# Problem Set 7a

## Problem 1: Defective coin

A defective coin minting machine produces coins whose probability of Heads is a random variable $Q$ with PDF

$$f_Q(q) = \begin{cases} 3q^2, & \text{if } q \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

A coin produced by this machine is tossed repeatedly, with successive tosses assumed to be independent. Let $A$ be the event that the first toss of this coin results in Heads, and let $B$ be the event that the second toss of this coin results in Heads.

1. $\mathbf{P}(A) =$
   (Your answer should be a number.)

2. Find the conditional PDF of $Q$ given event $A$. Express your answer in terms of $q$ using standard notation.

   For $0 \leq q \leq 1$, $f_{Q|A}(q) =$

3. $\mathbf{P}(B \mid A) =$
   (Your answer should be a number.)

**Problem 2: Hypothesis test between two coins**

Alice has two coins. The probability of Heads for the first coin is $1/3$, and the probability of Heads for the second is $2/3$. Other than this difference, the coins are indistinguishable. Alice chooses one of the coins at random and sends it to Bob. The random selection used by Alice to pick the coin to send to Bob is such that the first coin has a probability $p$ of being selected. Assume that $0 < p < 1$. Bob tries to guess which of the two coins he received by tossing it 3 times in a row and observing the outcome. Assume that for any particular coin, all tosses of that coin are independent.

1. Given that Bob observed $k$ Heads out of the 3 tosses (where $k = 0, 1, 2, 3$), what is the conditional probability that he received the first coin?

2. We define an error to have occurred if Bob decides that he received one coin from Alice, but he actually received the other coin. He decides that he received the first coin when the number of Heads, $k$, that he observes on the 3 tosses satisfies a certain condition. When one of the following conditions is used, Bob will minimize the probability of error. Choose the correct threshold condition.

3. For this part, assume that $p = 2/3$.

   (a) What is the probability that Bob will guess the coin correctly using the decision rule from part 2?

   (b) Suppose instead that Bob tries to guess which coin he received without tossing it. He still guesses the coin in order to minimize the probability of error. What is the probability that Bob will guess the coin correctly under this scenario?

4. Suppose that we increase $p$. Then does the number of different values of $k$ for which Bob decides that he received the first coin increase, decrease, or stay the same?

5. Find the values of $p$ for which Bob will never decide that he received the first coin, regardless of the outcome of the 3 tosses.
   $p$ is less than

**Problem 3: Hypothesis test with a continuous observation**

Let $\Theta$ be a Bernoulli random variable that indicates which one of two hypotheses is true, and let $\mathbf{P}(\Theta = 1) = p$. Under the hypothesis $\Theta = 0$, the random variable $X$ is uniformly distributed over the interval $[0, 1]$. Under the alternative hypothesis $\Theta = 1$, the PDF of $X$ is given by

$$f_{X|\Theta}(x \mid 1) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the MAP rule for deciding between the two hypotheses, given that $X = x$.

1. Suppose for this part of the problem that $p = 3/5$. The MAP rule can choose in favor of the hypothesis $\Theta = 1$ if and only if $x \geq c_1$. Find the value of $c_1$.
   $c_1 =$

2. Assume now that $p$ is general such that $0 \leq p \leq 1$. It turns out that there exists a constant $c$ such that the MAP rule always decides in favor of the hypothesis $\Theta = 0$ if and only if $p < c$. Find $c$.
   $c =$

3. For this part of the problem, assume again that $p = 3/5$. Find the conditional probability of error for the MAP decision rule given that the hypothesis $\Theta = 0$ is true.
   $\mathbf{P}(\text{error} \mid \Theta = 0) =$

4. Find the probability of error associated with the MAP rule as a function of $p$. Express your answer in terms of $p$ using standard notation.
   When $p \leq 1/3$, $\mathbf{P}(\text{error}) =$

   When $p \geq 1/3$, $\mathbf{P}(\text{error}) =$

**Problem 4: Trajectory estimation**

The vertical coordinate ("height") of an object in free fall is described by an equation of the form

$$x(t) = \theta_0 + \theta_1 t + \theta_2 t^2,$$

where $\theta_0$, $\theta_1$, and $\theta_2$ are some parameters and $t$ stands for time. At certain times $t_1, \ldots, t_n$, we make noisy observations $Y_1, \ldots, Y_n$, respectively, of the height of the object. Based on these observations, we would like to estimate the object's vertical trajectory.

We consider the special case where there is only one unknown parameter. We assume that $\theta_0$ (the height of the object at time zero) is a known constant. We also assume that $\theta_2$ (which is related to the acceleration of the object) is known. We view $\theta_1$ as the realized value of a continuous random variable $\Theta_1$. The observed height at time $t_i$ is $Y_i = \theta_0 + \Theta_1 t_i + \theta_2 t_i^2 + W_i$, $i = 1, \ldots, n$, where $W_i$ models the observation noise. We assume that $\Theta_1 \sim N(0, 1)$, $W_1, \ldots, W_n \sim N(0, \sigma^2)$, and all these random variables are independent.

In this case, finding the MAP estimate of $\Theta_1$ involves the minimization of

$$\theta_1^2 + \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \theta_0 - \theta_1 t_i - \theta_2 t_i^2)^2$$

with respect to $\theta_1$.

1. Carry out this minimization and choose the correct formula for the MAP estimate, $\hat{\theta}_1$, from the options below.

2. The formula for $\hat{\theta}_1$ can be used to define the MAP estimator, $\hat{\Theta}_1$ (a random variable), as a function of $t_1, \ldots, t_n$ and the random variables $Y_1, \ldots, Y_n$. Identify whether the following statement is true.

   The MAP estimator $\hat{\Theta}_1$ has a normal distribution.

3. Let $\sigma = 1$ and consider the special case of only two observations ($n = 2$). Write down a formula for the mean squared error $\mathbf{E}[(\hat{\Theta}_1 - \Theta_1)^2]$, as a function of $t_1$ and $t_2$. Enter 't1' for $t_1$ and 't2' for $t_2$.

   $\mathbf{E}[(\hat{\Theta}_1 - \Theta_1)^2] =$

4. Consider the "experimental design" problem of choosing when to make measurements. Under the assumptions of part (3), and under the constraints $0 \le t_1, t_2 \le 10$, find the values of $t_1$ and $t_2$ that minimize the mean squared error associated with the MAP estimator.

**Problem 5: Hypothesis test between two normals**

Let $T_1, T_2, \ldots, T_n$ be i.i.d. observations, each drawn from a common normal distribution with mean zero. With probability 1/2 this normal distribution has variance 1, and with probability 1/2 it has variance 4. Based on the observed values $t_1, t_2, \ldots, t_n$, we use the MAP rule to decide whether the normal distribution from which they were drawn has variance 1 or variance 4. The MAP rule decides that the underlying normal distribution has variance 1 if and only if

$$\left| c_1 \sum_{i=1}^{n} t_i^2 + c_2 \sum_{i=1}^{n} t_i \right| < 1.$$

Find the values of $c_1 \geq 0$ and $c_2 \geq 0$ such that this is true. Express your answer in terms of $n$, and use 'ln' to denote the natural logarithm function, as in 'ln(3)'.

$c_1 =$

$c_2 =$