Gauss elimination method

1. Core Concepts and Formula

Consider the following system of linear simultaneous equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad (1)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad (2)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad (3)$$

Gauss elimination is a popular technique for solving simultaneous linear algebraic equations. It reduces the coefficient matrix into an upper triangular matrix through a sequence of operations carried out on the matrix. The vector b is also modified in the process. The solution vector $\{x\}$ is obtained from a backward substitution procedure.

Two linear systems Ax = b and A′ x = b' of equations are said to be equivalent if any solution of one is a solution of the other. Also, let Ax = b is a linear non-homogeneous system of n equations. Suppose we subject this system to the system of following operations:

1. Multiplication of one equation by a non-zero constant.
2. Addition of a multiple of one equation to another equation.
3. Interchange of two equations.

If the sequence of operations produce the new system $\mathbf{A'x} = b'$, then both the systems $\mathbf{Ax} = b$ and $A'x = b'$ are equivalent. In particular, then $\mathbf{A}$ is invertible if $\mathbf{A'}$ is invertible. In Gauss elimination method, we adopt this and the elimination process is based on this theorem.

In Gauss elimination method, the unknowns are eliminated such that the elimination process leads to an upper triangular system and the unknowns are obtained by back substitution. It is assumed $a_{11} \neq 0$. The method can be described by the following steps:

Step 1: Eliminate x_1 from the second and third equations.

Using the first equation (1), the following operations are performed:

$$(2) - (a_{21}/a_{11,})*(1) \qquad \text{and} \qquad (3) - (a_{31}/a_{11,})*(1)$$

gives

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad (4)$$

$$a'_{22}x_2 + a'_{23}x_3 = b'_2 \quad (5)$$

$$a'_{32}x_2 + a'_{33}x_3 = b'_3 \quad (6)$$

Equation (4) is called the *pivotal equation* and the coefficient a_{11} is the *pivot*.

Step 2: Eliminate x_2 from the Eq. (6) using Eq. (5) by assuming $a'_{22} \neq 0$. We perform the following operation:

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$$(6) - (a'_{32}/a'_{22,})*(5)$$

to obtain $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ (7)

$$a'_{22}x_2 + a'_{23}x_3 = b'_2 \quad (8)$$

and $a^2_{33}x_3 = b''_3$ (9)

Here Eq. (8) is called the *pivotal equation* and the coefficient a'_{22} is the *pivot*.

Step 3: To find x_1, x_2 and x_3, we apply back substitution starting from Eq. (9) giving x_3, then x_2 from Eq. (8) and x_1 from Eq. (7).

Pivoting:

Gauss elimination method fails if any one of the pivots in the above equations (1) to (9) becomes zero. To overcome this difficulty, the equations are to be rewritten in a slightly different order such that the pivots are not zero.

Partial pivoting method:

Step 1: The numerically largest coefficient of x_1 is selected from all the equations are pivot and the corresponding equation becomes the first equation (1).

Step 2: The numerically largest coefficient of x_2 is selected from all the remaining equations as *pivot* and the corresponding equation becomes the second equation (5). This process is repeated till an equation into a simple variable is obtained.

Complete pivoting method:

In this method, we select at each stage the numerically largest coefficient of the complete matrix of coefficients. This procedure leads to an interchange of the equations as well as interchange of the position of variables.

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2. Example 1 :

Problem : Solve the following equations by Gauss elimination method: $2x + 4y - 6z = -4$

$x + 5y + 3z = 10$

$x + 3y + 2z = 5$

Solution:

$$2x + 4y - 6z = -4 \qquad \text{(E.1)}$$

$$x + 5y + 3z = 10 \qquad \text{(E.2)}$$

$$x + 3y + 2z = 5 \qquad \text{(E.3)}$$

To eliminate x from (E.2) and (E.3) using (E.1):

$$2x + 4y - 6z = -4$$

$$x + 5y + 3z = 10 \quad 1 \times (-2)$$

$$x + 3y + 2z = 5 \quad 1 \times (-2)$$

$$2x + 4y - 6z = -4$$

$$-2x - 10y - 6z = -20$$

$$-2x - 6y - 4z = -10$$

$$2x + 4y - 6z = -4$$

Row 1 + Row 2: $-6y - 12z = -24$ (E.6)

Row 1 + Row 3: $2y - 10z = -14 \quad 1 \times (-3)$ (E.5)

To eliminate y from (E.5) using (E.4):

$$2x + 4y - 6z = -4$$

$$-6y - 12z = -24$$

$$6y + 30z = 42$$

$$2x + 4y - 6z = -4$$

$$-6y - 12z = -24$$

Row 2 + Row 3: $18z = 18 \Rightarrow z=1$

Evaluation of the unknowns by back substitution:

$$-6y - 12z = -24$$

$$6y = 24 - 12z \quad \Rightarrow y = \frac{24 - 12 \times 1}{6} \quad \Rightarrow \boxed{y = 2}$$

$$2x + 4y - 6z = -4$$

$$2x = -4 - 4y + 6z \quad \Rightarrow x = \frac{-4 - 4 \times 2 + 6 \times 1}{2} \quad \Rightarrow \boxed{x = -3}$$

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2. Example 2 :

Problem : Use the method of Gaussian elimination to solve the following system of linear equations:
$x_1 + x_2 + x_3 – x_4 = 2$
$4x_1 + 4x_2 + x_3 + x_4 = 11$
$x_1 – x_2 – x_3 + 2x_4 = 0$
$2x_1 + x_2 + 2x_3 – 2x_4 = 2$

Solution:

Let , $2x_1 + x_2 + 2x_3 – 2x_4 = 2$ (E.1)
In the first step, eliminate x_1 terms from second, third and fourth equations of the set of equations (E.1) to obtain:
$x_1 + x_2 + x_3 – x_4 = 2$
$–3x_3 + 5x_4 = 3$
$–2x_2 – 2x_3 + 3x_4 = –2$
$–x_2 = –2$ (E.2)
Interchanging columns in Eq. (E.2) putting the variables in the order x_1, x_4, x_3 and x_2 as
$x_1 – x_4 + x_3 + x_2 = 2$
$–5x_4 – 3x_3 = 3$

$3x_4 – 2x_3 – 2x_2 = –2$
$–x_2 = –2$
(E.3)
In second step, eliminate x_4 term in third equation of the set of equations (E.3)
$x_1 – x_4 + x_3 + x_2 = 2$
$5x_4 – 3x_3 = 3$
$–1/5x_3 – 2x_2 = –19/5$
$–x_2 = –2$
Now, by the process of back substitution, we have
$x_2 = 2$, $x_3 = –1$, $x_4 = 0$, $x_1 = 1$.