Problem Set 5

Problem 1: Normal random variables

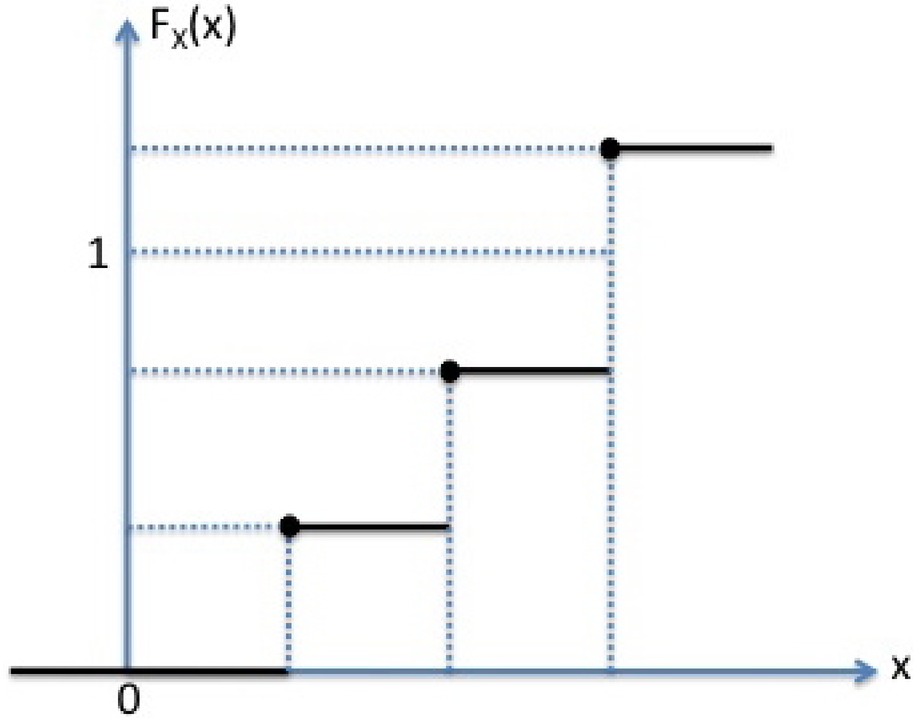
Let X and Y be normal random variables with means 0 and 2, respectively, and variances 1 and 9, respectively. Find the following, using the standard normal table. Express your answers to an accuracy of 4 decimal places.

1. $\mathbf{P}(X > 0.75) =$
2. $\mathbf{P}(X \leq -1.25) =$
3. Let $Z = (Y-3)/4$. Find the mean and the variance of Z.
   $\mathbf{E}[Z] =$
   $\mathrm{var}(Z) =$
4. $\mathbf{P}(-1 \leq Y \leq 2) =$

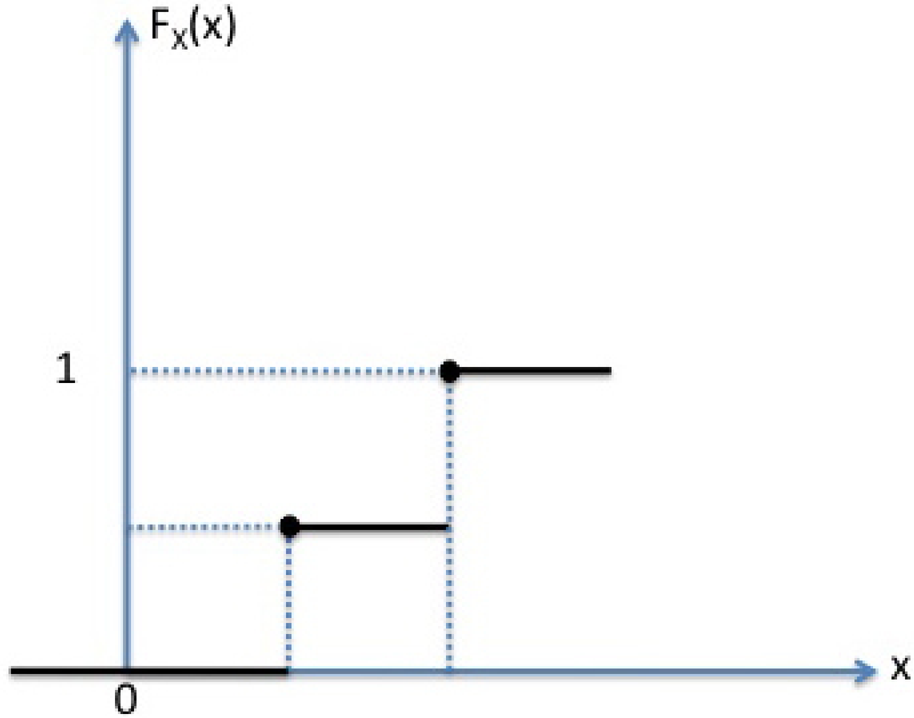
Problem 2: CDF

For each one of the following figures, identify if it is a valid CDF. The value of the CDF at points of discontinuity is indicated with a small solid circle.

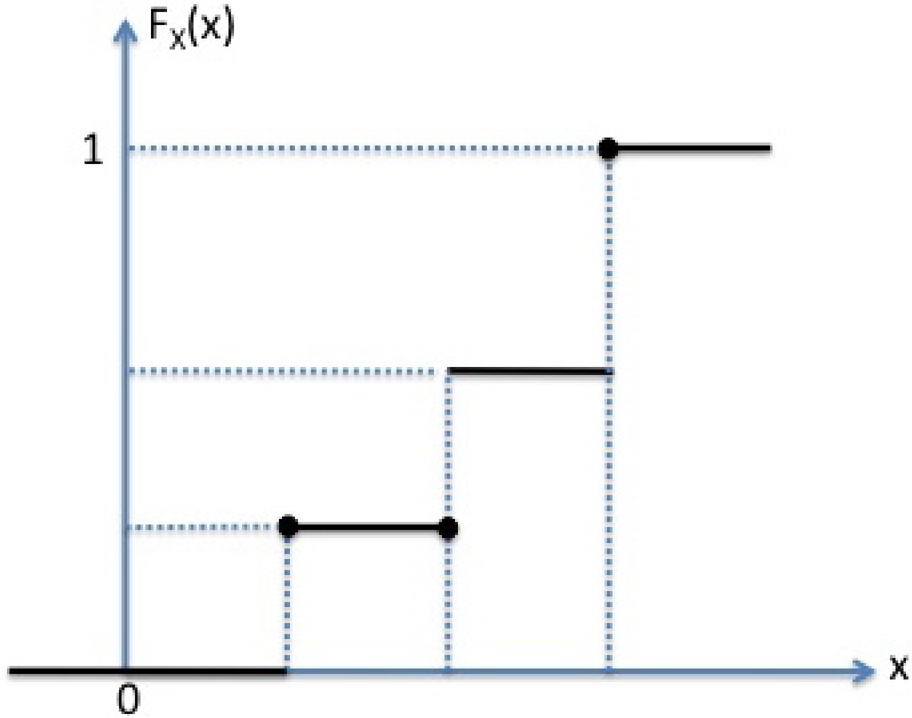
1.



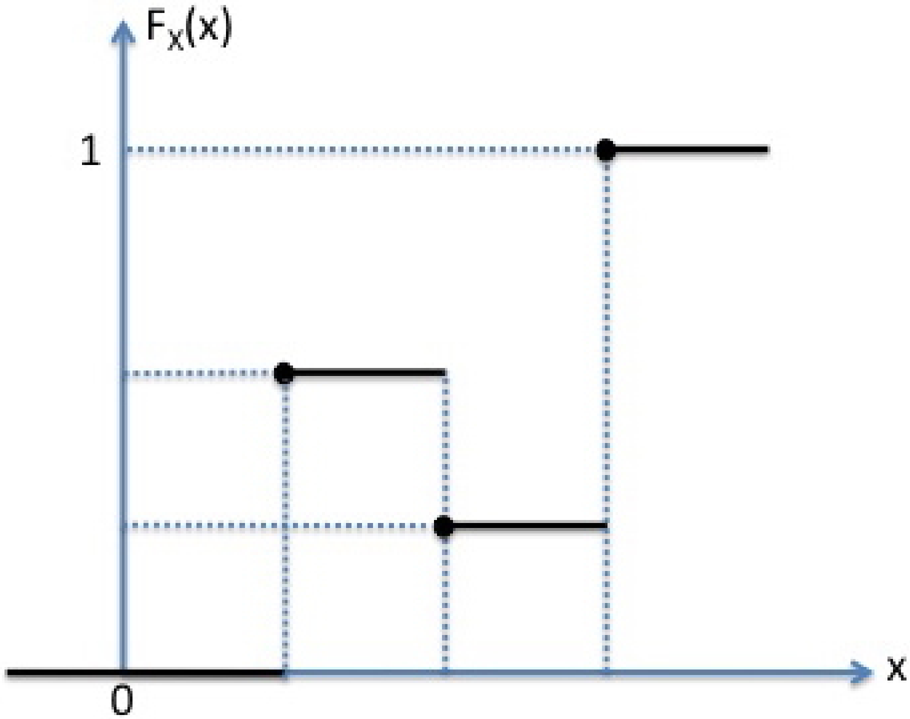
2.



3.



4.



Problem 3: A Joint PDF given by a simple formula

Random variables X and Y are distributed according to the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} ax, & \text{if } 1 \le x \le 2 \text{ and } 0 \le y \le x, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the constant a.
   $a =$

2. Determine the marginal PDF $f_Y(y)$. (Your answer can be either numerical or algebraic functions of y).

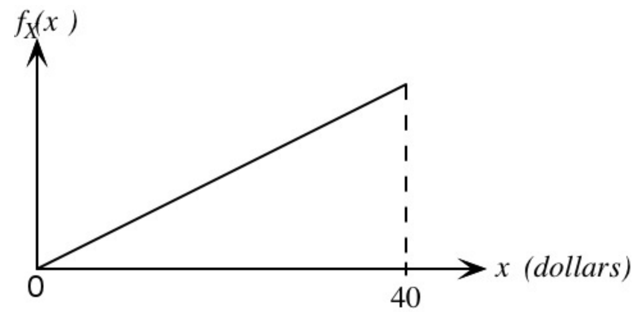
   For $0 \le y \le 1$,
   $f_Y(y) =$

   For $1 < y \le 2$,
   $f_Y(y) =$

3. Determine the conditional expectation of $1/X$ given that $Y = 3/2$.
   $\mathbf{E}[1/X \mid Y = 3/2] =$

Problem 4: Paul goes to the casino

Paul is vacationing in Monte Carlo. On any given night, he takes X dollars to the casino and returns with Y dollars. The random variable X has the PDF shown in the figure. Conditional on $X = x$, the continuous random variable Y is uniformly distributed between zero and $2x$.



1. Determine the joint PDF $f_{X,Y}(x, y)$.

   If $0 < x < 40$ and $0 < y < 2x$,
   $f_{X,Y}(x, y) =$

   If $y < 0$ or $y > 2x$,
   $f_{X,Y}(x, y) =$

2. On any particular night, Paul makes a profit of $Z = Y - X$ dollars. Find the probability that Paul makes a positive profit (i.e., $\mathbf{P}(Z > 0)$):

3. Find the PDF of Z. Express your answers in terms of z using standard notation. *Hint:* Start by finding $f_{Z|X}(z \mid x)$.

   If $0 < z < 40$, $f_Z(z) =$

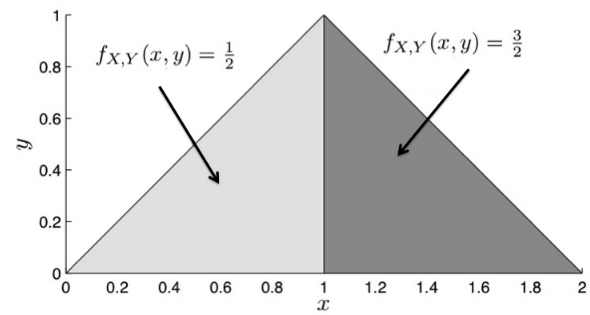
   If $-40 < z < 0$, $f_Z(z) =$

   If $z < -40$ or $z > 40$, $f_Z(z) =$

4. What is $\mathbf{E}[Z]$?
   $\mathbf{E}[Z] =$

Problem 5: A joint PDF on a triangular region

This figure below describes the joint PDF of the random variables X and Y. These random variables take values in $[0, 2]$ and $[0, 1]$, respectively. At $x = 1$, the value of the joint PDF is $1/2$.



1. Are X and Y independent?

2. Find $f_X(x)$. Express your answers in terms of x using standard notation.

   If $0 < x < 1$,
   $f_X(x) =$

   If $1 < x < 2$,
   $f_X(x) =$

3. Find $f_{Y|X}(y \mid 0.5)$.

   If $0 < y < 1/2$,
   $f_{Y|X}(y \mid 0.5) =$

4. Find $f_{X|Y}(x \mid 0.5)$.

If $1/2 < x < 1$,
$f_{X|Y}(x \mid 0.5) =$

If $1 < x < 3/2$,
$f_{X|Y}(x \mid 0.5) =$

5. Let $R = XY$ and let A be the event $\{X < 0.5\}$. Evaluate $\mathbf{E}[R \mid A]$.
   $\mathbf{E}[R \mid A] =$

Problem 6: True or False II

Determine whether each of the following statement is true (i.e., always true) or false (i.e., not always true).

1. Let X be a random variable that takes values between 0 and c only, for some $c \geq 0$, so that $\mathbf{P}(0 \leq X \leq c) = 1$. Then, $\text{var}(X) \leq c^2/4$.

2. X and Y are continuous random variables. If $X \sim N(\mu, \sigma^2)$ (i.e., normal with mean μ and variance σ^2), $Y = aX + b$, and $a > 0$, then $Y \sim N(a\mu + b, a\sigma^2)$.

3. The expected value of a non-negative continuous random variable X, which is defined by $\mathbf{E}[X] = \int_0^\infty x f_X(x) dx$, also satisfies $\mathbf{E}[X] = \int_0^\infty \mathbf{P}(X > t) \mathrm{d}t$.

Problem 7: Bayes' rule

Let K be a discrete random variable with PMF

$$p_K(k) = \begin{cases} 1/3, & \text{if } k = 1, \\ \\ 2/3, & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Conditional on $K = 1$ or 2, random variable Y is exponentially distributed with parameter 1 or 1/2, respectively.

Using Bayes' rule, find the conditional PMF $p_{K|Y}(k \mid y)$. Which of the following is the correct expression for $p_{K|Y}(2 \mid y)$ when $y \geq 0$?