(Another Non-Geometry and stolen) RMO Mock

AHAN CHAKRABORTY

§1 P1

Let $a_1, a_2, \cdots, a_{50}, b_1, b_2, \cdots, b_{50}$ be permutations of numbers $1, 2, \cdots, 100$ all distinct partitioned into two sets: the set A is such that $a_1 > a_2 > \cdots > a_{50}$ and the set B is such that $b_{50} > b_{49} > \cdots > b_1$. Prove that

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{50} - b_{50}|$$

is a perfect square.

§2 P2

Consider a round-robin tournament with $2n + 1$ teams, where each team plays each other team exactly one. We say that three teams X, Y and Z, form a "cyclo-phile triplet" if X beats Y, Y beats Z and Z beats X. There are no ties.

a)Determine the minimum number of cyclo-phile triplets possible.

b)Determine the maximum number of cyclo-phile triplets possible.

§3 P3

Show that if a, b, c are positive integers such that $\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)$ is an integer, then abc is a perfect cube.

§4 P4

Let $S = \{1, 2, \ldots, 2024\}$. For each non-empty subset $T \subseteq S$, one of its members is chosen as its representative. Find the number of ways to assign representatives to all non-empty subsets of S so that if a subset $D \subseteq S$ is a disjoint union of non-empty subsets $A, B, C \subseteq S$, then the representative of D is also the representative of one of A, B, C.

§5 P5

Call a number n "depressed for 0" if it has at least one multiple with 0 not used at any place as a digit. Find all "depressed for 0" numbers.

§6 P6

For $a, b, c \in \mathbb{R}^+$ such that

$$ab + bc + ca \le 3abc$$

then show that

$$a + b + c \le a^3 + b^3 + c^3$$