Homework 6

MATH 476

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Exercise 72

Show that $\mathbb{E}[S_n] = np$.
First note that we can break up S_n Bernoulli trials into n S_1 trials, ie the number of successes in n Bernoulli trials is the same as the number of successes in one Bernoulli trial n times, ie $S_n = S_1 + S_1 + ... + S_1$. By the linearity of expected value, we can write

$$\mathbb{E}[S_n] = \mathbb{E}[S_1] + \mathbb{E}[S_1] + \mathbb{E}[S_1] + ... + \mathbb{E}[S_1].$$

Let's assign a value of 1 to a success in a Bernoulli trial and a value of 0 to a failure. We know the expected value for S_1 is

$$\sum_{x \in \Omega} x \cdot p(x)$$

In this case, there are only two possible outcomes, with a success having probability p and failure $1 - p$. So the expected value is $(1 \cdot p) + (0 \cdot (1 - p)) = p$. Then we have

$$\mathbb{E}[S_n] = \mathbb{E}[S_1] + \mathbb{E}[S_1] + \mathbb{E}[S_1] + ... + \mathbb{E}[S_1] = p + p + p + ... + p = np$$

as desired. $\square$

Exercise 74

Show that $\mathbb{V}[S_n] = \sigma^2 = npq$.
First note $\mathbb{E}[X^2] = ((1^2)p + (0^2)(1 - p))$. Observe that for a Bernoulli trial, $\mu = p$, or $\mu^2 = p^2$. So then $\text{Var}(X) = p - p^2$ since variance $= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$. Then we have

$$\text{Var}(S_n) = \text{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{Var}(X_i) = n(p - p^2) = np(1 - p) = npq$$

as desired. $\square$

Exercise 75

Show that $\int_{-\infty}^{\infty} \phi(x)dx = 1$, where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.
Suppose that I is equal to this integral. We will switch to polar coordinates to prove the identity.

$$I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dxdy$$

Now we can make the substitution $x^2 + y^2 = r^2$ and change the bounds of integration accordingly.

$$\frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{1}{2}r^2} rdrd\theta$$

Since the integrand is entirely in r, we can separate the integrals and cancel the $d\theta$ integral from 0 to 2π with the $\frac{1}{2\pi}$ outside.

$$\int_0^\infty e^{-\frac{1}{2}r^2} r dr$$

Now we can use u-substitution to finish the improper integral.

$$u = \frac{1}{2}r^2, du = rdr \to \int_0^\infty e^{-u} du$$

$$-e^{-u}|_0^\infty = 0 + 1 = 1$$

as desired. $\square$