IOQM MOCK 3

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P1. Equilateral triangles ABF and BCG are constructed outside regular pentagon $ABCDE$. Compute $\angle FEG$.

P2. Find the greatest two-digit positive integer x such that for all three-digit (base 10) positive integers abc, if abc is a multiple of x, then the three-digit (base 10) number bca is also a multiple of x.

P3. Let $ABCD$ be a cyclic quadrilateral, with $AB = 7$, $BC = 11$, $CD = 13$, and $DA = 17$. Let the incircle of ABD hit BD at R and the incircle of CBD hit BD at S. What is RS?

P4. Positive integers a, b, c have the property that a^b, b^c, and c^a end in $4, 2$ and 9, respectively. Find the minimum possible value of $a + b + c$.

P5. Find sum of the 2 numbers in all pairs (p, q) of twin primes such that

$$(2p + q)^3 = p^3 + 2q^3 + 2018.$$

Note that twin prime means two primes with difference 2.

P6. Let $A_1A_2 \dots A_{19}$ be a regular nonadecagon. Lines A_1A_5 and A_3A_4 meet at X. Find the integer nearest to $\angle A_7XA_5$.

P7.

Two non-intersecting circles, not lying inside each other, are drawn in a plane. Two lines pass through a point P which lies outside each circle. The first line intersects the first circle at A and A' and the second circle at B and B'; here A and B are closer to P than A' and B' and P lies on segment AB. Analogously, the second line intersects the first circle at C and C' and the second circle at D and D'. If A, B, C, D are concyclic then find

$$\frac{PA \cdot PB \cdot PA' \cdot PB'}{PC \cdot PD \cdot PC' \cdot PD'}.$$

P8. Let (x, y) be the unique ordered pair of real numbers satisfying the system of equations

$$\frac{x}{\sqrt{x^2+y^2}} - \frac{1}{x} = 7 \quad \text{and} \quad \frac{y}{\sqrt{x^2+y^2}} + \frac{1}{y} = 4.$$

If $|x| = \frac{r}{s}$ and $|y| = \frac{t}{u}$ where r, s, t, u are positive integers such that $\gcd(r, s) = 1$ and $\gcd(t, u) = 1$, find $r + s - t - u$.

P9. In triangle ABC, points M and N are the midpoints of AB and AC, respectively, and points P and Q trisect BC. Given that A, M, N, P, and Q lie on a circle and $BC = 1$, let the area of triangle ABC be written as $\frac{\sqrt{m}}{n}$ where m, n are positive integers such that m, n are relatively prime. Find $m+n$.

P10. Let $f(x) = \frac{x+a}{x+b}$ satisfy $f(f(f(x))) = x$ for real numbers a, b. If the maximum value of a is $\frac{p}{q}$, where p, q are relatively prime integers, what is $|p| + |q|$.

P11.

Suppose we have a sequence $a_1, a_2, \ldots$ of positive real numbers so that for each positive integer n, we have that

$$\sum_{k=1}^{n} a_k a_{\lfloor \sqrt{k} \rfloor} = n^2.$$

Determine the first value of k so $a_k > 100$.

P12. Arnab has six distinct coins in a jar. Occasionally, he takes out three of the coins and adds a dot to each of them. If the number of orders in which Arnab can choose the coins so that, eventually, for each number $i \in \{0, 1, \ldots, 5\}$, some coin has exactly i dots on it, is N. Find last two digits of $N/100$

P13. Let $\mathcal{P}$ be the power set of $\{1, 2, 3, 4\}$ (meaning the elements of $\mathcal{P}$ are the subsets of $\{1, 2, 3, 4\}$). How many subsets S of $\mathcal{P}$ are there such that no two distinct integers $a, b \in \{1, 2, 3, 4\}$ appear together in exactly one element of S. Report last two digits of your answer

P14. Shounak has six boxes lined up in a straight line. Inside each of the first three boxes are a red ball, a blue ball, and a green ball. He randomly selects a ball from each of the three boxes and puts them into a fourth box. Then, he randomly selects a ball from each of the four boxes and puts them into a fifth box. Next, he randomly selects a ball from each of the five boxes and puts them into a sixth box, arriving at three boxes with 1, 3, and 5 balls, respectively. The probability that the box with 3 balls has each type of color is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

P15. In a square $ABCD$ of side length 1, two circular arcs are drawn, each with radius 1 and centres A, B respectively. 4 circles are drawn, each tangent to both of these arcs as well as one of the sides of the square, inside the square. Find the integer closest to 10 times the area of the quadrilateral obtained by joining their centres.

P16. Consider all positive integers a, b such that $\operatorname{lcm}(a^2 - 1, b^2 - 1) = 29^2 - 1$. Find the last two digits of the sum of all distinct values of $a + b$.

P17. Let $p(x)$ and $q(x)$ be two distinct polynomials with integer coefficients, of degree less or equal to 6, that satisfy $p(k) = q(k)$ and $p(-k) = -q(-k)$ for all integers $1 \leq k \leq 6$. What is the number of multiples of 6 which are also divisors of the minimum value of $p(0)^2 + q(0)^2$?

P18. Real numbers x, y, z are such that $x + \lfloor y \rfloor + \{z\} = 4.2$, $y + \lfloor z \rfloor + \{x\} = 3.6$, $z + \lfloor x \rfloor + \{y\} = 2.0$. What is the integer closest to $10x$?

P19. Consider all positive integers k for which there exists a positive integer n such that

$$n^4 + \frac{n^3 + n^2}{2} + n + 1 = k^2.$$

Find the greatest of all such k. Write the last two digits.

P20. A math class consists of five pairs of best friends who need to form five groups of two for an upcoming project. However, to promote synergy in the classroom, the teacher forbids any student to work with his best friend. In how many ways can the students pair up? Write the last two digits as your answer.

P21. For a positive integer N, let $r(N)$ be the number obtained by reversing the digits of N. For example, $r(2013) = 3102$. Find sum of all 3-digit numbers N for which $r(N)^2 - N^2$ is the cube of a positive integer. Report the nearest integer to the square root of your answer.

P22. Drohan, Subhankar, and Ritwick each flip a coin every day, starting from Day 1, until all three of them have flipped heads at least once. The last of them to flip heads for the first time does so on Day X. The probability that X is even can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

P23. In parallelogram $ABCD$, the circumcircle of $\triangle BCD$ has center O and intersects lines AB and AD at E and F, respectively. Let P and Q be the midpoints of AO and BD, respectively. Suppose that $PQ = 3$ and the height from A to BD has length 7. Find the last two digits of the value of $BF \cdot DE$.

P24. Arnab Sanyal and Arnab Palit are running for the position of "best Arnab in RSM". The election works as follows: There are 10 batches, each composed of 27 students exactly who will vote. The voters each cast their vote with equal probability to Sanyal or Palit. A majority of votes in a batch towards a candidate means they "win" the batch, and the candidate with a majority of wins batches becomes the best Arnab. Should both candidates win an equal number of batches, then whoever had the most votes cast for them wins.

Let the probability that they have an unresolved election, i.e., they both have a batch count of 5 and student vote count of 135, be $\frac{p}{q}$, and let m, n be the remainders when p, q, respectively, are divided by 100. Find $m + n$.

P25. Aishik has a bag with 16 marbles, each labeled with a distinct positive divisor of 210. Every minute, he draws two distinct marbles from the bag, uniformly at random. If they have labels a and b such that either $a|b$ or $b|a$, he keeps them and discards the marbles left in the bag. Otherwise, he replaces the marbles and repeats the procedure. The probability that Aishik keeps the marble labeled with a 1 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

P26. Over all ordered triples of positive integers (a, b, c) for which $a + b + c^2 = abc$, compute the sum of all values of $a^3 + b^2 + c$. Write the last 2 digits.

P27. An excircle of a triangle is a circle tangent to one of the sides of the triangle and the extensions of the other two sides. Let ABC be a triangle with $\angle ACB = 90°$ and let r_A, r_B, r_C denote the radii of the excircles opposite to A, B, C, respectively. If $r_A = 9$ and $r_B = 11$, then r_C can be expressed in the form $m + \sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

P28. In triangle ABC, $AB = 4, AC = 6$ and $\angle A = 60^\circ$. Define ω as the circumcircle of $\triangle ABC$, D as the midpoint of BC, E as the foot of B onto CA and F as the foot of C onto AB. Suppose AD intersects ω again at X and the circumcircle of $\triangle XEF$ intersects ω again at Y. Then, AY^2 can be expressed as $\frac{m}{n}$, where m and n are positive relatively prime integers. Find the last two digits of $m + n$.

P29. Let $\{x\} = x - \lfloor x \rfloor$. Consider a function f from the set $\{1, 2, \ldots, 2020\}$ to the half-open interval $[0, 1)$. Suppose that for all x, y, there exists a z such that $\{f(x) + f(y)\} = f(z)$. We say that a pair of integers m, n is valid if $1 \leq m, n \leq 2020$ and there exists a function f satisfying the above so $f(1) = \frac{m}{n}$. Determine the sum over all valid pairs m, n of $\frac{m}{n}$. Report the last two digits of your answer.

P30. In the country of LA HA Land, there are an infinite number of cities, connected by roads. For every two distinct cities, there is a unique sequence of roads that leads from one city to the other. Moreover, there are exactly three roads from every city. On a sunny morning in early July, n tourists from ISI have arrived at SN Bose Bhavan, the capital of LA HA Land. They repeat the following process every day: in every city that contains three or more tourists, three tourists are picked and one moves to each of the three cities connected to the original one by roads. If there are 2 or fewer tourists in the city, they do nothing. After some time, all tourists will settle and there will be no more changing cities. For how many values of n from 1 to 2020 will the tourists end in a configuration in which no two of them are in the same city?

In life, you will be challenged in each and every step. This mock is nothing in comparison. So, better suit up! For boosting your confidence here is a legendary specimen of what your attitude should be :