The Newton–Raphson method

1. Core Concepts and Formula

The Newton–Raphson method is one of the most widely used techniques for finding the root of a function f(x). It is simple, fast, and converges quickly when the initial guess is close to the actual solution. However, the method requires both the function f(x) and its derivative f′(x), which limits its use to problems where the derivative can be computed easily.

The method assumes that f(x) is continuous and differentiable, and that the equation has a root near the initial guess. The basic idea is to start from an initial point x_1 and draw the tangent to the curve y=f(x) at the point $(x_1,f(x_1))$. The point where this tangent intersects the x-axis becomes the next estimate x_c. Repeating this process gives successive approximations $x_3,x_4,\ldots$, moving closer to the actual root.

The slope of the tangent at x_1 is:

$$f'(x_1)=(f(x_1)-0)/(x_1-x_2)$$

Rearranging the equation gives the second approximation:

$$x_2=x_1-f(x_1)/f'(x_1)$$

In general, the Newton–Raphson formula for computing the next iteration is:

$$x_{i+1}=x_i-f(x_i)/f'(x_i)$$

This iterative process continues until the value of x becomes sufficiently close to the actual root

2. Graphical Representation and Convergence

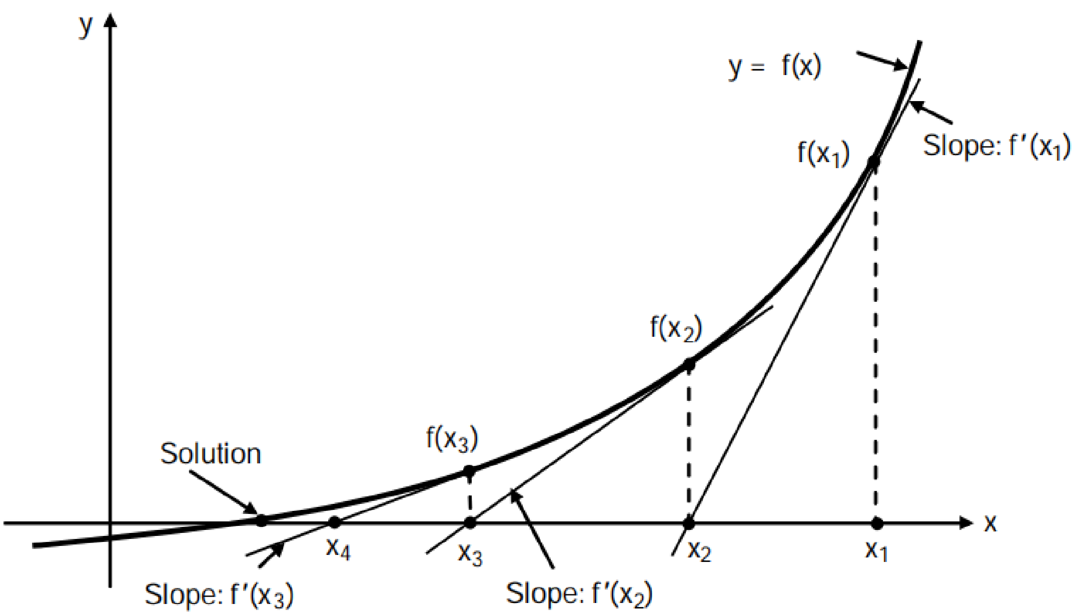

Figure : Newton-Raphson Method.

The Newton–Raphson method

3. Step-by-Step Algorithm for the method of False Position

Algorithm for Newton-Raphson Method:

1. Select a point x_1 as an initial guess of the solution.
2. For i=1,2,……, until the error is smaller than a specified value, compute x_{i+1} by using $x_{i+1}=x_i-f(x_i)/f'(x_i)$
3. .

Two error estimates that are generally used in Newton-Raphson method are given below:

The iterations are stopped when the estimated relative error

$$\left|\frac{x_{i+1}-x_i}{x_i}\right| \leq \in$$

The iterations are stopped when the absolute value of $f(x_i)$ is smaller than some number δ :

$$|f(x_i)| \leq \delta$$

The Newton-Raphson method, when successful, works well and converges fast. Convergence problems occur when the value of $f'(x)$ is close to zero in the vicinity of the solution, where $f(x)=0$. Newton-Raphson method generally converges when $f(x)$, $f'(x)$, and $f''(x)$ are all continuous, if $f'(x)$ is not zero at the solution and if the starting value x_1 is near the actual solution.

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4. Example 1 : $f(x)=x^4-11x+8=0$

Problem : Use Newton-Raphson Method to Find the Real Root Near 2 of the Equation x^4 - 11x + 8 = 0 Accurate to Five Decimal Places.

Solution:

Here,

f(x)= x^4 - 11*x + 81   f'(x)=4x^3 – 11

Let the initial guess $x_0=2$

We compute $f(x_0)$ and $f'(x_0)$ as follows:

$f(x_0)=f(2)=2^4-11*(2)+8=2$ $f'(x_0)=f'(2)=4*(2)^3-11=21$

Therefore,
$x_1=x_0-f(x_0)/f'(x_0)=2-2/21=1.90476$

Next, we compute x_2,x_3, and x_4 as follows:

$x_2=x_1-f(x_1)/f'(x_1)=1.90476$

$x_3=x_2-f(x_2)/f'(x_2)=1.89209$

$x_4=x_3-f(x_3)/f'(x_3)=1.89188$

Hence, the root of the equation is 1.89188.