(Probably a non-geometry and stolen) RMO Mock

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§1 P1

200 aops-ers participated in my RMO mock (which is one of my dreams). They had 6 problems to solve. As I make really easy mocks, each problem was correctly solved by at least 120 participants. Prove that there must be 2 (orz) participants such that every problem was solved by at least one of these two students.

§2 P2

Let $a, b, c > 0$ be real numbers. Prove that

$$\sum_{cyc} \frac{2a}{\sqrt{3a+b}} \leq \sqrt{3(a+b+c)}$$

§3 P3

On the circle, 99 points are marked, dividing this circle into 99 equal arcs. Shounak and Arnab play the game, taking turns. Shounak goes first; on his first move, he paints in red or blue any marked point. Then each player can paint on his own turn, in red or blue, any uncolored marked point adjacent to the already painted one. Arnab wins, if after painting all points there is an equilateral triangle, all three vertices of which are colored in the same color. Could Shounak prevent him?

§4 P4

Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorisation with $1 < p_1 < \cdots < p_k$. Prove that e_1 is an even number. A number n is perfect if $s(n) = 2n$, where $s(n)$ is the sum of the divisors of n.

§5 P5

Mr. Burns is very angry on Homer Simpson because of eating 120 donuts a day. He has ordered Homer to play a game with him in which if Homer loses, will be thrown in the mouth of a radioactive crocodile. Considering Homer a dumb, Burns allows Homer to choose in natural number $n \in \mathbb{N}$. Now they will take the polynomial $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$ and in turn, they choose one of the coefficients and assign a real value to it. Burns,being the boss, has the first move. Once a value is assigned to a coefficient, it cannot be changed any more.

The game ends after all the coefficients have been assigned values. Now Homer wins if the final polynomial is divisible by a polynomial given by Burns. Homer being an idiot has gone to Lisa for help. Lisa somehow knew from Burns' secretary Smithers that the polynomial will be $x^2 + 1$ (because Burns has bunked math classes too so he cannot do lengthy calculations). Find all n which Lisa should choose so that she can save her dad (to get a saxophone in return) and for that what strategy should she follow?

(I am so terrible at phrasing)

§6 P6

Let n be an odd positive integer, and let $x_1, x_2, \cdots, x_n$ be non-negative real numbers. Show that

$$\min_{i=1,\ldots,n} \left(x_i^2 + x_{i+1}^2\right) \leq \max_{j=1,\ldots,n} \left(2x_j x_{j+1}\right)$$

where $x_{n+1} = x_1$. ☺□

Best of luck, REMEMBER YOU,being the part of the infinite BRAHMAN, ARE THE BEST.