# Problem Set 6

## Problem 1: The PDF of exp(X)

Let $X$ be a random variable with PDF $f_X$. Find the PDF of the random variable $Y = e^X$ for each of the following cases:

1. For general $f_X$, when $y > 0$, $f_Y(y) =$

2. When $f_X(x) = \begin{cases} 1/3, & \text{if } -2 < x \leq 1, \\ 0, & \text{otherwise,} \end{cases}$

   we have $f_Y(y) = \begin{cases} g(y), & \text{if } a < y \leq b, \\ 0, & \text{otherwise.} \end{cases}$

   Give a formula for $g(y)$ and the values of $a$ and $b$ using standard notation. (In your answers, you may use the symbol 'e' to denote the base of the natural logarithm.)
   $g(y) =$
   $a =$
   $b =$

3. When $f_X(x) = \begin{cases} 2e^{-2x}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$

   we have $f_Y(y) = \begin{cases} g(y), & \text{if } a < y, \\ 0, & \text{otherwise.} \end{cases}$

   Give a formula for $g(y)$ and the value of $a$ using the standard notation.
   $g(y) =$
   $a =$

4. When $X$ is a standard normal random variable, we have, for $y > 0$, $f_Y(y) =$

**Problem 2: Functions of a standard normal**

The random variable $X$ has a standard normal distribution. Find the PDF of the random variable $Y$, where:

1. $Y = 3X - 1$.
2. $Y = 3X^2 - 1$. For $y \geq -1$,

**Problem 3: The PDF of the maximum**

Let $X$ and $Y$ be independent random variables, each uniformly distributed on the interval $[0, 1]$.

1. Let $Z = \max\{X, Y\}$. Find the PDF of $Z$. Express your answer in terms of $z$ using standard notation.
   For $0 < z < 1$, $f_Z(z) =$
2. Let $Z = \max\{2X, Y\}$. Find the PDF of $Z$. Express your answer in terms of $z$ using standard notation.
   For $0 < z < 1$, $f_Z(z) =$
   For $1 < z < 2$, $f_Z(z) =$

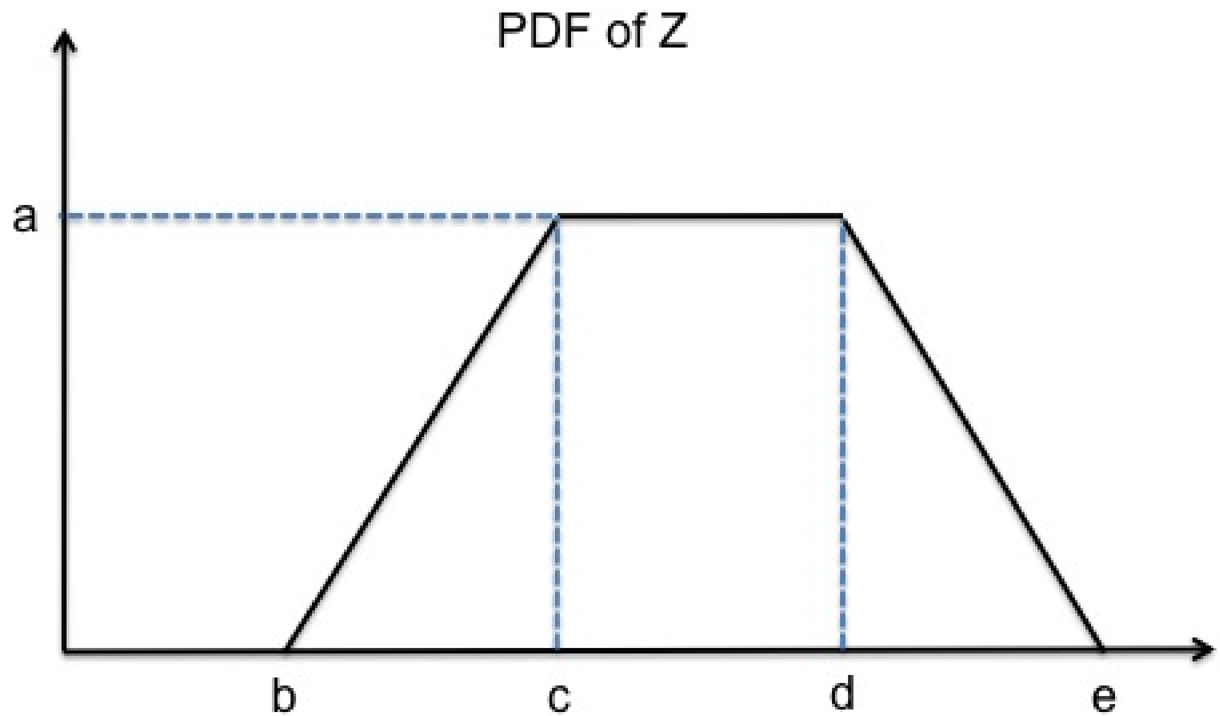
**Problem 4: Convolution calculations**

1. Let the discrete random variable $X$ be uniform on $\{0, 1, 2\}$ and let the discrete random variable $Y$ be uniform on $\{3, 4\}$. Assume that $X$ and $Y$ are independent. Find the PMF of $X+Y$ using convolution. Determine the values of the constants $a$, $b$, $c$, and $d$ that appear in the following specification of the PMF.

$$p_{X+Y}(z) = \begin{cases} a, & z = 3, \\ b, & z = 4, \\ c, & z = 5, \\ d, & z = 6, \\ 0, & \text{otherwise.} \end{cases}$$

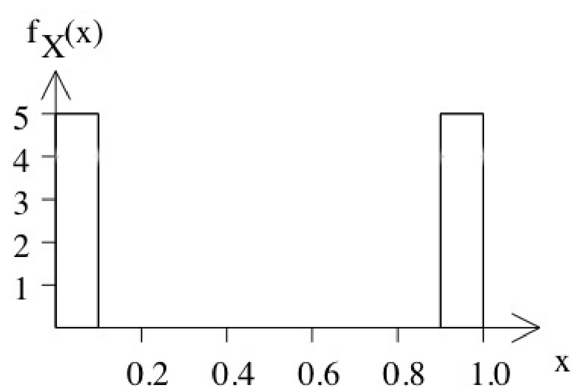
   $a =$
   $b =$
   $c =$
   $d =$

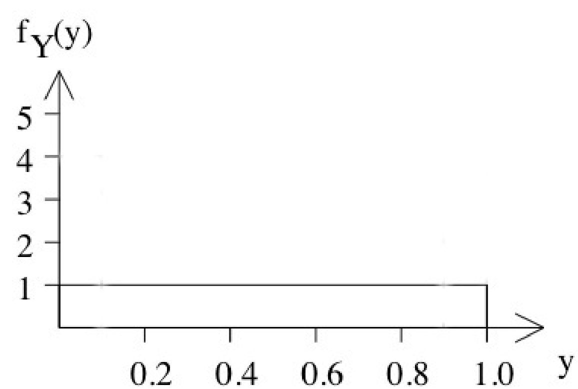
2. Let the random variable $X$ be uniform on $[0, 2]$ and the random variable $Y$ be uniform on $[3, 4]$. (Note that in this case, $X$ and $Y$ are continuous random variables.) Assume that $X$ and $Y$ are independent. Let $Z = X+Y$. Find the PDF of $Z$ using convolution. The following figure shows a plot of this PDF. Determine the values of $a$, $b$, $c$, $d$, and $e$.




$a =$
$b =$
$c =$
$d =$
$e =$

3. Let $X$and $Y$ be two independent random variables with the PDFs shown below. below.

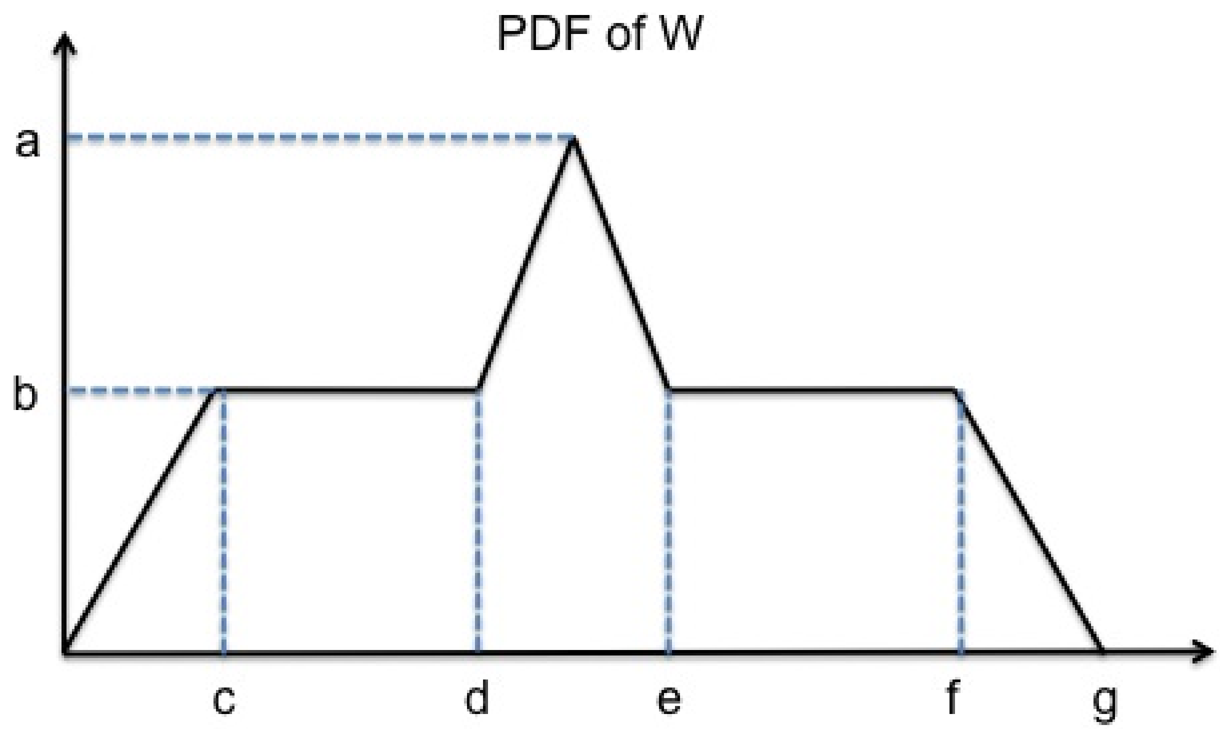


$$f_X(x) = \begin{cases} 5, & \text{if } 0 \le x \le 0.1 \text{ or } 0.9 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \begin{cases} 1, & \text{if } 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $W = X + Y$. The following figure shows a plot of the PDF of $W$. Determine the values of $a$, $b$, $c$, $d$, $e$, $f$, and $g$.




$a =$

$b =$

$c =$

$d =$

$e =$

$f =$

$g =$

**Problem 5: Covariance for the multinomial**

Consider $n$ independent rolls of a $k$-sided fair die with $k \geq 2$: the sides of the die are labelled 1, 2, . . . , $k$ and each side has probability $1/k$ of facing up after a roll. Let the random variable $X_i$ denote the number of rolls that result in side $i$ facing up. Thus, the random vector ($X_1$, . . . , $X_k$) has a multinomial distribution.

1. Which of the following statements is correct? Try to answer without doing any calculations.

$X_1$ and $X_2$ are uncorrelated.

$X_1$ and $X_2$ are positively correlated.

$X_1$ and $X_2$ are negatively correlated.

2. Find the covariance, $\text{cov}(X_1, X_2)$, of $X_1$ and $X_2$. Express your answer as a function of $n$ and $k$ using standard notation. *Hint:* Use indicator variables to encode the result of each roll.

$\text{cov}(X_1, X_2) =$

3. Suppose now that the die is biased, with a probability $p_i \neq 0$ that the result of any given die roll is $i$, for $i = 1, 2, \ldots, k$. We still consider $n$ independent tosses of this biased die and define $X_i$ to be the number of rolls that result in side $i$ facing up.

Generalize your answer to part 2: Find $\text{cov}(X_1, X_2)$ for this case of a biased die. Express your answer as a function of $n, k, p_1, p_2$ using standard notation. Write $p_1$ and $p_2$ as `p_1` and `p_2`, respectively, and wrap them in parentheses in your answer; i.e., enter `(p_1)` and `(p_2)`.

$\text{cov}(X_1, X_2) =$

**Problem 6: Correlation coefficients**

Consider the random variables $X$, $Y$ and $Z$, which are given to be pairwise uncorrelated (i.e., $X$ and $Y$ are uncorrelated, $X$ and $Z$ are uncorrelated, and $Y$ and $Z$ are uncorrelated). Suppose that

- $\mathbf{E}[X] = \mathbf{E}[Y] = \mathbf{E}[Z] = 0$,
- $\mathbf{E}[X^2] = \mathbf{E}[Y^2] = \mathbf{E}[Z^2] = 1$,
- $\mathbf{E}[X^3] = \mathbf{E}[Y^3] = \mathbf{E}[Z^3] = 0$,
- $\mathbf{E}[X^4] = \mathbf{E}[Y^4] = \mathbf{E}[Z^4] = 3$.

Let $W = a + bX + cX^2$ and $V = dX$, where $a, b, c$, and $d$ are constants, all greater than 0.

Find the correlation coefficients $\rho(X - Y, X + Y)$, $\rho(X + Y, Y + Z)$, $\rho(X, Y + Z)$ and $\rho(W, V)$.

1. $\rho(X - Y, X + Y) =$
2. $\rho(X + Y, Y + Z) =$
3. $\rho(X, Y + Z) =$
4. $\rho(W, V) =$

**Problem 7: Sum of a random number of r.v.'s**

A fair coin is flipped independently until the first Heads is observed. Let $K$ be the number of Tails observed **before** the first Heads (note that $K$ is a random variable). For $k = 0, 1, 2, \ldots,$ $K$, let $X_k$ be a continuous random variable that is uniform over the inter-val $[0, 3]$. The $X_k$'s are independent of one another and of the coin flips. Let the random
variable $X$ be defined as the sum of all the $X_k$'s generated before the first Heads. That is, $X = \sum_{k=0}^{K} X_k$. Find the mean and variance of $X$. You may use the fact that the mean and variance of a geometric random variable with parameter $p$ are $1/p$ and $(1-p)/p^2$, respectively.
$\mathbf{E}[X] =$

$\text{var}(X) =$