Fresnel's Equations for Reflection and Refraction

- Incident, transmitted, and reflected beams at interfaces
- Reflection and transmission coefficients
  - The Fresnel Equations
  - Brewster's Angle
  - Total internal reflection
- Power reflectance and transmittance
- Phase shifts in reflection
- The mysterious evanescent wave

Reflection and transmission for an arbitrary angle of incidence at one (1) interface

- Only Maxwell+Boundary conditions need. Gives Fresnels equations

Maxwell's eqns.

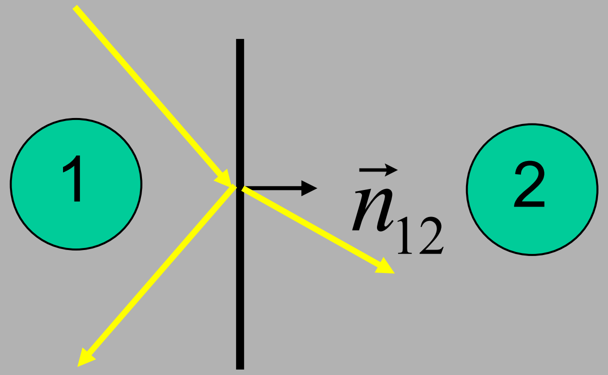
$$\boxed{\begin{aligned}
&\nabla \cdot \vec{D} = \rho_{ext} \\
&\nabla \cdot \vec{B} = 0 \\
&\nabla \times \vec{D} = -\frac{\partial \vec{B}}{dt} \\
&\nabla \times \vec{H} = \frac{\partial \vec{D}}{dt} + \vec{j}
\end{aligned}}$$

Boundary conditions of EM wave

- Tangential components of the:
  - **E** and **H** fields (from Gauss' theorem)
- Normal components of
  - **D** and **B** fields (from Stoke's theorem)

$$\vec{n}_{12} \times (\vec{E}^{(2)} - \vec{E}^{(1)}) = 0$$

$$\vec{n}_{12} \cdot (\vec{B}^{(2)} - \vec{B}^{(1)}) = 0$$



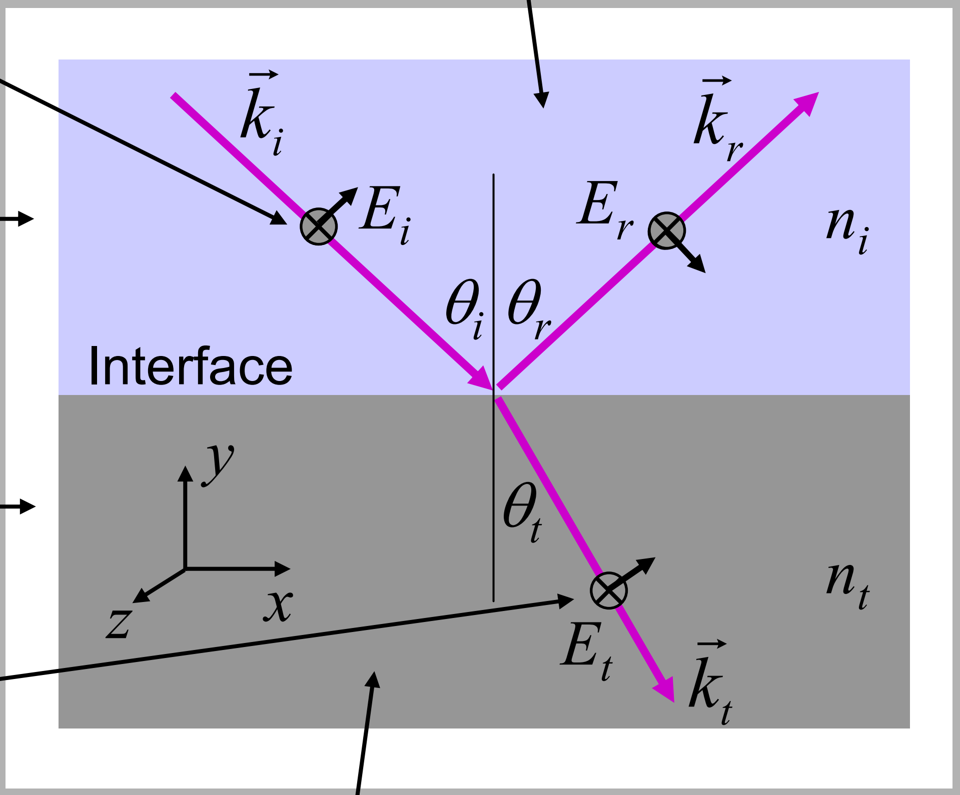
Definitions: Planes of Incidence and the Interface and the polarizations

Perpendicular ("S") polarization sticks out of or into the plane of incidence.

Plane of incidence (here the xy plane) is the plane that contains the incident and reflected k-vectors.

Parallel ("P") polarization lies parallel to the plane of incidence.

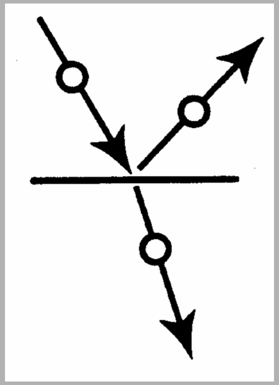
Incident medium



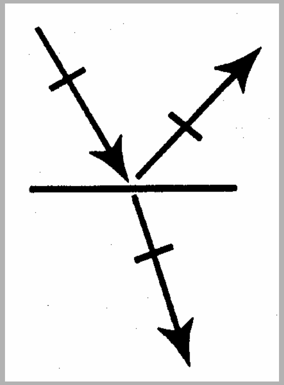
Transmitting medium

Shorthand notation for the polarizations

Perpendicular (S) polarization sticks up out of the plane of incidence.



Parallel (P) polarization lies parallel to the plane of incidence.



Fresnel Equations

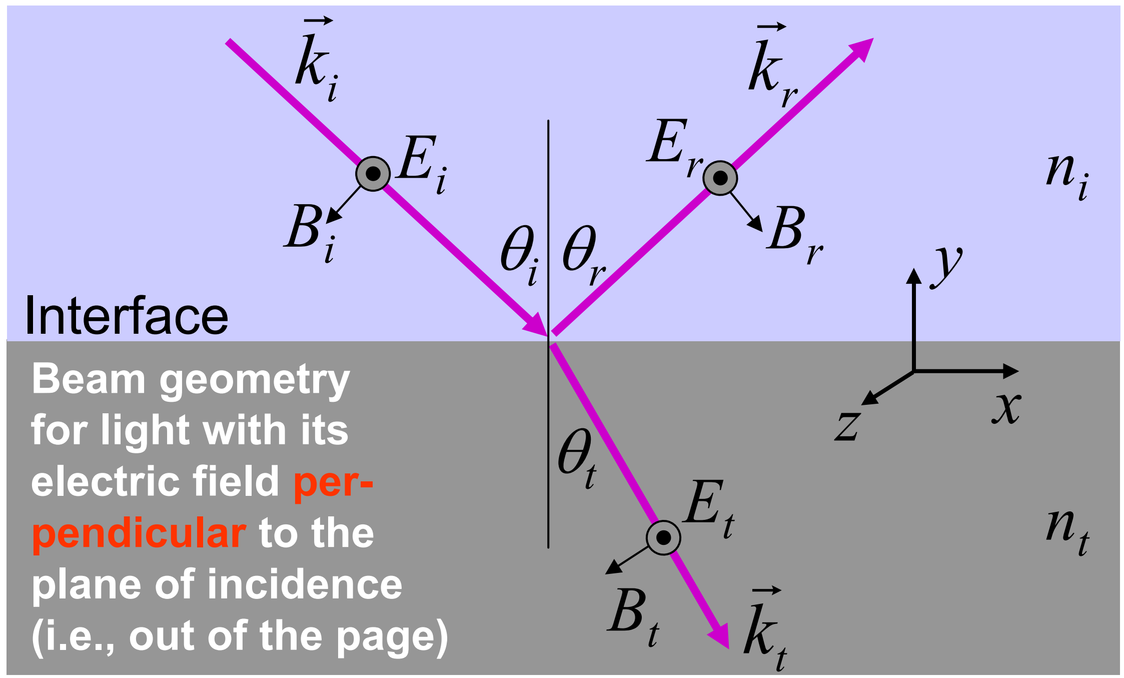
For example, $r_{\perp} = E_{0r} / E_{0i}$

$t_{\perp} = E_{0t} / E_{0i}$

We would like to compute the fraction of a light wave reflected and transmitted by a flat interface between two media with different refractive indices. Fresnel was the first to do this calculation.

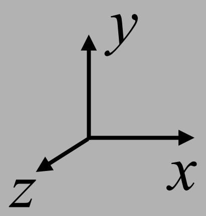
It proceeds by considering the boundary conditions at the interface for the electric and magnetic fields of the light waves.

We'll do the perpendicular polarization first.



Beam geometry for light with its electric field perpendicular to the plane of incidence (i.e., out of the page)

Boundary Condition for the Electric Field at an Interface

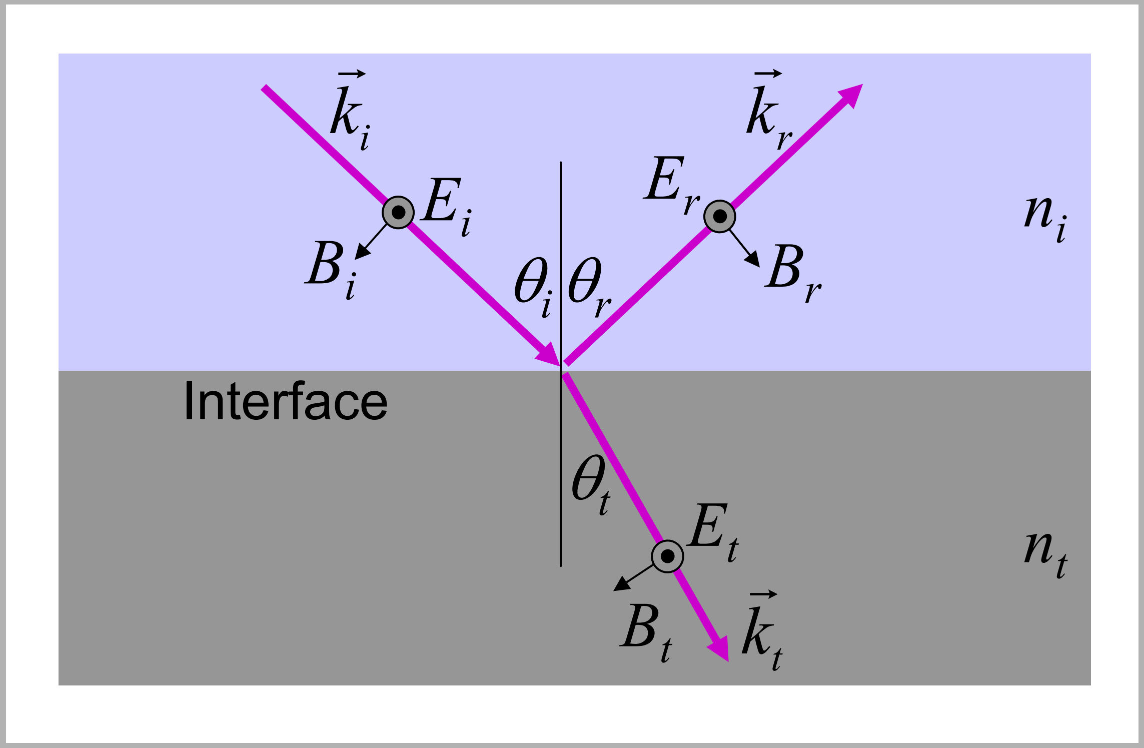


The Tangential Electric Field is Continuous

In other words:

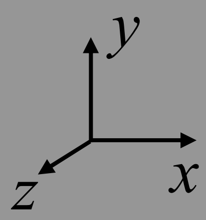
The total E-field in the plane of the interface is continuous.

Here, all E-fields are in the z-direction, which is in the plane of the interface (xz), so:



$$E_i(x, y = 0, z, t) + E_r(x, y = 0, z, t) = E_t(x, y = 0, z, t)$$

Boundary Condition for the Magnetic Field at an Interface

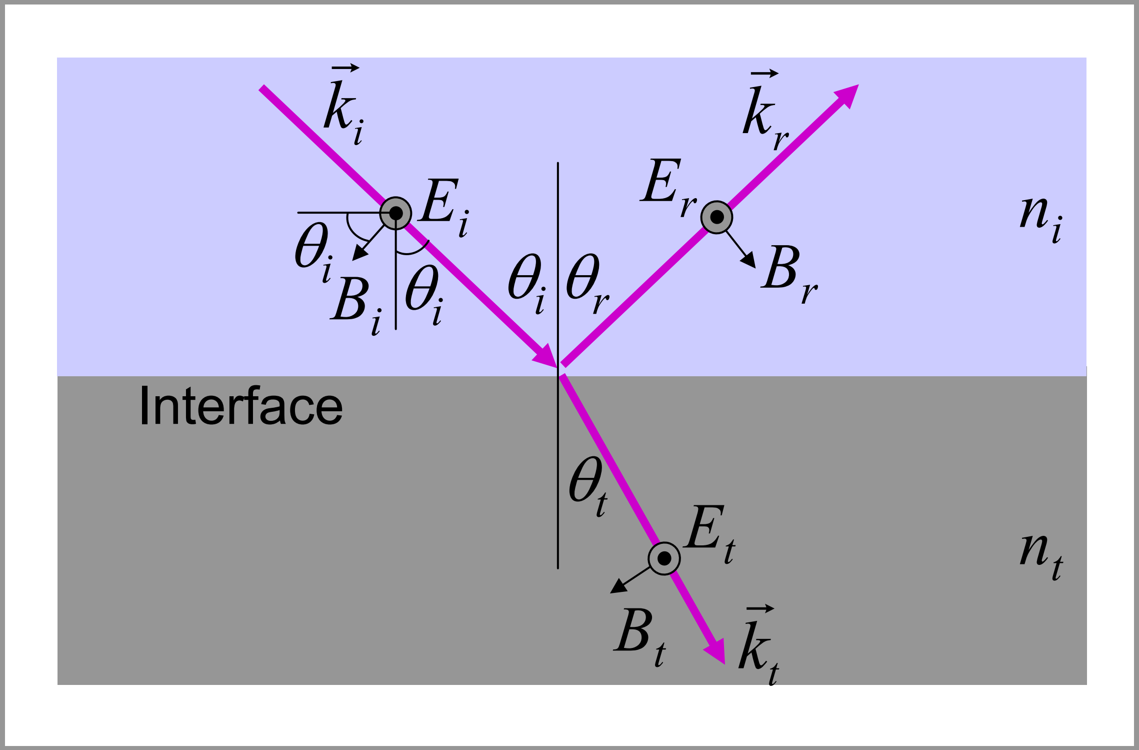


The Tangential Magnetic Field is Continuous*

In other words:

The total B-field in the plane of the interface is continuous.

Here, all B-fields are in the xy-plane, so we take the x-components:



$$-B_i(x, y=0, z, t)\cos(\theta_i) + B_r(x, y=0, z, t)\cos(\theta_r) = -B_t(x, y=0, z, t)\cos(\theta_t)$$

*It's really the tangential B/μ, but we're using $\mu = \mu_0$

Reflection and Transmission for Perpendicularly (S) Polarized Light

Canceling the rapidly varying parts of the light wave and keeping only the complex amplitudes:

$$E_{0i} + E_{0r} = E_{0t}$$

$$-B_{0i}\cos(\theta_i) + B_{0r}\cos(\theta_r) = -B_{0t}\cos(\theta_t)$$

But $B = E/(c_0/n) = nE/c_0$ and $\theta_r = \theta_i$:

$$n_i(E_{0r} - E_{0i})\cos(\theta_i) = -n_t E_{0t}\cos(\theta_t)$$

Substituting for E_{0t} using $E_{0i} + E_{0r} = E_{0t}$:

$$n_i(E_{0r} - E_{0i})\cos(\theta_i) = -n_t(E_{0r} + E_{0i})\cos(\theta_t)$$

Reflection & Transmission Coefficients for Perpendicularly Polarized Light

Rearranging $n_i(E_{0r} - E_{0i})\cos(\theta_i) = -n_t(E_{0r} + E_{0i})\cos(\theta_t)$ yields:

$$E_{0r}\left[n_i\cos(\theta_i) + n_t\cos(\theta_t)\right] = E_{0i}\left[n_i\cos(\theta_i) - n_t\cos(\theta_t)\right]$$

Solving for E_{0r} / E_{0i} yields the reflection coefficient :

$$\boxed{r_\perp = E_{0r} / E_{0i} = \left[n_i\cos(\theta_i) - n_t\cos(\theta_t)\right] / \left[n_i\cos(\theta_i) + n_t\cos(\theta_t)\right]}$$

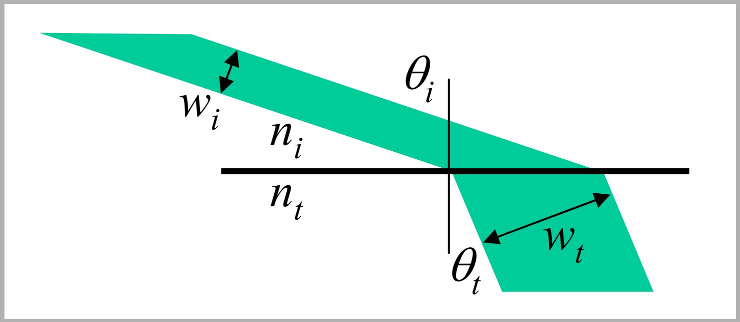
Analogously, the transmission coefficient, E_{0t} / E_{0i}, is

$$\boxed{t_\perp = E_{0t} / E_{0i} = 2n_i\cos(\theta_i) / \left[n_i\cos(\theta_i) + n_t\cos(\theta_t)\right]}$$

These equations are called the Fresnel Equations for perpendicularly polarized light.

Simpler expressions for $r_\perp$ and $t_\perp$

Recall the magnification at an interface, m:



$$m = \frac{w_t}{w_i} = \frac{\cos(\theta_t)}{\cos(\theta_i)}$$

Also let ρ be the ratio of the refractive indices, n_t / n_i.

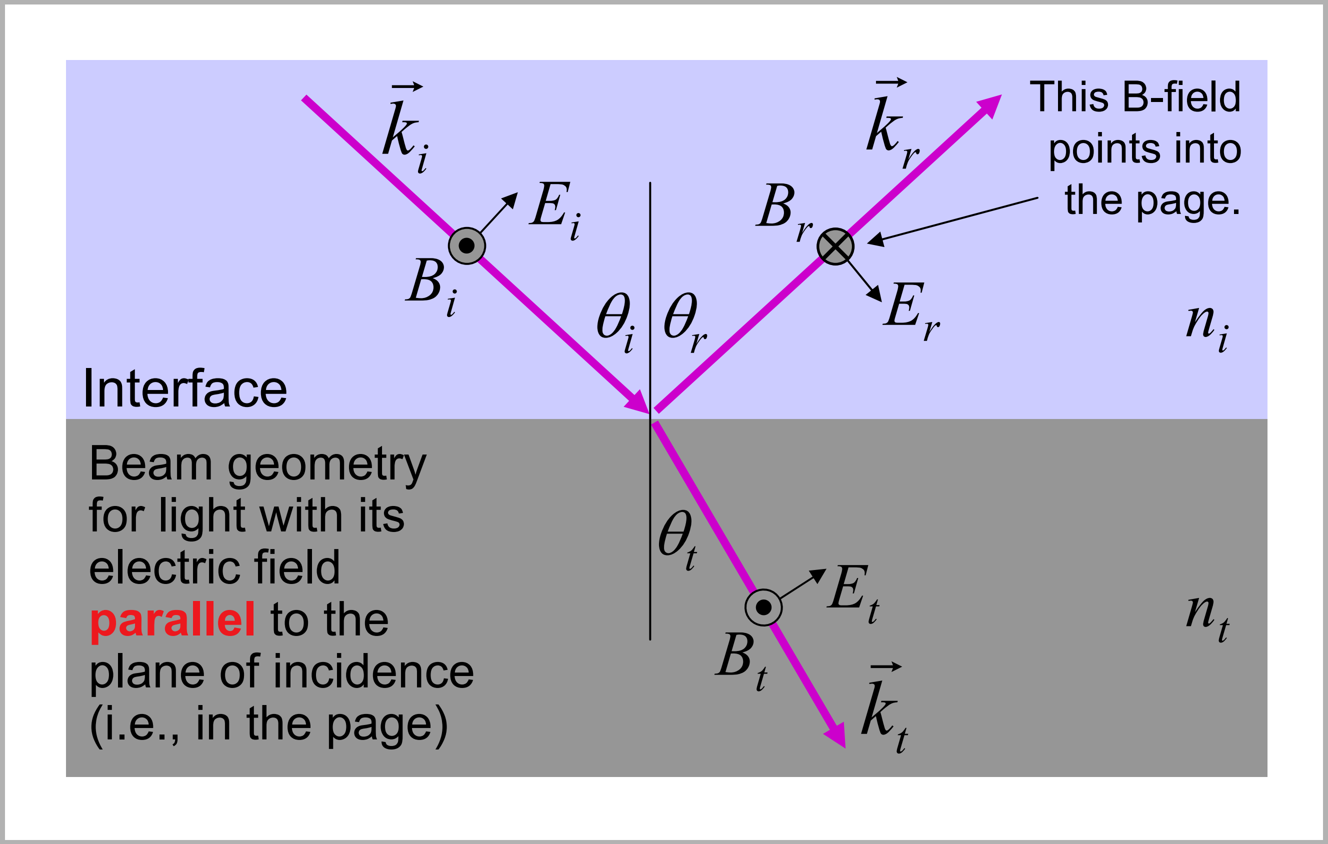
Dividing numerator and denominator of r and t by $n_i \cos(\theta_i)$:

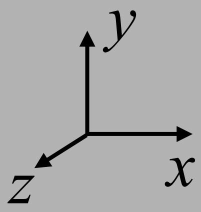
$$r_\perp = \left[n_i \cos(\theta_i) - n_t \cos(\theta_t)\right] / \left[n_i \cos(\theta_i) + n_t \cos(\theta_t)\right] = \left[1 - \rho m\right] / \left[1 + \rho m\right]$$

$$t_\perp = 2 n_i \cos(\theta_i) / \left[n_i \cos(\theta_i) + n_t \cos(\theta_t)\right] = 2 / \left[1 + \rho m\right]$$

$$\boxed{r_\perp = \frac{1 - \rho m}{1 + \rho m}} \qquad \boxed{t_\perp = \frac{2}{1 + \rho m}}$$

Fresnel Equations—Parallel electric field





Note that Hecht uses a different notation for the reflected field, which is confusing! Ours is better!

Note that the reflected magnetic field must point into the screen to achieve $\vec{E} \times \vec{B} \propto \vec{k}$. The x means “into the screen.”

Reflection & Transmission Coefficients for Parallel (P) Polarized Light

For parallel polarized light, $B_{0i} - B_{0r} = B_{0t}$

and $$E_{0i}\cos(\theta_i) + E_{0r}\cos(\theta_r) = E_{0t}\cos(\theta_t)$$

Solving for E_{0r} / E_{0i} yields the reflection coefficient, $r_{||}$:

$$\boxed{r_{||} = E_{0r} / E_{0i} = \left[n_i \cos(\theta_t) - n_t \cos(\theta_i)\right] / \left[n_i \cos(\theta_t) + n_t \cos(\theta_i)\right]}$$

Analogously, the transmission coefficient, $t_{||} = E_{0t} / E_{0i}$, is

$$\boxed{t_{||} = E_{0t} / E_{0i} = 2n_i \cos(\theta_i) / \left[n_i \cos(\theta_t) + n_t \cos(\theta_i)\right]}$$

These equations are called the Fresnel Equations for **parallel** polarized light.

Simpler expressions for $r_\|$ and $t_\|$

$$r_\| = E_{0r} / E_{0i} = \left[n_i \cos(\theta_t) - n_t \cos(\theta_i)\right] / \left[n_i \cos(\theta_t) + n_t \cos(\theta_i)\right]$$

$$t_\| = E_{0t} / E_{0i} = 2n_i \cos(\theta_i) / \left[n_i \cos(\theta_t) + n_t \cos(\theta_i)\right]$$

Again, use the magnification, m, and the refractive-index ratio, ρ .

And again dividing numerator and denominator of r and t by $n_i \cos(\theta_i)$:

$$r_\| = \left[m - \rho\right] / \left[m + \rho\right]$$

$$t_\| = 2 / \left[m + \rho\right]$$

$$\boxed{r_\| = \frac{m - \rho}{m + \rho}} \qquad \boxed{t_\| = \frac{2}{m + \rho}}$$

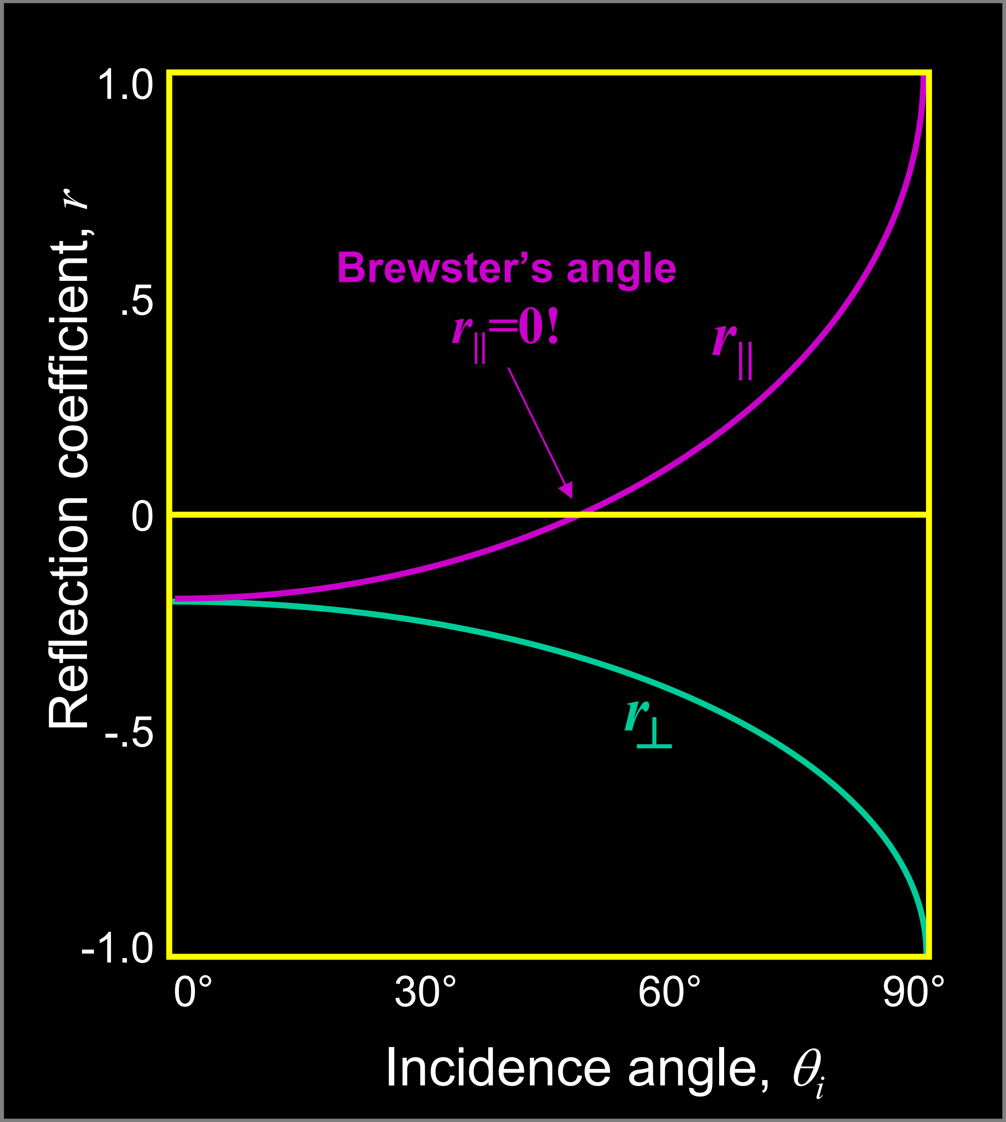
Reflection Coefficients for an Air-to-Glass Interface

$$n_{air} \approx 1 \;<\; n_{glass} \approx 1.5$$

Note that:

Total reflection at θ = 90° for both polarizations

Zero reflection for parallel polarization at **Brewster's angle** (56.3° for these values of n_i and n_t).

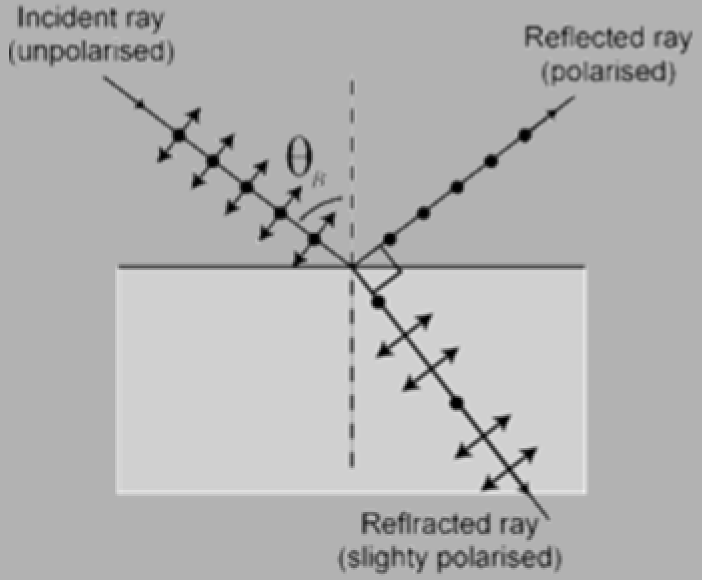


Condition for $r_{//}$ (r_p)=0

$\theta_1 + \theta_2 = 90°$

$$\boxed{\theta_B = \tan^{-1}\left(\frac{n_t}{n_i}\right)}$$

Brewster's angle



[pic from Wikipedia]

Brewster application: Sunglasses

- Polarized sunglasses use the principle of Brewster's angle to eliminate glare from from the sun. <u>In a large range of angles around Brewster's angle the reflection of p-polarized light is lower than s-polarized light.</u>

- Thus, if the sun is low in the sky mostly s-polarized light will reflect from water. Sunglasses made up of polarizers (e.g. polaroid film) aligned to block this light consequently block reflections from the water. To accomplish this, sunglass makers assume people will be upright while viewing the water and thus <u>align the polarizers to block the polarization which oscillates along the line connecting the sunglass ear-pieces (i.e. Horizontal).</u>

- Photographers use the same principle to remove reflections from water so that they might photograph objects beneath the surface. In this case, the polarizer filter camera attachment can be rotated to be at the correct angle (see figure).

Brewster angle

Photographs taken of mudflats with a camera polarizer filter rotated to two different angles. In the first picture, the polarizer is rotated to maximize reflections, and in the second, it is rotated 90° to minimize reflections - almost all reflected sunlight is eliminated

[pic from Wikipedia]

Reflection Coefficients for a Glass-to-Air Interface

$n_{glass} \approx 1.5 \; > \; n_{air} \approx 1$

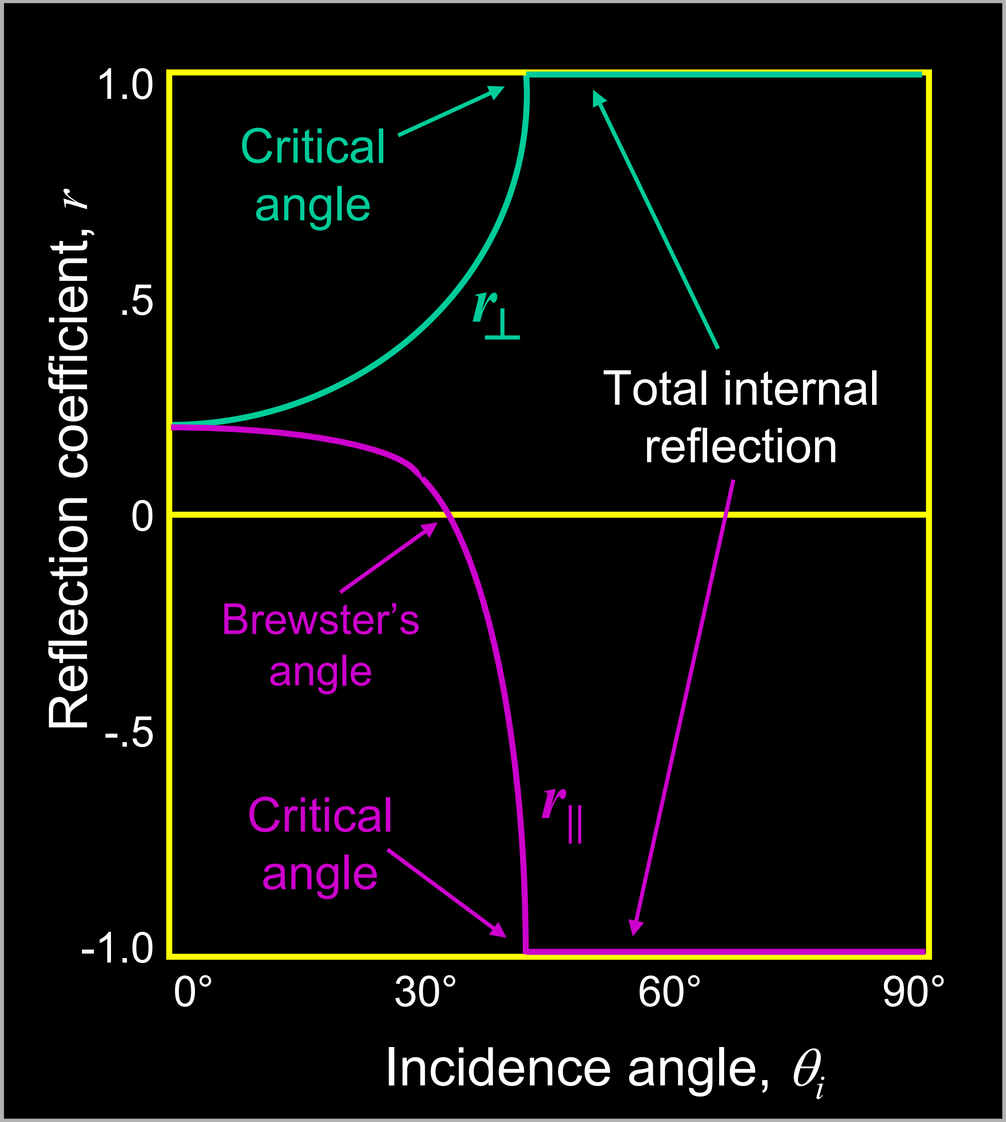
Note that:

Total internal reflection above the **critical angle**

$$\boxed{\theta_{crit} \equiv \sin^{-1}(n_t / n_i)}$$

(The sine in Snell's Law can't be > 1!):

$$\sin(\theta_{crit}) = n_t / n_i \sin(90°)$$



Relation between θ_B and θ_C

- Note $\tan\theta_B = \sin\theta_C$, $\tan\theta > \sin\theta \Rightarrow \theta_C > \theta_B$

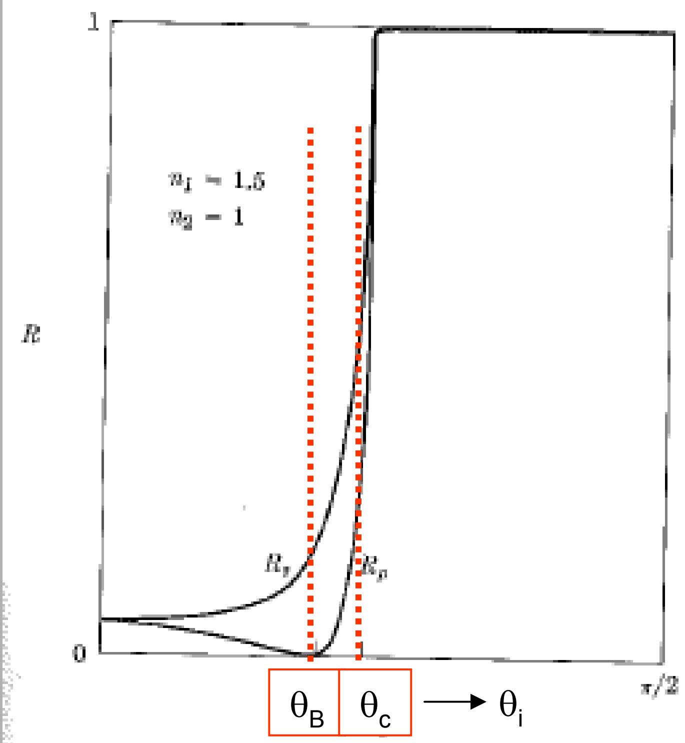


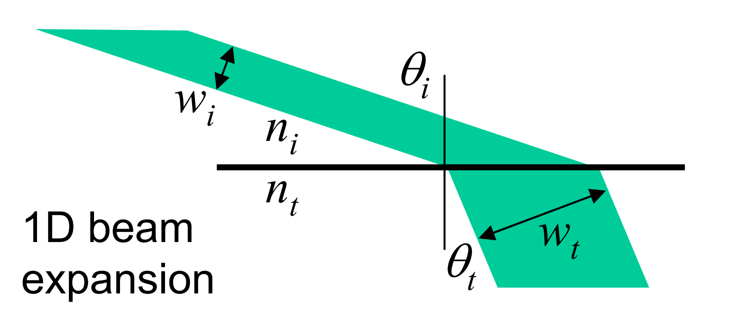
Figure 18–4 Reflectance for *s*- and *p*-polarization at a glass–air interface. Brewster's angle $\theta_B = 34°$ and the critical angle is $\theta_c = 42°$.

Transmittance (T)

$$I = \left(n \frac{\varepsilon_0 c_0}{2} \right) \left| E_0 \right|^2$$

$T \equiv$ Transmitted Power / Incident Power $= \dfrac{I_t A_t}{I_i A_i}$ A = Area

Compute the ratio of the beam areas:



$$\frac{A_t}{A_i} = \frac{w_t}{w_i} = \frac{\cos(\theta_t)}{\cos(\theta_i)} = m$$

The beam expands in one dimension on refraction.

$$\frac{\left| E_{0t} \right|^2}{\left| E_{0i} \right|^2} = t^2$$

$$T = \frac{I_t A_t}{I_i A_i} = \frac{\left(n_t \dfrac{\varepsilon_0 c_0}{2} \right) \left| E_{0t} \right|^2}{\left(n_i \dfrac{\varepsilon_0 c_0}{2} \right) \left| E_{0i} \right|^2} \left[\frac{w_t}{w_i} \right] = \frac{n_t \left| E_{0t} \right|^2 w_t}{n_i \left| E_{0i} \right|^2 w_i} = \frac{n_t}{n_i} t^2 \frac{\cos(\theta_t)}{\cos(\theta_i)}$$

$$\Rightarrow \boxed{T = \left[\frac{\left(n_t \cos(\theta_t) \right)}{\left(n_i \cos(\theta_i) \right)} \right] t^2 = \rho\, m\, t^2}$$

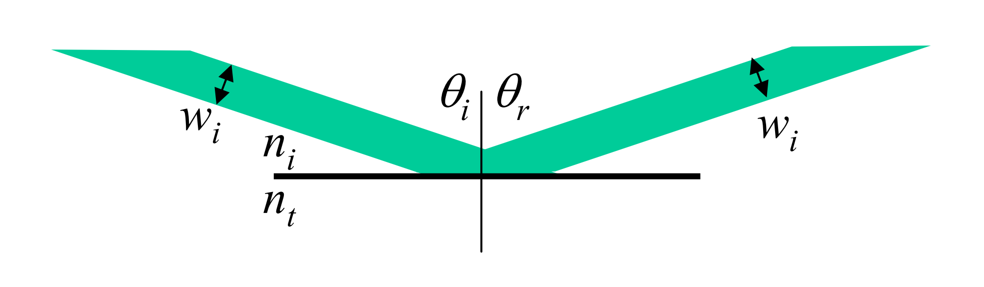
The Transmittance is also called the Transmissivity.

Reflectance (R)

$$I = \left(n \frac{\varepsilon_0 c_0}{2} \right) \left| E_0 \right|^2$$

$$R \equiv \text{Reflected Power / Incident Power} = \frac{I_r A_r}{I_i A_i}$$

A = Area



Because the angle of incidence = the angle of reflection, the beam area doesn't change on reflection.

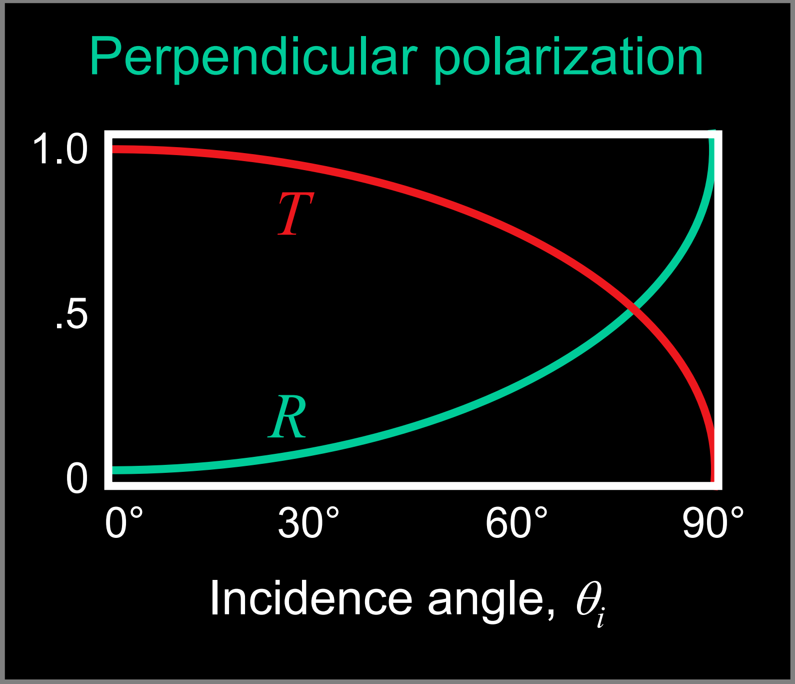
Also, n is the same for both incident and reflected beams.

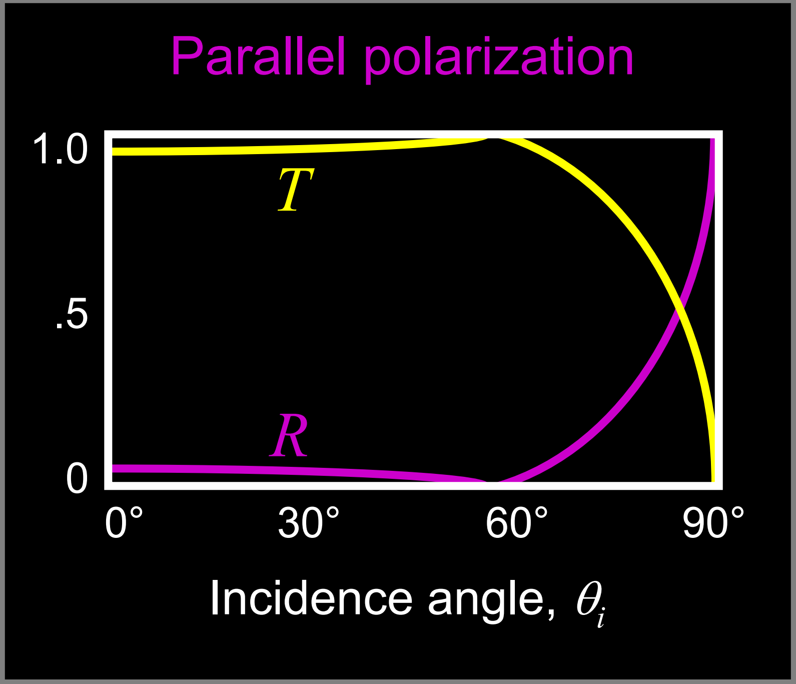
So:

$$\boxed{R = r^2}$$

The Reflectance is also called the Reflectivity.

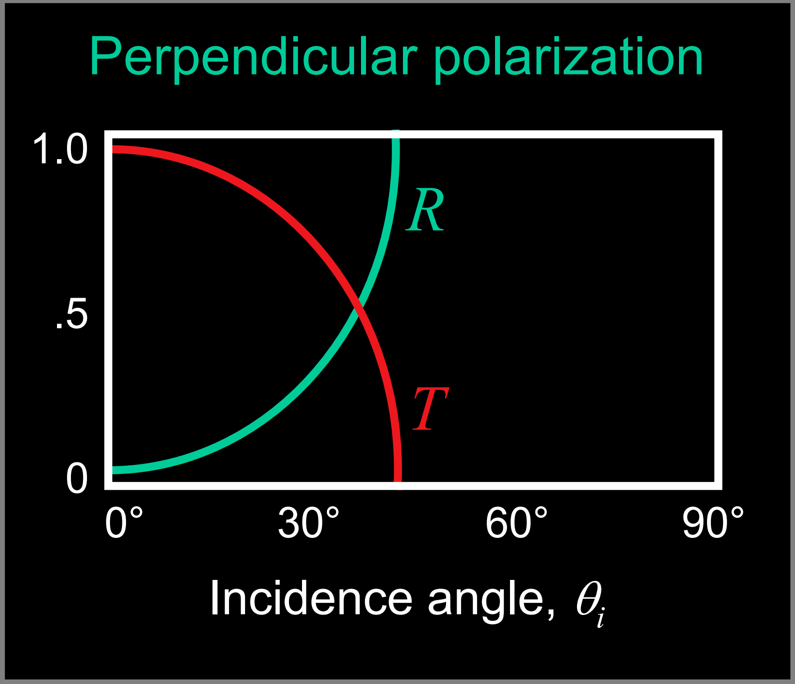
Reflectance and Transmittance for an Air-to-Glass Interface

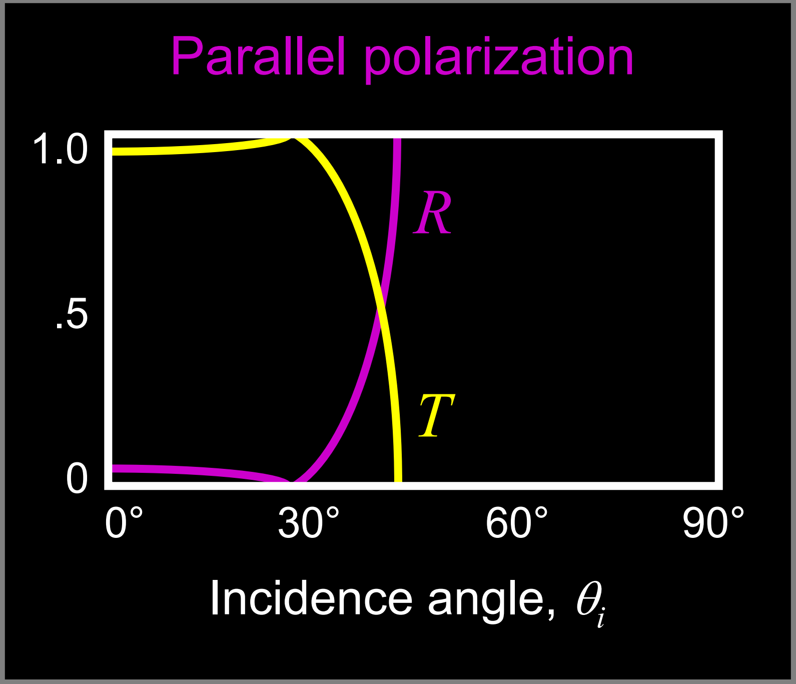




Note that $R + T = 1$

Reflectance and Transmittance for a Glass-to-Air Interface





Note that $R + T = 1$

Reflection at normal incidence

When $\theta_i = 0$,

$$R = \left(\frac{n_t - n_i}{n_t + n_i} \right)^2$$

and

$$T = \frac{4 n_t n_i}{\left(n_t + n_i \right)^2}$$

For an air-glass interface ($n_i = 1$ and $n_t = 1.5$),

R = 4% and T = 96%

The values are the same, whichever direction the light travels, from air to glass or from glass to air.

The 4% has big implications for photography lenses.

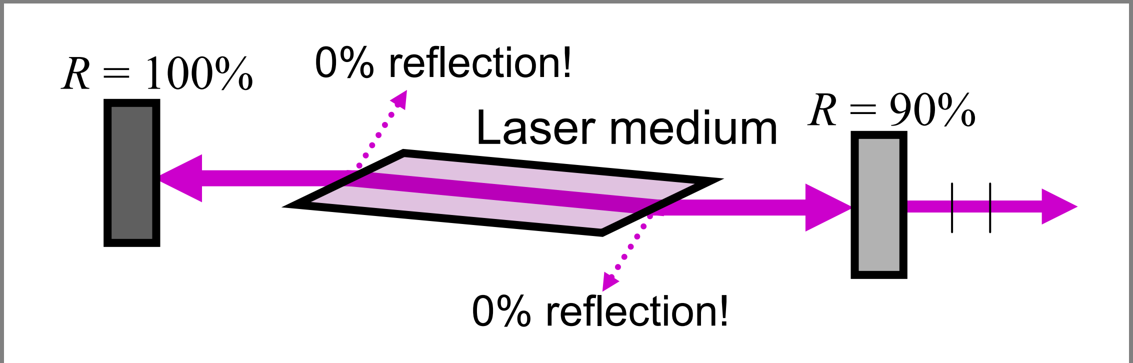
Practical Applications of Fresnel's Equations

Windows look like mirrors at night (when you're in the brightly lit room)

One-way mirrors (used by police to interrogate bad guys) are just partial reflectors (actually, aluminum-coated).

Disneyland puts ghouls next to you in the haunted house using partial reflectors (also aluminum-coated).

Lasers use Brewster's angle components to avoid reflective losses:



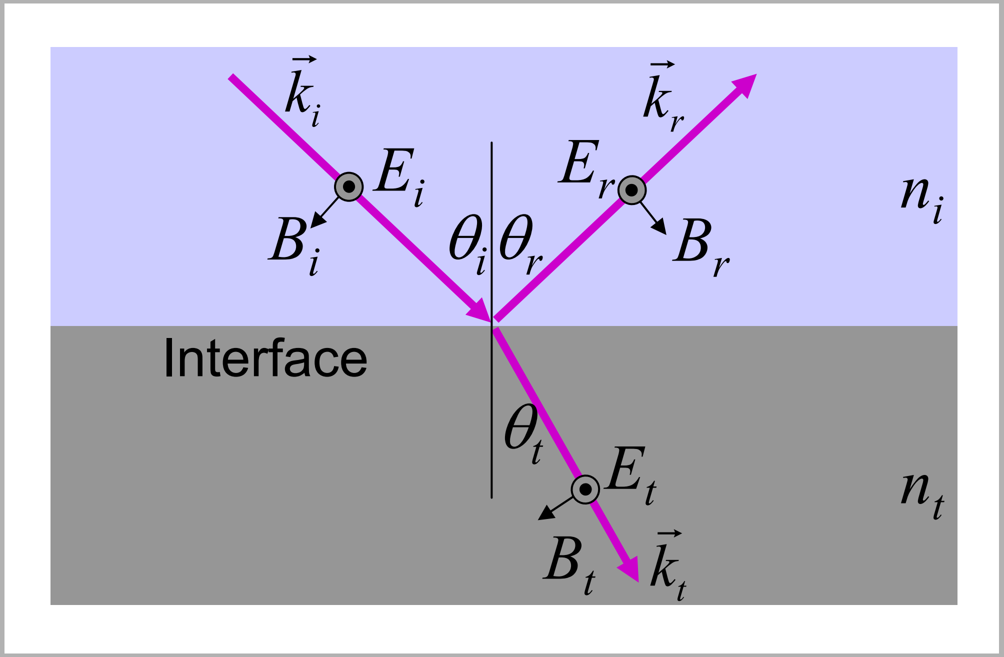
Optical fibers use total internal reflection. Hollow fibers use high-incidence-angle near-unity reflections.

Phase Shift in Reflection (for Perpendicularly Polarized Light)

$$r_\perp = E_{0r} / E_{0i} = \frac{\left[n_i \cos(\theta_i) - n_t \cos(\theta_t)\right]}{\left[n_i \cos(\theta_i) + n_t \cos(\theta_t)\right]}$$

When $\theta_i = 0, \;\; r = \dfrac{\left[n_i - n_t\right]}{\left[n_i + n_t\right]}$

If $n_i < n_t$ (air to glass), $r < 0$



So there will be destructive interference between the incident and reflected beams just before the surface.

Analogously, if $n_i > n_t$ (glass to air), $r_\perp > 0$, and there will be constructive interference.

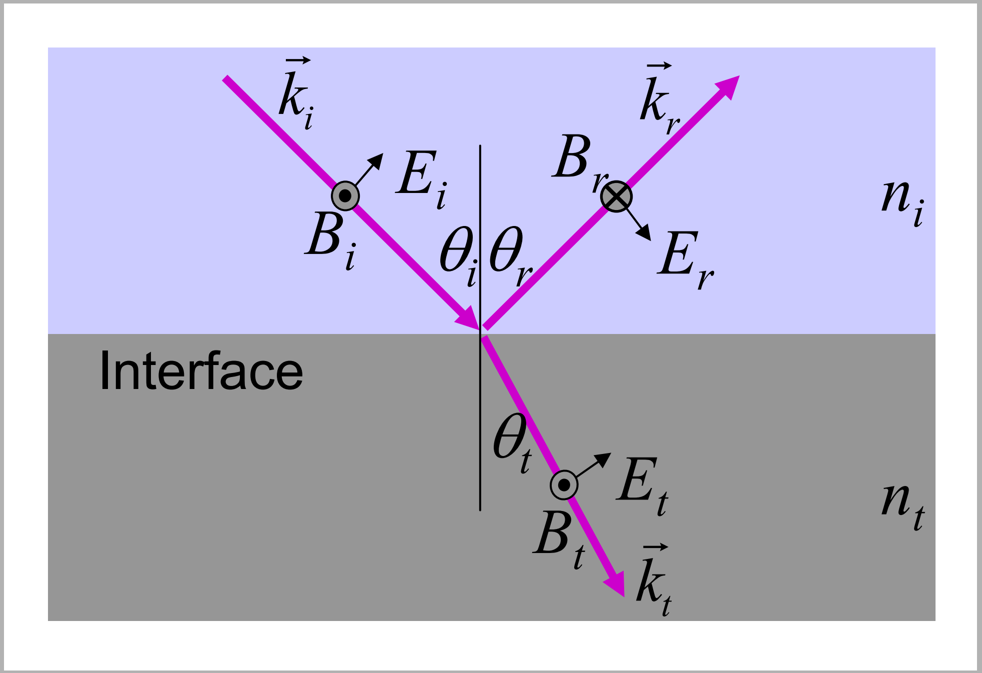
Phase Shift in Reflection (Parallel Polarized Light)

$$r_{\|} = E_{0r} / E_{0i} =$$

$$\frac{\left[n_i \cos(\theta_t) - n_t \cos(\theta_i)\right]}{\left[n_i \cos(\theta_t) + n_t \cos(\theta_i)\right]}$$

When $\theta_i = 0,\ r_{\|} = \dfrac{\left[n_i - n_t\right]}{\left[n_i + n_t\right]}$

If $n_i < n_t$ (air to glass), $r_{\|} < 0$



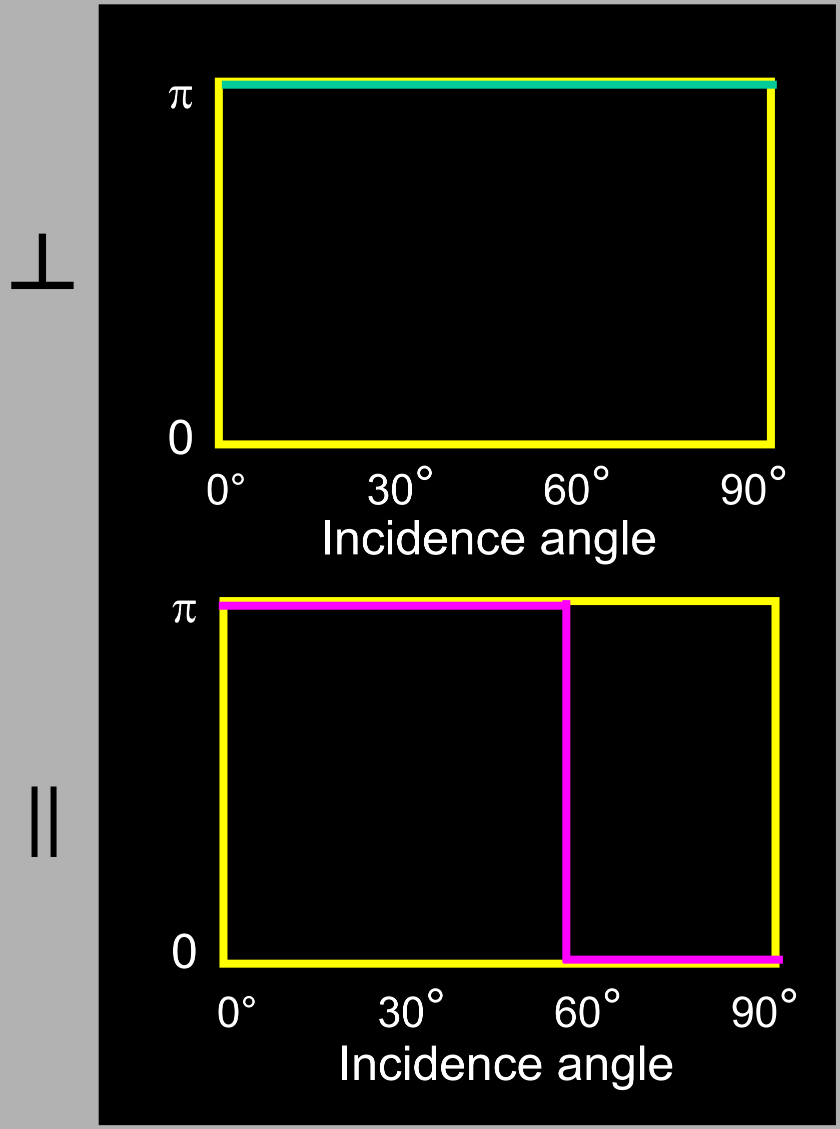
This also means destructive interference with incident beam.

Analogously, if $n_i > n_t$ (glass to air), $r_{\|} > 0$, and we have constructive interference just above the interface.

Good that we get the same result as for $r_{\perp}$; it's the same problem when $\theta_i = 0$! Also, the phase is opposite above Brewster's angle.

Phase shifts in reflection (air to glass)

$n_i < n_t$



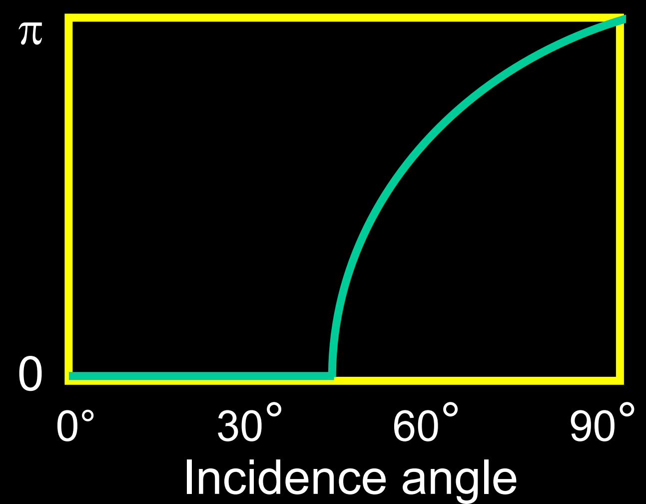
180° phase shift for all angles

180° phase shift for angles below Brewster's angle; 0° for larger angles

Phase shifts in reflection (glass to air)

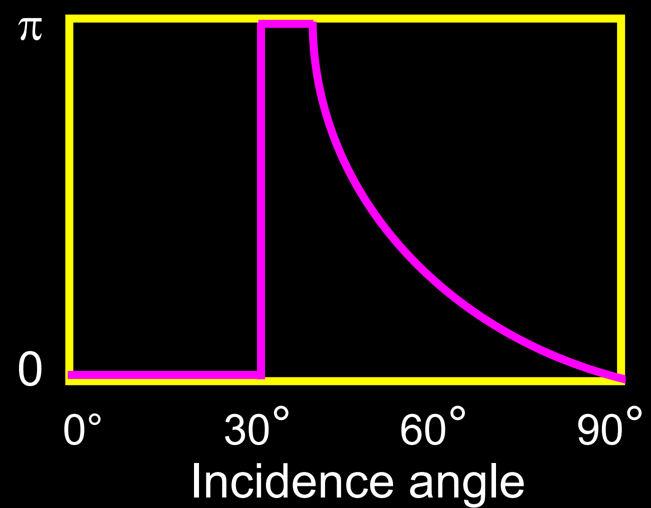
$n_t < n_i$

$\perp$



Interesting phase above the critical angle

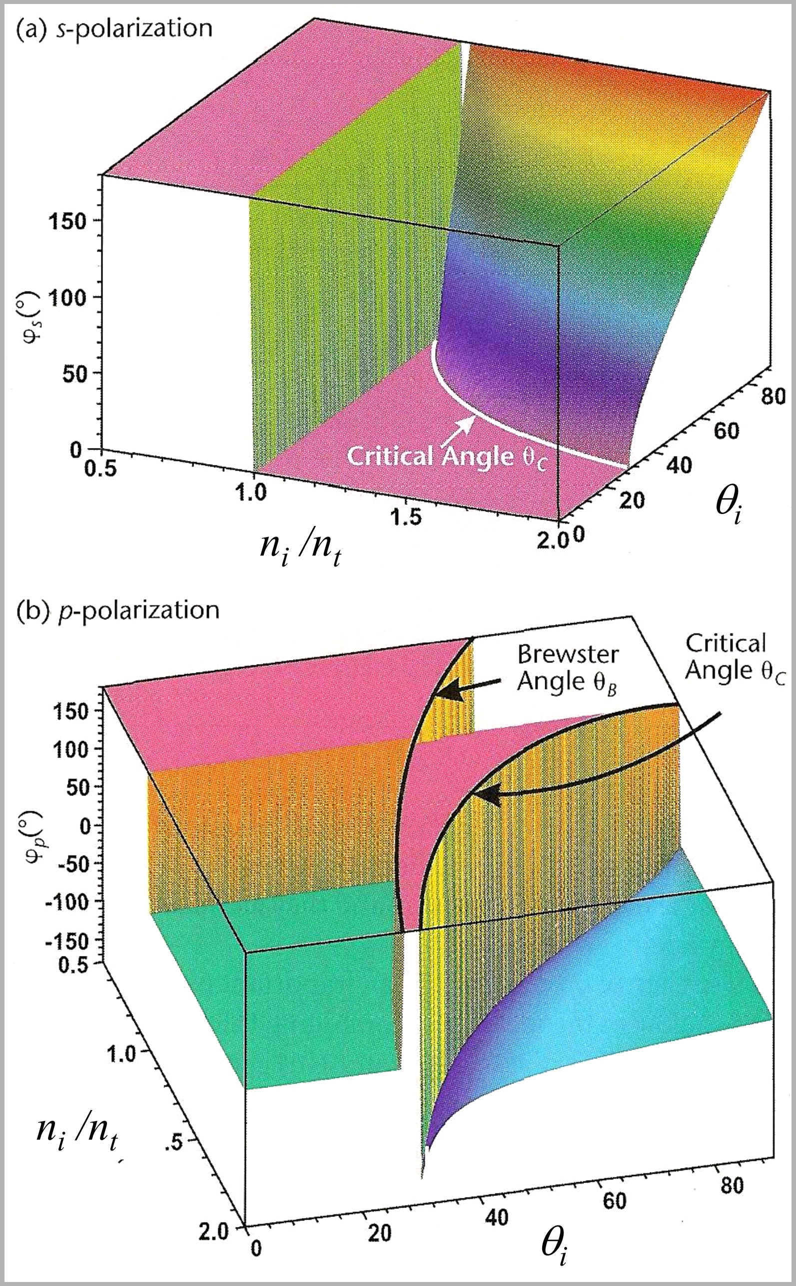
$||$



180° phase shift for angles below Brewster's angle; 0° for larger angles

Phase shifts vs. incidence angle and n_i/n_t

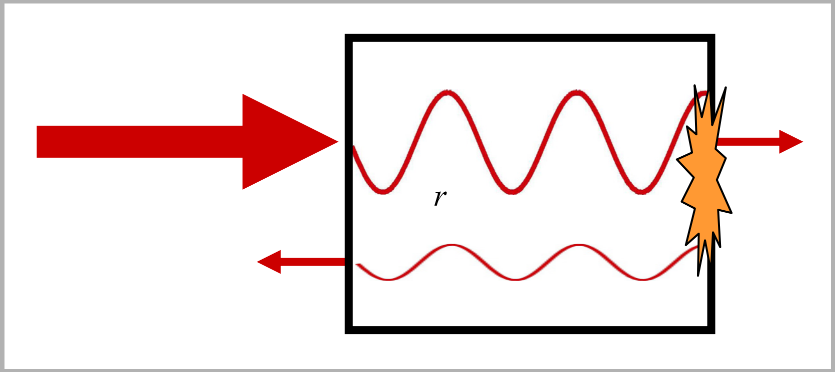
Note the general behavior above and below the various interesting angles…



Li Li, OPN, vol. 14, #9, pp. 24-30, Sept. 2003

If you slowly turn up a laser intensity incident on a piece of glass, where does damage happen first, the front or the back?

The obvious answer is the front of the object, which sees the higher intensity first.



But constructive interference happens at the back surface between the incident light and the reflected wave.

This yields an irradiance that is 44% higher just inside the back surface!

$(1+0.2)^2 = 1.44$ (glass to air), $r_{||} > 0 \Rightarrow r_{||} = [(1.5\text{-}1)/(1.5+1)]$

Phase shifts with coated optics

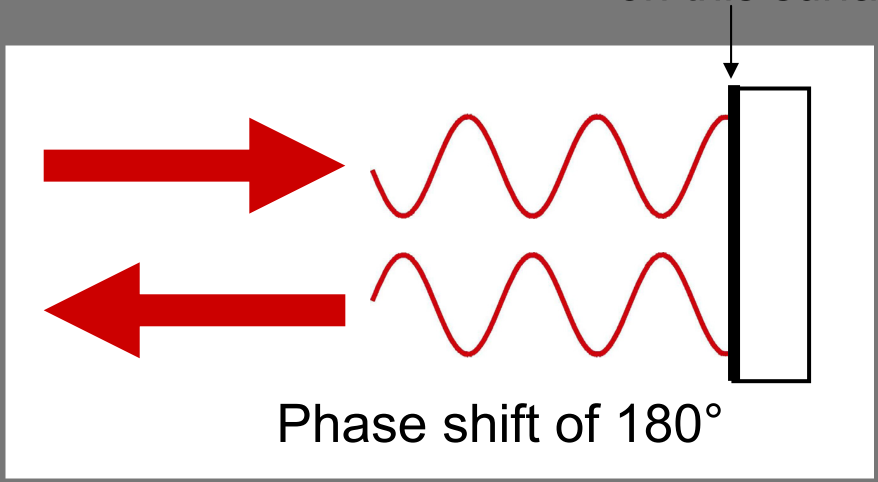
Reflections with different magnitudes can be generated using partial metallization or coatings. We'll see these later.

But the phase shifts on reflection are the same! For near-normal incidence:

180° if low-index-to-high and 0 if high-index-to-low.

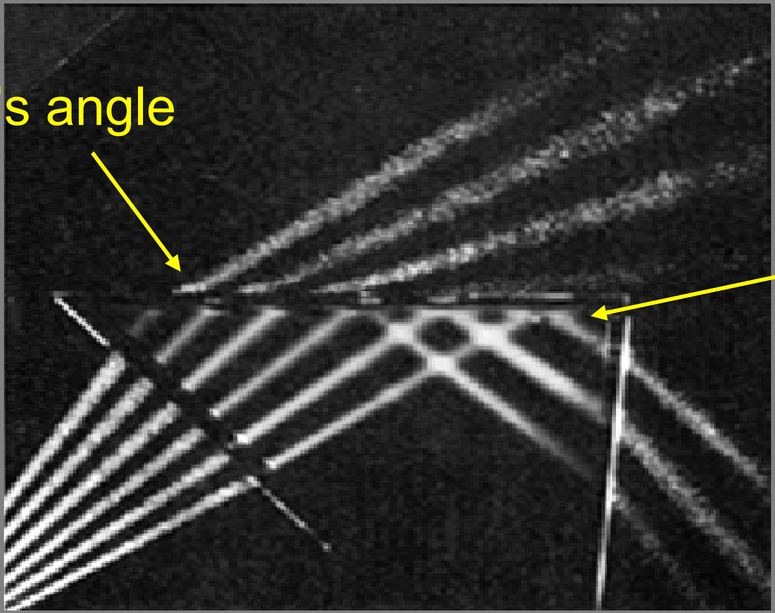
Example:

Laser Mirror



Total Internal Reflection occurs when $\sin(\theta_t) > 1$, and no transmitted beam can occur.

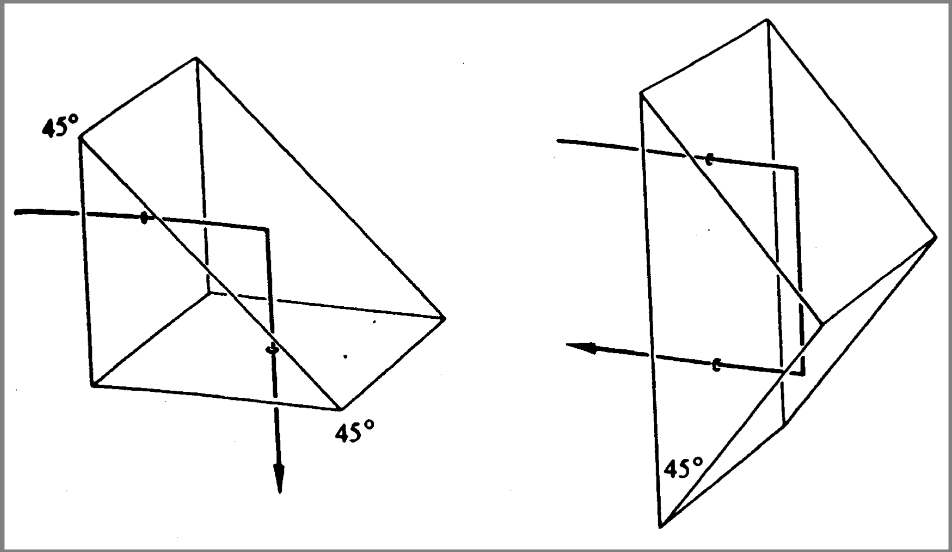
Note that the irradiance of the transmitted beam goes to zero (i.e., TIR occurs) as it grazes the surface.



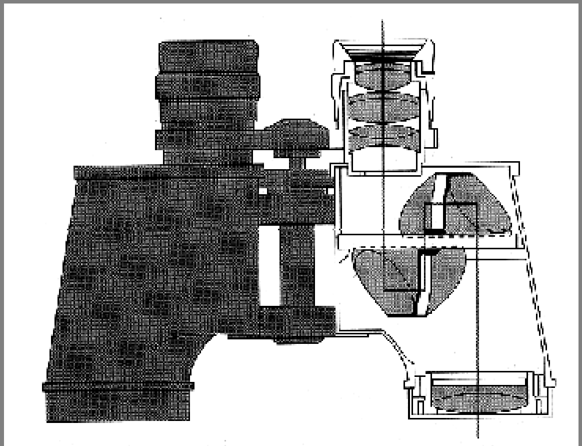
Total internal reflection is 100% efficient, that is, all the light is reflected.

Applications of Total Internal Reflection

Beam steerers

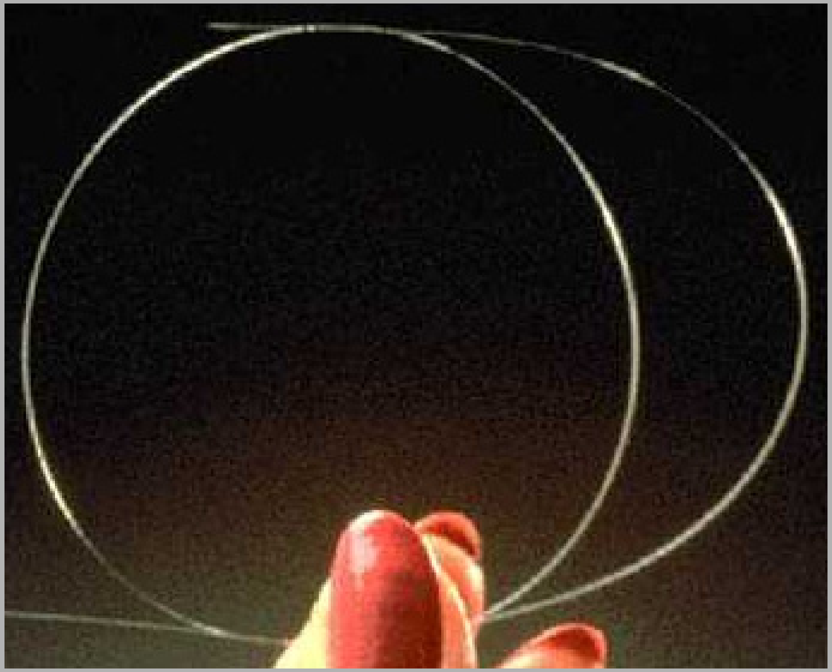


Beam steerers used to compress the path inside binoculars



Fiber Optics

Optical fibers use TIR to transmit light long distances.



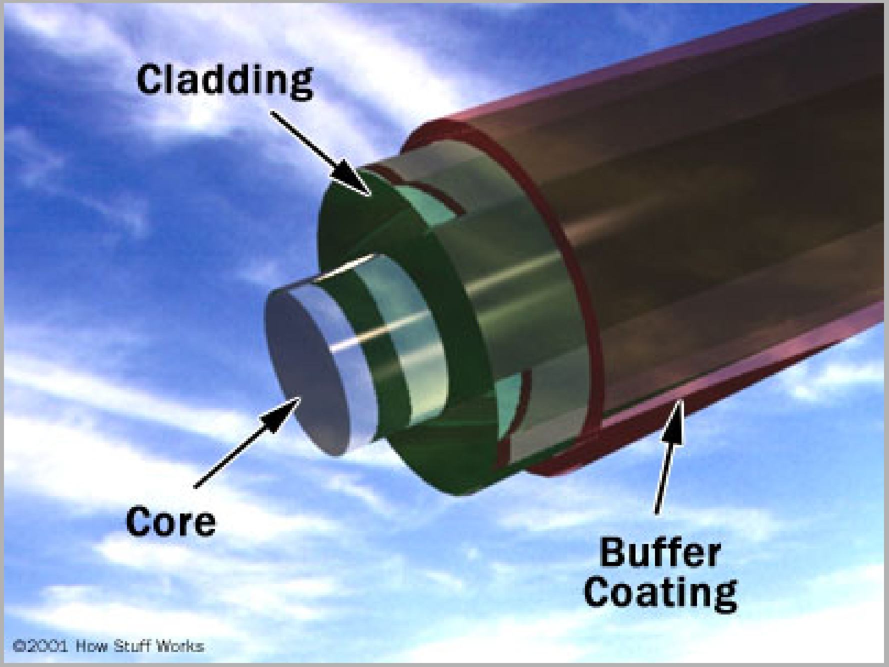
They play an ever-increasing role in our lives!

Design of optical fibers

Core: Thin glass center of the fiber that carries the light

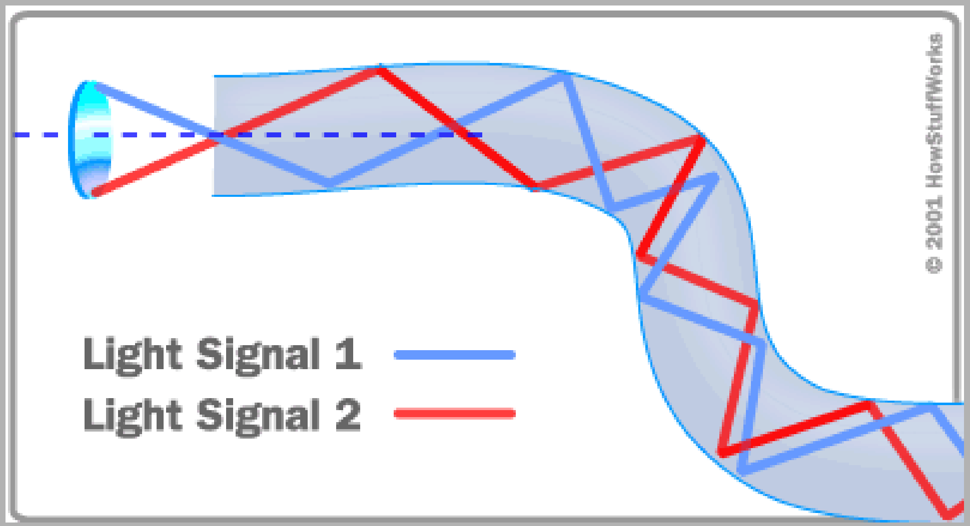
Cladding: Surrounds the core and reflects the light back into the core

Buffer coating: Plastic protective coating



$$n_{core} > n_{cladding}$$

Propagation of light in an optical fiber



Light travels through the core bouncing from the reflective walls. The walls absorb very little light from the core allowing the light wave to travel large distances.

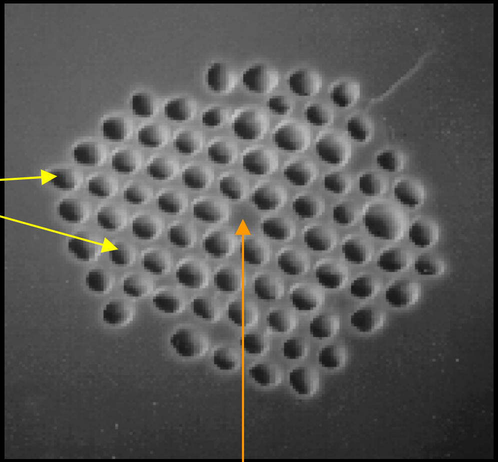
Some signal degradation occurs due to imperfectly constructed glass used in the cable. The best optical fibers show very little light loss -- less than 10%/km at 1,550 nm.

Maximum light loss occurs at the points of maximum curvature.

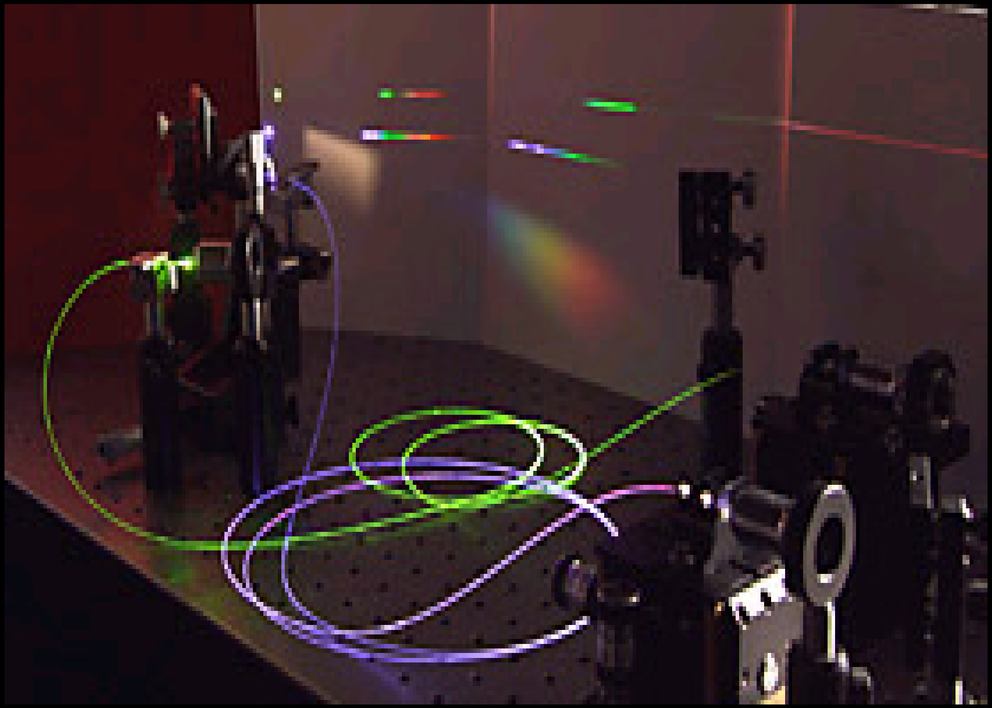
Microstructure fiber

In microstructure fiber, air holes act as the cladding surrounding a glass core. Such fibers have different dispersion properties.

Air holes



Core



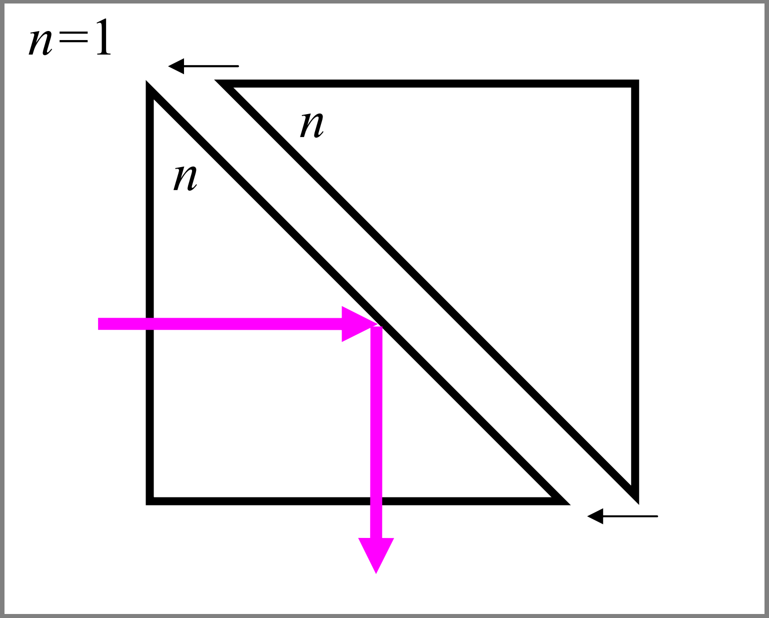
Such fiber has many applications, from medical imaging to optical clocks.

Photographs courtesy of Jinendra Ranka, Lucent

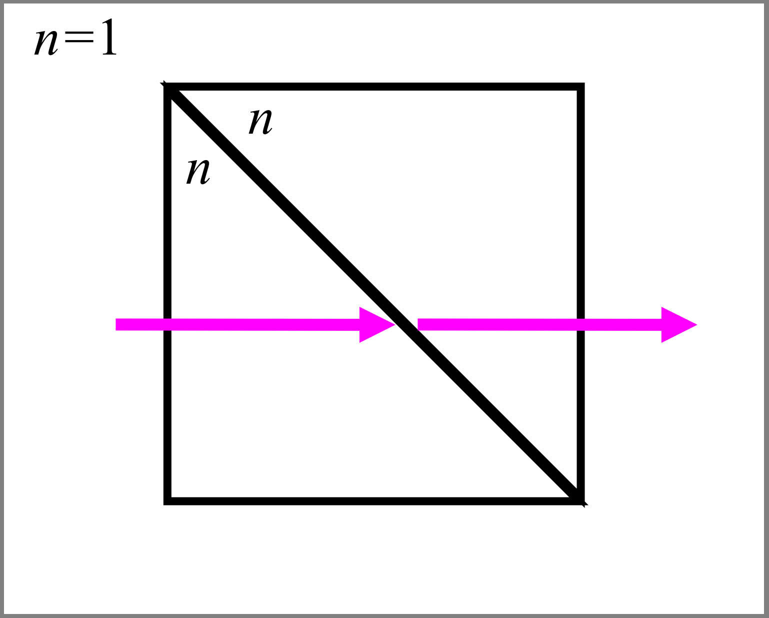
Frustrated Total Internal Reflection

By placing another surface in contact with a totally internally reflecting one, total internal reflection can be **frustrated**.

Total internal reflection



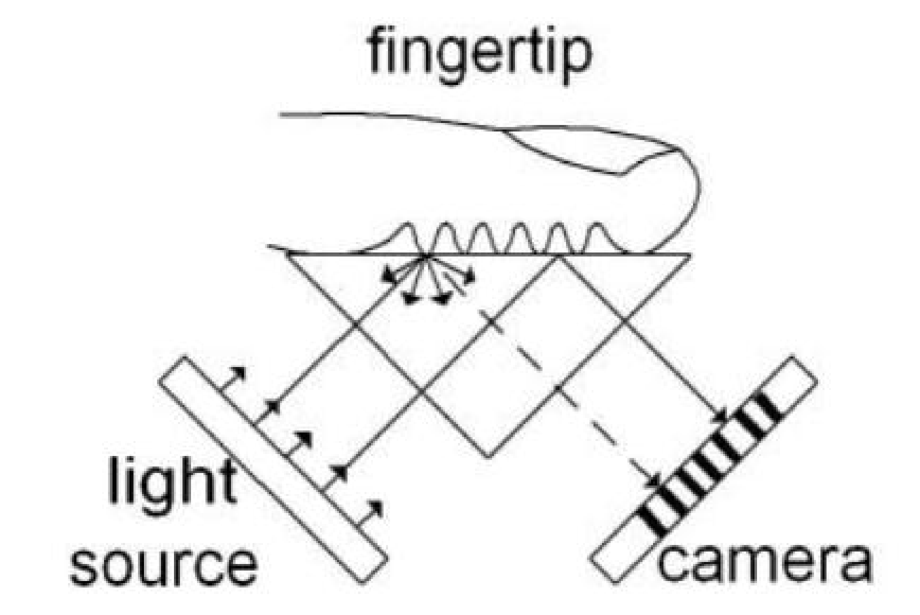
Frustrated total internal reflection

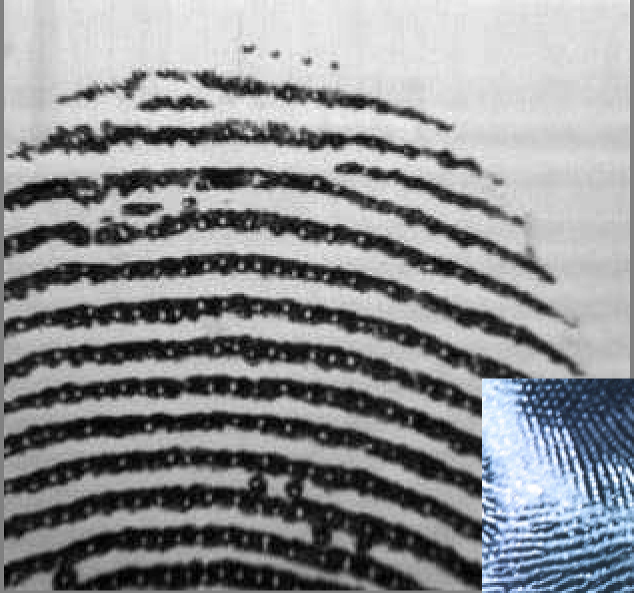


How close do the prisms have to be before TIR is frustrated?

This effect provides evidence for **evanescent fields**—fields that leak through the TIR surface–and is the basis for a variety of spectroscopic techniques.

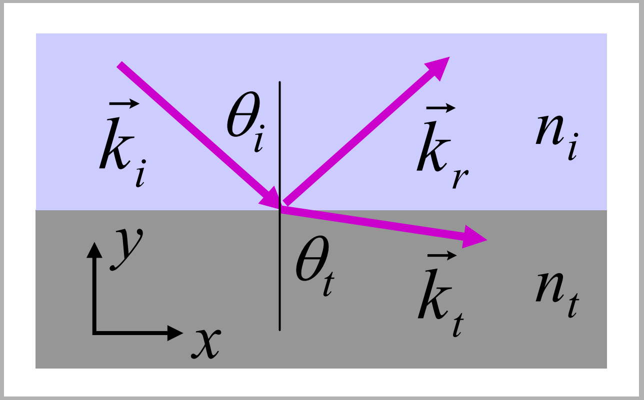
FTIR and fingerprinting





See TIR from a fingerprint valley and FTIR from a ridge.

The Evanescent Wave



The evanescent wave is the "transmitted wave" when total internal reflection occurs. A mystical quantity! So we'll do a mystical derivation:

$$r_\perp = \frac{E_{0r}}{E_{0i}} = \frac{\left[n_i \cos(\theta_i) - n_t \cos(\theta_t)\right]}{\left[n_i \cos(\theta_i) + n_t \cos(\theta_t)\right]}$$

Since $\sin(\theta_t) > 1$, θ_t doesn't exist, so computing $r_\perp$ is impossible.

Let's check the reflectivity, R, anyway. Use Snell's Law to eliminate θ_t :

$$\cos(\theta_t) = \sqrt{1-\sin^2(\theta_t)} = \sqrt{1-\left(\frac{n_i}{n_t}\right)^2 \sin^2(\theta_i)} = \sqrt{\text{Neg. Number}}$$

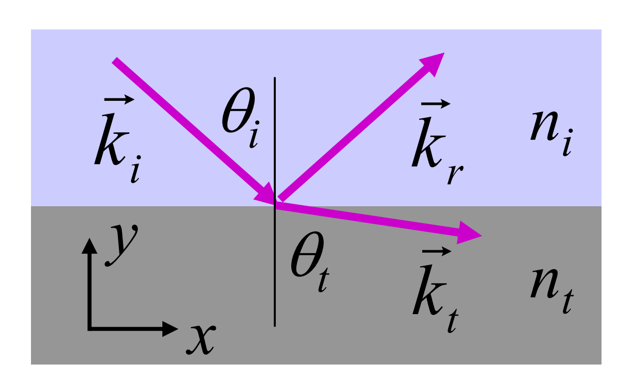
Substituting this expression into the above one for $r_\perp$ and

redefining R yields: $$R \equiv r_\perp r_\perp^* = \left(\frac{a-bi}{a+bi}\right)\left(\frac{a+bi}{a-bi}\right) = 1$$

So all power is reflected; the evanescent wave contains no power.

The Evanescent-Wave k-vector

The evanescent wave k-vector must have x and y components:



Along surface: $k_{tx} = k_t \sin(\theta_t)$

Perpendicular to it: $k_{ty} = k_t \cos(\theta_t)$

Using Snell's Law, $\sin(\theta_t) = (n_i / n_t) \sin(\theta_i)$, so k_{tx} is meaningful.

And again: $\cos(\theta_t) = [1 - \sin^2(\theta_t)]^{1/2} = [1 - (n_i / n_t)^2 \sin^2(\theta_i)]^{1/2}$

$$= \pm i\beta$$

Neglecting the unphysical $-i\beta$ solution, we have:

$$E_t(x,y,t) = E_0 \exp[-k\beta\, y] \exp i \,[\, k\,(n_i / n_t) \sin(\theta_i)\, x - \omega\, t \,]$$

The evanescent wave decays exponentially in the transverse direction.

Optical Properties of Metals

A simple model of a metal is a gas of free electrons

These free electrons and their accompanying positive nuclei can undergo "plasma oscillations" at frequency, ω_p.

where: $$\omega_p^2 = \frac{N e^2}{(\varepsilon_0 m_e)}$$

The refractive index for a metal is :

$$n^2 = 1 - \left(\frac{\omega_p}{\omega} \right)^2$$

When $n^2 < 0$, n is imaginary, and absorption is strong.

So for $\omega < \omega_p$ metals absorb strongly. For $\omega > \omega_p$ metals are transparent.

Reflection from metals

At normal incidence in air:

$$R = \frac{(n-1)^2}{(n+1)^2}$$

Generalizing to complex refractive indices:

$$R = \frac{(n-1)(n^*-1)}{(n+1)(n^*+1)}$$