Problem Set 3

Problem 1: Alice and Bob's card game

Alice plays the following game with Bob. First, Alice randomly chooses a set of 4 cards out of a 52-card deck, memorizes them, and places them back into the deck. (Any set of 4 cards is equally likely.) Then, Bob randomly chooses 8 cards out of the same deck. (Any set of 8 cards is equally likely.)

What is the probability that all 4 cards Alice chose were also among the 8 cards chosen by Bob?

Problem 2: 13 cards in a deck

A player is randomly dealt a sequence of 13 cards from a standard 52-card deck. All sequences of 13 cards are equally likely. In an equivalent model, the cards are chosen and dealt one at a time. When choosing a card, the dealer is equally likely to pick any of the cards that remain in the deck.

1. What is the probability the 13th card dealt is a King? **Note:** Your answer should be a number. Do not enter '!' or combinations in your answer.
2. Find the probability of the event that the 13th card dealt is the first King dealt. Identify the correct expression.

Problem 3: Splitting students into 3 classes

A group of 90 students is to be split at random into 3 classes of equal size. All partitions are equally likely. Joe and Jane are members of the 90-student group. Find the probability that Joe and Jane end up in the same class. **Note:** Your answer should be a number. Do not enter '!' or combinations in your answer.

Problem 4: A three-sided die

The newest invention of the 6.041x staff is a three-sided die. On any roll of this die, the result is 1 with probability $1/2$, 2 with probability $1/4$, and 3 with probability $1/4$.

Consider a sequence of six independent rolls of this die.

1. Find the probability that exactly two of the rolls results in a 3.

2. Given that exactly two of the six rolls resulted in a 1, find the probability that the first roll resulted in a 1. **Note:** Your answer should be a number. Do not enter '!' or combinations in your answer.

3. We are told that exactly three of the rolls resulted in a 1 and exactly three rolls resulted in a 2. Given this information, find the probability that the six rolls resulted in the sequence $(1, 2, 1, 2, 1, 2)$. **Note:** Your answer should be a number. Do not enter '!' or combinations in your answer.

4. The conditional probability that exactly k rolls resulted in a 3, given that at least one roll resulted in a 3, is of the form:

$$\frac{1}{1-(c_1/c_2)^{c_3}}\binom{c_3}{k}\left(\frac{1}{c_2}\right)^{k}\left(\frac{c_1}{c_2}\right)^{c_3-k}, \quad \text{for } k = 1, 2, \ldots, 6.$$

   Find the values of the constants c_1, c_2, and c_3:

   $c_1 =$
   $c_2 =$
   $c_3 =$

Problem 5: Hats in a box

Each one of n persons, indexed by 1, 2, . . . , n, has a clean hat and throws it into a box. The persons then pick hats from the box, at random. Every assignment of the hats to the persons is equally likely. In an equivalent model, each person picks a hat, one at a time, in the order of their index, with each one of the remaining hats being equally likely to be picked. Find the probability of the following events.

1. Every person gets his or her own hat back.

2. Each one of persons $1, \ldots, m$ gets his or her own hat back, where $1 \leq m \leq n$.

3. Each one of persons $1, \ldots, m$ gets back a hat belonging to one of the last m persons (persons $n - m + 1, \ldots, n$), where $1 \leq m \leq n$.

Now assume, in addition, that every hat thrown into the box has probability p of getting dirty (independently of what happens to the other hats or who has dropped or picked it up). Find the probability that:

4. Persons $1, \ldots, m$ will pick up clean hats.

5. Exactly m persons will pick up clean hats.