`Fatou.jl`: Complex Dynamics Visualization

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MATH 760
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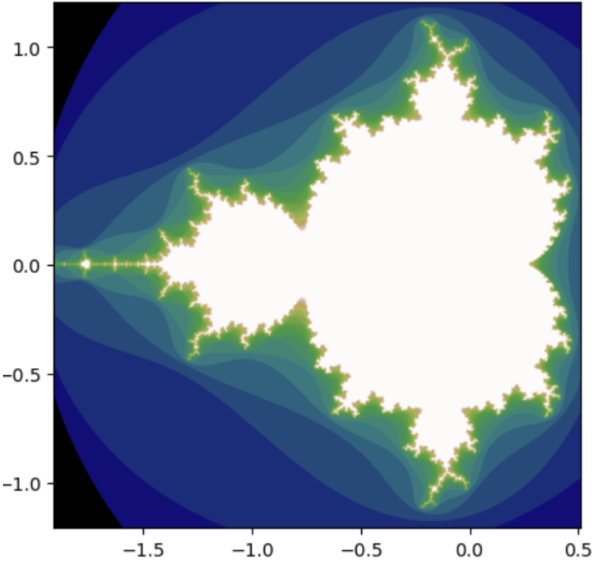
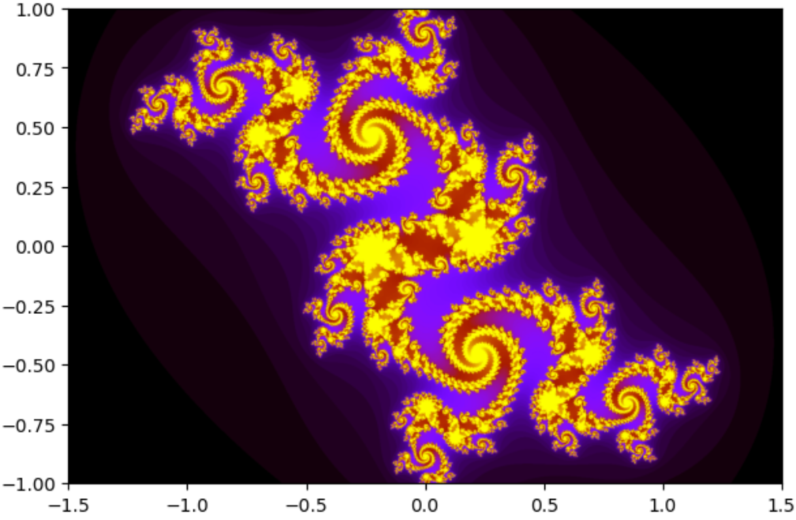
Outline

Purpose

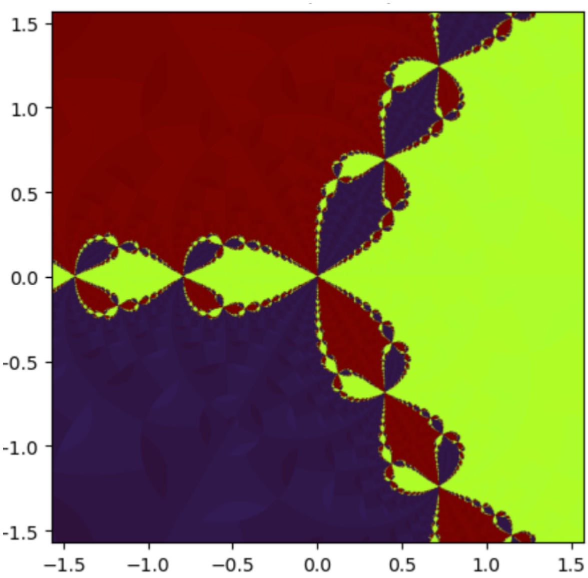
Visualizations



Visualizations



Visualizations



Complex Dynamics Background

Complex Analysis Refresh

This MATH 632 content and more can be found in [2].

Definition 2.1

Let $X, Y \subseteq \mathbb{C}$. A function $f : X \to Y$ is holomorphic if f is complex differentiable on a neighborhood U of every point $x \in X$.

Example 2.2

The function $f(z) = z^2$ is holomorphic.

Definition 2.3

Let $G \subset \mathbb{C}$ be a domain. The family (set) of all holomorphic functions $f : G \to \mathbb{C}$ is denoted $H(G)$.

Convergence & Normal Families

We can then discuss convergence of sequences of functions in $H(G)$.

- Pointwise = too "weak"
- Uniform = too "restrictive"

Definition 2.4

A sequence $(f_m) \in H(G)$ converges **uniformly on compact subsets** (or **locally uniformly**) if there exists $f \in H(G)$ such that given any compact set $K \subset G$, $(f_m) \to f$ uniformly on K.

Definition 2.5

A family $\mathcal{F} \subset H(G)$ is **normal** if for every sequence in $\mathcal{F}$ there is a subsequence which converges uniformly on compact subsets of G or which diverges uniformly to infinity on compact subsets of G.

Normal families can be viewed as an analogue of the Bolzano-Weierstrass Theorem.

Normality and Local Boundedness

Definition 2.6

A family $\mathcal{F} \subset H(G)$ **locally bounded** if for each $w \in G$ there are constants $M, r > 0$ such that for all $f \in \mathcal{F}$

$$|f(z)| \leq M, \ \ |z - w| < r.$$

In other words, $\mathcal{F}$ is locally bounded at each $w \in G$ if there is a disk centered at w on which $\mathcal{F}$ is uniformly bounded.

Theorem 2.7 (Montel's Theorem, [7])

If $\mathcal{F}$ is a locally bounded family of holomorphic functions on a domain G then $\mathcal{F}$ is a normal family in G.

Normal Family Example

Example 2.8

Determine whether the family $\mathcal{F} = \{z^{2^m} : m \in \mathbb{N}\}$ is normal on the unit disk $\mathbb{D}$.

Fatou and Julia Sets

The family $\mathcal{F} = \{f^m : m \in \mathbb{N}\}$ of iterates of a holomorphic function $f : G \to G$ will be of primary interest to us. Consider the following types of sets.

Definition 2.9

Let f be a non-constant rational function. The **Fatou set** is defined as

$$F(f) = \{z \in \mathbb{C} : \exists \,\mathrm{a\,nbhd}\, U \,\mathrm{of}\, z \,\mathrm{s.t.} \{f^m|_U : m \in \mathbb{N}\} \,\mathrm{is\,normal}\}$$

Definition 2.10

The **Julia set** of f is $J(f) = \mathbb{C} \setminus F(f)$.

The Fatou set is where we have stable behavior and the Julia set represents the chaotic behavior.

Filled in Julia Set and Connecting Definitions

Definition 2.11

The **Filled Julia Set** of f, denoted $K(f)$, is defined as

$$K(f) = \{z \in \mathbb{C} : f^m(z) \nrightarrow \infty \text{ as } m \to \infty\}$$

Definition 2.12

The **Julia set** of f is $J(f) = \partial K(f)$.

Question: How do all of these definitions relate?

Fatou and Julia Set Example

Example 2.13

Find the Fatou, Julia, and Filled Julia set for the polynomial $f(z) = z^2$.

Question

It was easy to find and draw the Fatou, Julia, and Filled in Julia set for $f(z) = z^2$. However, if we were to choose $f(z) = z^2 - 1$, it would not be as easy.

Question: Can we use computers to generate the the Julia and Fatou sets for more complicated polynomials?

Yes. This is seen in the first and third fractal images we saw at the beginning. These are called **Julia Fractals**.

Visualizations of Julia, Mandelbrot, and Newton Fractals

Package: Fatou.jl

Main Package Objective

The "Fatou.jl" package allows us to easily generate, explore, and share fractals of Julia, Mandelbrot, and Newton type to construct images of the Julia and Fatou set for a given polynomials.

- Language: Julia (Not the same "Julia" as Julia sets)
- Three main methods to construct fractals:
  1. `juliafill()`
  2. `madelbrot()`
  3. `newton()`
- Sources used for content in Jupyter Notebook: [1], [3], [4], [5], and [6]

Understanding The Fatou and Julia Sets

Orbits

Definition 4.1

Let $G \subset \mathbb{C}$ and $f : G \to G$ a function. The **forward orbit** of $z \in G$ under f is the set

$$O_f^+(z) = \{f^m(z) : m \in \mathbb{N}\}.$$

The **backward orbit** of $z \in G$ under f is defined as the set

$$O_f^-(z) = \{w \in \mathbb{C} \cup \{\infty\} : f^m(w) = z, m \in \mathbb{N}\}.$$

Fixed Point Theory

Once we understand where we have "stable" and "chaotic" behavior, we would like to know what the stable region looks like.

Definition 4.2

Let $f : G \to G$ non-constant polynomial and $z_0 \in G$.

(a) A point z_0 is called a **fixed point** of f if the forward orbit $O_f^+(z_0) = \{z_0\}$, or rather $f(z_0) = z_0$.

(b) A fixed point z_0 is called an **attracting fixed point** if there exists a neighborhood U of z_0 such that $f^m(w) \to z_0$ uniformly for all $w \in U$.

Basin of Attraction

Definition 4.3

Let $f : G \to G$ non-constant polynomial and suppose $z_0 \in G$ is an attracting fixed point of f. Then the **basin of attraction** for z_0 is the set of all points z such that $f^m(z) \to z_0$ locally uniformly as $m \to \infty$. Such set is denoted $\mathcal{A}_f(z_0)$. The **immediate basin of attraction** is the connected component of $\mathcal{A}_f(z_0)$ which contains z_0.

References I

[1] chakravala. *Fatou.jl Package*. URL: https://github.com/chakravala/Fatou.jl.

[2] J. B. Conway. *Functions of one complex variable I*. Vol. 11. Springer Science & Business Media, 1973.

[3] Robert L Devaney. *A first course in chaotic dynamical systems: theory and experiment*. CRC Press, 2018.

[4] IBM. *Benoit Mandelbrot*. URL: https://www.ibm.com/history/benoit-mandelbrot.

[5] Octogonapus. *PyPlot.jl Package*. URL: https://github.com/JuliaPy/PyPlot.jl.

[6] S. Rickman. *Quasiregular mappings*. Vol. 26. Springer Science & Business Media, 2012.

[7] J. L. Schiff. *Normal families*. Springer Science & Business Media, 1993.