A Mock

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§1 P1

Let $|U|, \sigma(U)$ denote the number of elements and the sum of elements of a set U (both 0 if empty set) . Let S be a finite set of positive integers. Prove that the sum

$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|}$$

is same as the product of the integers for any $m \geq \sigma(S)$.

(Nasty Calculations are punishable offence but recycling questions are not.)

§2 P2

Let $g(n)$ be the greatest odd divisor of n. Prove that

$$0 < \left(\sum_{k=1}^{n} \frac{g(k)}{k} \right) - \frac{2n}{3} < \frac{2}{3}.$$

(Hint: Don't ignore the scope of telescope but you may ignore the pun.)

§3 P3

For a positive integer n , denote by $S(n)$ the number of choices of the sign + or − such that $\pm 1 \pm 2 \pm 3 ... \pm n = 0$. Then prove that

$$S(n) = \frac{2^{n-1}}{\pi} \int_{0}^{2\pi} \cos t \cos 2t ... \cos nt \, dt$$

. (Hint: Use complex numbers if you want (you will need it trust me) and note that the problem is a combi problem(Yes! This is a hint))

§4 P4

Let $P(z)$ be a polynomial of degree n, whose all zeroes have absolute value 1 in the complex plane. Set $g(z) = \frac{P(z)}{z^{n/2}}$. Show that all roots of the equation $g'(z) = 0$ have absolute value 1.

§5 P5

Suppose that $a, b, c > 0$ such that $abc = 1$. Prove that

$$\frac{ab}{ab + a^5 + b^5} + \frac{bc}{bc + b^5 + c^5} + \frac{ca}{ca + c^5 + a^5} \leq 1.$$

§6 P6

Find the maximum value of

$$\int_0^1 [f'(x)^2]|f(x)|\frac{1}{\sqrt{x}}\, dx$$

over all continuously differentiable functions $f : [0, 1] \to \mathbb{R}\,; f(0) = 0\,; \int\limits_0^1 [f'(x)]^2\, dx \leq 1$.

(Hint: Just follow what your heart is telling you. Believe in yourself☺)

§7 P7

In a shop, there are infinitely many tickets with exactly one natural number written on each of them. Suppose, for any $n \geq 1$, there are n tickets exactly hosting a number d dividing n. For example if $n = 6$ the number of tickets with their number belonging to $\{1, 2, 3, 6\}$ is exactly 6. Show that

(a) Every natural number occurs at least on one ticket and on a finite number of tickets.

(b) No number occurs exactly on 2023 tickets.

(c) Say, $f(n)$ is the number of times n is appearing. Find a closed form of $f(n)$ and find all n such that $f(n)$ is odd.

(Don't tell anyone if you know the source.)

§8 P8

Construct a triangle using straight-edge and compass with proof ,where two ex-centres and the in-center are given.

Do not cheat yourself (though I don't care) and remember you are the BEST!