Homework 3
MATH 476

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Exercise 28

An investor buys for \$3 a 3-month European call with a strike price of \$30 and sells for \$1 a 3-month European call with a strike price of \$35. Find the profit from this bull spread in each of the following cases:

1. $S_T = \$25$. We know that $K_1 = \$30$ since it is the long position and that $K_2 = \$35$ since it is the short position. Then $S_T < K_1 < K_2$. Using the payoff form a bull spread from exercise 27, we know the payoff is 0. Then the profit is $-\$2$.

2. $S_T = \$34$. Then $K_1 < S_T < K_2$. So we know that the payoff is $S_T - K_1$. So the payoff is \$4. Then the profit is \$2.

3. $S_T = \$40$. Then $S_T > K_2$. So the payoff is $K_2 - K_1$, or \$5. Then the profit is \$3.

Exercise 31

Find the payoff from a straddle (long on an ECO and long on an EPO, both same strike price and expiry). We know the payoff from the long ECO is

$$\begin{cases} S_T - K & S_T > K \\ 0 & S_T \leq K \end{cases}$$

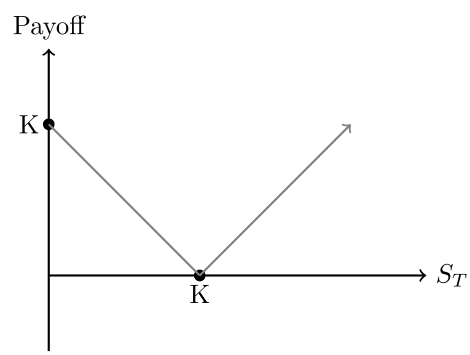
and the payoff from the long EPO is

$$\begin{cases} K - S_T & S_T < K \\ 0 & S_T \geq K \end{cases}$$

Then the total payoff is

$$\begin{cases} S_T - K & S_T > K \\ K - S_T & S_T \leq K \end{cases}$$

This can be represented with the following payoff diagram



Exercise 35

Initial deposit of \$100, annual interest of 10%, find value after two years when compounded

1. Annually: Then we have $V(2) = (1 + \frac{10}{1})^{2\cdot 1} \cdot 100 = \$121$
2. Monthly: Then we have $V(2) = (1 + \frac{10}{12})^{2\cdot 12} \cdot 100 = \$122.03$

Exercise 36

Show if $m < k$, then

$$(1 + \tfrac{r}{m})^m < (1 + \tfrac{r}{k})^k$$

Proof: Let $f(x) = (1 + \frac{r}{x})^x$ for $x > 0$. It is enough to show that $f(x)$ is increasing. Take the natural log of both sides and use log rules to get:

$$\ln(f(x)) = \ln((1 + \tfrac{r}{x})^x) = x\ln(1 + \tfrac{r}{x})$$

Now take the derivative of both sides and simplify:

$$\begin{gathered}\tfrac{1}{f(x)}f'(x) = \ln(1 + \tfrac{r}{x}) + x(\tfrac{1}{1+\frac{r}{x}})(-\tfrac{r}{x^2}) \\ f'(x) = f(x)\big(\ln(1 + \tfrac{r}{x}) - \tfrac{r}{x+r}\big)\end{gathered}$$

Observe that $f(x) > 0$, so we just need to show that the other term is positive.

$$\begin{gathered}\ln(1 + \tfrac{r}{x}) - \tfrac{r}{x+r} > 0 \\ (x + r) \cdot \ln(1 + \tfrac{r}{x}) > r \\ e^{x+r}(1 + \tfrac{r}{x}) > e^{x+r}\end{gathered}$$

We know this must be true, therefore $f'(x)$ is indeed greater than zero. So $f(x)$ is indeed increasing and we are done. □

Exercise 37

Solve $\frac{dV}{dt} = r \cdot V$

We can use separation of variables to solve this differential equation.

$$\tfrac{dV}{V} = r \cdot dt$$

Integrate both sides:

$$\begin{gathered}\int \tfrac{dV}{V} = \int r \cdot dt \\ \ln(|v|) = rt + C \\ V = e^{rt+C} = Ce^{rt}\end{gathered}$$

Now we can use our initial condition to find a specific solution.

$$\begin{gathered}V(0) = 0 \\ V(0) = Ce^0 \\ P = C\end{gathered}$$

Thus $V = Pe^{rt}$.

Exercise 38

Continous compounding is given by $V(t) = \lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{tm} P$.

1. Show that $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$. To start, we can take the natural log of both sides:

$$\begin{gathered}
\ln(e) = \ln(\lim_{x\to\infty}(1+\tfrac{1}{x})^{x}) \\
1 = \lim_{x\to\infty}(\ln(1+\tfrac{1}{x})^{x}) \\
1 = \lim_{x\to\infty}(x\ln(1+\tfrac{1}{x})) \\
1 = \lim_{x\to\infty}\left(\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}\right)
\end{gathered}$$

   Now we can use L'Hopital's Rule to evaluate this limit:

$$\begin{gathered}
1 = \lim_{x\to\infty}\left(\frac{\frac{1}{1+\frac{1}{x}}(\frac{-1}{x^2})}{(\frac{-1}{x^2})}\right) \\
1 = \lim_{x\to\infty}(\tfrac{1}{1+\frac{1}{x}}) = \tfrac{1}{1} = 1
\end{gathered}$$

   Thus we achieve the desired result.

2. Now we want to obtain a closed form expression for $V(t)$. By using the proof above, we can assert that

$$e^r = \lim_{x\to\infty}(1+\tfrac{r}{x})^{x}$$

   We know that $V(t) = \lim\limits_{m\to\infty}\left(1+\dfrac{r}{m}\right)^{tm} P$. We can rewrite this as

$$\begin{gathered}
\left(\lim_{m\to\infty}(1+\tfrac{r}{m})^{m}\right)^{t}\cdot P = \\
e^{rt}\cdot P = Pe^{rt}
\end{gathered}$$

   Thus the closed form for $V(t)$ is $V(t) = Pe^{rt}$, and this matches our result from solving the differential equation.