Homework 4

MATH 476

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Exercise 45

$S_0 = 19$, $c = 1$, $K = 20$, $r = 0.04$, expiry in 3 months means $T = 1/4$. We can use put-call parity to calculate the price of a put option with same strike price and expiry.

$$\begin{gathered} c + Ke^{-rT} = p + S_0 \\ 1 + 20e^{-0.04\cdot 0.25} = p + 19 \\ p = 1 + 20e^{-0.04*0.25} - 19 \\ p = 1.80 \end{gathered}$$

Exercise 46

$S_0 = 130$, expiry in one year means $T = 1$, $c = 20$, $p = 5$, $K = 120$. We can use put-call parity to calculate the risk-free interest rate.

$$\begin{gathered} c + Ke^{-rT} = p + S_0 \\ 20 + 120e^{-r\cdot 1} = 5 + 130 \\ e^{-r} = \tfrac{5+130-20}{120} \\ e^{r} = \tfrac{120}{115} \\ r = \ln(\tfrac{120}{115}) = 0.0426 \end{gathered}$$

Thus the risk-free interest rate is 4.26%.

Exercise 47

$S_0 = 31$, $c = 3$, $p = 2.25$, $K = 30$, $T = 0.25$, $r = 0.1$. Note that put-call parity does not hold here:

$$\begin{gathered} c + Ke^{-rT} = p + S_0 \\ 3 + 30e^{-0.1\cdot 0.25} = 2.25 + 31 \\ 32.26 \neq 33.25 \end{gathered}$$

Thus we should be able to construct an arbitrage opportunity. Consider a portfolio where we buy the call option and short-sell the put option and the stock. At $t = 0$, the cash flow is

$$-c + p + S_0 = \$30.25$$

Thus we have positive cash flow at $t = 0$. We can then invest this at the risk-free interest rate. At expiry, this will be worth \$31.02. At expiry, we have two cases.

1. $S_T \leq 30$. Then we let the ECO expire, the EPO will be exercised, and we return the stock we shorted. The payoff will be $0 - 30 = -30$, and since we started with \$31.02, we have \$1.02 profit.

2. $S_T > 30$. Then we exercise the ECO, the EPO expires worthless, and we return the stock we shorted. The payoff is $-K + S_T - S_T = -30$. Since we started with \$31.02 we have \$1.02 profit.

Thus in all cases, we make a profit with positive initial cash flow, and this was an arbitrage opportunity. □

Exercise 54

We have three call options with prices $c(K_1), c(K_2), c(K_3)$ with $K_1 < K_2 < K_3$. We also have three put options with prices $p(K_1), p(K_2), p(K_3)$. All of these options have the same expiry.

1. Show $c(K_1) \geq c(K_2)$. Suppose not for the sake of contradiction, ie $c(K_1) < c(K_2)$. Consider the portfolio where we are short in $c(K_2)$ and long in $c(K_1)$. The initial cash flow is $c(K_2) - c(K_1) > 0$, and let CF_0 represent this amount.

   (a) Case 1: $S_T < K_1 < K_2$. Then none of the call options will be exercised. The payoff is zero and the profit is $0 + CF_0 \cdot e^{rT}$, which is positive.

   (b) Case 2: $K_1 < S_T < K_2$. Then the $c(K_1)$ option will be exercised, which is our long position. So the payoff is $S_T - K_1$ and the profit is $S_T - K_1 + CF_0 \cdot e^{rT}$, which is positive.

   (c) Case 3: $K_1 < K_2 < S_T$. Then both options will be exercised. The payoff will be $S_T - K_1 - S_T + K_2 = K_2 - K_1$, which is positive, and thus the profit is $K_2 - K_1 + CF_0 \cdot e^{rT}$, which is positive.

   Thus in all cases, we have a risk free profit with no initial investment, ie an arbitrage opportunity. Contradition. Thus $c(K_1) \geq c(K_2)$. $\square$

2. Show $p(K_2) \geq p(K_1)$. Suppose not for the sake of contradiction, ie $p(K_2) < p(K_1)$ or $p(K_1) - p(K_2) > 0$. Consider the portfolio where we are short in $p(K_1)$ and long in $p(K_2)$. Then the initial cash flow is $p(K_1) - p(K_2) > 0$, and let CF_0 represent this amount.

   (a) Case 1: $S_T < K_1 < K_2$. Then both put options will be exercised. The payoff is $S_T - K_1 + K_2 - S_T = K_2 - K_1$. So the profit is $K_2 - K_1 + CF_0 \cdot e^{rT}$, which is positive.

   (b) Case 2: $K_1 < S_T < K_2$. Then the $p(K_2)$ option will be exercised. The payoff is $K_2 - S_T$ and the profit is $K_2 - S_T + CF_0 \cdot e^{rT}$, which is positive.

   (c) Case 3: $K_1 < K_2 < S_T$. Then none of the put options will be exercised and the payoff is zero. Then the profit is $CF_0 \cdot e^{rT}$, which is positive.

   Thus in all cases, we have a risk free profit with no initial investment, ie an arbitrage opportunity. Contradiction. Thus $p(K_2) \geq p(K_1)$. $\square$

3. Show $c(K_1) - c(K_2) \leq K_2 - K_1$. Suppose not for the sake of contradiction, ie $c(K_1) - c(K_2) > K_2 - K_1$. Consider the portfolio where we are short in $c(K_1)$ and long in $c(K_2)$, and also we hold cash equivalent to $(K_1 - K_2) \cdot e^{-rT}$. Then the initial cash flow is $c(K_1) - c(K_2) + (K_1 - K_2) \cdot e^{-rT}$, which we denote as CF_0, which is positive.

   (a) Case 1: $S_T < K_1 < K_2$. Then neither call option is exercised. The payoff is zero, and the profit is $CF_0 \cdot e^{rT}$.

   (b) Case 2: $K_1 < S_T < K_2$. Then $c(K_1)$ is exercised. The payoff is $K_1 - S_T$, and thus the profit is $K_1 - S_T + CF_0 \cdot e^{rT}$, which is positive.

   (c) Case 3: $K_1 < K_2 < S_T$. Then both call options are exercised. The payoff is $K_2 - K_1$, which when added to $CF_0 \cdot e^{rT}$ to get profit is positive.

   Thus in all cases, we have a risk free profit with no initial investment, ie an arbitrage opportunity. Contradition. Then $c(K_1) - c(K_2) \leq K_2 - K_1$. $\square$

4. Show $p(K_2) - p(K_1) \leq K_2 - K_1$. Suppose not for the sake of contradiction, ie $p(K_2) - p(K_1) > K_2 - K_1$. Consider the portfolio where we are short in $p(K_1)$ and long in $p(K_2)$, and also we hold cash equivalent to $(K_1 - K_2) \cdot e^{-rT}$. Then the initial cash flow is $p(K_1) - p(K_2) + (K_1 - K_2) \cdot e^{-rT}$, denoted as CF_0, which is positive.

   (a) Case 1: $S_T < K_1 < K_2$. Then both put options will be exercised. The payoff will be $K_2 - K_1$, which when added to $CF_0 \cdot e^{rT}$ to get profit, gives us a positive value.

   (b) Case 2: $K_1 < S_T < K_2$. Then $p(K_2)$ is exercised. The payoff will be $K_2 - S_T$, which is positive. When added to $CF_0 \cdot e^{rT}$, we get a positive profit.

   (c) Case 3: $K_1 < K_2 < S_T$. Then neither put option will be exercised. The payoff is zero and the total profit is $CF_0 \cdot e^{rT}$ which is positive.

   Thus in all cases we get a risk free profit will no initial investment, ie an arbitrage opportunity. Contradiction. Then $p(K_2) - p(K_1) \leq K_2 - K_1$. $\square$