The Bisection Method: Finding Roots Reliably

1. Introduction and Core Concepts

The Bisection Method, also known as the **interval halving method**, is the most reliable way to find the root of a function f(x) = 0.

It works based on a simple mathematical rule: if a function f(x) is **continuous** over an interval [a, b], and the function values f(a) and f(b) have **opposite signs** (one positive, one negative), then a root MUST exist somewhere between 'a' and 'b'.

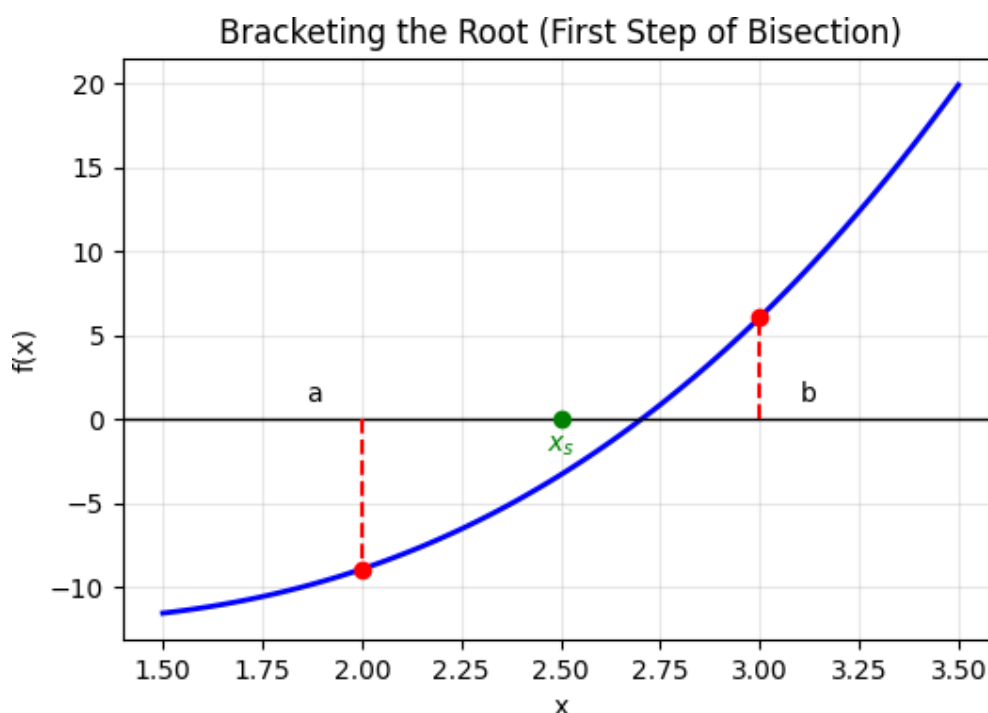The method constantly reduces the size of the interval until it is small enough to give an accurate root.

Figure 1: Finding a root by bracketing (Interval [a, b] is where f(a)f(b) < 0).

2. Algorithm for the Bisection Method

This procedure repeatedly halves the interval containing the root:

1. Compute the first estimate of the numerical solution x_s by:
   $x_s=(a+b)/2$.

2. Determine whether the true solution is between a and x_s or between x_s and b by checking the sign of the product $f(a)*f(x_s)$:.
   If $f(a)*f(x_s) < 0$, the true solution is between a and x_s .
   If $f(a)*f(x_s) > 0$, the true solution is between x_s and b.
   If b-c <= tolerace then accept c as the root and stop.

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3.   Choose the subinterval that contains the true solution (a to x_s or x_s to b) as the new interval (a, b) , and go back to step 1.

3. Example 3.1: Polynomial Equation

Problem: Use the Bisection method to find a root of the equation x^3 - 4x - 8.95 = 0 accurate to three decimal places.

Initial Check:
f(2) = 2^3 - 4(2) - 8.95 = -8.95 < 0
f(3) = 3^3 - 4(3) - 8.95 = 6.05 > 0
A root lies between a=2 and b=3.

n	a	b	Xs1	b-Xs1	f(Xs1)
1	2.0	3.0	2.5	0.5	-3.25
2	2.5	3.0	2.75	0.25	0.84688
3	2.5	2.75	2.625	0.125	-1.36211
4	2.75	2.625	2.6875	-0.0625	-0.28911
5	2.75	2.6875	2.71875	-0.03125	0.27092
6	2.6875	2.71875	2.70313	0.01563	-0.01108
7	2.71875	2.70313	2.71094	-0.00781	0.12942
11	2.71094	2.71191	2.71143	0.00049	0.13824

Result: The root is 2.711 accurate to three decimal places.

4. Worked Example E3.2: Transcendental Equation

Problem: Find one root of e^x - 3x = 0 correct to two decimal places. Initial
Check:
f(1.5) = e^(1.5) - 3(1.5) = -0.01831 < 0
f(1.6) = e^(1.6) - 3(1.6) = 0.15303 > 0
A root lies between a=1.5 and b=1.6.

n	a	b	Xs1	b-Xs1	f(Xs1)
1	1.5	1.6	1.55	0.05	0.06147
2	1.5	1.55	1.525	0.025	0.02014
3	1.5	1.525	1.5125	0.0125	0.00056
4	1.5	1.5125	1.50625	0.00625	-0.00896
5	1.50625	1.5125	1.50938	0.00313	-0.00422
6	1.50938	1.5125	1.51094	0.00156	-0.00184

Result: The root is x = 1.51 accurate up to two decimal places.