IOQM MOCK 1

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P1. Let the number of quadruples (a, b, c, d) of positive integers satisfying

$$12a + 21b + 28c + 84d = 2024$$

be M. Find the largest prime factor of M. [2]

P2. Let ABC be a triangle with side lengths $AB = 2011$, $BC = 2012$, $AC = 2013$. Create squares $S_1 = ABB'A''$, $S_2 = ACC''A'$, and $S_3 = CBB''C'$ using the sides AB, AC, BC respectively, so that the side $B'A''$ is on the opposite side of AB from C, and so forth. Let square S_4 have side length $A''A'$, square S_5 have side length $C''C'$, and square S_6 have side length $B''B'$. Let $A(S_i)$ be the area of square S_i.

Compute

$$\frac{A(S_4) + A(S_5) + A(S_6)}{A(S_1) + A(S_2) + A(S_3)}?$$

[2]

P3. How many triangles with positive area are there whose vertices are points in the xy-plane whose coordinates are integers (x, y) satisfying $1 \leq x \leq 4$ and $1 \leq y \leq 4$? Report the final two digits of your answer if it is more than two digits. [2]

P4. One boring morning, Arnab and Shounak are playing a game. There is a counter initially 0. They are thinking a number in $\{1, 2, 3, ..., 10\}$, then adding the number in their mind with the current value of the counter, then saying loud the new value of the counter. The person who says 2025 first, will win.

e.g. Say, Arnab first thinks 5, he wll say 5 loud, Shounak thinks 2, he will say 7 loud and so on.

Arnab is starting the game. Which number should he choose so that he can always ensure a win? If you think there is no such number, report 0. [2]

P5. Let AC be a diameter of a circle ω of radius 1, and let D be the point on AC such that $CD = \frac{1}{5}$. Let B be the point on ω such that DB is perpendicular to AC, and let E be the midpoint of DB. The line tangent to ω at B intersects line CE at the point X. Compute AX. [2]

P6. Let P be a polynomial satisfying $P(x + 1) + P(x - 1) = x^3$ for all real numbers x. Find the value of $P(6)$. [2]

P7. The side lengths of a scalene triangle are roots of the polynomial

$$x^3 - 20x^2 + 131x - 301$$

Find the square of the area of the triangle. [2]

P8. There are 8 leaves in ISI pond indexed $L_1, L_2, \cdots, L_8$. Dogesh starts on L_1. During the i^{th} day of the semester, Dogesh feels bored and shifts from L_i to L_{i+1}, without falling into the pond with a probability $\dfrac{1}{i+1}$. The probability that Dogesh ends up on L_8 without falling into the pond at any point can be written as $\dfrac{p}{q}$, where p, q are coprime integers. Find $p^2 + q^2$. [2]

P9. Consider a 6×6 grid of square tiles. Ritwick chooses four of these squares uniformly at random. The probability that the centers of these four squares form a square themselves is $\frac{p}{q}$ where p, q are co-prime naturals. Write the largest 2 digit odd factor of $q - p$. [2]

P.S. If you solve this problem correctly, Ritwick Pal will give you a treat (Terms and Conditions apply).

P10. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_n = \sum_{k=1}^{n} \sin(k)$, where k represents radian measure. Find the index of the 15th term for which $a_n < 0$. [2]

P11. Call a positive integer $n \geq 2$ "flexible" if for every k such that $2 \leq k \leq n$, n can be expressed as a sum of k positive integers that are relatively prime to n (although not necessarily relatively prime to each other). How many special integers are there less than 2025? Report the sum of digits of your answer. [3]

P12. In parallelogram $ABCD$, E is a point on $\overline{AD}$ such that $\overline{CE} \perp \overline{AD}$, F is a point on $\overline{CD}$ such that $\overline{AF} \perp \overline{CD}$, and G is a point on $\overline{BC}$ such that $\overline{AG} \perp \overline{BC}$. Let H be a point on $\overline{GF}$ such that $\overline{AH} \perp \overline{GF}$, and let J be the intersection of $\overrightarrow{EF}$ and $\overrightarrow{BC}$. Given that $AH = 8$, $AE = 6$, and $EF = 4$, compute CJ.

[3]

P13. For a permutation $p = (a_1, a_2, \ldots, a_9)$ of the digits $1, 2, \ldots, 9$, let $s(p)$ denote the sum of the three 3-digit numbers $a_1a_2a_3$, $a_4a_5a_6$, and $a_7a_8a_9$. Let m be the minimum value of $s(p)$ subject to the condition that the units digit of $s(p)$ is 0. Let n denote the number of permutations p with $s(p) = m$. Find largest odd divisor of $|m - n|$. [3]

P14. Let x and y be positive real numbers that satisfy $(\log x)^2 + (\log y)^2 = \log(x^2) + \log(y^2)$. Compute the maximum possible value of $(\log xy)^2$. [3]

P15. Let $f(x) = x^2 + 4x + 2$. Let r be the difference between the largest and smallest real solutions of the equation $f(f(f(f(x)))) = 0$. Then $r = a^{\frac{p}{q}}$ for some positive integers a, p, q so a is square-free and p, q are relatively prime positive integers. Compute $a + p + q$. [3]

P16. For a positive integer n, let $f(n) = \sum_{i=1}^{n} \lfloor \log_2 i \rfloor$. Let x be the largest $n < 2025$ such that $n \mid f(n)$. Find the last 2 digits of x. [3]

P17. For a positive integer n, let $f(n)$ be the number of (not necessarily distinct) primes in the prime factorization of n. For example, $f(1) = 0$, $f(2) = 1$, and $f(4) = f(6) = 2$. Let $g(n)$ be the number of positive integers $k \leq n$ such that $f(k) \geq f(j)$ for all $j \leq n$. Find the sum of distinct prime factors of

$$g(1) + g(2) + \cdots + g(100).$$

[3]

P18. In an election between A and B, during the counting of the votes, neither candidate was more than 2 votes ahead, and the vote ended in a tie, 6 votes to 6 votes. Two votes for the same candidate are indistinguishable. If votes could have been counted in n many orders, find sum of digits of n.
(One possibility is $AABBABBABABA$). [3]

P19. Archishman and Saikat each have 3 distinct coins. They put some or all of their coins into 3 identical boxes respectively (separately). Let the probability that they use the same number of boxes be $\frac{p}{q}$ where p, q are coprime integers. Find the largest prime factor of q. [3]

P20. In $\triangle ABC$, AD, BE, CF are cevians to BC, CA, AB respectively, such that $\frac{AF}{FB} = \frac{BD}{DC} = \frac{CE}{EA} = n \in \mathbb{N}$. Let $a_n = \frac{\Delta}{A}$ where Δ is the area of $\triangle ABC$, and A is the area of the triangle formed by the intersections of the cevians AD, BE, CF. Find the sum of all integral values of a_n. [3]

P21. Let n be a positive integer. Let $f(n)$ be the probability that, if divisors a, b, c of n are selected uniformly at random with replacement, then $\gcd(a, \operatorname{lcm}(b, c)) = \operatorname{lcm}(a, \gcd(b, c))$. Let $s(n)$ be the sum of the distinct prime divisors of n. If $f(n) < \frac{1}{2025}$, compute the smallest possible value of $s(n)$. [5]

P22. Let Γ be a circle with center A, radius 1 and diameter BX. Let Ω be a circle with center C, radius 1 and diameter DY, where X and Y are on the same side of AC. Γ meets Ω at two points, one of which is Z. The lines tangent to Γ and Ω that pass through Z cut out a sector of the plane containing no part of either circle and with angle 60°. If $\angle XYC = \angle CAB$ and $\angle XCD = 90^\circ$, then the length of XY can be written in the form

$$\frac{\sqrt{a} + \sqrt{b}}{c}$$

for integers a, b, c where $\gcd(a, b, c) = 1$. Find $a + b + c$. [5]

P23. Let $P(x) = x^3 - 6x^2 - 5x + 4$. Suppose that y and z are real numbers such that

$$zP(y) = P(y - n) + P(y + n)$$

for all reals n. Evaluate $|P(y)|$. [5]

P24. Let $p, q \leq 200$ be prime numbers such that $\frac{q^{p-1}}{p}$ is a square. Find the sum of $p + q$ over all such pairs. [5]

P25. Atmadeep, Souradeep, and Anshuman are playing a game with 6 cards numbered 1 through 6. Each player is dealt 2 cards uniformly at random. On each player's turn, they play one of their cards, and the winner is the person who plays the median of the three cards played. Anshuman goes last, so Atmadeep and Souradeep decide to tell their cards to each other, trying to prevent him from winning whenever possible. Let the probability that Anshuman wins regardless be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$. [5]

P26. The following 100 numbers are written on the board:

$$2^1 - 1, 2^2 - 1, 2^3 - 1, \dots, 2^{100} - 1.$$

Inesh chooses two numbers a, b, erases them and writes the number $\dfrac{ab - 1}{a + b + 2}$ on the board. He keeps doing this until a single number remains on the board.

If the sum of all possible numbers he can end up with is $\dfrac{p}{q}$ where p, q are coprime, then what is the value of $\sqrt{\log_2(p + q)}$? [5]

P27. For an arbitrary positive integer n, we define $f(n)$ to be the number of ordered 5-tuples of positive integers, $(a_1, a_2, a_3, a_4, a_5)$, such that $a_1 a_2 a_3 a_4 a_5 \mid n$. Compute the sum of all n for which $f(n)/n$ is maximized. Report the sum of digit of this number as your answer. [5]

P28. Let $SP_1P_2P_3EP_4P_5$ be a heptagon. A frog starts jumping at vertex S. From any vertex of the heptagon except E, the frog may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Find the number of distinct sequences of jumps of no more than 12 jumps that end at E. [5]

P29. Circles ω_1 and ω_2 intersect at two points P and Q, and their common tangent line closer to P intersects ω_1 and ω_2 at points A and B, respectively. The line parallel to line AB that passes through P intersects ω_1 and ω_2 for the second time at points X and Y, respectively. Suppose $PX = 10, PY = 14$, and $PQ = 5$. Then the area of trapezoid $XABY$ is $m\sqrt{n}$ where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$. [5]

P30. Call a permutation $a_1, a_2, \dots, a_n$ of the integers $1, 2, \dots, n$ quasi-increasing if $a_k \leq a_{k+1} + 2$ for each $1 \leq k \leq n - 1$. For example, 53421 and 14253 are quasi-increasing permutations of the integers $1, 2, 3, 4, 5$, but 45123 is not. Find the number of quasi-increasing permutations of the integers $1, 2, \dots, 7$. [5]

"You miss 100% of the shots you don't take. – Wayne Gretzky" – Michael Scott