Combi Advanced PROBLEM SET

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December 2024

§1 Problems

§1.1 P1

Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a balancing line if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.(can it be made stronger to 3?)

§1.2 P2

There exist three consecutive vertices A, B, C in every convex $n-$gon with $n \geq 3$ such that the circumcircle of ΔABC covers the whole $n-$gon. (PSS)

§1.3 P3

There are three cities each of which has exactly the same number of citizens, say n. Every citizen in each city has exactly a total of $(n+1)$ friends in the other two cities. Show that there exist three people, one from each city, such that they are friends. We assume that friendship is mutual (that is, a symmetric relation).

§1.4 P4

If a given equilateral triangle Δ of side length a lies in the union of five equilateral triangles of side length b, show that there exist four equilateral triangles of side length b whose union contains Δ.

§1.5 P5

In a competition, there are a contestants and b judges where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that for any two judges, their ratings coincide for at most k contestants. Prove that

$$k/a \geq (b-1)/2b$$

§1.6 P6

We have $n \geq 2$ lamps $L_1, ..., L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows: if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off; – otherwise, L_i is switched on. Initially all the lamps are off except the leftmost one which is on.

(a) Prove that there are infinitely many integers n for which all the lamps will eventually be off. (b) Prove that there are infinitely many integers n for which the lamps will never be all off.

§1.7 P7

Let S be a finite set of positive integers. Assume that there are precisely 2023 ordered pairs (x, y) in $S \times S$ so that the product xy is a perfect square. Prove that one can find at least four distinct elements in S so that none of their pairwise products is a perfect square.

Note: As an example, if $S = \{1, 2, 4\}$, there are exactly five such ordered pairs: $(1, 1)$, $(1, 4)$, $(2, 2)$, $(4, 1)$, and $(4, 4)$.

(Okay, this is not combi but a good GT solution exists.)

§1.8 P8

In a chess club, n people gathered to play chess against each other, as they pleased. No two people played against each other more than once. At the end of the day, it was observed that a total of 3n games had been played. Moreover, if we choose any two players, say A and B, there would be at most one other player who had played with both A and B. Prove that $n > 30$

§1.9 P9

Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two player, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins. (a) Does the game necessarily end? (This part was used in some movie) (b) Does there exist a winning strategy for the starting player?

§1.10 P10

Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is coloured red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

§1.11 P11

Define a k-clique to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

§1.12 P 12

There are 10001 students at an university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

i.) Each pair of students are in exactly one club.

ii.) For each student and each society, the student is in exactly one club of the society.

iii.) Each club has an odd number of students. In addition, a club with $2m + 1$ students (m is a positive integer) is in exactly m societies.

Find all possible values of k.

§1.13 P 13

A and B play a game, given an integer N, A writes down 1 first, then every player sees the last number written and if it is n then in his turn he writes $n + 1$ or $2n$, but his number cannot be bigger than N. The player who writes N wins. For which values of N does B win ?

Okay , I shall give you enough hints for this one. First try some numbers at hand note that B wins 2 then 8 and 10 and also A wins for all odds.

Then, try to find sort of a recursion

Then, try to find what if X jumps over the winning number of Y finally you have something with $\lfloor \frac{N}{4} \rfloor$ or something like this .Try to use binary to find its significance and don't forget that A wins for all odds.

§1.14 P14

In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.

§1.15 P15

Given are n red points and n blue points in the plane, no three collinear. Show that we can draw n nonintersecting segments connecting the blue points to the red points.

§1.16 P16

The Parliament of Penguins of North Pole consists of one house. Every member has at most three enemies among the remaining members. Note that a member cannot be his own enemy, and enmity is mutual. Show that one can split the house into two houses so that every member has one enemy at most in his house.

§1.17 P 17

The exact quantity of gas needed for a car to complete a single loop in a circular track is distributed among n containers placed along the track. Show that there exists a point from which the car can start with an empty tank and complete the loop by collecting gas from tanks it encounters along the way. (Assume that there is no limit to the amount of gas the car can carry)

§1.18 P18

Mantel's theorem : Of n points in a space no four lie in a plane. Some of the points are connected with lines.We get a graph with n vertices and k edges . If G has no triangles then prove that $k \leq \lfloor n^2/4 \rfloor$

§1.19 P19

Show that in a $G(n,k)$ minimum number of triangles is $\frac{k}{3n}[4k - n^2]$

Note that Mantel's theorem is a corollary of this. Again that is also a special case of Turan's Theorem. So , if you know these do not use them to prove the P18. Try to use basic things. You should not use missiles to kill mosquitoes.

§1.20 P20

Some useful lemmas in GT (should have added them earlier): With a graph with n vertices where degree of the i th vertex is d_i and total number of edges is k (A) $\sum d_i = 2k$ (B) $\sum d_i^2 \geq \frac{4k^2}{n}$ (C) $\sum \binom{d_i}{2} \geq \frac{2k^2}{n} - k$ (D) $\sum_{v_i v_j \in E}(d_i + d_j) = \sum\limits_{i=1}^{n} d_i^2$ where E is the set of edges.

§1.21 P21

In a tournament with n players , let the number of wins and losses of the i th player be W_i and L_i . Prove that $\sum W_i^2 = \sum L_i^2$

This is essentially a TOMATO problem with an easy solution. But I just wanted to highlight the insight that it is often pretty useful to interpret things combinatorically i.e. it is trivial that $\sum W_i = \sum L_i$ Now the given problem is equivalent to $\sum \binom{W_i}{2} = \sum \binom{L_i}{2}$ which does not suggest nothing. The next problem is also an excellent (and harder) example to this trick.

§1.22 P22

$n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let w_i and l_i be respectively the number of wins and losses of the i-th player. Prove that

$$\sum_{i=1}^{n} (w_i - l_i)^3 \geq 0.$$

§1.23 P23

There are n points on a unit circle. Show that at most $n^2/3$ pairs of these points are at a distance greater than $\sqrt{2}$

I am not very sure whether the bound is tight, I guess there may not be any 4 cycles in the graph where you join two vertex iff they are at least at a distance of $\sqrt{2}$ from each other and as far as I know there is a bound on the number of edges of a graph with no four cycles.

§1.24 P24

Let P_1, P_2, $\ldots$, P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1, R_2, $\ldots$, R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points $B_1, \ldots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering $R_1, \ldots, R_n$ of the red points.

§1.25 P25

There are $N > n^2$ stones on a table. A and B play a game. A begins, and they play alternate. In each turn a player can remove k stones where k is a positive integer that is either less than n or a multiple of n . The player who takes the last stone wins. Prove that A has a winning strategy.

Note:[A little background of Games specifically positional analysis is recommended (not necessarily required though) to do this problem. Let a position be marked as 0 if the player who has moved the game to that position has a winning strategy and marked 1 if the player who will move from that position has a winning strategy. Now note that (or at least convince yourself) from any 0 position all the moves will go to a 1 position and for any 1 position there exists at least a move to a 0 position. Now note that the last position is always a 0 position (we are considering that the game is deterministic and finite and so one and exactly one player has a winning strategy) Now if A starts from a 1 position he has the winning strategy. Let us assume the contrary. So, the first position is 0 we will write it as $f(N) = 0$(just for sake of ease) Hints: Note that $f(N - in) = 1$ for all $1 \leq i \leq n$ Let A moves from N to $N - in$ and then B goes to $N - in - jn$ Do you see any fallacy here?

So, B has to move to some $N - in - k(i)$ where $k(i) < n$. Now $1 \leq i \leq n$ and $k(i) \leq n - 1$. Does this remind you something?

If you have figured out the last part , can you find a $0 \to 0$ move to finish the problem?]

§1.26 P26

The game of double chess is played like regular chess, except each player makes two moves in their turn. As usual, white starts. Prove that white can always guarantee a win or a draw. (Double Chess / PSS)

§1.27 P 27

There are three empty jugs on a table. Arnab, Shounak, and Sayan put walnuts in the jugs one by one. They play successively, with the initial determined by a draw. Thereby Arnab plays either in the first or second jug, Shounak in the second or third, and Sayan in the first or third. The player after whose move there are exactly 1999 walnuts loses the games. Show that Arnab and Sayan can cooperate so as to make Shounak lose.

§1.28 P28

Arnab and Shounak clone are playing a game. Initially, $n \geq 3$ points are given on a circle. On a player's turn, that player must draw a triangle using three unused points as vertices, without creating any crossing edges. The first player who cannot move loses. If Shounak goes first and players alternate turns, who wins? (Your answer may be in terms of n .) (Hint: You can not cheat but you can copy!! So be a copycat.)

§1.29 P 29

On the circle, 99 points are marked, dividing this circle into 99 equal arcs. Arnab and Shounak play the game, taking turns. Arnab goes first; on his first move, he paints in red or blue any marked point. Then each player can paint on his own turn, in red or blue, any uncolored marked point adjacent to the already painted one. Shounak wins, if after painting all points there is an equilateral triangle, all three vertices of which are colored in the same color. Could Arnab prevent him?

§1.30 P30

Consider a round-robin tournament with $2n+1$ teams, where each team plays each other team exactly one. We say that three teams X, Y and Z, form a cycle triplet if X beats Y, Y beats Z and Z beats X. There are no ties. a)Determine the minimum number of cycle triplets possible. b)Determine the maximum number of cycle triplets possible.

§1.31 P 31

There are 10001 students at an university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

i.) Each pair of students are in exactly one club.

ii.) For each student and each society, the student is in exactly one club of the society.

iii.) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies.

Find all possible values of k.

§1.32 P 32

Let S be a set of n elements and $S_1,\ S_2, \ldots,\ S_k$ are subsets of S ($k \geq 2$), such that every one of them has at least r elements.

Show that there exists i and j, with $1 \leq i < j \leq k$, such that the number of common elements of S_i and S_j is greater or equal to: $r - \frac{nk}{4(k-1)}$

§1.33 P33

Let $a_1, a_2, ..., a_n$ be arbitrary real numbers. Prove that there exists a real number x such that each of the numbers $x + a_1, x + a_2, ..., x + a_n$ is irrational.

§1.34 P34

Prove that there is a Fibonacci number that ends with 2023 zeros.

§1.35 P 35

Let $(a_n)_{n\geq 1}$ be an infinite sequence of integers $a_{n+2} = 7a_{n+1}^2 + a_n$. where $a_1 = 1, a_2 = 25$. Find the remainder when $a_{2023!}$ is divided by 44.

(I know last three problems were not combi but they were PHP and that's why I added them here. This is very important idea often used to consider PHP on tuples.)

§1.36 P 36

On every square of a 1997×1997 board is written ± 1. For every row we compute the product $R - i$ of all the numbers written in that row and for every column we compute the product C_i of all numbers written in that column. Prove that $\sum_{i=1}^{1997}(R_i + C_i)$ is never equal to zero.

§1.37 P 37

Let S be a subset of $\{0, 1, 2, \ldots, 9\}$. Suppose there is a positive integer N such that for any integer $n > N$, one can find positive integers a, b so that $n = a + b$ and all the digits in the decimal representations of a, b (expressed without leading zeros) are in S. Find the smallest possible value of $|S|$

§1.38 P38

Show that any tournament (directed graph) has a $HAMILTONIAN$ Path.

In a sports tournament of n players, each pair of players plays exactly one match against each other. There are no draws. Prove that the players can be arranged in an order $P_1, P_2, ...P_n$ such that P_i defeats P_{i+1} for all $I \in [n-1]$

You might also be interested in counting the maximum number of such Hamiltonian Path or to bound the maximum number of such Hamiltonian paths in all the sets of the possible $2^{\binom{n}{2}}$ many $n-$ tournaments. The lower bound is easy to show and is $2^{(1-n)} \cdot n!$. (Challenge question if you want to solve it). However, the upper bound which is of the order $n^{1.5}2^{(1-n)} \cdot n!$ is beyond scope.

§1.39 P 39

Show that for any graph G where each vertex has degree at least k there exists a path which has at least $k + 1$ vertices.

§1.40 P40

Ore's theorem: Let G be a graph so that for all vertices i and j , $deg(i) + deg(j) \geq n$ Show that

(1.) G is connected

(2.) G is hamiltonian i.e. there exists a cycle spanning all the vertices.
(Dirac's theorem where minimum degree of $G, \delta(G) \geq \lceil n/2 \rceil$ is an immediate corollary to this.)

§1.41 P 41

A Magician and a Detective play a game. The Magician lays down cards numbered from 1 to 52 face-down on a table. On each move, the Detective can point to two cards and inquire if the numbers on them are consecutive. The Magician replies truthfully. After a finite number of moves, the Detective points to two cards. She wins if the numbers on these two cards are consecutive, and loses otherwise.

Prove that the Detective can guarantee a win if and only if she is allowed to ask at least 50 questions.

§1.42 P 42

Let n be a positive integer. Initially the sequence $0, 0, \cdots, 0$ (n times) is written on the board. In each round, Arnab chooses an integer t and a subset of the numbers written on the board and adds t to all of them. What is the minimum number of rounds in which Arnab can make the sequence on the board strictly increasing?

§1.43 P 43

Let $n \geq k$ be positive integers, and let $\mathcal{F}$ be a family of finite sets with the following properties: (i) $\mathcal{F}$ contains at least $\binom{n}{k} + 1$ distinct sets containing exactly k elements; (ii) for any two sets $A, B \in \mathcal{F}$, their union $A \cup B$ also belongs to $\mathcal{F}$. Prove that $\mathcal{F}$ contains at least three sets with at least n elements.

§1.44 P 44

Let $n \geq 3$ be an integer and let $A_1, A_2, \ldots, A_n$ be n distinct subsets of $S = \{1, 2, \ldots, n\}$. Show that there exists $x \in S$ such that the n subsets $A_i - \{x\}, i = 1, 2, \ldots n$ are also distinct.