Thesis

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Abstract

(what is accretion) When matter falls into black hole.. Accretion is efficient source of energy...would release XXX ergs of energy...so why don't we see this from black hole at center of galaxy? Such flows are called LLAGNs, ongoing theory. Typical calculations assume plasma is fluid model, But this type of accretion is hot, plasma is almost collisionless, so kinetic theory more appropriate. can we approximate the computationally-expensive kinetic theory with a fluid closure? We simulate using shearing box and find effective viscosity/resistivity....and find that [results]

Contents

Chapter 1

Introduction

Black holes are now a fixture in the popular imagination.

science fiction books and movies. From tiny primordial black holes (XXXXXX) to

but things rarely fall radially inward. Usually the matter has some angular momentum. Since angular momentum is conserved, a disk usually forms around the black hole as a repository for rotating matter. Processes within the disk then lead to turbulence and an effective friction that results in an outward transport of angular momentum and hence the falling in, or "accretion", of matter. For reasons discussed in Chapter **??**, the infalling matter (mostly hydrogen) heats up past the ionization threshold. We shall thus refer to the matter as a plasma; different types of plasma and the effect on accretion flows will be discussed in Chapter **??**.

For now, we motivate this paper by noting that gravitational binding energy is released as matter falls into a black hole, a process that is among the most efficient energy sources in the universe (several hundred times more efficient than fusion [**?**]). This energy presumably goes into radiation and thus

These processes and the debate surrounding them are discusssed throughout the paper, particularly

accretion=falling in. But how does accretion work?

Yay introduction.

1.1 AGN and Accretion Disk Observations

How explain low luminosity from AGN (Sgr A*), x-ray binaries in quiescence *rotation measure stability in Sgr A* (Sharma, Quataert, Stone 2008)

***move TYPES of accretion flows (currently in AFmech) here; Properties, etc.

1.2 Theories of Accretion

Kinetic vs. MHD...useful approximation? Motivation 2. Global sim here! Now zoom in to look at microphysics.

Chapter 2

Accretion Flow Mechanics

Accretion disks occur in many different types, from those are protostars to those in binary star systems and those around black holes. Depending on their context, these accretion flows have different properties. While much of the physics of accretion disks involves magnetohydrodynamics (including the mechanism behind accretion itself) as will be discussed in subsequent chapters, the classification of accretion flows is possible even at a more basic level: for example, hot vs. cold accretion flows.

This chapter clarifies some of the jargon in the extensive literature and characterizes the different types of accretion flows in Section 2.1, focusing on hot accretion flows (the subject of the original research in Chapter **??**) and their properties in Section 2.2. Some fundamental physics are presented in Sections **??** and **??**: respectively radiation processes and mechanics. These sections provide background for more complex calculations such as those in Chapter **??** as well as a connection back to observations.

2.1 Types of Accretion Flows

Accretion flows are generally characterized by their temperature, their radiative efficiency, and/or their thickness. These types and the relationships among them will be clarified and discussed soon enough: we first note that accretion does not have to lead to the formation of a disk. Indeed, some of the earliest studies on accretion concerned matter falling in radially and uniformly from all directions onto a compact object: spherical accretion [**?**]. However, such accretion flows are unlikely to occur in nature because matter will almost always be rotating with respect to the compact object and hence have angular momentum.

Another general type of accretion flows

2.2 Properties of Hot Accretion Flows

2.3 Radiation Processes

observations...

2.4 Mechanics

This section presents a series of calculations pertaining to important elements of accretion disks.

2.4.1 Eddington Luminosity

Radiation from a compact object generates a radiation pressure that begins to counter the gravitational force from the central object. When the luminosity is high enough, the object's gravity is no longer enough to keep it together. When the radiation pressure and gravity exactly cancel, the system is in equilibrium. We can derive the value of the luminosity at this equilibrium, the "Eddington luminosity" or "Eddington limit" as follows (following [**?**]xxxformatxx):

Assume the radiation has a flux F_{rad} and an opacity, or scattering cross-section times mass, of κ (for ionized hydrogen, $\kappa = \sigma_T/m_p$ where m_p is the mass of the proton and σ_T is the Thomson scattering cross-section of the electron). Then the force balance is given by:

$$\nabla\Phi = \frac{\kappa}{c}F_{\mathrm{rad}} \tag{2.1}$$

The luminosity is defined as the total energy output per time, so we can get it from the flux (amount of light per area per time) by integrating over a surface, which we assume to be spherical:

$$L = \int_S F_{\text{rad}} \cdot dS = \frac{c}{\kappa} \int_S \nabla\Phi \tag{2.2}$$

$$= \frac{c}{\kappa} \int_V \nabla^2\Phi = \frac{4\pi Gc}{\kappa} \int_V \rho dV \tag{2.3}$$

$$= \frac{4\pi GMc}{\kappa} \tag{2.4}$$

Note that the force balance equation does not include any forces other than pressure-gravity equilibrium and the radiation pressure; magnetic forces, for example, are excluded.

Black holes can exceed the Eddington luminosity because the energy contained in the infalling matter does not have to be radiated; it can simply be swallowed by the black hole and increase its mass. At high accretion rates, radiation could become trapped by the flow and advect into the black hole. This would lead to low observed luminosity.

2.4.2 Disk Scale Height

The scale height of a disk is a useful scale for defining distances. It comes from considering a stratified disk; that is, one that has a component of gravity in the vertical direction. As Figure **??** illustrates, the gravitational force in the vertical direction is proportional to the force along the purely horizontal ("R") direction and the ratio of the distances:

$$\frac{F_{gR}}{F_{gz}} = \frac{R}{z} \tag{2.5}$$

Introducing the Keplerian rotation speed $\Omega^2 = GM/R^3$, we have

$$\begin{aligned} F_{gz} = F_{gR}\frac{z}{R} &= \frac{GM}{R^2}\frac{z}{R} \\ &= \Omega^2 z \end{aligned}$$

Now relate F_{gz} to the disk density by considering the force balance equation for equilibrium in the vertical direction:

$$-\frac{1}{\rho}\frac{dP}{dz} = F_{gz}$$

Assuming an isothermal equation of state $P = \rho c^2$, this equation can be integrated:

$$-\frac{1}{\rho}c^2\frac{d\rho}{dz} = F_{gz} = \Omega^2 z$$
$$\rho(z) = \rho_0 e^{-\frac{\Omega^2}{c^2}\frac{z^2}{2}} = \rho_0 e^{-\frac{z^2}{2H^2}}$$

where $H^2 \equiv \frac{c^2}{\Omega^2}$. H can thus be interpreted as the distance a sound wave travels in a shear time (i.e., one rotation). Generally the thickness of a disk is taken to be $2H$. As will be discussed in Section **??**, it is often convenient to choose units such that $H = 1$, resulting in numerical domains with dimensions of $(H, 4H, H)$.

2.4.3 Rayleigh Stability Criterion

In hydrodynamics, disks with a Keplerian velocity profile are stable against perturbations. To see this, simply consider perturbations about a circular orbit in an azimuthally symmetric potential $\Phi = \Phi(r, z)$. In cylindrical coordinates we have

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \tag{2.6}$$
$$\hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y} \tag{2.7}$$

Upon taking the time derivatives, we find that

$$\begin{aligned}
\ddot{r} &= \frac{d}{dt}\dot{r} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\frac{d\hat{r}}{dt} + \dot{z}\hat{z}\right) \\
&= \ddot{r}\hat{r} + 2\dot{r}\frac{d\hat{r}}{dt} + r\frac{d^2\hat{r}}{dt^2} + \ddot{z}\hat{z} \\
&= \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} + \ddot{z}\hat{z} \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(\frac{1}{r}\frac{d}{dt}r^2\dot{\theta}\right)\hat{\theta} + \ddot{z}\hat{z}
\end{aligned}$$

From Newton's laws we have

$$F = -m\nabla\Phi = m\ddot{r} \tag{2.8}$$

leading to component-wise equilibrium equations:

$$-\frac{\partial\Phi}{\partial r} = \ddot{r} - r\dot{\theta}^2 \tag{2.9}$$

$$-\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = 0 = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) \tag{2.10}$$

$$-\frac{\partial\Phi}{\partial z} = \ddot{z} \tag{2.11}$$

Note that Eq. 2.10 gives conservation of specific momentum h_z: i.e., $h_z = r^2\dot{\theta}$ is a constant. For a circular orbit, $\ddot{r} = 0$ and we have $\frac{1}{r_0}\frac{\partial\Phi}{\partial r} = \dot{\theta}^2 = \Omega_0^2$, where we define $\Omega_0^2 \equiv \frac{1}{r_0}\frac{\partial\Phi}{\partial r}|_{r_0}$. Now perturbing this circular orbit, we have

$$r = r_0 + \delta r \tag{2.12}$$

$$\theta = \Omega_0 t + \delta\theta \tag{2.13}$$

$$z = 0 + \delta z \tag{2.14}$$

We obtain equations of motion by linearizing Eqns. 2.9-2.11 and using the perturbed variables:

$$\begin{aligned}
-\frac{\partial\Phi}{\partial r} &\approx -\left(\frac{\partial\Phi}{\partial r}\Big|_{r_0} + \delta r\frac{\partial^2\Phi}{\partial r^2}\Big|_{r_0}\right) = -\left(\Omega_0^2 r_0 + \delta r\frac{\partial^2\Phi}{\partial r^2}\Big|_{r_0}\right) \\
&\approx \delta\ddot{r} - r_0\Omega_0^2 - 2\Omega_0 r_0\delta\dot{\theta} - \delta r\Omega_0^2 && (2.15) \\
h_z &\approx r_0^2\Omega + r_0^2\delta\dot{\theta} + 2r_0\Omega_0\delta r \\
&= r_0^2\Omega_0 && (2.16) \\
-\frac{\partial\Phi}{\partial z} &\approx -\left(\delta z\frac{\partial\Phi}{\partial z}\Big|_{r_0,z_0}\right) \\
&\approx \delta\ddot{z} && (2.17)
\end{aligned}$$

From Eq. 2.16, we find that $r_0^2\delta\dot{\theta} = -2r_0\Omega_0\delta r$, and thus $\delta\dot{\theta} = -\frac{2}{r_0}\Omega_0\delta r = -\frac{2}{r_0^3}h_z\delta r$. Using this in the radial equation 2.15 yields

$$\begin{aligned}
\ddot{r} &= 2\Omega_0 r_0\left(-\frac{2}{r_0^3}h_z\delta r\right) + \delta r\left(\Omega_0^2 - \frac{\partial^2\Phi}{\partial r^2}\Big|_{r_0}\right) && (2.18) \\
&= -\left(3\Omega_0^2 + \frac{\partial^2\Phi}{\partial r^2}\Big|_{r_0}\right)\delta r && (2.19) \\
&= -\kappa_0^2\delta r && (2.20)
\end{aligned}$$

where $\kappa_0^2 \equiv 3\Omega_0^2 + \frac{\partial^2\Phi}{\partial r^2}|_{r_0}$ is the epicyclic frequency. The physical significance of this becomes clear upon writing the θ-equation of motion as a coupled harmonic oscillator with the radial equation: a particle will simply execute small circular orbits about its zeroth-order circular path as it travels. This will become an important mechanism distinguishing disk flow from shear flow, as energy can go into epicycles instead of turbulence. Section **??** illustrates this point in the context of accretion.

The epicyclic frequency appears often in astrophysical situations, although under many different guises. For example, remember that $\Omega^2 = \frac{1}{r}\frac{\partial \Phi}{\partial r}$. Then we have

$$\frac{d\Omega^2}{dr} = -\frac{1}{r^2}\frac{\partial \Phi}{\partial r} + \frac{1}{r}\frac{\partial^2 \Phi}{\partial r}$$

such that

$$\frac{\partial^2 \Phi}{\partial r^2} = r\frac{d\Omega^2}{dr} + \frac{1}{r}\frac{\partial \Phi}{\partial r} = r\frac{d\Omega^2}{dr} + \Omega^2$$

With this the epicyclic frequency becomes

$$\begin{aligned} \kappa^2 &= 3\Omega^2 + \frac{\partial^2 \Phi}{\partial r^2} && (2.21) \\ &= 4\Omega^2\left(1 + \frac{A}{\Omega}\right) && (2.22) \end{aligned}$$

where $A \equiv \frac{r}{4\Omega^2}\frac{d\Omega^2}{dr}$ is the Oort A-value, also expressed $A = -\frac{1}{2}d\Omega/d\ln r$. This can be further re-written:

$$\begin{aligned} \kappa^2 &= 4\Omega^2\left(1 + \frac{r}{4\Omega^2}\frac{d\Omega^2}{dr}\right) \\ &= \frac{1}{r^2}\left(4\Omega^2 r^3 + r^4\frac{d\Omega^2}{dr}\right) \\ &= \frac{1}{r^3}\frac{d}{dr}\left(\Omega^2 r^4\right) \end{aligned}$$

which is how it is written in, for example, [**?**]. For Keplerian rotation profiles, $A = -\frac{3}{4}\Omega$, and $\kappa^2 = \Omega^2$.

The stability of hydrodynamical disks can be done in multiple ways. Here, an intuitive argument is first presented showing that angular momentum must decrease outward for stability. This condition is put on firmer theoretical ground in the second method, which also introduces tools that will be used later in performing linear analysis of the magnetohydrodynamic equations (see Section **??**).

First, we look at a physically intuitive argument.

Equations for disk...continuity, momentum balance (will add B fields later) Du/Dt=d2u/dt2+du/dx dx/dt

2.4.4 Keplerian vs. Shear Flow

Chapter 3

Ideal MHD: Theory and Simulations

Understanding magnetohydrodynamics (MHD) is a topic worthy of a semester-long graduate class at least. This chapter briefly presents some basic plasma physics and physically motivates the driving equations of magnetohydrodynamics, which are fluid equations . The more rigorous derivation involves taking moments of the distribution function that will be introduced in Chapter **??**.

The ultimate goal of this thesis, however, is to study a particular phenomenon (the Magneto-Rotational Instability–MRI) in a particular regime (anisotropic viscosity) of non-ideal MHD. As such, for the sake of compactness we only summarize a portion of the necessary theory. This chapter explores ideal MHD (Section **??**) and introduces the physics of the MRI in this most basic setting and how they manifest in simulations (Sections **??** and **??**, respectively). Its results are extended to include viscosity and resistivity in Chapter **??**, and later in Chapter **??**, where the validity of the MHD approximations are evaluated via a comparison to the kinetic regime (also outlined in Chapter **??**). First, however, some context in general plasma physics is in order.

3.1 Properties of a Plasma

Plasma physics applies to a wide range of subject areas, from magnetic-confinement fusion pursuits like the tokamak ITER [**?**] and the stellarator Wendelstein 7-X [**?**] to a variety of astrophysical situations, including the sun's corona and protoplanetary disks. The uniting theme across these different disciplines is the plasma: so what exactly is a plasma?

A plasma is the so-called "fourth state of matter", coming after the gas phase in the increasing kinetic energy hierarchy solid-liquid-gas: that is, the inetic energy of a plasma particle is much greater than its potential energy. A plasma is made up of neutrals and the result of the neutrals' ionization: that is, ions and electrons. The basic physics of single-particle motion in electric and magnetic fields (for example, ∇B drift and $E \times B$ drift) apply to every single particle. Given the enormous quantity of particles (as defined by the plasma parameter Λ in Section 3.1.1, upwards of XXXXXXXXXX particles per cubed meter), working analytically or simulating such a situation for each individual particle is near impossible. Indeed, this is why the kinetic theory is so complicated and requires particle-in-cell simulations (Chapter **??**, Section **??**). The task of the Section **??** is to make this feasible via fluid equations.

3.1.1 Debye Length and Shielding

Since the ions and electrons in a plasma have opposite charge, they tend to attract, leading to the phenomenon of Debye shielding. Electrons of negative charge tend to cluster around a positive test charge and effectively shield the potential such that it is no longer Couloumbic ($V \propto 1/r^2$) but rather exponentially decays ($V \propto e^{-r/\lambda_D}/r$). The distance over which the potential is screened is the total Debye length, which is related to the electron and ion Debye lengths λ_e and λ_i by:

$$\lambda_D^{-2} = \lambda_e^{-2} + \lambda_i^{-2} \tag{3.1}$$

where each species Debye length λ_j is given by

$$\lambda_j = \sqrt{\frac{T_j}{4\pi n_0 e^2}} \tag{3.2}$$

where T_j is the species equilibrium temperature in units of energy (Boltzmann's constant $k_b = 1$), n_0 is the density of each species far away from the test charge, and e is the charge on an electron. The exponential decay comes from the Boltzmann distribution since the system is at equilibrium. Detailed derivations can be found in any standard plasma physics introduction [**?**, **?**].

Going back to our definition of a plasma, requiring that the potential energy of a particle is much less than its kinetic energy leads to the fact that the plasma parameter $\Lambda_s \equiv n_0 \lambda_s^3$ of each species is

$$\Lambda_s \gg 1 \tag{3.3}$$

This means that, in a plasma, there are many particles in a cube of the Debye length (alternative definitions define the plasma parameter as the number of particles within a sphere of radius the Debye length).

Other important quantities include the mean free path λ_{mfp} of a plasma particle, that is, how far it travels on average before it collides with another particle. This is related to the thermal velocity v_T and collision frequency ν by $\lambda_{mfp} = v_T/\nu$.

We can also define the gyrofrequency of a species Ω_s by looking at the equation of motion of a particle in a magnetic field, resulting in $\Omega_s = q_s B/m_s c$. Note that this is half of the Larmor frequency (XXXXXXXXXXX). From the gyrofrequency we define the mean gyroradius ρ_s as the radius of the circle of a particle traveling at the thermal velocity: $\rho_s \equiv v_{T_s}/\Omega_s$. This quantity in particular will prove important in later chapters (i.e. Chapter **??**), where ordering such that the gyroradius is XXX comparable toXXXXmuch less than the mean free path will lead to anisotropic viscosity along the magnetic field lines.

With the basic quantities of λ_{mfp} and ρ_s in mind, we can now turn to the assumptions behind ideal MHD.

3.2 Ideal MHD Equations

Ideal MHD treats the different species of electrons and ions as fluids. In the broader MHD context, each species is at a different temperature, has a different gyroradius, and so on. However, single-fluid MHD, where each species has the same temperature, is still often a useful framework. In these cases the number density adds linearly and other quantities such as velocity are redefined in center-of-mass coordinates.

Magnetohydrodynamics is a fluid formalism; this means that the individual particles' positions and velocities can be averaged out and the important quantities are bulk variables, like the mean flow of the fluid, density, and pressure. Treating each individual particle with a velocity distribution is characteristic of kinetic theory and the distribution function, as discussed in Chapter **??**. We are considering a continuous source rather than discrete particles that would manifest as delta functions. Formally, this is valid when the mean free path of the plasma particles is much less than the length scales of interest:

$$\lambda_{mfp} << L \tag{3.4}$$

Other assumptions include quasi-neutrality (that is, $\sum_s m_s n_s = 0$), XXXXXXXXX We now proceed to motivate the equations of ideal MHD by a combination of conservation arguments and Maxwell's equations. We consider for now a single species. First note that the standard continuity equation, saying that the fluid elements cannot be created or destroyed, leads to the first equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V}\right) = 0 \tag{3.5}$$

The change in the density ρ at any given point is due to the fluid flowing into or out of that region, with velocity $\vec{V}$.

We achieve a similar momentum conservation equation by considering a force balance: $\rho\dot{\vec{V}}(\vec{x},t) = F(\vec{x},t)$, where ρ is the mass density. V is the velocity of a fluid element at position $\vec{x}$ at time t; F is the net force per unit volume acting on the fluid element at position $\vec{x}$ at time t.

It is important to note that $\dot{\vec{V}}$ is not simply the partial derivative of the velocity with respect to time; the fluid velocity is not only changing in time but also spatially. The chain rule then gives

$$\begin{aligned} \dot{\vec{V}}(\vec{x},t) &= \frac{\partial \vec{V}}{\partial t} + \frac{dx_i}{dt}\frac{\partial \vec{V}}{dx_i} && (3.6) \\ &= \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla\right)\vec{V} && (3.7) \end{aligned}$$

This total derivative is often represented as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \nabla \tag{3.8}$$

where $\vec{V}$ is the background flow of the fluid.

The forces acting on the system will be the Lorentz force law $\vec{J} \times \vec{B}/c$ (note the Gaussian units) and the pressure gradient force $-\nabla P$, leading to a force per unit volume of:

$$\vec{F}(\vec{x}, t) = \frac{1}{c}\vec{J}(\vec{x}, t) \times \vec{B}(\vec{x}, t) - \nabla P(\vec{x}, t) \tag{3.9}$$

Altogether then, we have our equation of momentum conservation:

$$\rho \frac{D\vec{V}}{Dt} = \frac{1}{c}\vec{J} \times \vec{B} - \nabla P \tag{3.10}$$

Here, $\vec{J}$ comes from Maxwell's equation:

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{3.11}$$

However, we can neglect the displacement current since we are here concerned with non-relativistic MHD and the displacement current's electric field will be much smaller than the magnetic field.

The current can thus be written as

$$\vec{J} = \frac{c}{4\pi}\nabla \times \vec{B} \tag{3.12}$$

Using Eq. **??**, we can re-write the first term on the right hand side of Eq. 3.10 as

$$\frac{1}{4\pi}\left[(\nabla \times \vec{B}) \times \vec{B}\right] = \frac{1}{4\pi}\left[(\vec{B} \cdot \nabla)\vec{B} - \frac{1}{2}\nabla(B^2)\right] \tag{3.13}$$

$$\tag{3.14}$$

where $B^2 = \vec{B} \cdot \vec{B}$. These two terms can be interpreted as magnetic pressure and magnetic tension: magnetic pressure, which pushes field lines apart, is $\frac{1}{8\pi}B^2$ and the magnetic tension, which tries to uncurl magnetic field lines, is $(\frac{\vec{B}}{4\pi} \cdot \nabla)\vec{B}$. It is common to see the momentum equation in the form

$$\rho \frac{D\vec{V}}{Dt} = -\nabla \left(P + \frac{B^2}{8\pi}\right) + \left(\frac{\vec{B}}{4\pi} \cdot \nabla\right)\vec{B} \tag{3.15}$$

We can put the continuity equation 3.5 in a similar (non-conservative) form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V}\right) = 0 \tag{3.16}$$

$$= \frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} \tag{3.17}$$

$$= \frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} \tag{3.18}$$

$$\tag{3.19}$$

For the last ideal MHD equation, we consider Ohm's law:

$$0 = \vec{E} + \frac{1}{c}(\vec{v} \times \vec{B}) \tag{3.20}$$

XXXXxwhy

Using Faraday's law, we have

$$\frac{\partial \vec{B}}{\partial t} = -c\nabla \times \vec{E} \tag{3.21}$$

$$= \nabla \times \left(\vec{v} \times \vec{B}\right) \tag{3.22}$$

which is known as the ideal induction equation.

energy/entropy equation?

3.3 The MRI in Ideal MHD: Theory

As discussed in Chapter **??**, the source of turbulence in accretion disks remained a puzzle until linear analysis by Balbus and Hawley in 1991 [**?**]. This section outlines the derivation of the MRI: its dispersion relation, maximally growing mode, and other important features. The method used in this paper to simulate the MRI will be discussed in the next section (Section 3.4), with the actual simulation results and their interpretation coming in the section after that (Section 3.5).

3.3.1 MRI Spring Analogy

First, it is useful to have a bit of physical intuition for the MRI. Following [**?**, **?**], this section will look at the equations for a local frame: local in the sense that they describe small deviations from a circular orbit. Typical perturbations that might be considered are Eulerian perturbations: those deviations about a point in space. Now we consider Lagrangian deviations: deviations about a background state that follows a fluid element on its path in the background flow.

The Eulerian perturbations in density $\delta\rho$ are

$$\delta\rho = \rho_1(\vec{r}, t) - \rho_0(\vec{r})$$

where ρ_0 is the equilibrium density and ρ_1 is the new density at some later time. In contrast, the Lagrangian perturbations in density $\Delta\rho$ are

$$\begin{aligned}\Delta\rho &= \rho_1(\vec{r}, t) + \vec{\xi} \cdot \nabla\rho(\vec{r}) - \rho(\vec{r}) \\ &= \delta\rho + \vec{\xi} \cdot \nabla\rho(\vec{r})\end{aligned}$$

The two are different due to the background flow that carries fluid elements along in time (i.e. even in equilibrium their position would change. The lagrangian formalism takes that into account). Here, $\vec{\xi}$ is the displacement that the fluid element undergoes in time t.

We take a background flow $\vec{u} = R\Omega(R)\hat{\phi}$ and consider small radial and azimuthal deviations $\vec{\xi} = \xi_R\hat{R} + \xi_\phi\hat{\phi}$ in a shear flow. The Lagrangian velocity perturbation is

$$\Delta\vec{u} = \delta\vec{u} + \vec{\xi} \cdot \nabla\vec{u}$$

Expanding in polar coordinates, we have

$$\begin{aligned}\delta\vec{u} &= \frac{D\vec{\xi}}{Dt} - \vec{\xi} \cdot \nabla\vec{u} \\ &= \hat{R}\left(\frac{\partial\xi_R}{\partial t} + \Omega\frac{\partial\xi_R}{\partial\phi}\right) + \hat{\phi}\left(\frac{\partial\xi_\phi}{\partial t} + \Omega\frac{\partial\xi_\phi}{\partial\phi} - \xi_R R\frac{\partial\Omega}{\partial R}\right)\end{aligned} \tag{3.23}$$

We note that

$$(\vec{u} \cdot \nabla)\xi_R = \frac{u_\phi}{R}\frac{\partial\xi_R}{\partial\phi} = \Omega\frac{\partial\xi_R}{\partial\phi}$$

and

$$(\vec{u} \cdot \nabla)\xi_\phi = \frac{u_\phi}{R}\frac{\partial\xi_\phi}{\partial\phi} = \Omega\frac{\partial\xi_\phi}{\partial\phi}$$

Equation 3.23 can then be written as

$$\begin{aligned}\delta\vec{u} &= \delta u_R\hat{R} + \delta u_\phi\hat{\phi} \\ &= \hat{R}\frac{D\xi_R}{Dt} + \hat{\phi}\left(\frac{D\xi_\phi}{Dt} - \xi_R\frac{d\Omega}{d\ln R}\right)\end{aligned}$$

and we identify

$$\frac{D\xi_R}{Dt} = \delta u_R$$
$$\frac{D\xi_\phi}{Dt} = \delta u_\phi + \xi_R \frac{d\Omega}{d\ln R} \tag{3.24}$$

Looking back to Eq. 3.15, we can write the radial component of the momentum equation with perturbed velocity $\vec{u} + \delta\vec{u}$ as

$$\rho \frac{D}{Dt}\delta\vec{u} = \frac{D}{Dt}\delta u_R - 2\Omega\delta u_\phi = f_R$$

where now $D/Dt = \partial/\partial t + \vec{u}\cdot\nabla - \Omega\partial/\partial\phi$ to account for the system rotation. This is equal to the perturbed magnetic and pressure forces f_R.

Similarly, the azimuthal component

$$\rho \frac{D}{Dt}\delta u_\phi + \rho\left(\frac{d\Omega}{d\ln R} + 2\Omega\right)\delta u_R = f_\phi$$

where here the perturbed magnetic and pressure forces are f_ϕ. Calling $\frac{D\xi_R}{Dt} = \dot{\xi}_R$ and $\frac{D\xi_\phi}{Dt} = \dot{\xi}_\phi$ and plugging in Eqns. 3.24, we have the so-called "Hill equations":

$$\ddot{\xi}_R - 2\Omega\dot{\xi}_\phi + \xi_R \frac{d\Omega^2}{d\ln R} = f_R \tag{3.25}$$
$$\ddot{\xi}_\phi + 2\Omega\dot{\xi}_R = f_\phi \tag{3.26}$$

This set of equations may look familiar. Indeed, setting $f_R = Kx$ and $f_\phi = Ky$, they are the equations for two masses coupled together by a spring in circular orbits. Therefore, the MRI can be understood by analogy to point masses.

Let us explore this further. The situation is illustrated in Figure **??**: two masses orbit and are connected by a spring with spring constant K. Both masses follow a Keplerian rotation profile, meaning that the inner mass, m_i, rotates slightly faster than the outer mass m_o. The spring resists the masses' pulling apart by exerting forces on them to bring them closer together. In pulling the inner mass backwards, the spring causes the inner mass to lose angular momentum and fall to a closer orbit in accordance with Keplerian rotation. At the time, however, the outer mass is pulled forward and travels faster, meaning that it drifts outwards. As a result, the masses get even further apart, the spring pulls more, and the gap widens more. The process runs away as angular momentum is transported outward.

This analogy also demonstrates the importance of a *weak* magnetic field: if the spring is too strong, the masses will not be able to overcome the restoring force. The Hill equations 3.25 and 3.26 also provide an easy way to get the MRI stability condition: if the radial equation has $\frac{d\Omega^2}{d\ln R} - K > 0$, then the system is stable. The stability criterion is thus

$$-K > -\frac{d\Omega^2}{d\ln R}$$

Where does this restoring force come from in the actual MRI? It is the result of the magnetic field lines curling and the magnetic tension trying to unfurl them (see Figure **??**). In Fourier space, given perturbations of the form $\delta\vec{u} \propto e^{i\vec{k}\cdot\vec{x}}$, the tension $(\vec{B}\cdot\nabla)\vec{B} \propto (\vec{u}_A\cdot\vec{k})\vec{B}$ is proportional to $-(\vec{k}\cdot\vec{u}_A)^2$ where $\vec{u}_A = \vec{B}/(4\pi\rho)^{1/2}$ is the Alfven velocity. This leads to the stability condition

$$(\vec{k}\cdot\vec{u}_A)^2 > -\frac{d\Omega^2}{d\ln R} \tag{3.27}$$

This schematic derivation is done more rigorously in the next section (3.3.2). For now, it is interesting to note that it is always possible to find a wavenumber k such that the system is unstable unless $\frac{d\Omega^2}{d\ln R} > 0$, which would be very uncommon in astrophysical disks. Also note that if the magnetic field $B = 0$, then the Alfven velocity is also zero and Eq. 3.27 would have us believe that the hydrodynamic criterion for disk stability is $\frac{d\Omega^2}{d\ln R} > 0$. We know from, for example, Eq. **??** in Section 2.4.3 that hydrodynamic disks are stable if $\frac{1}{r^2}\frac{d}{dr}(\Omega^2 r^4) = 4\Omega^2 + \frac{d\Omega^2}{d\ln r} > 0$. The disagreement is due to the assumptions made in using the MHD equations, namely, that the mean free path of particles was much less than the length scales of interest. As k increases, we get down to such small scales that the scales of interest become comparable to the mean free path and thus this assumption is no longer valid. The conflict must be resolved through kinetic theory. This is another incentive to see if we can get the MHD equations to somehow approximate kinetic theory in Chapter **??**.

3.3.2 Linear Theory with the Boussinesq Approximation

The full general dispersion relation for the MRI can be derived straightfowardly by crunching the algebra and linearizing the MHD equations in Fourier space (see, for example, [**?**]). It is, however, more instructive to make an approximation that filters out the magnetosonic waves that are irrelevant in the regime of the MRI. This approximation, termed the "Boussinesq Approximation", effectively takes the sound speed to infinity. It manifests in the MHD equations by allowing us to drop pressure perturbations everywhere except the momentum equation, where they make sure the fluid is incompressible, and dropping density perturbations in the continuity equation, where they give buoyancy effects [**?**].

Our equations are then:

Entropy?

3.3.3 Max growth rate

3.4 Simulating the MRI: the Shearing-Box Method

In simulating accretion disks, it is much more computationally expensive to describe the disk as a whole, including pressure gradients and the overall global structure. A smaller-scale, local simulation is also effective at demonstrating the physics involved. As noted in Chapter **??**, the series of simulations in this chapter and the next zoom in to a typical region of the accretion disk and consider the MHD equations in that blown-up portion. We now explore exactly how such local simulations work and the relevant quantities.

The so-called "shearing-box" approximation has been in use since XXXXXX in studies examining the solar corona XXXXXXX etc. It was also used by Balbus and Hawley in their seminal 1991 papers showing the existence of the MRI instability [**?**, **?**, **?**].

3.5 The MRI in Ideal MHD: Simulations

The simulations in this section were modelled after [**?**], hereafter called HB3. Although the techniques used in that paper were slightly different (periodic radial conditions, for example, instead of periodic plus shear), it is instructive to replicate the results with the code that will be used **briefly 2D (just show dynamo decay), then HGB

3.5.1 Athena 4.2: An MHD Code....

The code in use for the simulations in this section and Chapter **??** and the MHD simulations of Chapter **??** is Athena4.2. XXXXXXXXXXXX

3.5.2 Parameters

3.5.3 Results

Chapter 4

Non-ideal MHD: Theory and Simulations

4.1 Non-ideal MHD: Theory

4.1.1 Resistivity: Physical Intuition

Smooth B field, field lines not frozen in

4.1.2 Viscosity: Physical Intuition

Smoothing velocity

4.2 Non-ideal MHD: Simulations

HGB!

Chapter 5

Kinetic Theory and Braginskii MHD

5.1 Kinetic Theory

Kinetic theory generalizes the brute-force method of applying Maxwell's equations (and the Lorentz Force Law) to many particles. Following [**?**], we consider first a single particle. Its density in phase space $N(\vec{x}, \vec{v}, t)$ is given by

$$N(\vec{x}, \vec{v}, t) = \delta(\vec{x} - \vec{x}_1(t))\delta(\vec{v} - \vec{v}_1(t))$$

Here, $\vec{x}$ and $\vec{v}$ are the coordinates themselves, while $\vec{x}_1$ and $\vec{v}_1$ are the locations of the particle (particle 1) at time t. It is straightforward to generalize this to many particles:

$$N(\vec{x}, \vec{v}, t) = \sum_{i=1}^{N_0} [\delta(\vec{x} - \vec{x}_i(t))\delta(\vec{v} - \vec{v}_i(t)]$$

where N_0 is the total number of particles (of one species, or summing over species if there is more than one). Remembering the chain rule, the time derivative of this phase density is given by

$$\begin{aligned}\frac{\partial N(\vec{x}, \vec{v}, t)}{\partial t} &= -\sum_{i=1}^{N_0} \dot{\vec{x}}_{ik} \frac{\partial}{\partial x_k} \delta(\vec{x} - \vec{x}_i(t))\delta(\vec{v} - \vec{v}_i(t) \\ &\quad - \sum_{i=1}^{N_0} \dot{\vec{v}}_{ik} \frac{\partial}{\partial v_k} \delta(\vec{x} - \vec{x}_i(t))\delta(\vec{v} - \vec{v}_i(t))\end{aligned} \tag{5.1}$$

with the sum over the index k for the three spatial directions. However, we know that

$$\dot{\vec{x}}_i(t) = \vec{v}_i(t)$$

and, from the Lorentz force law, that

$$m_i\dot{\vec{v}}_i = q_i\vec{\mathrm{E}}_i(\vec{x}(t), t) + \frac{q_i}{c}\vec{v}_i(t) \times \vec{\mathrm{B}}(\vec{x}_i(t), t)$$

where E and B are the electric charges produced by the other point particles that is acting on the particle of mass m_i and charge q_i. The phase density 5.1 becomes

$$\begin{aligned}\frac{\partial N(\vec{x}, \vec{v}, t)}{\partial t} &= -\sum_{i=1}^{N_0} \vec{v}_{ik}\frac{\partial}{\partial x_k}\delta(\vec{x} - \vec{x}_i(t))\delta(\vec{v} - \vec{v}_i) \\ &\quad - \sum_{i=1}^{N_0}\left[\frac{q_i}{m_i}\vec{\mathrm{E}}_{ik} + \frac{q_i}{m_i c}(\vec{v}_i \times \vec{\mathrm{B}})_k\right]\frac{\partial}{\partial v_k}\delta(\vec{x} - \vec{x}_i)\delta(\vec{v} - \vec{v}_i)\end{aligned}$$

However, due to the delta function property $a\delta(a - b) = b\delta(a - b)$, we can replace the $\vec{v}_{ik}$ in the first term of the sum with $\vec{v}_k$, which is independent of the sum. The E and B in the second term similarly simplify. We then notice that the sum is just the phase density again:

$$\begin{aligned}\frac{\partial N(\vec{x}, \vec{v}, t)}{\partial t} &= -\vec{v}_k\frac{\partial}{\partial x_k}\sum_{i=1}^{N_0}\delta(\vec{x} - \vec{x}_i(t))\delta(\vec{v} - \vec{v}_i) \\ &\quad - \left[\frac{q}{m}\vec{\mathrm{E}}_k + \frac{q}{mc}(\vec{v} \times \vec{\mathrm{B}})_k\right]\frac{\partial}{\partial v_k}\sum_{i=1}^{N_0}\delta(\vec{x} - \vec{x}_i)\delta(\vec{v} - \vec{v}_i) \\ &= -\vec{v}_k\frac{\partial N(\vec{x}, \vec{v}, t)}{\partial x_k} - \frac{q}{m}\left(\vec{\mathrm{E}} + \frac{\vec{v}}{c} \times \vec{B}\right)_k\frac{\partial N(\vec{x}, \vec{v}, t)}{\partial v_k} \qquad (5.2)\end{aligned}$$

here the mass m and charge q are the mass and charge of each particle in a certain species; for multiple species just sum over the species. Equation 5.2 is known as the Klimontovich equation. It is completely deterministic, given appropriate initial conditions and sufficient computing power.

Again, we are interested in the average properties of the plasma and not so much the orbits of each individual particle. Therefore we write the distribution function of a species s as $f_S = \langle N \rangle$, where the brackets denote an ensemble average over position and velocity space. This is equivalent to averaging over a length scale l such that the separation between particles is much less than l, but l is much less than the Debye length (Eq. **??**). We then have that the phase density is the distribution function plus some fluctuations δN whose average is zero. Similarly breaking up the electric field and magnetic field into their average and fluctuating pieces, we have

$$\begin{aligned} N(\vec{x},\vec{v},t) &= f(\vec{x},\vec{v},t) + \delta N \\ \tilde{\mathrm{E}}(\vec{x},\vec{v},t) &= \vec{E}(\vec{x},\vec{v},t) + \delta E \\ \tilde{\mathrm{B}}(\vec{x},\vec{v},t) &= \vec{B}(\vec{x},\vec{v},t) + \delta B \end{aligned}$$

With these definitions, the Klimontovich equation 5.2 becomes:

$$\begin{aligned} \frac{\partial f_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}_k \frac{\partial f_s}{\partial x_k} + \frac{q}{m}(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})_k \frac{\partial f_s}{\partial v_k} &= -\frac{q}{m} < (\delta E + \frac{\vec{v}}{c} \times \delta B)_k \frac{\partial \delta N}{\partial v_k} > \\ &= C[f_s] \end{aligned} \tag{5.3}$$

This is the Vlasov equation. The left side depends only smoothly-varying terms, whereas the right is spiky, being the average of products of delta functions. The righthand side represents the interactions between individuals particles, and we can lump all these effects into the so-called "collision operator" $C[f]$. Entire graduate courses can be taught on the collision operator so we will not delve too much into it here.

Important properties of the collision operator can be found in introductory plasma physics texts such asXX [**?**] orXX [**?**]. We note, for example, that conservation of particles is given by

$$\int d^3\vec{v} \frac{\partial f_s}{\partial t} = 0, \tag{5.4}$$

conservation of total momentum by

$$\int \sum_s d^3\vec{v} m_s \vec{v} \frac{\partial f_s}{\partial t} = 0, \tag{5.5}$$

and conservation of total energy by

$$\int \sum_s d^3\vec{v} \frac{m_s \vec{v}^2}{2} \frac{\partial f_s}{\partial t} = 0 \tag{5.6}$$

. This suggests a way to get the MHD equations discussed in Chapter **??**, and indeed they can be derived from Eq. 5.3, as described in the next section (5.1.1).

Note that it is usually collisions that facilitate exchange of energy between particles. Without collisions, the distribution function will move away from a Maxwellian more freely. This is an important idea in Braginskii MHD, explained in Section **??**.

5.1.1 The MHD Equations from Kinetic Theory

The MHD equations can be derived by taking moments of the Vlasov equation 5.3. First, however, it is easier to go to the frame of the plasma, writing the velocity peculiar to the mean flow $\vec{w}$ as $\vec{w} \equiv \vec{v} - \vec{u}_s(\vec{x}, t)$. This way, $m_s \int \vec{w}\vec{w} f_s d^3\vec{v} \equiv \overset{\leftrightarrow}{P}_s$.

Note that this pattern of always involving higher moments of the Vlasov equation does not simply disappear. Rather, it is a central problem of plasma physics known as the BBGKY hierarchy. It is various choices to "close" this loop of higher moments that defines different theories (Kunz Lecture 1) Hierarchy problem, closure.

5.2 Braginskii MHD: Theory

If we assume a different ordering, namely that the Larmor radius is not zero, we come to anisotropic viscosity. (Kunz Braginskii MHD) Adiabatic Invariants

In general, collisions push the distribution function back to a Maxwellian. When these collisions are weak or the plasma is collisionless, the distribution function deviates more substantially from a Maxwellian, influencing the pressure in different ways along the magnetic field lines and in the plane perpendicular to the field lines.

5.3 Kinetic Simulations of the MRI

5.3.1 Pegasus: A Particle-in-Cell Code

hybrid-kinetic MHD

5.4 Braginskii MHD Simulations of the MRI

yo. My research.

Chapter 6

Acknowledgements

MATT!

Appendix A

Identities and Derivations?

There are lots of appendices.

A.0.1 Angular Momentum Conservation in Ideal MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{A.1}$$

$$\rho \frac{\partial \vec{v}}{\partial t} + (\rho \vec{v} \cdot \nabla) \vec{v} = -\nabla \left(P + \frac{B^2}{8\pi} \right) - \rho \nabla \Phi + \left(\frac{\vec{B}}{4\pi} \cdot \nabla \right) \vec{B} \tag{A.2}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{v} \times \vec{B} \right) \tag{A.3}$$

Although it can be re-distributed, angular momentum is ultimately conserved in ideal MHD systems. Since conservation of angular momentum is a central idea to the rest of this paper, this section will show how to achieve it from Eq. A.2 (neglecting gravity). The basic idea is to multiply the ϕ-component by R and re-arrange terms into conservative form, as per [**?**]. Before re-arranging, the full equation is

$$R \left[\rho \frac{\partial \vec{v}}{\partial t} + (\rho \vec{v} \cdot \nabla) \vec{v} = -\nabla \left(P + \frac{B^2}{8\pi} \right) - \rho \nabla \Phi + \left(\frac{\vec{B}}{4\pi} \cdot \nabla \right) \vec{B} \right]_\phi \tag{A.4}$$

where the notation of $[\ ]_\phi$ indicates the ϕ-component of the expression in square brackets. It is best to continue term-by-term.

The density terms are rather straightforward:

$$\begin{aligned}
\left[R\rho\frac{\partial\vec{v}}{\partial t}\right]_{\phi} &= R\rho\frac{\partial v_{\phi}}{\partial t} \\
&= \frac{\partial}{\partial t}\left(R\rho v_{\phi}\right) - v_{\phi}R\frac{\partial\rho}{\partial t} \\
&= \frac{\partial}{\partial t}\left(R\rho v_{\phi}\right) + v_{\phi}R\nabla\cdot\left(\rho\vec{v}\right) \\
&= \frac{\partial}{\partial t}\left(R\rho v_{\phi}\right) + \nabla\cdot\left(R\rho v_{\phi}\vec{v}\right) - \rho\vec{v}\cdot\nabla\left(v_{\phi}R\right)
\end{aligned} \tag{A.5}$$

where in the second line, the continuity equation A.1 was used and in the fourth, the identity A.14 was used. Now we look more closely at the last term on the right:

$$\rho\vec{v}\cdot\nabla\left(v_{\phi}R\right) = \rho\left(v_R v_{\phi} + Rv_R\frac{\partial v\phi}{\partial R} + v_{\phi}\frac{\partial v_{\phi}}{\partial\phi} + Rv_z\frac{\partial v_{\phi}}{\partial z}\right) \tag{A.6}$$

However, this is exactly the ϕ-component of $\left(R\rho\vec{v}\cdot\nabla\right)\vec{v}$, as can be seen by Eq. A.16. Adding Eqns. A.5 and A.6, we have:

$$R\rho\frac{\partial v_{\phi}}{\partial t} + \left[R\left(\rho\vec{v}\cdot\nabla\right)\vec{v}\right]_{\phi} = \frac{\partial}{\partial t}\left(R\rho v_{\phi}\right) + \nabla\cdot\left(R\rho v_{\phi}\vec{v}\right) \tag{A.7}$$

which is in conservative form.

The pressure term is straightforward:

The magnetic terms require more insight. Here we introduce the poloidal magnetic field; that is, the components of the magnetic field in the $R-$ and $z-$directions. Thus $B^2 = B_p^2 + B_{\phi}^2$ and $B_p^2 = B_R^2 + B_z^2$. Beginning with the magnetic pressure term,

$$\begin{aligned}
\left[R\nabla B^2\right]_{\phi} &= \hat{e}_{\phi}\cdot\left[R\nabla B^2\right] \\
&= \hat{e}_{\phi}\cdot\left[R\nabla B_p^2 + R\nabla B_{\phi}^2\right] \\
&= \hat{e}_{\phi}\cdot\left[R\nabla B_p^2\right] + \frac{1}{R}\frac{\partial}{\partial\phi}\left(RB_{\phi}^2\right) \\
&= \hat{e}_{\phi}\cdot\left[\nabla\left(RB_p^2\right)\right] + 2B_{\phi}\frac{\partial B_{\phi}}{\partial\phi}
\end{aligned}$$

where we can move R into the derivative because $\partial R/\partial\phi = 0$. Now, we add zero via the term $\nabla \cdot \hat{e}_\phi$:

$$\begin{aligned}\left[R\nabla B^2\right]_\phi &= \hat{e}_\phi \cdot \left[RB_p^2 \; \nabla \cdot \hat{e}_\phi + \hat{e}_\phi \cdot \nabla \left(RB_p^2\right) + 2B_\phi \frac{\partial B_\phi}{\partial \phi}\right] \\ &= \nabla \cdot \left(RB_p^2 \hat{e}_\phi\right) + 2B_\phi \frac{\partial B_\phi}{\partial \phi} \qquad \text{(A.8)}\end{aligned}$$

where Eq. A.14 was used in the last line. We have thus achieved part of the conservative form, but with an extra $\partial B_\phi^2/\partial\phi$ term. We now turn to the magnetic tension term in hopes that it will cancel this extra term. First, however, we note using $\nabla \cdot \vec{B} = 0$ and the definition of $\vec{B}_p$ that

$$\begin{aligned}\nabla \cdot \vec{B}_p &= \nabla \cdot \left(\vec{B} - B_\phi \hat{e}_\phi\right) \\ &= -\nabla \cdot \left(B_\phi \hat{e}_\phi\right) \\ &= -\frac{1}{R}\frac{\partial B_\phi}{\partial \phi} \qquad \text{(A.9)}\end{aligned}$$

The magnetic tension term becomes

$$\begin{aligned}\left[R\left(\vec{B} \cdot \nabla \vec{B}\right)\right]_\phi &= RB_R \frac{\partial B_\phi}{\partial R} + B_\phi \frac{\partial B_\phi}{\partial \phi} + RB_z \frac{\partial B_\phi}{\partial z} + B_R B_\phi \\ &= \left[B_R \left(R\frac{\partial B_\phi}{\partial R} + B_\phi\right) + B_z R \frac{\partial B_\phi}{\partial z}\right] + B_\phi \frac{\partial B_\phi}{\partial \phi} \\ &= (B_R, 0, B_z) \cdot \left(\frac{\partial}{\partial R}(RB_\phi), 0, \frac{\partial}{\partial z}(RB_\phi)\right) + B_\phi \frac{\partial B_\phi}{\partial \phi} \\ &= \vec{B}_p \cdot \nabla \left(RB_\phi\right) + B_\phi \frac{\partial B_\phi}{\partial \phi} \qquad \text{(A.10)}\end{aligned}$$

Using Eq. A.9, we add zero and draw out a total divergence:

$$\begin{aligned}\left[R\left(\vec{B} \cdot \nabla \vec{B}\right)\right]_\phi &= \vec{B}_p \cdot \nabla \left(RB_\phi\right) - RB_\phi \nabla \cdot \vec{B}_p + 2B_\phi \frac{\partial B_\phi}{\partial \phi} \qquad \text{(A.11)} \\ &= \nabla \cdot \left(RB_\phi \vec{B}_p\right) + 2B_\phi \frac{\partial B_\phi}{\partial \phi} \qquad \text{(A.12)}\end{aligned}$$

where Eq. A.14 was used in the last line. Armed with each individual term, we combine them (Eqs. A.7, A.8, and A.12) to find the conservative form of the angular momentum equation:

$$
\begin{aligned}
&\left[R\rho\frac{\partial\vec{v}}{\partial t}+R\left(\rho\vec{v}\cdot\nabla\right)\vec{v}+R\,\nabla P+R\,\nabla\frac{B^2}{8\pi}-R\,\left(\frac{\vec{B}}{4\pi}\cdot\nabla\right)\vec{B}\right]_\phi= \\
&\qquad\frac{\partial}{\partial t}\left(R\rho v_\phi\right)+\nabla\cdot\left(R\rho v_\phi\vec{v}\right)+PRESSURE \\
&\qquad\qquad+\frac{1}{8\pi}\left[\nabla\cdot\left(RB_p^2\hat{e}_\phi\right)+2B_\phi\frac{\partial B_\phi}{\partial\phi}\right. \\
&\qquad\qquad\left.-2\nabla\cdot\left(RB_\phi\vec{B}_p\right)-2B_\phi\frac{\partial B_\phi}{\partial\phi}\right] \\
&\quad=\frac{\partial}{\partial t}\left(R\rho v_\phi\right)+\nabla\cdot R\left[\rho v_\phi\vec{v}+PRESSURE+\frac{B_p^2}{8\pi}\hat{e}_\phi-\left(\frac{B_\phi\vec{B}_p}{4\pi}\right)\right]
\end{aligned}
\tag{A.13}
$$

A.0.2 Alfv'en's Theorem (Flux Freezing)

In ideal MHD, the magnetic field lines thread a perfectly conducting fluid. They cannot diffuse without a finite resistivity; hence the field lines are "frozen" into the conducting medium. This can be seen mathematically

A.1 Vector Identities

This section is just for useful identities. A good reference is the NRL Plasma Formulary [**?**].

$$\nabla\left(f\vec{A}\right)=f\nabla\cdot\vec{A}+\vec{A}\cdot\nabla f \tag{A.14}$$

$$\nabla \tag{A.15}$$

A.2 Cylindrical Coordinates

Components of

$$\left[\left(\vec{A}\cdot\nabla\right)\vec{B}\right]_\phi= \tag{A.16}$$