Problem Set 8

Problem 1: Convergence in probability

For each of the following sequences, determine the value to which it converges in probability.

(a) Let X_1, X_2, . . . be independent continuous random variables, each uniformly dis-tributed between -1 and 1.

1. Let $U_i = \dfrac{X_1 + X_2 + \cdots + X_i}{i}, \quad i = 1, 2, \ldots.$
   What value does the sequence U_i converge to in probability?

2. Let $W_i = \max(X_1, X_2, \ldots, X_i), \quad i = 1, 2, \ldots.$
   What value does the sequence W_i converge to in probability?

3. Let $V_i = X_1 \cdot X_2 \cdot \cdots \cdot X_i, \quad i = 1, 2, \ldots.$
   What value does the sequence V_i converge to in probability?

(b) Let X_1, X_2, . . . be independent identically distributed random variables with $\mathbf{E}[X_i] = 2$ and $\text{var}(X_i) = 9$, and let $Y_i = X_i/2^i$.

1. What value does the sequence Y_i converge to in probability?

2. Let $A_n = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} Y_i$. What value does the sequence A_n converge to in probability?

3. Let $Z_i = \dfrac{1}{3}X_i + \dfrac{2}{3}X_{i+1}$ for $i = 1, 2, \ldots$, and let $M_n = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} Z_i$ for $n = 1, 2, \ldots.$
   What value does the sequence M_n converge to in probability?

Problem 2: Find the limits

Let S_n be the number of successes in n independent Bernoulli trials, where the probability of success for each trial is $1/2$. Provide a numerical value, to a precision of 3 decimal places, for each of the following limits. You may want to refer to the standard normal table.

1. $\lim_{n\to\infty} \mathbf{P}\left(\frac{n}{2} - 20 \leq S_n \leq \frac{n}{2} + 20\right) =$

2. $\lim_{n\to\infty} \mathbf{P}\left(\frac{n}{2} - \frac{n}{3} \leq S_n \leq \frac{n}{2} + \frac{n}{3}\right) =$

3. $\lim_{n\to\infty} \mathbf{P}\left(\frac{n}{2} - \frac{\sqrt{n}}{4} \leq S_n \leq \frac{n}{2} + \frac{\sqrt{n}}{4}\right) =$

Problem 3: The sample mean

Let X be a continuous random variable. We know that it takes values between 0 and 3, but we do not know its distribution or its mean and variance. We are interested in estimating the mean of X, which we denote by h. We will use 1.5 as a conservative value (upper bound) for the standard deviation of X. To estimate h, we take n i.i.d. samples $X_1, X_2, \ldots, X_n$, which all have the same distribution as X, and compute the sample mean

$$H = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

1. Express your answers for this part in terms of h and n using standard notation.

   $\mathbf{E}[H] =$

   Given the available information, the smallest upper bound for $\text{var}(H)$ is:

2. Calculate the smallest possible positive value of n such that the standard deviation of H is guaranteed to be at most 0.01.
   This minimum value of n is:

3. We would like to be at least 99% sure that our estimate is within 0.05 of the true mean h. Using the Chebyshev inequality, calculate the minimum value of n that will achieve this.
   This minimum value of n is:

4. Assume that X is uniformly distributed on $[0, 3]$. Using the Central Limit Theorem, identify the most appropriate expression for a 95% confidence interval for h.

Problem 4: Airline overbooking

For any given flight, an airline tries to sell as many tickets as possible. Suppose that on average, 10% of ticket holders fail to show up, all independent of one another. Knowing this, an airline will sell more tickets than there are seats available (i.e., overbook the flight) and hope that there is a sufficient number of ticket holders who do not show up to compensate for its overbooking. Using the Central Limit Theorem, determine n, the maximum number of tickets an airline should sell on a flight with 300 seats so that it can be approximately 99% confident that all ticket holders who do show up will be able to board the plane. Use the de Moivre-Laplace 1/2-correction in your calculations. *Hint:* You may have to solve numerically a quadratic equation.

Problem 5: Maximum likelihood estimation

The random variables $X_1, \ldots, X_n$ are independent Poisson random variables with a com-mon parameter λ. Find the maximum likelihood estimate of λ based on observed values $x_1, \ldots, x_n$.

$\hat{\lambda}_{ML} =$

Problem 6: Tossing a pair of coins

We have a white coin, for which $\mathbf{P}(\text{Heads}) = 0.4$ and a black coin for which $\mathbf{P}(\text{Heads}) = 0.6$. The flips of the same or of different coins are independent. For each of the following situations, determine whether the random variable N can be approximated by a normal. If yes, enter the mean and variance of N. If not, enter 0 in both of the corresponding answer boxes.

1. Let N be the number of Heads in 100 tosses of the white coin.
   mean:

   variance:

2. Let N be the number of Heads in 100 coin tosses. At each toss, one of the two coins is selected at random (either choice is equally likely), and independently from everything else.
   mean:

   variance:

3. Let N be the number of Heads in 50 tosses of the white coin followed by 50 tosses of the black coin (for a total of 100 tosses).
   mean:

   variance:

4. We select one of the two coins at random: each coin is equally likely to be selected. We then toss the selected coin 100 times, independently, and let N be the number of Heads.
   mean:

   variance: