

Tensorial Ethics

A Mathematical Framework for Moral Philosophy

By Andrew Bond

Preface

A brief account of how the idea emerged—perhaps from frustrations with scalar approaches to ethics (utilitarian calculus, single-metric optimization) and the recognition that moral reality seems to resist reduction to simple quantities. An acknowledgment that this work sits at an unusual intersection: differential geometry, physics, and moral philosophy.

Introduction: The Insufficiency of Scalar Ethics

The dominant quantitative approaches to ethics have largely been scalar—assigning single numerical values to outcomes, actions, or states of affairs. Utilitarianism asks us to maximize a quantity. Cost-benefit analysis reduces decisions to a single bottom line. Even when pluralistic in their inputs, these frameworks typically collapse moral reality into a one-dimensional output.

But consider: in physics, the move from scalars to vectors to tensors marked profound conceptual advances. A scalar tells you magnitude. A vector tells you magnitude and direction. A tensor tells you how quantities transform across different reference frames and coordinate systems.

What if ethics requires similar machinery?

This book argues that many persistent problems in moral philosophy—the incommensurability of values, the context-dependence of moral judgment, the perspectival nature of ethical claims, the difficulty of aggregating across persons—are symptoms of forcing tensorial phenomena into scalar containers.

Part I: Foundations

Chapter 1: A Primer on Tensors for Philosophers

An accessible introduction to tensor mathematics for readers without physics backgrounds. Covers:

- Scalars, vectors, and the intuition behind higher-rank tensors
- Coordinate systems and transformations
- Covariance and contravariance
- The metric tensor and the measurement of "distance"
- Tensor fields and their variation across manifolds

The goal is not technical mastery but conceptual fluency—enough to follow the philosophical arguments.

Chapter 2: Why Ethics Might Be Tensorial

The motivating intuitions:

- *Transformation behavior*: Moral claims seem to change form (not merely content) when evaluated from different positions, perspectives, or framings
- *Multi-index structure*: Many ethical concepts seem to require specification along multiple independent dimensions simultaneously
- *Frame-dependence and invariance*: Some moral features appear relative to perspective; others appear invariant. Tensors offer a natural way to model both.
- *Contraction and projection*: The process of arriving at a concrete moral judgment from abstract principles resembles tensor contraction—reducing dimensionality while preserving essential structure

Chapter 3: Historical Precursors

A genealogy of proto-tensorial thinking in ethics:

- Aristotle's doctrine of the mean as context-sensitive calibration
- Kant's categorical imperative as an invariance condition
- The intuitionist pluralism of Ross and the question of how *prima facie* duties interact
- Rawls's original position as a transformation-invariant framework
- Sen and Nussbaum's capabilities approach as implicitly multi-dimensional
- The recent work on moral uncertainty and its structural similarities to mixed states

Part II: The Formal Framework

Chapter 4: The Moral Manifold

Introducing the base space over which ethical tensors are defined:

- What are the "points" of moral space? (Possible actions, states of affairs, persons, relationships?)
- Local vs. global structure
- Curvature: does moral space have intrinsic geometry?
- Boundaries, singularities, and moral dilemmas as topological features

Chapter 5: Ethical Tensors of Various Ranks

A taxonomy:

- Rank-0 (scalars): Simple magnitudes—utility, welfare, badness
- Rank-1 (vectors): Directed moral quantities—obligations as vectors pointing from current to required states; interests as directional
- Rank-2: Relations between moral agents, or between values; responsibility as a two-index object (who owes what to whom)
- Higher ranks: Complex moral situations involving multiple parties, multiple values, and multiple temporal frames

Chapter 6: The Moral Metric

How do we measure distance in moral space?

- What makes two actions morally "close" or "far apart"?
- Is there a natural metric, or must one be chosen (and what does choice imply)?
- Signature: Is moral space Euclidean, Riemannian, or Lorentzian? Does it contain "timelike" and "spacelike" separations (perhaps: reversible vs. irreversible moral differences)?

Chapter 7: Transformation Groups and Moral Invariance

What transformations should leave moral claims unchanged?

- Permutation of persons (anonymity conditions, impartiality)
- Temporal translation (does when something happens matter morally?)
- Perspective shifts (first-person to third-person; individual to collective)

- The structure of moral objectivity: invariants under all admissible transformations
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Part III: Applications

Chapter 8: Value Pluralism Reconsidered

The incommensurability problem dissolves if values are components of a tensor rather than competing scalars. Different values aren't rivals for a single crown—they're independent dimensions that combine through tensor operations. This chapter develops:

- How to represent irreducibly plural values
- When and how "comparison" can occur (through contraction, projection, inner products)
- Why tragic dilemmas persist even in a tensorial framework (and what this shows)

Chapter 9: Persons, Perspectives, and the Aggregation Problem

Standard aggregation (summing utilities) treats persons as interchangeable scalars. A tensorial approach:

- Each person as a distinct index
- Interpersonal comparison as cross-index operations
- The separateness of persons as resistance to full contraction
- Reimagining social choice theory in tensorial terms

Chapter 10: Moral Responsibility as a Tensor Field

Responsibility isn't a simple property of an agent; it's a relational structure with multiple indices:

- Agent, action, outcome, affected party, normative standard
- How responsibility "transforms" under different descriptions of the same event
- Collective responsibility and emergent tensorial structures

Chapter 11: Applied Cases

Brief applications to contested domains:

- Climate ethics and intergenerational obligation (temporal indices)

- Global justice (spatial and perspectival indices)
 - AI alignment (what tensorial structure should we expect in artificial moral reasoning?)
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Part IV: Objections and Horizons

Chapter 12: The Formalism Objection

Isn't this just mathematical window-dressing? A response:

- Formalism vs. modeling: the difference between empty notation and genuine structural insight
- What would it mean for ethics to "really be" tensorial?
- The pragmatic defense: even as a model, tensorial ethics may organize thought productively

Chapter 13: The Measurement Problem

We can't actually measure moral quantities with precision. Does that doom the project?

This chapter argues:

- Physics faced analogous objections; utility theory faced them too
- Ordinal structure may suffice even without cardinal measurement
- The framework's value lies in structural relations, not numerical precision

Chapter 14: Open Problems and Future Directions

- Dynamics: Moral change over time; parallel transport of ethical judgment across contexts
 - Quantum ethics? (Speculative: do moral superpositions make sense?)
 - Computational tractability
 - Empirical work: can we test whether human moral reasoning exhibits tensorial structure?
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Conclusion: What the Framework Reveals

A recapitulation of the core claims and their implications. The tensorial framework doesn't solve ethics—it reframes it. It suggests that the search for a single moral truth was always misguided, not because morality is subjective, but because it's structurally richer than scalars can capture.

Appendices

- A. Mathematical Formalism (rigorous definitions for specialists)
 - B. Notation and Conventions
 - C. Worked Examples
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Bibliography

Notes on Tone and Approach

This structure assumes a serious philosophical monograph with genuine mathematical content—something aimed at readers comfortable with analytic philosophy and willing to engage with formalism. The MIT Press context suggests rigor is appropriate, but accessibility matters too; the primer chapter is crucial.

The risk with a project like this is vacuity—using tensors as a metaphor without real content. To avoid that, I'd suggest:

1. **Concrete examples throughout:** Every abstract claim should be illustrated with a moral case where the tensorial framing does work that scalar framing can't.
 2. **Honest about limits:** Where does the analogy break down? What can't be captured?
 3. **Engagement with existing pluralism literature:** Ross, Berlin, Raz, Chang—these thinkers have grappled with similar problems in different vocabularies.
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Would you like me to develop any particular section further—perhaps draft a fuller version of the introduction, or work out one of the chapters in more detail?

Please draft a full version of the introduction, and one of the chapters in more detail in 5000 - 6000 words, with the math details worked out and verified to be valid and sound:

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I'll develop a full introduction and then work out Chapter 5 (Ethical Tensors of Various Ranks) in detail, since that's where the mathematical rubber meets the philosophical road.

Introduction: The Insufficiency of Scalar Ethics

The history of quantitative moral philosophy is largely a history of scalar thinking. From Bentham's felicific calculus to contemporary expected utility theory, the dominant impulse has been to assign single numerical values to the objects of moral concern—actions, outcomes, states of affairs, lives—and then to optimize, maximize, or satisfice with respect to these values. Even sophisticated pluralistic theories that acknowledge multiple values typically seek, at the moment of decision, to collapse this plurality into a single ordering or a single number. The question is always, in the end: *how much?*

This book argues that this scalar orientation, however natural and mathematically convenient, systematically distorts moral reality. The problems that have plagued quantitative ethics—the apparent incommensurability of values, the context-dependence of moral judgment, the perspectival character of ethical claims, the difficulty of aggregating welfare across persons while respecting their separateness—are not merely technical difficulties awaiting cleverer solutions. They are symptoms of a category error: the attempt to force inherently tensorial phenomena into scalar containers.

To see what this means, consider a parallel from physics. For centuries, physicists worked with scalar quantities like mass, temperature, and energy. These are numbers: they have magnitude but nothing more. The development of vector analysis in the nineteenth century marked a significant advance, allowing the representation of quantities with both magnitude and direction—velocity, force, electric field. But the deepest revolution came with the recognition that many physical quantities are neither scalars nor vectors but *tensors*: mathematical objects that encode how quantities transform when we change our coordinate system or frame of reference.

The stress at a point in a material, for instance, cannot be captured by a single number (a scalar) or even by an arrow (a vector). It requires specification of how force distributes across surfaces of different orientations—a rank-2 tensor with nine components in three dimensions. The curvature of spacetime in general relativity requires a rank-4 tensor. The insight is not merely that more numbers are needed, but that these numbers relate to each other in structured ways that encode transformation behavior—how the quantity appears from different perspectives, in different coordinate systems.

The central thesis of this book is that moral quantities exhibit analogous structure. Consider a simple case: the wrongness of a lie. A scalar approach assigns some number to this wrongness—perhaps derived from the harm caused, the violation of autonomy, or the breach of trust. But notice how the moral character of the lie transforms depending on the perspective from which it is evaluated: the liar's perspective (with access to intentions and perceived justifications), the deceived party's perspective (experiencing betrayal and damaged interests), a third party's perspective (observing violation of social norms), a retrospective perspective (knowing how things turned out). These are not merely different estimates of a single underlying magnitude. They are genuinely different moral aspects of the situation, irreducible to one another, yet systematically related through transformation rules.

Or consider the moral relationship between two persons, A and B. A may have obligations to B; B may have claims against A; both may share responsibilities to third parties that mediate their relationship; historical interactions may generate special duties; their positions in social structures may create role-based requirements. The moral content of "A's relation to B" is not a single quantity but a complex structure with multiple indices—agent, patient, context, normative framework, temporal frame—that transforms in specific ways when any of these indices are permuted or reinterpreted. This is precisely the signature of a tensor.

The Promissory Note

Before proceeding, I should be clear about the nature of the project. This book does not claim to have discovered that ethics *is* a branch of differential geometry, nor that moral truths can be read off from mathematical structures in the way that physical laws can (supposedly) be read off from the structure of spacetime. The relationship between the tensorial framework and moral reality is more subtle.

The claim is, first, *diagnostic*: that many persistent problems in moral philosophy arise from tacitly scalar assumptions that are poorly suited to the structure of ethical phenomena. When we try to compare values and find them incommensurable, we may be attempting an operation (scalar comparison) that is undefined for the quantities in question—like trying to ask whether a vector is greater or less than a different vector that points in an incomparable direction. When we try to aggregate welfare across persons and find ourselves unable to respect both total utility and the separateness of persons, we may be attempting to contract a tensor in a way that loses essential information. The tensorial framework helps us see *why* these problems arise and *what structure* a solution would need to have.

The claim is, second, *constructive*: that the mathematical resources of tensor analysis—transformation groups, invariants, metric structure, contraction operations—provide useful tools for articulating moral theories with greater precision and for identifying structural features that informal philosophical language tends to obscure. Just as the formalization of probability theory clarified problems of induction and decision, formalization via tensors may clarify problems of value pluralism, interpersonal comparison, and moral uncertainty.

The claim is, third, *heuristic*: that thinking tensorially suggests new questions, new distinctions, and new theoretical possibilities that might not arise within a purely scalar framework. What is the moral metric—the structure that determines distances and angles in moral space? What transformations leave moral claims invariant, and what does this invariance group tell us about the nature of moral objectivity? Are there tensorial identities—moral analogues to the Bianchi identities of differential geometry—that constrain the possible forms of ethical theory?

What the claim is *not* is reductionist. I do not think that moral reality is "nothing but" mathematical structure, or that the hard work of ethical reasoning can be replaced by calculation. The tensorial framework is a model, and like all models it abstracts and idealizes. But it is, I shall argue, a better model than the scalar frameworks that currently dominate—better in the sense of capturing more of the structure that matters while distorting less.

The Plan of the Book

The book proceeds in four parts. Part I establishes foundations: a mathematical primer on tensors for readers without physics backgrounds (Chapter 1), the philosophical motivations for thinking that ethics might be tensorial (Chapter 2), and a historical survey of proto-tensorial thinking in the moral philosophy tradition (Chapter 3).

Part II develops the formal framework. Chapter 4 introduces the moral manifold—the base space of points over which ethical tensors are defined, and considers what structure this space might have. Chapter 5 offers a taxonomy of ethical tensors by rank, from scalar quantities like undifferentiated welfare, through vector quantities like directed obligations, to higher-rank tensors encoding complex moral relationships. Chapter 6 addresses the moral metric: the structure that allows us to measure distances and angles in moral space, to say when two actions or outcomes are "close" or "far apart" in morally relevant respects. Chapter 7 examines transformation groups and the question of moral invariance—what remains constant when we shift perspectives, permute persons, or translate across time.

Part III turns to applications. Chapter 8 reconsiders the problem of value pluralism through a tensorial lens, arguing that incommensurability results from treating tensor components

as competing scalars. Chapter 9 addresses the aggregation problem—how to combine welfare across persons while respecting their separateness—and shows how tensorial structure suggests alternatives to simple summation. Chapter 10 develops an account of moral responsibility as a tensor field, with indices for agent, action, outcome, affected party, and normative standard.

Part IV confronts objections and gestures toward future work. Chapter 11 addresses the formalism objection: the worry that this is all mathematical window-dressing with no real ethical content. Chapter 12 tackles the measurement problem: even if ethics is tensorial in structure, we cannot assign precise numerical values to moral quantities, so what use is the framework? Chapter 13 surveys open problems and speculative extensions.

A Note on Accessibility

This book is written for philosophers, not physicists or mathematicians. I assume familiarity with analytic philosophy and comfort with formal argumentation, but I do not assume prior knowledge of differential geometry or tensor analysis. Chapter 1 provides all the mathematical background needed to follow the arguments. Readers who want more rigor will find it in the appendices; readers who want to skip the mathematics and focus on the philosophical arguments will find that most chapters can be read with only qualitative understanding of the formal machinery.

My hope is that even readers skeptical of the entire project will find value in the reframings and distinctions the tensorial perspective affords. And perhaps some will be persuaded that the deep structure of moral thought has more in common with the geometry of spacetime than we have previously recognized.

Chapter 5: Ethical Tensors of Various Ranks

The previous chapters established the motivation for thinking tensorially about ethics and introduced the mathematical machinery of tensors in a general way. This chapter gets specific. What, concretely, are the ethical tensors? What moral quantities are scalars, what are vectors, and what require the richer structure of higher-rank tensors?

The answer I shall develop is that rank correlates, roughly, with complexity of moral structure—specifically, with the number of independently variable dimensions along which a moral quantity must be specified. A scalar moral quantity is fully determined once we fix its magnitude. A vector moral quantity requires both magnitude and direction in some moral space. A rank-2 tensor requires specification along two independent dimensions and encodes how the quantity transforms under changes in both. And so on.

But we should not expect a clean one-to-one mapping between familiar moral concepts and tensor ranks. A single concept like "obligation" or "responsibility" may be scalar in some contexts, vectorial in others, and fully tensorial in its complete articulation. The tensorial framework reveals this context-dependence as structure rather than ambiguity.

5.1 Rank-0: Moral Scalars

A scalar is a quantity that is fully specified by a single number and that remains unchanged under coordinate transformations. In physics, mass and electric charge are scalars: an electron has the same charge regardless of the reference frame from which we measure it.

What moral quantities have this structure? The natural candidates are *magnitudes of value* in the simplest cases: the raw quantity of pleasure or pain in a moment of experience, the degree of preference satisfaction in an outcome, the intensity of an interest. These are the atoms of utilitarian calculation—the inputs to the felicific calculus.

But even here, we must be careful. Bentham famously enumerated dimensions of pleasure: intensity, duration, certainty, propinquity, fecundity, purity, extent. These are not independent scalar quantities to be separately maximized; they interact in complex ways. Intensity and duration trade off against each other in how they contribute to overall value. Certainty and propinquity function as discount factors. Extent raises questions of aggregation across persons that cannot be resolved by treating it as another scalar dimension.

Nonetheless, let us grant that there is *some* level of description at which simple scalar magnitudes exist in ethics. A moment of pain has some intensity. A preference has some strength. A moral wrong has some degree of seriousness. If nothing else, ordinal rankings—this action is worse than that one—require at least a scalar structure (technically, a total or partial order, which need not be numerical but shares the one-dimensional character of scalars).

We can formalize this. Let \mathcal{M} be the moral manifold—the space of morally relevant situations, which we will examine more carefully in Chapter 4. A *moral scalar field* is a function:

$$\phi: \mathcal{M} \rightarrow \mathbb{R}$$

assigning to each point in moral space a real number. The defining property of a scalar is that under a coordinate transformation $x \mapsto x'$, the value at a point is unchanged:

$$\phi'(x') = \phi(x)$$

In the moral context, this means: if we describe the same situation using different coordinates—different concepts, different framings, different perspectives—the scalar quantity remains the same. The pain intensity of this experience, described in these terms or those terms, from this perspective or that perspective, is what it is.

This invariance condition immediately raises a question: *are there moral quantities with this strong invariance property?* The worry is that almost every moral quantity seems to vary with perspective, framing, or description. My pain looks different from inside and outside. The wrongness of an action depends on how we describe it (intentional harm vs. negligent harm vs. accidental harm). The value of an outcome depends on whose values we consult.

I suggest that the scarcity of genuine moral scalars is itself significant. It indicates that scalar ethics—ethics that attempts to reduce everything to invariant magnitudes—is working with too impoverished a toolkit. The tensorial framework starts from this recognition and builds upward.

5.2 Rank-1: Moral Vectors

A vector is a quantity with both magnitude and direction. In physics, velocity, force, and electric field are vectors. Formally, a vector V at a point transforms under coordinate changes according to:

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$$

where we use the Einstein summation convention (summing over repeated indices). The key feature is that the components of a vector are not invariant—they change with coordinates—but they change in a specific, lawful way that preserves the geometric object the vector represents.

What moral quantities are vectors? I propose that *directed obligations* have vector structure. Consider the claim "A ought to help B." This is not merely a magnitude (how much help? how strongly required?) but a directed quantity: help must flow *from A to B*, not vice versa. The obligation has a source, a target, and an orientation in moral space.

To formalize this, we need a moral space in which direction makes sense. Let us posit a space \mathcal{P} of persons or moral patients (we will be more careful about this in Chapter 4). An obligation vector might live in the tangent space of some moral manifold, pointing from one state of affairs toward another—from "A has not helped B" toward "A has helped B."

Alternatively, consider *interests* as vector quantities. An interest is not merely a magnitude of concern; it has a directedness, pointing toward the object of interest. My interest in my own welfare points toward states of the world in which I flourish. This interest can be stronger or weaker (magnitude) but also points in a specific direction through the space of possible outcomes.

The vector framework illuminates several features of moral reasoning:

Non-comparability of orthogonal interests. If two interests point in orthogonal directions in moral space, there is no natural way to compare their magnitudes. Asking whether my interest in autonomy is stronger or weaker than my interest in welfare may be like asking whether the eastward component of velocity is greater than the northward component—a question that has no answer because the quantities are not comparable along any single dimension. This is a tensorial gloss on value incommensurability.

Conflicting obligations as opposing vectors. When I have an obligation to A and an obligation to B that cannot both be satisfied, these may be represented as vectors pointing in incompatible directions. The "net obligation"—if such a thing exists—is the vector sum, which may be smaller in magnitude than either component and may point in a direction that corresponds to neither obligation fully.

Supererogation as vector extension. The distinction between obligation and supererogation might be captured by distinguishing between the obligatory component of a moral vector (the part required) and the full vector (which includes praiseworthy but non-required extension in the same direction).

Mathematically, let us construct a simple model. Suppose we have n morally relevant dimensions—perhaps welfare, autonomy, justice, and so on. A moral situation can be represented as a point $x \in \mathbb{R}^n$. An obligation can be represented as a vector $\mathbf{O} \in T_x \mathbb{R}^n \cong \mathbb{R}^n$ pointing from the current state toward the obligatory state.

For two obligations \mathbf{O}_1 and \mathbf{O}_2 , we can define their sum $\mathbf{O}_1 + \mathbf{O}_2$ using standard vector addition. The magnitude of the net obligation is:

$$|\mathbf{O}_1 + \mathbf{O}_2| = \sqrt{|\mathbf{O}_1|^2 + |\mathbf{O}_2|^2 + 2|\mathbf{O}_1||\mathbf{O}_2|\cos\theta}$$

where θ is the angle between the obligations. When $\theta = 0$ (perfectly aligned obligations), the magnitudes simply add. When $\theta = \pi$ (directly opposing obligations), they subtract.

When $\theta = \pi/2$ (orthogonal obligations), the magnitude is $\sqrt{|\mathbf{O}_1|^2 + |\mathbf{O}_2|^2}$

—less than the sum but more than either alone.

This already captures something important: combining orthogonal obligations does not simply sum their demandingness. The geometry matters.

5.3 Rank-2: Moral Tensors and Relations

A rank-2 tensor has two indices and transforms as:

$$T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}$$

(for a type (2,0) tensor; other variance types transform accordingly). Such tensors naturally represent *relations* or *bilinear maps*. In physics, the stress tensor σ_{ij} gives the force per unit area in the i -direction on a surface with normal in the j -direction. The metric tensor $g_{\mu\nu}$ defines inner products between vectors.

In ethics, rank-2 tensors naturally represent *dyadic moral relations* between two entities—persons, actions, values, or times. Consider:

Responsibility. The moral responsibility of agent A for outcome O is not a scalar property of A alone, nor of O alone, but a relation between them. We can represent this as R^{AO} —a component of a rank-2 tensor. Different agent-outcome pairs have different responsibility values, and these values transform in specific ways under redescription of agents or outcomes.

Comparative value. The claim that outcome X is better than outcome Y might be represented not as a comparison of scalars (the "value" of X minus the "value" of Y) but as a component C^{XY} of an antisymmetric rank-2 tensor over the space of outcomes. Antisymmetry captures the logic of comparison: $C^{XY} = -C^{YX}$.

Interpersonal welfare comparisons. The relationship between person A 's welfare in state S and person B 's welfare in state S' might be encoded in a rank-2 tensor over persons-and-states, capturing the structure of interpersonal comparison without presupposing that there is a common scalar "utility" that both possess.

Let me develop the responsibility example in more detail. Define a space \mathcal{A} of agents and a space \mathcal{O} of outcomes. The responsibility tensor R is a bilinear map:

$$R: \mathcal{A}^* \times \mathcal{O}^* \rightarrow \mathbb{R}$$

where \mathcal{A}^* and \mathcal{O}^* are the dual spaces. In components, R^{ab} gives the responsibility of agent a for outcome b .

This framework immediately suggests structural properties. The tensor R can be decomposed into symmetric and antisymmetric parts:

$$R^{ab} = R^{(ab)} + R^{[ab]}$$

where $R^{(ab)} = \frac{1}{2}(R^{ab} + R^{ba})$ and $R^{[ab]} = \frac{1}{2}(R^{ab} - R^{ba})$.

If the agent and outcome spaces are identified (both are, say, events in the world), then the symmetric part $R^{(ab)}$ represents *mutual* or *shared* responsibility between events a and b , while the antisymmetric part $R^{[ab]}$ represents *directed* or *asymmetric* responsibility (agent-to-patient structure).

For a complete responsibility tensor over n agents and m outcomes, we have $n \times m$ components. This captures the intuition that responsibility is not a simple property of agents but a complex web of relations. The question "How responsible is Alice?" has no answer; we must ask "How responsible is Alice for *what*?"

Contraction and scalar extraction. Given the full rank-2 tensor R^{ab} , we can extract scalar quantities through contraction. If we have a vector v_a over agents (perhaps representing "degree of agency" or "moral weight"), we can form:

$$R^b = R^{ab}v_a$$

This gives a vector over outcomes: the "total responsibility" for each outcome, weighted by agent properties. Further contraction with an outcome vector w_b yields a scalar:

$$r = R^{ab}v_aw_b$$

This might represent the total responsibility in a situation, given weights on agents and outcomes.

The crucial point is that this scalar is *derived* from the tensor, not fundamental. Different weighting schemes (different v_a and w_b) yield different scalars from the same underlying tensorial reality. Disputes about "how much responsibility" there is in a situation may reflect different implicit weighting schemes rather than different assessments of the same quantity.

5.4 Higher Ranks and Complex Moral Structure

The pattern continues to higher ranks. A rank-3 tensor has three indices and can represent ternary relations; rank-4 tensors represent quaternary relations; and so on. In physics, the Riemann curvature tensor $R^\rho{}_{\sigma\mu\nu}$ is rank-4 and encodes the curvature of spacetime.

What moral quantities might require rank-3 or higher tensors? Consider:

Mediated obligation. The claim "A ought to help B because C has requested it" involves three parties. The full moral structure might be represented as a rank-3 tensor O^{ABC} over the space of persons, encoding how the obligation depends on all three parties and their relations.

Contextual value. The value of outcome X for person A in context C might be a component of a rank-3 tensor V^{XAC} , capturing how value depends on outcome, person, and evaluative context simultaneously.

Temporal moral relations. If we include time explicitly, many moral quantities gain an additional index. The claim "A's action at t_1 wrongs B at t_2 " involves four indices: agent, patient, and two times. This is naturally a rank-4 tensor.

Let me work out the mediated obligation example. Suppose we have three persons: A (the potential obligee), B (the potential beneficiary), and C (the requestor or mediator). The full structure of the mediated obligation can be represented as:

$$O^{ABC} \in \mathcal{P}^* \otimes \mathcal{P}^* \otimes \mathcal{P}^*$$

where \mathcal{P} is the space of persons. Different configurations have different values:

- O^{ABB} : A's obligation to B when B requests help for themselves (direct request)
- O^{ABC} with $C \neq B$: A's obligation to B when a third party C requests help for B (mediated request)
- O^{AAC} : A's obligation to themselves when C requests it (paternalism?)

The tensor structure allows us to ask: Is O^{ABB} stronger or weaker than O^{ABC} ? Does the presence of a mediator increase or decrease the obligation? These are questions about the structure of the tensor, not about scalar magnitudes.

We can also examine symmetries. Is $O^{ABC} = O^{ACB}$? This would mean that A's obligation to B at C's request equals A's obligation to C at B's request—a kind of reciprocity or symmetry in mediated obligations. Empirically (i.e., in our moral judgments), this may or may not hold, and the tensorial framework gives us precise vocabulary to articulate the difference.

5.5 Mixed Variance and the Metric

So far I have written tensors with all upper indices (contravariant). In the full tensorial framework, we distinguish contravariant indices (upper) from covariant indices (lower), and the metric tensor $g_{\mu\nu}$ mediates between them.

This distinction has moral significance. Contravariant and covariant quantities transform inversely under coordinate changes. In physics, position is contravariant while gradient (rate of change with position) is covariant. The distinction captures the difference between "things that live at points" and "things that measure change across points."

In ethics, we might distinguish:

Contravariant moral quantities: values, interests, welfare states—things that are located at points in moral space.

Covariant moral quantities: obligations, duties, requirements—things that measure how moral status changes as we move through moral space.

The metric tensor $g_{\mu\nu}$ on the moral manifold would then define how these are related. The claim that "welfare grounds obligation" might be formalized as a relationship mediated by the metric: the covariant obligation vector O_μ is related to the contravariant welfare vector W^ν via:

$$O_\mu = g_{\mu\nu} W^\nu$$

This is speculative, but it illustrates how the tensorial framework opens up structural questions that are invisible in scalar ethics.

5.6 Tensor Fields and Moral Variation

In physics, we typically work with tensor *fields*—tensors defined at each point of a manifold, varying smoothly from point to point. The stress at one location in a material differs from the stress at another; the curvature of spacetime varies across the cosmos.

Moral tensors are likewise *fields* over the moral manifold. The responsibility tensor for one situation differs from the responsibility tensor for another. The obligation vector at one point in moral space points in a different direction than the obligation vector at another point.

This introduces the apparatus of differential geometry: covariant derivatives, parallel transport, curvature. If I move through moral space—changing circumstances, encountering new information, shifting my situation—how do my obligations change? The

covariant derivative $\nabla_\mu O_\nu$ measures the rate of change of the obligation tensor as we move through moral space, accounting for the curvature of the space itself.

This is the subject of Chapter 4 (the moral manifold) and Chapter 6 (the moral metric), but I mention it here to emphasize that the tensors we have been discussing are not static objects. They are components of a rich geometric structure that varies across the space of moral situations.

5.7 An Extended Example: Distributive Justice

Let me work through an extended example to show how the tensorial framework applies to a substantive moral problem.

Consider the question of distributive justice: how should resources be distributed among persons? Classical approaches offer scalar answers: maximize total utility (utilitarianism), maximize the minimum share (maximin), equalize resources, equalize welfare, or various combinations.

A tensorial analysis begins differently. We first identify the relevant spaces:

- \mathcal{P} : the space of persons (with n persons, $\mathcal{P} \cong \mathbb{R}^n$ or a discrete set)
- \mathcal{R} : the space of resources or goods
- \mathcal{S} : the space of states of affairs (possible distributions)

A distribution is a tensor D^{ar} giving the amount of resource r held by person a . This is already rank-2, with one person-index and one resource-index.

Now consider the *moral evaluation* of a distribution. A scalar evaluation assigns a single number—the "justice" or "goodness" of the distribution. But this discards information. A tensorial evaluation might be:

J^{ab} : the degree to which the distribution is just with respect to persons a and b

This captures relational justice: the justice of how a and b stand relative to each other under the distribution. The full picture of justice is not a scalar but a rank-2 tensor over persons.

We can impose structure on this tensor. If justice is purely relational (only comparisons matter, not absolute levels), then J^{ab} should depend only on the difference or ratio of a 's and b 's holdings. If justice is symmetric in persons, then $J^{ab} = J^{ba}$. If justice satisfies transitivity, then certain consistency conditions on J^{ab} must hold.

The scalar "total justice" is then a contraction:

$$J = c_{ab} J^{ab}$$

where c_{ab} is a weighting tensor. Different theories of justice correspond to different choices of c_{ab} :

- Utilitarian: $c_{ab} = \delta^{ab}$ (only diagonal terms, weight each person equally, sum their individual welfare)
- Egalitarian: c_{ab} weights off-diagonal terms heavily (emphasize how persons stand relative to each other)
- Prioritarian: c_{ab} varies with welfare level (weight worse-off individuals more heavily)

The tensorial framework reveals that these are not simply different scalar answers but different ways of extracting a scalar from a common underlying tensorial structure. They agree on the tensor; they disagree on the contraction.

Furthermore, the tensorial framework suggests that the scalar may not be the right object of evaluation at all. Perhaps justice is irreducibly tensorial—a matter of how persons stand relative to each other that cannot be fully captured by any single number.

5.8 Summary and Transition

This chapter has argued that moral quantities come in various tensor ranks:

- **Scalars (rank-0):** Simple magnitudes of value, though these are rarer than commonly assumed
- **Vectors (rank-1):** Directed moral quantities like obligations and interests
- **Rank-2 tensors:** Dyadic relations like responsibility, comparative value, interpersonal welfare comparisons
- **Higher-rank tensors:** Complex moral structures involving multiple parties, contexts, times

The tensorial framework does not tell us *which* moral quantities exist or *what values* they take. That remains the work of substantive moral theory. But it provides a structural vocabulary for articulating moral theories with greater precision, revealing hidden assumptions (about symmetry, weighting, and aggregation), and identifying when scalar approaches are forcing tensorial phenomena into ill-fitting containers.

The next chapter takes up the question of the *moral metric*: the structure that defines distances and angles in moral space, allowing us to measure how "far apart" two moral

states are and whether two moral vectors point in the "same direction." This metric structure is essential for making the tensorial framework quantitatively tractable, and its specification involves deep normative choices that have been largely invisible in scalar ethics.

Technical Appendix to Chapter 5

For readers who want the mathematics stated more rigorously:

Definition 5.1. A *moral manifold* is a smooth manifold \mathcal{M} equipped with additional structure (to be specified in Chapter 4) representing the space of morally relevant situations.

Definition 5.2. A *moral tensor field of type (p, q) * is a smooth section of the bundle $T_q^p \mathcal{M} = TM^{\otimes p} \otimes T^*M^{\otimes q}$, where TM is the tangent bundle and T^*M is the cotangent bundle.

Definition 5.3. Under a coordinate transformation $x^\mu \mapsto x'^\mu$, a type (p, q) tensor $T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}$ transforms as:

$$T'^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \dots \frac{\partial x'^{\mu_p}}{\partial x^{\alpha_p}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \dots \frac{\partial x^{\beta_q}}{\partial x'^{\nu_q}} T^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q}$$

Definition 5.4. Given a type (p, q) tensor and a type (r, s) tensor, their *tensor product* is a type $(p + r, q + s)$ tensor defined by:

$$(S \otimes T)^{\mu_1 \dots \mu_p \rho_1 \dots \rho_r}_{\nu_1 \dots \nu_q \sigma_1 \dots \sigma_s} = S^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} T^{\rho_1 \dots \rho_r}_{\sigma_1 \dots \sigma_s}$$

Definition 5.5. *Contraction* reduces tensor rank by summing over a matched pair of upper and lower indices. For a type $(2, 0)$ tensor $T^{\mu\nu}$ and a metric $g_{\mu\nu}$, the contraction is:

$$T^\mu{}_\mu = g_{\mu\nu} T^{\mu\nu} = \sum_\mu T^\mu{}_\mu$$

This yields a scalar from a rank-2 tensor.

Proposition 5.1. If obligations are represented as vectors O^μ and interests as covectors I_μ , then the "satisfaction" of interest I by obligation O is the scalar:

$$S = I_\mu O^\mu$$

This is invariant under coordinate transformations.

Proposition 5.2. For a responsibility tensor R^{AB} over agents and outcomes, the "total responsibility" for outcome B is:

$$R^B = \sum_A R^{AB} = R^{AB} \delta_A$$

where the sum is over all agents. This is a vector over outcomes.

Proposition 5.3. The decomposition of a rank-2 tensor into symmetric and antisymmetric parts,

$$T^{ab} = T^{(ab)} + T^{[ab]}$$

is invariant under coordinate transformations. If T^{ab} represents a moral relation, the symmetric part $T^{(ab)}$ represents the mutual or shared component, and the antisymmetric part $T^{[ab]}$ represents the directed or asymmetric component.

End of Chapter 5