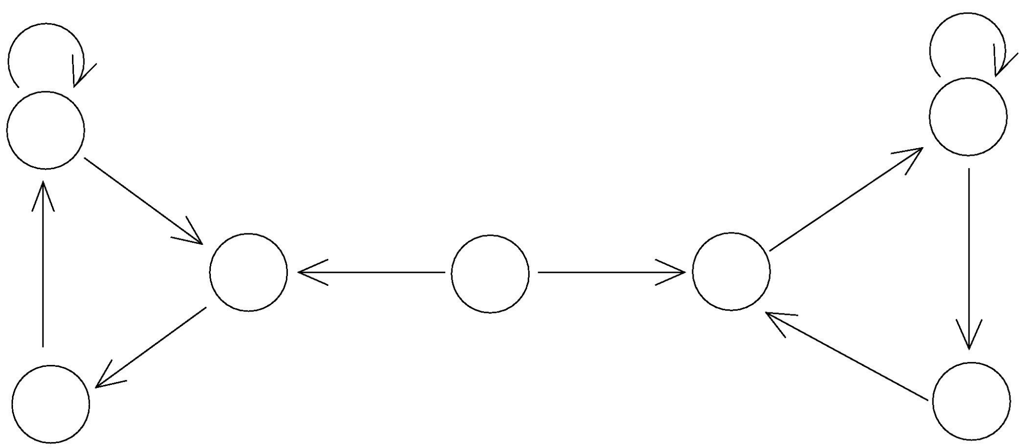
Problem Set 10

Problem 1: Steady-state convergence

Let $X_0, X_1, \ldots$ be a Markov chain, and let $r_{ij}(n) \equiv \mathbf{P}(X_n = j \mid X_0 = i)$.

1. Consider the Markov chain represented below. The circles represent distinct states, while the arrows correspond to positive (one-step) transition probabilities.

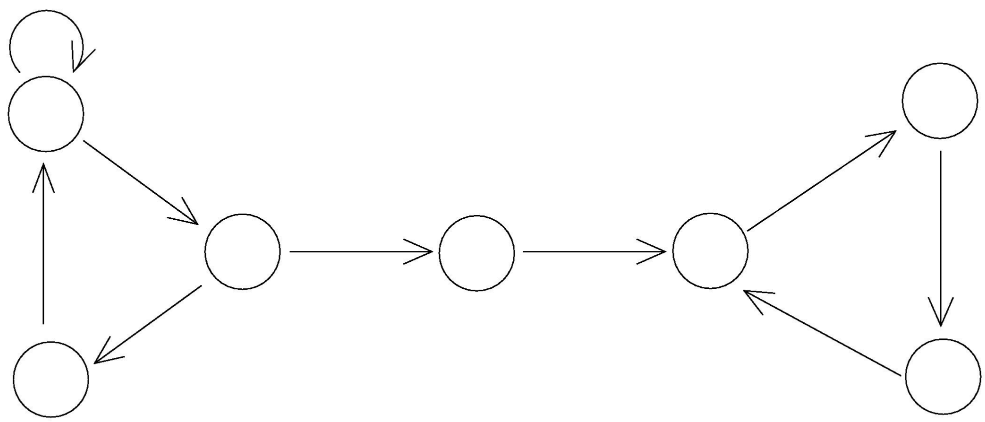


For this Markov chain, determine whether each of the following statements is true or false.

(a) For every i and j, the sequence $r_{ij}(n)$ converges, as $n \to \infty$, to a limiting value π_j, which does not depend on i.

(b) Statement (a) is true, and $\pi_j > 0$ for every state j.

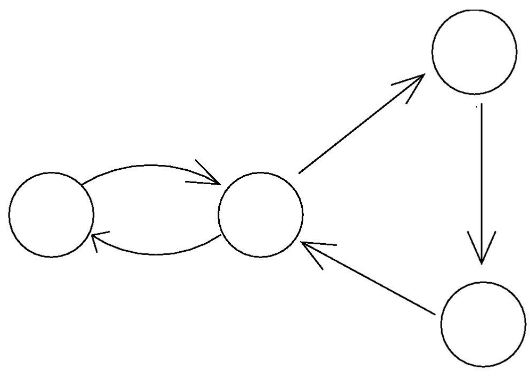
2. Consider the Markov chain represented below. The circles represent distinct states, while the arrows correspond to positive (one-step) transition probabilities.



(a) For every i and j, the sequence $r_{ij}(n)$ converges, as $n \to \infty$, to a limiting value π_j, which does not depend on i.

(b) Statement (a) is true, and $\pi_j > 0$ for every state j.

3. Consider the Markov chain represented below. The circles represent distinct states, while the arrows correspond to positive (one-step) transition probabilities.



(a) For every i and j, the sequence $r_{ij}(n)$ converges, as $n \to \infty$, to a limiting value π_j, which does not depend on i.

(b) Statement (a) is true, and $\pi_j > 0$ for every state j.

Problem 2: Oscar's running shoes

Oscar goes for a run each morning. When he leaves his house for his run, he is equally likely to use either the front or the back door; and similarly, when he returns, he is equally likely to use either the front or the back door. Assume that his choice of the door through which he leaves is independent of his choice of the door through which he returns, and also assume that these choices are independent across days.

Oscar owns only five pairs of running shoes, each pair placed at one of the two doors. If there is at least one pair of shoes at the door through which he leaves, he wears a pair for his run; otherwise, he runs barefoot. When he returns from his run, if he wore shoes for that run, he takes off the shoes after the run and leaves them at the door through which he returns.

We wish to determine the long-term proportion of time that Oscar runs barefoot.

1. We consider a Markov chain with states $\{0, 1, 2, 3, 4, 5\}$, where state i indicates that there are i pairs of shoes available at the front door in the morning, before Oscar leaves for his run. Specify the numerical values of the following transition probabilities.

   - For $i \in \{0, 1, 2, 3, 4\}$, $p_{i,i+1} =$

   - For $i \in \{1, 2, 3, 4, 5\}$, $p_{i,i-1} =$

   - For $i \in \{1, 2, 3, 4\}$, $p_{ii} =$

   - $p_{00} =$

   - $p_{55} =$

2. Determine the steady-state probability that Oscar runs barefoot.

Problem 3: Checking the Markov property

For each one of the following definitions of the state X_k at time k (for $k = 1, 2, \ldots$), determine whether the Markov property is satisfied by the sequence $X_1, X_2, \ldots$.

1. A fair six-sided die (with sides labelled $1, 2, \ldots, 6$) is rolled repeatedly and independently.

   (a) Let X_k denote the largest number obtained in the first k rolls. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

   (b) Let X_k denote the number of 6's obtained in the first k rolls, up to a maximum of ten. (That is, if ten or more 6's are obtained in the first k rolls, then $X_k = 10$.) Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

   (c) Let Y_k denote the result of the k^{th} roll. Let $X_1 = Y_1$, and for $k \geq 2$, let $X_k = Y_k + Y_{k-1}$. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

   (d) Let $Y_k = 1$ if the k^{th} roll results in an odd number; and $Y_k = 0$ otherwise. Let $X_1 = Y_1$, and for $k \geq 2$, let $X_k = Y_k \cdot X_{k-1}$. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

2. Let Y_k be the state of some Markov chain at time k (i.e., it is known that the sequence $Y_1, Y_2, \ldots$ satisfies the Markov property).

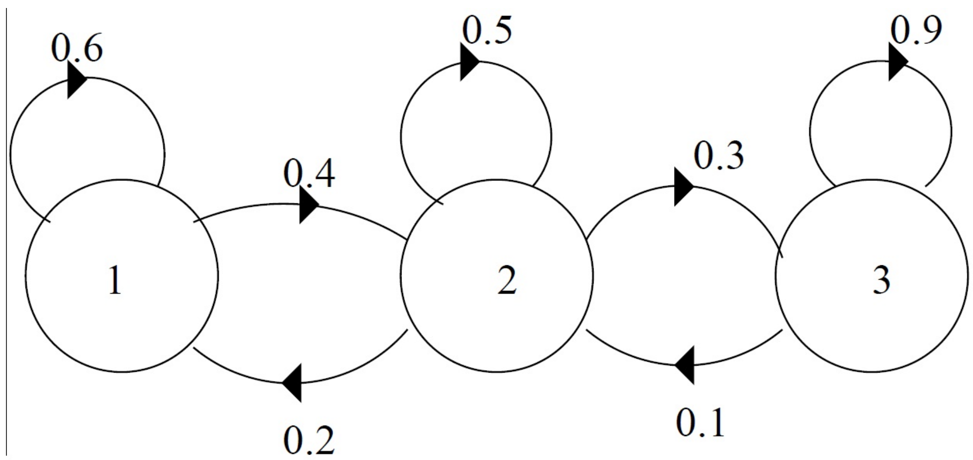
   (a) For a fixed integer $r > 0$, let $X_k = Y_{r+k}$. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

   (b) Let $X_k = Y_{2k}$. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

   (c) Let $X_k = (Y_k, Y_{k+1})$. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

Problem 4: A simple Markov chain

Consider a Markov chain $\{X_0, X_1, \ldots\}$, specified by the following transition probability graph.



1. $\mathbf{P}(X_2 = 2 \mid X_0 = 1) =$
2. Find the steady-state probabilities π_1, π_2, and π_3 associated with states 1, 2, and 3, respectively.
   - $\pi_1 =$
   - $\pi_2 =$
   - $\pi_3 =$

3. For $n = 1, 2, \ldots$, let $Y_n = X_n - X_{n-1}$. Thus, $Y_n = 1$ indicates that the nth transition was to the right, $Y_n = 0$ indicates that it was a self-transition, and $Y_n = -1$ indicates that it was a transition to the left. $\lim_{n\to\infty} \mathbf{P}(Y_n = 1) =$

4. Is the sequence $Y_1, Y_2, \ldots$ a Markov chain?

5. Given that the nth transition was a transition to the right ($Y_n = 1$), find (approximately) the probability that the state at time $n-1$ was state 1 (i.e., $X_{n-1} = 1$). Assume that n is large.

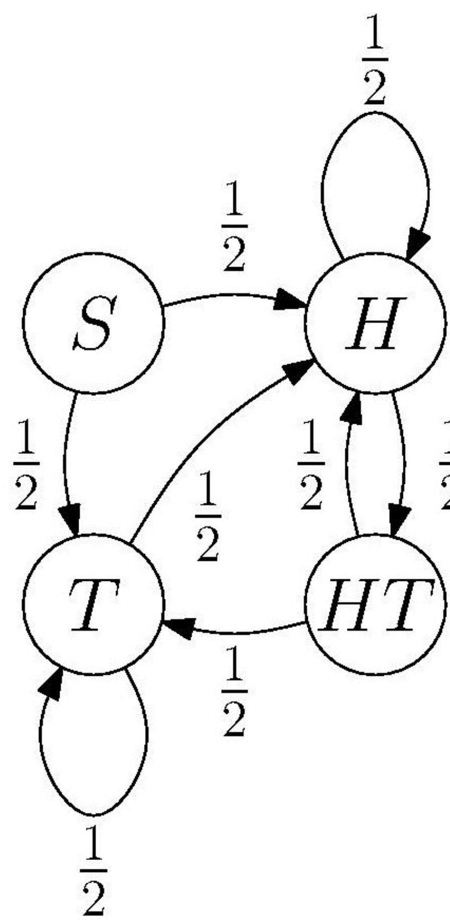
6. Suppose that $X_0 = 1$. Let T be the first *positive* time index n at which the state is equal to 1. $\mathbf{E}[T] =$

7. Does the sequence $X_1, X_2, X_3, \ldots$ converge in probability to a constant?

8. Let $Z_n = \max\{X_1, \ldots, X_n\}$. Does the sequence $Z_1, Z_2, Z_3, \ldots$ converge in probability to a constant?

Problem 5: Coin tosses revisited

A fair coin is tossed repeatedly and independently. We want to determine the expected number of tosses until we first observe Tails immediately preceded by Heads. To do so, we define a Markov chain with four states, $\{S, H, T, HT\}$, where S is a starting state, H indicates Heads on the current toss, T indicates Tails on the current toss (without Heads on the previous toss), and HT indicates Heads followed by Tails over the last two tosses. This Markov chain is illustrated below:



1. What is the expected number of tosses until we first observe Tails immediately preceded by Heads? *Hint:* Solve the corresponding mean first passage time problem for our Markov chain.

2. Assuming that we have just observed Tails immediately preceded by Heads, what is the expected number of additional tosses until we next observe Tails immediately preceded by Heads?

Next, we want to answer similar questions for the event that Tails is immediately preceded by Tails. Set up a new Markov chain from which you can calculate the expected number of tosses until we first observe Tails immediately preceded by Tails.

3. What is the expected number of tosses until we first observe Tails immediately preceded by Tails?

4. Assuming that we have just observed Tails immediately preceded by Tails, what is the expected number of additional tosses until we again observe Tails immediately preceded by Tails?