Double Counting

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1 Double Counting, Counting in 2 ways

1. In a tournament with n players , let the number of wins and losses of the i th player be W_i and L_i . Prove that $\sum W_i^2 = \sum L_i^2$
2. Suppose a class contains 100 students. Let, for $1 \leq i \leq 100$, the i^{th} student have a_i many friends. For $0 \leq j \leq 99$ let us define c_j to be the number of students who have strictly more than j friends. Show that
$$\sum_{i=1}^{100} a_i = \sum_{j=0}^{99} c_j$$
3. An olympiad has six problems and 200 contestants. The contestants are very skilled, so each problem is solved by at least 120 of the contestants. Prove that there exist two contestants such that each problem is solved by at least one of them.
4. Consider a 43 sided regular polygon, or each vertex, out of the line segments emerging from it, 20 are coloured red, 22 are coloured blue. Find the number of monochrome triangles.
5. In a chess club, n people gathered to play chess against each other, as they pleased. No two people played against each other more than once. At the end of the day, it was observed that a total of 3n games had been played. Moreover, if we choose any two players, say A and B, there would be at most one other player who had played with both A and B. Prove that $n > 30$
6. In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the two players earned 1/2 point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned against the ten players with the least number of points. (In particular, each of the ten lowest scoring players earned half of her/his points against the other nine of the ten.) What was the total number of players in the tournament?

7. There are 10001 students at an university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

   i.) Each pair of students are in exactly one club.

   ii.) For each student and each society, the student is in exactly one club of the society.

   iii.) Each club has an odd number of students. In addition, a club with $2m + 1$ students (m is a positive integer) is in exactly m societies.

8. Find all possible values of k. Let S be a set of n elements and $S_1,\ S_2, \ldots,\ S_k$ are subsets of S ($k \geq 2$), such that every one of them has at least r elements. Show that there exists i and j, with $1 \leq i < j \leq k$, such that the number of common elements of S_i and S_j is greater or equal to: $r - \frac{nk}{4(k-1)}$

9. Let $n \geq 3$ be an integer and let $A_1, A_2, \ldots, A_n$ be n distinct subsets of $S = \{1, 2, \ldots, n\}$. Show that there exists $x \in S$ such that the n subsets $A_i - \{x\}, i = 1, 2, \ldots n$ are also disjoint.

10. Show that a graph with n vertices and k edges has at least $k(4k–n^2)/3n$ triangles.

11. Consider a round-robin tournament with $2n + 1$ teams, where each team plays each other team exactly one. We say that three teams X, Y and Z, form a cycle triplet if X beats Y, Y beats Z and Z beats X. There are no ties. a)Determine the minimum number of cycle triplets possible. b)Determine the maximum number of cycle triplets possible.

12. There are n points in the plane such that no three of them are collinear. Prove that the number of triangles whose vertices are chosen from these n points and whose area is 1 is not greater than $\frac{2}{3}(n^2 - n)$.

13. In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that
$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

14. There are n people in a big party. Some of them know each other. It is given that there do not exist four people A, B, C, D such that A knows B, B knows C, C knows D, and D knows A. Prove that the number of acquaintances in the party is not greater than
$$\frac{n}{4}\left(1 + \sqrt{4n-3}\right).$$

15. Let X be a set of $2k$ elements and $\mathcal{F}$ a family of subsets of X each of cardinality k such that each subset of X of cardinality $(k-1)$ is contained in exactly one member of $\mathcal{F}$. Show that $(k+1)$ is a prime.

2 Games

1. The game of double chess is played like regular chess, except each player makes two moves in their turn. As usual, white starts. Prove that white can always guarantee a win or a draw.

2. There are three empty jugs on a table. Winnie the Pooh, Rabbit, and Piglet put walnuts in the jugs one by one. They play successively, with the initial determined by a draw. Thereby Winnie the Pooh plays either in the first or second jug, Rabbit in the second or third, and Piglet in the first or third. The player after whose move there are exactly 1999 walnuts loses the games. Show that Winnie the Pooh and Piglet can cooperate so as to make Rabbit lose.

3. Elmo and Elmo's clone are playing a game. Initially, $n \geq 3$ points are given on a circle. On a player's turn, that player must draw a triangle using three unused points as vertices, without creating any crossing edges. The first player who cannot move loses. If Elmo's clone goes first and players alternate turns, who wins? (Your answer may be in terms of n .)

4. On the circle, 99 points are marked, dividing this circle into 99 equal arcs. Petya and Vasya play the game, taking turns. Petya goes first; on his first move, he paints in red or blue any marked point. Then each player can paint on his own turn, in red or blue, any uncolored marked point adjacent to the already painted one. Vasya wins, if after painting all points there is an equilateral triangle, all three vertices of which are colored in the same color. Could Petya prevent him?