Law of Large Numbers and Central Limit Theorem

An non-technical overview

A. Haris Jameel

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Statistical inference about a mean, μ

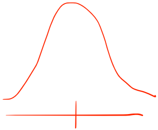
Aim:

Make inference about a population mean/characteristic, μ (Specifically, estimate μ and quantify uncertainty about estimates).

Approach:

Estimate μ using data, then appeal to the law of large numbers (LLN) and central limit theorem (CLT)...

Notation



X	random variable
μ	population mean
σ^2	population variance
$\mathrm{N}(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
$(X_1, X_2, X_3, \ldots, X_n)$	random sample of size n
$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$	sample mean

Law of large numbers

We can appeal to the LLN estimate the population mean, μ.

Law of large numbers

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n) \to \mu \text{ as } n \to \infty.$$

"When the sample is sufficiently large, the sample mean is approximately equal to the population mean".

LLN example

For a given coin, X is the outcome after a coin flip and $P(X = heads) = \mu$.

Suppose we flip the coin n times, for $n = 1, 2, 3, \ldots$.

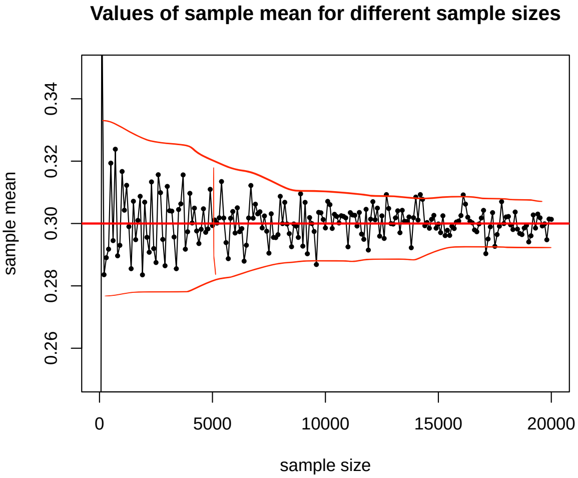
Define $X_i = 1$ to mean "heads is observed for the i^{th} coin flip" ($X_i = 0$ if tails observed). Then

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$

estimates $\mu = P(X = heads)$.

LLN example (cont'd)

Suppose $\mu = 0.3$. The figure below shows different values of $\bar{X}_n$ for $n = 1, 101, 201, \ldots, 20001$.



Central limit theorem

We can quantify uncertainty by using the CLT to describe the variation $\bar{X}_n$ about μ.

Central limit theorem

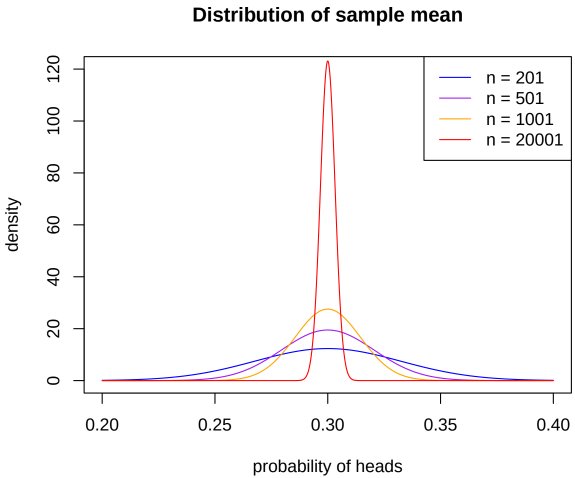
$$\sqrt{n}(\bar{X}_n - \mu) \to \mathrm{N}(0, \sigma^2) \text{ in distribution as } n \to \infty.$$

"When the sample is sufficiently large, the sample mean is distributed as (approximately) $\mathrm{N}(\mu, \frac{\sigma^2}{n})$".

A very powerful result (population needn't be normal-distributed)!

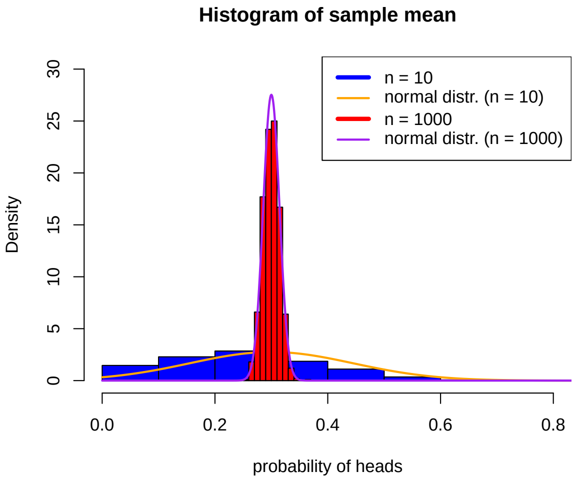
CLT example

Back to the coin example – the figure below shows the (asymptotic) normal distributions of $\bar{X}_n$ for values of $n = 201, 501, 1001, 20001$ based on the CLT.



CLT example (cont'd)

For each of $n = 10$ and $n = 1000$, compute $\bar{X}_n$ 1000 times and plot a histogram. The normal approximation is not good when the sample size is too small.



Notes and caveats

- There are many versions of LLN and CLT, each with different assumptions about the random sample or population distribution
- The LLN and CLT may not hold if some of the aforementioned assumptions are violated
- It can be difficult to determine how large n should be in practice

Summary

- The LLN is useful because sample means are "approximately equal" to respective population means for large enough sample size

- The CLT is useful in helping us quantify the precision of our estimates of the population mean, even if the underlying population is not normal-distributed.