Problem Set 7b

Problem 1: Determining the type of a lightbulb

The lifetime of a type-A bulb is exponentially distributed with parameter λ. The lifetime of a type-B bulb is exponentially distributed with parameter μ, where $\mu > \lambda > 0$. You have a box full of lightbulbs of the same type, and you would like to know whether they are of type A or B. Assume an *a priori* probability of $1/3$ that the box contains type-B lightbulbs.

1. You observe the value t_1 of the lifetime, T_1, of a lightbulb. A MAP decision rule implies that the lightbulb is of type A if and only if $t_1 \geq \alpha$.

   Assuming that $\mu \geq 2\lambda$, find α. Express your answer in terms of μ and λ. Use 'mu', 'lambda' and 'ln' to denote μ, λ, and the natural logarithm function, respectively. For example, $\ln \frac{2\mu}{\lambda}$ should be entered as 'ln((2*mu)/lambda)'.

   $\alpha =$

2. Assuming that $\mu \geq 2\lambda$, what is the probability of error of the MAP estimator?

3. Assume that $\lambda = 2$ and $\mu = 3$. Find the LMS estimate of T_2, the lifetime of another lightbulb from the same box, based on observing $T_1 = 2$. Assume that conditioned on the bulb type, bulb lifetimes are independent. (For this part, you will need a calculator. Provide an answer with an accuracy of two decimal places.)

   LMS estimate of $T_2 =$

Problem 2: Estimating the parameter of a geometric r.v.

We have k coins. The probability of Heads is the same for each coin and is the realized value q of a random variable Q that is uniformly distributed on $[0, 1]$. We assume that conditioned on $Q = q$, all coin tosses are independent. Let T_i be the number of tosses of the i^{th} coin until that coin results in Heads for the first time, for $i = 1, 2, \ldots, k$. (T_i includes the toss that results in the first Heads.)

You may find the following integral useful: For any non-negative integers k and m,

$$\int_0^1 q^k(1-q)^m dq = \frac{k!m!}{(k+m+1)!}.$$

1. Find the PMF of T_1. (Express your answer in terms of t using standard notation.)

   For $t = 1, 2, \ldots$, we have $p_{T_1}(t) =$

2. Find the least mean squares (LMS) estimate of Q based on the observed value, t, of T_1. (Express your answer in terms of t using standard notation.)

   $\mathbf{E}[Q \mid T_1 = t] =$

3. We flip each of the k coins until they result in Heads for the first time. Compute the maximum a posteriori (MAP) estimate $\hat{q}$ of Q given the number of tosses needed, $T_1 = t_1, \ldots, T_k = t_k$, for each coin. Choose the correct expression for $\hat{q}$.

Problem 3: Radiation from a remote star

Caleb builds a particle detector and uses it to measure radiation from a remote star. On any given day, the number of particles, Y, that hit the detector is distributed according to a Poisson distribution with parameter x. The parameter x is unknown and is modeled as the value of a random variable X that is exponentially distributed with parameter $\mu > 0$:

$$f_X(x) = \begin{cases} \mu e^{-\mu x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The conditional PMF of the number of particles hitting the detector is

$$p_{Y|X}(y \mid x) = \begin{cases} \frac{e^{-x}x^y}{y!}, & \text{if } y = 0, 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the MAP estimate of X based on the observed value y of Y. Express your answer in terms of y and μ. Use 'mu' to denote μ.

$\hat{x}_{\text{MAP}}(y) =$

(b) Our goal is to find the LMS estimate for X based on the observed particle count y.

1. We can show that the conditional PDF of X given Y is of the form

$$f_{X|Y}(x \mid y) = \frac{\lambda^{y+1}}{y!} x^y e^{-\lambda x}, \quad x > 0, y \geq 0.$$

   Express λ as a function of μ. You may find the following equality useful:

$$\int_0^\infty a^{y+1} x^y e^{-ax}\, dx = y!, \quad \text{for any } a > 0.$$

   $\lambda =$

2. Find the LMS estimate of X based on the observed particle count y. Express your answer in terms of y and μ. *Hint:* You may want to express $x f_{X|Y}(x \mid y)$ in terms of $f_{X|Y}(x \mid y+1)$.

   $\hat{x}_{LMS}(y) =$

Problem 4: LLMS estimation

Let $X = U + W$ with $\mathbf{E}[U] = m$, $\text{var}(U) = v$, $\mathbf{E}[W] = 0$, and $\text{var}(W) = h$. Assume that U and W are independent.

1. The LLMS estimator of U based on X is of the form $\hat{U} = a + bX$. Find a and b. Express your answers in terms of m, v, and h using standard notation.

   $a =$

   $b =$

2. Suppose we further assume that U and W are normal random variables and then construct $\hat{U}_{LMS}$, the LMS estimator of U based on X, under this additional assumption. Would $\hat{U}_{LMS}$ be the identical to $\hat{U}$, the LLMS estimator developed without the additional normality assumption in part (1)?

Problem 5: LLMS estimation with random sums

Let N be a geometric r.v. with mean $1/p$; let $A_1, A_2, \ldots$ be a sequence of i.i.d. random variables, all independent of N, with mean 1 and variance 1; let $B_1, B_2, \ldots$ be another sequence of i.i.d. random variable, all independent of N and of $A_1, A_2, \ldots$, also with mean 1 and variance 1. Let $A = \sum_{i=1}^{N} A_i$ and $B = \sum_{i=1}^{N} B_i$.

1. Find the following expectations using the law of iterated expectations. Express each answer in terms of p using standard notation.

   $\mathbf{E}[AB] =$

   $\mathbf{E}[NA] =$

2. Let $\hat{N} = c_1 A + c_2$ be the LLMS estimator of N given A. Find c_1 and c_2 in terms of p.

   $c_1 =$

   $c_2 =$

Problem 6: Estimating the parameter of a uniform r.v.

The random variable X is uniformly distributed over the interval $[\theta,\, 2\theta]$. The parameter θ is unknown and is modeled as the value of a continuous random variable Θ, uniformly distributed between zero and one.

1. Given an observation x of X, find the posterior distribution of Θ. Express your answers below in terms of θ and x. Use 'theta' to denote θ and 'ln' to denote the natural logarithm function. For example, $\ln(\theta)$ should be entered as 'ln(theta)'.

   For $0 \le x \le 1$ and $x/2 \le \theta \le x$, $f_{\Theta|X}(\theta \mid x) =$

2. Find the MAP estimate of Θ based on the observation $X = x$ and assuming that $0 \le x \le 1$. Express your answer in terms of x. For $0 \le x \le 1$, $\hat{\theta}_{\mathrm{MAP}}(x) =$

3. Find the LMS estimate of Θ based on the observation $X = x$ and assuming that $0 \le x \le 1$. Express your answer in terms of x.

   For $0 \le x \le 1$, $\hat{\theta}_{\mathrm{LMS}}(x) =$

4. Find the linear LMS estimate $\hat{\theta}_{\mathrm{LLMS}}$ of Θ based on the observation $X = x$. Specifically, $\hat{\theta}_{\mathrm{LLMS}}$ is of the form $c_1 + c_2 x$. Find c_1 and c_2.
   $c_1 =$
   $c_2 =$