Lecture : Introduction to the Normal Distribution

Last time:

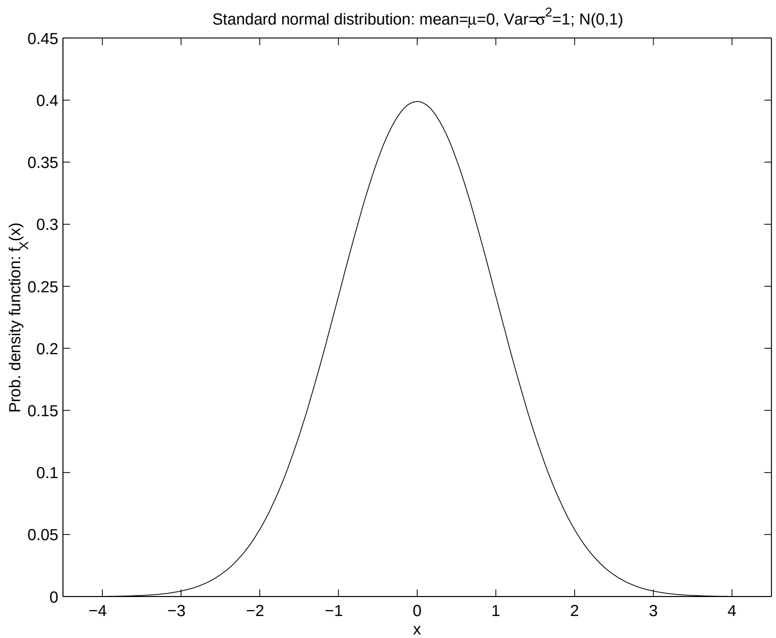
- Random variables; outcomes & probabilities
- expected value and standard deviation

This segment:

- continuous random variables, Normal distribution
- calculations using the Normal distribution
- Standardization

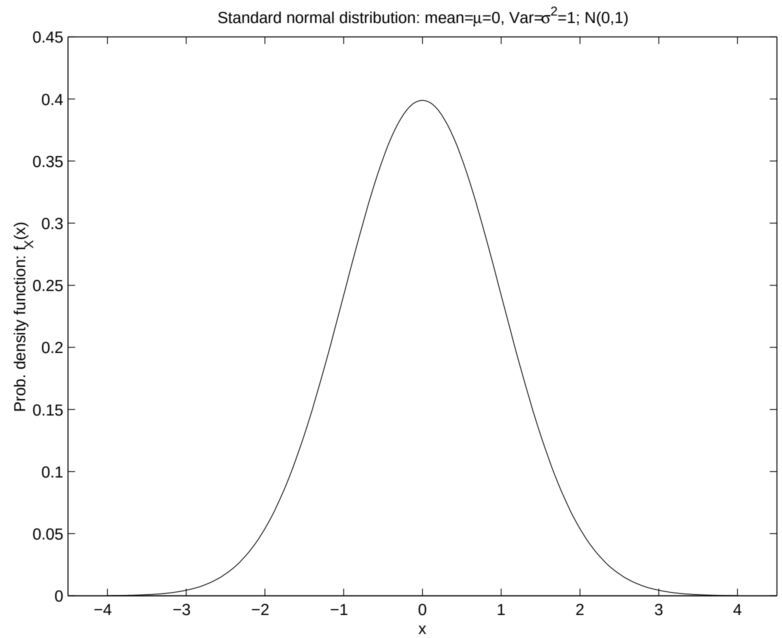
The Normal distribution

- Most important & popular distribution in statistics.
- Many problems can be (very well) approximated & solved using the normal distribution.
- Very good approximation for sum of large number of uncertain quantities



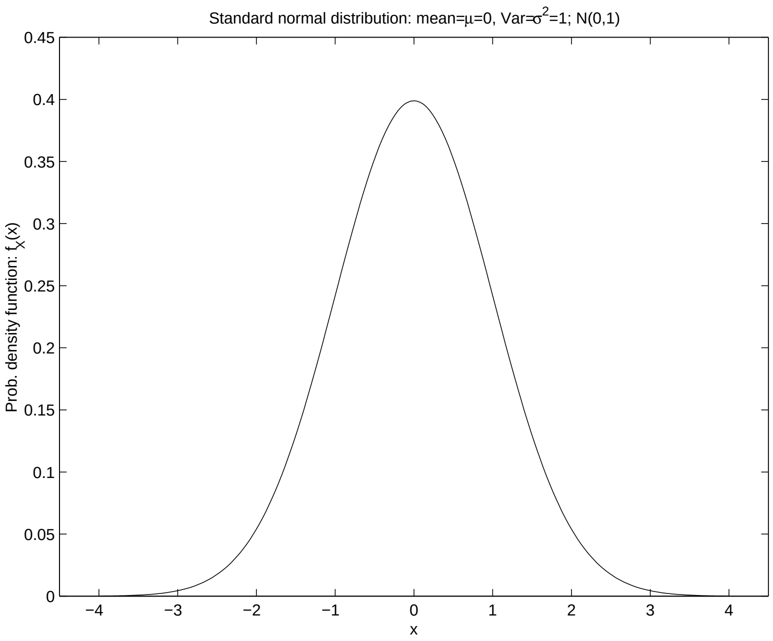
Notation: $N(\mu, \sigma^2)$; in figure: $\mu = 0$, $\sigma^2 = 1$.

Characteristics of normal distributions



- Continuous data
- Interpretation:
  - $P(X \in [x, x+dx]) \simeq f_X(x)dx$
  - $f_X(\cdot)$ is the probability density function
  - $P(a \leq X \leq b) =$ area under the curve between a, b.

Standard normal: $Z \sim N(0, 1)$



$P(Z \leq 1.30)$=?

Find z such that $P(Z \leq z) = .95$?

Fact: If $X \sim N(\mu, \sigma^2)$, then

$$\frac{X - \mu}{\sigma} = Z \sim N(0, 1),$$

Motivating Example

Consider the stocks:

Stock	Ann.return	Exp.ann.return	Stdev
A	X	$\mu_X = 15\%$	$\sigma_X = 10\%$
B	Y	$\mu_Y = 25\%$	$\sigma_Y = 30\%$

X, Y are Normally distributed. We want to compare two portfolios:

- Safe (S): 70% invested in A and 30% in B
- Risky (R): 30% invested in A and 70% in B

Expected return

Recap of the formula: portfolio standard deviation

$\mathsf{Var}[aX + bY] =?$

Independent case ($\Rightarrow \rho_{XY} = 0$):

$$\mathsf{Var}[aX + bY] = a^2\mathsf{Var}[X] + b^2\mathsf{Var}[Y]$$

Correlated case ($\rho_{XY} \neq 0$):

$$\mathsf{Var}[aX + bY] = a^2\mathsf{Var}[X] + b^2\mathsf{Var}[Y] + 2ab \cdot \mathsf{Cov}[X, Y]$$

Portfolio standard deviation calculation...

- Recall: $\sigma_X = 10\%$, $\sigma_Y = 30\%$ and X, Y Normal, and $\rho_{XY} = 0$
- $S = 0.7X + 0.3Y$ and $R = 0.3X + 0.7Y$

Which portfolio has higher probability of losing money?

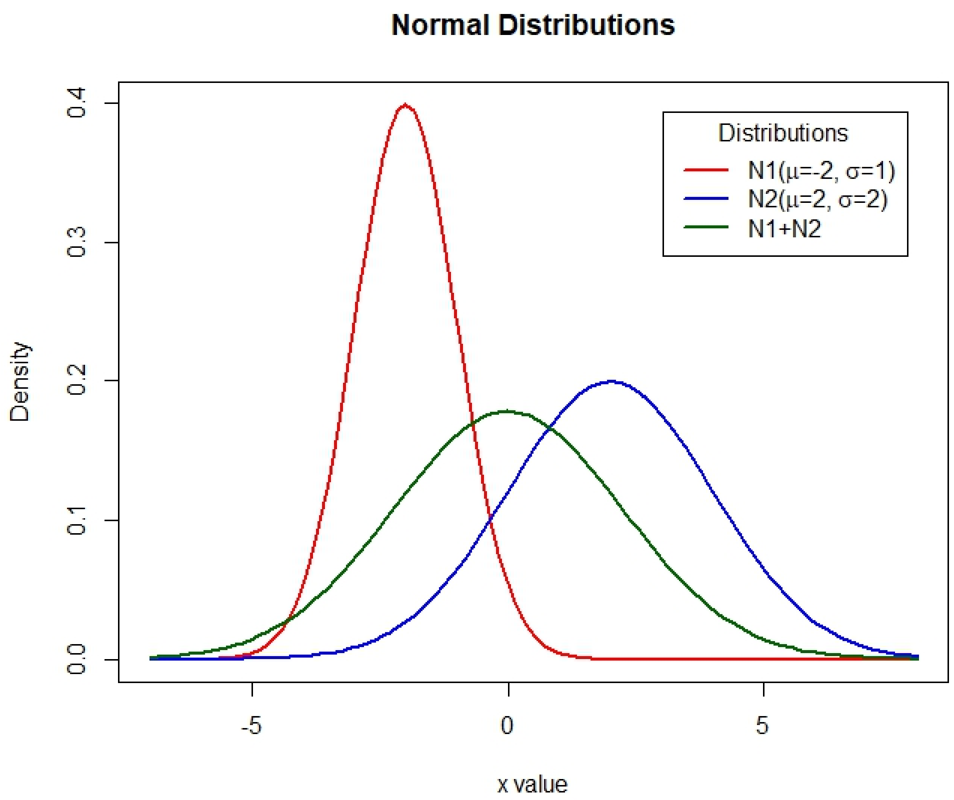
- $S = 0.7X + 0.3Y$

- $R = 0.3X + 0.7Y$

Distribution of sums of Normal random variables is Normal

Fact: If X, Y are normally distributed and *independent* then

- $aX + b$ is normal; i.e., linear transformation of normal is normal
- $Z = aX + bY$ is normal; sum of independent normals is normal
  - $Z \sim N(a\mu_X + b\mu_Y,\ a^2\sigma_X^2 + b^2\sigma_y^2)$



Joint Distributions

- Joint density function: $f : \mathbb{R}^2 \to \mathbb{R}$
- Interpretation:

$$P(X \in [x, x+dx], Y \in [y, y+dy]) \simeq f(x,y)dx \cdot dy \qquad \text{for all } (x,y)$$

- Properties:

$$f_{X,Y}(x,y)) \geq 0 \text{ for all } (x,y),$$
$$\int_x \int_y f_{X,Y}(x,y)dydx = 1$$

- Probability of any event

$$P((X,Y) \in B) = \int \int_{(x,y)\in B} f_{X,Y}(x,y)dydx$$

- $Marginal$ density function of X is defined as:

$$f_X(x) = \int_y f_{X,Y}(x,y)dy$$

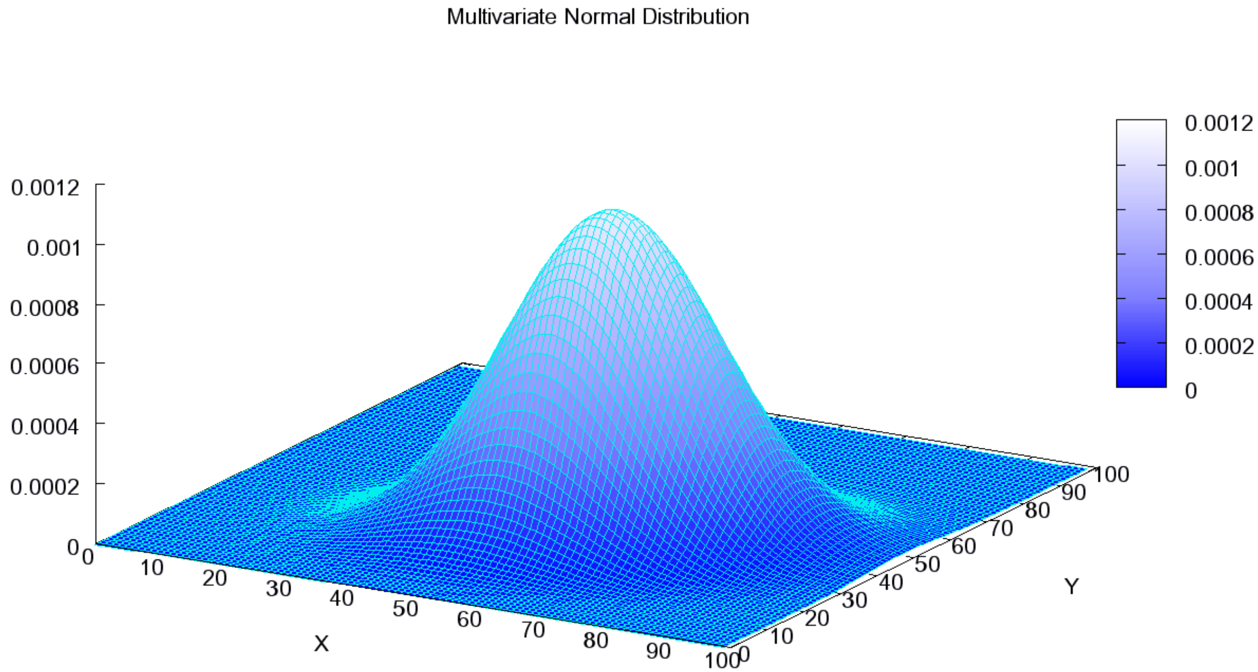
- If X and Y are **independent**:

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \qquad \text{(product of marginal densities)}$$

Distribution of sums of Normal random variables is Normal

Fact: If X, Y are *jointly* normally distributed then

- Any linear combination of X, Y also has a normal distribution



Two other portfolios

P_1 : 80% in A and 20% in B

P_2 : 90% in A and 10% in B

Positively correlated stocks: $\rho_{XY} = 0.1$

Q: can we construct better portfolios than S or R?

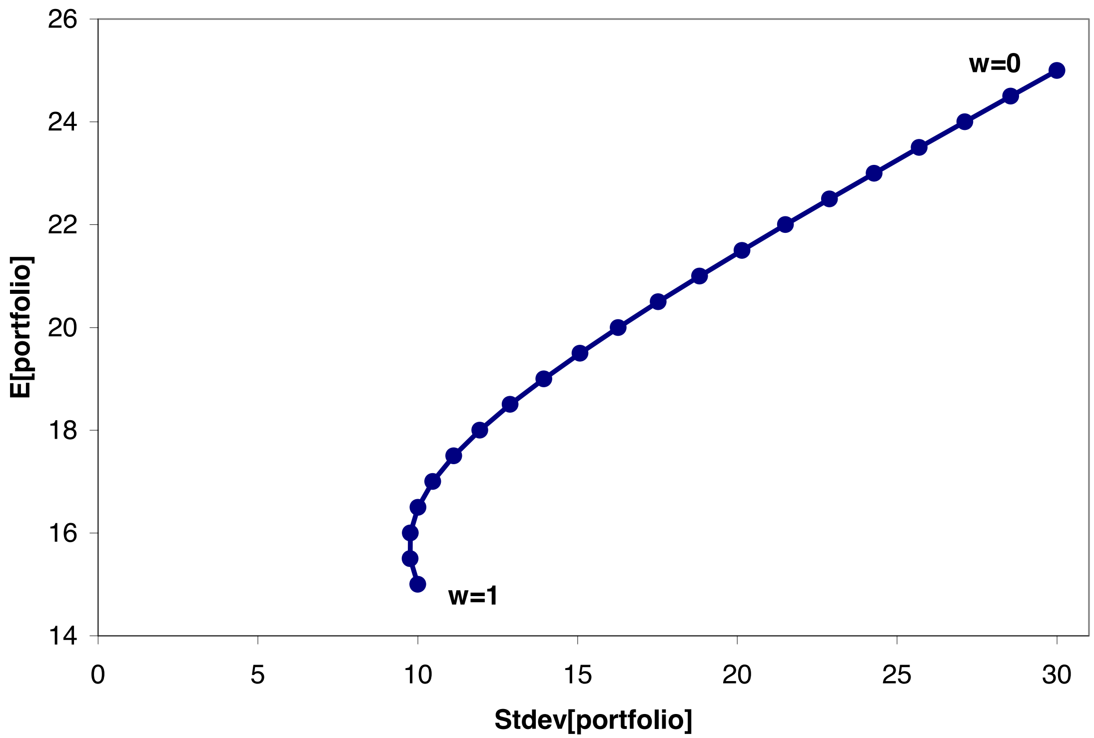
Proposed solution:

- invest fraction w of wealth in A and $(1 - w)$ in B
- expected returns?   standard deviations?

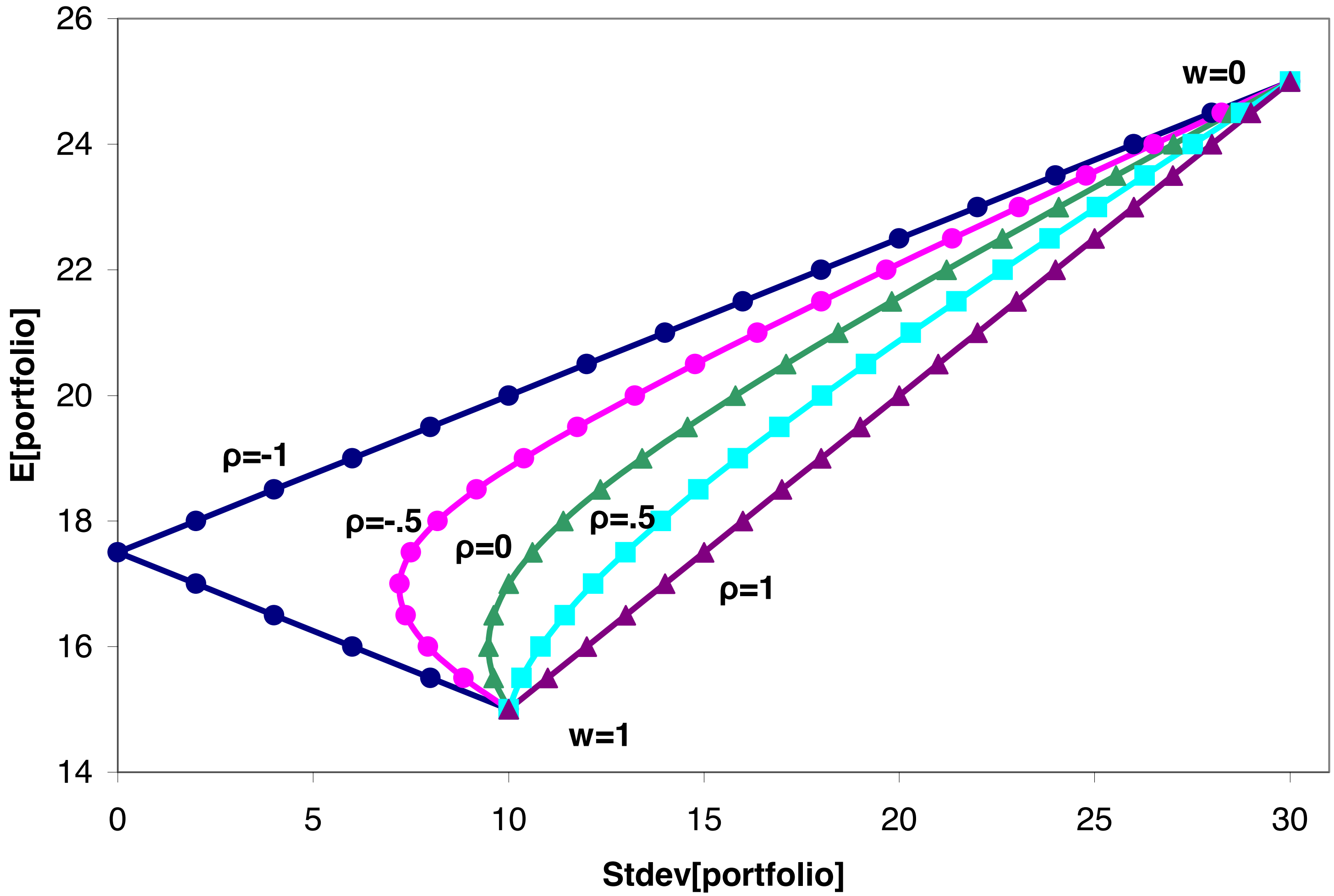
Portfolio diversification: $\rho_{XY} = 0.1$

Let's plot different portfolios:

Each point is a portfolio that invests a fraction w of wealth in A and $(1-w)$ in B

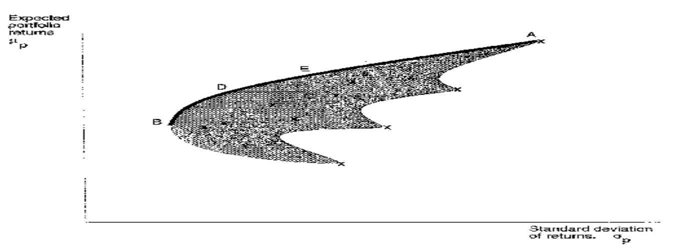


Portfolio returns for variable ρ



Portfolio returns with multiple stocks

- With multiple stocks, the best portfolio is more difficult to compute
- Basically, any point in region represents a portfolio
- *Efficient frontier*: first defined by Markowitz in his influential '52 paper that launched portfolio theory
  (he got the Nobel prize for that paper!)



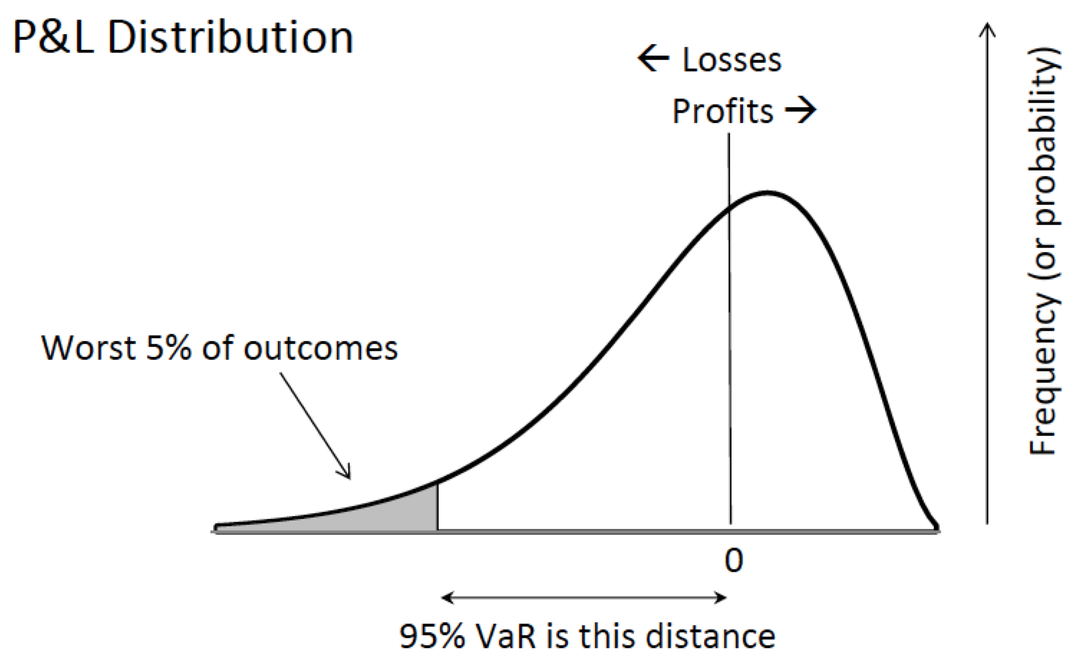
Value-at-Risk (VaR)

The 99% Value-at-Risk of an investment is the amount x, such that the returns from that investment over a fixed time period will be $\leq x$ with probability 1%.

What is the 99% VaR over one year for the S&P 500?
(Annual rate of return of S&P 500 is normal with $\mu = 8.79\%$ and $\sigma = 15.75\%$.)

Value-at-Risk: first glimpse

- VaR measures the risk of investments. Used by most banks
- VaR answers the following question: *How much can I lose with $x\%$ probability over a given horizon?*



VaR is reported as a positive dollar amount

The hard part is coming up with the right P&L distribution

Value-at-Risk: a simple example

You are managing a portfolio, say worth \$100M, with

average daily payoff $\bar{X} = \$0M$ and standard deviation of daily payoffs $\sigma = \$3M$

What is your 97.5% one-day Value-at-Risk? (Assume returns are Normally distributed.)

1. Plot a histogram of daily payoffs $\bar{X} = \$0M$ and $\sigma = \$3M$
2. From def'n of VaR: we want to find "x" such that 2.5% of days we lose x or more

Value-at-Risk: a simple example (2)

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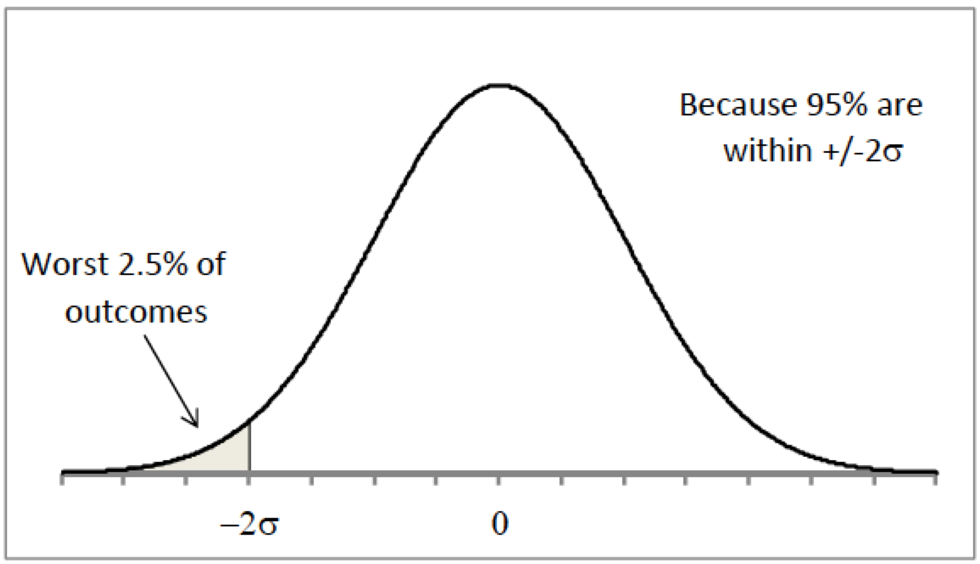


Summary

1. Standardize:
$$X \to \frac{X-\mu}{\sigma} = Z \sim N(0,1)$$

2. Rephrase question of interest for $X \sim N(\mu, \sigma^2)$ in terms of $Z \sim N(0,1)$; i.e., in # of StdDev. Translate solution back for $X \sim N(\mu, \sigma^2)$

3. **Fact:** If X, Y are jointly normally distributed then
   - $aX + b$ is normal; i.e., linear transformation of normal is normal.
   - $X + Y$ is normal; i.e., sum of jointly normally distributed random variables is normal.
   - $aX + bY$ is normal; combination of the above.

4. Formulas you should know:

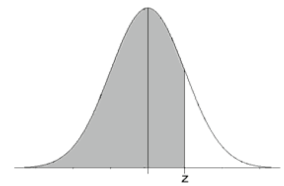
$$\boxed{\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]}$$

$$\text{Var}[aX + bY] = a^2\text{Var}[X] + b^2\text{Var}[Y] + 2ab \cdot \text{Cov}[X, Y]$$

or

$$\text{Var}[aX + bY] = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\rho_{XY}\,\sigma_X\sigma_Y$$

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998