PROBLEM SOLUTIONS

Chapter 1: Basic Properties of Numbers

1. Prove the following:

   (i) If $ax = a$ for some number $a \neq 0$, then $x = 1$.
   Proof: If $a \neq 0$ then there exists a^{-1} such that

   $$a \cdot a^{-1} = a^{-1} \cdot a = 1 \qquad \text{(P7)}$$

   Then:

   $$\begin{aligned} a^{-1} \cdot (a \cdot x) &= (a^{-1} \cdot a) \cdot x && \text{(P5)} \\ &= 1 \cdot x && \text{(P7)} \\ &= x && \text{(P6)} \end{aligned}$$

   and

   $$a^{-1} \cdot a = 1 \qquad \text{(P7)}$$

   Then $ax = a \Rightarrow x = 1$ ■

   (ii) $x^2 - y^2 = (x-y)(x+y)$.

   $$\begin{aligned} \underline{\text{Proof}}\text{: } (x-y)(x+y) &= x \cdot (x+y) + (-y) \cdot (x+y) && \text{(P9)} \\ &= x \cdot x + x \cdot y + (-y) \cdot x + (-y) \cdot y && \text{(P9)} \\ &= x \cdot x + x \cdot y + x \cdot (-y) + (-y) \cdot y && \text{(P4)} \\ &= x \cdot x + x \cdot (y + (-y)) + (-y) \cdot y && \text{(P9)} \\ &= x \cdot x + x \cdot 0 + (-y) \cdot y && \text{(P3)} \\ &= x \cdot x + (-y) \cdot y && \text{(P9)} \\ &= x \cdot x + -(y \cdot y) && \text{(P5)} \\ &= x^2 - y^2 \ ■ \end{aligned}$$

   (iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.
   Proof: Since $x^2 = y^2$ we have

   $$\begin{aligned} x^2 - y^2 &= y^2 - y^2 \\ &= 0 && \text{(P3)} \end{aligned}$$

   and $x^2 - y^2 = (x-y)(x+y)$ by 1(ii). Therefore

   $$(x-y)(x+y) = 0$$

Then if $x \neq -y$

$$\begin{aligned}((x-y)(x+y))\cdot(x+y)^{-1} &= (x-y)\cdot((x+y)(x+y)^{-1}) && \text{(P5)}\\ &= (x-y)\cdot 1 && \text{(P7)}\\ &= x-y && \text{(P6)}\end{aligned}$$

and since

$$0\cdot(x+y)^{-1} = 0 \qquad \text{(P9)}$$

we have $x-y=0 \Rightarrow x=y$ (P3). ■

If we start with $x \neq y$ a similar proof will show that $x=-y$.

(iv) $x^3 - y^3 = (x-y)(x^2+xy+y^2)$.

<u>Proof</u>:
$$\begin{aligned}(x-y)(x^2+xy+y^2) &= (x-y)\cdot x^2 + (x-y)\cdot xy + (x-y)\cdot y^2 && \text{(P9)}\\ &= x\cdot x^2 - y\cdot x^2 + x\cdot xy - y\cdot xy + x\cdot y^2 - y\cdot y^2 && \text{(P9)}\\ &= x^3 - yx^2 + x^2y + yxy + xy^2 - y^3\\ &= x^3 - x^2y + x^2y + xy^2 - xy^2 - y^3 && \text{(P5)}\\ &= x^3 - y^3 && \text{(P3)}\end{aligned}$$

(v) $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

<u>Proof</u>: $(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) =$

$$\begin{aligned}&= x\cdot x^{n-1} + x\cdot x^{n-2}y + \cdots + x\cdot xy^{n-2} + x\cdot y^{n-1}\\ &\qquad - y\cdot x^{n-1} - y\cdot x^{n-2}y - \cdots - y\cdot xy^{n-2} - y\cdot y^{n-1}\end{aligned}$$

$$\begin{aligned}&= x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1}\\ &\qquad - x^{n-1}\cdot y - x^{n-2}y\cdot y - \cdots - xy^{n-2}\cdot y - y^n \quad \text{(P5)}\end{aligned}$$

$$= x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1} - x^{n-1}y - x^{n-2}y^2 - \cdots - xy^{n-1} - y^n$$

$$\begin{aligned}&= x^n + x^{n-1}y - x^{n-1}y + \cdots + xy^{n-1} - xy^{n-1} - y^n && \text{(P1)}\\ &= x^n + 0 + \cdots + 0 - y^n && \text{(P3)}\\ &= x^n - y^n \ ■ && \text{(P2)}\end{aligned}$$

(vi) $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$.

Using the result of 1(iv) with $-y$ in place of y, we have

$$\begin{aligned}x^3 - (-y)^3 = x^3 - -y^3 = x^3 + y^3 &= (x-(-y))(x^2 + x(-y) + (-y)^2)\\ &= (x+y)(x^2 - xy + y^2) \ ■\end{aligned}$$

2. What is wrong with the following "proof"? Let $x = y$. Then

$$\begin{aligned} x^2 &= xy, \\ x^2 - y^2 &= xy - y^2, \\ (x+y)(x-y) &= y(x-y), \\ x + y &= y, \\ 2y &= y, \\ 2 &= 1. \end{aligned}$$

Answer: Between lines 3 and 4 we perform a division by $x - y$. But since $x = y$, this means we divided by 0, which is undefined.

3. Prove the following:

(i) $\frac{a}{b} = \frac{ac}{bc}$, if $b, c \neq 0$.
Proof: $\frac{a}{b} = a \cdot b^{-1}$ and $\frac{ac}{bc} = ac \cdot (bc)^{-1}$
Thus

$$ac \cdot (bc)^{-1} \cdot bc = ac$$

and

$$a \cdot b^{-1} \cdot bc = a \cdot (b^{-1}b) \cdot c = ac$$

and therefore

$$\frac{a}{b} = \frac{ac}{bc} \blacksquare$$

(ii) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ if $b, d \neq 0$.
Proof: Since $b, d \neq 0$ we can multiply each side by bd.

$$\begin{aligned} \frac{ad+bc}{bd} \cdot bd &= ((ad+bc) \cdot (bd)^{-1}) \cdot bd \\ &= (ad+bc) \cdot ((bd)^{-1} \cdot bd) \\ &= ad + bc \end{aligned}$$

and

$$\begin{aligned} \left(\frac{a}{b} + \frac{c}{d}\right) \cdot bd &= (a \cdot b^{-1} + c \cdot d^{-1}) \cdot bd \\ &= ab^{-1} \cdot bd + cd^{-1} \cdot db \\ &= a(b^{-1}b)d + c(d^{-1}d)b \\ &= ad + cb \\ &= ad + bc \end{aligned}$$

Therefore $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ $\blacksquare$

(iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$.

Proof: Since $a, b \neq 0$ we can multiply both sides by ba.

$$\begin{aligned} a^{-1}b^{-1} \cdot ba &= a^{-1}(b^{-1} \cdot b)a \\ &= a^{-1} \cdot a \\ &= 1 \end{aligned}$$

Since

$$\begin{aligned} (ba)^{-1} \cdot ba = 1 &\Rightarrow a^{-1}b^{-1} \cdot ba = (ba)^{-1} \cdot ba \\ &\Rightarrow a^{-1}b^{-1} = (ba)^{-1} = (ab)^{-1} \blacksquare \end{aligned}$$

(iv) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{db}$, if $b, d \neq 0$.

Proof:

$$\frac{a}{b} \cdot \frac{c}{d} = ab^{-1} \cdot cd^{-1} = acd^{-1}b^{-1} = ac(db)^{-1} = \frac{ac}{db} \blacksquare$$

(v) $\frac{a}{b} \big/ \frac{c}{d} = \frac{ad}{bc}$ if $b, c, d \neq 0$.

Proof:

$$\frac{ad}{bc} \cdot \frac{c}{d} = ad \cdot (bc)^{-1} \cdot cd^{-1} = ad \cdot b^{-1}c^{-1} \cdot cd^{-1} = ab^{-1}cc^{-1}dd^{-1} = ab^{-1} = \frac{a}{b}$$

therefore

$$\frac{ad}{bc} = \frac{a}{b} \cdot (cd^{-1})^{-1} = \frac{a}{b} \Big/ \frac{c}{d}$$

(vi) If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also determine when $\frac{a}{b} = \frac{b}{a}$.

Proof:

$$\begin{aligned} \frac{a}{b} = \frac{c}{d} &\Rightarrow \frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd \\ &\Rightarrow ab^{-1}bd = cd^{-1}db \\ &\Rightarrow ad = cb = bc. \end{aligned}$$

$$\begin{aligned} ad = bc &\Rightarrow ad \cdot d^{-1}b^{-1} = bc \cdot d^{-1}b^{-1} \\ &\Rightarrow add^{-1}b^{-1} = cbb^{-1}d^{-1} \\ &\Rightarrow ab^{-1} = cd^{-1} \\ &\Rightarrow \frac{a}{b} = \frac{c}{d} \blacksquare \end{aligned}$$

4. Find all numbers x for which

(i) $4 - x < 3 - 2x$

$$4 - x < 3 - 2x \Rightarrow x < -1$$

Ans: $\{x \in \mathbb{R} : x < -1\}$

(ii) $5 - x^2 < 8$

$$5 - x^2 < 8 \Rightarrow x^2 > -3$$

Ans: all $x \in \mathbb{R}$ satisfy this inequality.

(iii) $5 - x^2 < -2$

$$5 - x^2 < -2 \Rightarrow x^2 > 7 \Rightarrow (x > \sqrt{7}) \text{ OR } (x < -\sqrt{7})$$

Ans: $\{x \in \mathbb{R} : x > \sqrt{7}\} \cup \{x \in \mathbb{R} : x < -\sqrt{7}\}$

(iv) $(x - 1)(x - 3) > 0$

$$ab > 0 \Rightarrow (a > 0, b > 0) \text{ OR } (a < 0, b < 0)$$

Case 1: $a > 0, b > 0$:

$$x - 1 > 0 \Rightarrow x > 1$$

$$x - 3 > 0 \Rightarrow x > 3$$

$\{x \in \mathbb{R} : x > 1\} \cap \{x \in \mathbb{R} : x > 3\} = \{x \in \mathbb{R} : x > 3\}$.
Case 2: $a < 0, b < 0$:

$$x - 1 < 0 \Rightarrow x < 1$$

$$x - 3 < 0 \Rightarrow x < 3$$

Ans: $\{x \in \mathbb{R} : x < 1\} \cap \{x \in \mathbb{R} : x < 3\} = \{x \in \mathbb{R} : x < 1\}$.
Therefore the full answer is: $\{x \in \mathbb{R} : x > 3\} \cup \{x \in \mathbb{R} : x < 1\}$.

(v) $x^2 - 2x + 2 > 0$

$$x^2 - 2x + 2 = (x - 1)^2 + 1 > 0 \Rightarrow (x - 1)^2 > -1$$

If $(x - 1)$ is positive or negative, $(x - 1)^2 > 0 > -1$.
If it is zero, then clearly $(x - 1)^2 > -1$.
Ans: all $x \in \mathbb{R}$ satisfy this inequality.

(vi) $x^2 + x + 1 > 2$

$$x^2 + x + 1 > 2 \Rightarrow x^2 + x - 1 > 0$$

Employing the quadratic formula:

$$x > \frac{1 \pm \sqrt{5}}{2}$$

Ans: $\{x \in \mathbb{R} : x > \frac{1 \pm \sqrt{5}}{2}\}$

(vii) $x^2 - x + 10 > 16$

$$x^2 - x + 10 > 16 \Rightarrow x^2 - x - 6 > 0 \Rightarrow (x - 3)(x + 2) > 0$$

Then if $(x - 3) > 0, (x + 2) > 0 \Rightarrow x > 3$.
If $(x - 3) < 0, (x + 2) < 0 \Rightarrow x < -2$.
Ans: $\{x \in \mathbb{R} : x > 3\} \cup \{x \in \mathbb{R} : x < -2\}$

(viii) $x^2 + x + 1 > 0$

Ans: all $x \in \mathbb{R}$.

(ix) $(x - \pi)(x + 5)(x - 3) > 0$

Either all three terms are positive, else two are negative and one is positive.
Case 1: $(x - \pi) > 0, (x + 5) > 0, (x - 3) > 0$
$x > \pi$ and $x > -5$ and $x > 3 \Rightarrow x > \pi$.
Case 2: $(x - \pi) > 0, (x + 5) < 0, (x - 3) < 0$
$x > \pi$ and $x < -5$ and $x < 3 \Rightarrow x = \emptyset$
Case 3: $(x - \pi) < 0, (x + 5) > 0, (x - 3) < 0$
$x < \pi$ and $x > -5$ and $x < 3 \Rightarrow x > -5$ and $x < 3$.
Case 4: $(x - \pi) < 0, (x + 5) < 0, (x - 3) > 0$
$x < \pi$ and $x < -5$ and $x > 3 \Rightarrow x = \emptyset$.
Ans: $\{x : x > \pi\} \cup \{x : -5 < x < 3\}$

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$

Case 1: $(x - \sqrt[3]{2}) > 0, (x - \sqrt{2}) > 0$
$x > \sqrt[3]{2}$ and $x > \sqrt{2} \Rightarrow x > \sqrt{2}$
Case 2: $(x - \sqrt[3]{2}) < 0, (x - \sqrt{2}) < 0$
$x < \sqrt[3]{2}$ and $x < \sqrt{2} \Rightarrow x < \sqrt[3]{2}$
Ans: $\{x : x > \sqrt{2}\} \cup \{x : x < \sqrt[3]{2}\}$

(xi) $2^x < 8$

$$2^x < 8 \Rightarrow x < \log_2 8 = 3$$

Ans: $\{x : x < 3\}$

(xii) $x + 3^x < 4$

Take $x = 1$. Then

$$x + 3^x = 1 + 3 = 4 \not< 4.$$

Now take $x > 1$. Then

$$x + 3^x > 1 + 3 = 4 \not< 4.$$

Ans: $\{x : x < 1\}$

(xiii) $\frac{1}{x} + \frac{1}{1-x} > 0$

$$\frac{1}{x} + \frac{1}{1-x} > 0 \Rightarrow \frac{1}{x} > -\frac{1}{1-x} \Rightarrow 1 - x > -x \Rightarrow 1 > 0$$

Since this is a contradiction, there does not exist an x such that $\frac{1}{x} + \frac{1}{1-x} > 0$.
Ans: $\emptyset$

(xiv) $\frac{x-1}{x+1} > 0$

Due to the presence of $x + 1$ in the denominator, we are limited to $x \neq -1$. This permits us to multiply both sides of the inequality by $x + 1$.

$$\frac{x-1}{x+1} > 0 \Rightarrow x - 1 > 0 \Rightarrow x > 1$$

Ans: $\{x : x > 1\}$

5. Prove the following:

(i) If $a < b$ and $c < d$, then $a + c < b + d$.

$$a < b, c < d \Rightarrow 0 < b - a, 0 < d - c \Rightarrow 0 < (b - a) + (d - c) \Rightarrow a + c < b + d$$

(ii) If $a < b$, then $-b < -a$.

$$a < b \Rightarrow 0 < b - a \Rightarrow -b < -a$$

(iii) If $a < b$ and $c > d$, then $a - c < b - d$.

$$a < b, c > d \Rightarrow 0 < b - a, 0 < c - d \Rightarrow 0 < (b - a) + (c - d) \Rightarrow a - c < b - d$$

(iv) If $a < b$ and $c > 0$, then $ac < bc$.

$$a < b \Rightarrow 0 < b - a \Rightarrow 0 < c(b - a) = cb - ca = bc - ac \Rightarrow ac < bc$$

(v) If $a < b$ and $c < 0$, then $ac > bc$.

$$c < 0 \Rightarrow -c > 0$$

$$a < b \Rightarrow 0 < b - a \Rightarrow 0 < -c(b - a) \Rightarrow 0 < -cb + ca = -bc + ac \Rightarrow bc < ac$$

(vi) If $a > 1$, then $a^2 > a$.

$$a > 1 \Rightarrow a > 0$$

$$a > 1 \Rightarrow a - 1 > 0 \Rightarrow a(a - 1) = a^2 - a > 0 \Rightarrow a^2 > a$$

(vii) If $0 < a < 1$, then $a^2 < a$.

$$0 < a < 1 \Rightarrow a > 0, a - 1 < 0 \Rightarrow a(a - 1) = a^2 - a < 0 \Rightarrow a^2 < a$$

(viii) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.

If $a = 0$ or $c = 0$ then $ac = 0$.
Since $b > a = 0$ and $d > c = 0$, $bd > 0 = ac$.
Otherwise, we have $0 < a < b$ and $0 < c < d$.
Then

$$0 < a < b, c > 0 \Rightarrow 0 < ac < bc$$

and

$$0 < c < d, b > 0 \Rightarrow 0 < bc < bd$$

which gives us

$$0 < ac < bc < bd \Rightarrow ac < bd$$

(ix) If $0 \le a < b$, then $a^2 < b^2$.
If $a = 0$ then

$$b > 0 \Rightarrow b^2 > 0 = a^2.$$

Otherwise

$$0 < a < b \Rightarrow 0 < a^2 < ab \text{ and } 0 < ab < b^2 \Rightarrow 0 < a^2 < ab < b^2 \Rightarrow a^2 < b^2.$$

(Or we could just apply 5(viii) with $c = a$ and $d = b$.)

(x) If $a, b \ge 0$ and $a^2 < b^2$, then $a < b$.
If $a = 0$ and $a^2 = 0 < b^2$ then $0 \cdot b^{-1} < b^2 \cdot b^{-1} \Rightarrow 0 < b$.
Otherwise $a > 0$ and $b > 0$. Then

$$a^2 < b^2 \Rightarrow 0 < b^2 - a^2 = (b - a)(b + a).$$

Case 1: $b - a > 0, b + a > 0$.

$$b - a > 0 \Rightarrow b > a \text{ and } b + a > 0 \Rightarrow b > -a.$$

Since $a > 0 \Rightarrow a > -a \Rightarrow b > a$.
Case 2: $b - a < 0, b + a < 0$

$$b - a < 0 \Rightarrow b < a \text{ and } b + a < 0 \Rightarrow b < -a.$$

But since $a > 0 \Rightarrow -a < 0 \Rightarrow b < 0$. This is a contradiction. Case 2 never occurs.

6.

(a) Prove that if $0 \le x < y$, then $x^n < y^n$, $n = 1, 2, 3, \ldots$.
Proof: We can apply theorem 5(viii) any number of times with $a = c = x$ and $b = d = y$.

(b) Prove that if $x < y$ and n is odd, then $x^n < y^n$.
When $n = 1$, $x^1 < y^1 \iff x < y$ which is true by hypothesis. Assume the theorem holds for all odd n up to some k. That is, $x^n < y^n$ for $n = 1, 3, 5 \ldots, k$. Then $x^{k+2} < y^{k+2} \iff x^2 x^k < y^2 y^k \iff x^k < (y/x)^2 y^k$. Since $x < y$, we have $y/x > 1$ and so from 5(vi) we know $(y/x)^2 > y/x > 1$ and so $y^k < (y/x)^2 y^k$. Since we assume from the inductive step that $x^k < y^k$, we must have $x^k < y^k < (y/x)^2 y^k$.

(c) Prove that if $x^n = y^n$ and n is odd, then $x = y$.
Proof by contradiction. Assume $x^n = y^n$ for n odd but $x \ne y$. Take $x < y$. From 6(b), we have that $x^n < y^n$. Similarly, for $y < x$ we have $y^n < x^n$. By contradiction, we must have that $x = y$.

(d) Prove that if $x^n = y^n$ and n is even, then $x = y$ or $x = -y$.

Proof by contradiction. Assume $x^n = y^n$ for n even but $x \neq y$ and $x \neq -y$. Then necessarily $x < y$ or $x > y$. Take $x < y$.

Case 1: If $0 \leq x < y$ then we have from 6(a) that $x^n < y^n$ for any n, a contradiction.

Case 2: Assume $x < 0, y \geq 0$. Then we have $0 < -x \leq y$ or $0 \leq y < -x$. Then from 6(a), either $(-x)^n < y^n$ or $y^n < (-x)^n$, respectively. In either case, we have $(-x)^n \neq y^n$. Since n is even, we can write $n = 2k$ for some integer k. Then $(-x)^n = ((-x)^2)^k = (x^2)^k = x^{2k} = x^n$. Then we find that in either case $x^n \neq y^n$ which contradicts our assumption.

Case 3: Assume $x, y \leq 0$. Then $0 \leq -y < -x$. From 6(a) $(-y)^n < (-x)^n$ but since $(-y)^n = y^n$ and $(-x)^n = x^n$ from the argument above, we find that $y^n < x^n$, a contradiction.

Case 4: Assuming $x > 0, y \leq 0$ is a contradiction of $x < y$.

The proof for $x > y$ follows from symmetry arguments.

7. Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b.$$

Proof:

$$0 < a < b \Rightarrow 0 < a^2 < ab$$

$$0 < a < b \Rightarrow 0 < ab < b^2$$

$$0 < a^2 < ab < b^2 \Rightarrow 0 < a < \sqrt{ab} < b$$

$$\left(\frac{a+b}{2}\right)^2 = \frac{a^2 + 2ab + b^2}{2} = \frac{a^2}{2} + ab + \frac{b^2}{2} > ab \Rightarrow \frac{a+b}{2} > \sqrt{ab}$$

$$a - b < 0 \Rightarrow \frac{a-b}{2} < 0 \Rightarrow \frac{a-b}{2} + b < b \Rightarrow \frac{a-b}{2} + b = \frac{a+b}{2} < b$$

8. Although the basic properties of inequalities were stated in terms of the collection P of all positive numbers, and $<$ was defined in terms of P, this procedure can be reversed. Suppose that P10-P12 are replaced by

(P'11) For any numbers a and b one, and only one, of the following holds:

(i) $a = b$,

(ii) $a < b$,

(iii) $b < a$.

(P'12) For any numbers a, b, and c, if $a < b$ and $b < c$, then $a < c$.

(P'13) For any numbers a, b, and c, if $a < b$, then $a + c < b + c$.

(P'14) For any numbers a, b, and c, if $a < b$ and $0 < c$, then $ac < bc$.

Show that P10-P12 can then be deduced as theorems.

(P10) Applying P'10 with $b = 0$ we have that for any number a, either $a = 0$, $a < 0$ or $0 < a$. Since P is defined to be the collection of all numbers $a > 0$, this is equivalent to the statement of P10.

(P11) Let $0 < x, y$ (since using a and b would amount to an abusive confusion of variables in different scopes). Applying P'12 with $a = 0, b = x, c = y$, we have:

$$\begin{aligned} 0 + y &< x + y && \text{(P'12)} \\ y &< x + y && \text{(P2)} \\ 0 &< x + y && \text{(P'11)} \end{aligned}$$

The last step follows from the fact that $0 < y$. So we have shown that for numbers $0 < x$, $0 < y$, we have that $0 < x + y$, an equivalent statement to (P11).

(P12) Let $0 < x, y$. Applying P'13 with $a = 0, b = x, c = y$, we have:

$$\begin{aligned} 0 \cdot y &< x \cdot y && \text{(P'13)} \\ (x + (-x)) \cdot y &< x \cdot y && \text{(P3)} \\ y \cdot (x + (-x)) &< x \cdot y && \text{(P4)} \\ y \cdot x + y \cdot (-x) &< x \cdot y && \text{(P9)} \\ y \cdot x + (-y \cdot x) &< x \cdot y && \text{(P8)} \\ 0 &< x \cdot y && \text{(P3)} \\ & && (1) \end{aligned}$$

Therefore we've shown that when $0 < x$ and $0 < y$, we have $0 < x \cdot y$, an equivalent statement to (P12).

9. Express each of the following with at least one less pair of absolute value signs.

   (i) $|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}|$
   Since $7 > 5$ and $f(x) = \sqrt{x}$ is monotonically increasing from $x > 0$, $\sqrt{7} > \sqrt{5} \Rightarrow \sqrt{7} - \sqrt{5} > 0$. The total sum is positive so we can drop the absolute value signs and write $|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}| = \sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}$.

   (ii) $|(|a + b| - |a| - |b|)|$
   $|a + b| - |a| - |b| = |a + b| - (|a| + |b|)$. From the Triangle Inequality, $|a + b| \leq |a| + |b| \Rightarrow |a + b| - (|a| + |b|) \leq 0$. Since $|x| = -x$ for $x \leq 0$, we can write $|(|a + b| - |a| - |b|)| = -(|a + b| - |a| - |b|) = |a| + |b| - |a + b|$.

   (iii) $|(|a + b| + |c| - |a + b + c|)|$
   Let $d = a + b$. Then $|a+b| + |c| - |a+b+c| = |d| + |c| - |d+c|$. From the Triangle Inequality, $|d| + |c| \geq |d + c| \Rightarrow |d| + |c| - |d + c| \geq 0$ so we can drop the outermost absolute value signs and write $|(|a+b| + |c| - |a+b+c|)| = |a+b| + |c| - |a+b+c|$.

   (iv) $|x^2 - 2xy + y^2|$
   $x^2 - 2xy + y^2 = (x - y)^2 \Rightarrow |x^2 - 2xy + y^2| = |(x - y)^2| = (x - y)^2 = x^2 - 2xy + y^2$.

(v) $|(|\sqrt{2}+\sqrt{3}|-|\sqrt{5}-\sqrt{7}|)|$
Since $\sqrt{2}>0$ and $\sqrt{3}>0$, $\sqrt{2}+\sqrt{3}>0$ and so $|\sqrt{2}+\sqrt{3}|=\sqrt{2}+\sqrt{3}$. Since $\sqrt{5}<\sqrt{7}$, $\sqrt{5}-\sqrt{7}<0$ so $|\sqrt{5}-\sqrt{7}|=-(\sqrt{5}-\sqrt{7})=\sqrt{7}-\sqrt{5}$. Then $|(|\sqrt{2}+\sqrt{3}|-|\sqrt{5}-\sqrt{7}|)|=|\sqrt{2}+\sqrt{3}+\sqrt{5}-\sqrt{7}|$.

10. Express each of the following without absolute value signs, treating various cases separately when necessary.

(i) $|a+b|-|b|$
Case 1: $a+b\geq 0, b\geq 0 \iff b>0, a\geq -b \Rightarrow |a+b|=a+b, |b|=b \Rightarrow |a+b|-|b|=a+b-b=a$.
Case 2: $a+b\leq 0, b\geq 0 \iff b>0, a\leq -b \Rightarrow |a+b|=-(a+b), |b|=b \Rightarrow |a+b|-|b|=-(a+b)-b=-a-2b$.
Case 3: $a+b\geq 0, b\leq 0 \iff b\leq 0, a\geq -b \Rightarrow |a+b|=a+b, |b|=-b \Rightarrow |a+b|-|b|=a+b+b=a+2b$.
Case 4: $a+b\leq 0, b\leq 0 \iff b\leq 0, a\leq -b \Rightarrow |a+b|=-(a+b), |b|=-b \Rightarrow |a+b|-|b|=-(a+b)+b=-a$.

(ii) $|(|x|-1)|$
Case 1: $x\geq 0 \Rightarrow |x|=x \Rightarrow |(|x|-1)|=|x-1|$. Case 1a $x-1\geq 0 \iff x\geq 1 \Rightarrow |x-1|=x-1$. Case 1b $x-1\leq 0 \iff 0\leq x\leq 1 \Rightarrow |x-1|=-(x-1)=1-x$.
Case 2: $x\leq 0 \Rightarrow |x|=-x \Rightarrow |(|x|-1)|=|-x-1|$. Case 2a $-x-1\geq 0 \iff x\leq -1 \Rightarrow |-x-1|=-x-1$. Case 2b $-x-1\leq 0 \iff 0\geq x\geq -1 \Rightarrow |-x-1|=-(-x-1)=x+1$.

Summarizing the cases: (1) $x\geq 1: x-1$, (2) $0\leq x\leq 1: 1-x$, (3) $-1\leq x\leq 0: x+1$, (4) $x\leq -1: -x-1$.

(iii) $|x|-|x^2|$
Case 1: $x\geq 0 \Rightarrow |x|=x \Rightarrow |x|-|x^2|=x-x^2$.
Case 2: $x\leq 0 \Rightarrow |x|=-x \Rightarrow |x|-|x^2|=-x-x^2$.

(iv) $a-|(a-|a|)|$
Case 1: $a\geq 0 \Rightarrow |a|=a \Rightarrow a-|(a-|a|)|=a-|a-a|=a-0=a$.
Case 2: $a\leq 0 \Rightarrow |a|=-a \Rightarrow a-|(a-|a|)|=a-|a-(-a)|=a-|2a|=a-(-2a)=3a$.