Epsilon-Delta Definition of a Limit in Physics — Documentation

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Formal Definition of a Limit

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For a function $f(x)$ and a point c, we say:

$$\lim_{x \to c} f(x) = L$$

if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$$

Breaking Down the Definition:

- ε **(epsilon)** - Any positive tolerance around the limit value L
- δ **(delta)** - A corresponding neighborhood around the point c
- $0 < |x - c| < \delta$ - x is within δ of c but not equal to c
- $|f(x) - L| < \varepsilon$ - $f(x)$ is within ε of L

Key Insight: We can make $f(x)$ as close as we want to L by choosing x sufficiently close to c.

Proving a Limit Using Epsilon-Delta

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Example: Prove that $\lim_{x \to 2}(3x + 1) = 7$

Step 1: Set up the proof

We need to show: For every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$0 < |x - 2| < \delta \implies |(3x + 1) - 7| < \varepsilon$$

Step 2: Work backwards to find δ

Simplify the condition:

$$|(3x+1)-7| = |3x-6| = 3|x-2| < \varepsilon$$

Therefore: $|x-2| < \frac{\varepsilon}{3}$

Step 3: Choose δ

Choose: $\delta = \frac{\varepsilon}{3}$

Step 4: Verify the proof

If $0 < |x-2| < \delta = \frac{\varepsilon}{3}$, then:

$$|(3x+1)-7| = 3|x-2| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$$

Therefore: $\lim_{x\to 2}(3x+1) = 7 \checkmark$

Why Limits Matter in Physics

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The epsilon-delta definition of limits is fundamental to all of physics. Here are key applications:

Kinematics

- **Instantaneous Velocity:** $v = \lim_{\Delta t\to 0} \frac{\Delta x}{\Delta t}$
- **Acceleration:** $a = \lim_{\Delta t\to 0} \frac{\Delta v}{\Delta t}$
- **Angular Velocity:** $\omega = \lim_{\Delta t\to 0} \frac{\Delta \theta}{\Delta t}$

Electromagnetism

- **Electric Field:** $E = \lim_{q\to 0} \frac{F}{q}$
- **Current Density:** $J = \lim_{\Delta A\to 0} \frac{\Delta I}{\Delta A}$
- **Magnetic Field:** $B = \lim_{I\to 0} \frac{F}{I\cdot L}$

Thermodynamics

- **Density:** $\rho = \lim_{\Delta V\to 0} \frac{\Delta m}{\Delta V}$
- **Pressure:** $P = \lim_{\Delta A\to 0} \frac{\Delta F}{\Delta A}$
- **Heat Capacity:** $C = \lim_{\Delta T\to 0} \frac{\Delta Q}{\Delta T}$

Key Insight: All fundamental physics concepts rely on the rigorous foundation provided by epsilon-delta limits!

Epsilon-Delta Definition: Visual Representation

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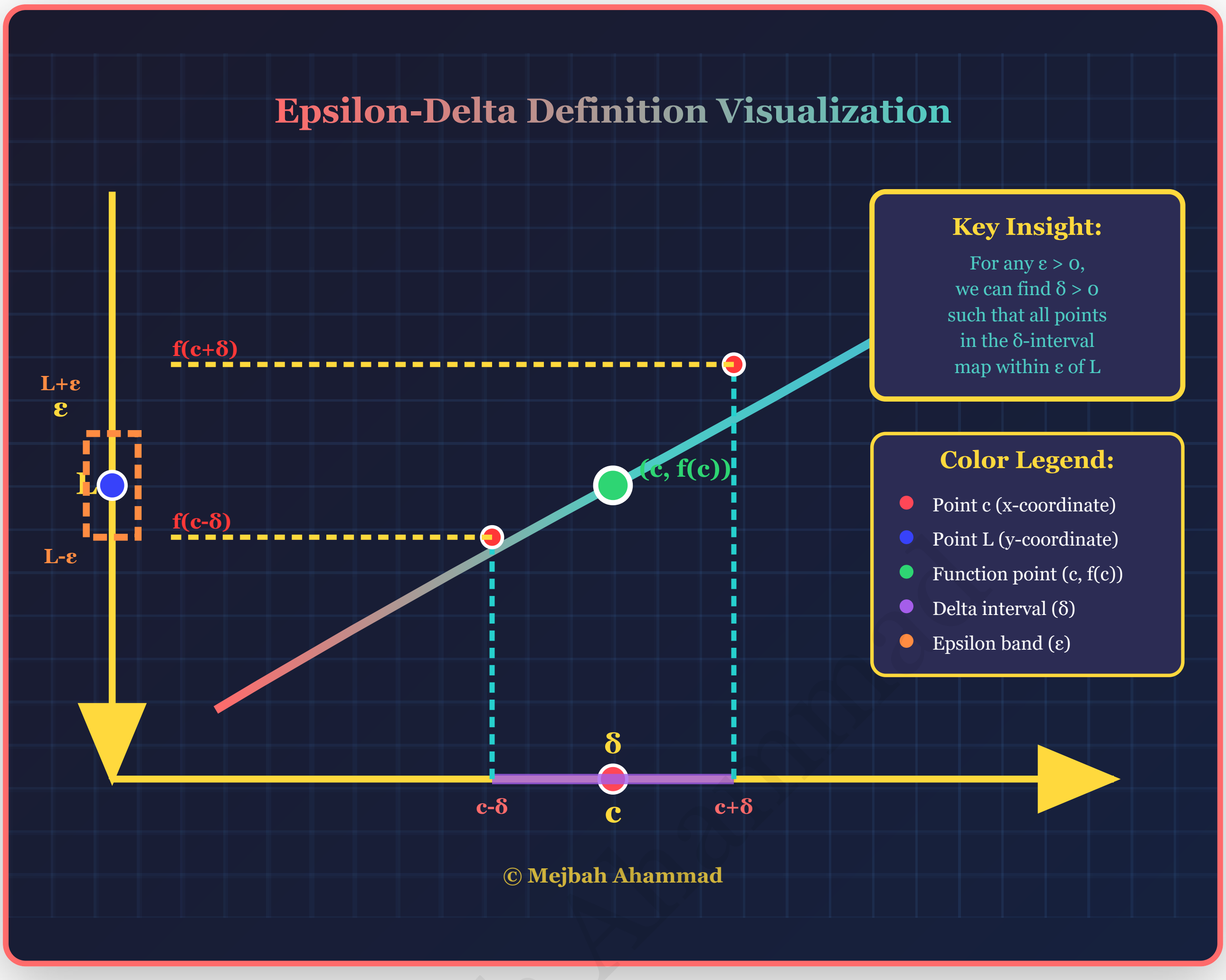
Understanding the geometric meaning of $\lim_{x \to c} f(x) = L$



Visual Interpretation:

- 🌈 **Gradient curve:** Function $f(x)$ with rainbow colors
- 🔴 **Red point:** Point c on x-axis
- 🔵 **Blue point:** Point L on y-axis
- 🟢 **Green point:** $(c, f(c))$ - the function point we're approaching
- 🟣 **Purple interval:** δ-neighborhood around c
- 🟠 **Orange band:** ε-tolerance around L
- 🔵 **Cyan lines:** Vertical projections from δ-interval to function
- 🟡 **Yellow lines:** Horizontal projections from function to ε-band
- 🔴 **Pink points:** Function values at boundaries $f(c-\delta)$ and $f(c+\delta)$

Mathematical Statement: $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$