Dynamics of learning and iterated games (2022-2023)

Project 1: permanent replicator dynamics

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1 Introduction

This assignment has two parts. In the first part, I will investigate the hypercycle equation given by
$\dot{x}_i = x_i(x_{i-1} - \sum_{j=1}^{n} x_j x_{j-1})$.
This system describes specific biological processes especially those related to RNA and DNA. We will first show that the unique Nash Equilibrium is stable when the dimension $n \leq 4$ and for $n \geq 5$ exhibits unstable or at least saddle behaviour. We show that the solutions don't go through the boundary and investigate the behaviour of solutions near the boundary.

The second part of the project is computational where we display the solutions within the simplex in a way that their higher dimension is projected in 2D. I will also investigate the chaos in the solutions.

Finally, in the mastery component, I will replicate some of the work of Sato et al. This system is interesting due to its closeness to the reinforcement model.

2 Answer to Question 2

We are trying to prove a test for a dynamical system to be permanent. This means that if $x_i(0) > 0$ for $i = 1, ..., n$ then

$$\lim_{t \to +\infty} inf x_i(t) > \delta \tag{1}$$

for all $i = 1, ..., n$. Where δ doesn't depend on the initial values $x_i(0)$. Hence permanence means that an orbit starts inside the simplex then it never reaches the boundary of the simplex.

Theorem1
Assume that P is continuous, P > 0 on the interior of Δ and for x in the interior of Δ define

$$\psi(x) := \frac{\dot{P}(x)}{P(x)} \tag{2}$$

Assume that ψ extends continuously to the boundary and that $x \in \partial\Delta$,

$$\int_0^T \psi(x(t))dt > 0 \text{ for some T } > 0 \tag{3}$$

Then the corresponding dynamical system is permanent as there exists $\delta > 0$ so that for each x in the interior Δ there exists T > 0 so that for t $\geq$ T, we have $x_i(t) \geq \delta$ for all $t \geq T$ and $i = 1, ..., n$

Note: P(x) measure the the distance of x from the boundary.

Proof [1]:

We choose T(x) to be a locally continuous function T(x), where T > 0 and can depend on x from (3). It's infimum τ is positive since $\partial\Delta$ is compact as all convergent subsequences of T must converge in Δ and T >0.

For h > 0, we define

$$U_h = \left\{x \in \Delta : \text{ there is a } T > \tau \text{ s.t. } \frac{1}{T}\int_0^T \psi(x(t))dt > h\right\} \tag{4}$$

For $x \in U_h$ we set

$$T_h(x) = inf\left\{T > \tau : \frac{1}{T}\int_0^T \psi(x(t))dt > h\right\} \tag{5}$$

Lets first show that U_h is open and T_h upper semicontinuous: i.e. if $x \in U_h$ and $\alpha > 0$ are given then for $y \in \Delta$ sufficiently close to x, we have

$$y \in U_h \text{ and } T_h(y) < T_h(x) + \alpha \tag{6}$$

So for α and x there is $T \in [\tau, T_h(x) + \alpha)$ s.t.

$$\epsilon = \frac{1}{T}\int_0^T \psi(x(t))dt - h > 0 \tag{7}$$

Since the solutions of the ODEs depend continuously on the initial values and x(t), y(t) are close $\forall t \in [0, T]$. Then if x and y are sufficiently close, the uniform continuity of ψ implies that $|\psi(x(t)) - \psi(y(t))| < \epsilon$ for $t \in [0, T]$, thus

$$\frac{1}{T}\int_0^T \psi(y(t))dt > \frac{1}{T}\int_0^T \psi(x(t))dt - \epsilon = h \tag{8}$$

$T > \tau$ and (8) implies that $y \in U_h$. Hence U_h is open. Also, due to the supremum of the set, we get T from $T_h(y) < T_h(x) + \alpha$

Therefore the family of nested sets U_h (with h >0) is an open covering of the compact set $\partial\Delta$. There exists an h >0 s.t. U_h is an open neighbourhood of $\partial\Delta$ (in Δ). Since Δ U_h is also compact, P reaches its minimum on this set. Choosing p >0 smaller than this minimum, then

$$I(p) = \left\{x \in \Delta : 0 < P(x) \le p\right\} \tag{9}$$

is contained in U_h. I(p) is a very thin layer extending inside Δ from the boundary if p is small. We will now show that if an orbit starts in I(p) then it will eventually leave I(p). On the contrary, x(t) would have to stay in U_h $\forall t > 0$. In this case, $\exists T \ge \tau$ s.t.

$$\frac{1}{T}\int_t^{t+T}\psi(x(s))ds > h \tag{10}$$

But since $\psi = log(P)'$ holds in , this implies

$$h < \frac{1}{T}\int_t^{T+t} log(P)'(x(s))ds = \frac{1}{T}[log(P)(x(T+t)) - log(P)(x(t))] \tag{11}$$

Which is

$$P(x(t+T)) > P(x(t))e^{hT} \geq P(x(t))e^{h\tau} \tag{12}$$

This contradicts the boundedness of P as there would exist a sequence t_n for which $P(x(t_n)) \to \infty$ as the exponential expression that is a lower bound to this explodes with the sequence.

Let $\tilde{I}(p)$ be the union of I(p) with $\partial\Delta$. Now we just need to show that $\exists\ q \in (0,p)$ s.t. $x(0) \notin \tilde{I}(p) \implies x(t) \notin I(q)$ for all $t \geq 0$.

The upper semicontinuous function T_h reaches its upper bound $\tilde{T}$ on the compact set $\tilde{I}(p)$. Let t_0 be the first time when x(t) reaches $\tilde{I}(p)$, in other words

$$t_0 = min\Big\{t > 0 : x(t) \in \tilde{I}(p)\Big\} \tag{13}$$

and let $x(t_0) = y$. Clearly P(y)=p. Let m be the minimum of ψ on . For $m \geq 0$, it's fine as P never decreases. In the case m < 0 we set $q = pe^{m\tilde{T}}$. For $t \in (0, \tilde{T})$

$$\frac{1}{t}\int_0^t \psi(y(s))ds \geq m \tag{14}$$

Hence

$$P(y(t)) \geq P(y)e^{mt} > pe^{m\tilde{T}} = q \tag{15}$$

This shows that the solution doesn't reach I(q) for t $\in (0, \tilde{T})$. Moreover, as $y \in I(p)$, there is a time $T \in [\tau, \tilde{T})$ s.t.

$$P(y(T)) \geq pe^{h\tau} \geq p \tag{16}$$

At time t+T, thus the orbit of x leaves I(p) without reaching the I(q). Repeating this argument it is clear that the orbit will never reach I(q).

3 Answer to question 3

Let $P(x) = x_1 ... x_{n-1}$. In order to show the permanence of the hypercycle for $n \geq 5$, we need to show that P is an average Lyapunov function or satisfies (3).

$$\begin{aligned}\psi &= \frac{\dot{P}}{P} \\ &= \sum_{i=1}^{n} \frac{\dot{x}_i}{x_i} \text{ (using product rule)}\end{aligned} \tag{17}$$

We know that the replicator dynamics is described by

$$\dot{x}_i = x_i\{x_{i-1} - \sum_{j=1}^{n} x_j x_{j-1}\} \tag{18}$$

Substituting (18) into (17). We get

$$\psi = 1 - n\sum_{j=1}^{n} x_j x_{j-1} \tag{19}$$

We need to show that $\forall x \in \partial\Delta$ there exits T >0 s.t.

$$\begin{aligned}0 &< \frac{1}{T}\int_0^T \psi(x(t))dt \\ &= \frac{1}{T}\int_0^T 1 - n\sum_{j=1}^{n} x_j x_{j-1} dt\end{aligned} \tag{20}$$

This implies

$$\frac{1}{T}\int_0^T \sum_{j=1}^{n} x_j x_{j-1} dt < \frac{1}{n} \tag{21}$$

This equivalent to showing that there doesn't exist a $x \in \partial\Delta$ s.t. $\forall T > 0$

This implies

$$\frac{1}{T}\int_0^T \sum_{j=1}^{n} x_j x_{j-1} dt \geq \frac{1}{n} \tag{22}$$

We will show this by first assuming there is such an x and show by induction that then

$$\lim_{t\to+\infty} x_i(t) = 0 \tag{23}$$

for i = 1,...,n. Since $x \in \partial\Delta$, there exists an index i_0 s.t. $x_{i_0} = 0$. If $x_i(t)$ converges to 0, then so does x_{i+1} but if $x_{i+1} > 0$ then using (18), we get

$$(logx_{i+1})' = \frac{\dot{x}_{i+1}}{x_{i+1}} = x_i - \sum_{j=1}^{n} x_j x_{j-1} \tag{24}$$

now integrating from 0 to T and dividing by T

$$\frac{logx_{i+1}(T) - logx_i(0)}{T} = \frac{1}{T}\int_0^T (logx_{i+1})' dt = \frac{1}{T}\int_0^T x_i(t)dt - \frac{1}{T}\int_0^T \sum_{j=1}^{n} x_j x_{j-1} dt \tag{25}$$

Since we assumed $x_i(t)$ converges to 0. It follows that

$$\frac{1}{T}\int_0^T x_i(t)dt < \frac{1}{2n} \tag{26}$$

for sufficiently large T. Using the upperbounds set in (26) and (21) to

$$logx_{i+1}(T) - logx_{i+1}(0) < \frac{T}{2n} - \frac{T}{n} = \frac{-T}{n} \tag{27}$$

or,

$$x_{i+1}(T) < x_{i+1}(0)exp(\frac{-T}{2n}) \tag{28}$$

This tells us that $x_{i+1}(t) \to 0$ as $T \to \infty$ hence (23) holds. This is a clear contradiction to $\sum x_i = 1$ as all components of x go to 0 as $T \to \infty$. Thus, P is a Lyapunov function and the dynamical system is permanent.

We know that for $n \geq 5$, it is not true that $\psi(x) > 0$ for all x close to the boundary of Δ (but P > 0 near e_i).

If $\psi(x) > 0$ on $\partial\Delta$ then (3) is satisfied. This also means that $\dot{P}(x) > 0$ for any x in the interior Δ near the boundary and so P would increase (repelled from the boundary).

If $\psi(x) = 0$ on the $\partial\Delta$ then $\dot{P}(x) = 0$. This means that the orbit stays at a fixed distance from the boundary and hence the direction of the orbit is tangent to the boundary.

If $\psi(x) < 0$ on the $\partial\Delta$ then $\dot{P}(x) < 0$. This means that the orbit will stay move closer to the boundary. However, in Answer to Question 2, we have proven that the orbit never reaches the boundary hence we can expect the orbit to fluctuate by going towards the boundary and away from the boundary. This makes sense as we know that p > 0 near the corners, which is where the orbit spends most of the time. So ψ isn't negative enough between the corner for the orbit to reach the boundary in a short time.

4 Answer to Question 4

$$A_0 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \tag{29}$$

Let $\hat{x}_1 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$. So $y \cdot A_0\hat{x}_1 = 0 \leq 0 = \hat{x}_1 \cdot A_0\hat{x}_1 \ \forall y \in \Delta$. Hence $\hat{x}_1$ is a NE of A_0.

Similarly let consider e_1. Then, $y \cdot A_0 e_1 = -y_2 - y_3 \leq 0 = e_1 \cdot A_0 e_1 \ \forall y \in \Delta$. Hence e_1 is a NE.

Lastly let $\hat{x}_2 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. This gives $A_0 \hat{x}_2 = \hat{x}_2$. Which gives $y \cdot A_0 x_2 = 0.5(y_2 + y_3) \leq 0.5 = x_2 \cdot A_0 x_2 \ \forall y \in \Delta$. Hence x_2 is a NE.

There are no other NE of A_0 after checking the standard basis and their linear combinations.

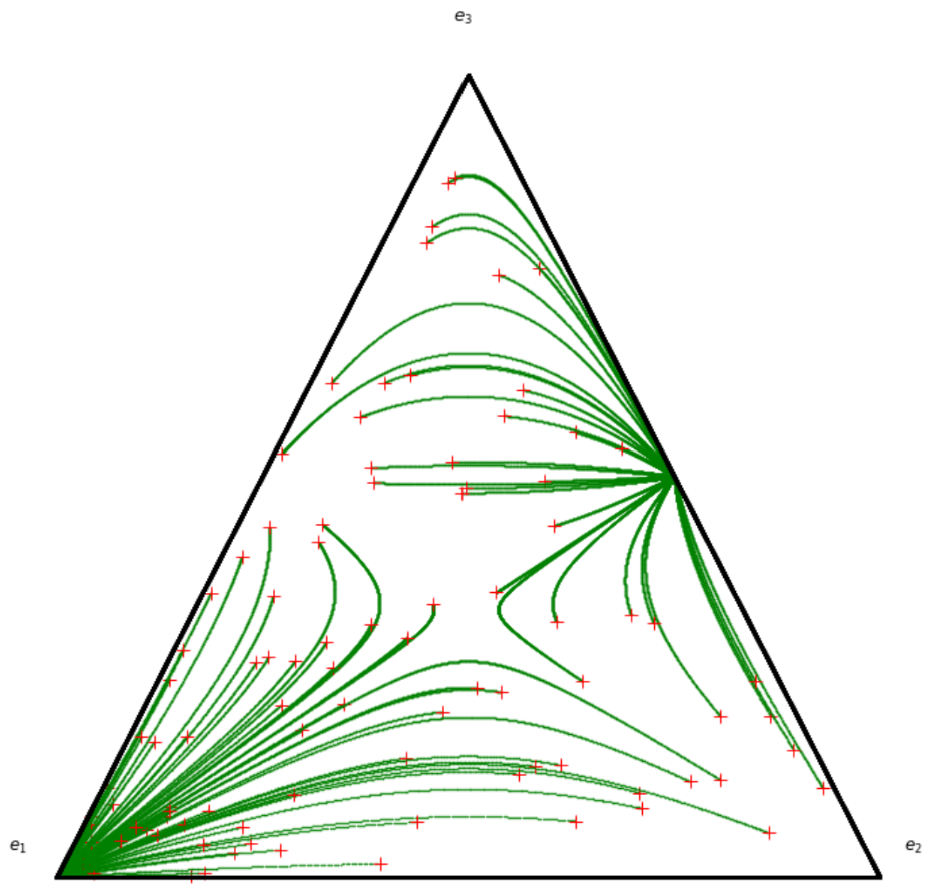


Figure 1: s asymptotic either to p or to a pat 75 random points in the simplex (marked as a cross) and plotted their motion with time. You can see that all the orbits have two omega-limits, they move towards two of the Nash equilibrium (e_1 or $\hat{x}_2$) but not towards the Nash equilibrium at $\hat{x}_1$. This means that e_1 and $\hat{x}_2$ are stable points however as observed by the motion around the middle of the simplex, it is clear that $\hat{x}_1$ is a saddle point.

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{30}$$

A_1 is cyclic in nature and hence only has one NE at $\hat{x} = \left[\frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}\right]$. This is because $y \cdot A_1\hat{x} = 0.2 \leq 0.2 = \hat{x} \cdot A_1\hat{x}\ \forall y$ in a 5 dimensional Δ.

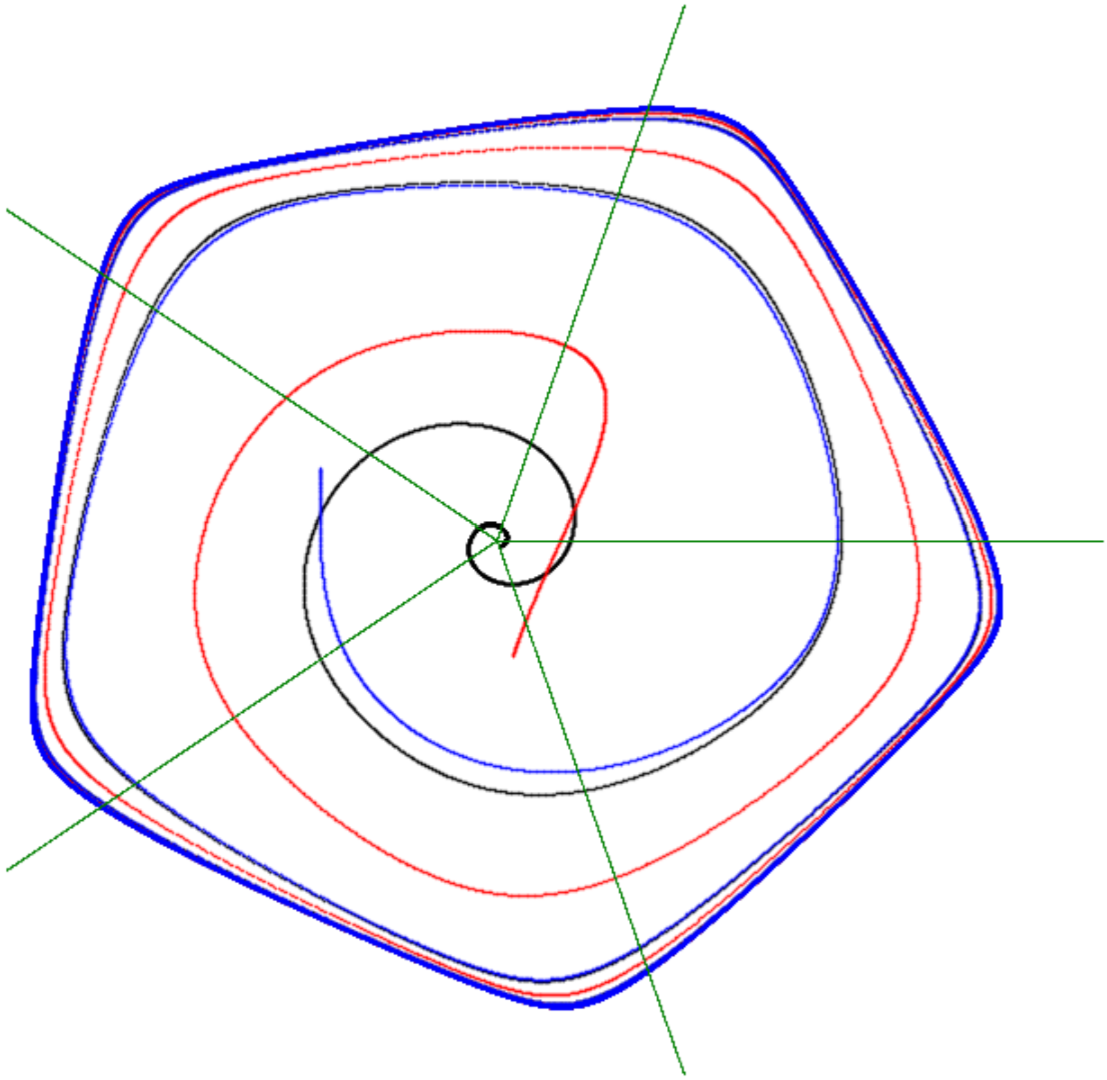
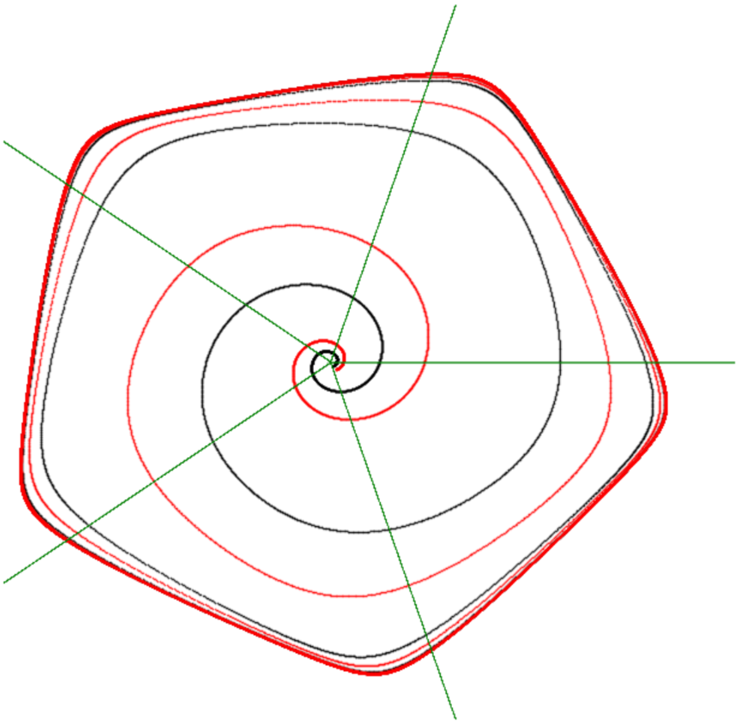
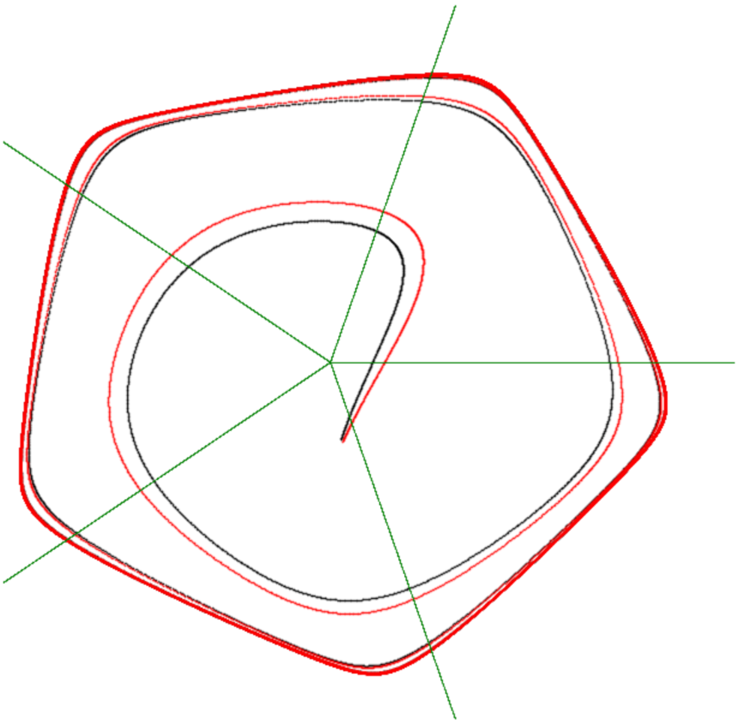
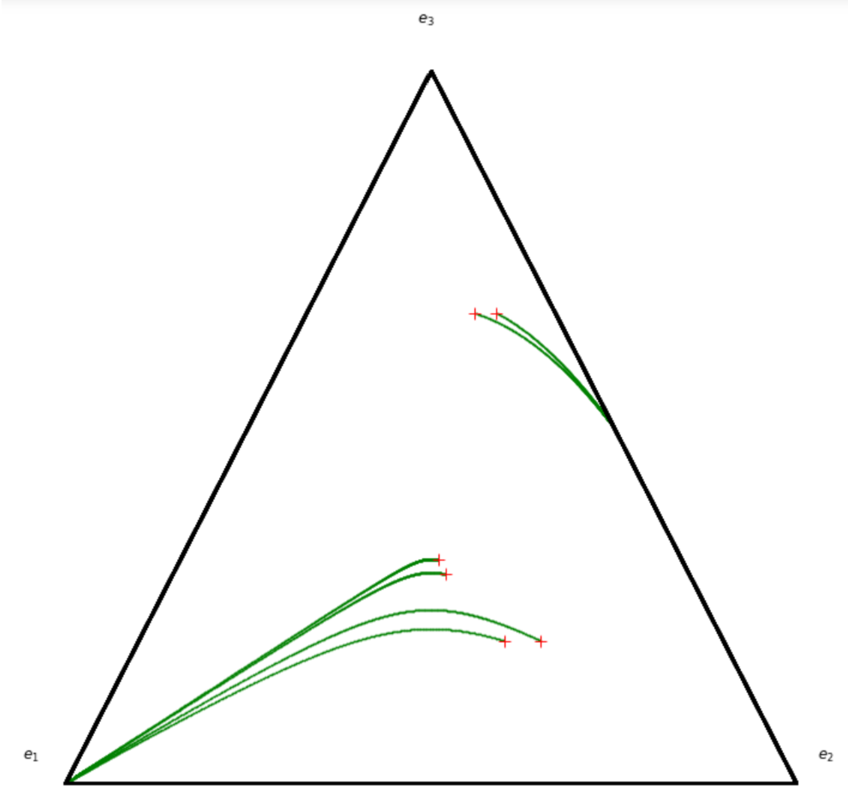


Figure 2: Here I started at 3 initial states and the graph shows the motion of the orbit with time. The green lines end at the standard basis. We can see that the motion of the orbits is an anti-clockwise rotation outwards getting closer and closer to the boundary but not reaching it. This follows from the Answer to question 3.

This shows that $\hat{x}$ is a repelling fixed point and that the omega-limit set is empty as the orbits aren't tending to a fixed point.







From the graph above and knowing that for $n \geq 5$, the orbits are a permanent hypercycle that never reaches the boundary. We can deduce that solutions to A_1 for $n \geq 5$ are chaotic, if we start from two very close initial points then the orbits taken are very different and even when its close to the limit hypercycle that it is convergent to, the position of the two orbits with time is different. We know that the solution of A_1 for $n \leq 4$ tends to a stable NE from [1]. This is supported by the triangle picture that for $n \leq 4$, if we start at two initial conditions close to each other, the two orbits will be similar and eventually converge. After sufficient time, the position of the orbits will be the same hence the solutions of A_1 for $n \leq 4$ are non-chaotic.

For a more mathematically rigorous explanation. We know from [8] that for solutions to be chaotic, they need to satisfy the following three conditions:
1) Deterministic system (only one future for each state)
2) Irregular spatial, temporal, or spatiotemporal patterns (a qualitative feature)
3) A positive largest Lyapunov exponent.

Clearly, condition 1 is satisfied as there is no randomness in our dynamical system. Condition 2 is satisfied because when $n \geq 5$ we have an orbit that tends towards a hypercycle but never reaches the hypercycle, however, condition 2 is not satisfied for $n \leq 4$ as the orbits eventually reach the stable NE.

I will now give an explanation of the Lyapunov exponent. Lyapunov exponent [9] characterises the rate of separation for two trajectories with initial separation vector δZ_0. This is given by $|\delta Z(t)| \approx e^{\lambda t}|\delta Z_0|$, where λ is the Lyapunov exponent

Finally, for condition 3, I approximated the largest Lyapunov constant (LLE) for A_1 for dimensions $n \in [2,6]$. The LLE was negative for $n \leq 4$ and positive for $n \geq 5$. This further support that the solutions for $n \geq 5$ are chaotic and solutions for $n \leq 4$ is non-chaotic.

In my code, as provided in the appendix. I use the following approximation:

$$\lambda = \frac{1}{n}\sum_{i=0}^{n-1} ln(\frac{seperation_i}{seperation_0}) \tag{31}$$

.

I chose n = 1000 with changes in time of 0.03 because after experimentation the LLE seems to have converged sufficiently to its true value for all the different dimensions of A_1 tested.

The results are printed below:
LLE for n = 2 is -0.0036310336078135726 (negative)
LLE for n = 3 is -0.004173579503592113 (negative)
LLE for n = 4 is -0.0002953247050468149 (negative)
LLE for n = 5 is 0.00019102576388430794 (positive)
LLE for n = 6 is 0.00016147016398595119 (positive)

According to the paper "Stable Periodic Solutions for the Hypercycle System", the solution of A_1 for $n \leq 4$ is non-chaotic based on the corollary under Theorem A because the boundary NE are global attractors hence asymptotically stable. The solutions to A_1 for $n \geq 5$ are chaotic based on the corollary because the internal NE is unstable and the solutions are orbitally asymptomatically stable and not asymptomatically stable.

The solutions associated with A_1 for $n \geq 5$ don't tend to the NE, as the central point is repelling.

Lets now consider:

$$A = \begin{bmatrix} \epsilon_x & -1 & 1 \\ 1 & \epsilon_x & -1 \\ -1 & 1 & \epsilon_x \end{bmatrix} \text{ and } B = \begin{bmatrix} \epsilon_y & -1 & 1 \\ 1 & \epsilon_y & -1 \\ -1 & 1 & \epsilon_y \end{bmatrix} \tag{32}$$

These matrices describe a two-player game with 3 actions. We can think of this game as a Rock-Paper-Scissors game.
Below I replicate the figures in the Sato et al.

In figure 3 and 4, $\epsilon_x = \epsilon_y = 0$, hence A and B are a zero sum game as $A + B^T = 0$. We can see from figure 4 that the system is non-chaotic as trajectories tend towards a neutrally stable periodic orbit.

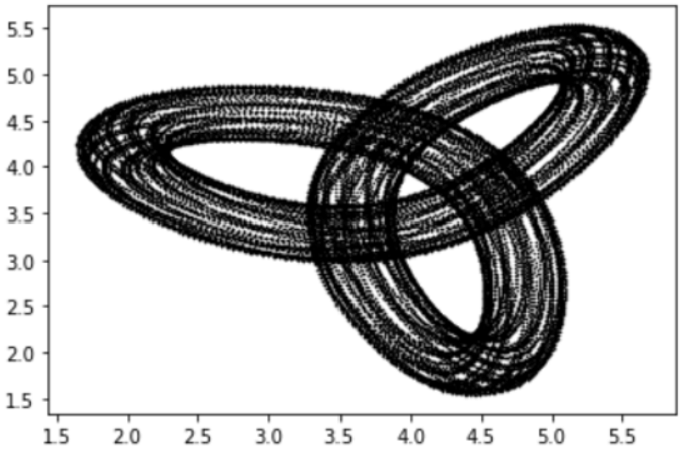


Figure 3: This figure shows the collective dynamic in the Δ with $\epsilon_x = -\epsilon_y = 0$. Where the initial condition is $\begin{pmatrix} 0.26 & 0.113333 & 0.626667 & 0.165 & 0.772549 & 0.062451 \end{pmatrix}$

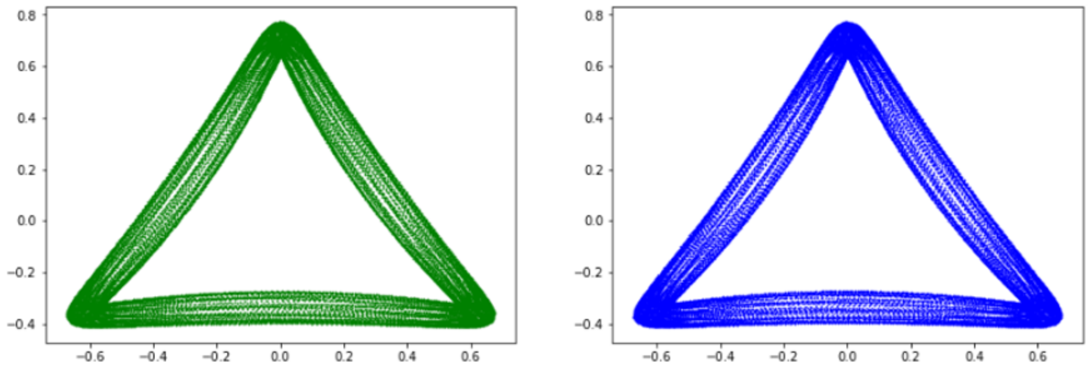


Figure 4: This shows the individual dynamic projected onto Δ_x and Δ_y for the conditions in the figure above.

In Figures 5 and 6 we plot the previous system but at different initial states. The result is a very different collective and individual dynamic. This is quite interesting as now the individual dynamic doesn't move closely along the boundary of the simplex in a cyclic manner (unlike in figs 3 and 4) but regularly diverges away from the boundary to more central areas of the simplex. This means that the strategy now is no longer to play the next action that the opponent previously played. For example, if the opponent played rock, unlike before now the strategy isn't necessary for us to play paper in the next move.

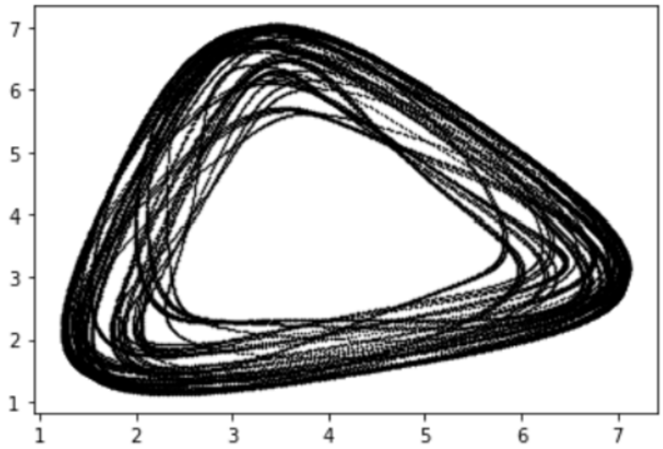


Figure 5: This figure shows the collective dynamic in the Δ with $\epsilon_x = -\epsilon_y = 0$. Where the initial condition is $\begin{pmatrix} 0.05 & 0.35 & 0.6 & 0.1 & 0.2 & 0.7 \end{pmatrix}$

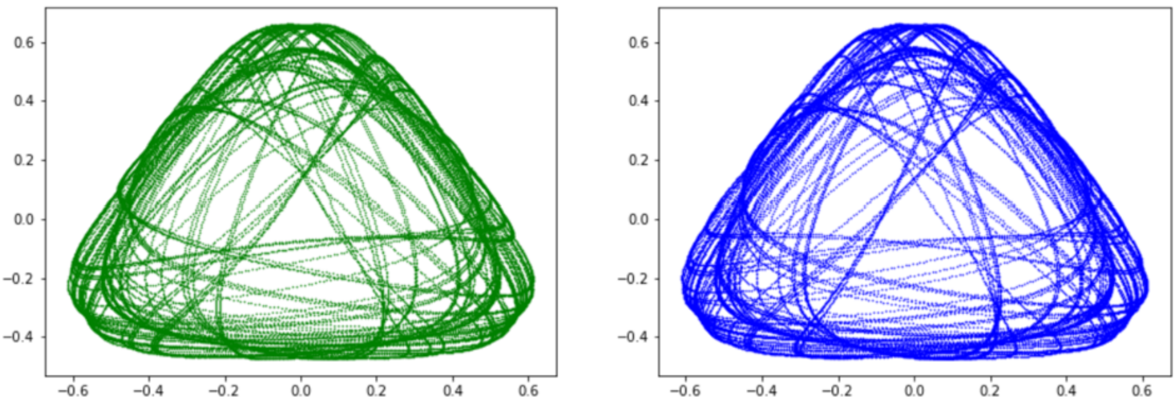


Figure 6: This shows the individual dynamic projected onto Δ_x and Δ_y for the conditions in the figure above.

For Figure 7 and 8 ($\epsilon_x = -\epsilon_y = 0$), the game is still zero-sum, however, now the collective dynamic starts off in a strange trajectory but then evolve into a more stable collective dynamic like the one for $\epsilon_x = -\epsilon_y = 0.5$. This loss of understanding of the centre to begin within the collective dynamic is due to perhaps now $\epsilon_x \neq \epsilon_y$.

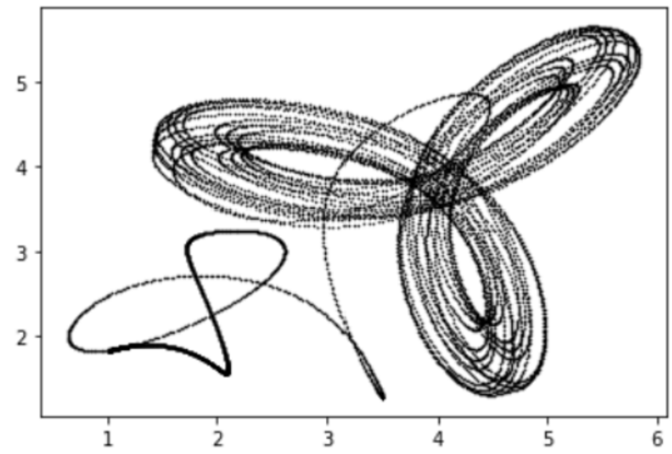


Figure 7: This figure shows the collective dynamic in the Δ with $\epsilon_x = -\epsilon_y = 0.5$. Where the initial condition is $\begin{pmatrix} 0.26 & 0.113333 & 0.626667 & 0.165 & 0.772549 & 0.062451 \end{pmatrix}$

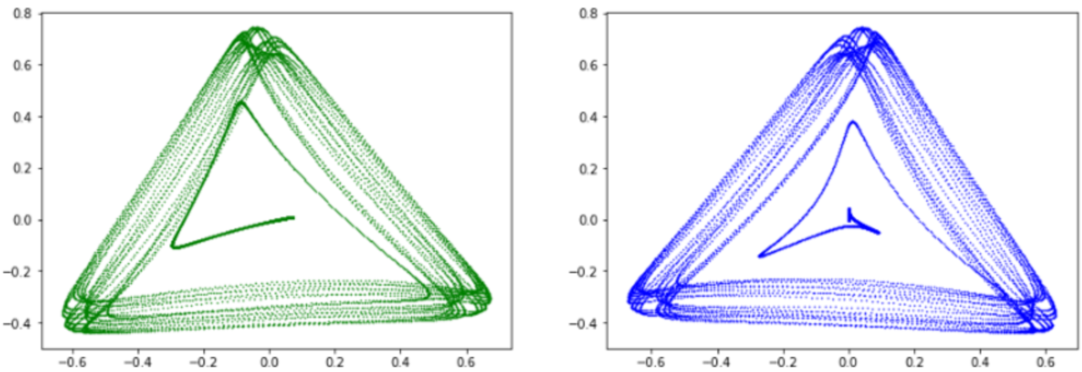


Figure 8: This shows the individual dynamic projected onto Δ_x and Δ_y for the conditions in the figure above.

Figure 9 and Figure 10 show a non-zero-sum game (with $\epsilon_x + \epsilon_y > 0$). The collective dynamics eventually diverge off the plot in a straight line, this could represent an eventual competition or cooperation state. The strategy of the two players from the individual dynamic looks out of phase with each other and they alternate winning each turn.

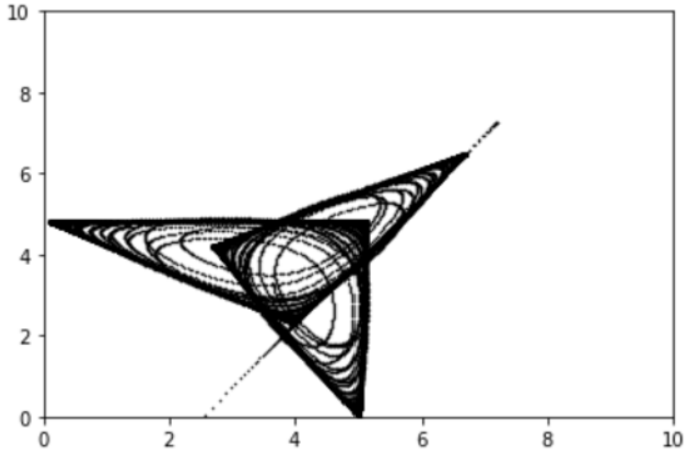


Figure 9: This figure shows the collective dynamic in the Δ with $\epsilon_x = -0.1$ and $\epsilon_y = 0.05$. Where the initial condition is $\begin{pmatrix} 0.26 & 0.113333 & 0.626667 & 0.165 & 0.772549 & 0.062451 \end{pmatrix}$

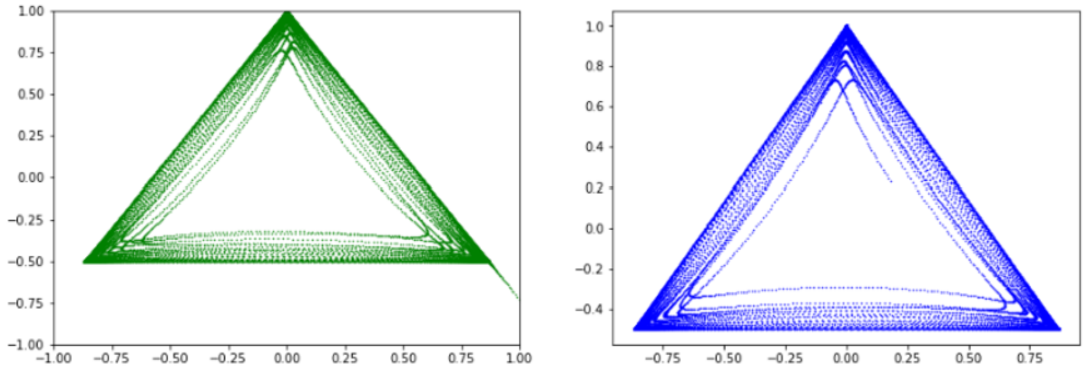


Figure 10: This shows the individual dynamic projected onto Δ_x and Δ_y for the conditions in the figure above.

Figures 11 and 12 show a chaotic orbit (here $\epsilon_x + \epsilon_y > 0$). According to Sato et al, Figure 12 depicts an infinitely persistent chaotic transient orbit as now player A can now choose a tie hence the cycles aren't closed.

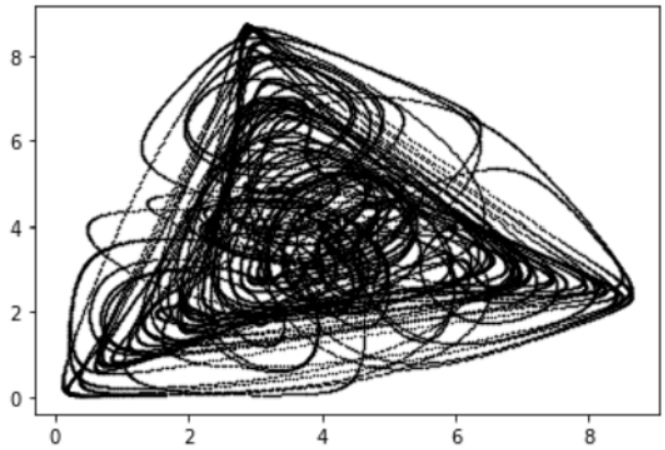


Figure 11: This figure shows the collective dynamic in the Δ with $\epsilon_x = 0.1$ and $\epsilon_y = -0.05$. Where the initial condition is $\begin{pmatrix} 0.26 & 0.113333 & 0.626667 & 0.165 & 0.772549 & 0.062451 \end{pmatrix}$

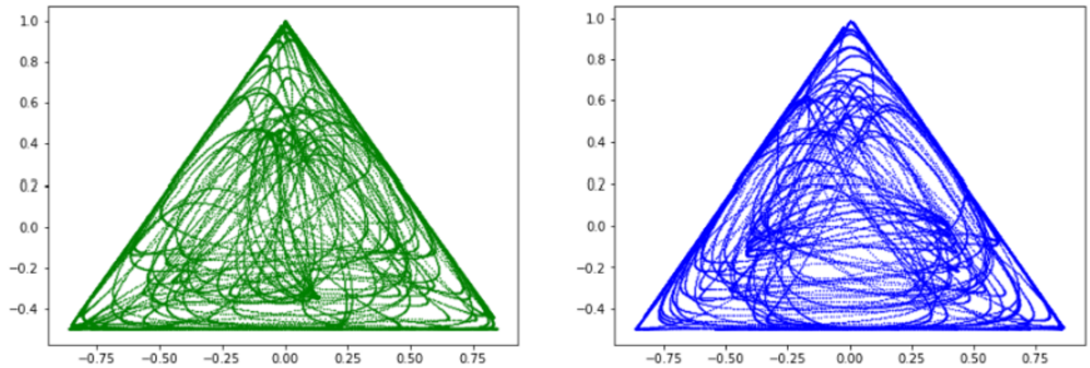


Figure 12: This shows the individual dynamic projected onto Δ_x and Δ_y for the conditions in the figure above.

5 Answer to mastery question 6

TheoremA

We are considering a dynamical system described by:

$$\dot{x}_i = x_i\Big[F_i(x_i, x_{i-1}) - \sum x_j F_j(x_j, x_{j-1})\Big] \tag{33}$$

Which is defined on the standard n-1 simplex in $\mathbb{R}^n_+$, which we denote to be S_{n-1}.

Our assumptions are

1) (33) is permanent on S_{n-1}. $\exists$ a positive number ρ s.t. every solution x(t), with $x(0) \in int\ S_{n-1}$ satisfies $x_i(t) > \rho$ for each i for large enough t.

2) $F_i(x_i, x_{i-1})$ are continuously differentiable functions defined for non-negative inputs satisfying

$$\frac{\partial F_i}{\partial x_{i-1}} > 0 \tag{34}$$

3) the dynamical system has a unique equilibrium point ρ in the interior of S_{n-1} and $Det\ DF(\rho) \neq 0$.

4) $F = F(F_1, F_2, ..., F_n)$ is homogeneous of degree $q > 0$:

$$F(sx) = s^q F(x) \tag{35}$$

where s≥ 0

The theorem states that if conditions 1) to 4) are met, then every orbit beginning in the interior of S_{n-1} is attracted to either ρ or to a non-trivial periodic orbit. If Diag(p)DF(p) has more than one eigenvalue with a positive real part (if ρ is unstable for (33)) then the system has a non-trivial periodic orbit. If F is analytic and on Int $\mathbb{R}^n_+$, then there can be at most finitely many periodic orbits in Int S_{n-1} and if ρ is linearly stable then at least one of these periodic orbits will be orbitally asymptotically stable.

5.1 Proof

To prove Theorem A we will not be investigating the solutions of (33) but of a similar system characterised by:

$$\dot{y}_i = y_i[F_i(y_i, y_{i-1} - K)] \tag{36}$$

where

$$K = \sum p_i F_i(p_i, p_{i-1}) = F_j(p_j, p_{j-1}) \tag{37}$$

We have assumed that 1) to 4) hold implying that (36) is a cooperative, irreducible and monotone cyclic feedback system. We note that ρ is an equilibrium of (36). Let y(t) be a solution for this system starting and always remaining in the Int $\mathbb{R}^n_+$. Computationally we can show that:

$$z(t) = Q[y(t)] = \frac{y(t)}{\sum y_i(t)} \in\ S_{n-1} \tag{38}$$

Which satisfies,

$$\dot{z}_i = \pi(t) z_i \Big[F_i(z_i, z_{i-1}) - \sum z_i F_j(z_j, z_{j-1})\Big] \tag{39}$$

where

$$\pi(t) = \Big[\sum y_i(t)\Big]^q \tag{40}$$

From our assumptions 1) and 4), we can imply that (36) has exactly one equilibrium, y= ρ in Int $\mathbb{R}^n_+$ and applying a modified version of theorem 4.1 from [10], we can conclude that the solution y(t) is asymptotic either to p or to a periodic orbit in Int $\mathbb{R}^n_+$.

Theorem4.1
a) Let $\dot{x}^i = f^i(x^i, x^{i-1}) \;\; for \;\; 1 \leq i \leq n$ where i(mod n)
be a monotone cyclic feedback system in $\mathbf{R}^n_+ = [0, \infty)^n$. Assume that $\mathbf{R}^n_+$ is positively invariant for $\dot{x}^i$ and that it contains a unique critical point x_*. If the forward orbit starting at initial condition x_0 is bounded then either:

(i) $\omega(x_0) = x_*$
(ii)$\omega(x_0)$ is a non-constant periodic orbit
(iii) if $\omega(x_0)$ contains x_* as well as many orbits that start and need at the same place.

Note that if $\Delta det(-Df(x_*)) < 0$, then (iii) can't occur.

(b) if $\Delta det(-Df(x_*)) = -1$, then sufficient conditions for ii) to occur for a bounded forward orbit are that Df(x_*) has at least 2 eigenvalues with positive real parts and x_0 is in a set which is a relatively open, positively invariant subset of $\mathbb{R}^n_+$ and is non-empty if either x_* is in the interior of R^n_+.

Using assumption 3) we can write the dynamical system in (33) to be:

$$\dot{x} = f(x) - [e^T f(x)]x \tag{41}$$

where $f_i(x) = x_i F_i(x)$, $e = (1, 1, ..., 1)^T$ and u^T denotes the transpose of the vector u. Similarly, we can re-write (36) as:

$$\dot{y} = f(y) - Ky \tag{42}$$

The corresponding Jacobian at y=ρ being Diag(ρ)DF(ρ).

The Jacobian for the right-hand side of (41) at y=ρ can be calculated as:

$$Df(\rho)p = (q+1)K\rho \tag{43}$$

$$Diag(\rho)DF(\rho) = Df(\rho) - KI \tag{44}$$

$$J(\rho) = Diag(\rho)DF(\rho) - \rho e^T[Diag(\rho)DF(\rho) + KI] \tag{45}$$

$$J(\rho)\rho = -KP \tag{46}$$

and

$$e^T J(\rho) = -Ke^T \tag{47}$$

Using this we can conclude that the stability of the equilibrium x= ρ for 33 on S_{n-1} is determined by n-1 eigenvalues of Diag(ρ)DF(ρ) distinct from Kq, whereas -K is the eigenvalue at ρ traversal to S_{n-1}

If k <0, then the remaining n-1 eigenvalues have a negative real part. ρ is globally stable for this case.

If k >0, then some eigenvalues may have non-negative real parts. H1) and [11] imply that the number of eigenvalues of J(ρ) with positive real parts must be even and using 3) we can conclude that Diag(ρ)DF()ρ will have an odd number of eigenvalues with positive real parts.

From now we will assume 1) to 4) hold for (36). Note that homogeneity of F_i implies that the infinite line through ρ is invariant with dynamics

$$\dot{s} = Ks(s^q - 1) \tag{48}$$

Where ρ is the only equilibrium point in Int$\mathbb{R}^n_+$, if $K \neq 0$.
If K <0, then ρ is globally stable for (36) on Int $\mathbb{R}^n_+$ and thus also for (33) on Int S_{n-1}.

This is Lemma 1.1 with a simple proof: F(0) = 0, which is repelling. But (48) shows that solutions on the invariant ray through ρ converge monotonically to ρ as t tends to infinity. For any $y \in$ Int $\mathbb{R}^n_+$, we can choose $r \in (0, 1)$ and $s > 1$ s.t. $y_1 = rp \leq y \leq sp = y_2$. Then by monotonicity, $y_1(t) \leq y(t) \leq y_2(t)\ \forall t \geq 0$, hence $y(t) \to \rho$ as $t \to \infty$.

For the rest of the report, we will assume that K >0. In this case, ρ repels the solution on the invariant ray through ρ. Hence we obtain the following results on the:

$$B(0) = \left\{\text{orbit tends to the initial state}\right\} \tag{49}$$

$$B(\infty) = \left\{\text{the sum of the components of the orbits tends to } \infty\right\} \tag{50}$$

Then : (**)
a) If $y \in B(0) \implies$ the set of points from origin to y is in B(0).
b) [0,ρ) is a subset of B(0).
c) [p, ∞] is a subset of B(0).

Lets also define a alternative for S_{n-1} for (36) :

$$M = \text{boundary of B(0) relative to Int } \mathbb{R}^n_+ \tag{51}$$

Theorem 1.4. The forward orbit of a point on M approaches either ρ or a non-trivial periodic orbit.

Proof: This is a result from theorem 4.1 [10] and that every positive orbit beginning on M, having a compact closure in M. On the other hand, y(0) $\in$ Int $\mathbb{R}^n_+$ belongs to an unbounded forward orbit if and only if $y_i(t) \to \infty$ as $t \to \tau$ for each i. This is because if y(t) were not to be bounded as $t \to \tau$, then $Y(t) = \sum y_i(t)$ is unbounded and hence $\liminf_{t\to\tau} z_i(t) = 0$ for some values of i, where $z(t) = \frac{y(t)}{Y(t)}$. This contradicts the permanence of (33), hence $y(0) \in B(\infty)$. If $y(0) \in M$, then y(t) is bounded since no other point in M than ρ can be related to ρ. (36) has an equilibrium on the boundary of the non-negative cone and hence we can apply Theorem 4.1 since the closure of the orbit is compact in M and M contains only the equilibrium ρ.
Now we will consider the case of the orbit of (36) not on M. We will use

$$\text{Int } \mathbb{R}^n_+ \subset B(0)UMUB(\infty) \tag{52}$$

We can see this if we suppose y(0) in not in B(0), M or $B(\infty)$. Then we know from before that y(0) must be bounded and be in the compact closure. Then through Theorem 1.4, its approaches ρ or a non-trivial periodic orbit. Then some scaled version by (0.1) of y(0) exists in M. Now using (39), we can re-parameterise the time in a monotonically increasing manner and set $y(t) = s(t)y_1[\tau(t)]$, where τ is the re-parameterisation and y_1 is the scaled version. If $y_1(t)$. Hence if y(t) tends to ρ, then y_1 must also tend to ρ. This leads to [y_1, y] being what is defined as a "trap" so using [11] hence implying that the leading eigenvalue at ρ is ≥ 0 hence contradicting K >0.
Hence the limit set must be a periodic orbit. We can show that these must be the same for y and y_1, hence again contradicts [11] as periodic orbits are linearly stable and the "most unstable manifold". QED.

Now we just need a few more ingredients to prove Theorem A.

Theorem 1.7 - if $Diag(\rho)DF(\rho)$ has more than eigenvalue then $\exists$ an open, positively invariant subset of M such that the limit set of the orbit of every point of U is a non-trivial orbit.

Proof: We know that the jacobian of (36) at y=ρ is Diag(ρ)DF(ρ). if K >0, then Det[-Diag(ρ)DF(ρ)] <0. Since we must have an odd number of eigenvalues, there must be at least 3 eigenvalues due to our assumption of more than one. Let's define:

U to be the set containing orbits that exist in M and N, where N is the integer values function N(z)= $\left\{i : z_i z_{i-1} < 0\right\}$ and N be the domain of N(z). Since N is open and N is continuous on N, we have that U is open on M. Since N can only decrease along a forward orbit and takes only even values, it follows that N[y(t)-ρ]=2 or N[y(t)-ρ]=0 (for large t). The latter cannot happen as then y(t) moves very far from ρ. Hence either $y(0) \in B(0)$ or $y(t) \to \infty$ leading to proving that U is positively invariant by (**). We can also show that U is non-empty. To

see this, set $z = (p_1, lp_2, ..., lp_n)$, where l >1 so $z \geq p = (p_1, p_2, ..., p_n)$. Then there is a unique 0 <r <1 (less than as orbit through z is bounded) s.t. rz is in M. However, $rlp_i > p_i$ must hold for $i \geq 2$ or else $rz \in B(0)$. It follows that rl >1 and N(rz-p)=2. Thus $rz \in U$. Hence for any initial condition starting in U, the orbit will tend towards a non-trivial periodic orbit.
Proposition 1.8: if F is analytic (a function that is locally given by the convergent power series) in Int$\mathbb{R}^n_+$, hence there are at most finitely periodic orbits in M.

Idea: the permanence of (33) implies that (36) has a compact set of attractors A, which is a subset of M. A contains the limit sets of all orbits of points of M. This attractor set can be used in the hypothesis of Theorem 4.3 [10] and the rest of the proof follows.
Assuming that Theorem 1.7 and Proposition 1.8 hold, then there is an orbitally asymptotically stable periodic orbit. The proof for this is omitted due to their complexity making it unfeasible to describe it without complete mathematical rigour.
Now all cases derived from K have been proved and we have proved theorem A for (33) by using a similar dynamical system (36) in the process also relying on the system.

6 References

[1] Hofbauer and Sigmund, Evolutionary Games and Population Dynamics, page 145 - 148, proof of theorem 12.2.1

[2] Hofbauer and Sigmund, Evolutionary Games and Population Dynamics, section 12.3, The permanence of the hypercycle

[3] Y. Sato, E. Akiyama and J. P. Crutchfield, Stability and diversity in collective adaptation, Pages 41-42

[4] Dynamics of Learning and Iterated Games lecture notes, code in Appendix - replicator-3x1

[5] Dynamics of Learning and Iterated Games lecture notes, code in Appendix - replicatorRPS-Sato

[6] Y. Sato, E. Akiyama and J. P. Crutchfield, Stability and diversity in collective adaptation, section 4

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[12] Hirsch, M. W. (1985). Systems of differential equations which are competitive or cooperative. II. Convergence almost everywhere. SlAM J.Math. Anal 16, 423-439

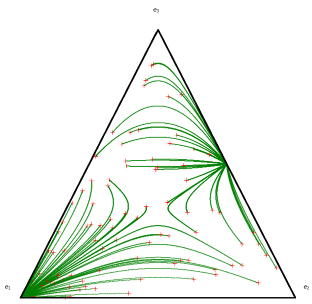
7 Appendix

7.1 Code for question 4

[4] The Poincar-Bendixson Theorem for Monotone Cyclic Feedback Systems, John Mallet and Hal Smith Theorem 4.1

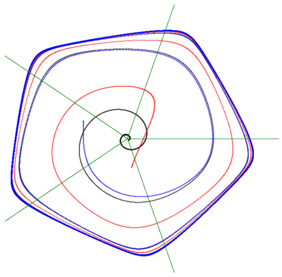
```
# Import the required modules
import numpy as np
import matplotlib.pyplot as plt
# This makes the plots appear inside the notebook
%matplotlib inline
from scipy.integrate import odeint
import random

# define a projection from the 3D simplex on a triangle
proj = np.array(
[[-1 * np.cos(30. / 360. * 2. * np.pi),np.cos(30. / 360. * 2. * np.pi),0.],
[-1 * np.sin(30. / 360. * 2. * np.pi),-1 * np.sin(30. / 360. * 2. * np.pi),1.]])
# project the boundary on the simplex onto the boundary of the triangle
ts = np.linspace(0, 1, 10000)
PBd1 = proj@np.array([ts,(1-ts),0*ts])
PBd2 = proj@np.array([0*ts,ts,(1-ts)])
PBd3 = proj@np.array([ts,0*ts,(1-ts)])

# choose game
A = np.array([[0, 1, -1], [-1, 0, 1], [-1, 1, 0]]) # row, 2nd row, 3rd row


points=[]

n=100

for i in range(n):
    a = random.random()
    b = (1-a)*random.random()
    c=1-a-b
    points.append(np.array([a, b, c]))


def replicator(x,t):
    return x * (A@x - np.transpose(x) @ (A@x))

# compute orbits
ts = np.linspace(0,100,10000)

xt= []
for i in range(n):
    xt.append(odeint(replicator, points[i], ts))

# project the orbits on the triangle
orbittriangle=[]

for i in range(n):
```

```
    orbittriangle.append(proj@xt[i].T)

ic=[]
for i in range(n):
    ic.append(proj@points[i])

f = plt.figure()
f.set_figwidth(12)
f.set_figheight(12)
plt.axis(False)

# plot the orbits, the initial values, the corner points, and the boundary points
for i in range(n):
    plt.plot(orbittriangle[i][0],orbittriangle[i][1],".",markersize=1,color='green')
    plt.plot(ic[i][0],ic[i][1],"+",markersize=10,color='red')

plt.text(-0.8660254-0.1, -0.5 +0.05 , "$e_1$",fontsize=12)
plt.text(+0.8660254+0.05, -0.5 +0.05 , "$e_2$",fontsize=12)
plt.text(0-0.03, 1 +0.1 , "$e_3$",fontsize=12)
plt.plot(PBd1[0], PBd1[1], ".",color='black',markersize=3)
plt.plot(PBd2[0], PBd2[1], ".",color='black',markersize=3)
plt.plot(PBd3[0], PBd3[1], ".",color='black',markersize=3)
```



```
# Import the required modules
import numpy as np
import matplotlib.pyplot as plt
# This makes the plots appear inside the notebook
%matplotlib inline
from scipy.integrate import odeint, solve_ivp, ode

# define a projection from the 5D simplex on 2D.
proj = np.array(
[[1, np.cos(2* np.pi/5),np.cos(4* np.pi/5),np.cos(6* np.pi/5),np.cos(8* np.pi/5)],
[0, np.sin(2* np.pi/5),np.sin(4* np.pi/5),np.sin(6* np.pi/5),np.sin(8* np.pi/5)]])

# choose game
A = np.array([[0, 0, 0, 0, 1], [1, 0, 0, 0, 0],[0, 1, 0, 0, 0],[0, 0, 1, 0, 0],[0, 0, 0, 1, 0]])

def replicator(t,x):
    return x*(A@x - np.transpose(x) @ (A@x))

```

```
t_final=1000
t_step = 0.03
time = np.arange(0, t_final, t_step)

x01 = np.array([0.21, 0.19, 0.2, 0.2, 0.2])
solx01=solve_ivp(replicator, [0, t_final], x01, method='DOP853', t_eval=time)
x02 = np.array([0.3990051 , 0.08843747, 0.01363782, 0.48547822, 0.01344139])
solx02=solve_ivp(replicator, [0, t_final], x02, method='DOP853', t_eval=time)
x03 = np.array([0.1067167 , 0.08228556, 0.51068711, 0.0910095 , 0.20930114])
solx03=solve_ivp(replicator, [0, t_final], x03, method='DOP853', t_eval=time)

PBd1 = proj@solx01.y
PBd2 = proj@solx02.y
PBd3 = proj@solx03.y

ts = np.linspace(0,1,10000)
xt1 = np.array([ts, 0., 0., 0., 0.])
xt2 = np.array([0., ts, 0., 0., 0.])
xt3 = np.array([0., 0., ts, 0., 0.])
xt4 = np.array([0., 0., 0., ts, 0.])
xt5 = np.array([0., 0., 0., 0., ts])

ic1=proj@xt1
ic2=proj@xt2
ic3=proj@xt3
ic4=proj@xt4
ic5=proj@xt5

f = plt.figure()
f.set_figwidth(12)
f.set_figheight(12)
plt.axis(False)

# plot the orbits, the initial values, the corner points, and the boundary points
plt.plot(PBd1[0],PBd1[1],".",markersize=1,color='black')
plt.plot(PBd2[0],PBd2[1],".",markersize=1,color='red')
plt.plot(PBd3[0],PBd3[1],".",markersize=1,color='blue')


plt.plot(ic1[0],ic1[1],"+",markersize=0.1,color='green')
plt.plot(ic2[0],ic2[1],"+",markersize=0.1,color='green')
plt.plot(ic3[0],ic3[1],"+",markersize=0.1,color='green')
plt.plot(ic4[0],ic4[1],"+",markersize=0.1,color='green')
plt.plot(ic5[0],ic5[1],"+",markersize=0.1,color='green')
```

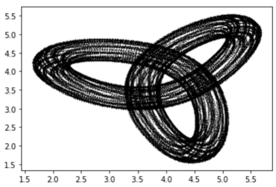


```
# below calculates Largest Lyapunov exponent for the specified n

n=4

A=np.diag(np.ones(n-1), -1)
A[0,-1] = 1

def replicator(t,x):
    return x*(A@x - np.transpose(x) @ (A@x))

x=np.ones((n,))/n
A_x = A@x
x_dot_A_x = x.T@A_x

t_final =1000
step=0.03
time = np.arange(0, t_final, step)

x1=step*np.random.rand(n)+x
x2=step*np.random.rand(n)+x
old_err = np.linalg.norm(x1-x2)

LLE = 0

for i in range(t_final):
    orbit1=solve_ivp(replicator, [0, t_final], x1, method='DOP853', t_eval=time)
    orbit2=solve_ivp(replicator, [0, t_final], x2, method='DOP853', t_eval=time)

    new_err = np.linalg.norm(orbit1.y[:, -1] - orbit2.y[:, -1])
    LLE += (1/ (t_final)*step)*np.log(new_err/old_err)

    x1=orbit1.y[:,-1]
    x2=x1+ ((orbit1.y[:, -1] - orbit2.y[:, -1])/(new_err/old_err))

print(LLE)
```

[5]

```
def replicator(t, x, A, B):
    dx = np.zeros(6)
    dx[:3] = x[:3] * (A@x[3:] - np.transpose(x[:3])@(A@x[3:]))
    dx[3:] = x[3:] * (B@x[:3] - np.transpose(x[3:])@(B@x[:3]))
    return dx

```

```
x0 = np.array([0.26, 0.113333, 0.626667, 0.165, 0.772549, 0.062451])

t_step=0.03
t_final = 1000
time = np.arange(0, t_final, t_step)

epsilonx=0
epsilony=-epsilonx

A=np.array([[epsilonx, 1, -1], [-1, epsilonx, 1], [1, -1, epsilonx]])
B=np.array([[epsilony, 1, -1], [-1, epsilony, 1], [1, -1, epsilony]])

sol = solve_ivp(replicator, [0, t_final], y0=x0, method='DOP853', t_eval = time,
    args=(A, B))
xx=sol.y

proj = np.array([[3.650, -1.350, 1.35, 5.35, 1.35, 1.45], [0.4, 0.4, 4.6,
    1.9,-0.4, 4.4]])
xy = proj@xx[:,:]
plt.plot(xy[0], xy[1], ".", markersize = 1, color='black')
```




```
proj = np.array([[-1*np.cos(np.pi/6), np.cos(np.pi/6), 0.], [-1*np.sin(np.pi/6),
    -1*np.sin(np.pi/6), 1]])
playA = xx[0, 100:], xx[1, 100:], xx[2,100:]
playB = xx[3, 100:], xx[4, 100:], xx[5,100:]
triangle1 = proj@playA
triangle2 = proj@playB

plt.figure(figsize=(15,5))
plt.subplot(121)
plt.plot(triangle1[0], triangle1[1], ".", markersize=1, color='green')
plt.subplot(122)
plt.plot(triangle2[0], triangle2[1], ".", markersize=1, color='blue')
```