Algebraic Topology

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Contents

Lecture 1: Introduction

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- First part: fundamental groups. We follow Munkres:
  - Chap 9 'Fundamental group'
  - Chap 11 'The Seifert-Van Kampen thm' and
  - Chap 12/13 'Classification covering spaces'
- Second part: Homology groups, via cudi, chapter of a book.
- 10 problems (solve them during the semester) No feedback during the semester, but asking questions is allowed. Working together is allowed.
- Exam (completely open book) First 4 questions: 1h30 prep. Last one 30 min, no prep
  1. Theoretical question (open book, explain the proof, . . .) /4

2. New problem (comparable to one of the 10 problems) /4
3. Explain your solution n-th problem solved at home /4
4. Explain your solution m-th problem solved at home /4
5. 4 small questions /4

After the exam, hand in your solutions of the other problems. After a quick look, the points of 3. and 4. can be ± 1 (in extreme cases ± 2)

- No exercise classes for this course!

Chapter 0

Introduction

0.1 What is algebraic topology?

Functor from category of topological spaces to the category of groups.

- Category: set of spaces and morphisms.
- Functor: $X \rightsquigarrow G_X$ and $f : X \to Y \rightsquigarrow f_* : G_X \to G_Y$ such that
  - $(f \circ g)_* = f_* \circ g_*$
  - $(1_X)_* = 1_{G_X}$

  Two systems we'll discuss:

- fundamental groups
- homology groups

Example. Suppose we have a functor. If $G_X \ncong G_Y$, then X and Y are not homeomorphic. If 'shadows' are different, then objects themselves are different too.

Proof. Suppose X and Y are homeomorphic. Then $\exists f : X \to Y$ and $g : Y \to X$, maps (maps are always continuous in this course), such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Then $f_* : G_X \to G_Y$ and $g_* : G_Y \to G_X$ such that $(g \circ f)_* = (1_X)_*$ and $(f \circ g)_* = (1_Y)_*$. Using the rules discussed previously, we get

$$g_* \circ f_* = 1_{G_X} \quad f_* \circ g_* = 1_{G_Y},$$

which means that $f_* : G_X \to G_Y$ is an isomorphism.

0.2 Fundamental group

Pick a base point x_0 and consider it fixed. (The fundamental group will not depend on it. We assume all spaces are path connected) $X \rightsquigarrow \pi(X)$.

- A loop based at $x_0 \in X$ is a map $f : I = [0,1] \to X$, $f(0) = f(1) = x_0$.
- Loops are equivalent if one can be deformed in the other in a continuous way, with the base point fixed.
- The fundamental group consists of equivalent classes of loops.

Example. Let $X = B^2$ (2 dimensional disk). Then $\pi(B^2) = 1$, because every loop is equivalent to the 'constant' loop.

Example. Let $X = S^1$ and pick x_0 on the circle. Two options:

- The loop is trivial equivalent to the constant loop
- The loop goes around the circle.
- The loop goes around the circle, twice.
- The loop goes around the circle, clockwise, once
- ...

$\pi(S^1) \cong \mathbb{Z}$ (proof will follow).

The composition of loops is simply pasting them. In the case of the circle, the loop $-1 \circ$ the loop 2 is the loop 1.

Suppose $\alpha : I \to X$ and $f : X \to Y$. Then we define

$$f_*[\alpha] = [f \circ \alpha].$$

Theorem 1 (Fixed point theorem of Brouwer). Any continuous map from a rectangle to itself has at least one fixed point.

Proof. Suppose there is no fixed point, so $f(x) \neq x$ for all $x \in B^2$. Then we can construct map $r : B^2 \to S_1$ as follows: take the intersection of the boundary and half ray between $f(x)$ and x.

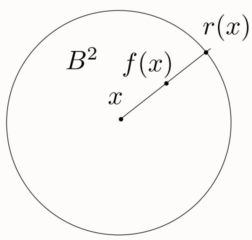

Figure 1: Proof of the Brouwer fixed point theorem

If x lies on the boundary, we have the identity map. This map is continuous. Then we have $S^1 \to B^2 \to S^1$, via the inclusion and r. Looking at the fundamental groups:

$$\pi(S^1) = \mathbb{Z} \to \pi(B^2) = 1 \to \pi(S^1) = \mathbb{Z}.$$

The map from $\pi(S^1) \to \pi(S^1)$ is the identity map, but the first map maps everything on 1. $\lightning$

Chapter 9

Fundamental group

9.51 Homotopy of paths

Definition 1 (Homotopy). Let $f, g : X \to Y$ be maps (so continuous). Then a homotopy between f and g is a continuous map $H : X \times I \to Y$ such that

- $H(x, 0) = f(x)$, $H(x, 1) = g(x)$
- For all $t \in I$, define $f_t : X \to Y : x \mapsto H(x, t)$

We say that f is homotopic with g, we write $f \simeq g$. If g is a constant map, we say that f is null homotopic.

Definition 2 (Path homotopy). Let $f, g : I \to X$ be two paths such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. Then $H : I \times I \to X$ is a path homotopy between f and g, if and only if

- $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$ (homotopy between maps)
- $H(0, t) = x_0$ and $H(1, t) = x_1$ (start and end points fixed)

Notation: $f \simeq_p g$.

Lemma 1. $\simeq$ and $\simeq_p$ are equivalence relations.

Proof.

- Reflective: $F(x, t) = f(x)$
- Symmetric: $G(x, t) = H(x, 1 - t)$
- Transitive: Suppose $f \simeq g$ and $g \simeq h$, with H_1, H_2 resp.

$$H(x, t) = \begin{cases} H_1(x, 2t) & 0 \le t \le \frac{1}{2} \\ H_2(x, 2t - 1) & \frac{1}{2} \le t \le 1 \end{cases}. \qquad \square$$

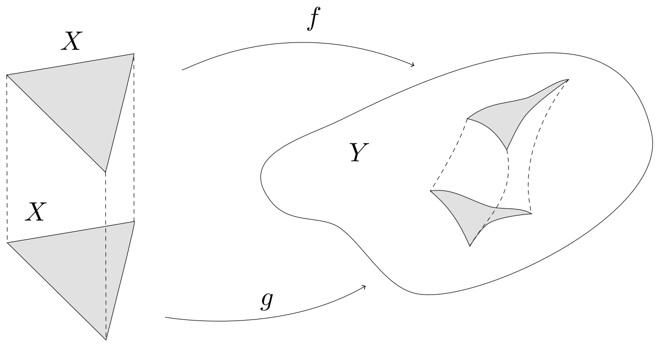

Figure 9.1: General example of a homotopy.

Example (Trivial, but important). Let $C \subset \mathbb{R}^n$ be a convex subset.

- Any two maps $f, g : X \to C$ are homotopic.
- Any two paths $f, g : I \to C$ with $f(0) = g(0)$ and $g(1) = f(1)$ are path homotopic.

Choose $H : X \times I \to C : (x, t) \to H(x, t) = (1 - t)f(x) + tg(x)$.

Product of paths

Let $f : I \to X$, $g : I \to X$ be paths, $f(1) = g(0)$. Define

$$f * g : I \to X : s \mapsto \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \le s \le 1. \end{cases}$$

Remark. If f is path homotopic to f' and g path homotopic to g' (which means that $f(1) = f'(1) = g(0) = g'(0)$), then $f * g \simeq_p f' * g'$.

So we can define $[f] * [g] := [f * g]$ with $[f] := \{g : I \to X \mid g \simeq_p f\}$

Theorem 2.

1. $[f] * ([g] * [h])$ is defined iff $([f] * [g]) * [h]$ is defined and in that case, they are equal.
2. Let e_x denote the constant path $e_x : I \to X : s \mapsto x$, $x \in X$. If $f(0) = x_0$ and $f(1) = x_1$ then $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.
3. Let $\overline{f} : I \to X : s \mapsto f(1-s)$. Then $[f]*[\overline{f}] = [e_{x_0}]$ and $[\overline{f}]*[f] = [e_{x_1}]$

Proof. First, two observations

- Suppose $f \simeq_p g$ via homotopy H, $f, g : I \to X$. Let $k : X \to Y$.

Then $k \circ f \simeq_p k \circ g$ using $k \circ H$.

- If $f * g$ (not necessarily path homotopic). Then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

Now, the proof

2. Take $e_0 : I \to I : s \mapsto 0$. Take $i : I \to I : s \mapsto s$ Then $e_0 * i$ is a path from 0 to $1 \in I$. The path i is also such a path. Because I is a convex subset, $e_0 * i$ and i are path homotopic, $e_0 * i \simeq_p i$. Using one of our observations, we find that

$$\begin{aligned} f \circ (e_0 * i) &\simeq_p f \circ i \\ (f \circ e_0) * (f \circ i) &\simeq_p f \\ e_{x_0} * f &\simeq_p f \\ [e_{x_0}] * [f] &= [f]. \end{aligned}$$

3. Note that $i * \bar{i} \simeq_p e_0$. Now, applying the same rules, we get

$$\begin{aligned} f \circ (i * \bar{i}) &\simeq_p f \circ e_0 \\ f * \overline{f} &\simeq_p e_{x_0} \\ [f] * [\overline{f}] &= [e_{x_0}]. \end{aligned}$$

1. Remark: Only defined if $f(1) = g(0), g(1) = h(0)$. Note that $f * (g * h) \neq (f * g) * h$. The trajectory is the same, but the speed is not.

   Assume the product is defined. Suppose $[a, b]$, $[c, d]$ are intervals in $\mathbb{R}$. Then there is a unique positive (positive slope), linear map from $[a, b] \to [c, d]$. For any $a, b \in [0, 1)$ with $0 < a < b < 1$, we define a path

$$\begin{aligned} k_{a,b} : [0,1] &\longrightarrow X \\ [0, a] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{f} X \\ [a, b] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{g} X \\ [b, 0] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{h} X \end{aligned}$$

   Then $f * (g * h) = k_{\frac{1}{2},\frac{3}{4}}$ and $(f * g) * h = k_{\frac{1}{4},\frac{1}{2}}$
   Let γ be that path $\gamma : I \to I$ with the following graphs:

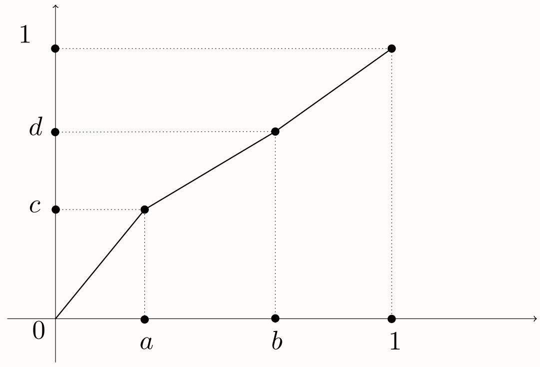

Figure 9.2: proof path

Note that $\gamma \simeq_p i$. Now, using the fact that composition of positive linear maps is positive linear.

$$k_{c,d} \circ \gamma \simeq_p k_{c,d} \circ i$$
$$k_{a,b} \simeq_p k_{c,d},$$

which is what we wanted to show.

□

9.52 Fundamental group

Definition 3. Let X be a space and $x_0 \in X$, then the fundamental group of X based at x_0 is

$$\pi(X, x_0) = \{[f] \mid f : I \to X, f(0) = f(1) = x_0\}.$$

(Also $\pi_1(X, x_0)$ is used, first homotopy group of X based at x_0)

For $[f], [g] \in \pi(X, x_0)$, $[f] * [g]$ is always defined, $[e_{x_0}]$ is an identity element, $*$ is associative and $[f]^{-1} = [\overline{f}]$. This makes $(\pi(X, x_0), *)$ a group.

Example. If $C \subset \mathbb{R}^n$, convex then $\pi(X, x_0) = 1$. E.g. $\pi(B^2, x_0) = 1$.

Remark. All groups are a fundamental group of some space.

Question: how does the group depend on the base point?

Theorem 3 (52.1)**.** Let X be a space, $x_0, x_1 \in X$ and $\alpha : I \to X$ a path from x_0 to x_1. Then

$$\begin{aligned}\hat{\alpha} : \pi(X, x_0) &\longrightarrow \pi(x, x_1)\\ [f] &\longmapsto [\overline{\alpha}] * [f] * [\alpha].\end{aligned}$$

is an isomorphisms of groups. Note however that this isomorphism depends on α.

Proof. Let $[f], [g] \in \pi_1(X, x_0)$. Then

$$\begin{aligned}\hat{\alpha}([f] * [g]) &= [\overline{\alpha}] * [f] * [g] * [\alpha]\\ &= [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] * [g] * [\alpha]\\ &= \hat{\alpha}[f] * \hat{\alpha}[g].\end{aligned}$$

We can also construct the inverse, proving that these groups are isomorphic. □

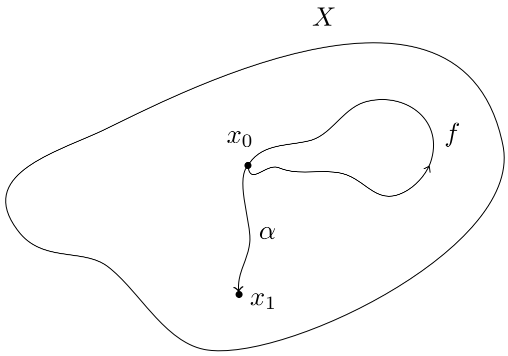

Figure 9.3: Construction of the group homomorphism

Remark. If $f : (x, x_0) \to (Y, y_0)$ is a map of pointed topology spaces ($f : X \to Y$ continuous and $f(x_0) = y_0$). Then

$$f_* : \pi(X, x_0) \to \pi(Y, y_0) : [\gamma] \mapsto [f \circ \gamma]$$

is a morphism of groups, because of the two 'rules' discussed previously, with

$$(f \circ g)_* = f_* \circ g_* \qquad (1_X)_* = 1_{\pi(X, x_0)}.$$

Lecture 2: Covering spaces

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Definition 4. Let X be a topological space, then X is simply connected iff X is path connected and $\pi_1(X, x_0) = 1$ for some $x_0 \in X$

Remark. If trivial for one base point, it's trivial for all base points.

Example. Any convex subset $C \subset \mathbb{R}^n$ is simply connected.

Example (Wrong proof of $\pi(S^2)$ being trivial)**.** Let f be a path from $[0,1] \to S^2$. Then pick $y_0 \notin \mathrm{Im}(f)$. Then $S^2 \setminus \{y_0\} \approx \mathbb{R}^2$. Then use $\mathbb{R}^2$.

This is wrong because we cannot always find $y_0 \notin \mathrm{Im}(f)$. Space filling loops! We'll see the correct proof later on.

Lemma 2. Suppose X is *simply connected* and $\alpha, \beta : I \to X$ two paths with same start and end points. Then $\alpha \simeq_p \beta$.

Proof. Simply connected implies loops are homotopic? Consider $\alpha * \overline{\beta} \simeq_p e_{x_0}$, since the space is simply connected.

$$([\alpha] * [\overline{\beta}]) * [\beta] = [e_{x_0}] * [\beta] = [\beta]$$
$$[\alpha] * ([\overline{\beta}] * [\beta]) = [\alpha] * [e_{x_0}] = [\alpha].$$

And therefore $\alpha \simeq_p \beta$. (Note: make sure end and start point matchs when using $*$) □

9.53 Covering Spaces

Definition 5 (Evenly covered)**.** Let $p : E \to B$, surjective map (so continuous). Let $U \subset B$ open. Then U is *evenly covered* iff $p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$. with

- V_α open in E
- $V_\alpha \cap V_\beta = \emptyset$ if $\alpha \neq \beta$
- $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

Remark. If $U' \subset U$, also open and U is evenly covered, then also U'.

Definition 6. Let $p : E \to B$ be a surjective map. Then p is a covering projection iff $\forall b \in B, \exists U \subset B$ open, containing b such that U is evenly covered by p. Then (E, p) is called a covering space.

Example. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Take $p : \mathbb{R} \to S^1 : t \mapsto e^{2\pi i t}$. Note that $\mathbb{R}$ is an easier space than S^1, and so will be π_1 (1 vs $\mathbb{Z}$).

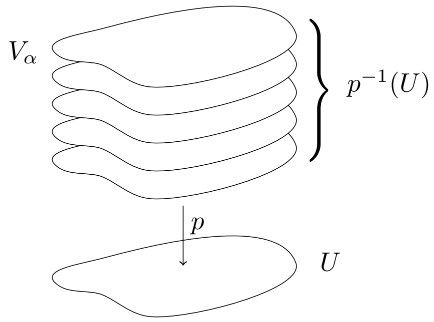


Figure 9.4: Evenly covered

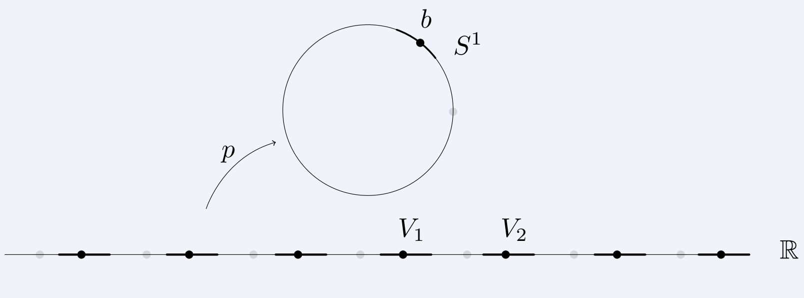


Figure 9.5: Example of a covering space

There are also other covering spaces of p. For example, $p' : S^1 \to S^1 : z \mapsto z^3$.

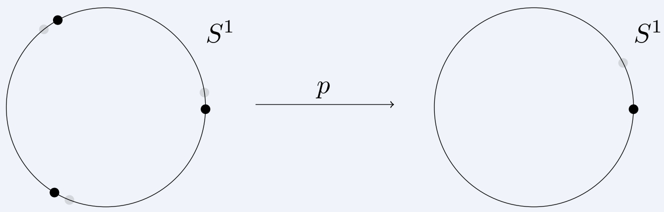


Figure 9.6: Second example of a covering space

Here we have three copies for each point. We say that the covering has 3 sheets. Note that this is independent of which point we take. This is always the case! We can show that these are the only coverings of S^1: $\mathbb{R}$ and $z \mapsto z^n$.

Proposition 1. A covering map is always a open map.

Proof. Exercise. □

Proposition 2. For any $b \in B$, $p^{-1}(b)$ is a discrete subset of E. (No accumulation point)

Proof. Indeed for any $\alpha \in I$, $V_\alpha \cap p^{-1}(b)$ is exactly one point. □

Remark. A covering is always local homeomorphism. But: there are surjective local homeomorphism which are not covering maps. *A covering map is more than a surjective local homeomorphism.*

For example, $p : \mathbb{R}_0^+ \to S^1 : t \mapsto e^{2\pi i t}$. Consider the inverse image of a neighbourhood around 1. When we restrict p to the part around 0, it is no longer a homeomorphism (we don't get the whole neighbourhood around one.)

Creating new covering spaces out of old ones.

- Suppose $p : E \to B$ is a covering and $B_0 \subset B$ is a subspace with the subspace topology. Let $E_0 = p^{-1}(B_0)$ and $p_0 = p|_{E_0}$. Then (E_0, p_0) is a covering of B_0.
- Suppose that (E, p) is a covering of B and (E', p') is a covering of B', then $(E \times E', p \times p')$ is a covering of $B \times B'$.

Example. Let $T^2 = S^1 \times S^1$.

- $p : \mathbb{R}^2 \to S^1 \times S^1 : (t, s) \mapsto (e^{ait}, e^{bis})$.
- $p' : \mathbb{R} \times S^1 \to T^2 : (t, z) \mapsto (e^{ait}, z^n)$
- $p' : S^1 \times S^1 \to T^2 : (z_1, z_2) \mapsto (z_1^n, z_2^m)$

These are the only types of coverings of the torus. We'll prove this later on.

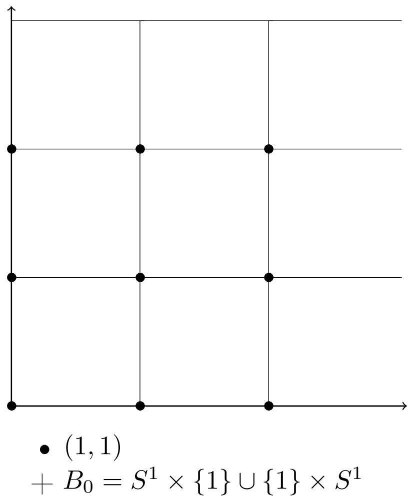


Figure 9.7: Covering of the torus. B_0 is the figure 8 space: two connected circles.

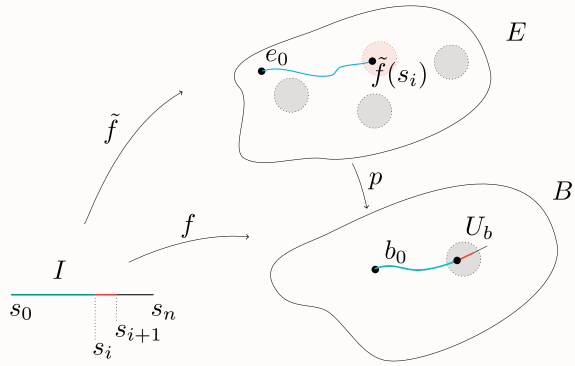
9.54 Fundamental group of the circle *(and more)*

Given f, when can f be 'lifted' to E? I.e. when does there exist an $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$? In this section, we'll only consider $X = [0,1], X = [0,1]^2$.

$$\begin{tikzcd} & E \arrow[d] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

Lemma 3 (Important result)**.** Suppose (E, p) is a covering of B, $b_0 \in B$ $e_0 \in p^{-1}(b_0)$. Suppose that $f : I \to B$ is a path starting at b_0. Then there exists a unique lift $\tilde{f} : I \to E$ of f with $\tilde{f}(0) = e_0$.

Proof. For any b of B, we choose an open U_b such that U_b is evenly covered by p. Then $\{f^{-1}(U_b) \mid b \in B\}$ is an open cover of I, which is compact. There is a number $\delta > 0$ such that any subset of I of diameter $\leq \delta$ is contained entirely in one of these opens $f^{-1}(U_b)$. (Lebesgue number lemma). Now, we divide the interval into pieces $0 = s_0 < s_1 < \ldots < s_n = 1$ such that $|s_{i+1} - s_i| \leq \delta$. For any i, we have that $f([s_i, s_{i+1}]) \subset U_b$ for some b.



We now construct $\tilde{f}$ by induction on $[0, s_i]$

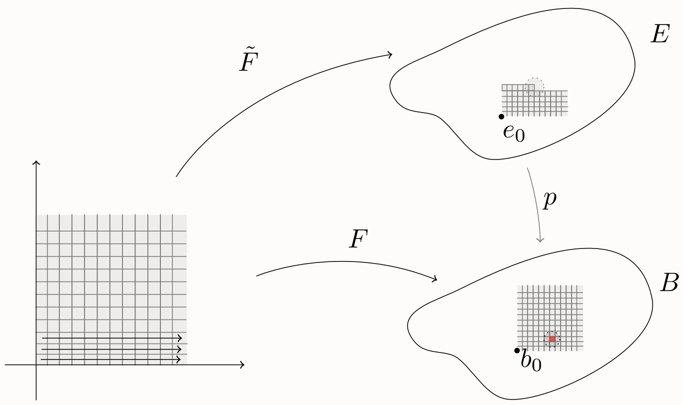
- $\tilde{f}(0) = e_0$
- Assume $\tilde{f}$ has been defined on $[0, s_i]$. Let U be an open such that $f[s_i, s_{i+1}] \subset U_b$.

  There is exactly one slice V_α in $p^{-1}(U_b)$ containing $\tilde{f}(s_i)$. We define $\forall s \in [s_i, s_{i+1}] : \tilde{f}(s) = (p|_{V_\alpha})^{-1} \circ f(s)$. By the pasting lemma, $\tilde{f}$ is continuous.
- In this way, we can construct $\tilde{f}$ on the whole of I.

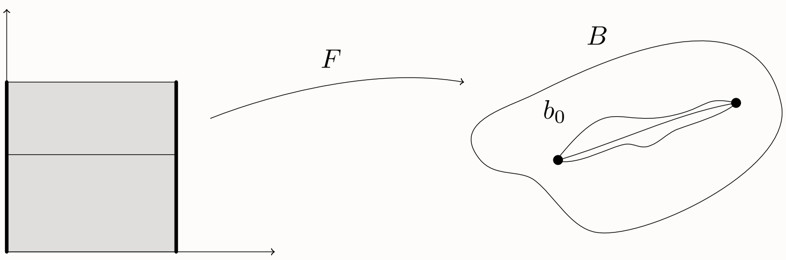
Uniqueness works in exactly the same way, by induction. □

Lemma 4 (54.2)**.** (E, p) is a covering of B, $b_0 \in B$, $e_0 \in E$, with $p(e_0) = b_0$. Suppose $F : I \times I \to B$ is a continuous map with $f(0,0) = b_0$, then there is a unique $\tilde{F} : I \times I \to E$. Moreover, if F is a path homotopy, then also $\tilde{F}$ is a path homotopy.

Proof. Same as in the one dimensional case.



'Moreover, if F is a path homotopy, then also $\tilde{F}$ is a path homotopy': Easy explanation in the book. Another explanation:



- $F(0,t) = b_0, F(1,t) = b_1$
- $\tilde{F}(0,\cdot) : I \to E : t \mapsto \tilde{F}(0,t)$ is a path starting at e_0, and is a lift of $F(0,\cdot)$ (the constant path at b_0).

  The path $k : I \to E : t \mapsto e_0$ is also a lift of the constant path at b_0 starting at e_0. Then $k = \tilde{F}(0,\cdot)$, as the lift is unique.

□

Theorem 4 (54.3)**.** Let (E,p) be a covering of B, $b_0 \in B$, $e_0 \in E$ with $p(e_0) = b_0$. Let f, g be two paths in B starting in b_0 s.t. $f \simeq_p g$ (so f and g end at the same point). Let $\tilde{f}, \tilde{g}$ be the unique lifts of f, g starting at e_0. Then $\tilde{f} \simeq_p \tilde{g}$, and so $\tilde{f}(1) = \tilde{g}(1)$.

Proof. $F : I \times I \to B$ is a path homotopy between f and g. Then $\tilde{F} : I \times I \to E$ with $\tilde{F}(0,0) = e_0$. Then $\tilde{F}$ is a path homotopy, by the previous result, between $\tilde{F}(\cdot,0)$ and $\tilde{F}(\cdot,1)$. Note that $p \circ \tilde{F}(t,0) = F(t,0) = f(t)$

and $p \circ \tilde{F}(t,1) = F(t,1) = g(t)$. By uniqueness $\tilde{F}(\cdot,0) = \tilde{f}$, $\tilde{F}(\cdot,1) = \tilde{g}$. $\square$

We've shown that homotopy from below lifts to above. The converse is easy. Now we're ready to discuss the relation between groups and covering spaces.

Definition 7. Let (E,p) be a covering of B. $b_0 \in B$, $e_0 \in E$ and $p(e_0) = b_0$. Then the lifting correspondence is the map

$$\begin{aligned} \phi : \pi(B, b_0) &\longrightarrow p^{-1}(b_0) \\ [f] &\longmapsto \tilde{f}(1), \text{where } \tilde{f} \text{ is the unique lift of } f, \text{ starting at } e_0. \end{aligned}$$

This is well-defined because $[f] = [g] \Rightarrow \tilde{f} \simeq_p \tilde{g} \Rightarrow \tilde{f}(1) = \tilde{g}(1)$. This ϕ depends on the choice of e_0.

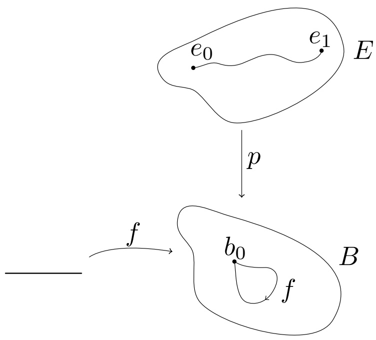


Figure 9.8: Lifting Correspondence

Theorem 5 (54.4)**.** If E is path connected, then ϕ is a surjective map. If E is simply connected, then ϕ is a bijective map.

Proof. Suppose E is path connected, and let $e_0, e_1 \in p^{-1}(b_0)$. Consider a path $\tilde{f} : I \to E$ with $\tilde{f}(0) = e_0$ and $\tilde{f}(1) = e_1$. This is possible because E is path connected. Let $f = p \circ \tilde{f} : I \to B$ with $f(0) = p(e_0) = b_0$ and $f(1) = p(e_1) = b_0$, so f is a loop based at b_0. So f is a loop at b_0 and its unique lift to E starting at e_0 is $\tilde{f}$. Hence $\phi[f] = \tilde{f}(1) = e_1$, which shows that ϕ is surjective.

Now assume that E is simply connected (group is trivial). Consider $[f], [g] \in \pi(B_0)$ with $\phi[f] = \phi[g]$. This implies $\tilde{f}(1) = \tilde{g}(1)$. These start at e_0. It follows from Lemma 52.3 that $\tilde{f} \simeq_p \tilde{g}$. $\square$

Example. Take the circle and the real line as covering space. Then $p^{-1}(1) = \mathbb{Z}$. So we know that as a set $\pi(S^1)$ is countable. Therefore, $p \circ \tilde{f} \simeq_p p \circ \tilde{g}$. This implies that $f \simeq_p g$, and therefore $[f] = [g]$.

Theorem 6. $\pi_1(S^1, 1) \cong (\mathbb{Z}, +)$.

Proof. Take $b_0 = 1$ and $e_0 = 0$ and $p : \mathbb{R} \to S^1 : t \mapsto e^{2\pi it}$. Then $p^{-1}(b_0) = \mathbb{Z}$. And since, $\mathbb{R}$ is simply connected, we have that $\phi : \pi(S, 1) \to \mathbb{Z} : [f] \mapsto \tilde{f}(1)$ is a bijection.

Now we'll show that it's a morphism. Let $[f]$ and $[g]$ elements of the fundamental group of S^1 and assume that $\phi[f] = \tilde{f}(1) = m$ and $\phi[g] = \tilde{g}(1) = n$.

We're going to prove that $\phi([f] * [g]) = \phi([f]) + \phi([g]) = m + n$. Define $\tilde{\tilde{g}} : I \to \mathbb{R} : t \mapsto \tilde{g}(t) + m$. Then $p \circ \tilde{\tilde{g}} = p \circ \tilde{g} = g$, as $p(s+m) = p(s)$ for all m. Now, look at $\tilde{f} * \tilde{\tilde{g}}$. This is a lift of $p \circ (\tilde{f} * \tilde{\tilde{g}}) = (p \circ \tilde{f}) * (p \circ \tilde{\tilde{g}}) = f * g$, which starts at 0. Hence, $\phi([f] * [g]) = \phi([f * g]) =$ the end point of $\tilde{f} * \tilde{\tilde{g}}$, so $\tilde{\tilde{g}}(1) = \tilde{g}(1) + m = n + m$. □

The following lemma makes the fact that the covering space is simpler than the space itself exact.

Lemma 5 (54.6)**.** Let (E, p) be a covering of B. $b_0 \in B$, $e_0 \in E$ and $p(e_0) = b_0$. Then

1. $p_* : \pi(E, e_0) \to \pi(B, b_0)$ is a monomorphism (injective).

2! Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence induces a well defined map[a]

$$\begin{aligned} \Phi : {}_H\backslash\pi_1(B, b_0) &\longrightarrow p^{-1}(b_0) \\ H * [f] &\longmapsto \phi[f], \end{aligned}$$

so ϕ is constant on right cosets. Dividing by H makes Φ always bijective, even when E is not simply connected.

3. Let f be a loop based at b_0, then $\tilde{f}$ is a loop at e_0 iff $[f] \in H$.

[a]different notation than the book

Proof.

1. Let $\tilde{f} : I \to E$ be a loop at e_0 and assume that $p_*[\tilde{f}] = 1$. (Then we'd like to show that f itself is trivial.) This implies $p \circ \tilde{f} \simeq_p e_{b_0}$. This implies that $\tilde{f} \simeq_p \tilde{e}_{b_0} = e_{e_0}$, or $[\tilde{f}] = 1$.

2. We have to prove two things:

   Well defined $H * [f] = H * [g] \Rightarrow \phi(f) = \phi(g)$.

   Assume $[f] \in H * [g]$, or $H * [f] = H * [g]$. This means that $[f] = [h] * [g]$, were $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ at e_0. In other words $[f] = [h * g]$, or $f \simeq_p h * g$. Let $\tilde{f}$ be the unique lift of f starting at e_0. Let $\tilde{g}$ be the unique lift of g starting at e_0. Then $\tilde{h} * \tilde{g}$ (which is allowed, $\tilde{h}$ is a loop) the unique lift of $h * g$ starting at e_0.
   $\tilde{f}(1) = \phi(f) = \phi(h * g) = (\tilde{h} * \tilde{g})(1) = \tilde{g}(1) = \phi(g)$. If the cosets are the same, then the end points of the lifts are also the same.

Injective $H * [f] = H * [g] \Leftarrow \phi(f) = \phi(g)$. The end points of f and g are the same

Now consider $\tilde{h} = \tilde{f} * \tilde{\bar{g}}$. Then $[\tilde{h}] * [\tilde{g}] = [\tilde{f}] * [\tilde{\bar{g}}] * [\tilde{g}] = [\tilde{f}]$. By applying p_*, $[h] * [g] = [f]$.

3. Trivial. Exercise. Apply 2. with the constant path.

□

Remark. $k : X \to Y$ induces a morphism k_*, we've proved that earlier. Here we only showed injectiveness.

Lecture 3: Retractions

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9.55 Retractions and Fixed points

Definition 8 (Retract, retraction)**.** Let $A \subset X$, then A is a *retract* of X iff there exists a map $r : X \to A$ such that $r|_A = 1|_A$, i.e. $r(a) = a$ for all $a \in A$. The map r is called a *retraction*

Example. Let X be the figure 8 space, and denote the right circle by A. Then it's easy to see that there exists a retract from the whole space to A by mapping the left circle onto the right.

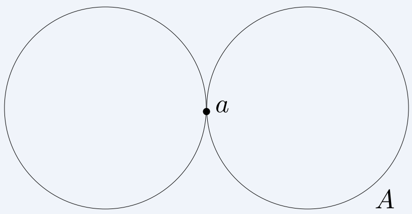


Figure 9.9: Figure 8 space

Lemma 6 (55.1)**.** If A is a retract of X, then $i : A \to X : a \mapsto a$ induces a monomorphism $i_* : \pi(A, a_0) \to \pi(X, a_0)$ with $a_0 \in A$.

Proof. Let $r : X \to A$ be a retraction. Then $r \circ i = 1_A$.

$$(A, a_0) \xrightarrow{i} (X, x_0) \xrightarrow{r} (A, a_0).$$

$$\pi(A, a_0) \xrightarrow{i_*} \pi(X, x_0) \xrightarrow{r_*} \pi(A, a_0).$$

As $r \circ i = 1_A$, we get that $r_* \circ i_* = (r \circ i)_* = (1_A)_* = 1_{\pi(A,a_0)}$. So i_* is injective, r_* is surjective, which completes the proof. □

Example (Theorem 55.2)**.** Let S^1 be the boundary of B^2. Then S^1 is *not* a retract of B^2. There is a surjective map from B^2 to S^1, but not one that is the identity on S^1.

Proof. Suppose S^1 is a retract. Then $i_* : \pi(S^1, x_0) \to \pi(B^2, x_0)$ is injective, but $i_* : \mathbb{Z} \to 1$. ↯

Theorem 7 (55.6, Brouwer fixed point theorem)**.** For any map $f : B^2 \to B^2$, there exists at least one fixed point.

Proof. Look at the proof of the first lecture. Now that we've actually proven that $\pi(S_1) = \mathbb{Z}$ and $\pi(B_2) = 1$, the proof is complete. □

Example. Let A be a 3×3 matrix with strict positive real entries. Then A has a positive real eigenvalue.

Proof. Let $B = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid 0 \leq x_1, x_2, x_3 \leq 1 \wedge x_1^2 + x_2^2 + x_3^2 = 1\}$, an octant of a 2-sphere. Note that $B \approx B^2$, a disk. Now, define $f : B \to B : x \mapsto \frac{Ax}{\|Ax\|}$. Note that this maps vectors from B to vectors of B, as A has positive entries. Note that f is continuous. By Brouwer fixed point theorem, there exists $x_0 \in B$, such that $f(x_0) = x_0$, or $Ax_0 = \|Ax_0\| x_0$.

Remark. Section 56 gives a proof of the theorem of algebra using fundamental proofs. Interesting, but not part of this course.

Remark. We'll skip the Borsak-Ulam theorem for now.

9.58 Deformation retracts and homotopy type

Lemma 7. Suppose $h, k : (X, x_0) \to (Y, y_0)$ and assume $H : X \times I : Y$ is a homotopy with

- $H(x, 0) = h(x)$, $H(x, 1) = k(x)$ (definition of homotopy)
- $H(x_0, t) = y_0$, for all $t \in I$

Then $h_* = k_* : \pi(X, x_0) \to \pi_1(Y, y_0)$.

Proof. We have to show that for all $f : I \to X$ with $f(0) = f(1) = x_0$ that $h \circ f \simeq_p k \circ f$, i.e. $h_*[f] = k_*[f]$.

$$\begin{array}{ccccc} G : I \times I & \longrightarrow & X \times I & \xrightarrow{H} & Y \\ (s, t) & \longmapsto & (f(s), t) & \longmapsto & H(f(s), t) \end{array}.$$

- Then G is continuous.

- $G(s, 0) = H(f(s), 0) = (h \circ f)(s)$
- $G(s, 1) = H(f(s), 1) = (k \circ f)(s)$
- $G(0, t) = H(f(0), t) = H(x_0, t) = y_0$
- $G(1, t) = H(f(1), t) = H(x_0, t) = y_0$

So G is a homotopy, and a path homotopy between the two loops. $\square$

Definition 9 (Deformation retract)**.** Let $A \subset X$, then A is a deformation retract of X, iff there exists

- $r : X \to A$, such that $r(a) = a$ for all $a \in A$. (normal retract)
- homotopy $H : X \times I \to X$ such that
  - $H(x, 0) = x$
  - $H(x, 1) = r(x)$
  - $H(a, t) = a$ for all $a \in A$

This means that 1_X is homotopic to $i \circ r$ via a homotopy leaving A invariant.

Example. Let $S^1 \subset \mathbb{R}^2 \setminus \{(0, 0)\}$. Then S^1 is a deformation retract of $\mathbb{R}^2 \setminus \{(0, 0)\}$. Using homotopy $H : \mathbb{R}^2_0 \times I \to \mathbb{R}^2_0 : x \mapsto (1 - t)x + t\frac{x}{\|x\|}$. (The same for S^n and $\mathbb{R}^n_0$)

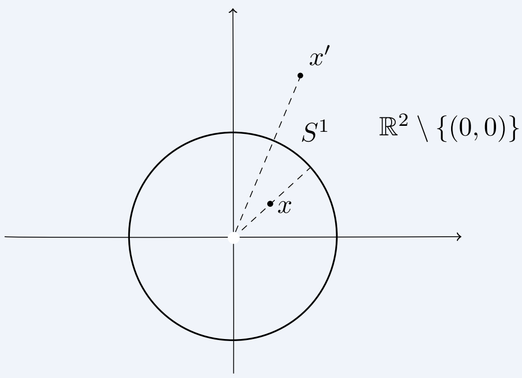


Figure 9.10: Example of a deformation retract

Example. Consider the figure 8 space. Claim: A is not a deformation retract of X. We'll prove this later on.

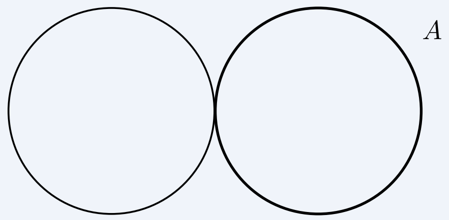


Figure 9.11: Example of a deformation rectract

Example. Consider the torus and a circle on the torus. Then it is a retract, but not a deformation retract.

Theorem 8. If A is a deformation retract of X, then $i : A \to X$ induces an *isomorphism* i_*. I.e. If you have a deformation retract, it's not only injective but also surjective.

Proof. Let $i : A \to X$ be the inclusion and $r : X \to A$ be the deformation retraction using H. Then $r \circ i = 1_A$, which gives $r_* \circ i_* = 1_{\pi(A,a_0)}$.

Now, $i \circ r \simeq_p 1_X$ using the homotopy of the previous lemma, i.e. H with $H(a_0, t) = a_0$. Call $h = i \circ r$, $k = 1_X$, and using the previous lemma, $(i \circ r)_* = (1_X)_* : \pi(X, x_0) \to \pi(X, x_0)$, which shows that $i_* \circ r_* = 1_{\pi(X,x_0)}$.

We conclude that both i_* and r_* are isomorphisms. □

Remark. This means that the fundamental group of $\mathbb{R}^2_0$ is the same as the one of S^1, which is $\mathbb{Z}$.

Example. The fundamental group of the figure 8 space and the θ-space are isomorphic. These spaces are not deformations of each other, but we can show that they are deformation retracts of $\mathbb{R}^2 \setminus \{p, q\}$. We say that these spaces are of the same homotopy type.

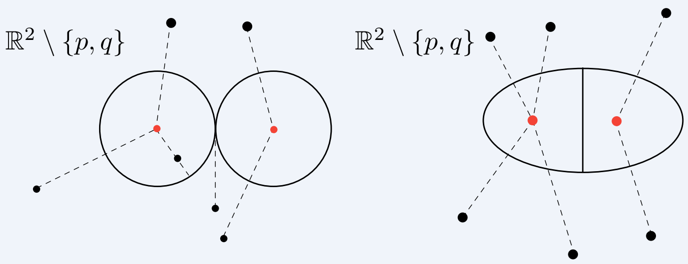


Figure 9.12: The figure eight and theta space seen as a deformation retract of $\mathbb{R}^2 \setminus \{p, q\}$

Definition 10. Let X, Y be two spaces, then X and Y are said to be of the same homotopy type iff there exists $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. We say that f, g are homotopy equivalences and are homotopy inverses of each other.

Remark. This is an equivalence relation.

We'll prove that spaces of the same homotopy type have the same fundamental group. For that, we'll prove the previous lemma in a more general form, not preserving the base point.

Lemma 8 (58.4)**.** Suppose $h, k : X \to Y$ with $h(x_0) = y_0$ and $k(x_0) = y_1$. Assume that $h \simeq k$ via a homotopy $H : X \times I \to X$, $(H(x, 0) = h(x)$, $H(x, 1) = k(x))$. Then $\alpha : I \to X : s \mapsto H(x_0, s)$ is a path starting in y_0 and ending in y_1 such that the following diagram commutes

$$\begin{tikzcd}
& \pi(Y, y_0) \arrow[dd, "\hat{\alpha}"] & \ni [g] \arrow[dd, mapsto] \\
\pi(X, x_0) \arrow[ur, "h_*"] \arrow[dr, "k_*"'] & & \\
& \pi(Y, y_1) & \ni [\overline{\alpha}] * [g] * [\alpha]
\end{tikzcd}$$

Proof. We need to show that $\hat{\alpha}(h_*[f]) = k_*[f]$, or $[\overline{\alpha}] * [h \circ f] * [\alpha] = [k \circ f]$, or $[h \circ f] * [\alpha] = [\alpha] * [k \circ f]$. We'll prove that these paths are homotopic. Using the picture, we see that $\beta_0 * \gamma_2 \simeq_p \gamma_1 * \beta_1$, because they are loops in a path connected space, $I \times I$. Therefore, $F \circ (\beta_0 * \gamma_2) \simeq_p F \circ (\gamma_1 * \beta_1)$. This is $f_0 * c \simeq_p c * f_1$. Now, if we apply H, we get $H \circ (f_0 * c) \simeq_p H \circ (c * f_1)$, so $(h \circ f) * \alpha \simeq_p \alpha * (k \circ f)$, which implies that $[h \circ f] * [\alpha] = [\alpha] * [k \circ f]$.

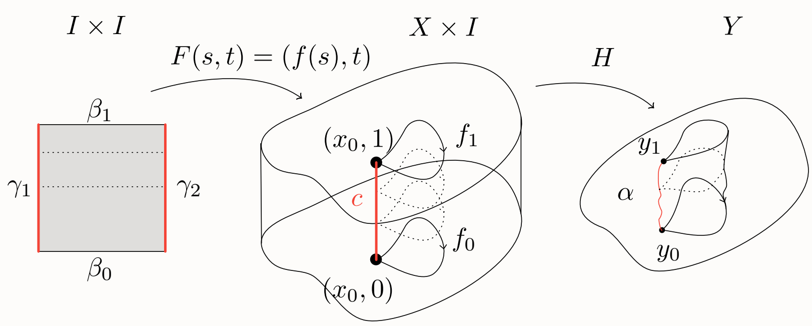


Figure 9.13: Proof of the Lemma

□

Theorem 9. Let $: X \to Y$ be a homotopy equivalence, with $f(x_0) = y_0$. Then $f_* : \pi(X, x_0) \to \pi(Y, y_0)$ is an isomorphism.

Proof. Let g be a homotopy inverse of f.

$$(X, x_0) \xrightarrow{\quad f \quad} (Y, y_0) \xrightarrow{\quad g \quad} (X, x_1) \xrightarrow{\quad f \quad} (Y, y_1) \cdots$$

$$\begin{tikzcd}
\pi(X, x_0) \arrow[r, "f_{*,x_0}"] \arrow[drr, "1_{\pi(X,x_0)}=(1_X)_*"'] & \pi(Y, y_0) \arrow[r, "g_{*,x_0}"] & \pi(X, x_1) \arrow[d, "\hat{\alpha}"] \\
& & \pi(X, x_0)
\end{tikzcd}$$

.

$$\begin{tikzcd}
\pi(Y, y_0) \arrow[r, "g_{*,x_0}"] \arrow[drr, "1_{\pi(Y,y_0)}=(1_Y)_*"'] & \pi(X, x_1) \arrow[r, "f_{*,x_1}"] & \pi(Y, y_1) \arrow[d, "\hat{\beta}"] \\
& & \pi(Y, x_0)
\end{tikzcd}$$

From the first diagram, $g_{y_0,*} \circ f_{x_0,*}$ is an isomorphism, $g_{y_0,*}$ is surjective. The second diagram gives that $f_{x_1,x} \circ g_{y_0,*}$ is an isomorphism, so $g_{y_0,*}$ is injective, so $g_{y_0,*}$ is an isomorphism. Now composing, we find that $g_{y_0,*}^{-1} \circ (g_{y_0,*} \circ f_{x_0,*}) = f_{x_0,*}$ is an isomorphism. $\square$

9.59 The fundamental group of S^n (and more)

Theorem 10 (59.1)**.** Let $X = U \cup V$, where U, V are open subsets of X, such that $U \cap V$ is path connected. Let $i : U \to X$ and $j : V \to X$ denote the natural inclusions and consider $x_0 \in U \cap V$. Then the images of i_* and j_* generate the whole group $\pi(X, x_0)$. In other words: any loop based at x_0 can be written as a product of loops inside U and V.

Proof. Let $[f] \in \pi(X, x_0)$ denote $f : I \to X$ is a loop based at x_0.

Claim: there exists a subdevision of $[0, 1]$ such that $f[a_i, a_{i+1}]$ lies entirely inside U or V and $f(a_i) \in U \cap V$. Proof of the claim: Lebesgue number lemma says that such a subdivision b_i exists. Now assume b_j is such that $f(b_j) \notin U \cap V$, for $0 < j < m$. Then either $f(b_j) \in U \setminus V$, or $f(b_j) \in V \setminus U$. The first one would imply that $f([b_{j-1}, b_j]) \subset U$ and $f([b_j, b_{j+1}]) \subset U$. So $f[b_{j-1}, b_{j+1}] \subset U$, so we can discard b_j. Same for the second possibility.

Let α_i be a path from x_0 to $f(a_i)$ and α_0 the constant path $t \mapsto x_0$, inside $U \cap V$ (which is possible, as it is path connected). Now define

$$f_i : I \to X : I \xrightarrow{\text{p.l.m.}} [a_{i-1}, a_i] \xrightarrow{f} X.$$

Then $[f] = [f_1] * [f_2] * \cdots * [f_n]$. Note that all f_i have images inside U or V. Now,

$$\begin{aligned}[f] &= [a_0] * [f_1] * [\overline{\alpha_1}] * [\alpha_1] * [f_2] * [\overline{\alpha_2}] * [\alpha_2] * [f_3] * \cdots * [\alpha_{n-1}] * [f_n] * [\overline{\alpha_n}] \\ &= [\alpha_0 * (f_1 * \overline{\alpha_1})] * [\alpha_1 * (f_2 * \overline{\alpha_2})] * \cdots .\end{aligned}$$

Every path of the form $\alpha_{i-1} * (f_i * \overline{\alpha_i})$ is a loop based at x_0 lying entirely inside U or V. This means that

$$[f] \in \operatorname{grp}\{i_*(\pi(U, x_0)), j_*(\pi(V, x_0))\}.$$

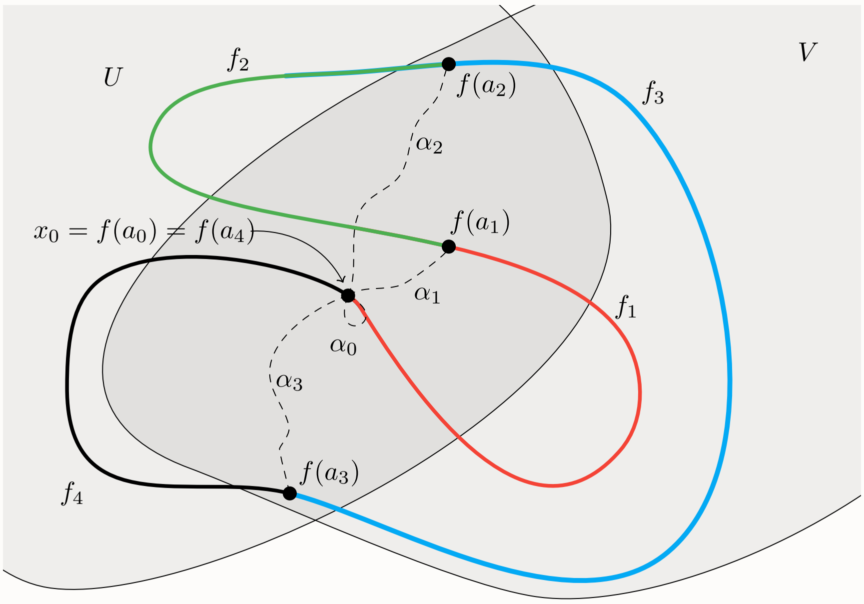


Figure 9.14: Proof of Theorem 59.1

□

Corollary. Let $n \geq 2$, then $\pi(S^n, x_0) = 1$

Proof. Consider S^n and N, S the north and south pole. Let $U = S^n \setminus \{N\}$ and $V = S^n \setminus \{S\}$. Then $U, V \approx \mathbb{R}^n$ and $U \cap V$ is path connected, which is easy to prove as it is simply homeo to $\mathbb{R}^n$ with points removed. Then $\pi(S^n, x_0)$ is generated by $i_*(\pi(U, x_0))$ and $j_*(\pi(V, x_0))$, which both are trivial. This proof doesn't work for S^1 because then the intersection is not path connected anymore! □

9.60 Fundamental group of some surfaces

Definition 11 (Surface)**.** Surface: compact two-dimensional topological manifold.

Theorem 11. Let X be a space and $x_0 \in X$. Let Y be a space and $y_0 \in Y$. Then $\pi(X \times Y, (x_0, y_0)) \cong \pi(X, x_0) \times \pi(Y, y_0)$.

Proof. Exercise. Idea: Let $f : I \to X \times Y$ be a loop based at (x_0, y_0). Then $f(s) = (g(s), h(s))$ where g is a loop in X based at x_0, similar for h, and conversely. $\square$

Example. $\pi_1(T^2, x_0) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z}^2$. We know that $\pi(S^2, x_0) = 1$, so the torus and the two sphere are not homeomorphic to each other, they aren't even homotopically equivalent.

Example. $\mathbb{R}\mathrm{P}^2 = \frac{S^2}{\sim}$ Then $p : S^2 \to \mathbb{R}\mathrm{P}^2$, which is by definition continuous by definition of the topology on the projective plane.

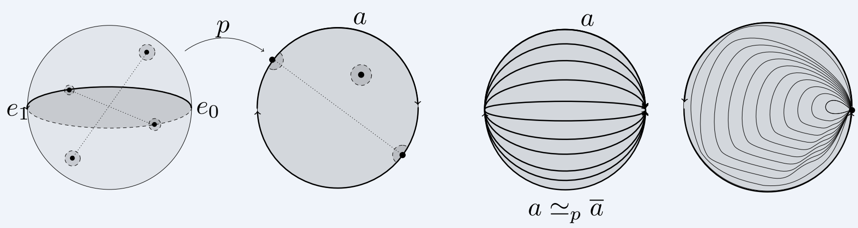


Figure 9.15: Example: projective plane.

This means that (S, p) is a covering of the projective plane. The lifting correspondence says that

$$\Phi : \pi(\mathbb{R}\mathrm{P}^2, x_0) \to p^{-1}(x_0) = \{\tilde{x_0}, -\tilde{x_0}\}$$

is a isomorphism. Therefore, $\pi_1(\mathbb{R}\mathrm{P}^2, x_0)$ is a group with 2 elements, so $\mathbb{Z}_2$.

This means, there exist loops which we cannot deform to the trivial loop, but when going around twice, they *do* deform to the trivial loop. E.g. consider the loop a. This is not homotopic equivalent with the trivial loop, as $e_1 \neq e_0$. (Or also you can see it because $\alpha = \overline{\alpha}$.) But pasting the loop it twice, we see that *is* possible. This means that the fundamental group of the projective space is different from all the one we've seen before.

Example. T^2 is the torus. $T^2 \# T^2$ is the connected sum of two tori (Remove small disc of both tori and glue together), in Dutch: 'tweeling zwemband'. This space has yet another fundamental group.

Example. Figure eight space: fundamental group is not abelian. Indeed, $[b * a] \neq [a * b]$.

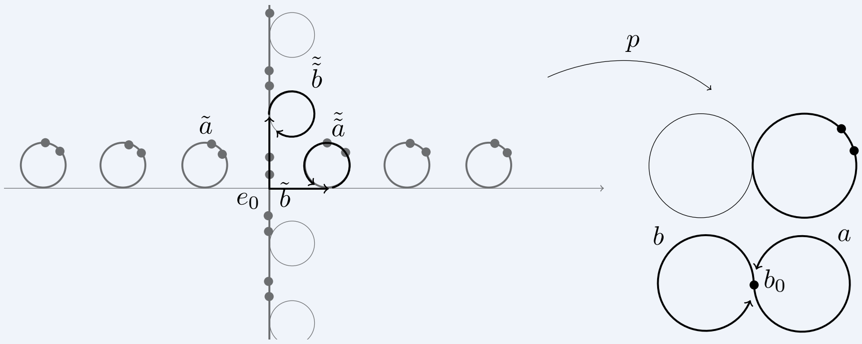


Figure 9.16: Figure 8 space is not Abelian

Example. Tweeling zwemband. The space retracts to the figure 8 situation, which shows that the group of the tweeling zwemband has a nonabelian component.

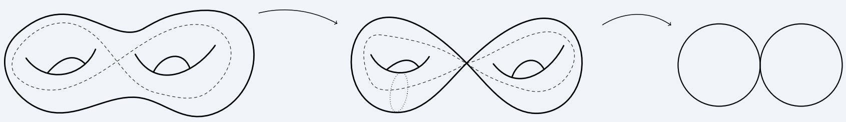


Figure 9.17: Tweeling zwemband: $T\#T$

Lecture 4: Seifert-Van Kampen theorem

di 22 okt 10:26

Chapter 11

Seifert-Van Kampen theorem

Note. This doesn't follow the book very well.

Definition 12. A free group on a set X consists of a group F_x and a map: $i : X \to F_X$ such that the following holds: For any group G and any map $f : X \to G$, there exists a unique morphism of groups $\phi : F_X \to G$ such that

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & F_x \arrow[d, "\exists!\phi"] \\ & G \end{tikzcd} .$$

Note. The free group of a set is unique. Suppose $i : X \to F_X$ and $j : X \to F'_X$ are free groups.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "j"'] & F_X \arrow[d, "\exists\phi"] \\ & F'_X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "i"'] & F'_X \arrow[d, "\exists\psi"] \\ & F_X \end{tikzcd} .$$

Then

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i"'] & F_X \arrow[d, "\psi\circ\phi"] \\ & F_X \end{tikzcd} .$$

Then by uniqueness, $\psi \circ \phi$ is 1_{F_X}, and likewise for $\phi \circ \psi$.

Note. The free group on a set always exists. You can construct it using "irreducible words"

Example. Consider $X = \{a, b\}$. An example of a word is $aaba^{-1}baa^{-1}bbb^{-1}a$.

This is not a irreducible word. The reduced form is $aaba^{-1}bba = a^2ba^{-1}b^2a$. Then F_X is the set of irreducible words.

Example. If $X = \{a\}$, then $F_x = \{a^z \mid z \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$. Exercise: check that $\mathbb{Z}$ satisfies the universal property.

Example. If $X = \emptyset$, then $F_X = 1$.

Definition 13 (Free product of a collection of groups)**.** Let G_i with $i \in I$, be a set of groups. Then the free product of these groups denoted by $*_{i\in I}G_i$ is a group G together with morphisms $j_i : G_i \to G$ such that the following universal property holds: Given any group H and a collection of morphisms $f_i : G_i \to H$, then there exists a unique morphism $f : G \to H$, such that for all $i \in I$, the following diagram commutes:

$$\begin{tikzcd} G_i \arrow[r, "j_i"] \arrow[dr, "f_i"'] & G \arrow[d, "\exists! f"] \\ & H \end{tikzcd}.$$

Note. Again, $*_{i\in I}G_i$ is unique.

Example. Construction is similar to the construction of a free group. Let $I = \{1, 2\}$ and $G_1 = G, G_2 = H$. Then $G * H$. Elements of $G * H$ are "words" of the form $g_1h_1g_2h_2g_3$, $g_1h_1g_2h_2$, or $h_1g_1h_2g_2h_3g_3$ or $h_1g_1h_2$, ... with $g_j \in G, h_j \in H$.

Note. $G * H$ is always infinite and nonabelian if $G \neq 1 \neq H$. Even if G, H are very small, for example $\mathbb{Z}_2 * \mathbb{Z}_2 = \{1, t\} * \{1, s\}$. Then $ts \neq st$ and the order of ts is infinite.

Note. $\mathbb{Z} * \mathbb{Z} = F_{a,b}$. In general: $F_X = *_{x\in X}\mathbb{Z}$

11.70 The Seifert-Van Kampen theorem

Theorem 12 (70.1)**.** Let $X = U \cup V$ where $U, V, U \cap V$ are open and path connected.[a] Let $x_0 \in U \cap V$. For any group H and 2 morphisms $\Phi_1 : \pi(U, x_0) \to H$ and $\Phi_2 : \pi(V, x_0) \to H$ such that that $\Phi_1 \circ i_1$ and $\Phi_2 \circ i_2$, there exists exactly one $\Phi : \pi(X, x_0) \to H$ making the diagram commute

$$\begin{tikzcd}
 & \pi(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\Phi_1"] & \\
\pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & \pi(x, x_0) \arrow[r, dashed, "\Phi"] & H \\
 & \pi(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\Phi_2"'] &
\end{tikzcd} \cdot$$

i_1, i_2, i, j_1, j_2 are induced by inclusions.

[a] Note that U, V should also be path connected!

Proof. Not covered in this class. You can have a look in the book, but not insightful. □

Theorem 13 (70.2, Seifert-Van Kampen (classical version)). Let $X = U \cup V$ as before ($U, V, U \cap V$, path connected) and $x_0 \in U \cap V$. Let $j : \pi(U, x_0) * \pi(V, x_0) \to \pi(X, x_0)$ (induced by j_1 and j_2). On elements of $\pi(U, x_0)$ it acts like j_1, on elements of $\pi(V, x_0)$ it acts like j_2.

$$\begin{tikzcd}
G_1 \arrow[d] \arrow[dr, "f_1"] & \\
G_1 * G_2 \arrow[r, dashed, "f"] & H \\
G_2 \arrow[u] \arrow[ur, "f_2"] &
\end{tikzcd} \cdot$$

Then j is surjective[a] and the kernel of j is the normal subgroup of $\pi(U, x_0) * \pi(U, x_0)$ generated by all elements of the form $i_1(g)^{-1} i_2(g)$, were $g \in \pi(U \cap V, x_0)$.

[a] This is the only place where algebraic topology is used. We've proved this last week. The groups U and V generate the whole group. The rest of this theorem follows from the previous theorem.

Proof.

- j is surjective (last week)
- Let N be the normal subgroup generated by $i_1(g)^{-1} i_2(g)$. Then we claim that $N \subset \ker(j)$ This means we have to show that $i_1(g)^{-1} i_2(g) \in \ker j$. $j(i_1(g)) = j_1(i_1(g))$ by definition of j. Looking at the diagram, we find that $j_1(i_1(g)) = j_2(i_2(g)) = i(g) = j(i_2(g))$. Therefore $j(i_1(g)^{-1} i_2(g)) = 1$, which proves that elements of the form $i_1(g)^{-1} i_2(g)$ are in the kernel.
- Since $N \subset \ker j$, there is an induced morphism

$$\begin{aligned} k : (\pi_1(U, x_0) * \pi_1(V, x_0))/N &\longrightarrow \pi_1(X, x_0) \\ gN &\longmapsto j(g). \end{aligned}$$

To prove that $N = \ker j$, we have to show that k is injective. Because this would mean that we've divided out the whole kernel of j.

Now we're ready to use the previous theorem. Let $H = (\pi(U) * \pi(V))/N$ Repeating the diagram:

$$\begin{tikzcd}
 & \pi(U,x_0) \arrow[d, "j_1"] \arrow[dr, "\Phi_1"] & \\
\pi(U\cap V,x_0) \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & \pi(X,x_0) \arrow[r, dashed, shift left, "\Phi"] & H \arrow[l, shift left, "k"] \\
 & \pi(V,x_0) \arrow[u, "j_2"'] \arrow[ur, "\Phi_2"] &
\end{tikzcd} .$$

Now, we define $\Phi_1 : \pi(U,x_0) \to H : g \mapsto gN$, and $\Phi_2 : \pi(V,x_0) \to H : g \mapsto gN$. For the theorem to work, we needed that $\Phi_1 \circ i_1 = \Phi_2 \circ i_2$. This is indeed the case: let $g \in \pi(U \cap V)$. Then $\Phi_1(i_1(g)) = i_1(g)N$ and $\Phi_2(i_2(g)) = i_2(g)N$ and $i_1(g)N = i_2(g)N$ because $i_1(g)^{-1}i_2(g) \in N$.

The conditions of the previous theorem are satisfied, so there exists a Φ such that the diagram commutes.

Note that we also have $k : H \to \pi(X)$. We claim that $\Phi \circ k = 1_H$, which would mean that k is injective, concluding the proof. It's enough to prove that $\Phi\circ k(gN) = gN$ for all $g \in \pi(U)$ and $\forall g \in \pi(V)$, as these g's generate the product of the groups. If a map is the identity on the generators, it is the identity on the whole group.

Let $g \in \pi(U)$. Then $(\Phi \circ k)(gN) = \Phi(k(gN)) = \Phi(j(g))$, per definition of k. On elements of $\pi(U)$, $j \equiv j_1$, so $\Phi(j(g)) = \Phi(j_1(g)) = \Phi_1(g)$ by looking at the diagram, and per definition of Φ_1, we find that $\Phi(g) = gN$. So we've proven that $(\Phi\circ k)(gN) = gN$. This means that N is the kernel, so we've proved that k is an isomorphism.

□

Corollary. Suppose $U\cap V$ is simply connected, so $\pi_1(U\cap V, x_0)$ is the trivial group. In this case $N = \ker j = 1$, hence $\pi(U,x_0) * \pi(V,x_0) \to \pi(X,x_0)$ is an isomorphism.

Corollary. Suppose U is simply connected. Then $\pi(X,x_0) \cong \pi(V,x_0)/N$ where N is the normal subgroup generated by the image of $i_2 : \pi(U\cap V) \to \pi(V,x_0)$.

Example. Let X be the figure 8 space.

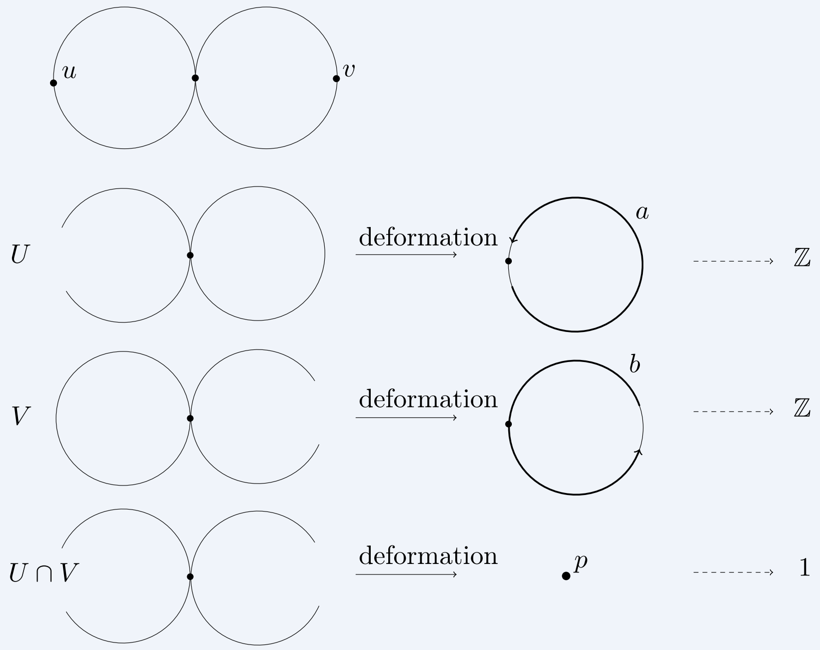


Figure 11.1: figure eight free product group

Conclusion: $\pi_1(X,p) \cong \mathbb{Z} * \mathbb{Z} = F_{\{[a],[b]\}}$

Example. By induction. Let W_n be the wedge of n circles. Then $\pi(W_n, p) = F_n$. This also holds for a wedge of infinite circles. (But be careful when choosing topology)

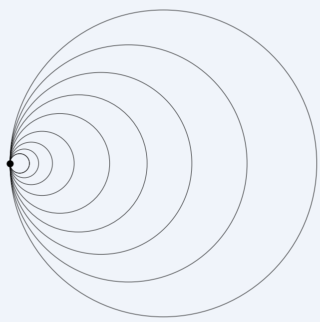
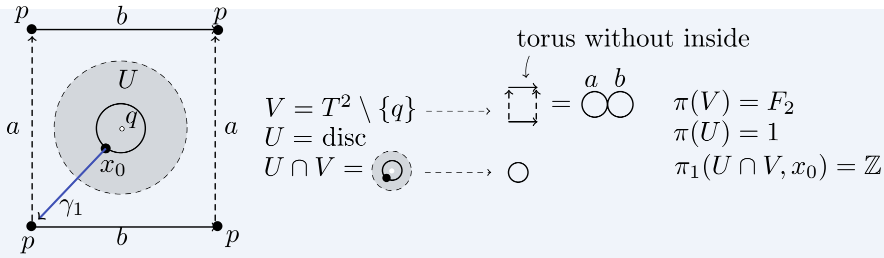


Figure 11.2: Wedge of circles

Example. Fundamental group of the torus.

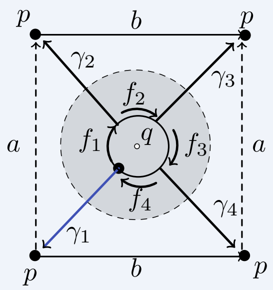


Define V, U as in the figure above.

$$\begin{aligned} \pi(T^2 \setminus \{q\}, x_0) &\longrightarrow \pi(T^2 \setminus \{q\}, p) \\ [f] &\longmapsto [\overline{\gamma_1}] * [f] * [\gamma_1]. \end{aligned}$$

Then $\pi(X, x_0) = \frac{\pi(V,x_0)}{N} \cong \frac{\pi(V,p)}{\hat{\gamma}(N)}$. With N the normal subgroup generated by the image of $i_2 : \pi(U \cap V, x_0) \to \pi(T^2 \setminus \{q\}, x_0)$.

$$\begin{aligned} i_2 : \pi(U \cap V, x_0) \cong \mathbb{Z} &\longrightarrow \pi(T^2 \setminus \{q\}, 0) \\ \langle [f_1] * [f_2] * [f_3] * [f_4] \rangle &\longmapsto \langle [f_1] * [f_2] * [f_3] * [f_4] \rangle \,. \end{aligned}$$



Now, defining γ_i as in the picture, we get

$$\begin{aligned} \hat{\gamma}([f_1] * [f_2] * [f_3] * [f_4]) = &\underbrace{[\overline{\gamma_1}] * [f_1] * [\gamma_2]}_{[a]} * \underbrace{[\overline{\gamma_2}] * [f_2] * [\gamma_3]}_{[b]} \\ &* \underbrace{[\overline{\gamma_3}] * [f_3] * [\gamma_4]}_{[a]^{-1}} * \underbrace{[\overline{\gamma_4}] * [f_4] * [\gamma_1]}_{[b]^{-1}} \,. \end{aligned}$$

Therefore,

$$\begin{aligned}\pi(X,p) &= \frac{\pi(V,p)}{\langle\langle [a][b][a^{-1}][b^{-1}]\rangle\rangle} \\ &= \frac{F_{\{[a],[b]\}}}{\langle\langle [a]*[b]*[a]^{-1}*[b]^{-1}\rangle\rangle} \\ &= \langle \alpha,\beta \mid \alpha\beta=\beta\alpha\rangle\,.\end{aligned}$$

Example. Fundamental group of a poolring for twins.

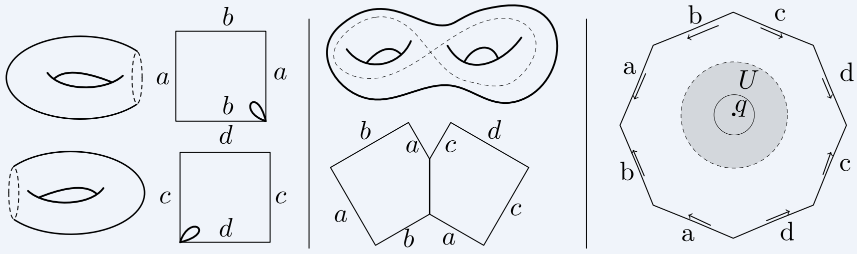


Figure 11.3: Fundamental group of a poolring for twins

Same idea. (Note, we've renamed the edges in the figure on the right.) $V = X \setminus \{a\}$. $\pi_1(V,x_0) = F_{\{[a],[b],[c],[d]\}}$, $\pi(U\cap V,x_0)=\mathbb{Z}$. Conclusion $\pi(X,x_0) = \langle \alpha,\beta,\gamma,\delta \mid [\alpha,\beta][\gamma,\delta]=1\rangle$

In this way, we can calculate the fundamental group of any surface, e.g. projective space ($\mathbb{Z}_2$), klein bottle ($\big\langle \alpha,\beta \mid \alpha\beta\alpha^{-1}\beta=1\big\rangle$, 'just read the boundary'), ...

End of Chapter 11.

Chapter 12

Classification of covering spaces

Note. This can be chapter 13 in some books

Lemma 9 (74.1! (General Lifting lemma))**.** Let $p : E \to B$ be a covering, Y a space. Assume B, E, Y are path connected, and locally path connected.[a]

Let $f : Y \to B$, $y_0 \in Y, b_0 = f(y_0)$. Let $e_0 \in E$ such that $p(e_0) = b_0$. Then $\exists \tilde{f} : Y \to E$ with $\tilde{f}(y_0) = e_0$ and $p \circ \tilde{f} = f$

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (B, b_0) \end{tikzcd}.$$

iff $f_*(\pi(Y, y_0) \subset p_*\pi(E, e_0)$. If $\tilde{f}$ exists then it is unique.

[a]From now on, all spaces are locally path connected: Every neighbourhood contains an open that is path connected

Example. Take $Y = [0, 1]$. Then f is a path, then we showed that every map can be lifted. And indeed, the condition holds: $f_*(\pi(Y, y_0)) = 1$, the trivial group, which is a subgroup of all groups.

Proof. $\Rightarrow$ Suppose $\tilde{f}$ exists. Then $p \circ \tilde{f} = f$, so $(p \circ \tilde{f})_*\pi(Y, y_0) = \pi(Y, y_0)$. The left hand side is of course $p_*(\tilde{f}_*(\pi(Y, y_0)) \subset p_*(\pi(E, e_0))$, so $p_*(\pi(E, e_0)) \subset f_*(\pi(Y, y_0))$.

$\Leftarrow$ First, we'll show the uniqueness. Suppose $\tilde{f}$ exists.

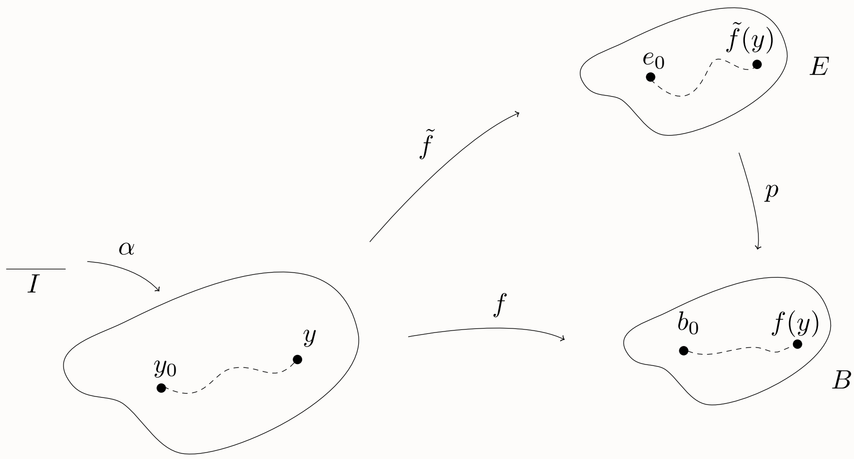


Figure 12.1: general lifting lemma

$p \circ (\tilde{f} \circ \alpha) = f \circ \alpha$, so $\tilde{f} \circ \alpha$ is the unique lift of $f \circ \alpha$ starting at e_0. Hence $\tilde{f}(y)$ the endpoint of the unique lift of $f \circ \alpha$ to E starting at e_0.

This also shows how you can define $\tilde{f}$: choose a path α from y_0 to y. Lift $f \circ \alpha$ to a path starting at e_0. Define $\tilde{f}(y) =$ the end point of this lift.

Problem: is this well defined? A second problem: is $\tilde{f}$ continuous?

- Well defined:

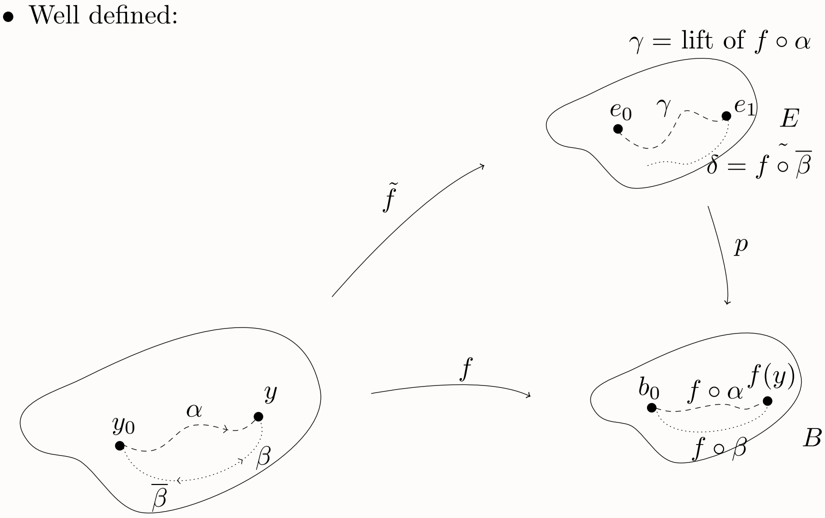


Figure 12.2: well defined general lifting lemma

As $[\alpha] * [\overline{\beta}] \in \pi(Y, y_0)$

$$f_*([\alpha] * [\overline{\beta}]) = ([f \circ \alpha] * [f \circ \overline{\beta}]) \in f_*(\pi_1(Y, y_0)),$$

which is by assumption a subgroup of $p_*(\pi(E, e_0)) = H$.

And now, by Lemma 54.6 (c), a loop in the base space lifts to a loop in E if the loop is in H. This lift is of course just $\gamma * \delta$, so the end points in the drawing should be connected! This means that $\overline{\delta}$ is *the* lift of $f \circ \beta$ starting at e_0, so the endpoint of the lift of $f \circ \beta$ is the endpoint of the lift of $f \circ \alpha$. Therefore $\tilde{f}(y)$ is well defined.

- Note that we didn't use the locally path connectedness yet. We'll need this for continuity. To be continued ...

□

Lecture 5: The general lifting Lemma

di 05 nov 10:22

Recap:

Lemma 10 (74.1! (General Lifting lemma))**.** Let $p : E \to B$ be a covering, Y a space. Assume B, E, Y are path connected, and locally path connected.[a]

Let $f : Y \to B$, $y_0 \in Y, b_0 = f(y_0)$. Let $e_0 \in E$ such that $p(e_0) = b_0$. Then $\exists \tilde{f} : Y \to E$ with $\tilde{f}(y_0) = e_0$ and $p \circ \tilde{f} = f$

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (B, b_0) \end{tikzcd}.$$

iff $f_*(\pi(Y, y_0) \subset p_*\pi(E, e_0)$. If $\tilde{f}$ exists then it is unique.

[a]From now on, all spaces are locally path connected: Every neighbourhood contains an open that is path connected

Proof. We prove that $\tilde{f}$ is continuous.

- Choose a neighborhood of $\tilde{f}(y_1)$, say N.
- Take U, a path connected open neighbourhood of $f(y_1)$ which is evenly covered, such that the slice $p^{-1}(U)$ containing $\tilde{f}(y_1)$ is completely contained in N.

  Can we do this? The inverse image of U is a pile of pancakes. One of these pancakes contains $\tilde{f}(y_1)$. Then, because N is a neighborhood of $\tilde{f}(y_1)$, we can shrink the pancake such that it is contained in N
- Choose a path connected open which contains y_1 such that $f(W) \subset U$. We can do this because of continuity of f.
- Take $y \in W$. Take a path β in W from y_1 to y. (Here we use that W is path connected) Now consider $p|_V$ and defined Then $\alpha * \beta$ is path from y_0 to y, $f \circ (\alpha * \beta) = (f \circ \alpha) * (f \circ \beta)$. Then $\widetilde{f \circ \alpha} * (p^{-1}|_V \circ f \circ \beta)$ is the lift of $f \circ (\alpha * \beta)$ starting at y_0. So by definitie of $\tilde{f}$, we have

that $\tilde{f}(y)$ is the endpoint of that lift, which belongs to $V \subset N$. This means that $\tilde{f}(W) \subset N$, which proves continuity.

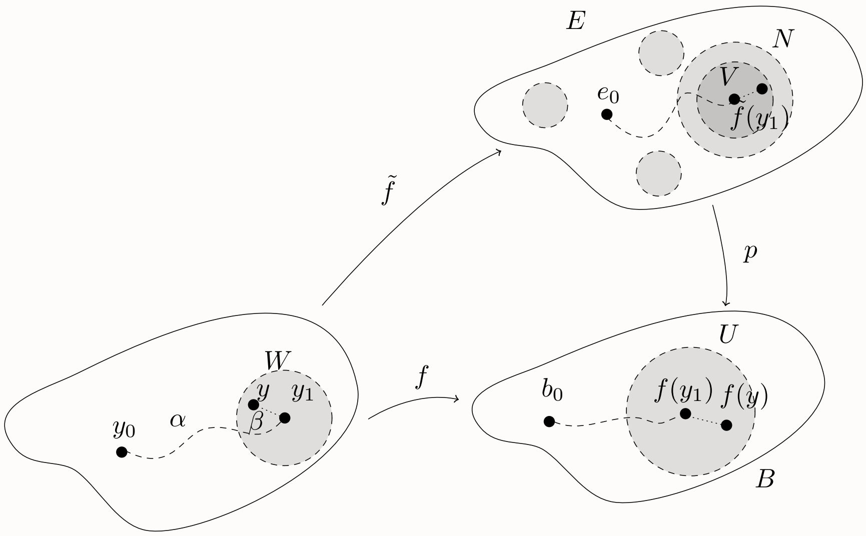


Figure 12.3: Proof of the continuity of the general lifting lemma

□

This is a powerful lemma: 'There exists a lift $\tilde{f}$ such that ...' is pure topology, and $f_*\pi(Y, y_0) \subset p_*\pi(E, e_0)$ is pure algebra. It's very unusual that two statements like this are completely equivalent.

Recap:

Lemma 11 (General lifting lemma, short statement)**.** Short statement:

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (B, b_0) \end{tikzcd}$$

$\exists! \tilde{f} \Leftrightarrow f_*\pi(Y, y_0) \subset p_*\pi(E, e_0)$.

Definition 14. Let (E, p) and (E', p') be two coverings of a space B. An equivalence between (E, p) and (E', p') is a homeomorphism $h : E \to E'$ such that

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[dr, "p"'] & E' \arrow[d, "p'"] \\ & B \end{tikzcd}$$

is commutative. $p' \circ h = p$.

Theorem 14 (74.2)**.** Let $p : (E, e_0) \to (B, b_0)$ and $p' : (E', e_0') \to (B, b_0)$ be coverings, and let $H_0 = p_*\pi(E, e_0)$ and $H_0' = p_*'\pi(E', e_0') \leq \pi(B, b_0)$. Then there exists an equivalence $h : (E, p) \to (E', p')$ with $h(e_0) = e_0'$ iff $H_0 = H_0'$. Not isomorphic, but really the same as a subgroup of $\pi(B, b_0)$. in that case, h is unique.

Proof. $\boxed{\Rightarrow}$ Suppose h exists. Then

$$\begin{tikzcd} (E, e_0) \arrow[r, "h"] \arrow[dr, "p"'] & (E', e_0') \arrow[d, "p'"] \\ & (B, b_0) \end{tikzcd}$$

Therefore $p_*\pi(E, e_0) = p_*'(h_*\pi(E, e_0))$. Since h is a homeomorphism, it induces an isomorphism, so $p_*'(h_*\pi(E, e_0)) = p *' (\pi(E', e_0')$

$\boxed{\Leftarrow}$

$$\begin{tikzcd} & (E', e_0') \arrow[d, "p'"] \\ (E, e_0) \arrow[ur, dashed, "k"] \arrow[r, "p"] & (B, b_0) \end{tikzcd}$$

By the previous lemma, there exists a unique k iff $p_*\pi(E, e_0) \subset p_*'\pi(E', e_0')$ or equivalently $H_0 \subset H_0'$, which is the case. Reversing the roles, we get

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (E', e_0') \arrow[ur, "l"] \arrow[r, "p'"] & (B, b_0) \end{tikzcd}$$

for the same reasoning, l exists. Now, composing the diagrams

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (E, e_0) \arrow[ur, "l \circ k"] \arrow[r, "p"] & (B, b_0) \end{tikzcd} \qquad \begin{tikzcd} & (E', e_0') \arrow[d, "p'"] \\ (E', e_0') \arrow[ur, "k \circ l"] \arrow[r, "p'"] & (B, b_0) \end{tikzcd}$$

But placing the identity in place of $l \circ k$ or $k \circ l$, this diagram also commutes! By unicity, we have that $l \circ k = 1_E$ and $k \circ l = 1_{E'}$. Therefore, k and l are homeomorphism $k(e_0) = e_0'$.

Uniqueness is trivial, because of the general lifting theorem.

□

Note that this doesn't answer the question 'is there a equivalence between two coverings', it only answers the question 'is there an equivalence between two coverings mapping $e_0 \to e_0'$'. So now, we seek to understand the dependence of H_0 on the base point.

Lemma 12. Let (E,p) be a covering of B. Let $e_0, e_1 \in p^{-1}(b_0)$. Let $H_0 = p_*\pi(E,e_0), H_1 = p_*\pi(E,e_1)$.

- Let γ be a path from e_0 to e_1 and let $p \circ \gamma = \alpha$ be the induced *loop* at b_0. Then $H_0 = [\alpha] * H_1 * [\alpha]^{-1}$, so H_0 and H_1 are conjugate inside $\pi(B,b_0)$.
- Let H be a subgroup of $\pi(B,b_0)$ which is conjugate to H_0, then there is a point $e \in p^{-1}(b_0)$ such that $H = p_*\pi(E,e)$.

So a covering space induces a conjugacy class of a subgroup of $\pi(B,b_0)$.

Proof.

- Let $[h] \in H_1$, so this means that $h = p \circ \tilde{h}$, where $\tilde{h}$ is a loop based at e_1. Then $(\gamma * \tilde{h}) * \overline{\gamma}$ is a loop based at e_0. This means that the path class $[p((\gamma * \tilde{h}) * \overline{\gamma})] \in H_0$. This means that $[p \circ \gamma] * [h] * [p \circ \overline{\gamma}] \in H_0$, or $[\alpha] * [h] * [\alpha]^{-1} \in H_0$. So we showed that if we take any element of H_1 and we conjugate it with α, we end up in H_0, so $[\alpha] * H_1 * [\alpha]^{-1} \subset H_0$

  For the other inclusion, consider $\overline{\gamma}$ going from $e_1 \to e_0$. The same argument shows that $[\alpha]^{-1} * H_0 * [\alpha] \subset H_1$, or $H_0 \subset [\alpha] * H_1 * [\alpha]^{-1}$. This proves that $H_0 = [\alpha] * H_1 * [\alpha]^{-1}$.

- Take $H = [\beta] * H_0 * [\beta]^{-1}$ for some $[\beta] \in \pi(B,b_0)$. So $H_0 = [\beta]^{-1} * H * [\beta]$. Take $\alpha = \overline{\beta}$. Then $H_0 = [\alpha] * H * [\alpha]^{-1}$, where α, β are loops based at b_0. Let γ be the unique lift of α starting at e_0. Take $e = \gamma(1)$, the end point of γ. (So $p(e) = b_0$) From the first bullet point, it follows that $p_*\pi(E,e) = H'$ satisfies $H_0 = [\alpha] * H' * [\alpha]^{-1}$. So we have both $H_0 = [\alpha] * H * [\alpha]^{-1} = H_0 = [\alpha] * H'[\alpha]^{-1}$. This implies that $H' = H$.

□

This completely answers the question: 'When are two covering spaces equivalent':

Corollary (Theorem 74.4)**.** Let (E,p) and (E',p') be two coverings, $e_0 \in E, e_0' \in E'$ with $p(e_0) = p(e_0') = b_0$. Let $H_0 = p_*\pi(E,e_0)$, $H_0' = p_*'\pi(E',e_0')$. Then (E,p) and (E',p') are equivalent *iff* H_0 and H_0' are conjugate inside $\pi(B,b_0)$.

Question: can we reach every possible subgroup? Answer: yes, in some conditions.

12.75 Universal covering space

Definition 15. Let B be a path connected and locally path connected space. A covering space (E,p) of B is called *the*[a] universal covering space *iff* E is simply connected, so $\pi(E,e_0) = 1$.

[a] upto equivalence

Remark. Any two universal coverings are equivalent. Even more, we can choose any base point we want.

$$\begin{tikzcd}
(E, e_0) \arrow[r, dashed, "h(e_0)=e_0'"] \arrow[dr, "p"] & (E', e_0') \arrow[d, "p'"] \\
& (B, b_0)
\end{tikzcd}$$

h exists because the groups of (E, e_0) and (E', e_0') are trivial.

Remark. We don't cover Lemma 75.1, and we only cover point (a) from 75.2

Lemma 13 (75.2)**.** Suppose

$$\begin{tikzcd}
X \arrow[dd, "p"] \arrow[dr, "q"] & \\
& Y \arrow[dl, "r"] \\
Z &
\end{tikzcd}$$

If p and r are covering maps, then also q is a covering map. (Also: if q and p are covering maps, then so is r. Not the case for $q, r \Rightarrow p$!)

Proof.

- q is a surjective map. Choose a base point in x_0, and call $y_0 = q(x_0)$, $z_0 = r(y_0)$. Certainly, y_0 lies in the image of q. Now, take $y \in Y$, and choose a path $\tilde{\alpha}$ from y_0 to y. Now, denote by α the projection of $\tilde{\alpha}$, a path from z_0 to $r(y)$. Let $\tilde{\tilde{\alpha}}$ be the unique lift of α to X starting at x_0. This is defined as we assume that p is a covering map. Then $q \circ \tilde{\tilde{\alpha}}$ is a path starting at $q(\tilde{\tilde{\alpha}}(0)) = q(x_0) = y_0$. Moreover, $q \circ \tilde{\tilde{\alpha}}$ is a lift of $\alpha = r \circ \tilde{\alpha}$ to Y. Indeed consider the projection, $r \circ q \circ \tilde{\tilde{\alpha}} = p \circ \tilde{\tilde{\alpha}} = \alpha$. Of course, $\tilde{\alpha}$ is also a lift from α starting at y_0. Since r is assumed to be a covering, and lifts of paths are unique, we get that $q \circ \tilde{\tilde{\alpha}} = \tilde{\alpha}$, so the end points are the same: $q(\tilde{\tilde{\alpha}}(1)) = \tilde{\alpha}(1) = y$, so y lies in the image of q.

  The only fact we've used is that q is a continuous map, so that $q \circ \tilde{\tilde{\alpha}}$ is again a path.

- Now we show that every point of y has a neighbourhood that is evenly covered. Choose $y \in Y$ and project it down to Z. $r(y)$ has a neighbourhood U that is evenly covered by p, and also by r. Now we can shrink it so that is is evenly covered by both covering maps. We can also choose it to be path connected.

  So $p^{-1}(U) = \bigcup_{\alpha \in I} U_\alpha$, and $r^{-1}(U) = \bigcup_{\beta \in J} V_\beta$. Let V be the slice containing Y. Then we claim that V will be evenly covered by U.

Consider a U_α. Then $q(U_\alpha)$ is connected and contained in $\bigcup_{\beta \in J} V_\beta$, but all these V_β are disjoint, so there is exactly one V_β such that $q(U_\alpha) \subset V_\beta$.

Now, let $I' = \{\alpha \mid q(U_\alpha) \subset V\}$. For any $\alpha \in I'$, we have the diagram

$$\begin{tikzcd} U_\alpha \arrow[dd, "p"] \arrow[dr, "q"] & \\ & V \arrow[dl, "r"] \\ U & \end{tikzcd}$$

As r and p is a homeomorphism, q is also a homeomorphism. Hence $q^{-1}(V) = \bigcup_{\alpha \in I'} U_\alpha$, and $q|_{U_\alpha} : U_\alpha \to V$is a homeomorphism.

This means that q is a covering projection.

□

Why is this useful? Because now we can say why the universal covering space is a universal covering space.

Theorem 15 (57.3)**.** Let (E, p) be a universal covering of B. Let (X, r) be a another covering of B. Then there exists a map $q : E \to X$ such that $r \circ q = p$ and q is a covering map.

$$\begin{tikzcd} E \arrow[dd, "p"] \arrow[dr, dashed, "q"] & \\ & X \arrow[dl, "r"] \\ B & \end{tikzcd}$$

Every covering space is itself covered by the universal covering space, *if* it exists.

Proof. Drawing the diagram differently,

$$\begin{tikzcd} & X \arrow[d, "r"] \\ E \arrow[ur, dashed, "q"] \arrow[r, "p"] & B \end{tikzcd}$$

Choose e_0, x_0 mapped to $b_0 \in B$. Then $\pi(E, e_0) = 1 \subset r_* \pi(X, x_0)$. Then there exists a map q by the general lifting lemma. So q makes the diagram commutative. By the previous result, q is a covering map. □

Covering transformations

Definition 16 (Group of covering transformation)**.** Let (E, p) be a covering of B. We define

$$C(E,p,B) = \{h : E \to E \mid h \text{ is an equivalence of covering spaces}\}$$

Elements of this set are homeomorphism h such that $p \circ h = p$. The composition of two such elements is again such an elements, same for inverse. This means that C is a group, the group of covering transformations, also called Deck-transformations.

Example. Consider the covering space $\mathbb{R} \to S^1 : t \mapsto e^{2\pi i t}$. For any $z \in \mathbb{Z}$, there is a map $h_z : \mathbb{R} \to \mathbb{R} : r \mapsto r + z$, which is a covering transformation. Indeed $e^{2\pi i t} = e^{2\pi i(t+z)}$. Claim: these are the only covering transformations. Conclusion: $C(\mathbb{R}, p, S^1) = (\mathbb{Z}, +)$

Proof. Suppose $h : \mathbb{R} \to \mathbb{R}$ is another covering transformation. We certainly have $h(0) = z$ for some $z \in \mathbb{Z}$. Therefore, $h(0) = h_z(0)$, from this follows immediately that $h \equiv h_z$.

Why? 'If two covering transformations agree in one point, the agree everywhere.' Indeed, $h_1, h_2 \in C(E,p,B)$ and $h_1(e) = h_2(e) \Rightarrow h_1 \equiv h_2$, because

$$\begin{tikzcd} & E \arrow[d] \\ E \arrow[ur, "h_1 \text{and} h_2"] \arrow[r, "p"] & B \end{tikzcd}$$

and, h_1 and h_2 are both lifts of p and there is a unique lift when fixing the base point, so h_1 and h_2 agree.

Goal: what is the structure of the group $C(E,p,B)$ in terms of fundamental groups? Let (E,p) be a covering of B. $p(e_0) = b_0$, $H_0 = p_*\pi(E, e_0)$. Remember:

$$\begin{aligned} \Phi : \pi(B,b_0)/H_0 &\longrightarrow p^{-1}(b_0) \\ H_0 * [\alpha] &\longmapsto \tilde{\alpha}(t) \end{aligned}$$

is a bijection, where $\tilde{\alpha}$ is the unique lift of α starting at e_0.

Now consider $\psi : C(E,p,B) \to p^{-1}(b_0) : h \mapsto h(e_0)$. ψ is injective. Reason: same as before, if they agree on one point, these are the same. In general ψ will not be surjective.

Lemma 14. $\operatorname{Im}\psi = \Phi\left(\dfrac{N_{\pi(B,b_0)}(H_0)}{H_0}\right)$, where

$$N_{\pi(B,b_0)}(H_0) = \{[\alpha] \in \pi(B,b_0) \mid [\alpha] * H_0 * [\alpha]^{-1} = H_0\},$$

which is the largest subgrouop of $\pi(B,b_0)$ in which H_0 is normal. 'Normaliser'.

We'll later show that this group has the same structure as $C(E,p,B)$.

Proof. Consider $H_0 * [\alpha]$.[a] Then $\Phi(H_0 * [\alpha]) = \tilde{\alpha}(1)$, where $\tilde{\alpha}$ is the lift of α starting at e_0. Let's denote $\tilde{\alpha}(1) = e_1$. Question: which of these elements lie in the image of ψ.

e_1 lies in the image of ψ iff there exists a covering transformation h sending e_0 to e_1, which is equivalent to $H_0 = H_1 = p_*\pi(E, e_1)$. On the other hand, we also know that $H_0 = [\alpha] * H_1 * [\alpha]^{-1}$. Conclusion: e_1 lies in the image of ψ iff $H_0 = [\alpha] * H_0 * [\alpha]^{-1}$, iff $[\alpha] \in N_{\pi(B,b_0)}(H_0)$. $\square$

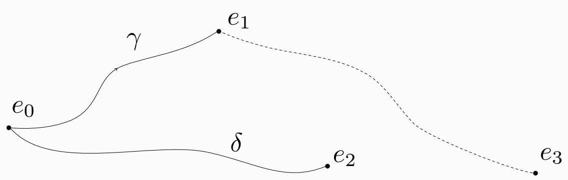
[a] In the book, they use $[\alpha] * H_0$. This is not wrong, as for elements in the normalizer, left and right cosets are the same, so writing $[\alpha] * H_0$ is allowed. But in general, we write $H_0 * [\alpha]$.

This means we have the following situation:

$$\begin{array}{ccccc} C(E,p,B) & \overset{\psi}{\rightarrowtail\!\!\!\twoheadrightarrow} & \operatorname{Im}\psi \subset p^{-1}(b_0) & \overset{\Psi^{-1}}{\rightarrowtail\!\!\!\twoheadrightarrow} & \dfrac{N_{\pi(B,b_0)}(H_0)}{H_0} \\ h & \longmapsto & h(e_0) = e_1 & \longmapsto & H_0 * [\alpha],\ \alpha = p \circ \gamma \end{array}$$

Theorem 16. The map $\Phi^{-1} \circ \psi : C(E,p,B) \to N_{\pi(B,b_0)}(H_0)/H_0$ is an isomorphism of groups.

Proof. Let $h, k \in C(E,p,B)$ with $h(e_0) = e_1$ and $k(e_0) = e_2$.



Then

$$(\Phi^{-1} \circ \psi)(h) = H_0 * [\alpha] \quad \alpha = p \circ \gamma$$
$$(\Phi^{-1} \circ \psi)(k) = H_0 * [\beta] \quad \beta = p \circ \delta.$$

Then call $h(k(e_0)) = h(e_2) = e_3$. Claim: $(h \circ \delta)(0) = h(e_1) = e_1$. $(h \circ \delta)(1) = h(e_2) = e_4$. Then $\varepsilon := \gamma * (h \circ \delta)$ is a path from e_0 to e_3. This implies that $(\Phi^{-1} \circ \psi)(h \circ k) = H_0 * [p \circ \varepsilon]$. Then $p \circ \varepsilon = (p \circ \gamma) * (p \circ h \circ \delta) = \alpha * (p \circ \delta) = \alpha * \beta$. This means that $(\Phi^{-1} \circ \psi)(h \circ k) = H_0 * [\alpha * \beta]$.

This shows that this is indeed a morphism. $\square$

What can we do with this? Some covering spaces are nice: e.g. when the normalizer is the entire group. (Which is the case when abelian)

Definition 17. A covering space (E, p) of B is called regular if H_0 is normal in $\pi(B, b_0)$, where $b_0 \in B, p(e_0) = b_0, H_0 = p_*\pi(E, e_0)$.

This is the case *iff* for every $e_1, e_2 \in p^{-1}(b_0)$, there exists an $h \in C(E,p,B)$ such that $h(e_1) = e_2$.

$$\begin{tikzcd}
(E, e_1) \arrow[r, "h"] \arrow[dr, "p"'] & (E, e_2) \arrow[d, "p"] \\
& B
\end{tikzcd}$$

This h exists iff $H_1 = H_2$. $H_1 = p_*\pi(E, e_1)$ and $H_2 = p_*\pi(E, e_2)$. If H_0 is normal. Then H_1 and H_2 are the same as H_0, because they are conjugate to H_0. So for H_1, H_2 the h exists. Conversely: exercise.

In this case, $(\Phi^{-1} \circ \psi) : C(E, p, B) \to \frac{\pi(B, b_0)}{H_0}$ is an isomorphism. Special case: Let (E, p) be the universal covering space, so $\pi(E, e_0) = 1$, or $H_0 = 1$. In this case, $\Phi^{-1} \circ \psi : C(E, p, B) \to \pi(B, b_0)$ is an isomorphism.

Example. $C(\mathbb{R}, p, S^1) \cong \mathbb{Z} \cong \pi(S^1, b_0)$.

Example. Consider $S^2 \xrightarrow{p} \mathbb{R}P^2$. $C(S^2, p, \mathbb{R}P^2) = \{1_{S^2}, -1_{S^2}\}$, because take a point $e_0 \in S^2$, then under a covering transformations, it can only be mapped to e_0 or to $-e_0$. So these are the only covering transformations. And indeed, $\pi(\mathbb{R}P^2) = \mathbb{Z}_2$

Note that $\mathbb{R}P^2 \cong \frac{S^2}{\sim}$, where $\sim$ is in a sense defined by the groupactions of 1_{S^2} and -1_{S^2}. More on this next week.

Lemma 15 (75.4)**.** If B admits a universal covering space (E, p), then any element b of B has a neighbourhood U such that

$$\begin{aligned} i_8 : \pi_1(U, b) &\longrightarrow \pi_1(B, b) \\ [\alpha] &\longmapsto [1]. \end{aligned}$$

is trivial. So the loop doesn't have to be trivial in E, but is is in B.

Proof. Take U evenly covered. $\tilde{\alpha} \simeq_p e_E$ in E, so $p \circ \tilde{\alpha} \simeq_p p \circ e_e$ in B so $\alpha \simeq_p e_b$, so $i_*[\alpha] = [e_b]$.

So this condition is needed for a universal covering spaces. And fun fact: this is sufficient. This is proved in section 77, but we don't need to know this.

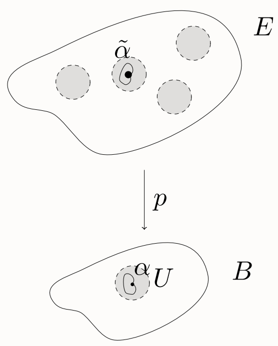


Figure 12.4: lemma universal covering space

□

We call this property semi-locally simply connected. The loops themself don't have to be simply connected, but in the bigger space they are.

Lecture 6

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Recap: $p : E \to B$ and $C(E, p, B) = \{h : E \to E \mid \text{h homeo}, p \circ h = h\}$. $C(E, p, B) \cong \frac{N_{\pi(B,b_0)}(H_0)}{H_0}$, where $H_0 = p_* \pi(E, e_0)$.

When $p : E \to B$ is regular (H_0 is normal in $\pi(B, b_0)$), then

$$C(E, p, B) \cong \frac{\pi(B, b_0)}{H_0}.$$

In particular when E is simply connected,

$$C(E, p, B) \cong \pi_1(B, b_0).$$

$p : E \to B$ is rigular iff (for all $e_1, e_2 \in E$ such that $p(e_1) = p(e_2) \Rightarrow \exists h \in C(E, p, B) : h(e_1) = e_2$)

Now let's talk about group actions. Let X be a space and $G \leq$ Homeo(X). G acts on X: $gx = g(x)$. We can consider X/G, the quotient space consisting of all orbits of this action with the quotient topology. Let $\pi : X \to X/G$ denote the natural projection: $x \mapsto Gx$. So $O \subset X/G$ is open iff $\pi^{-1}(O)$ is open in X

Definition 18. G acts *properly discontinuously (nice)* on X iff for all $x \in X$, there exists neighborhood $U(x)$ open such that $gU \cap U \neq 0 \Rightarrow g = 1$, or in other words, $g_1 U \cap g_2 U \neq \emptyset \Rightarrow g_1 = g_2$.

Remark. In the literature: many definition of non-equivalent properly discontinuous.

Remark. If G acts properly discontinuously, then G acts *freely*, i.e. $gx = x \Rightarrow g = 1$, i.e. there are no fixed points.

Theorem 17 (76.5, important!). Let $G \leq \text{Homeo}(X)$. Then $\pi : X \to X/G$ is a covering map iff G acts properly discontinuously. Moreover, when π is a covering, then it is a regular covering and G *is*[a] the group of covering transformations.

[a]not isomorphic to, it *is*

Proof. Remark that π is an open map. Indeed, let $U \subset X$ be open. Then $\pi(U)$ is open, since $\pi^{-1}(\pi(U)) = \bigcup_{g\in G} g(U)$ is open, as g is homeo.

Step 1. Assume G acts properly discontinuously on X. We have to show that π is a covering map. Let $\overline{x} = \pi(x)$ be an element of X/G. Let $U(x)$ be an open such that $gU \cap U = \emptyset$ if $g \neq 1$ (definition of properly discontinuous action) π is an open map, so $V = \pi(U)$ is an open containing $\overline{x}$, and $\pi^{-1}(V) = \bigcup_{g\in G} gU$. We want to show that V is evenly covered. Take as slices gU, which are disjoint since $g_1U \cap g_2U = \emptyset$ if $g_1 \neq g_2$. Consider $\pi : U \to V$

- Continuous
- Bijective (surjective, injective)
- Open,

which means it's an homeomorphism. Of course, $\pi : gU \to V$ is also a homomorphism, as it is a composition of g^{-1} and π, both of which are homeo. So we have a covering map

Step 3. Suppose π is a covering.

- Certainly, G is a subgroup of the group of covering transformations.
- Let $h \in C(x, \pi, X/G)$ and $x_1 \in X$. Then $\pi h(x_1) = \pi(x_1)$, so the orbit of $h(x_1)$ is the orbit of x_1, so there exists a $g \in G$ such that $g(x_1) = h(x_1)$. Both of them are covering transformations, $g(x_1) = h(x_1)$. From previous results, we conclude that $g = h$. Therefore $G = C(X, \pi, X/G)$.
- Let $x_1, x_2 \in X$ with $\pi(x_1) = \pi(x_2)$, then there exists a $g \in G$ such that $g(x_1) = x_2$, so π a regular covering.

Step 2. Reverse implication Suppose $X \to X/G$ is a regular covering. We have to show that the action is properly discontinuous. Let $x \in X$. Take $\overline{x} = \pi(x)$, and let V be a path connected open containing $\overline{x}$, which is evenly covered. $\pi^{-1}(V) = \bigsqcup_{\alpha\in I} U_\alpha$, and $\pi|_{U_\alpha} : U_\alpha \to V$ is homeo. Let U be the slice U_α which contains x. Then U is an open containing x, and we claim this is 'the good one' for the definition of properly discontinuous. Suppose $u \in g(U) \cap U$, so u is an element of U and also an element of the form gu' for some $u' \in U$. This implies

$\pi(u') = \pi(gu') = \pi(u)$. Since $\pi|_U$ is a homeomorphism, $u' = u$. This shows that $g(u) = u$. By Step 3, g is a covering transformation. Therefore g is the identity map, because that's the only covering transformation that has a fixed point.

□

Corollary. Let X be simply connected and $G \leq \mathrm{Homeo}(X)$ acting properly discontiously. Then $\pi : X \to X/G$ is a regular covering and $\pi_1(X/G) = C(x, \pi, X/G) = G$.

Remark (Exercise, Thm 76.6)**.** If $p : X \to B$ is a regular covering and $G = C(E, p, B)$, then B is homeomorphic to X/G.

If B admits a universal covering space, then $B = E/G$ where E is the universal covering space space and $G \cong \pi(B, b_0)$. So if a space has a universal covering space, it's always of the form E/G.

Example. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x+1, y)$. Let $\gamma : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x, y+1)$. Let $G = \mathrm{grp}\{\alpha, \gamma\}$, an Abelian group. Every element of G is of the form $\alpha^k\gamma^\ell(x, y) = (x + k, y + \ell)$, so $\alpha^k\gamma^\ell = \alpha^{k'}\gamma^{\ell'} \Leftrightarrow k = k', \ell = \ell'$, so $G \cong \mathbb{Z}^2$.

G acts properly discontinuously on $\mathbb{R}^2$. Therefore $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, x) \cong \mathbb{Z}^2$. $\mathbb{R}^2/\mathbb{Z}^2$ is the torus.

Example. Take $\alpha : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x + 1, y)$. Take $\beta : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \to (-x, y + 1)$. Note that $\beta^2 = \gamma^2$. Claim: $\alpha \circ \beta = \beta \circ \alpha^{-1}$. Note that $\alpha^{-1}\beta = \beta\alpha$, $\alpha\beta^{-1} = \beta^{-1}\alpha^{-1}$ and $\alpha^{-1}\beta^{-1} = \beta^{-1}\alpha$, so we can always move α to the front, which means that

$$G = \mathrm{grp}\{\alpha, \beta\} = \{\alpha^k\beta^l \mid k, l \in \mathbb{Z}\}.$$

note

$$\alpha^k\beta^\ell(x, y) = ((-1)^\ell x + k, y + \ell).$$

Therefore $\alpha^k\beta^l = \alpha^{k'}\beta^{l'} \Leftrightarrow k = k' \wedge l = l'$. As a set $G = \mathbb{Z}_2$, but not isomorphic.

Claim: G acts properly discontinuously on $\mathbb{R}^2$. Indeed consider $B(x, \frac{1}{4})$. So

$$\pi_1(\mathbb{R}^2/G) = G = \left\langle \alpha, \beta \mid \beta\alpha = \alpha^{-1}\beta \right\rangle.$$

This is the Klein bottle!

Remark. These two spaces are the only possible spaces you can create using affine actions. Projective plane? Not properly discontinuously?

Chapter 13

Singular Homology

This is the most general homology.

- A point is a 0-simplex
- A line segment is a 1-simplex
- A triangle (including interior) is a 2-simplex
- A filled in tetrahedron is a 3-simplex.

Definition 19 (Simplex)**.** In general a p-simplex is the convex hull of $p+1$ points in general position in $\mathbb{R}^n$ (affine independent)

Proposition 3 (1.1)**.** Let $x_0, \ldots, x_p \in \mathbb{R}^n$, $(p \geq 1)$ then the following are equivalent:

- The vectors $x_1 - x_0, \ldots, x_p - x_0$ are linearly independent
- If $\sum_{i=0}^p s_i x_i = \sum_{i=0}^p r_i x_i$ and $\sum_{i=0}^p s_i = \sum r_i$, then $s_i = r_i$ for all $i = 0, \ldots, p$.

Proof.

- Assume that $\sum_{i=0}^p s_i x_i = \sum_{i=0}^p r_i x_i$ and $\sum_{i=0}^p s_i = \sum r_i$. Then we can multiply the last expression by x_0. Then

$$\sum_{i=0}^{p} s_i(x_i - x_0) = \sum_{0}^{p} r_i(x_i - x_0)$$
$$\sum_{i=1}^{p} s_i(x_i - x_0) = \sum_{1}^{p} r_i(x_i - x_0).$$

Therefore for all $i = 1, \ldots, p \Rightarrow s_i = r_i$ and therefore also $s_0 = r_0$.

- Assume $\sum_{i=1}^p r_i(x_i - x_0) = 0$. Then $\sum_{i=1}^p r_i x_i = \sum_{i=1}^p r_i x_0$. This implies that $r_0 = 0$, $r_1 = \ldots = r_p = 0$ and $s_1 = \ldots = s_p = 0$.

□

Proposition 4. Let $x_0, \dots, x_p$ satisfy the conditions of the proposition. Then any point x in the convex hull of these points has a unique representation of the form

$$x = \sum t_i x_i \qquad t_i \geq 0 \text{ and } \sum t_i = 1.$$

This convex hull is the p-simplex spanned by $x_0, \dots, x_p$.

Proof. Exercise. Hint: consider $p = 0, p = 1$, and do the rest by induction. □

Definition 20. The standard p-simplex σ_p is the (ordered) p-simplex in $\mathbb{R}^{p+1}$ spanned by

$$\begin{aligned} x_0 &= (1, 0, 0, \dots, 0) \\ x_1 &= (0, 1, 0, \dots, 0) \\ x_p &= (0, 0, \dots, 0, 1). \end{aligned}$$

$$\sigma_p = \{(t_0, \dots, t_p) \mid t_i > 0 \text{ and } \sum t_i = 0\}.$$

Remark. Any p-simplex is homeomorphic to the standard p-simplex.

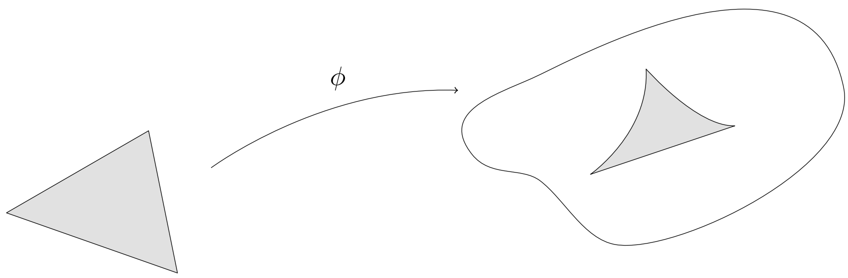
$$\begin{aligned} h : \sigma_p &\longrightarrow p\text{-simplex} \\ (t_0, \dots, t_p) &\longmapsto \sum t_i x_i. \end{aligned}$$

This is a homomorphism because this is a continuous map between Hausdorff, compact spaces.

Definition 21. A singular p-simplex in a topological space X is a continuous map

$$\phi : \sigma_p \longrightarrow X.$$

In the case of $p = 2$, we get the following:



Note that in the case of $p = 0$, $\phi : \sigma_0 = \{x_0\} \to X : x_0 \mapsto \phi(x_0) \in X$. So zero-simplices are identified with points of X.

In the case of $p = 1$, we get $\sigma_1 = I$, so a singular 1-simplex is a path in X.

For the moment, we don't assume spaces are connected, etc.

Definition 22. Let X be a topological space. Then $S_n(X)$ is the free abelian group on all singular n-simplices. This is a giant group. By the definition of a free abelian group, elements of $S_n(X)$ are of the form $\sum n_\phi \phi$, where

- the sum is finite,
- $n_\phi \in \mathbb{Z}$,
- $\phi : \sigma_n \to X$.

Let $\phi : \sigma_p \to X$ be a singular p-simplex, and $i \in \{0, \ldots, p\}$. Then the i-th boundary of ϕ, denoted by $\partial_i \phi$ is the singular $p - 1$ simplex.

$$\partial_i \phi : \sigma_{p-1} \to X : (t_0, \ldots t_{p-1}) \mapsto \phi(t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{p-1}).$$

This is ϕ restricted to the face opposite to x_i.

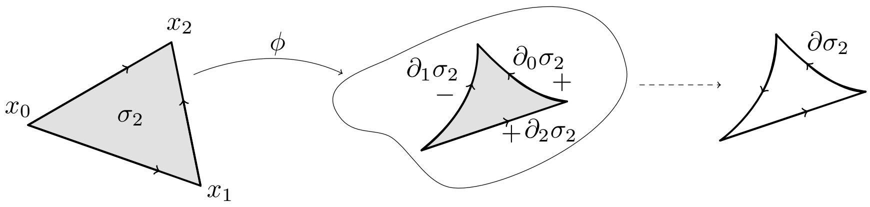


Figure 13.1: Boundary operator

Now, define

$$\partial : S_n(X) \to S_{n-1}(X) : \sum n_\phi \phi \mapsto \sum n_\phi (\partial \phi),$$

where

$$\partial \phi = \sum (-1)^i \partial_i \phi \in S_{n-1}(X).$$

Then certainly, ∂ is a morphism of abelian groups. In the case of the figure, we get

$$\partial \phi = \partial_0 \phi - \partial_1 \phi + \partial_2 \phi,$$

which really does goes around the simplex in the correct direction. The $(-1)^i$ fixes the direction of the simplices.

Example. Let $\phi : \sigma_1 \to X$ be a 1-simplex. Then $\partial \phi = \phi(x_1) - \phi(x_0)$.

Example. Let ϕ be a two simplex. Then $\partial(\partial \phi) = 0$.

Remark. We define $S_{-n} = 0$ for $n \geq 0$. Then $\partial : S_0(X) \to S_{-1}(X) = 0 : \psi \mapsto 0$.

Proposition 5 (1.3). $\partial \circ \partial = 0$.

Proof. Exactly the same thing as in the example. $\square$

Let X be a topological space.

$$\xrightarrow{\partial} S_{n+1}(X) \xrightarrow{\partial} S_n(X) \xrightarrow{\partial} S_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \xrightarrow{\partial} 0 \xrightarrow{\partial} 0 \xrightarrow{\partial} \cdots .$$

We do know that $\partial \circ \partial = 0$. Suppose we start with $\alpha \in S_{n+1}$. Then $\partial\alpha$ belongs to the kernel of the "next" ∂. This shows that the image of ∂ is a subset of the kernel of the "next" ∂. This holds at any spot. We call this a chain complex of abelian groups. Define

$$\begin{aligned} Z_n(X) &= \text{the } n\text{-cycles of } X = \text{kernel of } \partial : S_n \to S_{n-1} \\ B_n(X) &= \text{the } n\text{-boundaries of } X = \text{the image of } \partial S_{n+1} \to S_n. \end{aligned}$$

We claimed that $B_n \subset Z_n$.

Definition 23. The n-th singular homology group of X is $H_n(X) = \frac{Z_n(X)}{B_n(X)}$. Two cycles are the same if they differ with a boundary.

A cycle is more or less a loop. Then two cycles are the same when they differ with a boundary, in this case the boundary of the triangles indicated with the dashed lines.

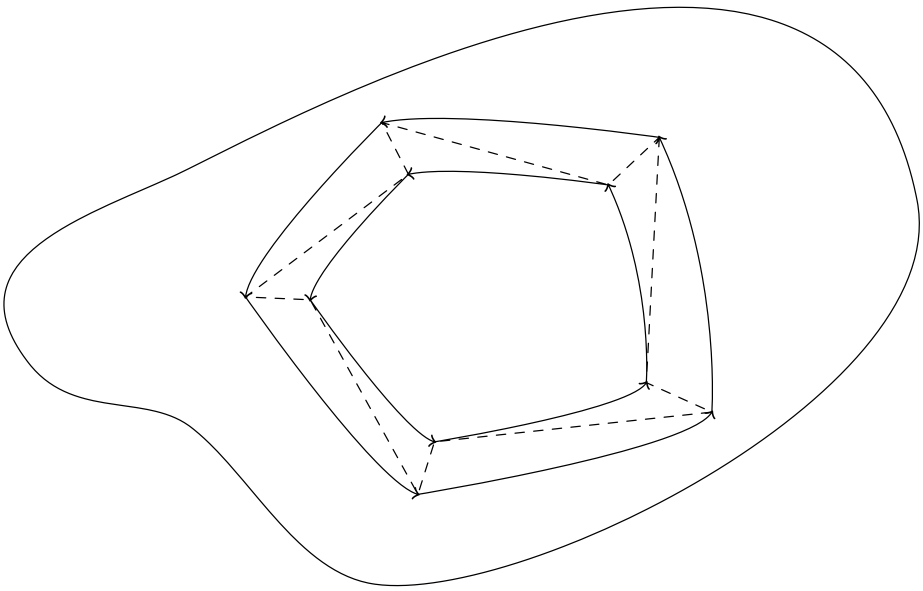


Figure 13.2: Intuition Homology

Lecture 7

di 19 nov 10:27

Consider the more general situation.

Definition 24. Chain complex:

$$C_* : \qquad \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \dots,$$

where C_n abelian group, ∂_n group morphism and $\partial_n \circ \partial_{n+1} = 0$

Definition 25. A chain map, $\Phi = \{\phi_i\}_{i\in\mathbb{Z}} : C_* \to D_*$ and each $\phi_n : C_n \to D_n$.

$$\begin{array}{lcccc}
C_* : & \cdots \to C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} C_{n-1} \to \dots \\
& \downarrow \phi_{n+1} & & \downarrow \phi_n & \\
D_* : & \cdots \to D_{n+1} & \xrightarrow{\partial'_{n+1}} & D_n & \xrightarrow{\partial'_n} D_{n-1} \to \dots
\end{array}$$

.

where $\phi_n \circ \partial_n = \partial'_n \circ \phi_n$

Define $Z_n = \operatorname{Ker}\partial_n \leq C_n$ and $B_n = \operatorname{Im}\partial_{n+1} \leq C_n$ and $B_n \leq Z_n$. Then we define

$$H_n(C_*) = \frac{Z_n(C_*)}{B_n(C_*)}.$$

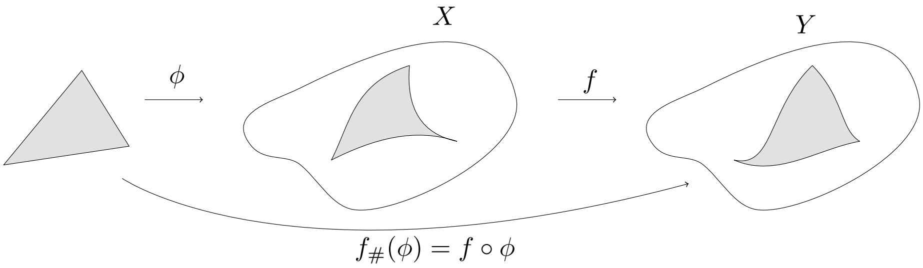
Definition 26. A sequence is exact if not only $B_n \leq Z_n$, but $B_n = Z_n$. In other words: the homology group at n is the trivial group. The homology group measures the 'exactness' of a chain complex.

Given a chain map Φ, it wil induce

$$\begin{aligned}
\Phi_* : H_n(C_*) &\longrightarrow H_n(D_*) \\
z + B_n(C_*) &\longmapsto \phi_n(z) + B_n(D_*).
\end{aligned}$$

We claim that this is well-defined. Indeed, first we check that $\phi_n(z)$ is a cycle. Indeed, using the commutativity of the diagram, $z \xrightarrow{\partial_n} 0 \xrightarrow{\phi_{n-1}} 0$ implies that $\phi_n(z) \xrightarrow{\partial'_n} 0$. Also, a boundary stays a boundary by the same reasoning.

We apply this in connection with a continuous map $f : X \to Y$. Consider the following:



Then we can extend $f_\#$ to a map from $S_n(X) \to S_n(Y)$. Then

$$\begin{tikzcd}
S_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & \phi \in S_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & \partial\phi \in S_{n-1} \arrow[d, "f_\#"] \\
S_{n+1}(Y) \arrow[r, "\partial"] & f_\#(\phi) \in S_n(Y) \arrow[r, "\partial"] & S_{n-1}
\end{tikzcd}$$

Now, the question is, is $\partial f_\#(\phi) = f_\#(\partial\phi)$. It's enough to check this for ∂_i:

$$\begin{aligned}
(f_\#(\partial_i\phi))(t_0, \dots, t_{n-1}) &= f(\phi(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})) \\
\partial_i(f_\#\phi)(t_0, \dots, t_{n-1}) &= f(\phi(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})),
\end{aligned}$$

which proves that we can interchange $f_\#$ and δ.

So f induces a map on Homology:

$$\begin{aligned}
f_* : H_n(X) &\longrightarrow H_n(Y) \\
z + B_n(x) &\longmapsto f_\#(z) + B_n(Y),
\end{aligned}$$

or, using different notation:

$$\begin{aligned}
f_* : H_n(X) &\longrightarrow H_n(Y) \\
\langle z \rangle &\longmapsto \langle f_\#(z) \rangle .
\end{aligned}$$

It's clear that

$$(f \circ g)_* = f_* \circ g_* \qquad (1_X)_* = 1_{H_n(X)} \qquad \forall n \in \mathbb{N},$$

so this is a functor.

Theorem 18. If $f : X \to Y$ is a homeomorphism, then f_* is an isomorphism for all n

Example. What are the homology groups of the space consisting of one point, $X = \{x_0\}$. Note that $\forall n \in \mathbb{N}$, there is exactly one singular n-simplex, $\phi_n : \sigma_n \to X : (t_0, \dots, t_n) \mapsto x_0$. S_n is the free abelian group on $\phi_n \cong \mathbb{Z}$ for all n.

$$\partial : S_0(X) \to S_{-1}(X) : \phi_0 \mapsto 0.$$

for $n > 0$, we have

$$\partial : S_n(X) \to S_{n-1}(X)$$
$$\phi_n \mapsto \sum(-1)^i \partial_i \phi_n = \sum(-1)^i \phi_{n-1} = \begin{cases} \phi_{n-1} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

So all odd maps are 0 and if odd, then it maps the generator of S_n to the generator of S_{n-1}:

$$\begin{array}{cccccccccccccccc} S_{2n+3} & \xrightarrow{0} & S_{2n+2} & \xrightarrow{1_{\mathbb{Z}}} & S_{2n+1} & \xrightarrow{0} & S_{2n} & \xrightarrow{1_{\mathbb{Z}}} & S_{2n-1} & \xrightarrow{0} & S_1 & \xrightarrow{0} & S_0 & \xrightarrow{0} & 0 \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & 0 \end{array}$$

Now,

$$H_0(X) = \frac{Z_0(X)}{B_0(X)} = \frac{\mathbb{Z}}{\langle 0 \rangle} \cong \mathbb{Z}$$
$$H_{2n+1}(X) = \frac{\mathbb{Z}}{\mathbb{Z}} \cong 0$$
$$H_{2n}(X) = \frac{0}{0} \cong 0.$$

Example. The other extreme example is computing H_0 of an arbitrary space, X. This is always possible. We'll assume that X is path connected.

$$S_1(X) \xrightarrow{\partial} S_0(X) \xrightarrow{\partial} 0$$

It's clear that $S_0(X) = Z_0(X)$, which consists of elements of the form $\sum n_i x_i$, where $x_i \in X$, a zero simplex.

Define $\alpha : Z_0(X) \to \mathbb{Z} : \sum n_i x_i \mapsto \sum n_i$. Then α is a morphism of abelian groups. Also $\operatorname{Im} \alpha = \mathbb{Z}$ (as $zx_0 \mapsto z$). This would imply that $\frac{Z_0}{\ker \alpha} \cong \mathbb{Z}$. Claim: $B_0(X) = \ker \alpha$, which would imply that $\frac{Z_0}{B_0} = H_0 \cong \mathbb{Z}$.

Proof:

- $B_0 \subset \ker \alpha$. Take $\phi : \sigma_1 \to X$. Then $\partial \phi \in S_0$ and $\partial \phi = \partial_0 \phi - \partial_1 \phi = \phi(1) - \phi(0)$. Then $\alpha(\partial \phi) = 1 - 1 = 0$. Since B_0 consist of linear combinations of things like $\partial \phi$, we get that $\alpha(B_0(\phi)) = 0$, so $B_0 \subset \ker \alpha$.
- Take any element $\sum n_i x_i \in \ker \alpha$. Then we have to show that this element is a boundary. Belonging to the kernel means that $\sum n_i = 0$.

  Choose any point $x_0 \in X$, fix one. For each i, choose a singular one simplex, a path $\phi : \sigma_1 \to X$, with starting point x_0 and end point x_i. Here we use the assumption that X is path-connected. Then

$$\partial(\sum n_i \phi_i) := \sum n_i \partial \phi_i = \sum n_i (x_i - x) = \sum n_i x_i - \left(\sum n_i\right) x = \sum n_i x_i,$$

so this element, which we've assumed lied in the kernel is also a boundary, i.e. $\sum n_i x_i \in B_0(X)$.

What happens if our space is not path-connected?

Suppose C_* is a chain complex and there exists an I such that for all n,

$$C_n = \bigoplus_{i \in I} C_n^i,$$

and $\partial : C_n \to C_{n-1}$ is of the form $\partial = \bigoplus \partial_i$, where for all $i \in I$, $\partial_i : C_n^i \to C_{n-1}^i$. If this is the case, we have that $\partial_i \circ \partial_i = 0$. Then it's easy to show that

$$H_n(C_*) = \bigoplus H_n(C_*^i).$$

Let X be a space such that $X = \bigsqcup_{\alpha \in A} X_\alpha$, where X_α are the path components of X. Now, let $\phi : \sigma_p \to X$ be any singular p-simplex. Then $\phi(\sigma_p) \subset X_\alpha$ for exactly one α. It's not so hard to see that $S_n(X) = \bigoplus S_n(X_\alpha)$. Moreover, $\partial(S_n(X_\alpha)) \subset S_{n-1}(X_\alpha)$, which means that ∂ is a direct sum. From this, it will follow that

$$H_n(X) = \bigoplus H_n(X_\alpha).$$

This shows that

Proposition 6. $H_0(X) = \bigoplus_\alpha H_0(X_\alpha) \cong \bigoplus_\alpha \mathbb{Z}$

Theorem 19. Let $X \subset \mathbb{R}^n$ be a convex subset and assume that $X \neq \emptyset$. Then $H_0(X) \cong \mathbb{Z}$, because a convex set is path connected, and $H_p(X) = 0$ for all $p > 0$.

Proof. We will define a map going from $S_n(X) \to S_{n+1}(X)$. Fix $x \in X$. Take $\phi : \sigma_n \to X$, and we'll build up a $\theta : \sigma_{n+1} \to X$.

Now consider the purple point, we can write is as a convex combination of the blue and the red point:

$$t_0 \underbrace{(1, 0, \ldots, 0)}_{\bullet} + (1 - t_0) \underbrace{\left(0, \frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}, \ldots, \frac{t_{p+1}}{1-t_0}\right)}_{\bullet}.$$

Then we define:

$$\theta(t_0, t_1, \ldots, t_{n+1}) = t_0 x + (1 - t_0)\phi\left(\frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}, \ldots, \frac{t_{p+1}}{1-t_0}\right)$$

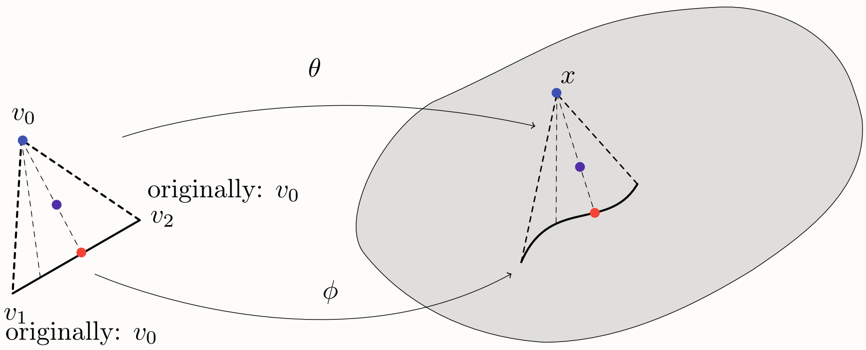
and if $t_0 = 1$, we simply define θ to be x.

We need to prove that θ is continuous.

$$\begin{aligned} \|\theta(t_0, t_1, \ldots, t_{n+1}) - x\| &= \|(1-t_0)\phi(\ldots) + t_0 x - x\| \\ &= \|(t - t_0)\phi(\ldots) - (1-t_0)x\| \\ &\leq |(1-t_0)|(\|\phi(\ldots)\| + \|x\|). \end{aligned}$$

Now, the second term is a constant. As σ_n is compact, $\phi(\sigma_n)$ is also compact, so $\|\phi(\sigma_n)\| \leq M$. So we have

$$\leq |(1-t_0)|(M+C) \xrightarrow{t_0 \to 0} 0.$$



This means we have a map $T : S_n(X) \to S_{n+1}(X)$, which is a morphism, as we defined it for all generators, $T(\phi) = \theta$. Have a look at the drawing and conclude that $\partial_0(T\phi) = \phi$. Also $\partial_1 T\phi = T\partial_0\phi$ and $\partial_2 T\phi = T\partial_1\phi$. In general, we claim that $\partial_i(T(\phi)) = T(\partial_{i-1}\phi)$, for any n-simplex with $n > 0$. Forgetting about $t_0 = 1$ (not a problem, consider this case separatly)

$$\begin{aligned}(\partial_i T(\phi))(t_0,\dots,t_n) &= \theta(t_0,t_1,\dots,t_{i-1},0,t_i,\dots,t_n)\\ &= (1-t_0)\phi\left(\frac{t_1}{1-t_0},\dots,\frac{t_{i-1}}{1-t_0},0,\frac{t_i}{1-t_0},\dots,\frac{t_n}{1-t_0}\right)+t_0x\\ &= T(\partial_{i-1}\phi)(t_0,t_1,\dots,t_n).\end{aligned}$$

This does not work if we're working with a 0-simplex. (Because then ∂_{i-1} becomes zero, and the only homomorphisms that we have are the identity, so ...)

Now, consider the total boundary,

$$\begin{aligned}\partial T\phi &= \partial_0 T\phi + \sum_{i=1}^{n}(-1)^i\partial_i T\phi\\ &= \phi + \sum_{i=1}^{n}(-1)^i T\partial_{i-1}\phi\\ &= \phi + \sum_{j=0}^{n-1}(-1)^{j+1}T\partial_j\phi\\ &= \phi + T\left(\sum_{j=0}^{n-1}(-1)^{j+1}\partial_j\phi\right)\\ &= \phi - T(\partial\phi).\end{aligned}$$

Have a look at the drawing to see this. Conclusion: $\phi = (\partial T + T\partial)\phi$.

For all $n > 0$. Let $\langle z \rangle \in H_n(X)$, with $z \in Z_n(X)$. Then $z = (\partial T + T\partial)z$, or $z = \partial T z + T\partial z$. Because z is a cycle, $z = \partial(Tz)$, so z is not only a cycle, but also a boundary. Hence $\langle z \rangle = \langle 0 \rangle$, so $H_n(X) = 0$. $\square$

Remark. The appearance of $\partial T + T\partial$ is a rather general phenomenon. Suppose we have two chain complexes.

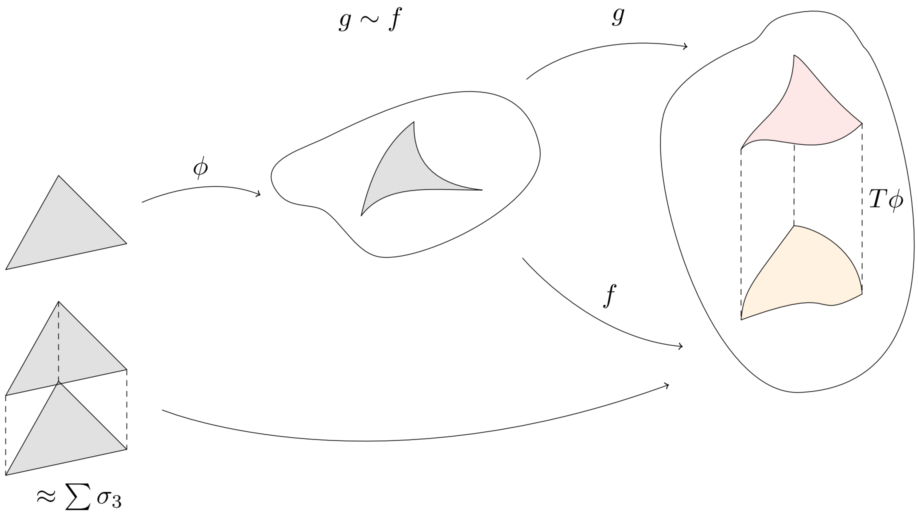
$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial"] \arrow[d, shift left, "f_{n+1}"] \arrow[d, shift right, "g_{n+1}"'] & C_n \arrow[r, "\partial"] \arrow[d, shift left, "f_n"] \arrow[d, shift right, "g_n"'] \arrow[dl, "T"] & C_{n-1} \arrow[d, shift left, "f_{n-1}"] \arrow[d, shift right, "g_{n+1}"'] \arrow[dl, "T"] \\
D_{n+1} \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] & D_{n-1}
\end{tikzcd}$$

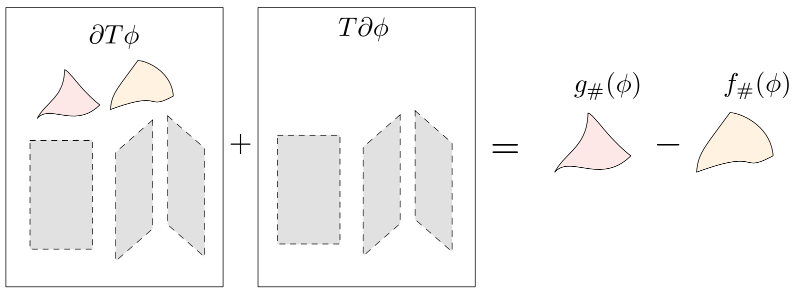
If $f - g = \partial T + T\partial$ for some T for some $T : C_n \to D_{n+1}$ for all n, where f and g are chain maps. Then $f_* = g_* : H_n(C_*) \to H_n(D_*)$. When $f - g = \partial T + T\partial$ is satisfied, we say that f and g are chain homotopic. This indeed has something to do with homotopic maps.

If you have two maps that are homotopic, then $f_\#$ and $g_\#$ will be chain homotopic maps.

There is a proof of the theorem in the book: long, not difficult. There is a better proof, see problem 8.

Idea of the proof:





This goes from two dimensional stuff to three dimensional stuff. But the prism on the left is not a simplex. But we can divide it into simplices. Now, we look at $\partial T\phi + T\partial\phi$, and we see that (being careful with signs) is $g_\#(\phi) - f_\#(\phi)$.

Conclusion: If $f \sim g : X \to Y$, then for all $n \geq 0$, we have that $f_* = g_* : H_n(X) \to H_n(Y)$.

Theorem 20. If X and Y have the same homotopy type, then they also have isomorphic homology groups.

Proof. Let $f : X \to Y$ be the homotopy equivalence and $g : Y \to X$ be the homotopy inverse, so $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. Then $f_* \circ g_* = 1_{H_n(Y)}$ and $f_* \circ g_* = 1_{H_n(X)}$. This means that f_* is an isomorphism with inverse g_*. $\square$

As we saw with homotopy groups, retracts and deformation retracts can be useful.

Recall:

Definition 27. Let $A \subset X$. Then A is a retract of X iff there exists a map $r : X \to A$ such that $r|_A = 1_A$.

We make the definition of deformation retract bit more general

Definition 28 (deformation retract *(version 2)*). A is a deformation retract iff there is a retract $r : X \to A$ such that $r \simeq 1_X$. So there exists r_t such that $r_0 = r$ and $r_1 = 1$. [a]

[a] And not necessarily $r_t|_A = 1_A$ for all t.

Remark. We sometimes call this a *weak deformation retract*

Proposition 7 (1.12).
- Let A be a retract of X. Then $i_* : H_n(A) \to H_n(X)$ (with $i : A \to X$) is injective and *moreover*, $i_*(H_n(A))$ is a direct summandof $H_n(X)$! So there exists a subgroup $B \leq H_n(X)$ such that $H_n(X) = i_* H_n(A) \oplus B$.
- If A is a deformation retract of X, then i_* is an isomorphism.

Proof.
- The last bullet point is trivial. This follows from induced maps from homotopic maps.
- Let $r : X \to A$ be a retraction. Then we have

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend right, "r\circ i=1_A"'] & X \arrow[r, "r"] & A \end{tikzcd}$$

So $(r \circ i)_* = (1_A)_* = 1_{H_n(A)}$ for all n. Therefore, i_* is injective. (Note that this proof holds for all functors!)

Claim: $H_n(X) = \operatorname{Im}(i_*) \oplus \ker r_*$.

- Let $\alpha \in \operatorname{Im}(i_*) \cap \ker r_*$. We show that $\alpha = 0$. So $\alpha = i_*(\beta)$ and $r_*(\alpha) = 0$. This means that $r_*(i_*(\beta)) = 0$, but $r_* \circ i_* = 1$, so $\beta = 0$. So $\alpha = i_* \beta = 0$.
- We show that any element can be written as a sum of elements of both groups. Let $\gamma \in H_n(X)$. Then

$$\gamma = \underbrace{i_*(r_*(\gamma))}_{\in \operatorname{Im} i_*} + \gamma - i_*(r_*(\gamma)),$$

and

$$\begin{aligned} r_*(\gamma - i_* r_* \gamma) &= r_* \gamma - r_* i_* r_* \gamma \\ &= r_* \gamma - r_* \gamma = 0. \end{aligned}$$

□

Lecture 8: Ill

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Lecture 9

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Recap:

$$0 \to H_1(S^1) = \mathbb{Z} \xrightarrow{inj} H_0(U \cap V).$$

and $\langle c + d \rangle \mapsto x - y$.

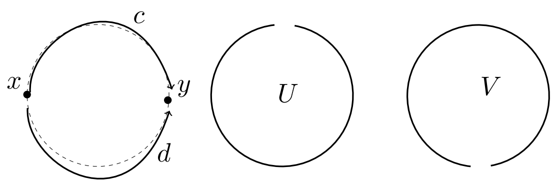


Figure 13.3: homology of circle

For S^n. Then we had

$$0 \to H_n(S^n) = \mathbb{Z} \xrightarrow{\Delta, \cong} H_{n-1}(U \cap V) \xrightarrow{r_*} (S^{n-1}) = \mathbb{Z}.$$

Corollary. There is no retraction of $D^n = B^n = n$-dimensional disc to S^{n-1}.

Proof. Suppose there is a retract. Then

$$\begin{tikzcd}
S^{n-1} \arrow[r, "1_{S^{n-1}}"] \arrow[dr, "i"'] & S^{n+1} \\
& D^n \arrow[u, "r"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z} = H_{n-1}(S^{n-1}) \arrow[r, "(1_{S^n})_* = 1_{H_{n-1}}"] \arrow[dr, "i*"] & H_{n-1}(S^{n-1}) = \mathbb{Z} \\
& H_{n-1}(D^n) = 0 \arrow[u, "r_*"']
\end{tikzcd}$$

□

Corollary (Brouwers fixed point theorem). Let $n \geq 1$. Then any continuous map $f : D^n \to D^n$ has a fixed point. (Note: this time, for any dimension!)

Maps from spheres to spheres

Let $f : S^n \to S^n$. Then f induces a map $f_* : H_n(S^n) \to H_n(S^n)$, so $f_* : \mathbb{Z} \to \mathbb{Z}$, or we can write $f_* : \langle\alpha\rangle \mapsto m\,\langle\alpha\rangle$, where α is the generator.

Definition 29. The degree of f is m. We write $d(f) = m$.

Proposition 8. Few properties of the degree:

1. $d(\mathrm{Id}) = 1$
2. $d(f \circ g) = d(f)d(g)$
3. f constant, or if f is not surjective, then $d(f) = 0$.
4. If $f \simeq g$, then $d(f) = d(g)$, because they induce the same map (Problem 8) (Deep theorem: $f \simeq g$ iff $d(f) = d(g)$.)
5. If f is a homotopy equivalence, then $d(f) = \pm 1$. (We have a homotopy inverse, and $f_* \circ g_* = 1$?, so the only possibilities are both 1 or -1.

Proof. We prove 3. Suppose $p \notin \operatorname{Im} f$. Then

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[dr, "f'"'] & S^n \\ & S^n \setminus \{p\} \arrow[u, "i"'] \end{tikzcd}$$

Then

$$\begin{tikzcd} H_n(S^n) \arrow[r, "f_*"] \arrow[dr, "f'_*"'] & H_n(S^n) \\ & H_n(S^n \setminus \{p\}) = 0 \arrow[u, "i_*"'] \end{tikzcd}$$

where f' is f restricted to the image. We see that $f_* = 0$. $\square$

Theorem 21. Let $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$ and $f : S^n \to S^n$

$$f : (x_1, x_2, \ldots, x_{n+1}) \mapsto (-x_1, x_2, \ldots, x_{n+1}).$$

Then $d(f) = -1$.

Proof. By induction on n.
Base case, $n = 1$. $S^1 = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$.

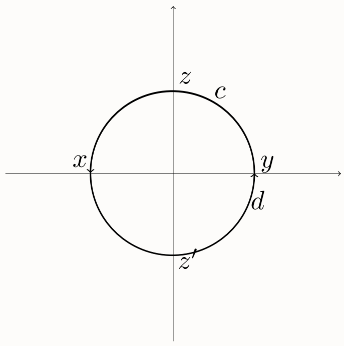


Figure 13.4: base case of theorem

Note that $f(x) = y$, $f(y) = x$. Denote $U = S \setminus \{z'\}$, $V = S \setminus \{z\}$. And $f(U) \subset U$ and $f(V) \subset V$.

Then

$$\begin{tikzcd} 0 \arrow[r] & H_1(S) \arrow[r, "\Delta"] \arrow[d, "f_*"] & H_0(U \cap V) \arrow[d, "(f|_{U\cap V})_*"] \\ 0 \arrow[r] & H_1(S) \arrow[r, "\Delta"] & H_0(U \cap V) \end{tikzcd}$$

TODO: right arrows tails

Then

$$(f|_{U\cap V})_*(x - y) = f(x) - f(y) = y - x = -(x - y).$$

So Δ is injective, we have that $f_*(\langle\alpha\rangle) = -\langle\alpha\rangle$, so $d(f) = -1$.

Now, assume that the theorem holds for S^{n-1}.

Define U, V in a similar way. $f(U) \subset U$ and $f(V) \subset V$. Again, we have a commutative diagram.

Question: is it true that $f_*(\langle\alpha\rangle)$ is just $-\langle\alpha\rangle$?

Now, we know that the f_* on the right is simply $\cdot(-1)$. Then, going the whole way around,

$$f_*(\langle\alpha\rangle) = \Delta^{-1}(i_*(-1(r_*(\Delta(\alpha))))).$$

This is

$$-\Delta^{-1}(i_* r_* \Delta(\langle\alpha\rangle)).$$

As i_* and r_* are inverses, we have that $d(f) = -1$. $\square$

Suppose $f : S^n \to S^n$. Then there is a map $\Sigma f : S^{n+1} \to S^{n+1}$ (the suspension of f), given by

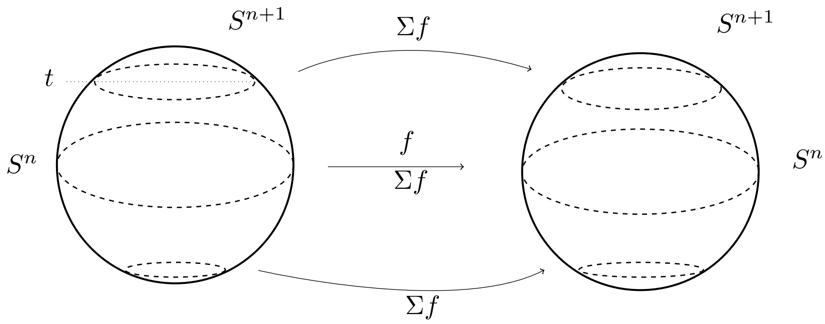


Figure 13.5: definition of suspension

Using formulas:

$$S^{n+1} = \{(x,t) : x \in \mathbb{R}^{n+1}, t \in \mathbb{R}, \|x\|^2 + t^2 = 1\}.$$

Then

$$\begin{aligned} \Sigma f : S^{n+1} &\longrightarrow S^{n+1} \\ (x,t) &\longmapsto \begin{cases} (x,t) & \text{if } x = 0, \text{ north and south pole} \\ \left(\|x\| f(\frac{x}{\|x\|}), t\right) & \text{if } x \neq 0 \end{cases}. \end{aligned}$$

Not too difficult to prove that Σf is continuous. Claim: $d(f) = d(\Sigma f)$.

Proof. Exactly the same technique as we used to prove the previous theorem. □

Remark. Suspension is a general technique. Let X be a topological space. Then ΣX, the suspension of X is defined as follows:

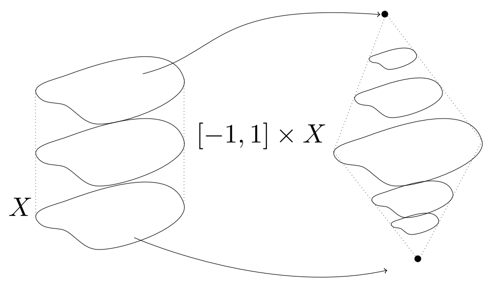


Figure 13.6: definition of suspension general

$\Sigma X = X \times [-1,1]/\sim$ with

$$(x,t) \sim (x',t') \Leftrightarrow t = t' = 1 \text{ or } t = t' = -1.$$

Check for yourself: $\Sigma S^n = S^{n+1}$ (pointy sphere).

Corollary. If $f_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1})$. Then $d(f) = -1$.

Proof. Let $h : S^n \to S^n$ be the map that exchanges the first coordinate and the ith coordinate. Then h is a homeomorphism: $h^{-1} = h$, so $h \circ h = 1_{S^n}$, so $d(h) = \pm 1$ Then $f_i = h \circ f \circ h$. So

$$d(f_i) = d(h)^2 d(f) = (\pm 1)^2(-1) = -1.$$

□

There is a big difference between even and odd dimensional spheres:

Corollary. Let $A : S^n \to S^n$ defined by

$$A(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_{n+1}).^a$$

Then $d(A) = (-1)^{n+1}$.

[a] Note the error in the book. A should have $n+1$ inputs. TODO

Proof.

$$d(A) = d(f_1)d(f_2)\cdots d(f_{n+1}) = (-1)^{n+1}.$$

□

So for even dimensional spheres, $d(A) = -1$ and for odd dimensional sphere, $d(A) = 1$. This is a *big* difference.

Theorem 22. Let $f : S^n \to S^n$ and $g : S^n \to S^n$. When $f(x) \neq g(x)$ for all $x \in S^n$. Then $f \simeq A \circ g$.

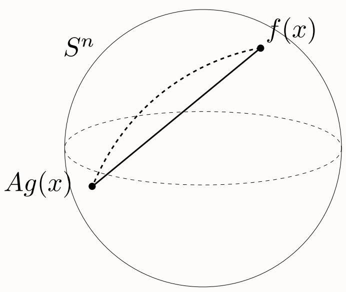


Figure 13.7: proof prop 123

Proof. The line does not go through the center, as $f(x) \neq g(x)$. Define a homotopy,

$$(x,t) \mapsto \frac{(1-t)f(x) + tAg(x)}{\|(1-t)f(x) + tAg(x)\| \neq 0}.$$

This is a homotopy between f and Ag □

Not in the text:

Corollary. If $|d(f)| \neq |d(g)|$, then there is a point x such that $f(x) = g(x)$.

Proof. The previous theorem says that $d(f) = d(A)d(g)$, so $|d(f)| = |d(g)|$. You can even make this better by making a distinction between odd and even dimensional spheres. □

Corollary. If $|d(f)| \neq 1$, then f has a fixed point, i.e. an x such that $f(x) = x = \mathrm{Id}(x)$.

Proof. If $|d(f)| \neq d(\mathrm{Id}) = 1$, then ... □

Theorem 23. Let $f : S^{2n} \to S^{2n}$. Then there exists an x in S^{2n} such that $f(x) = x$ or $f(x) = -x$.

Proof. Suppose there is no x such that $f(x) = x$ and $f(x) = -x$.

- Since $f(x) \neq x$ for all $x \in X$, we have that $f \simeq A$.
- Since $f(x) \neq -x = A(x)$ for all $x \in X$, we have that $f \simeq A \circ A = 1$.

From this it follows that

$$d(f) = (-1)^{2n+1} = -1 \qquad d(f) = 1,$$

which is a contradiction. □

Remark. This does not hold for odd dimensional spheres. For example, consider S_1, and make a small rotation, then no point is mapped onto itself or onto its antipodal point. Doing the same for S_2, we see that when we do a small rotation, the north and south pole are switched.

Remark. It's harder to show that for odd dimensional spheres, there always exist such a map.

Now, we prove the hairy ball theorem, which only holds in even dimensional spheres.

Corollary. There is no continuous map $f : S^{2n} \to S^{2n}$ such that x and $f(x)$ are orthogonal for all x.

Proof. By the previous theorem, there is a point for which $f(x) = \pm x$, and $x \not\perp \pm x$. □

Corollary. There exists no nonzero vector field ϕ on S^{2n}.

Proof. Suppose ϕ does exists. Then, map the tangent vector to a point on the sphere:

$$\psi : S^{2n} \to S^{2n} : x \mapsto \frac{\phi(x)}{\|\phi(x)\|}.$$

□

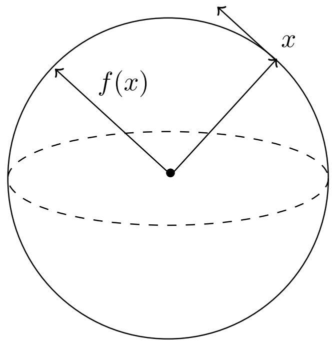


Figure 13.8: hairy ball theorem

Remark. For odd dimensional spheres, there always exists a such a vector field. We even know how many linearly independent vector fields exist!

Torus

Goal: $H_n(T^2)$.

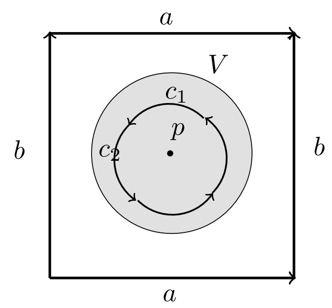


Figure 13.9: homology of torus

and , and $U \cap V = B^2 \setminus \{p\}$.

- Choose $U = T^2 \setminus \{p\}$ Then U retracts to the figure eight space (boundary of square) Last week, we saw that $H_0(U) = \mathbb{Z}$ and $H_1(U) = \mathbb{Z} \oplus \mathbb{Z}$, with generators a and b, and $H_n(U) = 0$ for $n \geq 2$.
- $V = B^2$. As B^2 is path connected, $H_0(U) = \mathbb{Z}$ and $H_n(U) = 0$ for all $n \geq 1$
- $U \cap V = B^2 \setminus \{p\}$, which retracts to a circle, so $H_0(U \cap V) = H_1(U \cap V) = \mathbb{Z}$.

$$H_2(U) \oplus H_2(U) \to H_2(T^2) \to H_1(U \cap V) \to H_1(U) \oplus H_1(U) \to H_1(T^2) \to H_0(U \cap V) \to H_0(U) \oplus H_0(U) \to H_0(T^2)$$

So we have

$$\to 0 \oplus 0 \to H_2(T^2) \to \mathbb{Z} \to \mathbb{Z}^2 \oplus 0 \to H_1(T^2) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0.$$

Now, $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is injective, as the kernel of $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is non-trivial. (1 gets mapped to something non-trivial) So we can shorten the sequence

$$\to 0 \oplus 0 \to H_2(T^2) \to \mathbb{Z} \xrightarrow{g_*} \mathbb{Z}^2 \oplus 0 \to H_1(T^2) \to 0.$$

Now, we want to know how g_* works. We need a generator of $\mathbb{Z}$ and look at its embedding? Then $\mathbb{Z} = \langle c_1 + c_2 + c_3 + c_4 \rangle$. Then

$$U \cap V \xrightarrow{\quad i \quad} U \xrightarrow{\quad r \quad} \text{figure 8}$$

$$H_1(U \cap V) \xrightarrow{\quad i_r \quad} H_1(U) \xrightarrow{\quad r_* \quad} H_1(\text{figure 8})$$

$$\langle c_1 + c_2 + c_3 + c_4 \rangle \longrightarrow \langle c_1 + c_2 + c_3 + c_4 \rangle \longrightarrow \langle r \circ c_1 + r \circ c_2 + r \circ c_3 + r \circ c_4 \rangle$$

$$\langle \overline{a} + \overline{b} + a + b \rangle$$

Now, as these are all cycles, we have that this is

$$\langle\overline{a}\rangle + \langle\overline{b}\rangle + \langle a\rangle + \langle b\rangle \,.$$

Now, using Problem 6, we find that this is

$$-\langle a\rangle - \langle b\rangle + \langle a\rangle + \langle b\rangle = 0.$$

Therefore, g_* is the zero map.

$$\to 0 \oplus 0 \to H_2(T^2) \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \oplus 0 \to H_1(T^2) \to 0,$$

so the kernel of $\mathbb{Z}^2 \oplus 0 \to H_1(T^2)$ is an injective map. As $H_1(T^2) \to 0$, this implies that $\mathbb{Z}^2 \oplus 0 \to H_1(T^2)$ is a surjective map. Therefore, $H_1(T^2) = \mathbb{Z}^2$. Now, $H_2(T^2) \to H_1(U \cap V)$ is surjective, and it is also injective, so $H_2(T^2) = \mathbb{Z}$.

Now, for $n \geq 3$. Then

$$H_n(U) \oplus H_n(V) \to H_n(T^2) \to H_{n-1}(U \cap V)..$$

This simplifies to

$$0 \oplus 0 \to H_n(T^2) \to 0.,$$

so $H_n(T^2) = 0$. Conclusion:

$$H_0(T^2) = \mathbb{Z} \quad H_1(T^2) = \mathbb{Z}^2 \quad H_2(T^2) = \mathbb{Z} \quad H_n(T^2) = 0 \text{ for all } n >= 3.$$

Remark. $H_2(S^1 \times S^1) \neq H_2(S^1) \times H_2(S^1)$.

Remark. $H_n(T^k) = \mathbb{Z}^{\binom{n}{k}}$, where $\binom{n}{k} = 0$ when $k > n$.

Klein bottle

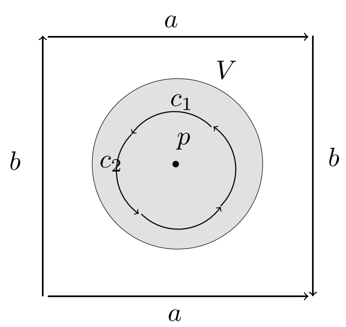


Figure 13.10: klein bottle homology

- $U = K \setminus \{p\}$, which retracts to the figure eight space (boundary of the square) So $H_0 = \mathbb{Z}$ and $H_1 = Z^2 = \text{grp}(\langle a \rangle, \langle b \rangle)$, and $H_n = 0$ for $n \geq 2$.
- $V = B^2$ So $H_0 = \mathbb{Z}$ and $H_n = 0$ for $n \geq 1$.
- $U \cap V = B^2 \setminus \{p\}$, which retracts to a circle. So $H_0 = \mathbb{Z}$ and $H_1 = \mathbb{Z} = \text{grp}(\langle c_1 + c_2 + c_3 + c_4 \rangle)$

$$\to H_2(U) \oplus H_2(V) \to H_2(K) \to H_1(U \cap V) \xrightarrow{g_*} H_1(U) \oplus H_1(V) \to H_1(K) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(K) \to 0$$

Klein bottle is path connected, so

$$\to 0 \oplus 0 \to H_2(K) \to \mathbb{Z} \xrightarrow{g_*} \mathbb{Z}^2 \oplus 0 \to H_1(K) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

We can cut of the sequence because all spaces are path connected.

$$\to 0 \oplus 0 \to H_2(K) \to \mathbb{Z} \xrightarrow{g_*} \mathbb{Z}^2 \oplus 0 \to H_1(K) \to 0$$

Now, we have to understand what g_* is
Then

$$U \cap V \xrightarrow{\quad i \quad} U \xrightarrow{\quad r \quad} \text{figure 8}$$

$$H_1(U \cap V) \xrightarrow{\quad i_r \quad} H_1(U) \xrightarrow{\quad r_* \quad} H_1(\text{figure 8})$$

$$\langle c_1 + c_2 + c_3 + c_4 \rangle \longrightarrow \langle c_1 + c_2 + c_3 + c_4 \rangle \longrightarrow \langle r \circ c_1 + r \circ c_2 + r \circ c_3 + r \circ c_4 \rangle$$

$$\langle \overline{a} + \overline{b} + a + \overline{b} \rangle$$

So we get

$$- \langle a \rangle - \langle b \rangle + \langle a \rangle - \langle b \rangle = -2 \langle b \rangle .$$

So $g_* : \mathbb{Z} \to \mathbb{Z}^2 : 1 \mapsto (0, -2)$, or $z \to (0, -2z)$. So, the image of $g_* = 0 \times 2\mathbb{Z} \leq \mathbb{Z} \times \mathbb{Z}$. The kernel of g_* is $\{0\}$. So g_* is injective. And $0 \times 2\mathbb{Z}$ is the kernel of h_*

$$\to 0 \oplus 0 \to H_2(K) \to \mathbb{Z} \xrightarrow{g_*} \mathbb{Z}^2 \xrightarrow{h_*} H_1(K) \to 0$$

Now, the zero on the right implies that h_* is surjective. so $(H_1(U) \oplus 0)/\operatorname{Ker} h_*$, which is $\frac{\mathbb{Z}^2}{0 \times 2\mathbb{Z}}$, which is $\mathbb{Z} \oplus \mathbb{Z}_2$.

Now, on the left. As g_* is injective, so $H_2(K) \to H_1(U \cap V) = \mathbb{Z}$ is the zero map. On the other hand, we have zeros on the left, so $H_2(K) \to H_1(U \cap V) = \mathbb{Z}$ is also injective. The only way the zero map can be injective is when the spaces are zero. Therefore, $H_2(K) = 0$.

Higher dimensional: trivial.

Let $n \geq 3$

$$0 \oplus 0 = H_n(U) \oplus H_n(V) \to H_n(K) \to H_{n-1}(U \cap V) = 0,$$

so $H_n(K) = 0$ for all $n \geq 3$.
Conclusion:

$$H_0(K) = \mathbb{Z} \quad H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2 \quad H_2(K) = 0 \quad H_{\geq 3}(K) = 0.$$

Remark. Fact: if X is an n-dimensional manifold, which is non orientable, then H_2 is always zero. If it's orientable, then $H_2(X) = \mathbb{Z}$.

Remark. Fact: The $H_1(X) = \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]}$. Abelianized version of fundamental group. Remember, we did compute the fundamental group of the klein bottle! $\pi_1(K) = \langle \alpha, \beta \mid \beta\alpha = \alpha^{-1}\beta \rangle$. Making this group commutative, we have

$$H_1(K) = \langle \alpha, \beta \mid \beta\alpha = \alpha^{-1}\beta, \alpha\beta = \beta\alpha \rangle .$$

From this is follows that $\alpha^{-1}\beta = \alpha\beta$, so $\alpha^{-1} = \alpha$, so $\alpha^2 = 1$. Therefore, we can rewrite this as

$$H_1(K) = \langle \alpha, \beta \mid \alpha\beta = \beta\alpha, \alpha^2 = 1 \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}.$$

Another way is to actually compute commutators.

$$[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta = \alpha^{-1}\alpha^{-1}\beta^{-1}\beta^{-1} = \alpha^{-2}.$$

So, we mod out α^2.

Remark (Exam). $H_n(\mathbb{R}P^2)$. The answer should be:

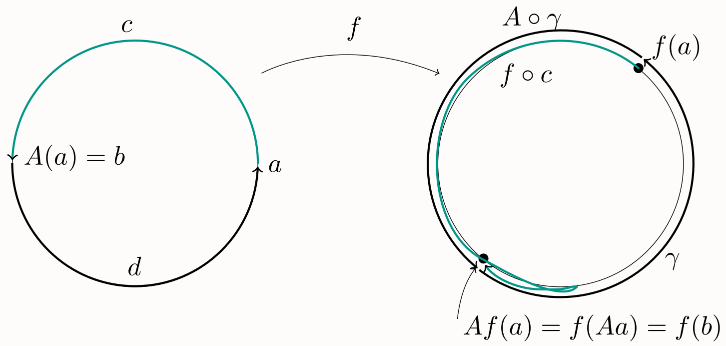
- $H_0 = \mathbb{Z}$ (path connected)
- $H_1 = \mathbb{Z}_2$ (abelianized version)
- $H_2 = 0$ (orientable)
- $H_{n \geq 3} = 0$

Lecture 10: Ham sandwich theorem

di 17 dec 10:26

Theorem 24. Let $f : S^1 \to S^1$ such that f is antipode preserving: $f(Ax) = Af(x)$. Then $d(f)$ is odd.

Proof. Picture:



What is $f_* : H_1(S^1) \to H_1(S^1)$? $H_1(S^1) \cong \mathbb{Z} = \langle \alpha \rangle = \langle c + d \rangle$. Note that $d = A \circ c$.

Now, Let's look at the image of c under f. The only thing we know, is that it starts at $f(a)$ and ends at $f(b) = Af(a)$. Let γ be the path that is the half circle, counterclockwise from $f(b)$ to $f(a)$. Then $f \circ c + \gamma \in Z_1(S^1)$. Indeed, $\partial(f \circ c + \gamma) = f(b) - f(a) + f(a) - f(b) = 0$. Then $\langle f \circ c + \gamma \rangle \in H_1(S^1)$. This has to be equal to $m \langle \alpha \rangle$ for some $m \in \mathbb{Z}$.

Now, $A_* : H_1(S^1) \to H_1(S^1)$. We know that $d(A_*) = (-1)^2 = 1$, so this is just the identity map:

$$A_*(\langle \alpha \rangle) = A_* \langle c + d \rangle = \langle A \circ c + A \circ d \rangle = \langle d + c \rangle = \langle \alpha \rangle .$$

So

$$m \langle \alpha \rangle = A_* \langle f \circ c + \gamma \rangle = \langle A \circ f \circ c + A \circ \gamma \rangle .$$

Now, $A \circ \gamma$ is the path from $f(a)$ to $f(b)$, going counter clockwise. Now, we assume $A \circ f = f \circ A$, so

$$m \langle \alpha \rangle = \langle f \circ d + A \circ \gamma \rangle .$$

Now, also

$$\begin{aligned} 2m \langle a \rangle &= \langle f \circ c + \gamma \rangle + \langle f \circ d + A \circ \gamma \rangle \\ &= \langle f \circ c + f \circ d + \gamma + A \circ \gamma \rangle \end{aligned}$$

$\gamma + A \circ \gamma$ is a one cycle, and $f \circ c + f \circ d$ is also a one cycle

$$\begin{aligned} &= \langle f \circ c + f \circ d \rangle + \langle \gamma + A \circ \gamma \rangle \\ &= f_* \langle c + d \rangle + \langle \alpha \rangle \\ &= f_* \langle \alpha \rangle + \langle \alpha \rangle . \end{aligned}$$

So

$$f_* \langle \alpha \rangle = (2m - 1) \langle \alpha \rangle ,$$

for some integer m. □

TODO: In problem: say that one cycles

Remark. Exercise: even maps have even degrees: $f(Ax) = f(x)$.

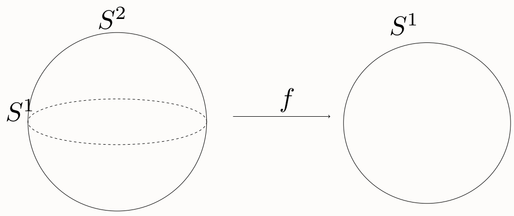
Remark. We don't use $-x$ for Ax because it could be misleading: this $-$ has nothing to do with the $-$ from the abelian homology group.

Theorem 25. Let $f : S^n \to S^n$ such that $f(Ax) = Af(x)$. Then $d(f)$ is odd.

Proof. We don't prove this. □

Theorem 26. There is no continuous antipodal preserving map from $S^2 \to S^1$.

Proof. Suppose there is such a map.



$$\begin{tikzcd} S^2 \arrow[r, "f"] & S^1 \\ S^1 \arrow[u, "i"] \arrow[ur, "f|_{S^1}"'] & \end{tikzcd}$$

Then we have

$$\begin{tikzcd} H_1(S^2) = 0 \arrow[r, "f_*"] & H_1(S^1) = \mathbb{Z} \\ H_1(S^1) = \mathbb{Z} \arrow[u, "i_*"] \arrow[ur, "\cdot(2k+1)"'] & \end{tikzcd}$$

But $(f_* \circ i_*)(1) = 0$, but $(f|_{S^1})_*(1) = 2k+1$, which is odd, and certainly not 0. This works for all higher dimensional spheres to S^1. □

Theorem 27 (Borsuk-Ulam theorem)**.** *There are two antipodal points on the earth such that both the temperature and air pressure are the same.* If $f : S^2 \to \mathbb{R}^2$ is continuous. Then there exists $x \in S^2$ such that $f(x) = f(-x)$.

Proof. Suppose $f(x) \neq f(-x)$ for all $x \in S^2$. Then we define

$$g : S^2 \to \mathbb{R}^2 : x \mapsto \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|} \in S^1.$$

Then $g(-x) = -g(x)$, which is a contradiction with the previous theorem. □

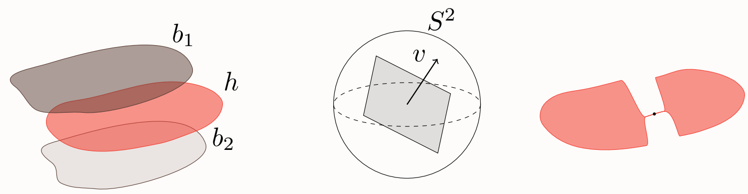
Remark. Also holds for $S^n \to \mathbb{R}^n$.

Theorem 28 (Ham Sandwich theorem). *Suppose you give me two pieces of break and 1 slice of ham. Then it is possible to divide both the pieces of bread and the slice of ham in equal pieces by 1 straight cut of knife.*

Proof. Consider for each $v \in S^2$ a plane $P_v \subset \mathbb{R}^3$. $P_v \perp v$ and P_v cuts the slice of ham exactly in two. We define the "upper side" of the plane to be the half to which v is pointing to.

If you have some weird ham which which you can cut in multiple places in half, then you take the middel of the line segment. This makes it unique.

Note that $P_v = P_{-v}$.



Now, consider

$$f : S^2 \to \mathbb{R}^2 : v \mapsto (f_1(v), f_2(v)).$$

Then $f_1(v)$ is the volume of bread b_1 above P_v. Then $f_2(v)$ is the volume of bread b_2 above P_v.

Now, you should believe that f_1 and f_2 are continuous. (Proving this precisely needs measure theory etc.) So, now, we can use the Borsak Ulam theorem. So there exists a $v \in S^2$ such that $f(v) = f(-v)$. So $f_1(v) = f_1(-v)$, so volume of bread b_1 above P_v is the volume of bread b_1 below P_v, and similar for f_2. This proves the Ham sandwich theorem. $\square$

Last result.

Theorem 29. Let A_1, A_2, A_3 be three closed subsets of S^2 such that $S^2 = A_1 \cup A_2 \cup A_3$. Then at least one of the A_i contains a pair of antipodal points.

Proof. Let $f : S^2 \to \mathbb{R}^2$ mapping $x \mapsto (d(x, A_1), d(x, A_2))$.

$$d(x, A_i) = \min_{y \in A_i} d(x, y).$$

Then, there exists a point $x \in S^2$ such that $f(x) = f(-x)$. So

$$d(x, A_1) = d(-x, A_1) \qquad d(x, A_2) = d(-x, A_2).$$

Possibility 1 $d(x, A_1) = 0$. Then $x \in A_1$ and $-x \in A_1$.

Possibility 2 $d(x, A_2) = 0$. Then $x \in A_2$ and $-x \in A_2$.

Possibility 3 $d(x, A_2) \neq 0 \neq d(x, A_1)$: $x, -x \notin A_1$ and $x, -x \notin A_2$. As $x \in S^2$, $x, -x \in A_3$.

$\square$

Remark. This also works for 2 (Just take $A_3 = \emptyset$), but this doesn't work for 4.

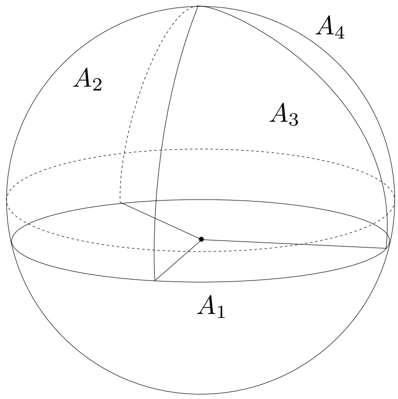


Figure 13.11: Example with four elements

Remark. Time on schedule is when you have to enter the room. Hand over problems after exam.