonzero? It is when $S_0 u^j d^{n-j} - K > 0$, otherwise the max term makes everything zero. Let's move K over the inequality and take the natural log:

$$\ln(S_0 u^j d^{n-j}) > \ln(K)$$
$$\ln(S_0) + j\ln(u) + (n-j)\ln(d) > \ln(K) \text{ since } u = e^{\sigma\sqrt{T/n}} \text{ and } d = e^{-\sigma\sqrt{T/n}}$$
$$\ln(S_0/K) > -j\sigma\sqrt{T/n} - (n-j)(-\sigma\sqrt{T/n})$$

Thus the terms of the sum are nonzero iff $\ln(S_0/K) > n\sigma\sqrt{T/n} - 2j\sigma\sqrt{T/n}$, or if $j > \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$. Let that quantity equal a. Then we have achieved the desired result. It follows that $c = e^{-rT} \sum_{j>a} \binom{n}{j} p^j (1-p)^{n-j} (S_0 u^j d^{n-j} - K) = e^{-rT}(S_0 U_1 - K U_2)$. $\square$

Exercise 81

Compute the following limits

1. $\lim_{n\to\infty} p(1-p) = \frac{1}{4}$

   We know that $p = \frac{e^{r\Delta t} - d}{u - d}$ where $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$. Then

$$1 - p = 1 - \frac{e^{r\Delta t} - d}{u - d} = \frac{u - e^{r\Delta t}}{u - d}. \text{ Then}$$
$$p(1-p) = \frac{(e^{r\Delta t} - d)(u - e^{r\Delta t})}{(u-d)^2}$$

In our limit, as $n \to \infty$, $\Delta t \to 0$, $d \to 1$, $u \to 1$. This means that the limit is $\frac{0}{0}$, an indeterminate form. Note that we can use power series representations to find the limit. Consider

$$\begin{aligned} e^{r\Delta t} - d &= (1 + r\Delta t + ...) - (1 - \sigma\sqrt{\Delta t} + ...) = \sigma\sqrt{\Delta t} + O(\Delta t) \\ u - e^{r\Delta t} &= (1 + \sigma\sqrt{\Delta t} + ...) - (1 + r\Delta t + ...) = \sigma\sqrt{\Delta t} + O(\Delta t) \end{aligned}$$

Then $N(\Delta t) = (\sigma\sqrt{\Delta t} + O(\Delta t))^2 = \sigma^2\Delta t + O(\Delta t^2)$ and $(u - d) = 4\sigma^2\Delta t + O(\Delta t^2)$. Then finally,

$$\lim_{\Delta t \to 0} \frac{N(\Delta t)}{(u-d)} = \lim_{\Delta t \to 0} \frac{\sigma^2\Delta t + O(\Delta t^2)}{4\sigma^2\Delta t + O(\Delta t^2)} = \frac{\sigma^2}{4\sigma^2} = \frac{1}{4}$$

as desired. □

2. $\lim_{n\to\infty} \sqrt{n}(p - \frac{1}{2}) = \frac{(r-\sigma^2/2)\sqrt{T}}{2\sigma}$

   Again, we can start by substituting $p = \frac{e^{r\Delta t} - d}{u-d}$ where $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$. This results in

$$\sqrt{n}\left(\frac{e^{r\Delta t} - d}{u-d} - \frac{1}{2}\right)$$

   As $n \to \infty$, we have $\Delta t \to 0, d \to 1, u \to 1$, which results in the limit having an indeterminate form. We can use a similar trick as in part 1 as follows

$$\begin{aligned} e^{r\Delta t} - d &= (1 + r\Delta t + ...) - (1 - \sigma\sqrt{\Delta t} + ...) = \sigma\sqrt{\Delta t} + O(\Delta t) \\ u - d &= (1 + \sigma\sqrt{\Delta t} + ...) - (1 - \sigma\sqrt{\Delta t} + ...) = 2\sigma\sqrt{\Delta t} + O(\Delta t) \end{aligned}$$

   Then when we divide, we have

$$\begin{gathered} \sqrt{n}\left(\frac{\sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}} - \frac{1}{2}\right) \\ \sqrt{n}\left(\frac{\sigma\sqrt{\Delta t} - 2\sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}}\right) \\ \left(\frac{\sigma\sqrt{T} - 2\sigma\sqrt{T}}{2\sigma}\right) \\ \frac{(-\sigma)\sqrt{T}}{2\sigma} \\ \frac{(r-\sigma^2/2)}{2\sigma} \end{gathered}$$

   as desired. □

Exercise 82

Use the Central Limit Theorem to show that $U_2 = N(d_2)$

Consider that $U_2 = \lim_{n\to\infty} \sum_{j>\alpha} \binom{n}{j} p^j (1-p)^{n-j}$. Let S_n equal the number of successes in n Bernoulli trials with probability of success p. Then $U_2 = P(S_n > \alpha) = 1 - P(S_n \leq \alpha)$. Recall that $\mathbb{E}[S_n] = np$, $V[S_n] = np(1-p)$. So then $P(S_n \leq \alpha) = P(\frac{S_n - np}{\sqrt{np(1-p)}} \leq \frac{\alpha - np}{\sqrt{np(1-p)}}) = N(\frac{\alpha - np}{\sqrt{np(1-p)}})$ by the Central Limit Theorem. Finally,

$$U_2 = P(S_N > \alpha) = 1 - P(S_N \leq \alpha) = 1 - N(\frac{\alpha - np}{\sqrt{np(1-p)}}) =$$

$$N(\frac{np - \alpha}{\sqrt{np(1-p)}}). \text{ Now we can substitute } \alpha \text{ in to get}$$

$$U_2 = N(d_2) \text{ where } d_2 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

as desired. □

Exercise 83

Show that $U_1 = [pu + (1-p)d]^n \sum j > \alpha \binom{n}{j} (p^\star)^j (1-p^\star)^{n-j} = e^{rT} \sum j > \alpha \binom{n}{j} (p^\star)^j (1-p^\star)^{n-j}$

It is enough to show that $[pu + (1-p)d]^n$ is equal to e^{rT}. Let's make the appropriate substitutions and simplify

$$[pu+(1-p)d]^n = [\tfrac{e^{r\cdot T/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}} \cdot e^{\sigma\sqrt{T/n}} + \tfrac{e^{\sigma\sqrt{T/n}} - e^{r\cdot T/n}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}} \cdot e^{-\sigma\sqrt{T/n}}]^n$$
$$[\tfrac{e^{r\cdot T/n + \sigma\sqrt{T/n}} - 1}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}} + \tfrac{1 - e^{r\cdot T/n - \sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}]^n$$
$$[\tfrac{e^{r\cdot T/n + \sigma\sqrt{T/n}} - e^{r\cdot T/n - \sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}]^n$$
$$[\tfrac{e^{r\cdot T/n (e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}})}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}]^n$$
$$[e^{rT/n}]^n =$$
$$e^{rT}$$

as desired. $\square$

Exercise 84

Conclude that $c = S_0 N(d_1) - Ke^{-rT} N(d_2)$.

We know that $c = e^{-rT}(S_0 U_1 - KU_2)$ from exercise 80. We also know that $U_2 = N(d_2)$ from exercise 82. So already we have $c = e^{-rT}(S_0 U_1 - KN(d_2))$, or equivalently, $c = S_0 \cdot e^{-rT} U_1 - Ke^{-rT} N(d_2)$. When multiplying e^{-rT} and U_1, observe that the e^{rT} term outside of the sum is cancelled away and we are left with the expression for U_2, just with $p^\star$. We can write this as $N(d_1)$, so finally we get $c = S_0 N(d_1) - Ke^{-rT} N(d_2)$.

Exercise 85

Show that the price of a European put option on a non-dividend paying stock with strike price K and expiry T is

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1).$$

Proof: We will prove this using put-call parity. Recall that put-call parity states

$$c + Ke^{-rT} = p + S_0.$$

We also know that $c = S_0 N(d_1) - Ke^{-rT} N(d_2)$. Plugging in this value of c into the formula for put-call parity and isolating p, we get

$$S_0 N(d_1) - Ke^{-rT} N(d_2) + Ke^{-rT} = p + S_0$$
$$p = S_0 N(d_1) - Ke^{-rT} N(d_2) + Ke^{-rT} - S_0$$
$$p = S_0(N(d_1) - 1) + Ke^{-rT}(1 - N(d_2))$$
$$p = S_0(-N(-d_1)) + Ke^{-rT}(N(-d_2))$$
$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

as desired. $\square$

Exercise 86

$S_0 = 41, \sigma = 0.3, r = 0.08, K = 40, T = 3$. Find the price of the ECO.

We use the Black-Scholes-Merton formula with the above values to get $c = 3.399$.

Exercise 87

Find the price of an EPO with the same characteristics as the previous exercise.

We can use put-call parity and isolate p to get the value of the EPO.

$$c + Ke^{-rT} = p + S_0$$
$$3.399 + 40e^{-0.08 \cdot 0.25} = p + 41$$
$$p = 1.607$$

Exercise 88

Find the binomial approximation for the call option in exercise 86 with $n = 1, 2, 10, 12, 100$. What happens as n increases?

Observe that with the given values, we have

$$u = e^{\sigma\sqrt{T/n}} = 1.162$$
$$d = e^{-\sigma\sqrt{T/n}} = 0.861$$
$$p = \frac{e^{0.08\cdot 0.25} - 0.861}{1.162 - 0.861} = 0.5297$$
$$1 - p = 0.4703$$

Then, using the binomial approximation for c,

$$c = e^{-0.08\cdot 0.25}\sum_{j=0}^{1}\binom{1}{j}0.5297^j(0.4703)^{1-j}\max\left(41\cdot(1.162)^j(0.861)^{1-j} - 40, 0\right)$$
$$c = 3.964$$

For other values of n, we get

$$n = 2 \to c = 3.331$$
$$n = 10 \to c = 3.427$$
$$n = 12 \to c = 3.427$$
$$n = 100 \to c = 3.400$$

As n increases, the calculated value of c approaches the value that BSM gives us.

Exercise 89

$S_0 = 120, r = 0.08, \sigma = 0.3, K = 100, T = 1$

1. We use BSM to calculate c. First, observe $d_1 = \frac{ln(S_0/K)+(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = 1.024$ and $d_2 = d_1 - \sigma\sqrt{T} = 0.724$. Then $N(d_1) = 0.8461$ and $N(d_2) = 0.7642$. Then $c = 120(0.8461) - 100e^{-0.08\cdot 1}(0.7642) = 30.984$
2. Using several values of T, we get

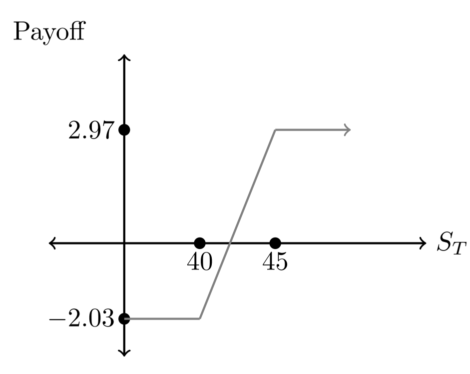
$$T = 50 \to c = 118.35$$
$$T = 100 \to c = 119.97$$
$$T = 1000 \to c = 120$$

As $T \to \infty$, the price of the call option approaches S_0.

Exercise 90

Consider a bull spread where we buy a 40-strike call and sell a 45-strike call. $S_0 = 40, \sigma = 0.3, r = 0.08, T = 0.5$.

Using BSM, we can calculate that $c_1 = 4.16, c_2 = 2.13$, so there is a difference of -2.03.



Exercise 91

Suppose that $\sigma = 0.3, S_0 = 50$. Find the standard deviation of the stock price in 1 week and 4 weeks.

We know that the standard deviation of the stock return is $\sigma\sqrt{T}$.

1. For 1 week, we have $T = 1/52$. So then the standard deviation is 4.16%
2. For 4 weeks, we have $T = 4/52$. So then the standard deviation is 8.32%

Exercise 92

We have a table of values representing the daily returns for 21 days. If we assume there are 252 trading days in a year, then $T = 21/252$. Given the data, we can calculate its standard deviation to get 0.0557. Then we have $0.0557 = \sigma\sqrt{21/252}$. Simplifying, we get $\sigma = 0.193$ or 19.3%

Exercise 93

We are given stock closing prices for 15 consecutive weeks. We assume that there are 52 trading weeks in a year. Using the data, we calculate the standard deviation to be 0.1116. Then $0.1116 = \sigma\sqrt{15/52}$, so $\sigma = 0.2079$ or 20.79%

Exercise 94

A call option on a non-dividend paying stock has a price of 2.50. The stock is \$15, the strike price is 13, time to maturity is 3 months, and the risk free rate is 5%. Find the implied volatility.

We can use the BSM formula and isolate σ to find the implied volatility.

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2)$$
$$2.50 = 15N\left(\frac{ln(15/13)+(0.05+\frac{\sigma^2}{2})0.25}{\sigma\sqrt{0.25}}\right) - 13e^{-0.05\cdot 0.25}N(d_1 - \sigma\sqrt{0.25})$$
$$\sigma = 0.3960 \text{ or } 39.60\%$$

Exercise 95

Calculating implied volatility of AAPL from 21 consecutive days of trading between May 8 2025 and June 6 2025. Standard deviation is 3.59, and $\sqrt{T} = \sqrt{21/252} = 0.288$. So then the implied volatility $\sigma = 12.46\%$

Exercise 96

$S_0 = 100, K = 50, r = 0.06, T = 0.01$

1. Find c for $\sigma \in \{0.05, 0.1, 0.15, ..., 1\}$. For each value of σ, we calculate (using BSM) $c \approx 50.029$
2. Since the value of T is so small, buying this call option is just like buying the stock itself.
3. If $\sigma = 5$, then $c = 51.23$.

Exercise 97

$\sigma = 0.3$, expected return of 15% per year with continuous compounding. Find the process that describes ΔS over one week.

Observe that $\Delta t = 1/52$. Then $\Delta S = 0.15(S)(1/52) + \epsilon(S)(0.3)(\sqrt{1/52})$, or $\Delta S = 0.00288(S) + 0.0416(S)(\epsilon)$.

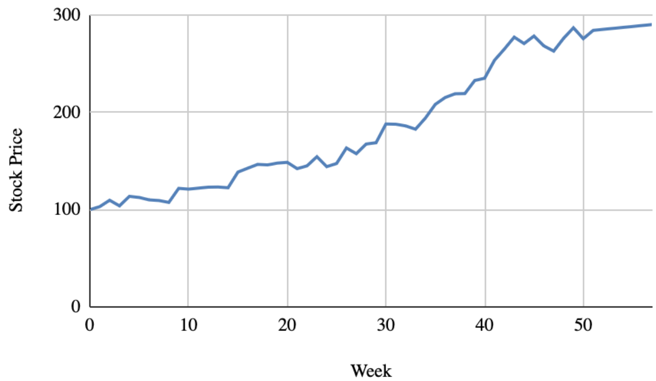
Exercise 98

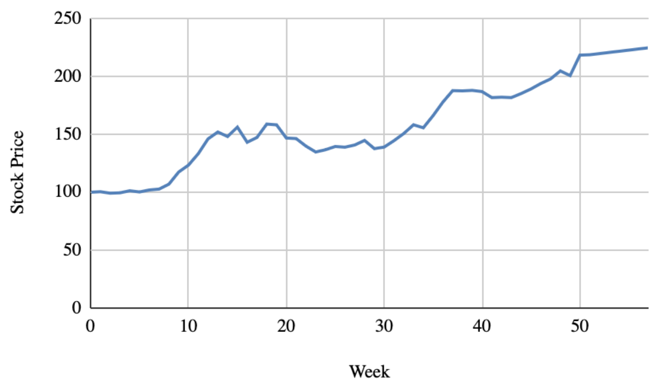
$\mu = 0.1, \sigma = 0.3, S_0 = 100, T = 1/252$

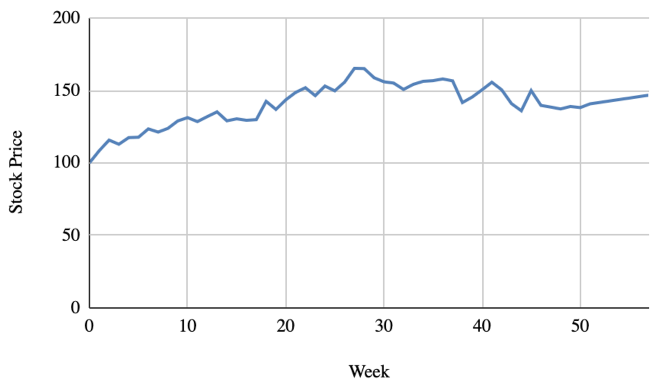
1. $S_0 e^{\mu T} = 100e^{0.1 \cdot 1/252} = 100.08$
2. $\sigma\sqrt{\Delta T} = 0.3\sqrt{1/252} = 0.02$
3. Mean $\pm$ 2 standard deviations = 100.04 - 100.12

Exercise 99

Consider a stock that pays no dividends, has a volatility of 30% per annum, and provides an expected return of 15% per annum with continous compounding. The initial stock price is \$100. Use Monte Carlo simulation to simulate the stock price during one week periods for one year.
Using the Monte Carlo method three separate times, we get







With each simulation, we get different results due to the random nature of the way we sample the normal distribution. The first simulation, the stock price nearly touches \$300, in the second, it barely cracks \$220, and in the last it fails to finish the year above \$150.

Exercise 100

The Ito process for the stock price is $\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}$. Explain the difference between this model and each of the following.

1. $\Delta S = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}$. This model is indifferent to the stock price, meaning that with this model, a stock worth \$5 and a stock worth \$50000 would move by the same amounts, which is not accurate.

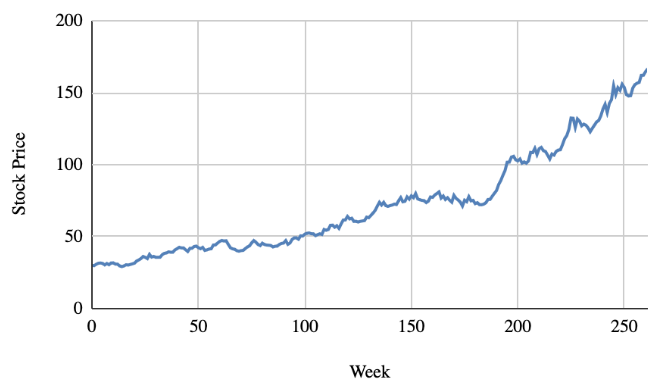
2. $\Delta S = \mu S \Delta t + \sigma \epsilon \sqrt{\Delta t}$. Although this model will take into account the stock's initial price, it does not adjust the drift term over time, meaning that in the long run, stock price changes will not vary. For example, if a stock starts at \$5, this model may work for some time, but as the stock increases to many multiples of 5, the model will become less and less accurate.

3. $\Delta S = \mu \Delta t + \sigma S \epsilon \sqrt{\Delta t}$. With this model, ΔS will always be constant (plus or minus a "chaos" term). This would not be accurate in the long term as the stock price changes.

It is also important to note that all three of these models violate the no-arbitrage principle. Consider a stock worth \$0. Then all three of these models imply that ΔS will be a nonzero value, which violates the no-arbitrage principle.

Exercise 101

Using the Monte Carlo simulation with initial price 30, return 9%, and volatility 20%, we get the following



Exercise 102

Show that $\Delta(call) = N(d_1)$

We know that $\Delta = \frac{\partial C}{\partial S}$. We apply this to BSM to get

$$\Delta(call) = \frac{\partial}{\partial S}[S_0 N(d_1) - Ke^{-rT}N(d_2)]$$

Since the d_1 and d_2 terms both have S terms in them, we have to employ the chain rule. Observe that

$$\frac{\partial d_1}{\partial S} = \frac{\partial}{\partial S}[\frac{ln(S)}{\sigma\sqrt{T}} - \frac{ln(K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}] = \frac{1}{S\sigma\sqrt{T}}$$
$$\frac{\partial d_2}{\partial S} = \frac{\partial}{\partial S}[d_1 - \sigma\sqrt{T}] = \frac{1}{S\sigma\sqrt{T}}$$
$$\text{Thus, } \Delta = N(d_1) + SN'(d_1)(\frac{1}{s\sigma\sqrt{T}}) - Ke^{-rT}N'(d_2)(\frac{1}{S\sigma\sqrt{T}}) =$$
$$= N(d_1) + \frac{N'(d_1)}{\sigma\sqrt{T}} - \frac{Ke^{-rT}N'(d_2)}{S\sigma\sqrt{T}}$$

At this point, it suffices to show that the second and third terms in that expression are equal, so that we are left with $N(d_1)$ So, we have

$$SN'(d_1) = Ke^{-rT}N'(d_2)$$
$$S\frac{1}{\sqrt{2\pi}}e^{-d_1^2/2} = Ke^{-rT}\frac{1}{\sqrt{2\pi}}e^{-d_2^2/2}$$
$$\ln(Se^{-d_1^2/2}) = \ln(Ke^{-rT}e^{-d_2^2/2})$$

$$\begin{gathered}
\ln(S) + \ln(e^{-d_1^2/2}) = \ln(Ke^{-rT}) + \ln(e^{-d_2^2/2}) \\
\ln(S) - d_1^2/2 = \ln(K) - rT - d_2^2/2 \\
\ln(S) - \ln(K) + rT = \frac{d_1^2 - d_2^2}{2} = \frac{(\sigma\sqrt{T})(2d_1 - \sigma\sqrt{T})}{2} = d_1\sigma\sqrt{T} - \frac{\sigma^2\sqrt{T}}{2} \\
\ln(S) - \ln(K) + rT = \ln(S) - \ln(K) + rT + \frac{\sigma^2 T}{2} - \frac{\sigma^2 T}{2} = \ln(S) - \ln(K) + rT
\end{gathered}$$

Thus $\frac{N'(d_1)}{\sigma\sqrt{T}} = \frac{Ke^{-rT}N'(d_2)}{S\sigma\sqrt{T}}$. So then $\Delta = N(d_1)$ as desired. □

Exercise 103

Show that $\Delta(put) = N(d_1) - 1$.

Observe that using BSM we know $p = Ke^{-rT}N(-d_2) - S_0 N(-d_1)$. We can approach this the same way as the previous exercise.

$$\begin{gathered}
\Delta p = \frac{\partial}{\partial S}[Ke^{-rT}N(-d_2) - S_0 N(-d_1)] = \\
= -N(-d_1) - \frac{Ke^{-rT}\phi(d_2)}{S_0\sigma\sqrt{T}} + \frac{\phi(d_1)}{\sigma\sqrt{T}} = \\
= -N(-d_1) = -(1 - N(d_1)) = N(d_1) - 1
\end{gathered}$$

as desired. □

Exercise 104

Show that $\Theta(\text{call}) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$.

From BSM, we know $c = S_0 N(d_1) - Ke^{-rT}N(d_2)$. Then

$$\begin{gathered}
\Theta(c) = \frac{\partial}{\partial t}[S_0 N(d_1) - Ke^{-rT}n(d_2)] = \\
S_0 N'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-rT}N(d_2) - Ke^{-rT}N'(d_2)\frac{\partial d_2}{\partial t} = \\
(S_0 N'(d_1) - rKe^{-rT}N'(d_2))\frac{\partial d_1}{\partial t} - rKe^{-rT}N(d_2) = Ke^{-rT}N'(d_2)\frac{\sigma}{2\sqrt{T}} \\
\Theta(\text{call}) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)
\end{gathered}$$

as desired. □

Exercise 105

Consider a call option where $S_0 = 49, K = 50, r = 0.05, T = 0.3846, \sigma = 0.2$. Find the theta of the option.

Using the formula derived in the previous exercise, we have

$$\begin{gathered}
\Theta(\text{call}) = -\frac{49 \cdot N'(d_1) \cdot 0.2}{2\sqrt{0.3846}} - 0.05 \cdot 50e^{-0.05 \cdot 0.3846}N(d_2) \\
d_1 = \frac{\ln(49/50) + (0.05 + 0.2^2/2)0.3846}{0.2\sqrt{0.3846}} \\
d_2 = d_1 - 0.2\sqrt{0.3846} \\
\Theta = -4.31
\end{gathered}$$

Exercise 106

Show that $\Gamma(call) = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}$.

Recall that gamma is the second derivative with respect to S, or the derivative of delta. From exercise 102, we know that $\Delta(call) = N(d_1)$. Then

$$\begin{gathered}
\Gamma(call) = \frac{\partial}{\partial S}\Delta(call) \\
\frac{\partial}{\partial S}(N(d_1)) = N'(d_1) \cdot \frac{\partial}{\partial S}d_1 \\
= N'(d_1) \cdot \frac{1}{S_0\sigma\sqrt{T}} = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}
\end{gathered}$$

as desired. □

Exercise 107

Consider a call option where $S_0 = 49, K = 50, r = 0.05, T = 0.3846, \sigma = 0.2$. Find the gamma of the option.

Using the formula from the previous exercise, we have

$$\Gamma(call) = \frac{N'(d_1)}{49 \cdot 0.2\sqrt{0.3846}}$$
$$d_1 = \frac{\ln(49/50) + (0.05 + 0.2^2/2)0.3846}{0.2\sqrt{0.3846}}$$
$$\Gamma(call) = 0.066$$

Exercise 108

Consider a call option where $S_0 = 49, K = 50, r = 0.05, T = 0.3846, \sigma = 0.2$. Find the vega of the option.

Observe that vega is the derivative of the option price with respect to volatility. So

$$\nu(call) = \frac{\partial}{\partial \sigma}[S_0 N(d_1) - Ke^{-rT}N(d_2)] =$$
$$= S_0 N'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-rT}N'(d_2)\frac{\partial d_2}{\partial \sigma} =$$
$$= S_0 \sqrt{\frac{T}{2\pi}} e^{-(\ln(S_0/K) + (r + \sigma^2/2)T)^2/(2\sigma^2 T)} =$$
$$= S_0 \sqrt{T}\phi(d_1)$$

Then using the given quantities,

$$\nu = 49 \cdot \sqrt{0.3846}\phi(d_1)$$
$$d_1 = \frac{\ln(49/50) + (0.05 + 0.2^2/2)0.3846}{0.2\sqrt{0.3846}}$$
$$\nu = 12.1$$

Exercise 109

Show that $\rho(call) = KTe^{-rT}N(d_2)$

Recall that $c = S_0 N(d_1) - Ke^{-rT}N(d_2)$ and ρ is the derivative of option price with respect to r.

$$\frac{\partial d_1}{\partial r} = \frac{T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\sqrt{T}}{\sigma}$$
$$\frac{\partial c}{\partial r} = S_0 N'(d_1)\frac{\sqrt{T}}{\sigma} - KTe^{-rT}N(d_2) + Ke^{-rT}N'(d_2)\frac{\sqrt{T}}{\sigma} =$$
$$S_0 N'(d_1) = Ke^{-rT}N'(d_2) = KTe^{-rT}N(d_2)$$

as desired. □

Exercise 110

Use put-call parity to derive the relationship between the Greeks.

Recall that delta is the derivative with respect to S, gamma is the second derivative with respect to S, vega is with respect to σ, and theta with respect to T. Also recall that $c + Ke^{-rT} = p + S_0$

1. Δ: $\Delta c + 0 = \Delta p + 1$, so $\Delta c = \Delta p + 1$
2. Γ: $\Gamma c = \Gamma p$
3. ν: $\nu c + 0 = \nu p + 0$, so $\nu c = \nu p$
4. Θ: $\Theta_c - Kre^{-r(T-t)} = \Theta_p + 0$, so $\Theta_c = \Theta_p - Kre^{-r(T-t)}$

Exercise 111

Find the mean, standard deviation, and variance of Δx

$\Delta x = a\Delta t + b\Delta z$. Then $\mathbb{E} = \mathbb{E}[a\Delta t] + \mathbb{E}[b\Delta z] = a\Delta t$. Also, $V[\Delta x] = b^2\Delta t$. So then the standard deviation equals $b\sqrt{\Delta t}$.

Exercise 112

Find $S(t)$ if $\sigma = 0$.

If volatility is zero, then there is no random "chaos" term. Then dS is determined entirely by the expected return and the stock price, ie μS. Then we can integrate to find $S(t)$ and we get $\mu/2S^2$.

Exercise 113

Find the process followed by a function G of the stock price S and time t.

If $G(S,t)$ is a function of stock price and time, then $dG = (\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial G}{\partial S}\sigma S dz$ with

$$\frac{dS}{S} = \mu dt + \sigma dz \text{ and } dz = \epsilon\sqrt{dt}$$

Exercise 114

The Lognormal property

$G(S,t) = \ln(S)$, then $\ln(S_T) \sim \phi[\ln(S_0) + (\mu - \sigma^2/2)T, \sigma^2 T]$. Observe that in this case, $\ln(S_T)$ is normally distributed with mean $\ln(S_0) + (\mu - \sigma^2/2)T$ and standard distribution $\sigma^2 T$.

Exercise 115

$S_0 = 40, \mu = 0.16, \sigma = 0.2, T = 0.5$

From the previous exercise, we have $\ln(S_0) + (\mu - \sigma^2/2)T$. Plugging in our given values, we have $\ln(40) + (0.16 - (0.2)^2/2)(0.5) = 3.759$. Also, $\sigma\sqrt{T} = 0.2\sqrt{0.5} = 0.141$. To find the 95% interval, we add and subtract 2 standard deviations from the mean to get $[3.477, 4.041]$. But we have to remember that we need to exponentiate in order to find the actual interval of the stock price itself, which is $[32.36, 56.88]$

Exercise 116

Suppose that a stock price S follows the Brownian motion process $dS = \mu S dt + \sigma S dz$. Find the process by the variable S^n and its expected value.

First we apply Itô's Lemma. Let $G = S^n$ to obtain

$$dG = f'(S)dS + \tfrac{1}{2}f''(S)(dS)^2$$
$$f(S) = S^n, \quad f'(S) = nS^{n-1}, \quad f''(S) = n(n-1)S^{n-2}$$

Then

$$\begin{aligned} d(S^n) &= nS^{n-1}\,dS + \tfrac{1}{2}n(n-1)S^{n-2}\cdot(dS)^2 \\ &= nS^{n-1}(\mu S\,dt + \sigma S\,dz) + \tfrac{1}{2}n(n-1)S^{n-2}\cdot\sigma^2 S^2\,dt \\ &= n\mu S^n\,dt + n\sigma S^n\,dz + \tfrac{1}{2}n(n-1)\sigma^2 S^n\,dt \\ &= S^n\left(n\mu + \tfrac{1}{2}n(n-1)\sigma^2\right)dt + n\sigma S^n\,dz \end{aligned}$$

and finally,

$$d(S^n) = S^n\left(n\mu + \tfrac{1}{2}n(n-1)\sigma^2\right)dt + n\sigma S^n\,dZ$$

Now we can calculate the expected value. Since we know that $S(T)$ follows geometric Brownian motion, $S(t) = S(0)e^{\left((\mu-\frac{1}{2}\sigma^2)t+\sigma z_t\right)}$. Then

$$\begin{aligned}
S^n(t) &= S(0)^n e^{\left(n(\mu-\frac{1}{2}\sigma^2)t+n\sigma z_t\right)} \\
\mathbb{E}[S^n(t)] &= S(0)^n \exp\left(n\left(\mu-\tfrac{1}{2}\sigma^2\right)t\right)\mathbb{E}\left[\exp(n\sigma Z_t)\right] \\
&= S(0)^n \exp\left(n\left(\mu-\tfrac{1}{2}\sigma^2\right)t\right)\exp\left(\tfrac{1}{2}n^2\sigma^2 t\right) \\
&= S(0)^n \exp\left(n\mu t+\tfrac{1}{2}n(n-1)\sigma^2 t\right)
\end{aligned}$$

Thus, the expected value is

$$\mathbb{E}[S^n(t)] = S(0)^n e^{\left(n\mu t+\frac{1}{2}n(n-1)\sigma^2 t\right)}.$$

Exercise 117

Find the expected value and variance of S_T

As mentioned in class, we can simply state the following:

$$\begin{aligned}
\mathbb{E}[S_T] &= S_0 e^{\mu T} \\
V[S_T] &= S_0^2 e^{2\mu T}(e^{\sigma^2 T-1})
\end{aligned}$$

Exercise 118

Consider a stock with $S_0 = 20, \mu = 0.2, \sigma = 0.4$. Find expected value and variance at $T = 1$.

Using the formulas from the previous exercise, we have

$$\begin{aligned}
\mathbb{E}[S_T] &= S_0 e^{\mu T} = 20 \cdot e^{0.2\cdot 1} = 24.43 \\
V[S_T] &= S_0^2 e^{2\mu T}(e^{\sigma^2 T-1}) = 20^2 \cdot e^{2\cdot 0.2\cdot 1}(e^{0.4^2\cdot 1-1}) = 103.54
\end{aligned}$$

Exercise 119

Show that $\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t$.

Recall that $\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$. Then, with some algebra,

$$\begin{aligned}
-\left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt - \frac{\partial f}{\partial S}\sigma S dz + \frac{\partial f}{\partial S}\left(\mu S\Delta t + \sigma S\Delta z\right) = \\
= \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt
\end{aligned}$$

and so $\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t$ as desired. $\square$

Exercise 120

Show that $\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$.

Consider $r\Pi\Delta t$. From the previous exercise, this gives us

$$-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2} = r(-f + \frac{\partial f}{\partial S}S)$$

Then,

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$$

For an ECO, we have $f(S,T) = \max(S_T - K, 0)$ and for an EPO, we have $f(S,T) = \max(K - S_T, 0)$.

Exercise 121

Show the price of a forward contract with delivery price K, $f = S - Ke^{-r(T-t)}$, satisfies the BSM PDE.

Recall that the BSM PDE tells us $\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$. From our given f, we calculate

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}$$
$$\frac{\partial f}{\partial S} = 1$$
$$\frac{\partial^2 f}{\partial S^2} = 0$$

We plug these values into the BSM PDE to get

$$-rKe^{-r(T-t)} + rS(1) + \frac{1}{2}\sigma^2 S^2(0) = r(S - Ke^{-r(T-t)})$$
$$rf = rS - rKe^{-r(T-t)}$$
$$f = S - Ke^{-r(T-t)}$$

as desired. □

Exercise 122

Show that $c = S_0 N(d_1) - Ke^{-rT}N(d_2)$ satsfies the BSM PDE.

1. Find $N'(x)$. From previous exercises, we know that $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.
2. Show that $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$. Refer to exercise 102.
3. Find $\frac{\partial d_1}{\partial S}$ and $\frac{\partial d_2}{\partial S}$. From exercise 102, we know that $\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T}}$ and $\frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T}}$
4. Show that if $c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$, then $\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$. This is the same as calculating the theta of an option. Refer to exercise 104.
5. Show that $\frac{\partial c}{\partial S} = N(d_1)$. This is the same as calculating the delta of a call option. Refer to exercise 102.
6. Show that c satisfies the BSM PDE. We have

   $$\frac{\partial c}{\partial S} = N(d_1)$$
   $$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$
   $$\frac{\partial^2 c}{\partial S^2} = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}$$

   Then we plug this into the BSM PDE to get

   $$-rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} + rSN(d_1) + \frac{1}{2}\sigma^2 S^2 \cdot \frac{N'(d_1)}{S_0\sigma\sqrt{T}} = rf$$

   With some algebra, we can achieve

   $$rf = rf$$

   and so c satisfies the BSM PDE as desired. □
7. Show that c satisfies the boundary conditions for an ECO, ie $c = \max(S-K, 0)$ as $t \to T$.

   $$\text{As } t \to T,\ Ke^{-r(T-t)} \to 1, \text{ so } c = \max(S-K, 0)$$

   as desired. □

Exercise 123

1. Find the price of the derivative at t. We have $dS = \mu S dt + \sigma S dz$ and $\mathbb{E}[\ln(S_T)] = \ln(S_0) + (\mu - \sigma^2/2)T$. In a risk-neutral valuation, $\mu = r$, so then $\mathbb{E}[\ln(S_T)] = \ln(S_0) + (r - \sigma^2/2)T$. Then

$$\mathbb{E}[\ln(S_T)] = \ln(S_T) + (r - \sigma^2/2)(T-t) = f(S,T)$$
$$f(S,T) = e^{-r(T-t)}\mathbb{E}[\ln(S_t)] = e^{-r(T-t)}[\ln(S_T) + (r - \sigma^2/2)(T-t)]$$

2. Show that this result satisfies the BSM PDE. We have

$$\frac{\partial f}{\partial t} = re^{-r(T-t)}[\ln(S) + (r - \sigma^2/2)(T-t)] + e^{-r(T-t)}(r - \sigma^2/2)(-1)$$
$$\frac{\partial f}{\partial S} = e^{-r(T-t)} \cdot \frac{1}{S}$$
$$\frac{\partial^2 f}{\partial S^2} = e^{-r(T-t)} \cdot \frac{-1}{S^2}$$

Then we plug these values into the PDE to get

$$re^{-r(T-t)}[\ln(S) + (r - \sigma^2/2)(T-t)] + e^{-r(T-t)}(r - \sigma^2/2)(-1) + rS \cdot e^{-r(T-t)} \cdot \frac{1}{S} + \frac{1}{2}\sigma^2 S^2 \cdot e^{-r(T-t)} \cdot \frac{-1}{S^2} = rf$$

With a lot of algebra, we can conclude that the left side of this equation is equal to rf and thus the PDE is satisfied, as desired. $\square$

Exercise 124

Show that $\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$.

We can replace each derivative in the BSM PDE with its respective Greek, ie $\frac{\partial f}{\partial t}$ is theta, $\frac{\partial f}{\partial S}$ is delta, and $\frac{\partial^2 f}{\partial S^2}$ is gamma. Substituting into the BSM PDE, we have $\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$ as desired. $\square$

Exercise 125

Show that $\lim_{S_0 \to \infty} c = S_0 - Ke^{-rT}$.

Recall that $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{(-1/2)t^2} dt$, $d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$. Then observe that $\lim_{S_0 \to \infty} d_1 = \infty$, and $\lim_{S_0 \to \infty} d_2 = \infty$. Then note that $\lim_{S_0 \to \infty} N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(-1/2)t^2} dt = 1$ and $\lim_{S_0 \to \infty} N(d_2) = 1$. Then, $c = S_0 N(d_1) - Ke^{-rT} N(d_2)$, so $\lim_{S_0 \to \infty} c = S_0(1) - Ke^{-rT}(1) = S_0 - Ke^{-rT}$ as desired. $\square$

Exercise 126

Show that $\lim_{S_0 \to \infty} p = 0$.

Recall that $p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$. Then $p = Ke^{-rT}(1 - N(d_2)) - S_0(1 - N(d_1))$. Then finally, $\lim_{S_0 \to \infty} p = Ke^{-rT}(1-1) - S_0(1-1) = 0$ as desired. $\square$

Exercise 127

Show that $\lim_{\sigma \to 0} c = \max(S_0 - Ke^{-rT}, 0)$.

Recall that $c = S_0 N(d_1) - Ke^{-rT} N(d_2)$, $d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$, and $d_2 = d_1 - \sigma\sqrt{T}$. So then $c = S_0 N(\lim_{\sigma \to 0} d_1) - Ke^{-rT} N(\lim_{\sigma \to 0} d_2)$. We need to consider three cases:

1. $\ln(S_0/K) + rT > 0$. Then $d_1, d_2 \to \infty$.
2. $\ln(S_0/K) + rT < 0$. Then $d_1, d_2 \to -\infty$.
3. $\ln(S_0/K) + rT = 0$. Then $d_1, d_2 \to 0$.

Then we can look at c in each case:

1. $\lim_{\sigma \to 0} c = S_0(1) - Ke^{-rT}(1) = S_0 - Ke^{-rT}$.
2. $\lim_{\sigma \to 0} c = S_0(0) - Ke^{-rT}(0) = 0$.
3. $\lim_{\sigma \to 0} c = S_0(1/2) - Ke^{-rT}(1/2) = \frac{1}{2}(S_0 - Ke^{-rT})$. In this case, $e^{\ln(S_0/K)+rT} = e^0$, so $Ke^{-rT} = 0$.

Thus, $\lim_{\sigma \to 0} c = \max(S_0 - Ke^{-rT}, 0)$ as desired. $\square$