Incorporation, Selection and Firm Dynamics: A Quantitative Exploration

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Abstract

This paper studies how incorporation, which provides limited liability to firm owners, affects firm dynamics and macroeconomy. I propose an equilibrium model of firm dynamics with endogenous entry and exit, where firms spend resources to improve their productivity and choose whether to incorporate or not. Incorporation provides liability protection which ensures that firm value is bounded from below, at the expense of high set-up and maintaining cost. An important model feature is that firms have heterogeneous (high and low) types which differ in their capacity to improve productivity. This heterogeneity allows for the possibility of selection as high-type firms, who have higher growth potential, benefit more from incorporation. I estimate the model by using firm-level data, specifically exploiting the heterogeneity in exit rates by age conditional on size to identify firm types in growth potential and therefore selection. The estimation results suggest that accounting for firm heterogeneity in growth potential is quantitatively important in explaining the observed better performance of incorporated firms. Upon entry, 90% (15%) of the incorporated (unincorporated) firms are high-types, which are estimated twice as efficient as low-types in improving their productivity. This underlines a significant selection effect which is more pronounced among incumbents as the exit rate of high-type firms is lower. In a counterfactual economy where the incorporation decision is randomized within firm types, the productivity growth decreases from 3% to 2.7% and the difference in the average size of incorporated and unincorporated firms decreases by 32%. I find significant welfare gains from subsidizing incorporated firms and large welfare losses from removing incorporation choice. These welfare results are largely driven by the change in the degree of selection, i.e. the change in the composition of firm types.

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1. Introduction

An extensive empirical literature has documented striking differences across firms. While many firms fail in their early years and most of those that survive do not grow, others grow rapidly and significantly contribute to job creation and aggregate productivity growth. This reflects substantial firm heterogeneity in many aspects, one of which is the legal form they choose to operate in. For example in the U.S. roughly half of all business owners prefer to shield themselves against the downside risks by attaining limited liability through incorporating their businesses. How does limited liability affect firm behavior and the macroeconomy? How does the choice of legal form interact with ex-ante and ex-post firm heterogeneity?

To answer these questions, this paper proposes a macroeconomic model of firm dynamics with endogenous entry and exit, where entrepreneurs choose whether or not to incorporate their firms. In the model, firms invest in resources to improve their productivities which determine their profitability and contribute to economic growth. Firms have heterogeneous (high and low) types that differ in their efficiency to improve productivity. In other words, firms are heterogeneous in terms of their growth potential. Successful entrepreneurs increase their firm productivity and stay in the economy, whereas unsuccessful ones end up exiting the economy, either endogenously due to the deterioration in their profitability or due to exogenous shocks that render the firm unproductive. Firms are subject to an exit cost which is proportional to their size. Due to this cost, the firm value falls below zero in the case of exit. By paying a sunk cost, entrepreneurs can incorporate their firms to ensure that their losses are limited to the initial cost of setting up the firm. In other words, incorporation provides insurance to the owner by bounding the firm value from below.

This environment underlines two main effects that generate differences in firm dynamics between incorporated and unincorporated firms. The first one is a treatment effect of incorporation: since incorporation protects firms from downside risks, it incentivizes them to invest more in improving their productivity, subsequently grow large and exit less often. The second one is a selection effect due to the presence of firm heterogeneity: entrepreneurs with higher growth potential (i.e. more efficient in improving productivity) are more likely to choose incorporation as it is more valuable to large firms. The strength of this selection effect is determined by the interplay between endogenous entry, investment, and exit decisions.

To quantify the importance of these channels and study their macroeconomic implications, I calibrate the model to firm-level micro data from Denmark. Calibration targets several key empirical moments of firm dynamics for incorporated and unincorporated firms. Specifically, the calibrated model is able to quantitatively match the observed differences between incorporated and unincorporated firms: incorporated firms have higher employment upon entry, grow faster, and exit less often conditional on their size and age, compared to unincorporated firms. Furthermore, to validate the model, I show that a variety of moments that are not targeted in the estimation are in line with the data.

My estimation strategy exploits the heterogeneity in firms exit rates by age conditional on size to identify firm types in growth potential. The model implies that without this firm type heterogeneity, the likelihood of exit would be independent of age conditional on size. In data, however, such conditional exit rates are strongly decreasing in firm age. My framework rationalizes this pattern through the interaction between firm heterogeneity and endogenous selection in that the share of firms with high growth potential, which have lower exit rates conditional on size, increases within a given cohort as the cohort ages.

The quantitative results suggest that both treatment and selection effects are important and accounting for firm heterogeneity is quantitatively relevant in explaining the observed better performance of incorporated firms. Conditional on the firm type, incorporated firms choose an expansion rate, the rate at which firms improve their productivity, 50% higher than unincorporated firms do on average. This indicates a significant positive treatment effect of incorporation on firm-level productivity growth. Among entrants, 90% (15%) of the firms that choose (not) to incorporate are high-types, highlighting a significant selection effect upon entry. Among incumbents, the selection effect becomes more pronounced where the share of high-types rises to 99% within incorporated firms. To further explore the importance of selection effect, I consider a counterfactual economy where the incorporation decision is randomized within firm types, while keeping the distribution of firm types upon entry constant. In this counterfactual economy, the difference in the average size of incorporated and unincorporated firms decreases by 32% compared to the baseline economy and the aggregate productivity growth decreases from 3% to 2.7%. Aggregate productivity growth declines mainly because the randomization of legal form decisions deteriorates the equilibrium composition of firm types.

Finally, I use the model to conduct two experiments to assess the value of incorporation. First, I consider a case where the option of incorporation is not available to the firms. The absence of incorporation not only eliminates the positive treatment effect on firms expansion rates but also mitigates the selection of high-growth potential firms in the economy. Consequently, the growth rate decreases to 2.49% and welfare decreases by 4.6% (in consumption equivalent terms). Eliminating incorporation choice also affects the firm entry and number of incumbent firms negatively. On the other hand, subsidizing the incorporated firms provides significant welfare gains. This last result is largely driven by the change in the degree of selection, i.e. the change in the composition of firm types.

The rest of the paper is organized as follows: In Section 2, I describe the theoretical model. Section 3 summarizes the data that I use in the quantitative analysis and discusses the identification of the model. In Section 4, I present the calibration results, and assess the model fit based on various out-of-sample moments. In Section 5, I provide the main analysis to quantify the importance of treatment and selection channels on firm dynamics and the aggregate economy. Section 6 concludes. All proofs and additional details are contained in the Appendix.

2. Model

2.1 Preferences, Technology, and Static Allocations

The economy is in continuous time and admits a representative household with per-period log utility function

$$U_0 = \int e^{-\rho t} \ln C(t) dt \tag{1}$$

where $C(t)$ is consumption at time t and $\rho > 0$ is the discount rate.The household is populated by a continuum of individuals with measure one. Each member is endowed with one unit of labor that is supplied inelastically.

The individuals consume a unique final good $Y(t)$, which is also used for other purposes as will be discussed below. The final good is produced competitively by labor $L(t)$ and a continuum of intermediate goods over the set $\mathcal{N}(t)$, with measure Φ_t, following the production technology

$$Y(t) = \frac{L(t)^{\beta}}{1-\beta} \int_{\mathcal{N}(t)} q_j(t)^{\beta} y_j(t)^{1-\beta} dj \tag{2}$$

where $q_j(t)$ and $y_j(t)$ are the quality and quantity of intermediate good j, respectively. The measure of intermediate goods produced in the economy is determined endogenously through entry and exit decisions. The price of the final good is normalized to be one in every period without loss of generality. In what follows, I will drop the time subscript t whenever it does not cause any confusion.

Each intermediate good $j \in \mathcal{N}$ is produced by a single firm which monopolistically competes against other firms active in the economy. Therefore index j also refers to the firm that produces intermediate good j. These firms have access to a linear technology of the form

$$y_j = \bar{q} l_j \tag{3}$$

where l_j is the amount of labor that firm j hires for the production, and $\bar{q} \equiv \frac{\int_{\mathcal{N}} q_j}{\Phi}$ is the average quality in the economy. In addition to the labor cost, production requires also a fixed cost of operation $\psi\bar{q}$ at every period in terms of the final good. As will be discussed later, this fixed cost is allowed to be different for different legal structures chosen by the firm.

The maximization problem of the final goods producer generates the inverse demand $p_j = L^{\beta} q_j^{\beta} y_j^{-\beta}$. Given the production technology, each firm is faced with a constant marginal cost of production given by $w/\bar{q}$, where w is the wage rate in the economy. Therefore, for a given level of quality q_j, we can write firm j's static profit maximization problem as

$$\pi(q_j) = \max_{y_j \geq 0} \left\{ L^{\beta} q_j^{\beta} {y_j}^{1-\beta} - \frac{w}{\bar{q}} y_j \right\}.$$

where $\pi(q_j)$ is the per-period profit of firm j (before paying the fixed cost of operation) with quality q_j. The price and output level of firm follow from this maximization as

$$p_j = \frac{1}{(1-\beta)} \frac{w}{\bar{q}} \text{ and } y_j = \left[(1-\beta) \frac{\bar{q}}{w} \right]^{\frac{1}{\beta}} L q_j \tag{4}$$

implying that the price is a constant markup over the marginal cost, and firm's optimal output and labor is linear in quality. Therefore quality can be considered as a summary statistics for the size of the firm.

The resulting equilibrium profits can then be written as

$$\pi(q_j) = \Pi q_j, \tag{5}$$

where $\Pi = \beta \left[(1-\beta) \right]^{\frac{1-\beta}{\beta}} \left(\frac{\bar{q}}{w} \right)^{\frac{1-\beta}{\beta}}$, i.e. profits are increasing in quality q_j. Therefore firms have profit incentives to improve their product quality, which is the source of firm growth and will be discussed next.

2.2 Evolution of Firm Quality

Quality at the firm level evolves over time depending on the firm's investments in improving its quality. This process is modeled as a controlled stochastic process as in Akcigit and Kerr (2018) and Atkeson and Burstein (2010). In particular, I assume that by investing R in terms of final good, an incumbent firm with quality q_j improves its quality at the Poisson flow rate x_j

$$x_j = \theta \left(\frac{R}{q_j} \right)^{\eta} \tag{6}$$

where $\eta \in (0, 1)$ and θ is the *efficiency* of the investment technology. This particular investment technology assumes that the cost required to increase the quality scales with the size of the firm. This implies that, for sufficiently large firms (large quality), their growth rate is independent of their size, consistent with Gibrat's law.

When the investment is successful, the current quality of the firm improves from q_j to $q_j + J(q_j, \bar{q})$ where

$$J(q_j, \bar{q}) = \lambda \left[\omega \bar{q} + (1-\omega) q_j \right], \quad \omega \in [0,1]. \tag{7}$$

i.e. improvement in the quality is a convex combination of current quality of the firm and the average quality in the economy. This formulation is a generalization of Acemoglu et al. (2018), where quality improvements depend only on average quality in the economy, and Akcigit and Kerr (2018), where quality improvements are proportional to current quality of the firm.[1]

Firm Heterogeneity Firms are heterogeneous in how *efficient* they are at improving their quality. It is this heterogeneity across firms that gives rise to the possibility of selection. Formally, I assume that firms differ in their efficiency of the investment technology θ and can be either low-type (θ_L) or high-type (θ_H). A firm's type is persistent and determined upon entry. New entrant draws its type from a Bernoulli distribution

$$\theta = \begin{cases} \theta_L & \text{with probability } \alpha \\ \theta_H & \text{with probability } 1-\alpha \end{cases}. \tag{8}$$

where $\alpha \in (0,1)$ and $\theta_H > \theta_L > 0$. As it will be discussed later in detail, allowing this heterogeneity is not only important in quantifying the scope of firm selection into different legal forms, but also quantitatively relevant in accounting for the firm growth and exit heterogeneity within legal forms.[2]

2.3 Entry and Exit

A unit mass of potential entrants attempts to enter the economy at any point in time. They use a similar investment technology as incumbent firms, where the flow rate of

[1] Having average quality in (7) introduces spillovers between firms: each firm's improvement in its quality adds to the average quality, which in turn provides bigger quality improvement for all the firms in the economy. Therefore the parameter ω controls the extend of this spillover.

[2] For the relevance of this type of heterogeneity in other settings, see Acemoglu et al. (2018) in the context of optimal industrial policies, and Jones and Kim (2018) in the context of top income inequality.

entry x_e is related to the spending on entry efforts R_e according to $x_e = \theta_e \left(\frac{R_e}{\bar{q}}\right)^{\eta}$. Following a successful entry, the entrant first draws its initial quality from a distribution $\Psi(q)$ and its type, $\theta \in \{\theta_L, \theta_H\}$, then decides whether to incorporate its firm or not. This description implies the following optimization problem for entrants:

$$\max_{x_e} \{x_e \mathbb{E}\left(v(q,\theta)\right) - c_e(x_e, \theta_E)\} \tag{9}$$

where $\mathbb{E}\left(v(q,\theta)\right)$ is the expected value of entry (and the expectation is over the quality the successful entrants will obtain and firm type θ). Given that there is a unit measure of potential entrants, x_e is also equal to the total entry flow rate.

Exit of a firm happens either due to (i) an exogenous death shock at Poisson rate $\kappa > 0$ or (ii) firm choosing to exit *endogenously*: firms will voluntarily shut down when their quality is low enough such that they are no longer sufficiently profitable relative to the fixed cost of operation. When firms exit, they stop producing and their flow profits drop to zero. Importantly, I assume that firms are subject to an *exit cost* that is proportional to the quality of the firm at the time of exit. This cost can be considered as a liquidation cost or firing cost as in Hopenhayn and Rogerson (1993) and Poschke (2009). Recall that firm's optimal output and amount of labor are proportional to its quality as in equation (4), motivating the exit cost assumption being proportional to the quality. The presence of this cost introduces the main advantage of incorporation: XXX

2.4 Incorporation Choice

New entrants choose their legal form (incorporate or not) after they learn their initial quality (q) and their type (θ_L vs θ_H). Incumbent firms have the option to switch between legal forms at arrival rate μ. Incorporation entails a sunk setup cost C_I. In this setting, a firm with quality q and type θ chooses to incorporate if and only if

$$v_I(q;\theta) - C_I > v_U(q;\theta) \tag{10}$$

where v_I and v_U denote the value of an incorporated and unincorporated firm, respectively. Incorporation provides liability protection which ensures that firm owner does not suffer any losses beyond the value of the firm.[3] In other words, in the case of exit, incorporated firms do not suffer loses due to liquidation or firing costs that derive the firm value below zero. This benefit comes at the expense of set-up and maintaining cost. In short, incorporation trade-offs exit cost, which is proportional to the firm size, with higher fixed cost of operation and setup cost. This trade-off and its implications on firm behavior will be more clear in the next section.

2.5 Firm Decision and Value Functions

I normalize all the growing variables by average quality in the economy, $\bar{q}(t)$ to keep the stationary equilibrium values constant. Let us denote the relative value of a generic variable X by $\hat{X}$. Moreover, let g denote the growth rate of average quality, which is also the aggregate growth rate in the economy endogenously determined in equilibrium. Then the stationary equilibrium value function for incorporated firms with relative quality $\hat{q}$ and type θ can be written as

$$\rho v_I(\hat{q};\theta) = \max \left\{ 0, \max_{x_I \geq 0} \left[\begin{array}{c} \pi\hat{q} - c(x,\hat{q},\theta) - \psi_I \\ -g\hat{q}\frac{\partial v_I(\hat{q};\theta)}{\partial \hat{q}} \\ +x_I\left[v_I\left(\hat{q}_+;\theta\right) - v_I(\hat{q};\theta)\right] \\ +\kappa\left[0 - v_I(\hat{q};\theta)\right] \\ +\mu\left[\max\{v_I(\hat{q};\theta), v_U(\hat{q};\theta)\} - v_I(\hat{q};\theta)\right] \end{array} \right] \right\} \tag{11}$$

where $\hat{q}_+$ denotes the new level of relative quality after a successful investment. Notice that the type of the firm affects the firm value through the quality enhancing investment cost function $c(x,\hat{q},\theta)$ implied by equation (6). This value function implicitly defines a threshold level of relative quality at which firms choose to exit and firms' optimal rate of expansion x_I which determines the rate of quality growth at the firm level.

The value function above can be interpreted as follows. Given discounting at the rate ρ, the left-hand side is the flow value of firm with relative productivity $\hat{q}$. The right-hand side includes the components that make up this flow value. The outer maximiza-

[3] Initial costs of starting a firm such as the cost of entry given by (9) and the setup cost of incorporation C_I are considered as sunk.

tion underlines the endogenous exit decision of the firm. The first line includes the instantaneous profits, minus the cost of quality enhancing investment and the fixed costs of operation. The second line reflects the change in firm value due to the increase in the average quality in the economy, which happens at the rate g. This term accounts for the fact that as the average quality increases, the relative quality at which the firm operates declines, leading to the erosion of profits.The third line expresses the change in firm value when the firm is successful with its investment at the rate x_I. The fourth line shows the change in value when the firm has to exit due to an exogenous death shock at the rate κ. Notice that in the case of incorporation, firm value drops to zero. The last line adds the the change in the value of the firm if firm decides to switch to being unincorporated.

The value function for unincorporated firms is given by

$$\rho v_U(\hat{q};\theta) = \max\left\{ -c_E\hat{q}, \max_{x_U\geq 0}\left[\begin{array}{c} \pi\hat{q} - c(x,\hat{q},\theta) - \psi_U \\ -g\hat{q}\frac{\partial v_U(\hat{q};\theta)}{\partial \hat{q}} \\ +x_U\left[v_U\left(\hat{q}_+;\theta\right) - v_U(\hat{q};\theta)\right] \\ +\kappa(-c_E\hat{q} - v_U(\hat{q};\theta)) \\ +\mu\left[\max\left\{v_I(\hat{q};\theta) - C_I, v_U(\hat{q};\theta)\right\} - v_U(\hat{q};\theta)\right] \end{array}\right]\right\} \tag{12}$$

The main difference here is that the presence of exit cost creates the possibility to drive firm value below zero. This happens either due to firms experiencing exogenous death shock at the rate κ or firm choosing to exit when their relative quality is too low to be profitable.

The optimal expansion decision of a firm with legal form $l \in \{I, U\}$ and firm type $\theta \in \{\theta_L, \theta_H\}$is given by

$$x_{l,\theta}(\hat{q}) = \frac{v_l\left(\hat{q}_+;\theta\right) - v_l(\hat{q};\theta)}{\theta}. \tag{13}$$

Intuitively, the incentives to invest on quality depend on the marginal returns of doing so $v_l\left(\hat{q}_+;\theta\right) - v_l(\hat{q};\theta)$. Importantly, higher the efficiency of the investment technology (higher θ), higher the resulting the expansion rate chosen by the firm, for a given marginal return.

As discussed in Appendix in detail, the value functions are convex and conditional on legal form, high-type firms have a lower threshold quality under which they exit.

Lemma 1 *Given the value functions in (11) and (12), the followings are true:*

(1) For a firm with legal form $l \in \{I, U\}$ and type $\theta \in \{\theta_L, \theta_H\}$, there exist a threshold relative quality $\hat{q}_{min,l,\theta}$ such that firm endogenously exits the economy if its relative quality decreases below this threshold.

(2) $\hat{q}_{min,l,\theta_L} > \hat{q}_{min,l,\theta_H}$

(3) For a type θ firm, there exist a relative quality level $\hat{q}_\theta$ such that

$$v_I(\hat{q};\theta) - C_I \geq v_U(\hat{q},\theta), \quad for\ \hat{q} \geq \hat{q}_\theta. \tag{14}$$

(4) $\hat{q}_{\theta_L} > \hat{q}_{\theta_H}$

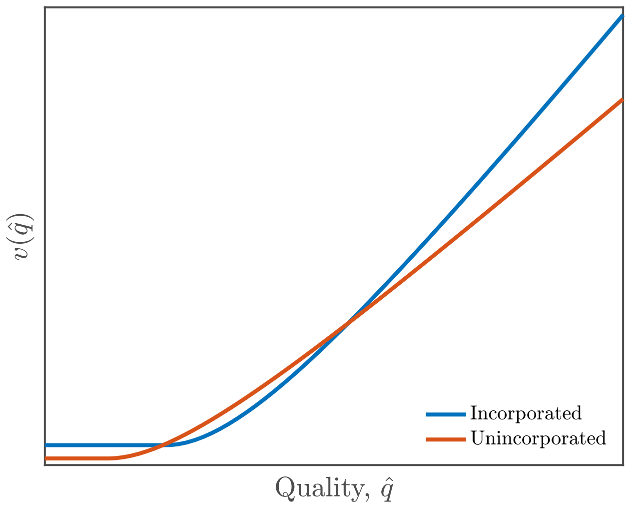
(5) $v_I(\hat{q};\theta_H) - v_U(\hat{q},\theta_H) > v_I(\hat{q};\theta_L) - v_U(\hat{q},\theta_L)$

Above lemma suggest that when firms reach a certain size, they want to be incorporated. Importantly, this threshold size is smaller for high type firms. In other words, the probability of being incorporated is increasing in relative quality and higher for high type firms. Figure 1 depicts a comparison of value function for incorporated and unincorporated firms.

2.6 Quality Distributions

As the quality improvements are stochastic in nature, firms (within each legal form and type category) are heterogeneous in terms of quality. Along balance growth path, the stationary distribution of relative qualities emerges as the result of the expansion and exit decisions of all firms, and characterizes the long-run state of the economy. For a given legal form (incorporated vs unincorporated) and firm type (θ_L vs θ_H), the distri-

Figure 1: Value Functions: Incorporated vs Unincorporated



bution of relative qualities in stationary equilibrium satisfies

$$gqf'(\hat{q}) = (x_{\hat{q}} + \kappa - g)f(\hat{q}) - x_{\hat{q}-\eta}f(q-\eta)\frac{1}{1+\lambda(1-\omega)} - x_e\Psi(\hat{q}) \tag{15}$$

where $f()$ is the density with $f(q) = 0$ for $q < q_{min}$ where q_{min} is solved from value function. Integrating over the domain $[q_{min}, \infty)$, we get

$$gq_{min}f(q_{min}) + \kappa\Phi = \gamma \tag{16}$$

under the assumption that the density is integrable, i.e. $\lim_{q\to\infty} f(q) = 0$. Above equation simply implies that the amount of qualities going under q_{min} plus exits due to κ should be equal to the measure of firms entering to the economy so that total mass is stable in stationary equilibrium.

By assuming that entry distribution has a thinner tail, i.e.$\lim_{q\to\infty} \frac{\psi(q)}{Cq^{-\rho-1}} = 0$, we get

$$[(1+\lambda(1-\omega))^{\rho} - 1] = \frac{\rho g + \kappa}{\bar{x}}. \tag{17}$$

Here one solution for ρ is zero which yields a degenerate solution. The next result partially characterize the non-degenerate solution.

Lemma 2 *The solution to ρ described in (26) is non-decreasing in ω and g and non-increasing in λ and τ for $\rho \geq 1$. Moreover $\rho = 1$ is a solution whenever $\lambda(1-\omega)\bar{x} = g + \kappa$ is satisfied. Finally $\lim_{\omega \to 1} \rho(\omega) = \infty$.*

2.7 Dynamic Equilibrium

Given the above description of the environment, I can now formally define the full dynamic equilibrium for this economy.

Definition 1 Consider the environment described above. A stationary equilibrium of this economy is a tuple yj,pj,lj,Vl,Vh,qh,min,ql,min,xh,xl,xentry,hh,hl,hentry,h,l,np,Fl(q),Fh(q),ws,wu,g,r such that (i) yj and pj maximize profits as in (11) and the labor demand lj satisfies (5); (ii) Vl and Vh are given by the low-type and high-type value functions in (15) and (16); (iii) (qh, min, ql, min) satisfy the threshold rule in (19); (iv) xh and xl are given by the R&D policy functions in (18) and xentry solves the entrants problem in (14); (v) the skilled worker demands hh, hl, and hentry satisfy (8) and (13); (vi) the sta- tionary equilibrium productivity distributions (Fl(q),Fh(q)) and the product line shares (h , l , np) satisfy Lemma 3; (vii) the growth rate is given by (23); (viii) the interest rate satisfies the Euler equation (4); and (ix) ws and wu are consis- tent with labor market-clearing for unskilled and skilled workers as given by (20) and (22).

3. Data and Calibration

3.1 Data

The quantitative analysis of the model uses both firm and individual level data for the years between 2001 and 2015. To measure properties of the process of firm dynamics, I rely on micro data for the population of non-farm and non-financial businesses from the Danish Business Statistics Register. The variables used in each year include the two-digit industry identifier, employment level (in FTE) and legal form of the business.

Since the focus of the paper is how limited liability affects the incorporation decision and firm dynamics, I restrict the sample to firms which there is a single owner who owns and operates the business. The restriction to solely owned businesses allows us to isolate the insurance motives for incorporation by mitigating the influence of other benefits, such as issuing equity. Therefore, for the purpose of this paper, we define an entrepreneur as an individual who owns the entirety of the business and has an active management role in that business; and, we will call sole-proprietors unincorporated entrepreneurs and single-owner corporations incorporated entrepreneurs. As a result of the restriction to solely owned businesses, the sample size is reduced to 1,917 firms, of which 41% are incorporated.

The calibration uses legal form switching behavior of the firms.

The empirical and quantitative analysis in this project is built on detailed micro-level data from Statistics Danmark (DST). For data on workers, we rely on the Integrated Database for Labor Market Research (IDA). In this dataset, each individual is assigned a unique identifier. We observe individuals on an annual basis. From different subsets of this dataset, we can leverage information on workers highest completed education, background family characteristics (e.g. parental income), employment status, occupation, and income. In addition, the FIDA dataset links individuals to their place of employment (a unique employer identifier) each year. In addition to the individual level data in IDA, there is additional data on individuals academic pursuits. The PHD dataset contains detailed information on individuals who enrolled in a PhD program. We can link individuals to all other observables by their unique identifier. In addition, PHD contains information on most students subject of PhD, date of enrollment and date of graduation. We combine the standard administrative datasets with IQ data provided from the Danish military test, Borge Priens Prove, which is required for conscripts at age 18-19. This test data goes back to 1995. With tests mostly taken at age 18, this provides data on half of the males (conscripted males) approximately age 36 and under in 2013. This data informs us on the links between talent, education, and innovation. While the data is somewhat limited in the scope of the population, it still delivers approximately 500,000 males with IQ data which is more than enough to discuss the links between IQ, education, innovation, and parental income. DST also provides details on project funding related to R&D as the Danish government expanded their support for R&D starting in 2002. In particular, we observe both funding through the broad program Innovation Danmark and R&D tax credits. The reason for R&D subsidies and tax credits are denoted by each firm. The innovation program also has information on funding to education, which will be a key component of our discussion when it comes to subsidies. We combine DST data with innovation data made available through patents at the European Trademark Office (EPO). These are used to measure the degree of innovation at the firm and individual level. We use a disambiguation algorithm provided by the DST to match on about 75% of inventors on patents from Denmark. This data will be our primary link to innovation. These data allow for the ingredients of previously discussed mechanisms: evaluating the role of IQ in the education/innovation decision, the innovative rate and occupation of PhDs, and the team structure of innovation. In addition, because of a host of policy tools the Danish government has utilized we have variation related to policy instruments that the model can encompass.

This leaves us with approximately 32 million individual matched employer-employee observations, 10,000 inventors,

We then combine this information with individual-level census records from both countries to measure the importance of managerial employment. Here we briefly describe the main data sources

In this section, we describe our strategy to estimate relevant parameters of the models. We calibrate moments from the model to their counterpart in the data in order to back out fundamental parameters. We start with 14 parameters of interest as noted in Table 1. We refer to the literature for 3 parameters, and directly match 2 with the data. For the remaining 9 parameters, we select 9 informative moments, ME, that help us identify underlying parameters in the data. These moments come from quantitative aspects of the facts that we will listed in the next subsection 4.3. We then utilize simulated method of moments (SMM) to jointly calibrate the 9 remaining parameters. We minimize the distance between model-simulated moments, M(), and their empirical counterparts, ME, by searching over the parameter space , using a simulated annealing algorithm, as follows: Parameter

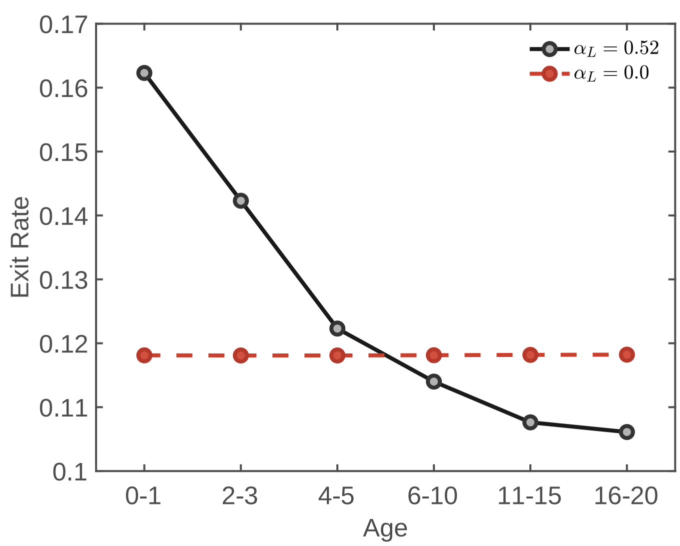
min (MiE Mi())2.

3.2 Identification

To identify the share of low-type firm upon entry, I focus on the age-profile of exit rates conditional on firm size. Without type heterogeneity, the likelihood of exit would be independent of age conditional on size. In the data, however, such conditional exit rates are strongly decreasing in firm age. Through the lens of our model, this pattern is rationalized through endogenous selection, whereby the share of low-type firms within a given cohort declines as the cohort ages. This is shown in Panel B of Figure 4, where we display the exit rate of small firms by age for different values of share of low types upon entry, α. Without any heterogeneity, i.e. $\alpha = 1$, the conditional exit hazard is flat.

4. Calibration

Figure 2: Exit rate by age, conditional on size (Incorporated)



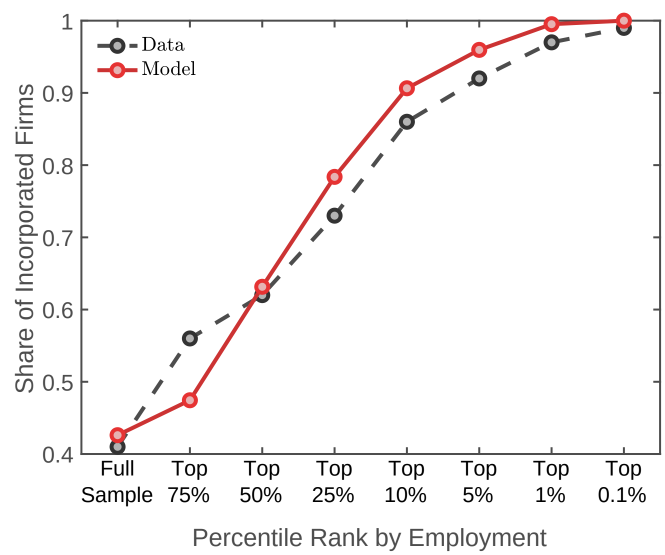
4.1 Non-targeted Moments

5. Quantitative Results

Table 1: Parameter Estimates

Parameter		Value
Fixed cost (I)	c_I	0.250
Fixed cost (U)	c_U	0.043
Exogenous death rate	κ	0.036
Exit cost	c_E	2.061
Step size	λ	0.156
Share of low types on entry	α	0.678
Expansion efficiency (high type)	θ_H	0.932
Expansion efficiency (low type)	θ_L	0.392
Scale of entry cost	θ_E	0.388
Incorp. set-up cost (scale)	$\tilde{C}_I$	0.014
Incorp. set-up cost (shape)	ζ	0.125
Entry dist. (rate)	χ	11.24
Switching rate	μ	0.022

Figure 3: Share of Incorporation by Firm Size



5.1 Incentive Channel

Table 4 presents the average expansion rates for the legal form and firm type categories. It suggests that for both low- and high- types, incorporation increases expansion rates

Table 2: Targeted Moments

	Model	Data
Growth	0.030	0.029
Entry rate	0.078	0.085
Empl. share of entrants	0.031	0.022
Employment at age 20 (I)	4.14	4.28
Employment at age 20 (U)	1.51	1.54
Share of I at age 0	0.22	0.23
Share of I at age 10	0.39	0.38
Exit rate of small firms at age 0 (I)	0.08	0.06
Exit rate of small firms at age 0 (U)	0.17	0.12
Exit ratio of small firms, age 0 to 20 (I)	1.42	1.53
Exit ratio of small firms, age 0 to 20 (U)	2.46	2.67
Exit rate of large firms	0.036	0.036
Tail of firm size dist.	2.04	2.04

Table 3: Non-targeted Moments

	Model	Data
Average firm size (I/U)	3.97	4.92
Share of switchers from U to I (cond. on switching)	0.93	0.99
Standard dev. of log employment age 10 (I)	1.49	1.32
Standard dev. of log employment age 10 (U)	1.12	0.95

by 50%, which can be considered as the direct effect of incorporation. This is due to the fact that incorporation limits the losses due to exit.

5.2 Selection

To study the extend of selection effects, Table 5 provides the share of firm for each legal form and firm type categories both upon entry and among incumbents. The former provides a measure of selection of types into different legal forms upon entry, whereas the latter emphasize the selection through competition and exit behavior of firms. The results imply that selection is strong among entrants: 90% (15%) of the incorporated (unincorporated) firms are high-types, which are estimated twice as efficient

Table 4: Average Expansion rates (x)

Unincorp.		Incorp.	
Low type	High type	Low type	High type
0.03	0.20	0.047	0.31

as low-types in improving their productivity. This selection becomes more pronounced among incumbents: almost all firms within incorporation are high types.

Table 5: Share of Firms

	Unincorp.		Incorp.	
	Low type	High type	Low type	High type
Shares among Entrants	0.67	0.12	0.02	0.19
Shares among Incumbents	0.36	0.22	0.005	0.42

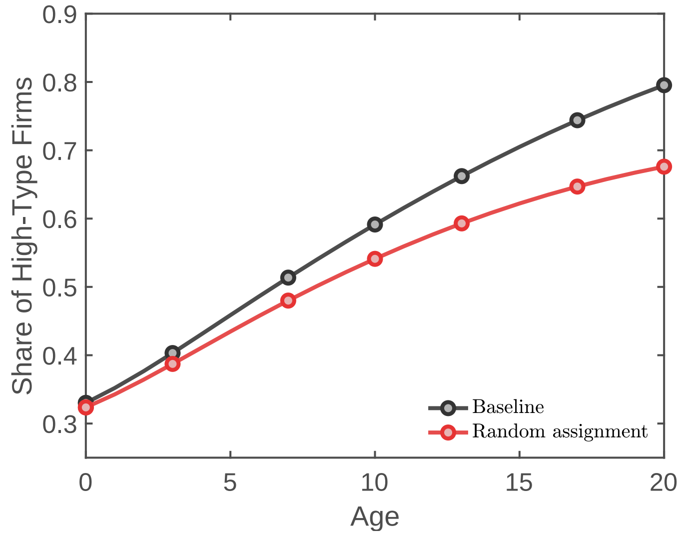
To understand the importance of the selection on the dynamics, I consider a counterfactual economy where the incorporation decision is randomized within firm types. Figure 4 shows the resulting effect of this counterfactual on selection pattern by presenting the share of high-type firms by age. In both baseline and counterfactual economy, initial entrants have the same type heterogeneity. However as the cohort gets older, the share of high-types grows slower under the counterfactual economy. This is because a lower share of high-types benefits from the direct benefit of incorporation due to the randomization. This decreases the growth rate in the economy from 3% to 2.2%, and decreases the exit rate of low-types. Moreover, the difference in the average size of incorporated and unincorporated firms decreases by 32% under the counterfactual economy.

5.3 Policies

XXX

6. Conclusion

Figure 4: Counterfactual: Random Assignment of Legal Status



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7. Appendix

7.1 Derivations

Labor market clearing

$$L + \tilde{L} = \bar{L}, \quad \tilde{L} = \int_{\mathcal{N}} l_j dj \tag{18}$$

$$\begin{aligned} (1-\beta)L^{\beta} q_j^{\beta} k_j{}^{-\beta} &= \frac{w}{\bar{q}} \\ k_j &= \left[(1-\beta)\frac{\bar{q}}{w}\right]^{\frac{1}{\beta}} Lq_j \end{aligned} \tag{19}$$

which implies

$$p_j = \frac{1}{(1-\beta)}\frac{w}{\bar{q}}$$

The realized price is a constant markup $(1/(1-\beta))$ over the marginal cost and is independent of the individual product quality. The profit is then

$$\begin{aligned} \pi(q_j) &= \left(\frac{1}{(1-\beta)}\frac{w}{\bar{q}} - \frac{w}{\bar{q}}\right) \times k_j \\ &= \frac{\beta}{1-\beta}\frac{w}{\bar{q}}\left[(1-\beta)\frac{\bar{q}}{w}\right]^{\frac{1}{\beta}} Lq_j \\ &= \beta\left[(1-\beta)\right]^{\frac{1-\beta}{\beta}} \left(\frac{\bar{q}}{w}\right)^{\frac{1-\beta}{\beta}} Lq_j \\ &= \pi q_j \end{aligned}$$

where $\pi = \beta\left[(1-\beta)\right]^{\frac{1-\beta}{\beta}} \left(\frac{\bar{q}}{w}\right)^{\frac{1-\beta}{\beta}}$, i.e profits are linear in quality.

By using the maximization problem in the final goods sector, together with optimal price and quantity at the intermediate firm level, we can get a relation between wage rate and average quality:

$$
\begin{aligned}
Y &= \frac{L^\beta}{1-\beta}\int_{\mathcal{N}} q_j \left(\left[(1-\beta)\frac{\bar{q}}{w}\right]^{\frac{1}{\beta}} Lq_j\right)^{1-\beta} dj \\
Y &= L(1-\beta)^{\frac{1-2\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}}\bar{q}\Phi
\end{aligned} \tag{20}
$$

Also final goods' producer profit needs to be zero (with aggregate price index normalized to 1)

$$
\begin{aligned}
Y - \int_{\mathcal{N}} p_j k_j dj - wL = 0 & \\
Y &= \frac{1}{(1-\beta)}\frac{w}{\bar{q}}\left[(1-\beta)\frac{\bar{q}}{w}\right]^{\frac{1}{\beta}} L\int_{\mathcal{N}} q_j dj + wL \\
Y &= (1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}} L\bar{q}\Phi + wL
\end{aligned} \tag{21}
$$

Using 20 and 21

$$
\begin{aligned}
L(1-\beta)^{\frac{1-2\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}}\bar{q}\Phi &= (1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}} L\bar{q}\Phi + wL \\
(1-\beta)^{\frac{1-2\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}}\bar{q}\Phi - (1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}}\bar{q}\Phi &= w \\
(1-\beta)^{\frac{1-2\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}}\bar{q}\Phi\left[1-(1-\beta)\right] = w & \\
\beta(1-\beta)^{\frac{1-2\beta}{\beta}}\Phi &= \left[\frac{w}{\bar{q}}\right]^{\frac{1}{\beta}} \\
\frac{w}{\bar{q}} &= \beta^\beta(1-\beta)^{1-2\beta}\Phi^\beta \\
w &= \tilde{\beta}\bar{q}
\end{aligned} \tag{22}
$$

where $\tilde{\beta} = \beta^\beta(1-\beta)^{1-2\beta}\Phi^\beta$. So wage is proportional to average quality in the economy. Incorporating the equilibrium wage rate, the profits simplify to

$$
\begin{aligned}
\pi(q_j) &= \beta\left[(1-\beta)\right]^{\frac{1-\beta}{\beta}}\left(\beta^{\beta}(1-\beta)^{1-2\beta}\Phi^{\beta}\right)^{\frac{\beta-1}{\beta}} L q_j \\
\pi(q_j) &= \beta^{\beta}(1-\beta)^{(1-2\beta)\frac{\beta-1}{\beta}-\frac{1-\beta}{\beta}}\Phi^{\beta-1} L q_j \\
\pi(q_j) &= \beta^{\beta}(1-\beta)^{2-2\beta}\Phi^{\beta-1} L q_j \\
\pi(q_j) &= \frac{(1-\beta)}{\Phi}\beta^{\beta}(1-\beta)^{1-2\beta}\Phi^{\beta} L q_j \\
\pi(q_j) &= \frac{(1-\beta)}{\Phi}\tilde{\beta} L q_j \\
&= \frac{\beta^{\beta}(1-\beta)^{2-2\beta}}{\Phi^{1-\beta}} L q_j
\end{aligned}
$$

This last expression makes it clear that higher the firm mass, lower the profits. This is because more firms implies higher wages, given the constant supply of workers, so that it reduces the profits

Furthermore, by combining 20 with 23, we can show that output is linear in $\bar{q}$

$$
\begin{aligned}
Y &= L(1-\beta)^{\frac{1-2\beta}{\beta}}\left[\beta^{\beta}(1-\beta)^{1-2\beta}\Phi^{\beta}\right]^{\frac{\beta-1}{\beta}}\bar{q}\Phi \\
&= (1-\beta)^{\frac{1-2\beta}{\beta}}\left[\beta^{\beta}(1-\beta)^{1-2\beta}\Phi^{\beta}\right]^{\frac{\beta-1}{\beta}}\Phi L\bar{q} \\
&= \beta^{\beta-1}(1-\beta)^{(\beta-1)\frac{1-2\beta}{\beta}+\frac{1-2\beta}{\beta}}\Phi^{\beta-1}\Phi L\bar{q} \\
&= \frac{(1-\beta)^{1-2\beta}}{\beta^{1-\beta}}\Phi^{\beta} L\bar{q}
\end{aligned}
$$

Finally, by combining **??**, 18, 19 and 23, we can find L as

$$
\begin{aligned}
\tilde{L} &= \int_{\mathcal{N}} l_i dj \\
&= \int_{\mathcal{N}}\left[(1-\beta)\frac{\bar{q}}{w}\right]^{\frac{1}{\beta}} L q_j \frac{1}{\bar{q}} dj \\
&= \left[(1-\beta)\frac{1}{\tilde{\beta}}\right]^{\frac{1}{\beta}} L\Phi.
\end{aligned}
$$

Labor market clearing implies that

$$
\begin{aligned}
1 &= L + \tilde{L} \\
1 &= L + \left[(1-\beta) \frac{1}{\tilde{\beta}} \right]^{\frac{1}{\beta}} L\Phi \\
L &= \frac{\beta}{\beta + (1-\beta)^2}
\end{aligned}
$$

Note that mass of firms does not affect L. If we substitute this to the profit, we get

$$
\pi(q_j) = \frac{\beta^{\beta}(1-\beta)^{2-2\beta}}{\Phi^{1-\beta}} \frac{\beta}{\beta + (1-\beta)^2} q_j
$$

7.2 Quality Distributions

The density $f()$ is given by

$$
0 = (gqf)_q + x_{q-\eta} f(q-\eta) \frac{1}{1+\lambda(1-\omega)} - x_q f(q) + \gamma\psi(q) - \kappa f(q) \tag{23}
$$

$$
gqf_q = (x_q + \kappa - g) f(q) - x_{q-\eta} f(q-\eta) \frac{1}{1+\lambda(1-\omega)} - \gamma\psi(q) \tag{24}
$$

with $f(q) = 0$ for $q < q_{min}$ where q_{min} is solved from value function. Integrating over the domain $[q_{min}, \infty)$, we get

$$
gq_{min} f(q_{min}) + \kappa\Phi = \gamma \tag{25}
$$

under the assumption that the density is integrable, i.e. $\lim_{q\to\infty} f(q) = 0$. Above equation simply implies that the amount of qualities going under q_{min} plus exits due to κ should be equal to the amount entering the system so that total mass is stable in stationary distribution.

Lets look at the tail of the distribution, which will help us to solve the distribution. First note that as q goes to infinity, x_q becomes constant.

We start with guessing that the distribution tail has a Pareto shape of the form $Cq^{-\rho-1}$ as $q \to \infty$. Substituting this guess into the equation for the density delivers

$$-\rho g C q^{-\rho-1} + \bar{x}\left[C\left(\frac{q-\lambda\omega\bar{q}}{1+\lambda(1-\omega)}\right)^{-\rho-1}\frac{1}{1+\lambda(1-\omega)} - Cq^{-\rho-1}\right] + \gamma\psi(q) - \kappa C q^{-\rho-1} = 0$$

$$-\rho g - \kappa + \bar{x}\left[q^{\rho+1}\left(\frac{q-\lambda\omega\bar{q}}{1+\lambda(1-\omega)}\right)^{-\rho-1}\frac{1}{1+\lambda(1-\omega)} - 1\right] + \gamma\frac{\psi(q)}{Cq^{-\rho-1}} = 0$$

Now assume that entry distribution has a thinner tail, i.e.$\lim_{q\to\infty}\frac{\psi(q)}{Cq^{-\rho-1}} = 0$. Then we have

$$[(1+\lambda(1-\omega))^{\rho} - 1] = \frac{\rho g + \kappa}{\bar{x}}. \tag{26}$$

Here one solution for ρ is zero which yields a degenerate solution. The next result partially characterize the non-degenerate solution.

Lemma 3 *The solution to ρ described in (26) is non-decreasing in ω and g and non-increasing in λ and τ for $\rho \geq 1$. Moreover $\rho = 1$ is a solution whenever $\lambda(1-\omega)\bar{x} = g + \kappa$ is satisfied. Finally* $\lim_{\omega\to 1}\rho(\omega) = \infty$.

7.3 Derivation of Value Functions

The value function is given by

$$rV(q) - \frac{dV(q)}{dt} = \max\{0, \max_x\{\pi q - \eta\chi x^{\frac{1}{\eta}} q - c_F\bar{q} + x\left[V\left(q + \lambda\left(\omega\bar{q} + (1-\omega)q\right)\right) - V(q)\right] + \kappa(0 - V(q))\}\}$$

First define $V(q) = v(\hat{q})\bar{q}$ where $v(.)$ is the normalized value function and $\hat{q} = \frac{q}{\bar{q}}$. Then divide both sides by $\bar{q}$ to get

$$\rho v(\hat{q}) - \frac{dv(\hat{q})}{dt} = \max\{0, \max_x\{\pi\hat{q} - \eta\chi x^{\frac{1}{\eta}}\hat{q} - c_F + x\left[v\left(\hat{q} + \lambda\left(\omega + (1-\omega)\hat{q}\right)\right) - v(\hat{q})\right] + \kappa(0 - v(\hat{q}))\}\}.$$

where growth rate of $\bar{q}$ is g, $\frac{\dot{\bar{q}}}{\bar{q}} = g$ and we use the fact that $r = \rho + g$ from the representative consumer problem.

Next we look at $\frac{dv(\hat{q})}{dt}$

$$\begin{aligned}\frac{dv(\hat{q})}{dt} &= \frac{\partial v(\hat{q})}{\partial \hat{q}}\frac{\partial \hat{q}}{\partial \bar{q}}\frac{\partial \bar{q}}{\partial t} \\ &= \frac{\partial v(\hat{q})}{\partial \hat{q}} \times \left(-\frac{q}{\bar{q}^2}\right) \times \bar{q}g \\ &= \frac{\partial v(\hat{q})}{\partial \hat{q}} \times \left(-\frac{\hat{q}}{\bar{q}}\right) \times \bar{q}g \\ &= -g\hat{q}\frac{\partial v(\hat{q})}{\partial \hat{q}}.\end{aligned}$$

Therefore the final (stationary) value function is

$$\begin{aligned}\rho v(\hat{q}) = \max\{0, \max_x\{&\pi\hat{q} - \eta\chi x^{\frac{1}{\eta}}\hat{q} - c_F \\ &- g\hat{q}\frac{\partial v(\hat{q})}{\partial \hat{q}} \\ &+ x\left[v\left(\hat{q} + \lambda\left(\omega + (1-\omega)\hat{q}\right)\right) - v(\hat{q})\right] \\ &+ \kappa(0 - v(\hat{q}))\}\}.\end{aligned}$$

Above value function is for the incorporated firms. For unincorporated firms, the difference is that when firm exit with κ arrival rate, the value goes to a negative value, instead of zero. For this, if we assume that this is proportional quality i.e.

$$rV(q) - \frac{dV(q)}{dt} = \max\{0, \max_x\{\pi q - \eta\chi x^{\frac{1}{\eta}} q - c_F\bar{q} + x\left[V\left(q + \lambda\left(\omega\bar{q} + (1-\omega)q\right)\right) - V(q)\right] + \kappa(-c_E \times q - V(q))\}\}$$

This will result in the following normalized value function

$$\begin{aligned}\rho v(\hat{q}) = \max\{0, \max_x\{(\pi - \kappa c_E)\hat{q} - \eta\chi x^{\frac{1}{\eta}}\hat{q} - \psi_F \\ - g\hat{q}\frac{\partial v(\hat{q})}{\partial \hat{q}} \\ + x\left[v\left(\hat{q} + \lambda\left(\omega + (1-\omega)\hat{q}\right)\right) - v(\hat{q})\right] \\ + \kappa(0 - v(\hat{q}))\}\}.\end{aligned}$$

i.e. just like a proportional decrease in the per period profits.

7.4 Welfare

As derived above, the final output is given by

$$Y = A\Phi^{\beta}\bar{q}$$

where A is constant that depends only on β. Final good is used for consumption, fixed cost, innovation, entry and exit purposes and incorporation cost (should we include this last thing). Therefore we have

$$C = Y - C_F - RD - C_{Entry} - C_E - C_M$$

where

$$\begin{aligned} C_F &= c_F\bar{q}\Phi \\ RD &= \sum_{i\in\{l,h\}}\sum_{k\in\{I,U\}} \eta\chi_i \int x_{q,i,k}^{\frac{1}{\eta}} q f_{i,k}(q)dq \\ C_{Entry} &= \frac{1}{\eta}\chi_e x_e^{\eta}\bar{q} \\ C_E &= \kappa c_E\Phi\bar{q} \\ C_M &= z_{LI}\frac{\zeta}{\zeta+1}\left[\mathbb{E}v_{LI}(q) - \mathbb{E}v_{LU}(q)\right] + z_{HI}\frac{\zeta}{\zeta+1}\left[\mathbb{E}v_{HI}(q) - \mathbb{E}v_{HU}(q)\right]\end{aligned}$$

where $f_{i,k}$ is the unnormalized quality distribution for type i firm with legal structure k. Note that at BGP, the only object that grows is $\bar{q}$. We normalize initial value of $\bar{q}$ to 1, $\bar{q}_0 = 1$. The welfare can be obtained as

$$
\begin{aligned}
U_0(C_0, g) &= \int_0^\infty e^{-\rho t} \ln C_t dt \\
&= \int_0^\infty e^{-\rho t} \ln \left(C_0 e^{gt} \right) dt \\
&= \ln C_0 \int_0^\infty e^{-\rho t} dt + \int_0^\infty e^{-\rho t} g t dt \\
&= \ln C_0 \left[-\frac{e^{-\rho t}}{\rho} |_0^\infty \right] + \left[-\frac{g e^{-\rho t} (\rho t + 1)}{\rho^2} |_0^\infty \right] \\
&= \frac{g}{\rho^2} + \frac{\ln C_0}{\rho}
\end{aligned}
$$

7.5 Growth Rate

Let Q_t be the sum of qualities in the economy, which is the relevant measure for growth rate.

$$
\begin{aligned}
Q_t &= \int q_t \\
\bar{q}_t \equiv \mathbb{E}(q_t) &= \frac{Q_t}{N} \\
Q_{t+\Delta t} &= Q_t + \int x_q \Delta t \lambda \left[\omega \bar{q}_t + (1-\omega) q_t \right] f(q_t) dq_t \\
&- \kappa \Delta t Q_t \\
&- g q_{min,t} \Delta t f(q_{min,t}) \\
&+ \gamma \Delta t \mathbb{E}(q_t^{entry})
\end{aligned}
$$

where N is the measure of active qualities and entry distribution is exogenous. If we assume exponential distribution for entry with minimum parameter $\hat{q}_{min}$

$$
\mathbb{E}(q_t^{entry}) = (\hat{q}_{min} + 1/\zeta) \bar{q}_t
$$

then, we can derive the growth rate as

$$
\begin{aligned}
Q_{t+\Delta t} &= Q_t + \int x_q \Delta t \lambda \left[\omega \bar{q}_t + (1-\omega) q_t\right] f(q_t) dq_t - \kappa \Delta t Q_t - g q_{min,t} \Delta t f(q_{min,t}) + \gamma \Delta t \mathbb{E}(q_t^{en} \\
&= Q_t + \Delta t \lambda \omega \frac{Q_t}{N} N \mathbb{E}(x_q) + \int x_q \Delta t \lambda (1-\omega) q_t f(q_t) dq_t - \kappa \Delta t Q_t - g q_{min,t} \Delta t f(q_{min,t}) \\
g \equiv \frac{Q_{t+\Delta t} - Q_t}{\Delta t Q_t} &= \lambda \omega \mathbb{E}(x_q) + \lambda(1-\omega) \frac{\int x_q q_t f(q_t) dq_t}{Q_t} - \kappa - g f(q_{min,t}) \frac{q_{min,t}}{Q_t} + \gamma \frac{\mathbb{E}(q_t^{entry})}{Q_t} \\
g \left(1 + f(q_{min,t}) \frac{q_{min,t}}{Q_t}\right) &= \lambda \omega \mathbb{E}(x_q) + \lambda(1-\omega) \frac{\int x_q q_t f(q_t) dq_t}{Q_t} - \kappa + \gamma \frac{\mathbb{E}(q_t^{entry})}{Q_t} \\
g &= \frac{\lambda \omega \mathbb{E}(x_q) + \lambda(1-\omega) \frac{\int x_q q_t f(q_t) dq_t}{Q_t} - \kappa + \gamma \frac{\mathbb{E}(q_t^{entry})}{Q_t}}{1 + f(q_{min,t}) \frac{q_{min,t}}{Q_t}}
\end{aligned}
$$

Note that we solve the distribution for the relative qualities. Suppose we normalize the qualities with average quality $\bar{q}_t \equiv \frac{Q_t}{N}$, then above growth expression can be written as

$$
\begin{aligned}
g &= \frac{\lambda \omega \mathbb{E}(x_q) + \lambda(1-\omega) \frac{\int x_q q_t f(q_t) dq_t}{\bar{q}_t N} - \kappa + \gamma \frac{\mathbb{E}(q_t^{entry})}{\bar{q}_t N}}{1 + f(q_{min,t}) \frac{q_{min,t}}{\bar{q}_t N}} \\
&= \frac{\lambda \omega \mathbb{E}(x_q) + \lambda(1-\omega) \frac{\int x_q \hat{q} f(\hat{q}) d\hat{q}}{N} - \kappa + \gamma \frac{(\hat{q}_{min} + 1/\zeta)}{N}}{1 + f(\hat{q}_{min}) \frac{\hat{q}_{min}}{N}} \\
&= \frac{\lambda \omega \mathbb{E}(x_q) + \lambda(1-\omega) \mathbb{E}(x_q \hat{q}) - \kappa + \gamma \frac{(\hat{q}_{min} + 1/\zeta)}{N}}{1 + f(\hat{q}_{min}) \frac{\hat{q}_{min}}{N}}
\end{aligned}
$$

Notice that, in this case, $\mathbb{E}(\hat{q}) = 1$.

This is convenient since it allows $\bar{\hat{q}}$ in step size equal to 1.