

# Homework 3

## MATH 476

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### Exercise 28

An investor buys for \$3 a 3-month European call with a strike price of \$30 and sells for \$1 a 3-month European call with a strike price of \$35. Find the profit from this bull spread in each of the following cases:

1.  $S_T = \$25$ . We know that  $K_1 = \$30$  since it is the long position and that  $K_2 = \$35$  since it is the short position. Then  $S_T < K_1 < K_2$ . Using the payoff from a bull spread from exercise 27, we know the payoff is 0. Then the profit is  $-\$2$ .
2.  $S_T = \$34$ . Then  $K_1 < S_T < K_2$ . So we know that the payoff is  $S_T - K_1$ . So the payoff is \$4. Then the profit is \$2.
3.  $S_T = \$40$ . Then  $S_T > K_2$ . So the payoff is  $K_2 - K_1$ , or \$5. Then the profit is \$3.

### Exercise 31

Find the payoff from a straddle (long on an ECO and long on an EPO, both same strike price and expiry). We know the payoff from the long ECO is

$$\begin{cases} S_T - K & S_T > K \\ 0 & S_T \leq K \end{cases}$$

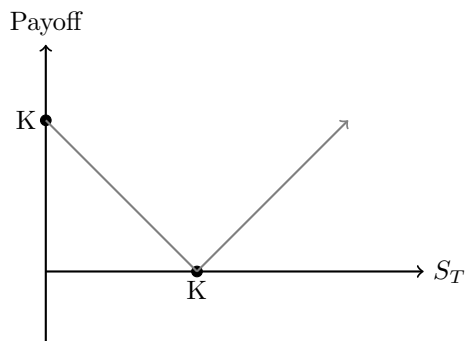
and the payoff from the long EPO is

$$\begin{cases} K - S_T & S_T < K \\ 0 & S_T \geq K \end{cases}$$

Then the total payoff is

$$\begin{cases} S_T - K & S_T > K \\ K - S_T & S_T \leq K \end{cases}$$

This can be represented with the following payoff diagram



## Exercise 35

Initial deposit of \$100, annual interest of 10%, find value after two years when compounded

1. Annually: Then we have  $V(2) = (1 + \frac{10}{1})^{2 \cdot 1} \cdot 100 = \$121$
2. Monthly: Then we have  $V(2) = (1 + \frac{10}{12})^{2 \cdot 12} \cdot 100 = \$122.03$

## Exercise 36

Show if  $m < k$ , then

$$(1 + \frac{r}{m})^m < (1 + \frac{r}{k})^k$$

Proof: Let  $f(x) = (1 + \frac{r}{x})^x$  for  $x > 0$ . It is enough to show that  $f(x)$  is increasing. Take the natural log of both sides and use log rules to get:

$$\ln(f(x)) = \ln((1 + \frac{r}{x})^x) = x \ln(1 + \frac{r}{x})$$

Now take the derivative of both sides and simplify:

$$\begin{aligned} \frac{1}{f(x)} f'(x) &= \ln(1 + \frac{r}{x}) + x(\frac{1}{1+\frac{r}{x}})(-\frac{r}{x^2}) \\ f'(x) &= f(x)(\ln(1 + \frac{r}{x}) - \frac{r}{x+r}) \end{aligned}$$

Observe that  $f(x) > 0$ , so we just need to show that the other term is positive.

$$\begin{aligned} \ln(1 + \frac{r}{x}) - \frac{r}{x+r} &> 0 \\ (x+r) \cdot \ln(1 + \frac{r}{x}) &> r \\ e^{x+r}(1 + \frac{r}{x}) &> e^{x+r} \end{aligned}$$

We know this must be true, therefore  $f'(x)$  is indeed greater than zero. So  $f(x)$  is indeed increasing and we are done.  $\square$

## Exercise 37

Solve  $\frac{dV}{dt} = r \cdot V$

We can use separation of variables to solve this differential equation.

$$\frac{dV}{V} = r \cdot dt$$

Integrate both sides:

$$\begin{aligned} \int \frac{dV}{V} &= \int r \cdot dt \\ \ln(|v|) &= rt + C \\ V &= e^{rt+C} = Ce^{rt} \end{aligned}$$

Now we can use our initial condition to find a specific solution.

$$\begin{aligned} V(0) &= 0 \\ V(0) &= Ce^0 \\ P &= C \end{aligned}$$

Thus  $V = Pe^{rt}$ .

## Exercise 38

Continuous compounding is given by  $V(t) = \lim_{m \rightarrow \infty} \left(1 + \frac{r}{m}\right)^{tm} P$ .

1. Show that  $e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$ . To start, we can take the natural log of both sides:

$$\begin{aligned}\ln(e) &= \ln\left(\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x\right) \\ 1 &= \lim_{x \rightarrow \infty} \ln\left(1 + \frac{1}{x}\right)^x \\ 1 &= \lim_{x \rightarrow \infty} \left(x \ln\left(1 + \frac{1}{x}\right)\right) \\ 1 &= \lim_{x \rightarrow \infty} \left(\frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}\right)\end{aligned}$$

Now we can use L'Hopital's Rule to evaluate this limit:

$$\begin{aligned}1 &= \lim_{x \rightarrow \infty} \left(\frac{\frac{1}{1+\frac{1}{x}} \left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right) \\ 1 &= \lim_{x \rightarrow \infty} \left(\frac{1}{1+\frac{1}{x}}\right) = \frac{1}{1} = 1\end{aligned}$$

Thus we achieve the desired result.

2. Now we want to obtain a closed form expression for  $V(t)$ . By using the proof above, we can assert that

$$e^r = \lim_{x \rightarrow \infty} \left(1 + \frac{r}{x}\right)^x$$

We know that  $V(t) = \lim_{m \rightarrow \infty} \left(1 + \frac{r}{m}\right)^{tm} P$ . We can rewrite this as

$$\begin{aligned}\left(\lim_{m \rightarrow \infty} \left(1 + \frac{r}{m}\right)^m\right)^t \cdot P &= \\ e^{rt} \cdot P &= P e^{rt}\end{aligned}$$

Thus the closed form for  $V(t)$  is  $V(t) = P e^{rt}$ , and this matches our result from solving the differential equation.