ISI MOCK

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§1 P1

Call a cubic polynomial $P(x)$ to be "Giggity Poly" if $P(x)$ has three distinct integer roots and also P' has at least one integer roots as well. Prove or disprove that there exists infinitely many giggity poly polynomials.

Hint: $3+6 \mid 3 \times 6$

§2 P2

Let $f(x) = x^5$. For $x_1 > 0$, Let $P_1 = (x_1, f(x_1))$. Draw tangent at P_1 and it cuts the graph again at P_2. Thus construct P_n.Show that the ratio

$$\frac{(\Delta P_n P_{n+1} P_{n+2})}{(\Delta P_{n+1} P_{n+2} P_{n+3})}$$

is constant.

§3 P3

(a)Bart and Lisa are playing a game. Bart gives Lisa two distinct lines which are non-parallel mutually and a point other than their point of intersection. Show that Lisa can always find a unique parabola touching those two lines and having focus at the point given by Bart.

(b)Say Bart gives the point (5,5) and the two lines $y = x + 1$ and $y = -x + 1$. Find the equation of the parabola.

§4 P4

Call two strictly increasing linear functions $f(x)$ and $g(x)$ "emotionally attached linear pair" if $f(x)$ is an integer iff $g(x)$ is an integer. Find all "emotionally attached linear pair".

§5 P5

$f : [a, b] \to [a, b]$ (a, b may not be attained) is a differentiable function with continuous and positive first derivative. Prove that $\exists c \in (a, b)$ such that

$$f(f(b)) - f(f(a)) = (b - a)(f'(c))^2$$

§6 P6

Three quadratic polynomials $x^2 + 2ax + b; x^2 + 2bx + c; x^2 + 2cx + a$ with natural number co-efficients are said to be "mio-amore triplet" ifall their roots are integer. Find all "mio-amore triplet"

§7 P7

Find the positive integer solutions to the equation

$$5^x + 12^y = z^2$$

§8 P8

Let $|U|, \sigma(U)$ denote the number of elements and the sum of elements of a set U (both 0 if empty set) . Let S be a finite set of positive integers. Prove that the sum

$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|}$$

is same as the product of the integers for any $m \geq \sigma(S)$.

(Nasty Calculations are punishable offence)

§9 P9

Let $f : [0, 1] \to \mathbb{R}$ is a continuous function such that there exusts $a \in (0, 1]$ so that $\int\limits_0^a f(x)\, dx = 0$. Prove that

$$|\int_0^1 f(x)\, dx| \leq \frac{1-a}{2} M$$

where $M = \sup_{x \in (0,1)} f'(x)$

§10 P10

(a) Prove that for any positive integer x and non-negative integer n

$$\sum_{k=0}^{n} \binom{k+x}{k}^{-1} = \frac{x}{x-1} \left(1 - \frac{1}{\binom{x+n}{n+1}}\right)$$

(b) Prove that for $r \geq 2$,

$$\sum_{n=r}^{\infty} \frac{1}{\binom{n}{r}} = \frac{r}{r-1}$$