Derivation of the Governing Equations of Nonlinear Deflections Subject to Several Applied Forces

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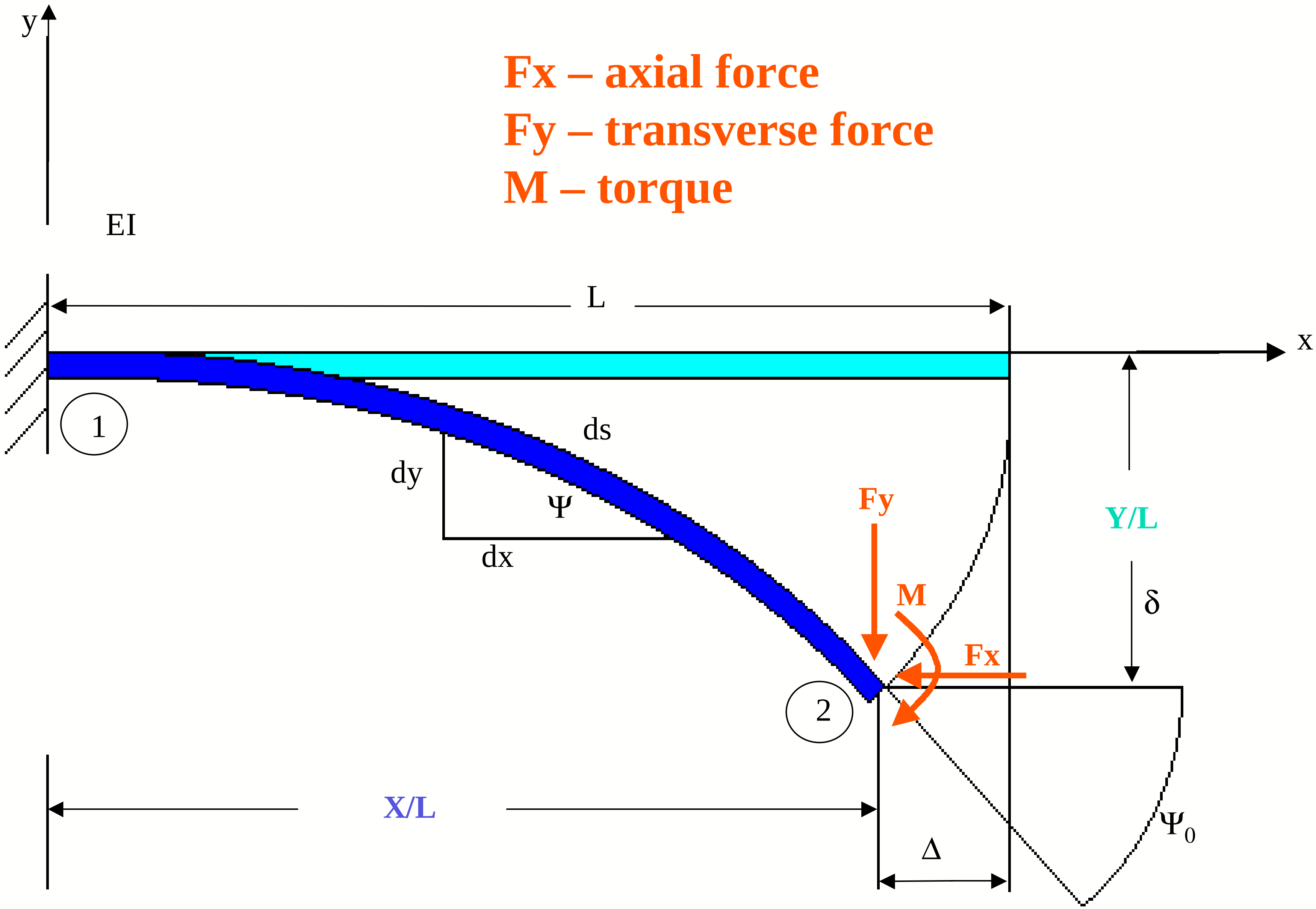
Nomenclature: Nonlinear Deflection due to simultaneous forces

Fx – axial force
Fy – transverse force
M – torque



Derivation:

Beam Curvature due to M and F

The curvature at each point *s* along the beam is defined to be

$$curvature \equiv \frac{1}{r(s)} = \frac{-d^2 y}{dx^2}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\left(\frac{-2}{3}\right)} = \frac{d\psi(s)}{ds} = \frac{M(s)}{EI}$$

If an external transverse force is applied at the node 2, the curvature at s is

$$\frac{M(s)}{EI} = \frac{F_o(L-x-\Delta\ )}{EI} \tag{1}$$

Change of variables

Equation (1) contains two independent variables, s and x. Eliminate x by differentiating (1) with respect to s, then re-integrate it.

Noting $dx/ds=\cos(\psi)$, differentiating (1) gives

$$\frac{d^2\psi}{ds^2} = \frac{-F_o}{EI}\frac{dx}{ds} = \frac{-F_o}{EI}\cos\psi \tag{2}$$

Applying the identity $\int \frac{d^2\psi}{ds^2} d\psi = \frac{1}{2}\left(\frac{d\psi}{ds}\right)^2$, the integration of (2) leads to

$$\frac{1}{2}\left(\frac{d\psi}{ds}\right)^2 = \frac{-F_o}{EI}\sin\psi + C \tag{3}$$

Boundary conditions

To solve for the integration constant in (3) apply the following boundary condition at the end node 2

$$\left.\frac{d\psi}{ds}\right|_{\substack{\psi=\psi_o \\ s=L}} = 0 \qquad (4)$$

since the moment vanishes at node 2. Therefore, equation (3) becomes

$$\frac{d\psi}{ds} = \sqrt{\frac{2F_o}{EI}\left(\sin\psi_o - \sin\psi\right)} \qquad (5)$$

$$\Rightarrow$$

$$\int_0^{\psi_0} ds = L = \int_0^{\psi_0} \frac{d\psi}{\sqrt{\frac{2F_o}{EI}\left(\sin\psi_o - \sin\psi\right)}} \qquad (6)$$

But we need to end up with a solution of the form $\psi_0 = \psi_0\left(F_o, EI, L\right)$

Transformation to Elliptic Integrals

Elliptic integrals have been well characterized (see M. Abramowitz, A. Stegun, Handbook of Mathematical Functions). Transforming equation (6) to an elliptic integral form provides us with their benefits.

Introducing two new variables p and ϕ defined as

$$p^2 \equiv \frac{1}{2}\left(1+\sin\psi_o\right) \qquad (7) \qquad \text{and} \qquad \sin^2\phi \equiv \frac{1+\sin\psi}{1+\sin\psi_o} \qquad (8)$$

Seek substitutions for $d\Psi$, $sin(\Psi)$, and $sin(\Psi_0)$ in equation (6).

Find the differential element $d\Psi$ by the differentiation of equation (8) with respect to ϕ, and then using the trigonomic identity

$$\cos\psi = \sqrt{1-\sin^2\psi} = 2p\sin\phi\sqrt{1-p^2\sin^2\phi} \qquad (9)$$

This leads to

$$d\psi = \frac{4p^2\sin\phi\cos\phi}{2p\sin\phi\sqrt{1-p^2\sin^2\phi}}\,d\phi \qquad (10)$$

And from (7) & (8)

$$\sin\psi_o = 2p^2-1 \qquad (11) \qquad\qquad \sin\psi = 2p^2\sin^2\phi-1 \qquad (12)$$

The Elliptic Integral Forealization of Force and Displacement, SIMO

Substitution of equations (10-12) into equation (6) results in

$$\sqrt{\frac{F_0 L^2}{EI}} = \int_{\phi_1(p)}^{\phi_2(p)} \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} \qquad (13)$$

where the lower and upper limits of the integral are determined from equation (11)

$$\psi = 0 \Rightarrow \phi_1 = \sin^{-1}\left(\frac{1}{\sqrt{2}\,p}\right) \qquad \psi = \psi_0 \Rightarrow \phi_2 = \frac{\pi}{2} \qquad (14)$$

Associating equation (13) with the complete and incomplete elliptic integrals of the first kind, we have

$$\sqrt{\frac{F_0 L^2}{EI}} = K(p) - F(p, \phi_1) \qquad (15)$$

Realization of Force and Displacement, SIMO

Given ψ_0, it's required force is found by plugging (7) & (14) into (15).

Equation (15) is the *nondimensionalized* external force at node 2.

After finding the nondimensional force, the relationship $dy = ds\sin(\psi)$ gives us the nondimensional transverse deflection, $\delta/L=\delta(\Psi_0)/L$

$$\frac{\delta}{L} = \int_0^{\psi_0} \frac{1}{\sqrt{2}}\sqrt{\frac{EI}{F_0L^2}}\frac{\sin\psi d\psi}{\sin(\psi_0)-\sin(\psi)} \tag{16}$$

Equation (1) with the BCs $\psi\big|_{x=0} = 0$, and $\left.\frac{d\psi}{ds}\right|_{x=0} = \frac{M_0}{EI} = \sqrt{\frac{2F_0\sin\psi_0}{EI}} = F_0(L-\Delta_x)$

gives the nondimensional projected beam shortening as

$$\frac{\Delta}{L} = 1-\sqrt{\frac{EI}{F_0L^2}}\sqrt{2(2p^2-1)} \tag{17}$$

Nonlinear StiffnesPlot of Nonlinear Stiffness vs Theory

The outputs of equations (15-17) as functions of the input Ψ_0 are plotted as “O’s” below. The solid curves are 3rd order, piecewise continuous, polynomial fits.
To obtain nonlinear stiffness, we first assume that the curves can be approximated by a third order polynomial of the form

$$\sqrt{\frac{F_0 L^2}{EI}} = A + B\frac{q}{L} + C\left(\frac{q}{L}\right)^2 + D\left(\frac{q}{L}\right)^3 \qquad (18)$$

were q stands for θ, x, y, etc. Seeing that the solution has odd symmetry, we only need to keep constants B and D, which also eases iterative computations.
Absorbing the material and geometric terms into B and D, respectively K_1 and K_2, we find that

$$F_0 = K_{1,i}q + K_{2,i}q^3 \qquad (19)$$

The coefficients of these polynomial curves are associated with the linear stiffnesses $K_{1,i}$ and the cubic nonlinearities $K_{2,i}$.

In order to maintain accuracy, (19) is applied in a continuous piecewise fashion by dividing the total physical range into, say, 4 intervals, were *i(q)=[1,4]*. (Fig 2)

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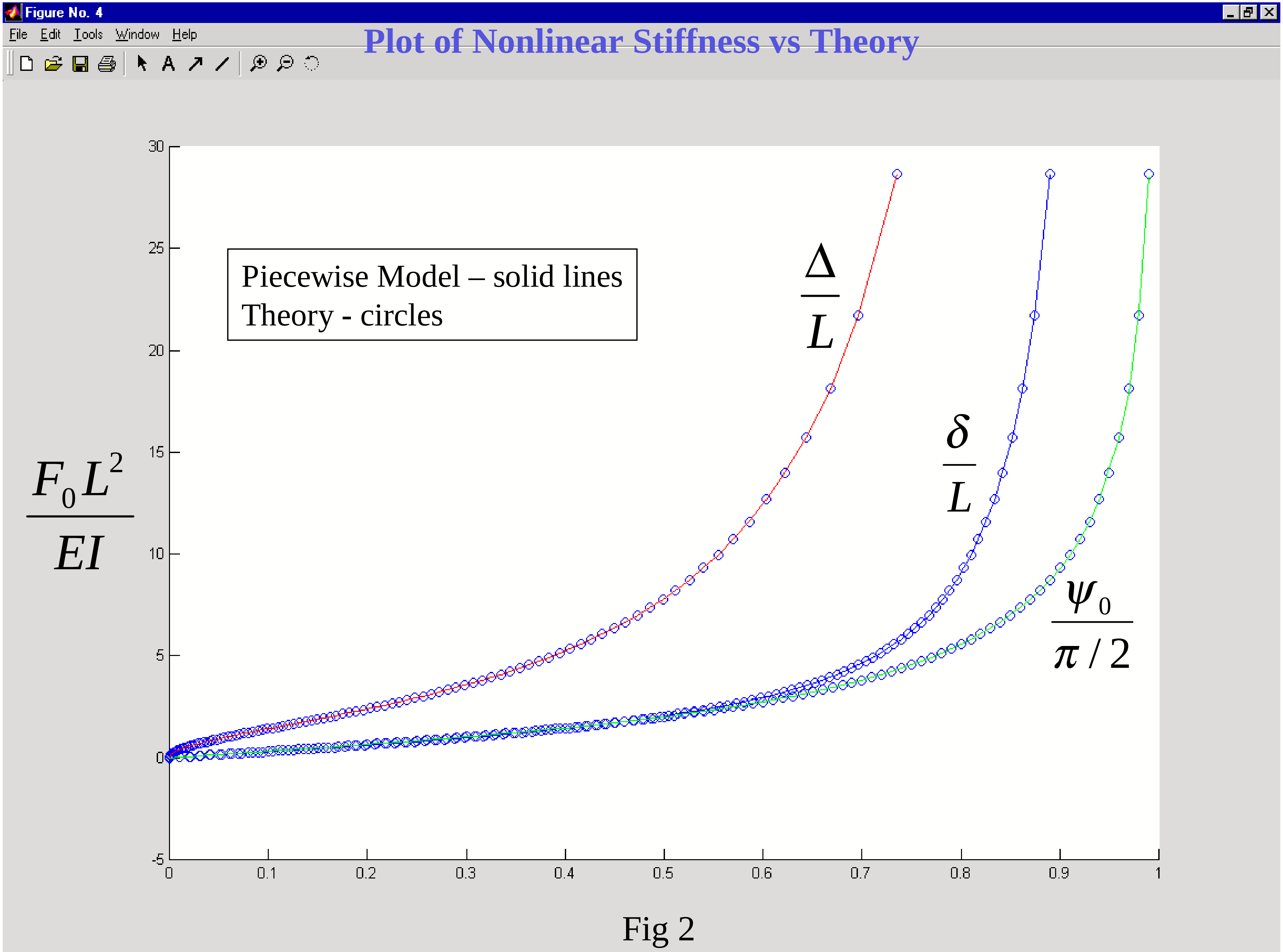
Figure No. 4
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Plot of Nonlinear Stiffness vs Theory
Piecewise Model – solid lines
Theory - circles
$\frac{F_0 L^2}{EI}$
$\frac{\Delta}{L}$
$\frac{\delta}{L}$
$\frac{\psi_0}{\pi/2}$
30
25
20
15
10
5
0
-5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Fig 2

Intuition can be gained by a geometrical representation: Principle of Elastic Similarity

Using the geometric properties of the above elliptic integrals together with the principle of elastic similarity we find an intuitive relationship. This type of analysis was first reported by [R. Fay, "A new approach to the analysis of the deflection of thin cantilevers," Journal of Applied Mechanics 28, Trans. ASME, 83, Ser. E (1961) 87].

From similarity:

From Fig 1, using your imagination, assume that the beam, which remains anchored at node 1, continues arching to the left of node 1 until it reaches a new node 0. Node 0 is such that it is oriented normal to the horizontal axis (see Fig 3). The length of the section from node 0 to 1 is L_2. Also imagine that the shape of L_2 is the shape that would be required if L_1 is to remain unaltered when node 1 is no longer fixed.

Geometric Properties

From geometry: From the geometric properties of the elliptic functions, we make the following identifications.

$$L = L_1 + L_2 = K(p)/k$$

$$\sin(\theta_B/2) = p\sin(\phi_B)$$

$$p = \sin(\gamma/2)$$

$$\gamma = \pi/2 + \Psi_0$$

$$L_2 = \frac{1}{k}\int_0^{\phi_B} \frac{d\phi}{\sqrt{1-p^2\sin^2(\phi)}}$$

$$k = \sqrt{\frac{F_0}{EI}}$$

$$L_1 = L - L_2 = \frac{1}{k}\{K(p) - F(p,\phi_B)\}$$

$$h = 2p/k$$

$$\Delta = L_1 - h\cos(\phi_B)$$

$$\delta = \{[2E(p,\phi_B) - F(p,\phi_B)] - [2E(p) - K(p)]\}/k \Rightarrow b - \delta$$

Identification of terms

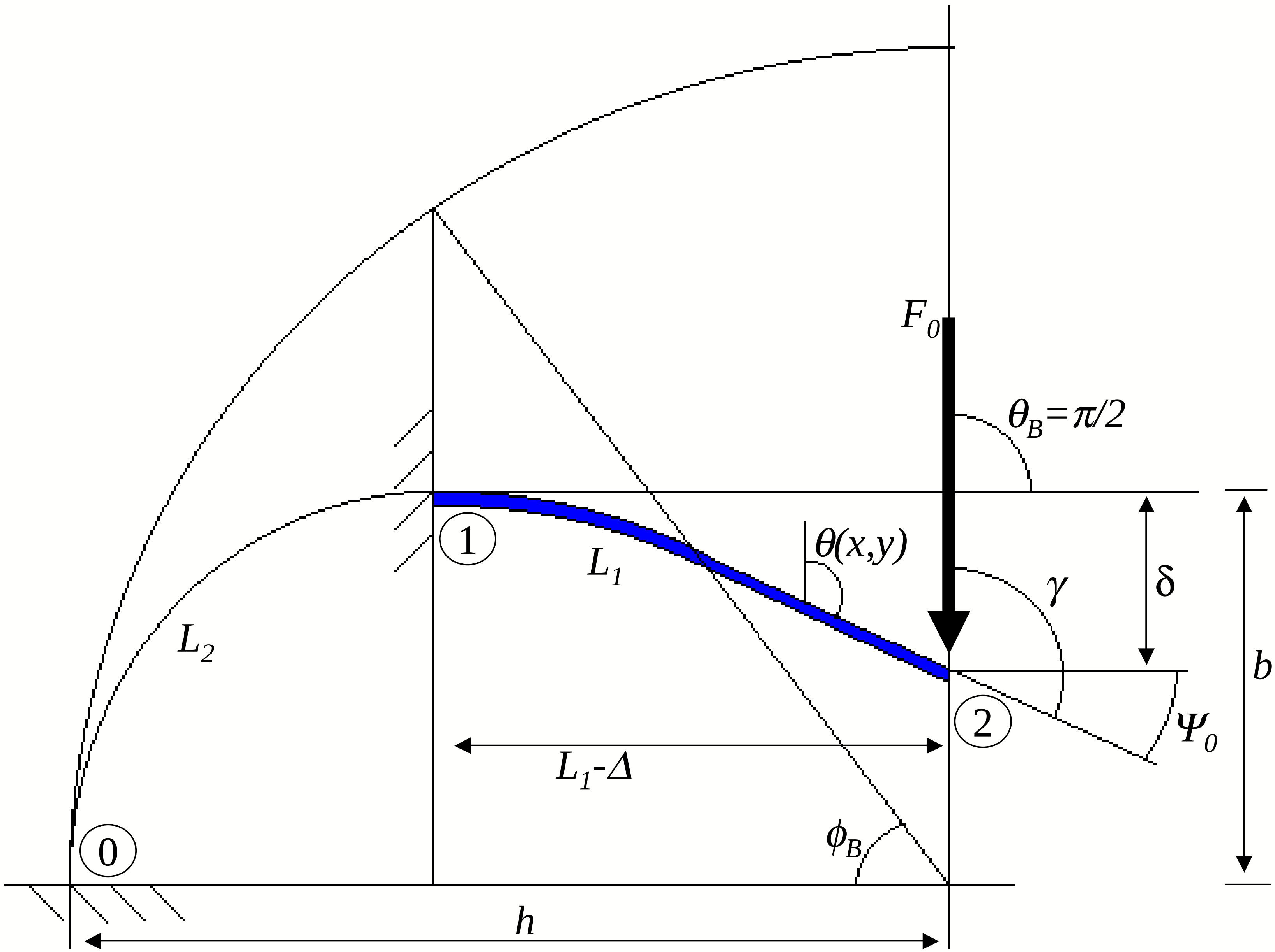


Fig 3

Nonlinear Deflection - *F_x, F_y, & M*

Here we attempt answer the question:
Given the angles of node2, determine θ, x, y, F_x, F_y, and M.

Method of derivation:

By intuition, jumping straight to elastic similarity and the geometric properties of elliptic integrals.

End result:

θ, x, & y as a function of F_x, F_y, and M

$F = K_1 q + K_2 * q^3 \quad => \quad F=F(q)$

Here's my modification to analyzing nonlinear deflections with simultaneous Fx, Fy, and M Forces.

I now present a way to extend the above method to a more general situation where Fx, Fy, &M are simultaneously applied.
The geometric quantities listed below are self-described in the new geometric representation shown in Fig 4.

Given Ψ_0 and θ_B, the following holds (see fig 4)

$$L_3 = h\cos(\phi_0)$$

$$p = \sin(\theta_B/2)/\sin(\phi_B) = \sin\left(\frac{\Psi_0 + \theta_B}{2}\right)$$

$$L_2 k = F(p,\phi_0) - F(p,\phi_B) = \sqrt{\frac{F_0 L_2^{\,2}}{EI}}$$

$$M = L_3 F_0$$

$$F_x = F_0\cos(\theta_B)$$

$$F_y = F_0\sin(\theta_B)$$

Here's My Modified Geometric Representation for Fx,Fy,M

$M=F*L3$
$Fx=F_0\cos(\theta_b)$
$Fy=F_0\sin(\theta_b)$

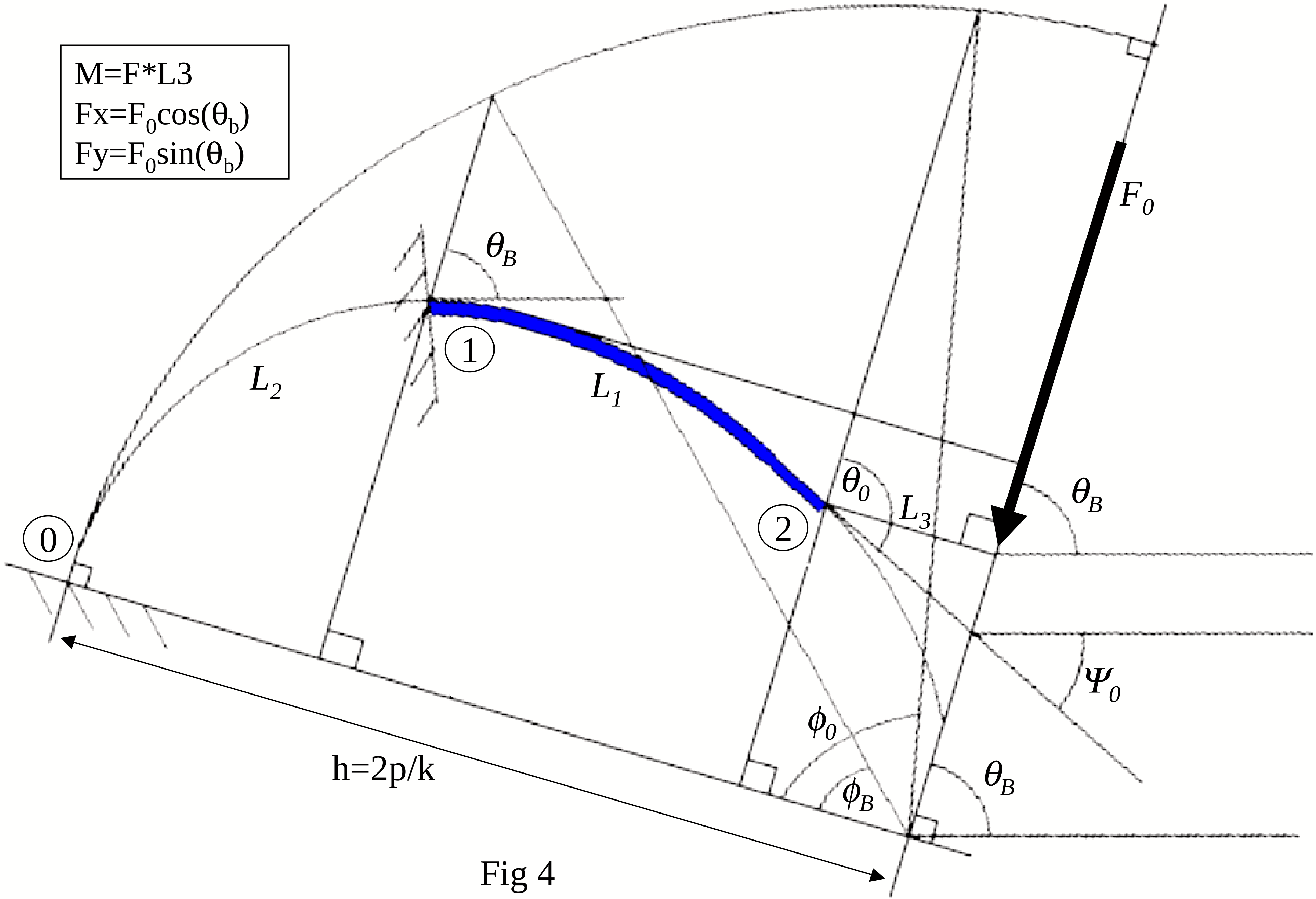
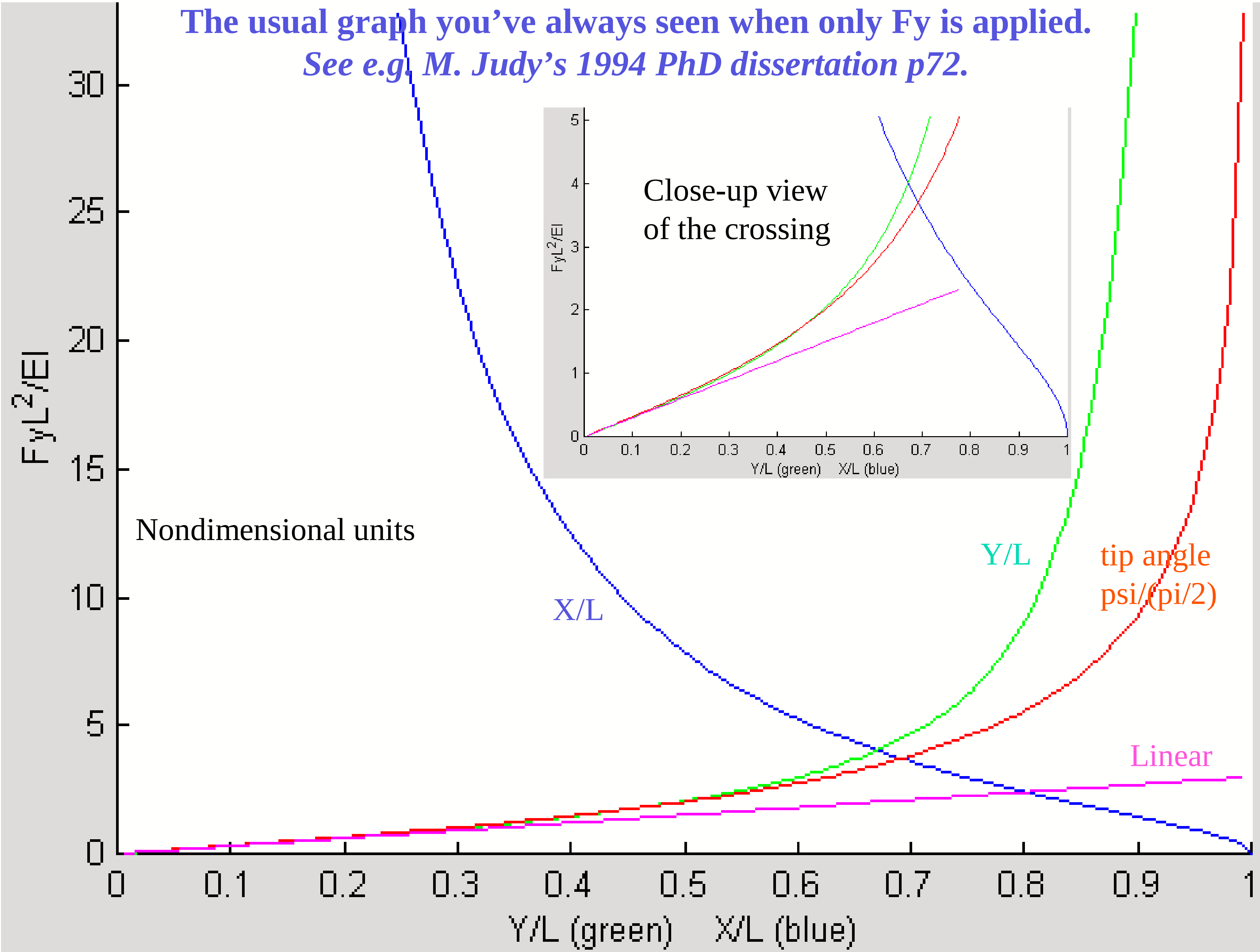
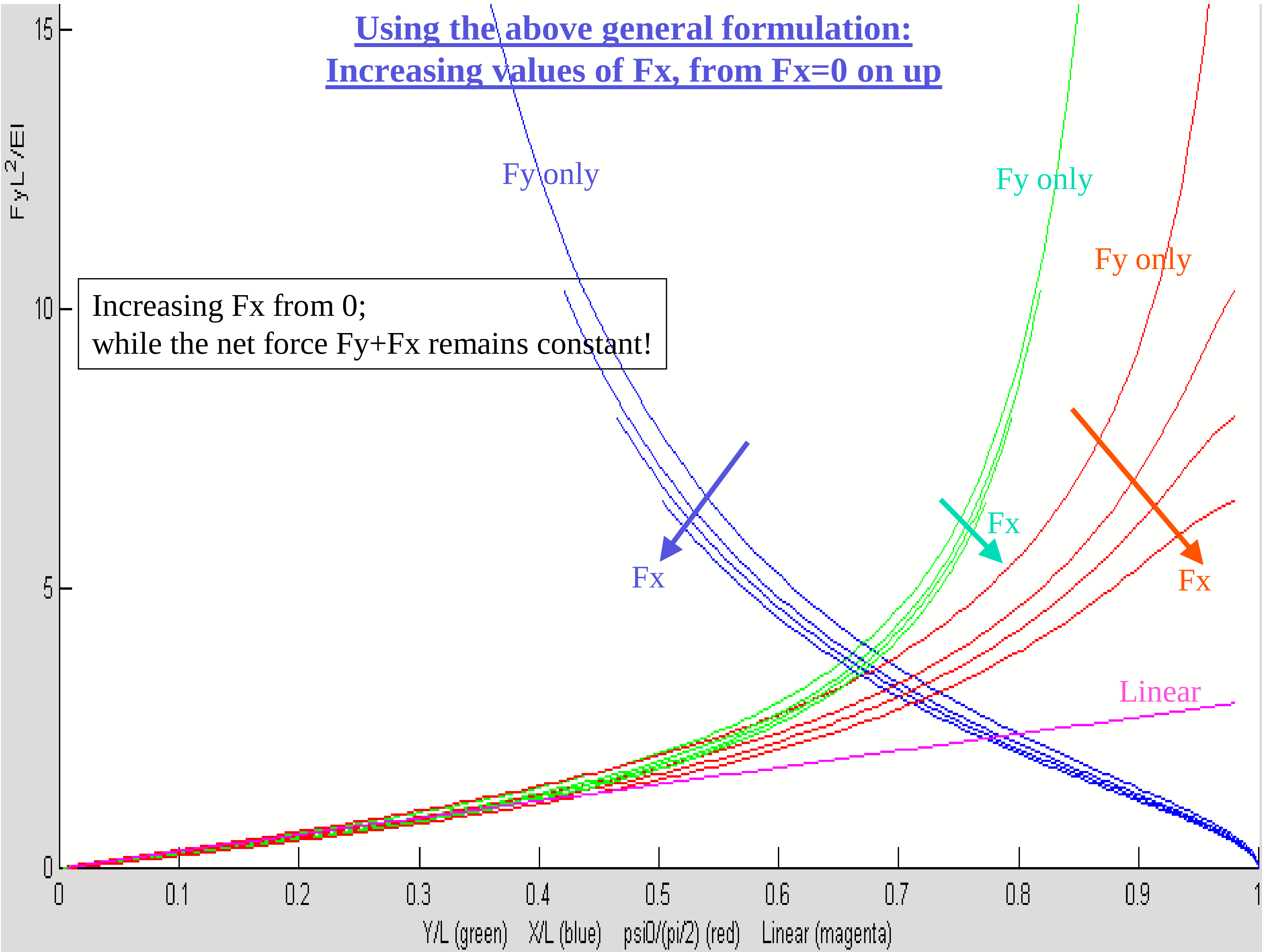


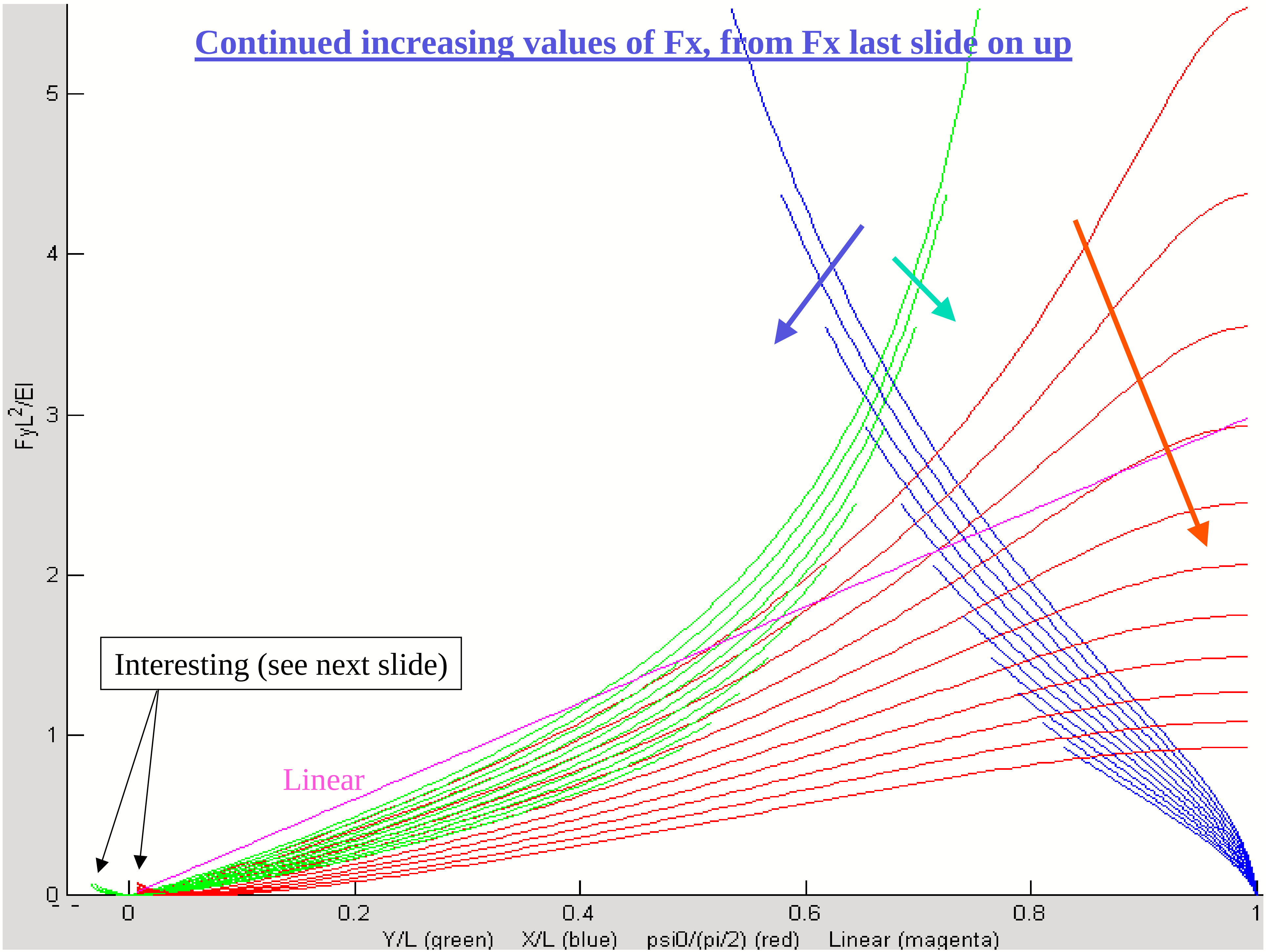
Fig 4

The usual graph you've always seen when only Fy is applied.

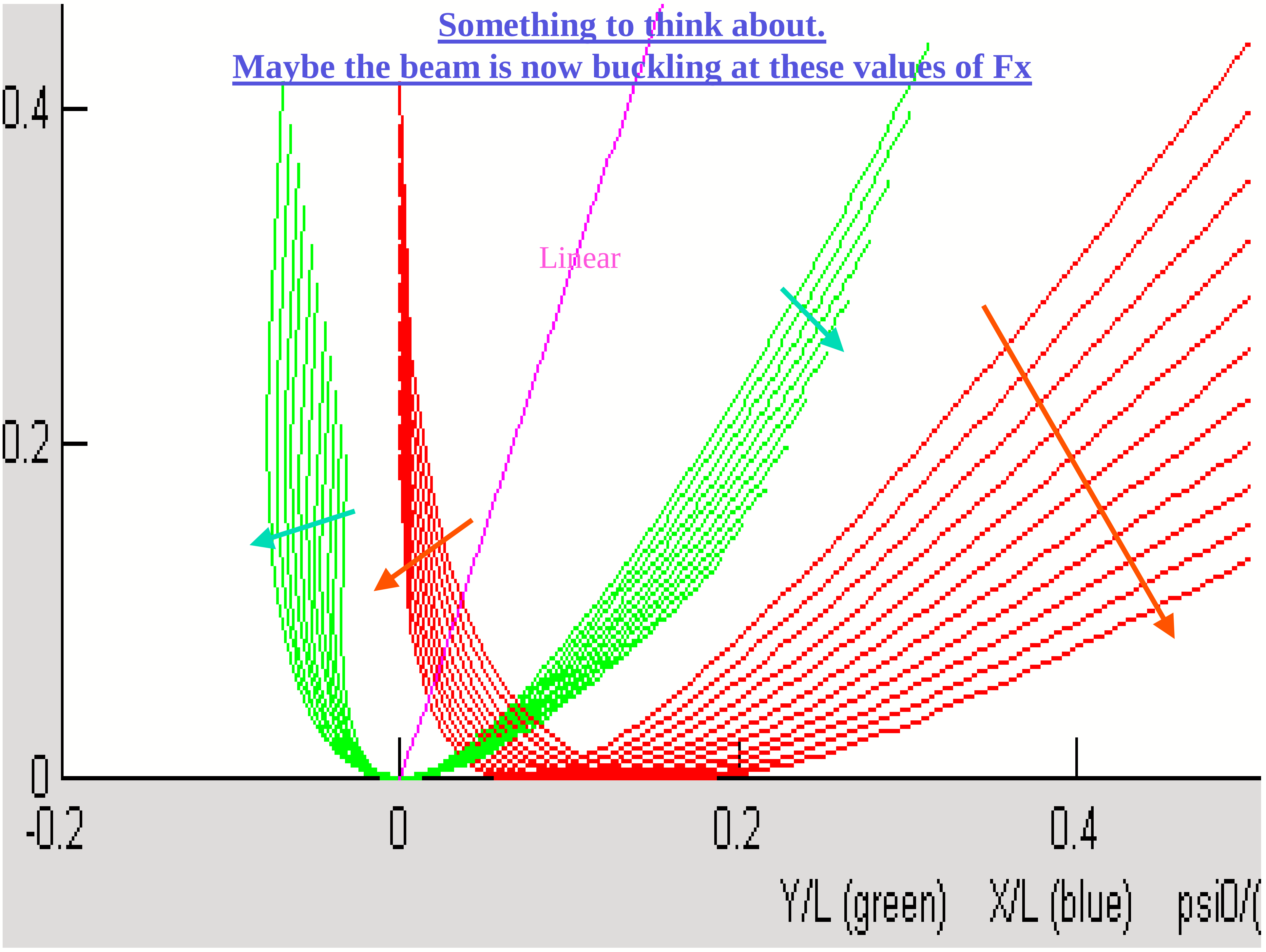
See e.g. M. Judy's 1994 PhD dissertation p72.



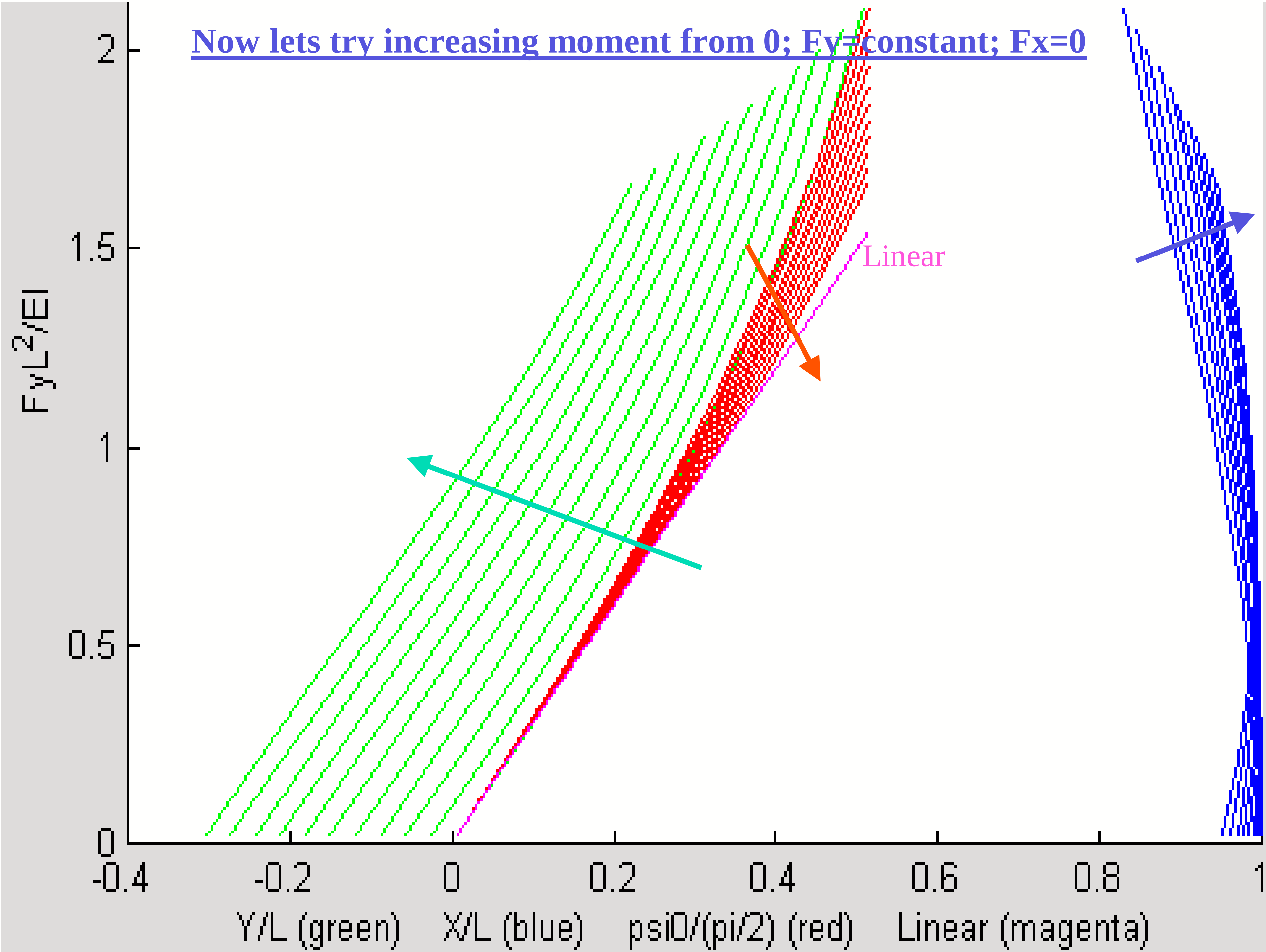

Using the above general formulation:
Increasing values of Fx, from Fx=0 on up
Increasing Fx from 0;
while the net force Fy+Fx remains constant!
Fy only
Fy only
Fy only
Fx
Fx
Fx
Linear
Fy L^2/EI
15
10
5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Y/L (green) X/L (blue) psi0/(pi/2) (red) Linear (magenta)

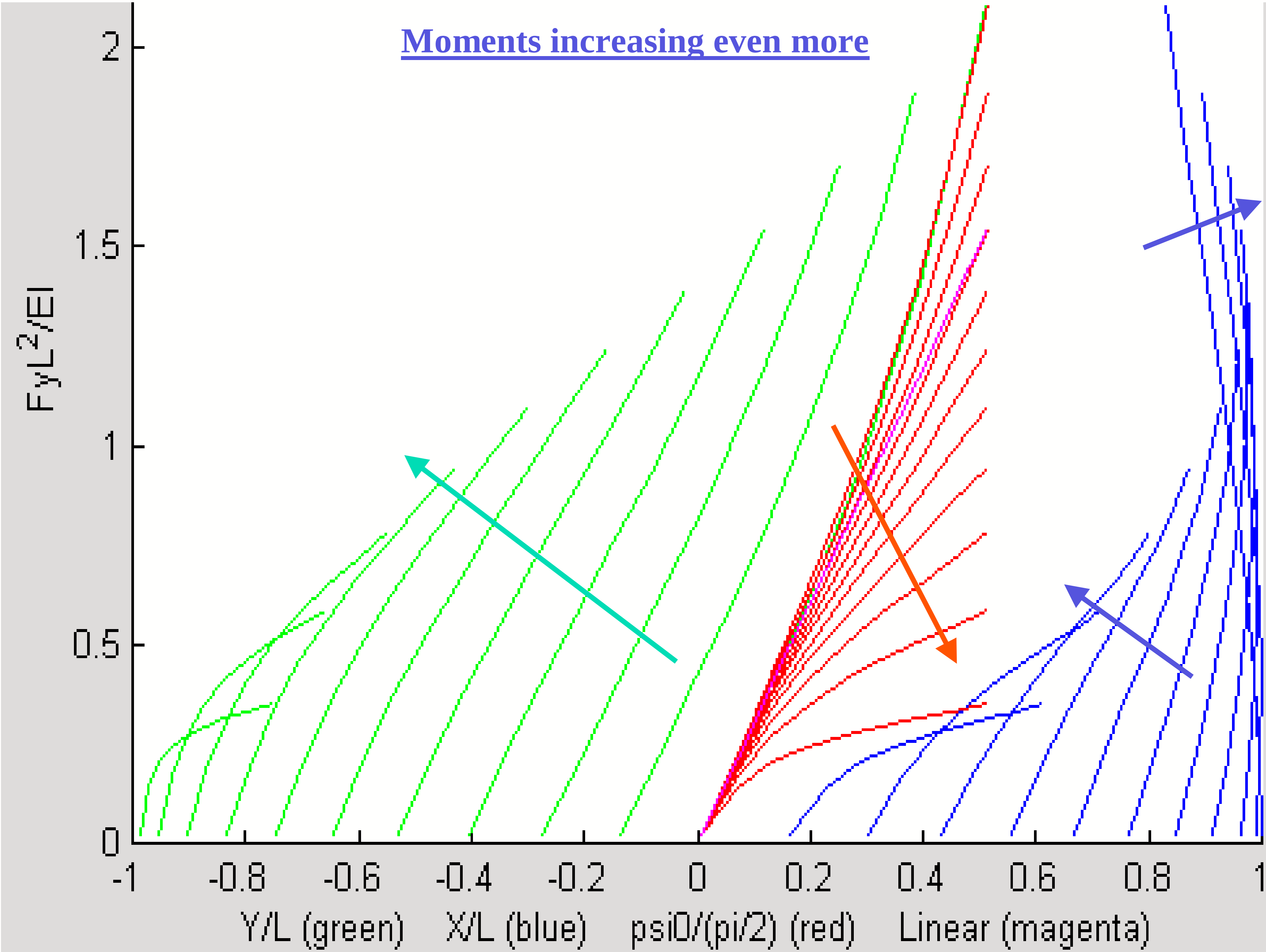


Continued increasing values of Fx, from Fx last slide on up
Interesting (see next slide)
Linear
F_yL^2/EI
0
1
2
3
4
5
0
0.2
0.4
0.6
0.8
1
Y/L (green) X/L (blue) psi0/(pi/2) (red) Linear (magenta)



Something to think about.
Maybe the beam is now buckling at these values of Fx
Linear
0.4
0.2
0
-0.2
0
0.2
0.4
Y/L (green)
X/L (blue)
psiO/(


Now lets try increasing moment from 0; Fy=constant; Fx=0
Linear
FyL2/EI
0
0.5
1
1.5
2
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Y/L (green)   X/L (blue)   psi0/(pi/2) (red)   Linear (magenta)



Moments increasing even more
$F_y L^2/EI$
0
0.5
1
1.5
2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Y/L (green) X/L (blue) psi0/(pi/2) (red) Linear (magenta)

So, to get the nonlinear results, this is what I do:

Input: Ψ_0 and θ_B angles

Output: Fx, Fy, M, X/L, & Y/L

$\theta_0 = \theta_b + \Psi_0$;

p=sin(θ_0 /2);

L3k=2*p*cos(ϕ_0); %associated with moment

ϕ_b=asin(sin(θ_b/2)/p);

%elliptic integration

Ephib=quad8('elliptic',0,ϕ_b,[],[],p,2); %elliptic integral of the second kind

Fphib=quad8('elliptic',0,ϕ_b,[],[],p,1); %elliptic integral of the fist kind

Ephi0=quad8('elliptic',0,ϕ_0,[],[],p,2); %elliptic integral of the second kind

Fphi0=quad8('elliptic',0,ϕ_0,[],[],p,1); %elliptic integral of the first kind

L1k=Fphi0-Fphib; %associated with force

L1minusXoverL1=2*p*(cos(ϕ_b)-cos(ϕ_0))/L1k; %beam shortening

YoverL1=(2*Ephib-Fphib)/L1k - (2*Ephi0-Fphi0)/L1k; %beam transverse deflection

FL2EIx=L1k^2*cos(θ_b); %nondimensional force

FL2EIy=L1k^2*sin(θ_b); %nondimensional force

MLEI=L3k*L1k; %nondimensional moment