Problem Set 4

Problem 1: Tosses of a biased coin

Consider 10 independent tosses of a biased coin with the probability of Heads at each toss equal to p, where $0 < p < 1$.

1. Let A be the event that there are 6 Heads in the first 8 tosses. Let B be the event that the 9th toss results in Heads.

   Find $\mathbf{P}(B \mid A)$ and express it in terms of p using standard notation.

2. Find the probability that there are 3 Heads in the first 4 tosses and 2 Heads in the last 3 tosses. Express your answer in terms of p using standard notation. Remember not to use factorials or combinations in your answer.

3. Given that there were 4 Heads in the first 7 tosses, find the probability that the 2nd Heads occurred at the 4th toss. Give a numerical answer.

4. We are interested in calculating the probability that there are 5 Heads in the first 8 tosses and 3 Heads in the last 5 tosses. Give the numerical values of a, b, c, d, e, and f that would match the answer $ap^7(1-p)^3 + bp^c(1-p)^d + ep^f(1-p)^f$.

   $a =$
   $b =$
   $c =$
   $d =$
   $e =$
   $f =$

Problem 2: Three-sided dice

We have two fair three-sided dice, indexed by $i = 1, 2$. Each die has sides labelled 1, 2, and 3. We roll the two dice independently, one roll for each die. For $i = 1, 2$, let the random variable X_i represent the result of the ith die, so that X_i is uniformly distributed over the set $\{1, 2, 3\}$. Define $X = X_2 - X_1$.

1. Calculate the numerical values of following probabilities, as well as the expected value and variance of X:

   $\mathbf{P}(X = 0) =$
   $\mathbf{P}(X = 1) =$
   $\mathbf{P}(X = -2) =$
   $\mathbf{P}(X = 3) =$
   $\mathbf{E}[X] =$
   $\text{var}(X) =$

2. Let $Y = X^2$. Calculate the following probabilities:

   $\mathbf{P}(Y = 0) =$
   $\mathbf{P}(Y = 1) =$
   $\mathbf{P}(Y = 2) =$

Problem 3: PMF, expectation, and variance

The random variables X and Y have the joint PMF

$$p_{X,Y}(x,y) = \begin{cases} c \cdot (x+y)^2, & \text{if } x \in \{1,2,4\} \text{ and } y \in \{1,3\}, \\ 0, & \text{otherwise.} \end{cases}$$

All answers in this problem should be numerical.

1. Find the value of the constant c.
   $c =$

2. Find $\mathbf{P}(Y < X)$.
   $\mathbf{P}(Y < X) =$

3. Find $\mathbf{P}(Y = X)$.
   $\mathbf{P}(Y = X) =$

4. Find the following probabilities.

   $\mathbf{P}(X = 1) =$
   $\mathbf{P}(X = 2) =$
   $\mathbf{P}(X = 3) =$
   $\mathbf{P}(X = 4) =$

5. Find the expectations $\mathbf{E}[X]$ and $\mathbf{E}[XY]$.

   $\mathbf{E}[X] =$
   $\mathbf{E}[XY] =$

6. Find the variance of X.

   $\text{var}(X) =$

Problem 4: Joint PMF

The joint PMF, $p_{X,Y}(x, y)$, of the random variables X and Y is given by the following table:

$y = 1$	$4c$	0	$2c$	$8c$
$y = 0$	$3c$	$2c$	0	$2c$
$y = -1$	$2c$	0	c	$4c$
	$x = -2$	$x = -1$	$x = 0$	$x = 1$

1. Find the value of the constant c. $c =$
2. Find $p_X(1)$. $p_X(1) =$
3. Consider the random variable $Z = X^2Y^3$. Find $\mathbf{E}[Z \mid Y = -1]$. $\mathbf{E}[Z \mid Y = -1] =$
4. Conditioned on the event that $Y \neq 0$, are X and Y independent?
5. Find the conditional variance of Y given that $X = 0$. $\text{var}(Y \mid X = 0) =$

Problem 5: Indicator variables

Consider a sequence of independent tosses of a biased coin at times $k = 0, 1, 2, \ldots, n$. On each toss, the probability of Heads is p, and the probability of Tails is $1 - p$.

A reward of one unit is given at time k, for $k \in \{1, 2, \ldots, n\}$, if the toss at time k resulted in Tails and the toss at time $k - 1$ resulted in Heads. Otherwise, no reward is given at time k.

Let R be the sum of the rewards collected at times $1, 2, \ldots, n$.

We will find $\mathbf{E}[R]$ and $\mathrm{var}(R)$ by carrying out a sequence of steps. Express your answers below in terms of p and/or n using standard notation. Remember to write "*" for all multiplications and to include parentheses where necessary.

We first work towards finding $\mathbf{E}[R]$.

1. Let I_k denote the reward (possibly 0) given at time k, for $k \in \{1, 2, \ldots, n\}$. Find $\mathbf{E}[I_k]$. $\mathbf{E}[I_k] =$
2. Using the answer to part 1, find $\mathbf{E}[R]$. $\mathbf{E}[R] =$

The variance calculation is more involved because the random variables $I_1, I_2, \ldots, I_n$ are not independent. We begin by computing the following values.

3. If $k \in \{1, 2, \ldots, n\}$, then $\mathbf{E}[I_k^2] =$
4. If $k \in \{1, 2, \ldots, n - 1\}$, then $\mathbf{E}[I_k I_{k+1}] =$
5. If $k \geq 1$, $\ell \geq 2$, and $k + \ell \leq n$, then $\mathbf{E}[I_k I_{k+\ell}] =$
6. Using the results above, calculate the numerical value of $\mathrm{var}(R)$ assuming that $p = 3/4$, $n = 10$. $\mathrm{var}(R) =$

Problem 6: True or False

For each of the following statements, determine whether it is true (meaning, always true) or false (meaning, not always true). Here, we assume all random variables are discrete, and that all expectations are well-defined and finite.

1. Let X and Y be two binomial random variables.

   a) If X and Y are independent, then $X + Y$ is also a binomial random variable.

   b) If X and Y have the same parameters, n and p, then $X + Y$ is a binomial random variable.

   c) If X and Y have the same parameter p, and are independent, then $X + Y$ is a binomial random variable.

2. Suppose that $\mathbf{E}[X] = 0$. Then, $X = 0$.

3. Suppose that $\mathbf{E}[X^2] = 0$. Then, $\mathbf{P}(X = 0) = 1$.