# On the Effect of Dual Weights in Computer Aided Design of Rational Motions

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*In recent years, it has become well known that rational Bézier and B-spline curves in the space of dual quaternions correspond to rational Bézier and B-spline motions. However, the influence of weights of these dual quaternion curves on the resulting rational motions has been largely unexplored. In this paper, we present a thorough mathematical exposition on the influence of dual-number weights associated with dual quaternions for rational motion design. By deriving the explicit equations for the point trajectories of the resulting motion, we show that the effect of real weights on the resulting motion is similar to that of a rational Bézier curve and how the change in dual part of a dual-number weight affects the translational component of the motion. We also show that a rational Bézier motion can be reparameterized in a manner similar to a rational Bézier curve. Several examples are presented to illustrate the effects of the weights on rational motions.* [DOI: 10.1115/1.1906263]

## 1 Introduction

The problem of motion approximation and interpolation has generated a lot of interest in robotics, CNC tool path planning, computer graphics, computer aided geometric design as well in as task specification in mechanism synthesis. In recent years, it has been well established that rational Bézier and B-spline based curve representation schemes can be combined with dual-quaternion representation of spatial displacements to obtain rational Bézier and B-spline motions. However, the influence of weights of the dual quaternion curves on the resulting rational motions has been largely unexplored. In this paper, we present a thorough mathematical exposition on the influence of dual-number weights associated with dual quaternions for computer aided geometric design of rational motions. It is shown that these dual-number weights, like their counterparts in curve and surface design, serve as additional shape parameters for adjusting the path of a rational Bézier or B-spline motion.

There has been a great deal of research in applying the principles of Computer Aided Geometric Design (CAGD) (Farin [1,2], Hoschek and Lasser [3], Piegl and Tiller [4]) to the problem of computer aided motion design. Ge and Ravani [5–7] developed a new framework for geometric constructions of spatial motions by combining the concepts from kinematics and CAGD. Their work was built upon the seminal paper of Shoemake [8], in which he used the concept of a quaternion (Bottema and Roth [9]) for rotation interpolation. The approach advocated by Ge and Ravani generates Bézier-type spatial motions by repeated screw motions. If constant-speed and constant-pitch (In this paper, any motion about a fixed screw axis is called a screw motion, whether its pitch is constant or not.) screw motions are used, then the resulting Bézier spatial motion is a straightforward extension of Shoemake's Bézier spherical motion. If rational screw motions are used, then the resulting Bézier motion is a rational motion. While the Bézier motion resulting from constant-pitch screw motion has the advantage that its speed is more uniform, the rational Bézier motion, is compatible with NURBS based CAD systems.

Jüttler [10], Jüttler and Wagner [11], and Wagner [12,13] used the quaternion-based formulation to develop new spatial rational motions by essentially treating rotation and translation separately. Their research gave rise to a rich set of theoretical tools and provided insight into the rational motion design problem. In a parallel and related development, Park and Ravani [14] and Žefran and Kumar [15,16] pursued the motion design problem in the framework of Lie groups and Riemannian geometry and Etzel and McCarthy [17] and Mullineux [18] investigated the spatial interpolation problem using the Clifford Algebra of $E^4$. Recently, Eberharter and Ravani [19] defined a new distance metric for rigid body displacements using stereographic projection that may provide new methods for motion synthesis. Rossignac et al. [20] outline some important prerequisites of motion design problem and Röschel [21] provides a comprehensive survey on the rational motion design. More recently, Jüttler and Wagner [22] summarized their research results on Kinematics and Animation in a handbook of CAGD, which contains an exhaustive list of references on the subject.

Despite these advances in the field of rational motion design, the effect of real and dual weights on the rational motion design has not been studied in a generalized and complete fashion. Ge and Ravani [7] discussed the effect of dual weights on rational screw motions. Wagner [12] touched upon the issue of real weights for his formulation of rational motions from a purely CAGD perspective. In this paper, we present a comprehensive treatment on the effect of weights (including both real and dual weights) on rational motions.

The organization of the paper is as follows: Sec. 2 discusses several representations of spatial displacements with an emphasis on dual quaternions based representation. It is the dual quaternion representation that we use to formulate and develop our exposition. Section 3 recasts the dual quaternion representation of Bézier and B-spline motions in matrix form and in the process establishes the mathematical framework required for investigating the effects of the weights. Section 4 discusses the effect of real part and the dual part of the dual weights separately on rational motion and presents various graphical illustrations before drawing conclusions in the final section.

## 2 Representations of Spatial Displacements

A spatial displacement of a rigid body is commonly represented by the following transformation of a moving frame $M$ attached to the moving body with respect to a fixed frame $F$ attached to the fixed space:

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$$\begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \begin{bmatrix} [R] | \mathbf{d} \\ 000 | 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}. \tag{1}$$

where, $[R]$ is an orthogonal matrix representing a rotation and $\mathbf{d}$ is a vector representing a translation: $\mathbf{X}$ and $\mathbf{x}$ are vectors whose scalar components are the Cartesian coordinates of the point as measured in $F$ and $M$, respectively. The use of such matrix representation, however, is not convenient when dealing with the problem of synthesizing a rational motion that interpolates or approximates a set of displacements. One of the main obstacles is to the issue of preserving the orthogonality of the rotation matrix in the interpolation/approximation process (Fillmore [23], Röschel [21]). It has been recognized that an effective way of dealing with the problem is to use quaternions (Shoemake [8]) and dual quaternions (Ge and Ravani [6]). In what follows, we review the concepts of quaternions and dual quaternions in so far as necessary for the development of the current paper.

**2.1 Unit Quaternion as Rotation and Scalars as Weights.** Let a unit vector $\mathbf{s}=(s_1,s_2,s_3)$ and an angle $\theta$ represent the axis and angle of a rotation, respectively. The components of a unit quaternion $\mathbf{q}=(q_1,q_2,q_3,q_4)$ representing rotation are given by the so-called Euler–Rodrigues parameters:

$$q_1 = s_1 \sin\frac{\theta}{2},\ q_2 = s_2 \sin\frac{\theta}{2},\ q_3 = s_3 \sin\frac{\theta}{2}, \quad q_4 = \cos\frac{\theta}{2}. \tag{2}$$

The four parameters satisfy the relation

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1. \tag{3}$$

Therefore, $\mathbf{q}$ can be thought of as a point lying on a unit 3-sphere ($S^3$) embedded in a four-dimensional space. The rotation matrix $[R]$ can be recovered from the Euler–Rodrigues parameters using (see Bottema and Roth [9]):

$$[R] = \frac{1}{S^2}\begin{bmatrix} q_4^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_4q_3) & 2(q_1q_3+q_4q_2) \\ 2(q_2q_1+q_4q_3) & q_4^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_4q_1) \\ 2(q_3q_1-q_4q_2) & 2(q_3q_2+q_4q_1) & q_4^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}. \tag{4}$$

where $S^2=q_1^2+q_2^2+q_3^2+q_4^2$.

From the above, it is clear that the rotation matrix $[R]$ remains the same after multiplying each one of the Euler-Rodrigues parameters by a scalar $w$ ($w \neq 0$). This means that $\mathbf{q}$ can be treated as homogeneous coordinates of a rotation. In this paper, we let $\mathbf{Q}=w\mathbf{q}$ denote the nonunit quaternion of homogeneous coordinates. The homogeneous quaternion $\mathbf{Q}$ represents one and the same rotation regardless of the choice of nonzero weight $w$.

**2.2 Dual Quaternion as Spatial Displacement and Dual numbers as Dual Weights.** Similar to the homogeneous quaternion $\mathbf{Q}$, the translation vector $\mathbf{d}$ can also be homogenized and written in a quaternion form to obtain $\mathbf{D}=(w_0\mathbf{d},w_0)$. Jüttler [10], Jüttler and Wagner [11], and Wagner [12,13] used this set of eight homogeneous parameters $(\mathbf{Q},\mathbf{D})$ for computer aided design of rational motions. While this formulation allows direct application of the existing CAGD techniques to motion design, the resulting motions are not completely reference-frame invariant and depend on the choice of the origins of the reference frames. In this paper, we follow McCarthy [24] and Ge and Ravani [7] and use, a slightly modified version of Study's Soma parameters (Bottema and Roth [9], Study [25]) to represent the spatial displacement. Study's parameters are given by another set of eight homogeneous parameters $(\mathbf{Q},\mathbf{Q}^0)$, where $\mathbf{Q}=(Q_1,Q_2,Q_3,Q_4)$ represents the quaternion of homogeneous Euler parameters of rotation and $\mathbf{Q}^0=(Q_1^0,Q_2^0,Q_3^0,Q_4^0)$ is another quaternion whose components are given by

$$\begin{bmatrix} Q_1^0 \\ Q_2^0 \\ Q_3^0 \\ Q_4^0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -d_3 & d_2 & d_1 \\ d_3 & 0 & -d_1 & d_2 \\ -d_2 & d_1 & 0 & d_3 \\ -d_1 & -d_2 & -d_3 & 0 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}. \tag{5}$$

The translation vector $\mathbf{d}=(d_1,d_2,d_3)$ can be recovered from Eq. (5) in terms of $(\mathbf{Q},\mathbf{Q}^0)$ by using the following:

$$\mathbf{d} = -\frac{2}{S^2}\begin{bmatrix} Q_4^0Q_1 - Q_1^0Q_4 + Q_2^0Q_3 - Q_3^0Q_2 \\ Q_4^0Q_2 - Q_2^0Q_4 + Q_3^0Q_1 - Q_1^0Q_3 \\ Q_4^0Q_3 - Q_3^0Q_4 + Q_1^0Q_2 - Q_2^0Q_1 \end{bmatrix}. \tag{6}$$

where, $S^2=Q_1^2+Q_2^2+Q_3^2+Q_4^2$.

It is instructive to note here that $(\mathbf{Q},\mathbf{Q}^0)$ serve as homogeneous coordinates of spatial displacements since multiplying them by a nonzero scalar yields the same rotation matrix and translation vector $\mathbf{d}$.

Study's parameters can also be written in dual quaternion form as $\hat{\mathbf{Q}}=\mathbf{Q}+\varepsilon\mathbf{Q}^0$, where $\varepsilon$ denotes the dual unit (see Bottema and Roth [9] for details on dual number). In quaternion form, Eqs. (5) and (6) can be written more concisely as follows, respectively:

$$\mathbf{Q}^0 = (1/2)\mathbf{d}\mathbf{Q}, \tag{7}$$

$$\mathbf{d} = \frac{(\mathbf{Q}^0)\mathbf{Q}^* - \mathbf{Q}(\mathbf{Q}^0)^*}{\mathbf{Q}\mathbf{Q}^*}, \tag{8}$$

where, $\mathbf{d}$ is a vector quaternion, which has no scalar part, and $\mathbf{Q}^*=(-Q_1,-Q_2,-Q_3,Q_4)$ is the conjugate of $\mathbf{Q}$ such that $\mathbf{Q}\mathbf{Q}^*=Q_1^2+Q_2^2+Q_3^2+Q_4^2$. Note that Eq. (6) or Eq. (8) can be used to recover $\mathbf{d}$ from $\mathbf{Q}$ and $\mathbf{Q}^0$ even when they do not satisfy the well-known Plücker condition:

$$Q_1Q_1^0 + Q_2Q_2^0 + Q_3Q_3^0 + Q_4Q_4^0 = 0. \tag{9}$$

However, when the dual quaternion components satisfy the above Plücker condition, Eq. (8) reduces to the following well-known equation

$$\mathbf{d} = \frac{2(\mathbf{Q}^0)\mathbf{Q}^*}{\mathbf{Q}\mathbf{Q}^*}$$

which follows directly from Eq. (7).

An alternative way of defining a dual quaternion is based on the concept of screw displacements. In this case, a dual vector is used to define the screw axis and a dual angle defines the angle of rotation about the axis and the distance of translation along the axis. Let the unit dual vector be denoted as $\hat{\mathbf{s}}=(\hat{s}_1,\hat{s}_2,\hat{s}_3)$, and let the dual angle be denoted by $\hat{\theta}=\theta+\varepsilon h$. Then the four dual-components of a dual quaternion can be given by the so-called dual Euler parameters (Bottema and Roth [9], McCarthy [24]):

$$\hat{q}_1 = \hat{s}_1 \sin\frac{\hat{\theta}}{2}, \quad \hat{q}_2 = \hat{s}_2 \sin\frac{\hat{\theta}}{2}, \quad \hat{q}_3 = \hat{s}_3 \sin\frac{\hat{\theta}}{2}, \quad \hat{q}_4 = \cos\frac{\hat{\theta}}{2}. \tag{10}$$

where,

$$\hat{q}_1^2 + \hat{q}_2^2 + \hat{q}_3^2 + \hat{q}_4^2 = 1. \tag{11}$$

The resulting dual quaternion, $\hat{\mathbf{q}}=(\hat{q}_1,\hat{q}_2,\hat{q}_3,\hat{q}_4)$, is therefore a unit dual quaternion. Ravani and Roth [26] showed that an important advantage of dual-quaternion formulation of spatial motions is that it is invariant with respect to change of both the moving and the fixed reference frames.

The dual orthogonal matrix $[\hat{R}]$ (McCarthy [24]) can be parameterized with dual Euler parameters by

$$[\hat{R}] = \frac{1}{\hat{S}^2}\begin{bmatrix} \hat{q}_4^2+\hat{q}_1^2-\hat{q}_2^2-\hat{q}_3^2 & 2(\hat{q}_1\hat{q}_2-\hat{q}_4\hat{q}_3) & 2(\hat{q}_1\hat{q}_3+\hat{q}_4\hat{q}_2) \\ 2(\hat{q}_2\hat{q}_1+\hat{q}_4\hat{q}_3) & \hat{q}_4^2-\hat{q}_1^2+\hat{q}_2^2-\hat{q}_3^2 & 2(\hat{q}_2\hat{q}_3-\hat{q}_4\hat{q}_1) \\ 2(\hat{q}_3\hat{q}_1-\hat{q}_4\hat{q}_2) & 2(\hat{q}_3\hat{q}_2+\hat{q}_4\hat{q}_1) & \hat{q}_4^2-\hat{q}_1^2-\hat{q}_2^2+\hat{q}_3^2 \end{bmatrix}. \tag{12}$$

where, $\hat{S}^2=\hat{q}_1^2+\hat{q}_2^2+\hat{q}_3^2+\hat{q}_4^2$. Equation (12) can also be obtained directly by applying the principle of transference (Hsia and Yang [27], Ravani and Roth [26]) on Eq. (4).

A general dual quaternion $\hat{\mathbf{Q}}$ can be obtained by multiplying a unit dual quaternion with a nonpure dual number, $\hat{w}=w+\varepsilon w^0$, such that $w$ is a nonzero scalar. That is, we have $\hat{\mathbf{Q}}=\hat{w}\hat{\mathbf{q}}$, where $\hat{\mathbf{q}}=(\hat{q}_1,\hat{q}_2,\hat{q}_3,\hat{q}_4)$ is a unit dual quaternion. It is clear from Eq. (12) that the two dual quaternions $\hat{\mathbf{Q}}$ and $\hat{\mathbf{q}}$ represent one and the same orthogonal matrix, and therefore, they represent the same spatial displacement. Ravani and Roth [26] considered $\hat{\mathbf{Q}}=(\hat{Q}_1,\hat{Q}_2,\hat{Q}_3,\hat{Q}_4)$ as a set of four homogeneous dual coordinates that define a point in a projective dual three-space, called the image space of spatial displacements. Thus, we may refer to $\hat{\mathbf{Q}}$ as a homogeneous dual quaternion.

## 3 Rational Bézier and B-Spline Motions

Let $\hat{\mathbf{q}}=\mathbf{q}+\varepsilon\mathbf{q}^0$ denote a unit dual quaternion. A homogeneous dual quaternion may be written as a pair of quaternions, $\hat{\mathbf{Q}}=\mathbf{Q}+\varepsilon\mathbf{Q}^0$, where $\mathbf{Q}=w\mathbf{q}$, $\mathbf{Q}^0=w\mathbf{q}^0+w^0\mathbf{q}$. This is obtained by expanding $\hat{\mathbf{Q}}=\hat{w}\hat{\mathbf{q}}$ using dual-number algebra. To examine the effects of changing weights on the motion of an object, we now recast Eq. (1) in terms of dual quaternions and the homogeneous coordinates of a point $\mathbf{P}$: $(P_1,P_2,P_3,P_4)$ of the object (see Appendix and Ge and Sirchia [28]):

$$\widetilde{\mathbf{P}} = \mathbf{QPQ}^* + P_4[(\mathbf{Q}^0)\mathbf{Q}^* - \mathbf{Q}(\mathbf{Q}^0)^*] \tag{13}$$

where, $\mathbf{Q}^*$ and $(\mathbf{Q}^0)^*$ are conjugates of $\mathbf{Q}$ and $\mathbf{Q}^0$, respectively, and $\widetilde{\mathbf{P}}$ denotes homogeneous coordinates of the point after the displacement.

Given a set of unit dual quaternions and dual weights $\hat{\mathbf{q}}_i,\hat{w}_i;(i=0,\ldots,n)$ respectively, the following represents a rational Bézier curve in the space of dual quaternions.

$$\hat{\mathbf{Q}}(t) = \sum_{i=0}^{n} B_i^n(t)\hat{\mathbf{Q}}_i = \sum_{i=0}^{n} B_i^n(t)\hat{w}_i\hat{\mathbf{q}}_i \tag{14}$$

where, $B_i^n(t)$ are the Bernstein polynomials. The Bézier dual quaternion curve given by Eq. (14) defines a rational Bézier motion of degree $2n$, as is obvious from Eq. (13). It is noted here that in light of Eq. (13), the real and dual parts of $\hat{\mathbf{Q}}(t)$ resulting from Eq. (14) are not required to satisfy the Plücker condition as given by Eq. (9).

Similarly, a B-Spline dual quaternion curve, which defines a NURBS motion of degree $2p$, is given by,

$$\hat{\mathbf{Q}}(t) = \sum_{i=0}^{n} N_{i,p}(t)\hat{\mathbf{Q}}_i = \sum_{i=0}^{n} N_{i,p}(t)\hat{w}_i\hat{\mathbf{q}}_i \tag{15}$$

where, $N_{i,p}(t)$ are the $p$th-degree B-spline basis functions.

A representation for the rational Bézier motion and rational B-spline motion in the Cartesian space can be obtained by substituting Eq. (14) and Eq. (15) in Eq. (13), respectively. In what follows, we deal with the case of rational Bézier motion. The trajectory of a point undergoing rational Bézier motion obtained by substituting Eq. (14) in Eq. (13) is given by,

$$\widetilde{\mathbf{P}}^{2n}(t) = [H^{2n}(t)]\mathbf{P}, \tag{16}$$

$$[H^{2n}(t)] = \sum_{k=0}^{2n} B_k^{2n}(t)[H_k], \tag{17}$$

where, $[H^{2n}(t)]$ is the matrix representation of the rational Bézier motion of degree $2n$ in Cartesian space. The following matrices $[H_k]$ (also referred to as Bézier Control Matrices) define the *affine control structure* (Jüttler and Wagner [11]) of the motion:

$$[H_k] = \frac{1}{C_k^{2n}}\sum_{i+j=k} C_i^n C_j^n w_i w_j [H_{ij}^*], \tag{18}$$

where,

$$[H_{ij}^*] = [H_i^+][H_j^-] + [H_j^-][H_i^{0+}] - [H_i^+][H_j^{0-}] + (\alpha_i - \alpha_j)[H_j^-][Q_i^+]. \tag{19}$$

In the above equations, $C_i^n$ and $C_j^n$ are binomial coefficients and $\alpha_i=w_i^0/w_i$, $\alpha_j=w_j^0/w_j$ are the weight ratios and

$$[H_j^-] = \begin{bmatrix} q_{j,4} & -q_{j,3} & q_{j,2} & -q_{j,1} \\ q_{j,3} & q_{j,4} & -q_{j,1} & -q_{j,2} \\ -q_{j,2} & q_{j,1} & q_{j,4} & -q_{j,3} \\ q_{j,1} & q_{j,2} & q_{j,3} & q_{j,4} \end{bmatrix}, \tag{20}$$

$$[Q_i^+] = \begin{bmatrix} 0 & 0 & 0 & q_{i,1} \\ 0 & 0 & 0 & q_{i,2} \\ 0 & 0 & 0 & q_{i,3} \\ 0 & 0 & 0 & q_{i,4} \end{bmatrix}, \tag{21}$$

$$[H_i^{0+}] = \begin{bmatrix} 0 & 0 & 0 & q_{i,1}^0 \\ 0 & 0 & 0 & q_{i,2}^0 \\ 0 & 0 & 0 & q_{i,3}^0 \\ 0 & 0 & 0 & q_{i,4}^0 \end{bmatrix}, \tag{22}$$

$$[H_j^{0-}] = \begin{bmatrix} 0 & 0 & 0 & -q_{j,1}^0 \\ 0 & 0 & 0 & -q_{j,2}^0 \\ 0 & 0 & 0 & -q_{j,3}^0 \\ 0 & 0 & 0 & -q_{j,4}^0 \end{bmatrix}, \tag{23}$$

$$[H_j^{0-}] = \begin{bmatrix} 0 & 0 & 0 & -q_{j,1}^0 \\ 0 & 0 & 0 & -q_{j,2}^0 \\ 0 & 0 & 0 & -q_{j,3}^0 \\ 0 & 0 & 0 & q_{j,4}^0 \end{bmatrix}, \tag{24}$$

$$[H_i^+] = \begin{bmatrix} q_{i,4} & -q_{i,3} & q_{i,2} & q_{i,1} \\ q_{i,3} & q_{i,4} & -q_{i,1} & q_{i,2} \\ -q_{i,2} & q_{i,1} & q_{i,4} & q_{i,3} \\ -q_{i,1} & -q_{i,2} & -q_{i,3} & q_{i,4} \end{bmatrix}. \tag{25}$$

In above matrices, $(q_{i,1},q_{i,2},q_{i,3},q_{i,4})$ are four components of the real part $(\mathbf{q}_i)$ and $(q_{i,1}^0,q_{i,2}^0,q_{i,3}^0,q_{i,4}^0)$ are four components of the dual part $(\mathbf{q}_i^0)$ of the unit dual quaternion $(\hat{\mathbf{q}}_i)$.

## 4 The Effects of the Weights

This section studies the effects of the weights of the dual quaternion curve on the path and parameterization (i.e., speed) of the resulting motion. Specifically, we seek to answer the following questions:

(1) How does the path of the motion change when the real and dual parts of the weights are changed, individually and together?
(2) Is the change in motion intuitive?

(3) What, if any, are such changes in the weights that leave the path of the motion invariant?In other words, can the motion be reparameterized?

**4.1 The Effect of the Real Weight and Reparameterization of a Rational Bézier Motion.** We first consider the case of real weights only. For a rational Bézier motion of degree $2n$, changing real weights, in general, leads to a change in both the parameterization and the path of the motion. However, it is shown in this section that similar to rational Bézier curves in CAGD, there exists a reparameterization of a rational Bézier motion that leaves the path of the motion unchanged. In CAGD, such a reparameterization corresponds to rational linear parameter transformations (Farin [2]; Chap. 13) of the following type:

$$t = \frac{t'}{\rho(1-t')+t'}. \tag{26}$$

It has been shown by Patterson [29] that such a projective transformation of the parameter space leaves the shape of a rational curve invariant if each weight $w_i$ is replaced by $\rho^{n-i}w_i$, where $\rho$ is a nonzero real number. Thus, if the real weights of a dual quaternion curve are transformed in the same way, then the shape of the dual quaternion curve remains invariant. In what follows, we show that the path of the resulting motion is also invariant. Consider the following weight transformation: $\hat{v}_i=\lambda^{n-i}\hat{w}_i$, where $\lambda$ is any nonzero *real* scalar and $\hat{v}_i$ can be either real or dual weights.

Expanding above, we obtain

$$v_i = \lambda^{n-i}w_i$$
$$v_i^0 = \lambda^{n-i}w_i^0 \tag{27}$$

where, $v_i$ and $v_i^0$ are the real and dual part of the new weight $\hat{v}_i$.

Substituting Eq. (27) in Eq. (18), we get new control matrix $[\tilde{H_k}]$

$$[\tilde{H_k}] = \sum_{i+j=k} \frac{{}^nC_i\,{}^nC_j}{{}^{2n}C_k}\lambda^{2n-(i+j)}w_iw_j[H_{ij}^*] = \lambda^{2n-k}[H_k]. \tag{28}$$

The point trajectory is now given by

$$\tilde{\mathbf{P}}^{2n}(t) = \sum_{k=0}^{2n} B_k^{2n}(t)\lambda^{2n-k}[H_k]\mathbf{P}. \tag{29}$$

The above equation shows that transforming the weights via Eq. (27) is equivalent to multiplying the Bézier control positions ($[H_k]$) of the motion by a scalar $\lambda^{2n-k}$. This means that the path of the trajectory of any point $\mathbf{P}$ under the transformed motion is invariant. It follows that the weight change as defined by Eq. (27) does not change the path of the motion. However, the speed of the resulting motion is in general different from the original motion.

*4.1.1 Rational Screw Motion*. For the simplest case of $n=1$, we have the following linear interpolation in dual quaternion space, which corresponds to a rational screw motion:

$$\hat{\mathbf{Q}}(t) = (1-t)\hat{w}_0\hat{\mathbf{q}}_0 + t\hat{w}_1\hat{\mathbf{q}}_1. \tag{30}$$

Ge and Ravani [6] have shown that the above represents a quadratic rational motion that has a fixed screw axis with varying angular speed and pitch. Now consider real weights only, i.e., let $\hat{w}_i=w_i$ for all $i$. In this case, Eq. (27) reduces to $v_0=\lambda w_0$ and $v_1=w_1$, or equivalently, $v_0/v_1=\lambda w_0/w_1$. It follows that the change of real weights of the linear interpolation leaves the path of the screw motion invariant. In this case, Eq. (29) becomes

$$\tilde{\mathbf{P}}^{2n}(t) = (\lambda^2B_0^2(t)[H_0] + \lambda B_1^2(t)[H_1] + B_2^2(t)[H_2])\mathbf{P}. \tag{31}$$

Equation (31) is a reparameterized equation that can also be obtained by a rational linear parameter transformation, which leaves the shape of a rational curve and in this case, path of the motion, invariant. Figure 1(*a*) shows the screw motion with unit real weights and Fig. 1(*b*) shows another screw motion with a different set of real weights. These two screw motions have the same path but different speed (or parameterization). Doubling the second real weight causes slowing down when the object approaches the second position. [In all the figures, given control positions/key frames are marked as $\mathbf{C}_i(i=0,\ldots,n)$].

**Fig. 1 (a) Screw motion with unit real weights $\hat{w}_i=1+\varepsilon 0$; $i=0,1$. (b) Screw motion with nonunit real weights $\hat{w}_0=1+\varepsilon 0$, $\hat{w}_1=2+\varepsilon 0$.**

*4.1.2 A Rational Bézier Motion*. Consider a rational Bézier motion of degree 6 as defined by a cubic dual quaternion Bézier curve:

$$\hat{\mathbf{Q}}(t) = \sum_{i=0}^{3} B_i^3(t)\hat{w}_i\hat{\mathbf{q}}_i. \tag{32}$$

Figure 2(*a*) shows this motion for a given set of four different control configurations (marked by $\mathbf{C}_i$) each of which are associated with unit real weights. Also shown is the affine control structure connected by solid lines and indicated by label $H_i$. Figure 2(*b*) shows degree 6 rational Bézier motion for the same set of control configurations but with different real weights associated with them. It is clear that a change in real weight causes a change in the affine control structure as defined by the control matrices $[H_i]$, and therefore changes the path of the resulting motion.

Figure 3(*a*) and 3(*b*) illustrate a rational Bézier motion of degree 6 before and after a reparameterization, respectively as described in Sec. 4.1. As shown below, when the weights are scaled according to Eq. (27), the path of the motion does not change, although the speed along the path does change.

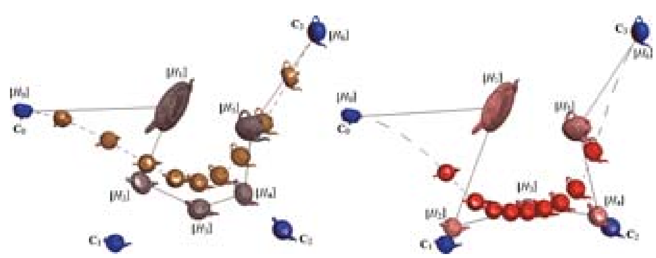

**Fig. 2 (a) A rational Bézier motion of degree 6 with unit real weights, $\hat{w}_i=1+\varepsilon 0$; $i=0,\ldots,3$. (b) A rational Bézier motion with nonunit real weights $\hat{w}_i=1+\varepsilon 0$; $i=0,3$ and $\hat{w}_i=4+\varepsilon 0$; $i=1,2$.**

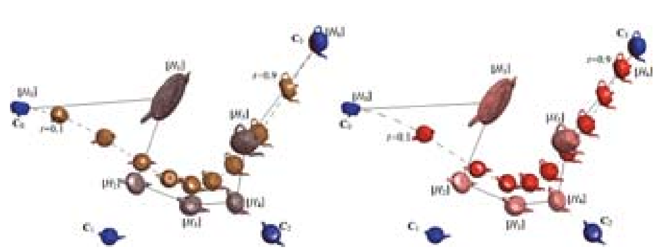

**Fig. 3 (a) A rational Bézier motion of degree 6 $\hat{w}_i=1+\varepsilon 1$; $i=0,\ldots,3$. (b) A reparameterized rational Bézier motion $\hat{w}_i=\lambda^i+\varepsilon\lambda^i$; $\lambda=2$ and $i=0,\ldots,3$.**

**4.2 The Effect of the Dual Part of a Weight on a Rational Bézier Motion.** We now examine how the change in the dual part of the weights affects a rational Bézier motion. In general, an $n$th degree Bézier curve in the space of dual quaternion is given by,

$$\hat{\mathbf{Q}}(t) = \sum_{i=0}^{n} B_i^n(t)\hat{w}_i\hat{\mathbf{q}}_i. \tag{33}$$

Let, for a particular $k$, $\hat{w}_k = w_k + \varepsilon w_k^0$, where we impose the condition that $w_k^0 \neq 0$, i.e., we assume that there is only one dual weight $\hat{w}_k$ that has a nonzero dual part. The intent is to separate the effect of change in just the dual part of the weight and we do so by restricting ourselves to just one such weight.

Hence, $w_i^0 = 0$ for all $i \neq k$. Then,

$$\hat{\mathbf{Q}}(t) = \sum_{i=0}^{n} B_i^0(t) w_i \hat{\mathbf{q}}_i + \varepsilon B_k^n(t) w_k^0 \mathbf{q}_k. \tag{34}$$

Based on this, a new translation vector $\mathbf{d}'$ can be found from Eq. (8) as

$$\mathbf{d}'(t) = \frac{(\mathbf{Q}^0)\mathbf{Q}^* - \mathbf{Q}(\mathbf{Q}^0)^*}{\mathbf{Q}\mathbf{Q}^*} + \Delta\mathbf{d}(t), \tag{35}$$

where,

$$\Delta\mathbf{d}(t) = \frac{w_k^0 B_k^n(t)(\mathbf{q}_k\mathbf{Q}^* - \mathbf{Q}\mathbf{q}_k^*)}{\mathbf{Q}\mathbf{Q}^*}, \tag{36}$$

and,

$$\mathbf{Q} = \sum_{i=0}^{n} B_i^n(t) w_i \mathbf{q}_i, \quad \mathbf{Q}^0 = \sum_{i=0}^{n} B_i^n(t) w_i \mathbf{q}_i^0.$$

Equation (35) implies that the introduction of the nonzero part of a dual weight $(w_k^0)$ adds a translation component $\Delta\mathbf{d}$ to the rational Bézier motion, while leaving the rotation component unchanged.

In general, if there is more than one weight, which have a nonzero dual part, $\Delta\mathbf{d}(t)$ in Eq. (36) changes accordingly to:

$$\Delta\mathbf{d}(t) = \frac{\sum_{i=k(k\geq 0)}^{m(m\leq n)} w_i^0 B_i^n(t)(\mathbf{q}_i\mathbf{Q}^* - \mathbf{Q}\mathbf{q}_i^*)}{\mathbf{Q}\mathbf{Q}^*}; \; w_i^0 = 0 \text{ if } i \notin [k,m]. \tag{37}$$

where, $(m-k)+1$ are the number of consecutive weights with a nonzero dual part. In particular, since at $t=0$, we have $B_i^n(t)=0$ for $i \neq 0$, it follows that

$$\Delta\mathbf{d} = \frac{w_0^0[\mathbf{q}_0(w_0\mathbf{q}_0)^* - (w_0\mathbf{q}_0)\mathbf{q}_0^*]}{w_0 w_0 \mathbf{q}_0\mathbf{q}_0^*} = 0.$$

The same result hold true at $t=1$. This indicates that even though introducing dual weights gives rise to an extra translation component, it does not violate the end point interpolation property, as commonly and desirably found in Bézier curves. Since this extra translation component: $\Delta\mathbf{d}$ is time dependent, its effect is to change the motion via translation that varies in direction and magnitude along the path of the motion.

*4.2.1 Rational Screw Motion.* As an example, let us examine the effect of dual part of the weight on a rational screw motion that we considered earlier also. Let $\hat{w}_0 = 1 + \varepsilon 0$; $\hat{w}_1 = 1 + \varepsilon 2$. Then we have

$$\hat{\mathbf{Q}}(t) = (1-t)\hat{\mathbf{q}}_0 + t(1+\varepsilon 2)\hat{\mathbf{q}}_1,$$

or,

$$\hat{\mathbf{Q}}(t) = \{(1-t)\hat{\mathbf{q}}_0 + t\hat{\mathbf{q}}_1\} + \varepsilon 2t\hat{\mathbf{q}}_1.$$

It can be easily seen that at $t=1$, $\hat{\mathbf{Q}}(t) = (1+\varepsilon 2)\hat{\mathbf{q}}_1$, which as discussed in Sec. 2.2 represents the same spatial displacement as $\hat{\mathbf{q}}_1$, which is a unit dual quaternion.

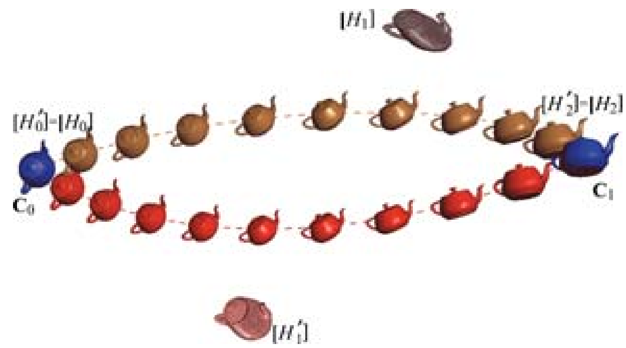


**Fig. 4 The effect of dual weights on screw interpolation.**

Figure 4 shows that not only the motion passes through the end positions but also that the change in the motion is related by a simple translation only. In this case, since the real part of the weights does not change, the orientation of the teapots at same instants (for the same value of $t$) does not change. Furthermore, it can be shown that, in terms of the matrix representation of its point trajectory, the dual part of the weight translates the middle control matrix from $[H_1]$ to $[H_1']$ and thus changes the path of the screw motion.

*4.2.2 A Rational Bézier Motion.* It has already been shown via Eq. (37) in Sec. 4.2 that introducing dual weights while keeping the real part of the weights invariant changes the motion at most by a translation only. This is exemplified in Fig. 5, where it is visible that the trajectories of the motion differ but for the same instant (at the same value of $t$) the orientation of each teapot matches with its counterpart on the other trajectory (notice the pouring mouth and the handle of the teapots at $t=0.9$ and at $t=0.1$). The given spatial positions are labeled $\mathbf{C}_i$.

*4.2.3 A Rational B-spline Motion.* Since a rational B-spline motion has a piecewise rational Bézier form, the effect of weights on a rational B-spline motion is similar to that of a rational Bézier motion. We now present an example of interpolating B-spline motions. Figure 6(*a*) shows five key-frames (labeled $\mathbf{C}_i$) and the path followed by the object undergoing a rational $C^2$ spline interpolating motion through them. Figure 6(*b*) shows the interpolating motion itself for different choice of weights.

As can be seen in Fig. 6(*b*), the paths (A and B) exhibit more pronounced difference from key frames $\mathbf{C}_1$ to $\mathbf{C}_4$ in order to achieve $C^2$ continuity at the key-frames, while being under the influence of weights with nonzero dual components for the two key frames $\mathbf{C}_i$ $(i=2,3)$ in the middle.

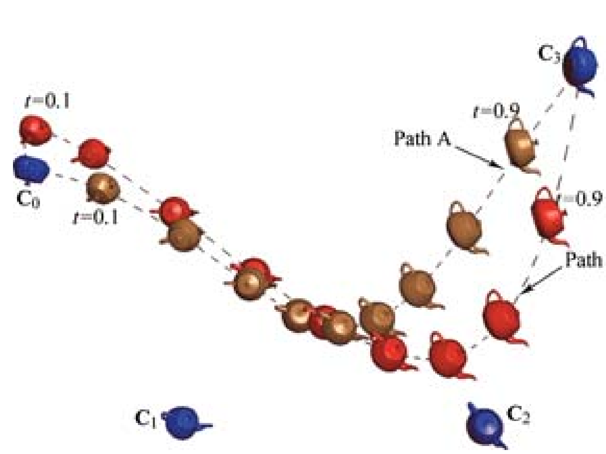


**Fig. 5 The effect of dual weights on a rational Bézier motion.**

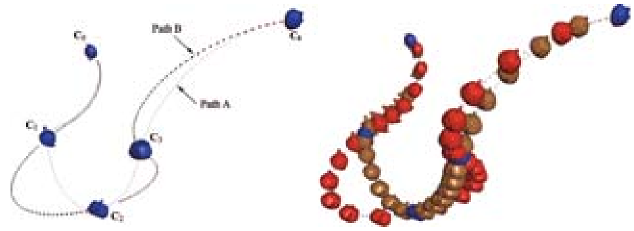


**Fig. 6 (a) Key frames and the trajectory. (b) $C^2$ spline interpolating motion for key-frames of (a).**

## 5 Conclusion

In this paper, we explored the effect of weights on rational Bézier and B-spline motions. We showed that the effects of real weights on rational motions are similar to that for rational curves, i.e., one can fine-tune a rational motion by adjusting the real weights. Furthermore, real weights can be used to reparameterize a rational motion by changing its speed while keeping the path unchanged. The use of dual weights leads to interesting changes in the motion. Their effect is to introduce a translational component that varies in direction and magnitude along the path of the motion leading to an alternate path.

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## Appendix: Derivation of Equation (13)

Given a dual quaternion $(\mathbf{Q},\mathbf{Q}^0)$, a general displacement in three-dimensional Euclidean space can be written as

$$\tilde{\mathbf{p}} = \frac{\mathbf{Q}\mathbf{p}\mathbf{Q}^*}{\mathbf{Q}\mathbf{Q}^*} + \frac{[(\mathbf{Q}^0)\mathbf{Q}^* - \mathbf{Q}(\mathbf{Q}^0)^*]}{\mathbf{Q}\mathbf{Q}^*}. \tag{38}$$

where $\mathbf{p}=p_1 i+p_2 j+p_3 k$ and $\tilde{\mathbf{p}}=\tilde{p}_1 i+\tilde{p}_2 j+\tilde{p}_3 k$ are vector quaternions whose components are Cartesian coordinates of a point before and after the displacement, respectively. The symbols, $i$, $j$, and $k$, denote quaternion units. While the first part of the sum in Eq. (38) is the well-known representation of the rotation in quaternion form (Bottema and Roth [9], McCarthy [24]), the second part represents the translation as given by Eq. (8).

Let $\mathbf{P}=P_4(\mathbf{p}+1)$ and $\tilde{\mathbf{P}}=\tilde{P}_4(\tilde{\mathbf{p}}+1)$ denote the quaternions whose components are homogeneous coordinates of the same point before and after the displacement, respectively. Substituting for $\mathbf{p}$ and $\tilde{\mathbf{p}}$ from the previous two expressions in Eq. (38), one obtains:

$$\tilde{\mathbf{P}} = \frac{\tilde{P}_4}{P_4}\frac{\mathbf{Q}\mathbf{P}\mathbf{Q}^*}{\mathbf{Q}\mathbf{Q}^*} + \frac{\tilde{P}_4[(\mathbf{Q}^0)\mathbf{Q}^* - \mathbf{Q}(\mathbf{Q}^0)^*]}{\mathbf{Q}\mathbf{Q}^*}, \tag{39}$$

$\tilde{\mathbf{P}}$ being a homogeneous coordinate, multiplying by a scalar $P_4\mathbf{Q}\mathbf{Q}^*/P_4$ yields Eq. (13):

$$\tilde{\mathbf{P}} = \mathbf{Q}\mathbf{P}\mathbf{Q}^* + P_4[(\mathbf{Q}^0)\mathbf{Q}^* - \mathbf{Q}(\mathbf{Q}^0)^*]$$

## References

[1] Farin, G., 1995, *Nurbs Curves and Surfaces: From Projective Geometry to Practical Use*, A. K. Peters.
[2] Farin, G., 1996, *Curves and Surfaces for Computer-Aided Geometric Design: A Practical Guide*, 4th ed., Academic, New York.
[3] Hoschek, J., and Lasser, D., 1993, *Fundamentals of Computer Aided Geometric Design*, A. K. Peters.
[4] Piegl, L., and Tiller, W., 1995, *The Nurbs Book*, Springer.
[5] Ge, Q. J., and Ravani, B., 1993, "Computational Geometry and Motion Approximation," *Computational Kinematics*, J. Angeles, eds. et al., KAP.
[6] Ge, Q. J., and Ravani, B., 1994, "Computer-Aided Geometric Design of Motion Interpolants," J. Mech. Des., **116**(3), pp. 756–762.
[7] Ge, Q. J., and Ravani, B., 1994, "Geometric Construction of Bezier Motions," J. Mech. Des., **116**(3), pp. 749–755.
[8] Shoemake, K., 1985, "Animating Rotation with Quaternion Curves," in Proceedings of the 12th Annual Conference on Computer Graphics and Interactive Techniques, ACM press, pp. 245–254.
[9] Bottema, O., and Roth, B., 1979, *Theoretical Kinematics*, North-Holland, Amsterdam.
[10] Juttler, B., 1994, "Visualization of Moving-Objects Using Dual Quaternion Curves," Comput. Graph., **18**(3), pp. 315–326.
[11] Juttler, B., and Wagner, M. G., 1996, "Computer-Aided Design with Spatial Rational B-Spline Motions," J. Mech. Des., **118**(2), pp. 193–201.
[12] Wagner, M. G., 1994, "A Geometric Approach to Motion Design," Ph.D. dissertation, Technische Universität, Wien.
[13] Wagner, M. G., 1995, "Planar Rational B-Spline Motions," Comput.-Aided Des., **27**(2), pp. 129–137.
[14] Park, F. C., and Ravani, B., 1995, "Bezier Curves on Riemannian-Manifolds and Lie-Groups with Kinematics Applications," J. Mech. Des., **117**(1), pp. 36–40.
[15] Zefran, M., and Kumar, V., 1998, "Interpolation Schemes for Rigid Body Motions," Comput.-Aided Des., **30**(3), pp. 179–189.
[16] Zefran, M., Kumar, V., and Croke, C. B., 1998, "On the Generation of Smooth Three-Dimensional Rigid Body Motions," IEEE Trans. Rob. Autom., **14**(4), pp. 576–589.
[17] Etzel, K. R., and McCarthy, J. M., 1999, "Interpolation of Spatial Displacements Using the Clifford Algebra of $E^4$," J. Mech. Des., **121**(1), pp. 39–44.
[18] Mullineux, G., 2004, "Modeling Spatial Displacements Using Clifford Algebra," J. Mech. Des., **126**(3), pp. 420–424.
[19] Eberharter, J. K., and Ravani, B., 2004, "Local Metrics for Rigid Body Displacements," J. Mech. Des., **126**(5), pp. 805–812.
[20] Rossignac, J. R., and Kim, J. J., 2001, "Computing and Visualizing Pose-Interpolating 3d Motions," Comput.-Aided Des., **33**(4), pp. 279–291.
[21] Roschel, O., 1998, "Rational Motion Design—a Survey," Comput.-Aided Des., **30**(3), pp. 169–178.
[22] Jüttler, B., and Wagner, M., 2002, "Kinematics and Animation," *Handbook of Computer Aided Geometric Design*, eds. et al., Elsevier, New York.
[23] J., P. F., 1984, "A Note on Rotation Matrices," IEEE Comput. Graphics Appl., **4**(2), pp. 30–33.
[24] McCarthy, J. M., 1990, *Introduction to Theoretical Kinematics*, MIT.
[25] Study, E., 1903, *Die Geometrie Der Dynamen*, Verlag Teubner.
[26] Ravani, B., and Roth, B., 1984, "Mappings of Spatial Kinematics," ASME J. Mech., Transm., Autom. Des., **106**(3), pp. 341–347.
[27] Hsia, L. M., and Yang, A. T., 1981, "On the Principle of Transference in Three-Dimensional Kinematics," ASME J. Mech. Des., **103**(3), pp. 652–656.
[28] Ge, Q. J., and Sirchia, M., 1999, "Computer Aided Geometric Design of Two-Parameter Freeform Motions," J. Mech. Des., **121**(4), pp. 502–506.
[29] Patterson, R. R., 1985, "Projective Transformations of the Parameter of a Bernstein-Bezier Curve," ACM Trans. Graphics, **4**(4), pp. 276–290.