Problem Set 9

Problem 1: Fred promotes dog food

Fred is giving away samples of dog food. He makes visits door to door, but he gives a sample away (one can of dog food) only on those visits for which the door is answered *and* a dog is in residence. On any visit, the probability of the door being answered is 3/4, and the probability that any given household has a dog is 2/3. Assume that the events "Door answered" and "A dog lives here" are independent and also that events related to different households are independent.

1. What is the probability that Fred gives away his first sample on his third visit?
2. Given that he has given away exactly four samples on his first eight visits, what is the conditional probability that Fred will give away his fifth sample on his eleventh visit?
3. What is the probability that he gives away his second sample on his fifth visit?
4. Given that he did not give away his second sample on his second visit, what is the conditional probability that Fred will give away his second sample on his fifth visit?
5. We will say that Fred "needs a new supply" immediately *after* the visit on which he gives away his last sample. If he starts out with two samples, what is the probability that he completes at least five visits before he needs a new supply?
6. If he starts out with exactly 10 samples, what is the expected value of the number of homes with dogs where Fred visits but leaves no samples (because the door was not answered) before he needs a new supply?

Problem 2: Laptop failures

Suppose that you have two laptops, both of which you begin using at time 0. Each laptop will eventually fail, and we model each one's lifetime as exponentially distributed with the same parameter λ. The lifetimes of the two laptops are independent. One of the laptops will fail first, followed by the other. Define T_1 as the time of the first failure and T_2 as the time of the second failure.

In parts 1, 2, 4, and 5 below, your answers will be algebraic expressions. Enter 'lambda' for λ and use 'exp()' for exponentials. Follow standard notation.

1. Determine the PDF of T_1. For $t > 0$, $f_{T_1}(t) =$
2. Let $X = T_2 - T_1$. Determine the conditional PDF $f_{X|T_1}(x \mid t)$. For $x, t > 0$, $f_{X|T_1}(x \mid t) =$
3. Is X independent of T_1?
4. Determine the PDF $f_{T_2}(t)$. For $t > 0$, $f_{T_2}(t) =$
5. $\mathbf{E}[T_2] =$

Problem 3: Shuttles

In parts 1, 3, 4, and 5 below, your answers will be algebraic expressions. Enter 'lambda' for λ and 'mu' for μ. Follow standard notation.

1. Shuttles bound for Boston depart from New York every hour on the hour (e.g., at exactly one o'clock, two o'clock, etc.). Passengers arrive at the departure gate in New York according to a Poisson process with rate λ per hour. What is the expected number of passengers on any given shuttle? (Assume that everyone who arrives between two successive shuttle departures boards the shuttle immediately following his/her arrival.)

2. Now, and for the remaining parts of this problem, suppose that the shuttles are not operating on a deterministic schedule. Rather, their interdeparture times are independent and exponentially distributed with common parameter μ per hour. Shuttle departures are independent of the process of passenger arrivals. Is the sequence of shuttle departures a Poisson process?

3. Let us say that an "event" occurs whenever a passenger arrives or a shuttle departs. What is the expected number of "events" that occur in any one-hour interval?

4. If a passenger arrives at the gate and sees 2λ people waiting (assume that 2λ is an integer), what is his/her expected waiting time until the next shuttle departs?

5. Find the PMF, $p_N(n)$, of the number, N, of people on any given shuttle. Assume that $\lambda = 20$ and $\mu = 2$.

   For $n \geq 0$, $p_N(n) =$

Problem 4: Ships

All ships travel at the same speed through a wide canal. Each ship takes t days to traverse the length of the canal. Eastbound ships (i.e., ships traveling east) arrive as a Poisson process with an arrival rate of λ_E ships per day. Westbound ships (i.e., ships traveling west) arrive as an independent Poisson process with an arrival rate of λ_W ships per day. A pointer at some location in the canal is always pointing in the direction of travel of the most recent ship to pass it.

In each part below, your answers will be algebraic expressions in terms of $\lambda_E, \lambda_W, x, t, v$ and/or k. Enter 'LE' for λ_E and 'LW' for λ_W, and use 'exp()' for exponentials. Do **not** use 'fac()' or '!' for factorials; instead, calculate out the numerical value of any factorials. Follow standard notation.

For parts (1) and (2), suppose that the pointer is currently pointing west.

1) What is the probability that the next ship to pass will be westbound?

2) Determine the PDF, $f_X(x)$, of the remaining time, X, until the pointer changes direction.

   For $x \geq 0$, $f_X(x) =$

For the remaining parts of this problem, we make no assumptions about the current direction of the pointer.

3) What is the probability that an eastbound ship does not pass by any westbound ships during its eastward journey through the canal?

4) Starting at an arbitrary time, we monitor the cross-section of the canal at some fixed location along its length. Let V be the amount of time that will have elapsed (since we began monitoring) by the time we observe our seventh eastbound ship. Determine the PDF of V.

   For $v \geq 0$, $f_V(v) =$

5) What is the probability that the next ship to arrive causes a change in the direction of the pointer?

6) If we begin monitoring a fixed cross-section of the canal at an arbitrary time, determine the probability mass function $p_K(k)$ for K, the total number of ships we observe up to and including the seventh eastbound ship we see. The answer will be of the form $p_K(k) = \binom{a}{6} \cdot b$, for suitable algebraic expressions in place of a and b.

$a =$

$b =$

Problem 5: Arrivals during overlapping time intervals

Consider a Poisson process with rate λ. Let N be the number of arrivals in $(0, t]$ and M be the number of arrivals in $(0, t+s]$, where $t > 0$, $s \geq 0$.

In each part below, your answers will be algebraic expressions in terms of λ, t, s, m and/or n. Enter 'lambda' for λ and use 'exp()' for exponentials. Do **not** use 'fac()' or '!' for factorials. Follow standard notation.

1. For $0 \leq n \leq m$, the conditional PMF $p_{M|N}(m \mid n)$ of M given N is of the form $\frac{a}{b!}$ for suitable algebraic expressions in place of a and b.

   $a =$

   $b =$

2. For $0 \leq n \leq m$, the joint PMF $p_{N,M}(n, m)$ of N and M is of the form $\frac{c}{n!d!}$ for suitable algebraic expressions in place of c and d.

   $c =$

   $d =$

3. For $0 \leq n \leq m$, the conditional PMF $p_{N|M}(n|m)$ of N given M is of the form $f \cdot \frac{g!}{n!h!}$ for suitable algebraic expressions in place of f, g, and h.

   $f =$

   $g =$

   $h =$

4. $E[NM] =$

Problem 6: Random incidence under Erlang interarrivals

A single dot is placed on a very long length of yarn at the textile mill. The yarn is then cut into pieces. The lengths of the different pieces are independent, and the length of each piece is distributed according to the same PDF $f_X(x)$. Let R be the length of the piece that includes the dot. Determine the expected value of R in each of the following cases.

In each part below, express your answer in terms of λ using standard notation. Enter "lambda" for λ.

1. Suppose that $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$

   $\mathbf{E}[R] =$

2. Suppose that $f_X(x) = \begin{cases} \frac{\lambda^3 x^2 e^{-\lambda x}}{2}, & x \geq 0, \\ 0, & x < 0. \end{cases}$

   $\mathbf{E}[R] =$

Problem 7: Sampling families

We are given the following statistics about the number of children in the families of a small village.

There are 100 families: 10 families have no children, 40 families have 1 child each, 30 families have 2 children each, 10 families have 3 each, and 10 families have 4 each.

1. If you pick a family at random (each family in the village being equally likely to be picked), what is the expected number of children in that family?

2. If you pick a child at random (each child in the village being equally likely to be picked), what is the expected number of children in that child's family (including the picked child)?

3. Generalize your approach from part 2: Suppose that a fraction p_k of the families have k children each. Let K be the number of children in a randomly selected family, and let $a = \mathbf{E}[K]$ and $b = \mathbf{E}[K^2]$. Let W be the number of children in a randomly chosen child's family. Express $\mathbf{E}[W]$ in terms of a and b using standard notation.

   $\mathbf{E}[W] =$