Formalising Mathematics (2022-2023)

Coursework 1

(01495449)

Imperial College London

February 3, 2023

1 Introduction

In my project, I aimed to prove that all Cauchy sequences converge and all convergent sequences are Cauchy.

2 Definitions

Below are all the definitions I defined for my project. I used definitions from the Analysis I module in Year 1.

```
/-- a(n) is a convergent sequence iff the below is satisfied -/
def convergent (a : ℕ → ℝ) : Prop :=
∃t : ℝ, ∀ ε > 0, ∃ B : ℕ, ∀ n, B ≤ n → |a n - t| < ε

/-- Definition of a cauchy sequence a(n) -/
def cauchy (a : ℕ → ℝ) : Prop :=
∀ ε > 0, ∃ B : ℕ, ∀ n, ∀ m, B ≤ n ∧ B ≤ m → |a n - a m| < ε

/--Definition for a sequence to be bounded-/
def bounded_seq (a : ℕ → ℝ) : Prop := ∃ M, ∀ m, |a m| ≤ M

/--Definition for a sequence to have a lower bound-/
def lower_bounded_seq (a : ℕ → ℝ) : Prop :=
∃ M, ∀ n, M ≤ a n

/--Definition for a sequence to be monotonically decreasing-/
def mono_dec_seq (a : ℕ → ℝ) : Prop :=
∀ {m n}, m ≤ n → a n ≤ a m

/--Definition for a set to have a lower bound-/
def lower_bbd_set (S : set ℝ) (x : ℝ) : Prop :=
∀ i ∈ S, x ≤ i
```

3 Convergent sequences are Cauchy

I follow the below proof from the Analysis notes in my code:

Proposition 3.17. If $a_n \to a$ then (a_n) is Cauchy.
Proof. $a_n \to a \implies \forall \epsilon > 0 \exists N$ such that $n \geq N \implies |a_n - a| < \frac{\epsilon}{2}$. $(*)$

So $m \geq N \implies |a_m - a| < \frac{\epsilon}{2}$ (†). Combining these, for $m, n \geq N$ we have

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < \underbrace{\epsilon/2}_{(*)} + \underbrace{\epsilon/2}_{(\dagger)} = \epsilon.$$

Below is my proof that is complete and compiles well (have I cropped out the proof of Cauchy sequences are convergent).

```
/--Needed in the proof of cauchy implies convergence.-/
theorem expansion {a b : ℝ} (x : ℝ) : |a - b| = |(a - x) + (x - b)|:=
begin
  norm_num,
end

/--Proof that if a sequences is convergent then it is cauchy and if a
 sequence is cauchy it is convergent -/
theorem convergent_iff_cauchy {a : ℕ → ℝ} : convergent a ↔ cauchy a :=
begin
  split,
  {--Proving convergent sequences are cauchy
  intro ha, -- hypothesis that a is convergent
  rw convergent at ha,
  cases ha with t ht, --deconstructs the hypothesis ha into two hypothesis
  rw cauchy at ⊢,
  intros ε hε, -- assumes everything before the first implies in the goal
  specialize ht (ε/2) (by linarith), --replaces ε with ε/2 in defintion of convergence
  --...to replicate start and dagger in proof.
  simp_rw expansion t, --replaces |a n - a m)| by |a n - t + (t - a m)| in goal
  cases ht with B hB,
  use B,
  intros n m hnm,
  cases hnm,
  have h := hB n hnm_left,  --uses hypothesis n and hn to argue forward in hB
  have h' := hB m hnm_right,
  rw abs_sub_comm at h', --replaces |a m - t| with |t - a m| as |.| is commutative
  --abs_add applies traingle hypothesis on |a n - t + (t - a m)|
  have h'' := abs_add (a n - t) (t - a m),
  --brings in star and dagger like in the Prop 3.17 proof last step
  have h''' := add_lt_add h h',
  linarith,  --proof my contradiction by simple arithmetic on not-goal and h'''
  },
```

This proof required the following manipulation:

$$|an - am| = |an - t + (t - am)| \leq |an - t| + |t - am| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

This was done in the last via the last 3 'have' statements.

```
/--Proof if a sequence is cauchy then it is bounded-/
theorem cauchy_then_bounded {a : ℕ → ℝ} : cauchy a → bounded_seq a :=
begin
  intro ha,
  rw cauchy at ha,
  specialize ha 1, --gives the input that ε is 1
  specialize ha _, --makes goal that ε = 1 is valid
  norm_num,
  cases ha with B hB,
  rw bounded_seq at ⊢,
  specialize hB _,
  use B, -- taking n = B
```

4 Cauchy sequences are convergent

The proof that Cauchy sequences imply convergence needs us to first prove a few lemmas. I was able to show:
1) All Cauchy sequences are bounded
2) Monotonically decreasing sequences that are bounded below are convergent.

However, I ran out of time after this and wasn't able to bring 1) and 2) together into a complete proof of Cauchy sequences are convergent.

4.1 All Cauchy sequences are bounded

The proof for this is fair simple to write down but is fairly lengthy to code hence I will break up and explain my code. Lets first begin with the proof:

Lemma 3.18. (a_n) is Cauchy $\implies (a_n)$ is bounded.
Proof. Pick $\epsilon = 1$, then $\exists B$ such that $\forall n, m \geq B, |a_n - a_m| < 1$.
In particular, taking $n = B$ gives $|a_m| < 1 + |a_B| \, \forall m \geq B$, so

$$|a_m| \leq \max\{|a_1|, |a_2|, \dots |a_{B-1}|, 1 + |a_B|\} \quad \forall m \in \mathbb{N}$$

The first line of the proof is taken care of:
I have also fixed $n = B$. Next, I want $|a_m| < 1 + |a_B| \, \forall m \geq B$, I do this by first showing the following via a few 'have' statements and their proof in my code:

$$|a_m| - |a_B| \leq ||a_B| - |a_m|| \leq |a_B - a_m| < 1, \forall m \geq B$$

and then rearranging $|a_m| < 1 + |a_B|, \forall m \geq B$. The rearranging is done through 'have hB1:', 'have hB3' and 'have hB4'. The proof for these is fairly simple and hence I won't include it

here.
Using Prof Buzzard's proof ('have maximal:...') we showed that the set $S = |a_0|, ..., |a_B|$ has a maximal element. I have commented on the code and a summary of it follows as:
1) We show S is non-empty. Assume $|a_0|$ is in S, this is equivalent to there exists $n \leq B$ in natural numbers such that $|a_0| = |a_n|$. Which is itself equivalent to $0 \leq B$, which is definitely true. Hence by backward equivalence $|a_0|$ is in S and S is non-empty.
2) Then Prof proves that the maximum of S exists in S. This is true from the fact that S is non-empty.
This gives us : $\exists(x \leq B), \forall m \leq B, |a_m| \leq |a_x|$.
so $|a_m| < |a_0|, ..., |a_B|, \forall m \leq B$. Combining this with the earlier $|a_m| < 1 + |a_B| \, \forall m \geq B$.
My code shows that get that

$$|a_m| \leq \max\left\{|a_1|, |a_2|, ... |a_{B-1}|, 1 + |a_B|\right\} \quad \forall m \in \mathbb{N}$$

This proof is complete and works.

4.2 Monotonically decreasing sequences that are bounded below are convergent

I used the following method in my code to prove this:
I started with a set $S = \{a_n | n \in \mathbb{N}\}$. S is non-empty. Since sequence a is bounded below, then the set S is also bounded below.

```
/-Proof that a monotonically decreasing and lower bounded sequence is convergent.-/
theorem mono_dec_bdd_imp_convergent (a : ℕ → ℝ) (ha : mono_dec_seq a)
(hb : lower_bounded_seq a) : convergent a :=
begin
  set S := {x | ∃ i, x = a i}, --defines a set that contains elements of sequence a.

  have hS : S.nonempty, -- i prove S isnt empty as it has a 0 in it
  {
  use a 0,
  simp only [S],
  simp,
  },

  have h : ∃ x, lower_bbd_set S x, --there exists a lower bound of the set S
  {
  rw lower_bounded_seq at hb,
  cases hb with M hM,
  use M,  --lower bound of the sequence is also a lower bound of the set S
  intros i hi,
  have hc : ∃ (n : ℕ), i = a n, --i in S is of the form a n for some n
  {
    simp only [set.mem_set_of_eq] at hi,
    exact hi,
  },
  rcases hc with ⟨n, rfl⟩,
  apply hM,
  },
```

Hence S has an infimum (let inf = infimum (S)). I sorried the proof for this because it's complicated as it requires completeness.

I then fixed ϵ >0. Clearly $inf < inf + \epsilon$.
By definition of infinium, $\exists a_x \in S$ s.t. $a_x < inf + \epsilon$.

```
have h_inf_ε_not_lb : ∃ (x : ℕ), a x < inf + ε,
  {
    obtain ⟨i, hi⟩ := hS, --since S is non empty i is in S
    specialize hinf i,

    have k : inf ≤ i ∧ ∀ (y j : ℝ), j ∈ S → y ≤ j → y ≤ inf,
      {exact hinf hi,},

    by_contra, --going to show ∃ (x : ℕ), a x < inf + ε by contradiction
    simp only [not_exists, not_lt] at h, --rewrites the contradiction hypothesis

    set y := inf + ε, --sets y so we can get a contradiction with inf + ε < inf later
    cases k with hP hQ,

    specialize hQ y,
    specialize hQ i,

    have h_one : y ≤ i → y ≤ inf,
      {apply hQ,
      exact hi,},

    have h_i_ax :∃ (x : ℕ),  i = a x,
      {exact hi,},

    cases h_i_ax with x hj,
    specialize h_1 x,

    have h_two : y ≤ i,
      {rwa hj,},

    have h_three : y ≤ inf,
      {exact h_one h_two,},

    linarith,
  },
```

By contradiction the goal becomes false in the above image, hence I am disproving $\nexists a_x \in S$ s.t. $a_x < inf + \epsilon$, which is the same as $\forall x, inf + \epsilon \geq a_x$. Then $inf + \epsilon$ is a lower bound of the sequence. I have a hypothesis 'hinf' that I have proven earlier that says all lower bounds of the sequence are less than inf. Hence using our contradiction assumption we get $inf + \epsilon < \epsilon$. This is clearly wrong and hence we satisfy the false goal.
From here the proof follows easily. First, we show that $\forall n > x, a_n - inf \leq a_x - inf < \epsilon$.

```
use x,  --sets B as x in definition of convergence
intros n h_x_less_n, --x ≤ n

have h_an_ax_ε : a n - inf ≤ a x - inf ∧ a x - inf < ε,
  {
  split,

  simp,
  rw mono_dec_seq at ha, --pulls the definition of monotonically decreasing seq
  -- a n - inf ≤ a x - inf → a n ≤ a x → n ≤ x (as decreasing)
  apply ha,
  have h_first_goal : a x - inf < ε,
    {--below changes goal a x - inf < ε to a x < ε + inf, which is same as hx.
      apply sub_left_lt_of_lt_add hx,},
  exact h_x_less_n, -- left inequality proved

  exact sub_left_lt_of_lt_add hx, --a x < inf + ε → a x - inf < ε
  },
```

Next, I show that since $S = \{a_n | n \in \mathbb{N}\}$, then by definition of greatest lower bound $\forall n, inf \leq a_n$. This part was a little tricky and I needed to break it down first by showing that $a_n \in S$. I also had earlier proved that $\forall i \in S, inf \leq i$. Hence I substituted a_n for 'i' (done by a command provided by jazzon, labeled in code) and proved the result.

```
have h_final : a n ∈ S,
  {use n,},

have h_inf_an : inf ≤ a n,
  {-- Since a n is in S, we sub a n for i in hinf : ∀ (i : ℝ), i ∈ S → (inf ≤ i ∧ ...)
  --from that we can then retrieve the goal
    exact (hinf (a n) h_final).1,  --jazzon provided this command
    },
```

From here the work was really simple, hence I won't include the code.
We know $\forall n > x, a_n - inf \leq a_x - inf < \epsilon$. Then
$\forall n > x, 0 = inf - inf \leq a_n - inf \leq a_x - inf < \epsilon$.

Now that we know, $\forall n > x, 0 \leq a_n - inf < \epsilon$. This $\implies |a_n - inf| < \epsilon, \forall n > x$. Hence a_n is convergent by definition.

5 Credit

I would like to thank Prof. Kevin Buzzard for providing proof that:

$$\exists (x \leq B), \forall m \leq B, |a_m| \leq |a_x|$$

This is referred to as 'have maximal' in the code for Cauchy sequences are bounded. I gave it my best but couldn't how to define $S = |a_0|, ..., |a_{B-1}|$. Learning some of the commands through Prof. Buzzard's code, I was able to later work with finite sets in the theorem which shows monotonically decreasing and bounded below sequences, converge. Additionally, thank you for the lecture on finiteness. It was extremely helpful.

I would also like to thank 77Tigers and Haruna. They were helpful in telling me what I can try to fix my code. They introduced me to the tactic 'obtain' and were just, in general, helpful in explaining why I was getting certain types of errors in trying some commands.

Yazzon also provided me with the command $exact(hinf(a_n)h_{final}).1,$. I wasn't sure how to code this and am impressed at the elegance of it.