Homework 7

MATH 476

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Exercise 78

X a normal distribution with $\mu = 70$ and $\sigma = 10$

1. $P(X > 50) = 1 - P(X \leq 50) = 1 - P(z \leq \frac{50-70}{10}) = 1 - P(z \leq -2) = 0.9773$
2. $P(X < 60) = P(z < \frac{60-70}{10}) = P(z < -1) = N(-1) = 1 - N(1) = 0.1587$
3. $P(X > 90) = 1 - P(X \leq 90) = 1 - P(z \leq \frac{90-70}{10}) = 1 - P(z \leq 2) = 1 - 0.9973 = 0.0027$
4. $P(60 < X < 80)$
   (a) $P(X < 80) = P(z < \frac{80-70}{10}) = N(1) = 0.8413$
   (b) $P(60 < X) = 1 - P(X \leq 60) = N(-1) = 1 - N(1) = 0.1587$

   Thus $P(60 < X < 80) = P(X < 80) - P(X \leq 60) = 0.6826$

Exercise 80

Show that the terms of $c = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max\left(S_0 u^j d^{n-j} - K, 0\right)$ are nonzero if and only if $j > a = \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$.

When are the terms of this sum nonzero? It is when $S_0 u^j d^{n-j} - K > 0$, otherwise the max term makes everything zero. Let's move K over the inequality and take the natural log:

$$\ln(S_0 u^j d^{n-j}) > \ln(K)$$
$$\ln(S_0) + j\ln(u) + (n-j)\ln(d) > \ln(K) \text{ since } u = e^{\sigma\sqrt{T/n}} \text{ and } d = e^{-\sigma\sqrt{T/n}}$$
$$\ln(S_0/K) > -j\sigma\sqrt{T/n} - (n-j)(-\sigma\sqrt{T/n})$$

Thus the terms of the sum are nonzero iff $\ln(S_0/K) > n\sigma\sqrt{T/n} - 2j\sigma\sqrt{T/n}$, or if $j > \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$. Let that quantity equal a. Then we have achieved the desired result. It follows that $c = e^{-rT} \sum_{j>a} \binom{n}{j} p^j (1-p)^{n-j} (S_0 u^j d^{n-j} - K) = e^{-rT}(S_0 U_1 - K U_2)$. $\square$

Exercise 81

Compute the following limits

1. $\lim_{n\to\infty} p(1-p) = \frac{1}{4}$

   We know that $p = \frac{e^{r\Delta t} - d}{u-d}$ where $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$. Then

$$1 - p = 1 - \frac{e^{r\Delta t} - d}{u-d} = \frac{u - e^{r\Delta t}}{u-d}. \text{ Then}$$
$$p(1-p) = \frac{(e^{r\Delta t} - d)(u - e^{r\Delta t})}{(u-d)^2}$$

In our limit, as $n \to \infty$, $\Delta t \to 0$, $d \to 1$, $u \to 1$. This means that the limit is $\frac{0}{0}$, an indeterminate form. Note that we can use power series representations to find the limit. Consider

$$\begin{aligned} e^{r\Delta t} - d &= (1 + r\Delta t + ...) - (1 - \sigma\sqrt{\Delta t} + ...) = \sigma\sqrt{\Delta t} + O(\Delta t) \\ u - e^{r\Delta t} &= (1 + \sigma\sqrt{\Delta t} + ...) - (1 + r\Delta t + ...) = \sigma\sqrt{\Delta t} + O(\Delta t) \end{aligned}$$

Then $N(\Delta t) = (\sigma\sqrt{\Delta t} + O(\Delta t))^2 = \sigma^2\Delta t + O(\Delta t^2)$ and $(u - d) = 4\sigma^2\Delta t + O(\Delta t^2)$. Then finally,

$$\lim_{\Delta t \to 0} \tfrac{N(\Delta t)}{(u-d)} = \lim_{\Delta t \to 0} \tfrac{\sigma^2\Delta t + O(\Delta t^2)}{4\sigma^2\Delta t + O(\Delta t^2)} = \tfrac{\sigma^2}{4\sigma^2} = \tfrac{1}{4}$$

as desired. □

2. $\lim_{n\to\infty} \sqrt{n}(p - \frac{1}{2}) = \frac{(r-\sigma^2/2)\sqrt{T}}{2\sigma}$

   Again, we can start by substituting $p = \frac{e^{r\Delta t} - d}{u-d}$ where $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$. This results in

$$\sqrt{n}\left(\tfrac{e^{r\Delta t}-d}{u-d} - \tfrac{1}{2}\right)$$

   As $n \to \infty$, we have $\Delta t \to 0, d \to 1, u \to 1$, which results in the limit having an indeterminate form. We can use a similar trick as in part 1 as follows

$$\begin{aligned} e^{r\Delta t} - d &= (1 + r\Delta t + ...) - (1 - \sigma\sqrt{\Delta t} + ...) = \sigma\sqrt{\Delta t} + O(\Delta t) \\ u - d &= (1 + \sigma\sqrt{\Delta t} + ...) - (1 - \sigma\sqrt{\Delta t} + ...) = 2\sigma\sqrt{\Delta t} + O(\Delta t) \end{aligned}$$

   Then when we divide, we have

$$\begin{gathered} \sqrt{n}\left(\tfrac{\sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}} - \tfrac{1}{2}\right) \\ \sqrt{n}\left(\tfrac{\sigma\sqrt{\Delta t} - 2\sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}}\right) \\ \left(\tfrac{\sigma\sqrt{T} - 2\sigma\sqrt{T}}{2\sigma}\right) \\ \tfrac{(-\sigma)\sqrt{T}}{2\sigma} \\ \tfrac{(r-\sigma^2/2)}{2\sigma} \end{gathered}$$

   as desired. □