Lecture 3: Random Variables, Expected Value, Stdev

So far:

- Descriptive Statistics:
  mean, median, standard deviation, covariance, correlation coef.

This segment:

- Random variables (RV)
- Introduction to probability
- Expected values and standard deviation

$\boxed{\text{Why? RVs extend descriptive stats to do predictions}}$

Random variables

A **random variable** maps outcomes of a random process to numerical values

Toss a Coin → Heads $1 = X$
→ Tails $0 = X$

Examples:

- the outcome of the roll of a die
- the number of infected individuals in a random sample of 100,000 people
- the number of GM cars to be sold next month
- the % change in Dow Jones/your portfolio over the next month
- the number of Republicans in the House after the next elections

→ "random" means outcome is uncertain not "anything is possible"

Examples of random variables

You invest \$10,000 in each of two different stocks.
Let X_i = % change of the i^{th} stock over one day.

% change	-1	0	1	
Prob. for X_1	.27	.35	.38	= 1
Prob. for X_2	.31	.40	.29	= 1

This table describes the **distributions** of X_1, X_2; e.g.,

$$P_{X_1}(x_1) = \mathbb{P}(X_1 = x_1).$$

$$\begin{aligned} Z &= \%\text{ change for portfolio over one day} \\ &= (.5 \cdot X_1 + .5 \cdot X_2;) \end{aligned}$$

i.e., Z is a function of the observed outcome.

X_1, X_2, Z discrete r.v.

- X_1, X_2, Z are all random variables
- X is a **discrete** r.v. if it takes on only discrete values
- distributions of r.v.'s can be described as histograms

$X_1, X_2 \in \{-1, 0, 1\}$

$Z \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$

Next: extend notions from descriptive statistics to r.v.'s.

Expected value

... \$10,000 in each stock and $X_i =$ % daily change of i^{th} stock.

% change	-1	0	1	
Prob. for X_1	.27	.35	.38	= 1
Prob. for X_2	.31	.40	.29	

What is the expected return of X_1 over the next day?

- what are the possible outcomes: -1 0 1

- what are the probabilities associated with these outcomes?

$P(X_1 = -1) = 0.27, \quad P(X_1 = 0) = 0.35, \quad P(X_1 = 1) = 0.38$

$$E[X_1] = P(X_1 = -1) \cdot (-1) + P(X_1 = 0) \cdot 0 + P(X_1 = 1) \cdot 1$$
$$= -0.27 + 0.38 = 0.11$$

$$E[X_2] = P(X_2 = -1)(-1) + P(X_2 = 1)(1) = -0.31 + 0.29$$
$$= -0.02$$

Recall the interpretation of $\bar{X}$ as $\sum$ (probability) $\times$ (outcomes).

Expected value (2)

% change	-1	0	1
Prob. for X_1	.27	.35	.38
Prob. for X_2	.31	.40	.29

$$
\begin{aligned}
Z &= \text{\% change for portfolio over one day} \\
&= (.5 \cdot X_1 + .5 \cdot X_2;)
\end{aligned}
$$

$X_1 \in \{-1, 0, 1\}$ 3 outcomes

$X_2 \in \{-1, 0, 1\}$ 3 outcomes

9 outcomes

$$\mathbb{E}[Z] = \sum_{i=-1}^{1} \sum_{j=-1}^{1} P(X_1 = i \text{ and } X_2 = j) \frac{(i+j)}{2}$$

$$= \underbrace{\sum_{i=-1}^{1} P(X_1 = i) \frac{i}{2}} + \sum_{j=-1}^{1} P(X_2 = j) \frac{j}{2}$$

$$= \frac{1}{2} \mathbb{E}[X_1] + \frac{1}{2} \mathbb{E}[X_2]$$

X_1 \ X_2	-1	0	1
-1	Z=1 -1, -1		
0		Z=0 (0,0)	Z=1/2 0, 1
1			

Linearity of Expected value: $\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$

More on expected value

Daily portfolio returns: $W = (\$20,000)Z/100 = 200Z$

What are the expected portfolio returns?

$$\mathbb{E}[W] = \mathbb{E}[200Z] = \sum_{i=-1}^{1} \sum_{j=-1}^{1} P(X_1 = i \text{ and } X_2 = j) \frac{(i+j)}{2}(200)$$

$$= 200 \Bigg(\underbrace{\sum_{i=-1}^{1} \sum_{j=-1}^{1} P(X_1 = i \text{ and } X_2 = j) \frac{(i+j)}{2}}_{\mathbb{E}[Z]} \Bigg)$$

$$= 200\, \mathbb{E}[Z]$$

$$\mathbb{E}[aX] = a\,\mathbb{E}[X]$$

Example: Investing in an early stage company

Crowdfunding allows early-stage startups to raise much needed seed capital from many, small ("retail") investors. As a retail investor, you are considering investing \$10K in one such early stage opportunity (we refer to this as "first round seed"). The overall statistics for such early stage opportunities is as follows: 50% of them will go bankrupt after 6 to 12 months (resulting in a loss of the initial \$10K investment); 30% will need an additional early stage capital infusion to prolong the runway, which is typically fulfilled by the early stage investors, raising your total investment to \$20K (which we refer to as "second round seed"); and finally, the remaining 20% will be sufficiently promising to the extent that after the 6–12 month milestone they raise significant venture funding from institutional investors (VC) (we will refer this as "VC investment"). In addition to this latter category, half of the opportunities that receive additional crowd funding in the second seed round will go bankrupt (resulting in a loss of \$20K) while the other half will eventually proceed to receive VC funding. Finally, one third of the firms that eventually received VC funding will have a profitable exit that would result in a payout of \$150K on your total investment, whereas the remaining two thirds will go bankrupt resulting in no eventual payouts to the investors.

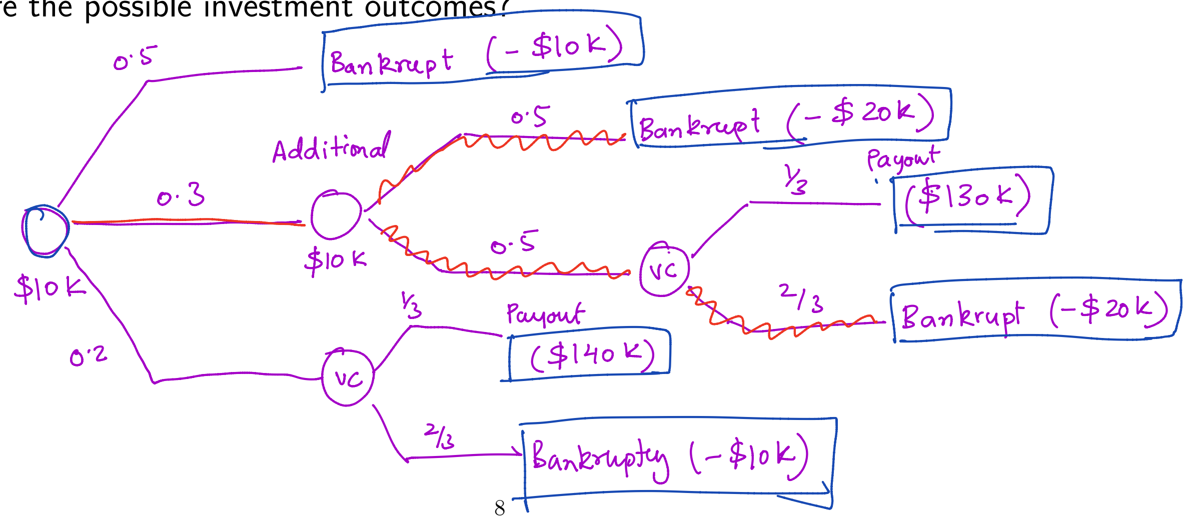
- Suppose you make one such investment. What is the probability that it is successful?
- What is the expected payout?
- What is the probability that a startup that got 2^{nd} round financing will be successful?
- What is the annualized rate of return of this investment?

Possible investment outcomes

The overall statistics for such early stage opportunities is as follows: 50% of them will go bankrupt after 6 to 12 months (resulting in a loss of the initial $10K investment); 30% will need an additional early stage capital infusion to prolong the runway, which is typically fulfilled by the early stage investors, raising your total investment to $20K (which we refer to as "second round seed"); and finally, the remaining 20% will be sufficiently promising to the extent that after the 6–12 month milestone they raise significant venture funding from institutional investors (VC) (we will refer this as "VC investment"). In addition to this latter category, half of the opportunities that receive additional crowd funding in the second seed round will go bankrupt (resulting in a loss of $20K) while the other half will eventually proceed to receive VC funding. Finally, one third of the firms that eventually received VC funding will have a profitable exit that would result in a payout of $150K on your total investment, whereas the remaining two thirds will go bankrupt resulting in no eventual payouts to the investors.

$\{-20k, -10k, 130k, 140k\}$

- What are the possible investment outcomes?

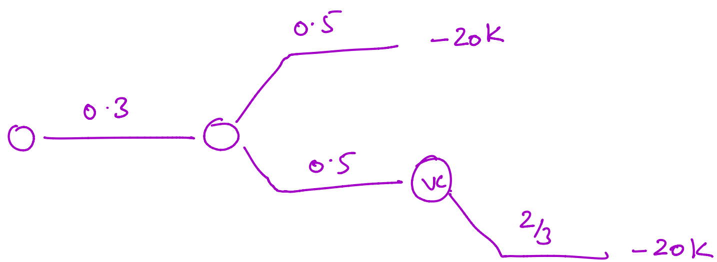


Outcomes & probabilities

The overall statistics for such early stage opportunities is as follows: 50% of them will go bankrupt after 6 to 12 months (resulting in a loss of the initial $10K investment); 30% will need an additional early stage capital infusion to prolong the runway, which is typically fulfilled by the early stage investors, raising your total investment to $20K (which we refer to as "second round seed"); and finally, the remaining 20% will be sufficiently promising to the extent that after the 6–12 month milestone they raise significant venture funding from institutional investors (VC) (we will refer this as "VC investment"). In addition to this latter category, half of the opportunities that receive additional crowd funding in the second seed round will go bankrupt (resulting in a loss of $20K) while the other half will eventually proceed to receive VC funding. Finally, one third of the firms that eventually received VC funding will have a profitable exit that would result in a payout of $150K on your total investment, whereas the remaining two thirds will go bankrupt resulting in no eventual payouts to the investors.

- What are the possible investment outcomes? What are the corresponding probabilities?

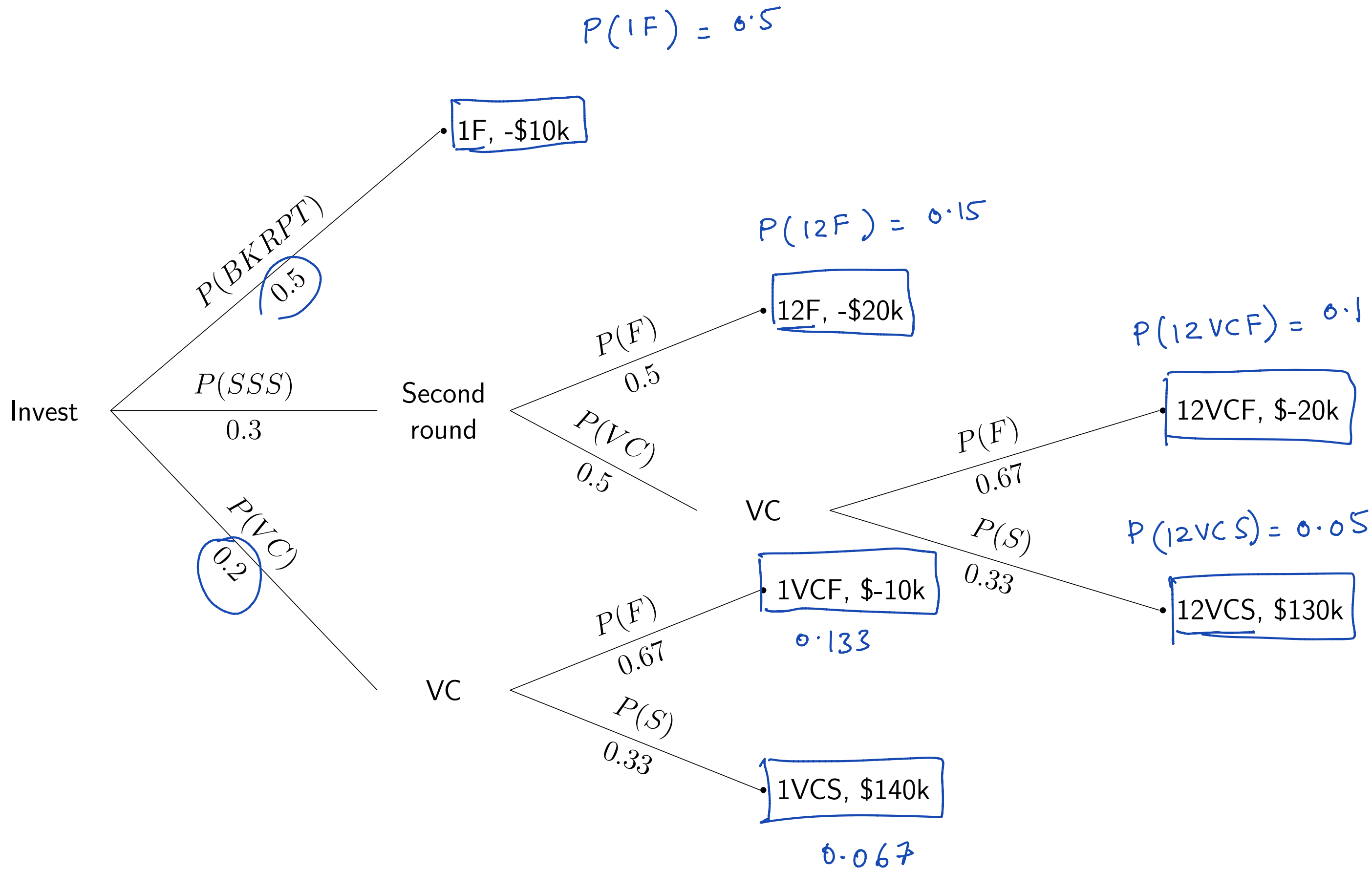
P (payout = -20k)



$$P(\text{payout} = -20K) = (0.3)(0.5) + (0.3)(0.5)\left(\frac{2}{3}\right)$$

$$= 0.15 + 0.1 = 0.25$$

Probability tree



Example: Investing in an early stage company

The overall statistics for such early stage opportunities is as follows: 50% of them will go bankrupt after 6 to 12 months (resulting in a loss of the initial \$10K investment); 30% will need an additional early stage capital infusion to prolong the runway, which is typically fulfilled by the early stage investors, raising your total investment to \$20K (which we refer to as "second round seed"); and finally, the remaining 20% will be sufficiently promising to the extent that after the 6–12 month milestone they raise significant venture funding from institutional investors (VC) (we will refer this as "VC investment"). In addition to this latter category, half of the opportunities that receive additional crowd funding in the second seed round will go bankrupt (resulting in a loss of \$20K) while the other half will eventually proceed to receive VC funding. Finally, one third of the firms that eventually received VC funding will have a profitable exit that would result in a payout of \$150K on your total investment, whereas the remaining two thirds will go bankrupt resulting in no eventual payouts to the investors.

- What are the possible outcomes? What are the corresponding probabilities?

Outcome	1F	12F	12VCF	1VCF	1VCS	12VCS
\$ Payout net of investment	-10	-20	-20	-10	140	130
Probability	.5	.15	.1	.133	.067	.05

- $\mathbb{P}$(investment is successful) = $P(\text{payout} > 0) = P(1VCS) + P(12VCS)$

$= 0.067 + 0.05 = 0.117$

Expected payout calculation

Outcome	1F	12F	12VCF	1VCF	1VCS	12VCS
$ Payout net of investment	-10	-20	-20	-10	140	130
Probability	.5	.15	.1	.133	.067	.05

$\mathbb{E}(\text{Payout}) =$ 4.55 ($K) = $4,550

Q: How do we measure the risk of the payout, i.e. quantify its variability around the mean? The payout is slightly positive but couldn't some variability make it negative?

Definition: variance and standard deviation

Suppose X is a random variable:

$$\mathrm{Var}[X] = \sum_{\text{all outcomes}} (\text{outcome} - \mathbb{E}[X])^2 \, \mathbb{P}(\text{outcome})$$

(It is computed as follows:

- calculate the expected value of the random variable: $\mathbb{E}[X]$
- for each outcome, compute its squared deviation from the expected value by $(outcome - \mathbb{E}[X])^2$
- weigh the square deviation of each outcome by the corresponding probability and sum up all terms.)

An alternative formula for the variance is

$$\mathrm{Var}[X] = \sum_{\text{all outcomes}} (\text{outcome})^2 \cdot \mathbb{P}(\text{outcome}) - (\text{Expected value of } X)^2$$

Standard deviation is defined by $\sigma_X = \sqrt{\mathrm{Var}[X]}$

Standard deviation of the payout

Outcome	1F	12F	12VCF	1VCF	1VCS	12VCS
\$ Payout net of investment	-10	-20	-20	-10	140	130
Probability	.5	.15	.1	.133	.067	.05

What is the variability around this payout? $\mathbb{E}[X] = 4.55$ (\$K)

$$\mathrm{Var}[X] = \sigma_X^2 = \sum_{\text{all outcomes}} (\text{outcome} - \mathbb{E}[\text{payout}])^2 \mathbb{P}(\text{outcome})$$

$$\begin{aligned}\mathrm{Var}(\text{Payout}) &= (-10-4.55)^2(.5) + (-20-4.55)^2(.15) + \cdots + (130-4.55)^2(.05) \\ &= \underline{2,301}(\$K)(\$K)\end{aligned}$$

Standard deviation $= \sqrt{\mathrm{Var}} = \sqrt{2,301} = \$47.97K$ = \$47,970 > 10 $\mathbb{E}[X]$

What would happen if you invested twice as much money in that opportunity? (i.e., start with \$20K, and also double your second round seed, if it took place, ending with twice the equity)

\$300K

Standard deviation of the payout

What would happen if you invested twice as much money in that opportunity? (i.e., start with $20K, and also double your second round seed, if it took place, ending with twice the equity)

$20 K

$300 K

(instead of $150k)

$\mathbb{E}$[net payout], Var[net payout] ?

$\mathbb{E}$[net payout] = 2 (4.55) = 9.1 ($k)

Var[net payout] = 4 Var = 4 (2301) = 9204 $(\$k)^2$

Basic concepts in probability

Sample space: set of possible outcomes; e.g.,

1. Startup Investing: All 6 Possible investment outcomes;
2. Roll of a die: $\{1, 2, 3, 4, 5, 6\}$;

Event: a subset of the sample space; e.g.,

$\{$ 1VCS, 1VCF $\}$

1. A=$\{$Receive VC funding in round one $\leftarrow$ *VC*$\}$,
   B=$\{$Startup succeeds after receiving VC funding in round one $\leftarrow$ *1VCS*$\}$; $\{$1VCS$\}$
2. C=$\{$2,4,6$\}$ $\leftarrow$ the event "die comes up even," $\{2, 4, 6\}$
   D=$\{$1,2$\}$;

Probability: for any event A we assign a number $P(A)$ between 0 and 1 that is interpreted as

"frequentist"

P(A) = sum of probabilities of basic outcomes in event A

- the long-run proportion of times that A occurs,
- a subjective assessment of the chance that A occurs.

"bayesian"

e.g.,

1. $P(A) = 0.2$ and $P(B) = 0.067$;
2. $P(C) = 1/2$ and $P(D) = 1/3$.

Complement, union, and intersection

Complement of an event A, denoted by $\bar{A}$, is the event that A does *not* occur; e.g.,

$\bar{A}$

1. not A={not receive VC funding in round 1} $\leftarrow$ startup goes bankrupt or goes to round 2;
2. not C={1,3,5} $\leftarrow$ the event "die comes up odd."

$A = \{1VCS, 1VCF\}$

$\bar{A} = \{1F, 12F, 12VCF, 12VCS\}$

$A \cup B$

Union: the event (A or B) occurs if either A or B occurs (notation $A \cup B$); e.g.,

1. (A or B)={Get VC funding in round 1 *or* Succeed after getting VC funding in round 1};
2. (C or D)={1,2,4,6}.

$B = \{1VCS\}$

$(A \text{ or } B)$ $\quad A \cup B = \{1VCS, 1VCF\}$

$A \cap B$

Intersection: the event (A and B) occurs if both A and B occur (notation $A \cap B$); e.g.,

1. (A and B)={Get VC funding in round 1 *and* Succeed afterwards};
2. (C and D)={2}.

$A \cap B = \{1VCS\}$

Mutually exclusive events & examples

Two events A and B are **mutually exclusive** if they cannot occur simultaneously; e.g.,

1. A={Get VC Funding in round one} and B={Go Bankrupt in round one}; $A \cap B = \phi$
2. C={even} and D={odd}.
   roll of a die

The events $A_1, \ldots, A_n$ are **collectively exhaustive** if collectively they cover every possible outcome; i.e., the entire sample space,

$$(A_1 \cup A_2 \cup \cdots \cup A_n) = \text{sample space}.$$

Example: roll a 6-sided die ; sample space = $\{1, 2, \ldots, 6\}$

$A_1 = \{1, 3, 5\}$

$A_2 = \{2, 3, 4, 5, 6\}$

$A_1 = \{\text{die comes up odd}\}$, $A_2 = \{\text{die comes up} \geq 2\}$.

- $(A_1 \cup A_2) =$ $\{1, 2, 3, 4, 5, 6\}$
- $(A_1 \cap A_2) =$ $\{3, 5\}$
- $(A_1 \cap \bar{A}_2) =$ $\{1\}$
- $(A_1 \cap A_2) \cup (A_1 \cap \bar{A}_2) =$ $A_1 = \{1, 3, 5\}$

Basic rules of probability

event $A \subseteq$ sample space

$$P(A) = \sum_{\text{basic outcomes} \in A} p(\text{basic outcome})$$

Total probability: $P(\text{sample space}) = 1$;
e.g., $P(\{1,2,3,4,5,6\}) = 1$.

Complement: $P(\bar{A}) = 1 - P(A)$; e.g.,

1. P(not receive VC funding in round 1) = 1 - P(receive VC funding in round 1) = 0.8;
2. $P(\text{not } \{1,2\}) = P(\{3,4,5,6\}) = 2/3$.

Addition rule:

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B);}$$

e.g.,

$A = \{\text{die comes up odd}\} = \{1,3,5\}$ and $B = \{\text{die comes up } \geq 5\} = \{5,6\}$

then

$P(A) = \frac{1}{2}$

$P(B) = \frac{1}{3}$

$$A \cup B = \{\text{odd or } \geq 5\} =? \quad \{1,3,5,6\} \qquad P(A \cup B) = \frac{4}{6} = \frac{2}{3}$$

$$A \cap B = \{\text{odd and } \geq 5\} =? \quad \{5\} \qquad P(A \cap B) = \frac{1}{6}$$

We want to verify that $P(A \cup B) =?$ $2/3$

$$P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3} = P(A \cup B)$$

What is $P(A \cup B)$ for mutually exclusive events? $A \cap B = \emptyset$ $\quad P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

Conditional probability

The **conditional probability** of A **given** B, denoted by $\boxed{P(A|B)}$, is the probability that A occurs given that B is known to occur; e.g.,

seed

Suppose you want to co-invest in the **second** round of the startup. As such, you are interested in knowing: Given that startup succeeds, what is the probability that it does so after round 2? That would be,

success = { 1VCS, 12VCS }

round 2 = { 12F, 12VCF, 12VCS }

$$\boxed{P(\text{round 2}|\text{sucess})} = \left(\frac{P(\text{round 2 and sucess})}{P(\text{sucess})}\right) = \frac{P(\{12VCS\})}{P(\{1VCS, 12VCS\})}$$

Note the difference between $P(\text{round 2}|\text{sucess})$ and $P(\text{success}|\text{round 2})$(which could directly be obtained from probability tree).

$\Downarrow$ = 1/6

$2 \xrightarrow{1/2} VC \xrightarrow{1/3} 12VCS$

In general:

$$\boxed{P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}.}$$

Multiplication rule

Next, the multiplication rule. Recall our calculation of

$P(\text{round } 2 \mid \text{success})$ success = {1VCS, 12VCS}

round 2 = {12F, 12VCF, 12VCS}

$P(\text{sucess and round 2}) =$ $P(\{12VCS\})$

The **multiplication rule** is that

$$P(A \text{ and } B) = P(A|B)P(B) = P(B|A)P(A).$$

This follows from the definition of the conditional probability as $P(A|B) = P(A \text{ and } B)/P(B)$. Right?

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

Multiplication rule: Example

A bank classifies borrowers as high-risk (HR) and low-risk(LR). Only 15% of its loans are made to HR borrowers. Of all its loans, 5% are in default (D), and 40% of those in default are to HR borrowers. What is the probability that a HR borrower will default?

HR: loan to high risk borrower

$P(HR) = 0.15$

$P(D) = 0.05$

$P(HR \mid D) = 0.4$

multiplication rule

$$P(D \mid HR) = \frac{P(D \cap HR)}{P(HR)} = \frac{P(D) \cdot P(HR \mid D)}{P(HR)}$$

$$= \frac{(0.05)\,(0.4)}{(0.15)} = 0.1333$$

Bayes' rule

Let's go back to the startup investing example and evaluate the probability expression (previously listed in the slide of *conditional probability*

round 2 = {12F, 12VCF, 12VCS}
round 1 = {1F, 1VCF, 1VCS}
P(success | round i)   i = 1, 2

$$\rightarrow P(\text{round 2}|\text{success}) = \frac{P(\text{round 2 and success})}{P(\text{sucess})}$$
$$= \frac{P(\text{sucess}|\text{round 2})P(\text{round 2})}{P(\text{sucess}|\text{round 2})P(\text{round 2}) + P(\text{sucess}|\text{round 1})P(\text{round 1})}$$

We have just derived *Bayes' Rule* to evaluate this probability expression.

$$P(B_j|A) = \frac{P(B_j \cap A)}{P(A)} = \frac{P(B_j) \cdot P(A|B_j)}{P(A)}$$

$P(B_i)$ $P(A|B_i)$

$$P(A) = \sum_{j=1}^{k} P(A \cap B_j) = \sum_{j=1}^{k} P(B_j) \cdot P(A|B_j)$$

Bayes' Rule:
Assume that $\{B_1, B_2, \cdots, B_k\}$ is a partition of Ω (as such that $P(B_i) > 0$, for $i = 1, 2, \cdots, k$. Then,

$$\boxed{P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}}$$

$P(A|B_j)$ ✓

$\boxed{P(B_j|A)}$

Independence

When does event A carry information about event B?
The S&P500 is up 1% today. Does this carry any information about how the Dow Jones index?

- A woman's first child is a boy. What is the probability that the second child is also a boy?

$\boxed{BB,\ BG,}\ GB,\ GG$

- A woman tells you that she has 2 children and that at least one is a boy. What is the probability that the other is also a boy?

$BB,\ BG,\ GB,\ GG$

$A = \{\text{at least one is a boy}\} = \{BB, BG, GB\}$

$$P(\text{other is also boy} \mid A) = \frac{P(\{BB\})}{P(A)} = \frac{1}{3}$$

Independence

Question: You have been observing patterns of roulette outcomes, or lotto winning numbers. How would you use this information?

More on independent events

Consequences of Independence:

Let A and B be two independent events. Then:

- $P(B|A) = P(B)$. Why?

$P(A|B) = P(A)$

- $P(B \cap A) = P(A)P(B)$

$P(B \cap A) = P(B|A)\,P(A)$
$= P(B)\,P(A)$

Independent Random Variables

Two random variables X and Y are **independent** iff

- realization of one does not affect the probability distribution of other
- Formally, events $\{X \leq a\}$ and $\{Y \leq b\}$ are independent for all a, b

Suppose X and Y are independent random variables

- $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

- Covariance, $Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] =$ $E[XY + E[X]E[Y] - YE[X] - XE[Y]]$

$= E[XY] + E[X]E[Y] - 2E[X]E[Y]$

$= 0$

- Correlation, $\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} =$ 0

X, Y independent $\Rightarrow \rho_{x,y} = 0$

Independent Random Variables and Correlation

Suppose X and Y are two r.v. with $\rho_{X,Y} = 0$. Are they independent?

NOT necessarily

$x \in \{1, 2 \ldots 6\}$

Example. Let X be the random outcome of a roll of a die. Let W be a random variable that takes values -1 and 1 with probability $1/2$ each independent of X. Let $Y = WX$.

- What is $\rho_{X,Y}$? $= 0$

$$\mathbb{E}[X] = \frac{1}{6}(1+2+\cdots 6) = \frac{21}{6} = \frac{7}{3}$$

$$\mathbb{E}[Y] = \mathbb{E}[WX] = \mathbb{E}[W]\,\mathbb{E}[X] = 0$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = \mathbb{E}\left[XY - Y\,\mathbb{E}[X]\right]$$

$$= \mathbb{E}[WX^2] - \mathbb{E}[Y]\,\mathbb{E}[X]$$

$$= 0$$

- Are X and Y independent?

$(\{X = 1\}) \longrightarrow Y \in \{-1, 1\}$

$Y \in \{-6, -5, \ldots 6\}$

More on Independence

Q: What is the connection between independent and mutually exclusive events?

A, B are mutually exclusive $\quad A \cap B = \phi$

Q a) Can A & B be independent?

No. $\quad P(B|A) = 0$

$P(A|B) = 0$

Summary

Random variable: outcome $\xrightarrow{\text{map}}$ numerical value

- what are the possible values? what are their probabilities?
- distribution: $\mathbb{P}(X = \text{outcome})$ for every possible outcome

Basic concepts of probability:

- Sample space: set of all possible outcomes.
- Event: subset of the sample space.
- Union: A or B occurs when either A or B occur.
- Intersection: A and B occurs when A and B occur.
- Complement: $\bar{A}$ is the event that A does not occur $\rightarrow P(\bar{A}) = 1 - P(A)$.

Summary

Axioms of Probability

- $0 \leq P(A) \leq 1$.
- Total probability: $P(\text{sample space}) = 1$.
- Addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Rules of probability:

- Conditional probability: $P(A|B) = P(A \cap B)/P(B)$.
- Multiplication rule: $P(A \text{ and } B) = P(A)P(B|A)$. Baye's rule
- Independence: $P(A|B) = P(A)$.

Summary

Expected value:

- Calculation: $\mathbb{E}[X] = \sum_{\text{all outcomes}} (\text{outcome}) \cdot \mathbb{P}(X = \text{outcome})$
- Properties:
  1. $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
  2. $\mathbb{E}[a+bX] = a + b\mathbb{E}[X]$

Variance & standard deviation:

$$\mathrm{Var}[X] = \sum_{\text{all } x} (x - \mathbb{E}[X])^2 \cdot \mathbb{P}(X = x)$$

$$\sigma_x = \sqrt{\mathrm{Var}[X]}$$

σ_x measures variability of the distribution around $\mathbb{E}[X]$.

$$\mathrm{Var}[a+bX] = b^2 \mathrm{Var}[X]$$

$$\text{Stdev}(a+bX) = |b| \cdot \text{Stdev}(X).$$