Formalising Mathematics (2022-2023)

Coursework 2

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1 Introduction

In this project, I will prove the following theorem by following the proof below at a higher level:

Theorem 1.7. The derivative, if it exists, is unique.

Proof. Suppose $\Omega \subset \mathbb{R}^n$ is open, $f : \Omega \to \mathbb{R}^m, p \in \Omega$ and that K and K' satisfy:

$$\lim_{h \to 0} \frac{\|f(p+h) - f(p) - K[h]\|}{\|h\|} = \lim_{h \to 0} \frac{\|f(p+h) - f(p) - K'[h]\|}{\|h\|} = 0. \tag{1}$$

Let e be an arbitrary vector in $\mathbb{R}^n$ with $\|e\| = 1$. Then for any real number $\alpha \neq 0$ we have

$$\frac{K[\alpha e]}{\alpha} = K[e] \tag{2}$$

Now, let $(\alpha_j)_{j=0}^{\infty}$ be a sequence of non-zero real numbers tending to 0 as $j \to \infty$. By adding and subtracting identical terms, we see that

$$\begin{aligned}
&\|K[e] - K'[e]\| && (A)\\
&= \left\| \frac{K\left[\alpha_j e\right]}{\alpha_j} - \frac{K'\left[\alpha_j e\right]}{\alpha_j} \right\| && (B)\\
&= \lim_{j \to \infty} \frac{\left\|K\left[\alpha_j e\right] - K'\left[\alpha_j e\right]\right\|}{\left\|\alpha_j e\right\|} && (C)\\
&= \lim_{j \to \infty} \underbrace{\frac{\left\|-f\left(p + \alpha_j e\right) + f(p) + K\left[\alpha_j e\right] + f\left(p + \alpha_j e\right) - f(p) - K'\left[\alpha_j e\right]\right\|}{\left\|\alpha_j e\right\|}}_{**} && (D)\\
&\leq \lim_{j \to \infty} \underbrace{\frac{\left\|f\left(p + \alpha_j e\right) - f(p) - K\left[\alpha_j e\right]\right\|}{\left\|\alpha_j e\right\|}}_{\text{(E)}} + \lim_{j \to \infty} \underbrace{\frac{\left\|f\left(p + \alpha_j e\right) - f(p) - K'\left[\alpha_j e\right]\right\|}{\left\|\alpha_j e\right\|}}_{\text{(F)}}\\
&= 0.
\end{aligned}$$

For the last equality in the above equation we have used that $\alpha_j e \to 0$ as $j \to \infty$. By the above equation, for any unit vector e we have $K[e] = K'[e]$, which implies that (as linear maps) $K = K'$.

2 Definitions

```
variables {m n : ℕ} (Ω : set (fin n → ℝ)) (f : (fin n → ℝ) → (fin m → ℝ))
variables {Ω} (p : fin n → ℝ) {K K': (fin n → ℝ) →ₗ[ℝ] (fin m → ℝ)}

/-Given Ω ⊂ R^n is open, f : Ω → R^m, p ∈ Ω, and Λ. Defines what it means for Λ to be the derivative of f.-/
def deriv_exists (D : (fin n → ℝ) →ₗ[ℝ] (fin m → ℝ)) : Prop :=
∀ (ε : ℝ), 0 < ε → ∃ (R : ℝ), 0 < R ∧ (∀ (h : fin n → ℝ) , (0 < ‖h‖) ∧ (‖h‖ ≤ R) → ‖f (p + h) - f p - D h‖ / ‖h‖ < ε)

/-Defines a sequence whose limit is 0 and where each term of the sequence is non-zero.-/
def non_zero_convergent_zero (a : ℕ → ℝ) : Prop :=
(∀ j, a j ≠ 0) ∧ (∀ (ε : ℝ), 0 < ε → ∃ B : ℕ, ∀ j, B ≤ j → |a j| < ε)
```

3 Proof

I split my proof such that the main theorem that I am calling calls on lemmas in its body. This helps in breaking down the large proof and makes it faster when running my code.

Throughout this proof:
- f is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- K and K' is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- e is an arbitrary $\mathbb{R}^n$ unit vector.
- a_j is a sequence with a limit of 0 and no term of the sequence is 0.

3.1 Part 1

I start by first showing that (A)=(D):

$$\|K[e]-K'[e]\| = \lim_{j\to\infty} \frac{\left\|-f\left(p+\alpha_j e\right)+f(p)+K\left[\alpha_j e\right]+f\left(p+\alpha_j e\right)-f(p)-K'\left[\alpha_j e\right]\right\|}{\left\|\alpha_j e\right\|} \quad (3)$$

```
/-This proves that for a (D) is equal to (A) from the proof description.-/
lemma h_limit_is_diff_linear_maps_norm (a : ℕ → ℝ) (h_non_zero_convergent_zero : non_zero_convergent_zero a) :
∀ (e : fin n → ℝ), ‖e‖ = 1 → (∀ (ε : ℝ), 0 < ε → ∃ (N : ℕ), ∀ z , N ≤ z →
|(‖- f (p + ((a z) • e)) + (f (p)) + K ((a z) • e) + (f (p + ((a z) • e))) - (f p) - K' ((a z) • e)‖ / ‖ (a z) • e ‖)
- ‖K e - K' e‖ | < ε) :=
begin
  cases h_non_zero_convergent_zero with h_nonzero h_converge,
  intros e he,

  intros ε hε,
  specialize h_converge ε,

  specialize h_converge _,
  exact hε,
  cases h_converge with B hx,

  use B,
  intro z,
  intro h_b_geq_z,

  rw norm_smul, --rewrite things similar to ‖a z • e‖ in goal to ‖a z‖ * ‖e‖
  rw he,
  norm_num,

  rw add_assoc,
  rw add_comm,
  rw add_assoc,
  rw add_neg_cancel_left, -- cancels out (f (p + a z • e) + (-f (p + a z • e)))
  rw add_sub_cancel, -- cancel outs f(p) - f(p)

  rw ← smul_sub, --rewrites a z • ⇑K e - a z • ⇑K' e in goal to a z • (⇑K e - ⇑K')
  rw norm_smul, --rewrites ‖a z • (⇑K e - ⇑K')‖ in goal to ‖a z‖ * ‖⇑K e - ⇑K' e‖
  norm_num,
  rw mul_comm,
  simp [h_nonzero], --|a z| / |a z| cancels to 1 because we know from h_nonzero: ∀ j, a j ≠ 0
  exact hε,
end
```

This proof is fairly simple we first show that some terms of the left-hand side of the below equation cancel to:

$$\frac{\left\|-f\left(p+\alpha_j e\right)+f(p)+K\left[\alpha_j e\right]+f\left(p+\alpha_j e\right)-f(p)-K'\left[\alpha_j e\right]\right\|}{\left\|\alpha_j e\right\|}=\frac{\left\|K\left[\alpha_j e\right]-K'\left[\alpha_j e\right]\right\|}{\left\|\alpha_j e\right\|} \tag{4}$$

The RHS was then manipulated as below since K and K' are linear maps hence preserve scalar manipulation.

$$\begin{aligned}
&\frac{\left\|K\left[\alpha_j e\right]-K'\left[\alpha_j e\right]\right\|}{\left\|\alpha_j e\right\|} \\
&\qquad=\frac{\left\|\alpha_j * K[e]-\alpha_j * K'[e]\right\|}{\left\|\alpha_j e\right\|} \\
&\qquad=\frac{\left\|\alpha_j *\left(K[e]-K'[e]\right)\right\|}{\left\|\alpha_j e\right\|} \\
&\qquad=\frac{\left\|\alpha_j\right\| * \left\|\left(K[e]-K'[e]\right)\right\|}{\left\|\alpha_j\right\| * \|e\|} \\
&\qquad=\left\|\left(K[e]-K'[e]\right\|\right.
\end{aligned} \tag{5}$$

3.2 Part 2

This part involves showing that (**) $\leq$ (E) + (F):

$$\begin{aligned}
&\frac{\left\|-f\left(p+\alpha_j e\right)+f(p)+K\left[\alpha_j e\right]+f\left(p+\alpha_j e\right)-f(p)-K'\left[\alpha_j e\right]\right\|}{\left\|\alpha_j e\right\|} \\
&\leq \frac{\left\|f\left(p+\alpha_j e\right)-f(p)-K\left[\alpha_j e\right]\right\|}{\left\|\alpha_j e\right\|}+\frac{\left\|f\left(p+\alpha_j e\right)-f(p)-K'\left[\alpha_j e\right]\right\|}{\left\|\alpha_j e\right\|}
\end{aligned} \tag{6}$$

This is a triangular inequality formulation for norms.

```
/-This proves that (**) ≤ (E) + (F).-/
lemma h_norm_frac_traingle_ineq (a : ℕ → ℝ) (h_non_zero_convergent_zero : non_zero_convergent_zero a)  :∀ (e : fin n → ℝ), ‖e‖ = 1 →
(∀ (z : ℕ), (‖- f (p + ((a z) • e)) + (f (p)) + K ((a z) • e) + (f (p + ((a z) • e))) - (f p) - K' ((a z) • e)‖ / ‖ (a z) • e ‖)
≤ (‖f (p + ((a z) • e)) - f p - K ((a z) • e)‖ / ‖((a z) • e)‖) + (‖f (p + ((a z) • e)) - f p - K' ((a z) • e)‖ / ‖((a z) • e)‖)):=
begin
  cases h_non_zero_convergent_zero with h_nonzero h_converge,
  intros e he,
  intro z,
  have h_factoring_out_minus : ‖ - (f (p + ((a z) • e)) - f p - K ((a z) • e))‖ = ‖- f (p + ((a z) • e)) + f (p) + K ((a z) • e)‖,
    {
      ring_nf, -- simplifies by applying the distributive property, collecting like terms, and simplifying constants
      },

  have h_norm_neg : ‖f (p + ((a z) • e)) - f p - K ((a z) • e)‖ = ‖- f (p + ((a z) • e)) + (f (p)) + K ((a z) • e)‖,
    {
      rw ← h_factoring_out_minus,
      rw norm_neg,
      },

  rw h_norm_neg,
  set r := (-f (p + a z • e) + f p + K (a z • e)), --labelling multiple terms into one making it easier to apply library items

  /-Putting brackets in the right places so I can later set specific collective terms as g.-/
  have h_put_bracket_on : ‖r + f (p + a z • e) - f p - K' (a z • e)‖ = ‖r + (f (p + a z • e) - f p - K' (a z • e))‖,
    {
      rw [add_sub, sub_eq_add_neg],
      rw add_sub,
      refl,
      },

  rw h_put_bracket_on,
  set g := (f (p + a z • e) - f p - K' (a z • e)),

  rw norm_smul, --rewrites ‖a z • e‖ to ‖a z‖ * ‖e‖
  rw he,
  norm_num,

  rw div_add_div_same, --rewrites ‖r‖ / |a z| + ‖g‖ / |a z| to (‖r‖ + ‖g‖) / |a z|
  rw div_le_div_right, --since both sides of the inequality have the same positive denominator we can cancel them
  apply norm_add_le, --this closes one of the goals by applying the traingle inequality for norms
  rw abs_pos, -- 0 < |a z| is equivalent to a z ≠ 0
  apply h_nonzero,
end
```

3.3 Part 3

If we have a derivative K and K' of f:
$\forall \epsilon_1 > 0, \exists R_1 > 0$ s.t. $0 < \|h\| \leq R_1 \rightarrow \frac{\|f(p+h)-f(p)-K[h]\|}{\|h\|} < \epsilon_1)$
$\forall \epsilon_2 > 0, \exists R_2 > 0$ s.t. $0 < \|h\| \leq R_2 \rightarrow \frac{\|f(p+h)-f(p)-K[h]\|}{\|h\|} < \epsilon_2)$

And that a_j is a sequence with a limit of 0 and no term of the sequence is 0:
$(\forall j, aj \neq 0) \wedge (\forall \delta > 0, \exists B \in \mathbb{N}$ such that $\forall j \geq B, |a_j| < \delta)$

The aim of this part is to prove that:
$\forall \epsilon_1 > 0, \forall \epsilon_2 > 0, \exists N_{final}$ s.t. $\forall z \geq N_{final}$

$$\frac{\|f(p+\alpha_j e) - f(p) - K[\alpha_j e]\|}{\|\alpha_j e\|} + \frac{\|f(p+\alpha_j e) - f(p) - K'[\alpha_j e]\|}{\|\alpha_j e\|} \leq \epsilon_1 + \epsilon_2 \tag{7}$$

The way to prove this is to set $\delta = min(R_1, R_2)$ in the definition of a_j converges to 0 and to set $h = \alpha_j e$. Then we can use B from a_j converges to 0 definition as N_{final} and the rest

of the proof follows.
This is proven in the notes:

```
/-Proves part 3 from the document explaining the proof-/
lemma h_using_derivs_sequence (a : ℕ → ℝ) (h_non_zero_convergent_zero : non_zero_convergent_zero a) (h_deriv_K : deriv_exists f p K)
(h_deriv_K_dash : deriv_exists f p K') :
∀ (e : fin n → ℝ), ‖e‖ = 1 → (∀ (ε_one ε_two : ℝ), 0 < ε_one ∧ 0 < ε_two → ∃ (N_final : ℕ), ∀ z , N_final ≤ z →
‖f (p + ((a z) • e)) - f p - K ((a z) • e)‖ / ‖((a z) • e)‖ + ‖f (p + ((a z) • e)) - f p - K' ((a z) • e)‖ / ‖((a z) • e)‖ < ε_one + ε_two):=
```

3.4 Main body of proof

First of all, we bring in all the lemmas from Part 1, Part 2 and Part 3 as have statements into our main theorem proof. We give all these lemmas the inputs they require.
Next, I prove that:

$$\lim_{j\to\infty} \frac{\left\|-f\left(p+\alpha_j e\right)+f(p)+K\left[\alpha_j e\right]+f\left(p+\alpha_j e\right)-f(p)-K'\left[\alpha_j e\right]\right\|}{\left\|\alpha_j e\right\|}=0 \qquad (8)$$

```
have h_upper_bound_lim_zero : ∀ (e : fin n → ℝ), ‖e‖ = 1 → (∀ (ε : ℝ), 0 < ε →
(∃ (H : ℕ), ∀ (z : ℕ), H ≤ z → | ‖-f (p + a z • e) + f p + K (a z • e) + f (p + a z • e) - f p - K' (a z • e)‖ / ‖a z • e‖ | < ε)),
```

We already have from earlier that the LHS of equation (8) is equal to $\|K[e]-K'[e]\|$ from (3). Now we will prove that $\|K[e]-K'[e]\|=0$ as a convergent sequence can't have two distinct limits.

```
/-Suppose a convergent sequence has two limits, this shows that the limits are equal-/
have h_fix_ε_for_contra : ∀ (e : fin n → ℝ), ‖e‖ = 1 → (‖K e - K' e‖ = 0),
```

The proof of this uses contradiction. Suppose M $=0$ and L $=\|K[e]-K'[e]\|$ and set $a_n = \frac{\|-f(p+\alpha_n e)+f(p)+K[\alpha_n e]+f(p+\alpha_n e)-f(p)-K'[\alpha_n e]\|}{\|\alpha_n e\|}$. Then I proof L = M by:

Suppose to the contrary that $L \neq M$. Let $\epsilon = \frac{|L-M|}{10}$.
There is an N_1 such that if $n > N_1$ then $|a_n - L| < \epsilon$.
There is an N_2 such that if $n > N_2$ then $|a_n - M| < \epsilon$.
Let $N = \max(N_1, N_2)$. If $n > N$ then $|a_n - L| < \epsilon$ and $|a_n - M| < \epsilon$
But then by the Triangle inequality $|L-M| \leq |a_n - L| + |M - a_n| < \frac{2}{10}|L-M|$. This is impossible. Hence the assumption $L \neq M$ is false and $L = M$.

Now that I have proven $\|K[e]-K'[e]\|=0$. I need to show that K = K' which will complete the proof that if a derivative exists it is unique.

We aim to show that the output of the linear map of both K and K' are the same for each input vector. I first do a case where the input vector is 0.

```
--Adam Topaz gave me 'linear_map.ext' command. I was unable to find this in documentation.
apply linear_map.ext,
intro v,

by_cases hv: v=0,
{
  rw hv,
  simp [linear_map.map_zero], --this library tells us that all linear maps at 0 equal 0.
},
```

Now we are left with the case where v isn't equal to the zero vector. Then we have that $\hat{v} = \frac{v}{\|v\|}$, where $\hat{v}$ is a unit vector.
In my proof, I then use this to show:

$$K(v) = K(\|v\| * \hat{v}) = \|v\|K(\hat{v}) = \|v\|K'(\hat{v}) = K'(\|v\| * \hat{v}) = K'(v) \tag{9}$$

This completes our proof as K and K' give the same outputs for all input vectors.

4 Credits

Prof. Buzzard introduced me to the 'swap' and 'ext' tactics. Adam Topaz on Zulip introduced me to 'linear_map.ext', I was looking for this but wasn't able to search the right name in the documentation.