EXTREMAL PRINCIPLE

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1 Introduction

In this problem set, we shall see some problems, in which, as the name suggests, you may have to use some Extremal Argument, i.e. push some configuration to its Extreme state, consider some maximal set or consider a special element may be minimal or maximal if convenient of a set. The vague theme of the problems will be almost this.

2 Problems

1. There are n points in a plane and any three of them form a triangle of area less or equal to 1. Then prove that they lie inside a triangle with area ≤ 4.

2. Rooks are placed on a $n \times n$ chessboard satisfying this following condition. If the (i, j) th square is free, then at least n rooks are on the $i - th$ row and jth column together. Show that there are at least $n^2/2$ rooks on the board.

3. Each point is a midpoint of two other points in S. Show that S must be infinite.

4. In a chess-board a number is written on each square. And the number written in a square is equal to the average of the numbers written on the adjacent(side sharing) squares. Prove that all the numbers written on the square are equal.

5. The Parliament of Sikinia consists of one house. Every member has at most three enemies among the remaining members. Note that a member cannot be his own enemy, and enmity is mutual. Show that one can split the house into two houses so that every member has one enemy at most in his house.

6. Given are n red points and n blue points in the plane, no three collinear. Show that we can draw n nonintersecting segments connecting the blue points to the red points.

7. The exact quantity of gas needed for a car to complete a single loop in a circular track is distributed among n containers placed along the track. Show that there exists a point from which the car can start with an empty tank and complete the loop by collecting gas from tanks it encounters along the way. (Assume that there is no limit to the amount of gas the car can carry)

8. In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

9. Let A and B be two towns, with B to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way.

10. Prove that there is exactly one town that cannot be swept away by any other one.

11. There are three cities each of which has exactly the same number of citizens, say n. Every citizen in each city has exactly a total of $(n+1)$ friends in the other two cities. Show that there exist three people, one from each city, such that they are friends. We assume that friendship is mutual (that is, a symmetric relation).

    Graph Theory Examples:

12. G is a graph where each vertex has degree at least 2. Then show that there must exist at least one cycle.

    An immediate result of this should be, if G is a connected and an acyclic graph (It has a special name, Tree) then there must exist one vertex with degree exactly 1 (Such a vertex is called a leaf).

13. Every tournament has a Hamiltonian path. **Number Theory examples:**

14. Show that for every $n > 1$, the number $2^n - 1$ is not a multiple of n.

15. Find all integers $n \geq 3$ so that , if we list the divisors of $n!$, including 1 and itself, in increasing order then we have

$$d_2 - d_1 \leq d_3 - d_2 \leq ... \leq d_k - d_{k-1}$$

. **Combi Geo Examples:**

Often taking the convex hull and trying some sweeping arguments, maybe a bit IVP-ish arguments should often come handy.

16. Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a balancing line if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

17. (2n+3) points on a plane, no three on a line, no four on a circle. Prove that we can choose three of the points and draw a circle through them, such that n are inside and n are outside the circle.

18. Originally, every square of 8 8 chessboard contains a rook. One by one, rooks which attack an odd number of other rooks are removed. (Rooks may not jump over other rooks.) Find the maximal number of rooks that can be removed. (A rook attacks another rook if they are on the same row or column and there are no other rooks between them.)

19. Finitely many points in the plane are given, not all collinear. Prove that there exists a line passing through exactly two of them.

20. A finite set S of points in the plane has the property that any line through two of them passes through a third. Show that all the points lie on a line.

21. Every road in Sikinia is one-way. Every pair of cities is connected exactly by one direct road. Show that there exists a city which can be reached from every city directly or via at most one other city

22. Let M be the largest distance among six distinct points of the plane, and let m be the smallest of their mutual distances. Prove that $\frac{M}{m} \geq \sqrt{3}$.