Tensorial Ethics

A Mathematical Framework for Moral Philosophy

By Andrew Bond

Preface

A brief account of how the idea emerged—perhaps from frustrations with scalar approaches to ethics (utilitarian calculus, single-metric optimization) and the recognition that moral reality seems to resist reduction to simple quantities. An acknowledgment that this work sits at an unusual intersection: differential geometry, physics, and moral philosophy.

Introduction: The Insufficiency of Scalar Ethics

The dominant quantitative approaches to ethics have largely been scalar—assigning single numerical values to outcomes, actions, or states of affairs. Utilitarianism asks us to maximize a quantity. Cost-benefit analysis reduces decisions to a single bottom line. Even when pluralistic in their inputs, these frameworks typically collapse moral reality into a one-dimensional output.

But consider: in physics, the move from scalars to vectors to tensors marked profound conceptual advances. A scalar tells you magnitude. A vector tells you magnitude and direction. A tensor tells you how quantities transform across different reference frames and coordinate systems.

What if ethics requires similar machinery?

This book argues that many persistent problems in moral philosophy—the incommensurability of values, the context-dependence of moral judgment, the perspectival nature of ethical claims, the difficulty of aggregating across persons—are symptoms of forcing tensorial phenomena into scalar containers.

Part I: Foundations

Chapter 1: A Primer on Tensors for Philosophers

An accessible introduction to tensor mathematics for readers without physics backgrounds. Covers:

- Scalars, vectors, and the intuition behind higher-rank tensors
- Coordinate systems and transformations
- Covariance and contravariance
- The metric tensor and the measurement of "distance"
- Tensor fields and their variation across manifolds

The goal is not technical mastery but conceptual fluency—enough to follow the philosophical arguments.

Chapter 2: Why Ethics Might Be Tensorial

The motivating intuitions:

- *Transformation behavior*: Moral claims seem to change form (not merely content) when evaluated from different positions, perspectives, or framings
- *Multi-index structure*: Many ethical concepts seem to require specification along multiple independent dimensions simultaneously
- *Frame-dependence and invariance*: Some moral features appear relative to perspective; others appear invariant. Tensors offer a natural way to model both.
- *Contraction and projection*: The process of arriving at a concrete moral judgment from abstract principles resembles tensor contraction—reducing dimensionality while preserving essential structure

Chapter 3: Historical Precursors

A genealogy of proto-tensorial thinking in ethics:

- Aristotle's doctrine of the mean as context-sensitive calibration
- Kant's categorical imperative as an invariance condition
- The intuitionist pluralism of Ross and the question of how prima facie duties interact
- Rawls's original position as a transformation-invariant framework
- Sen and Nussbaum's capabilities approach as implicitly multi-dimensional
- The recent work on moral uncertainty and its structural similarities to mixed states

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Part II: The Formal Framework

Chapter 4: The Moral Manifold

Introducing the base space over which ethical tensors are defined:

- What are the "points" of moral space? (Possible actions, states of affairs, persons, relationships?)
- Local vs. global structure
- Curvature: does moral space have intrinsic geometry?
- Boundaries, singularities, and moral dilemmas as topological features

Chapter 5: Ethical Tensors of Various Ranks

A taxonomy:

- Rank-0 (scalars): Simple magnitudes—utility, welfare, badness
- Rank-1 (vectors): Directed moral quantities—obligations as vectors pointing from current to required states; interests as directional
- Rank-2: Relations between moral agents, or between values; responsibility as a two-index object (who owes what to whom)
- Higher ranks: Complex moral situations involving multiple parties, multiple values, and multiple temporal frames

Chapter 6: The Moral Metric

How do we measure distance in moral space?

- What makes two actions morally "close" or "far apart"?
- Is there a natural metric, or must one be chosen (and what does choice imply)?
- Signature: Is moral space Euclidean, Riemannian, or Lorentzian? Does it contain "timelike" and "spacelike" separations (perhaps: reversible vs. irreversible moral differences)?

Chapter 7: Transformation Groups and Moral Invariance

What transformations should leave moral claims unchanged?

- Permutation of persons (anonymity conditions, impartiality)
- Temporal translation (does when something happens matter morally?)
- Perspective shifts (first-person to third-person; individual to collective)

- The structure of moral objectivity: invariants under all admissible transformations

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Part III: Applications

Chapter 8: Value Pluralism Reconsidered

The incommensurability problem dissolves if values are components of a tensor rather than competing scalars. Different values aren't rivals for a single crown—they're independent dimensions that combine through tensor operations. This chapter develops:

- How to represent irreducibly plural values
- When and how "comparison" can occur (through contraction, projection, inner products)
- Why tragic dilemmas persist even in a tensorial framework (and what this shows)

Chapter 9: Persons, Perspectives, and the Aggregation Problem

Standard aggregation (summing utilities) treats persons as interchangeable scalars. A tensorial approach:

- Each person as a distinct index
- Interpersonal comparison as cross-index operations
- The separateness of persons as resistance to full contraction
- Reimagining social choice theory in tensorial terms

Chapter 10: Moral Responsibility as a Tensor Field

Responsibility isn't a simple property of an agent; it's a relational structure with multiple indices:

- Agent, action, outcome, affected party, normative standard
- How responsibility "transforms" under different descriptions of the same event
- Collective responsibility and emergent tensorial structures

Chapter 11: Applied Cases

Brief applications to contested domains:

- Climate ethics and intergenerational obligation (temporal indices)

- Global justice (spatial and perspectival indices)
- AI alignment (what tensorial structure should we expect in artificial moral reasoning?)

Part IV: Objections and Horizons

Chapter 12: The Formalism Objection

Isn't this just mathematical window-dressing? A response:

- Formalism vs. modeling: the difference between empty notation and genuine structural insight
- What would it mean for ethics to "really be" tensorial?
- The pragmatic defense: even as a model, tensorial ethics may organize thought productively

Chapter 13: The Measurement Problem

We can't actually measure moral quantities with precision. Does that doom the project? This chapter argues:

- Physics faced analogous objections; utility theory faced them too
- Ordinal structure may suffice even without cardinal measurement
- The framework's value lies in structural relations, not numerical precision

Chapter 14: Open Problems and Future Directions

- Dynamics: Moral change over time; parallel transport of ethical judgment across contexts
- Quantum ethics? (Speculative: do moral superpositions make sense?)
- Computational tractability
- Empirical work: can we test whether human moral reasoning exhibits tensorial structure?

Conclusion: What the Framework Reveals

A recapitulation of the core claims and their implications. The tensorial framework doesn't solve ethics—it reframes it. It suggests that the search for a single moral truth was always misguided, not because morality is subjective, but because it's structurally richer than scalars can capture.

Appendices

A. Mathematical Formalism (rigorous definitions for specialists)
B. Notation and Conventions
C. Worked Examples

Bibliography

Notes on Tone and Approach

This structure assumes a serious philosophical monograph with genuine mathematical content—something aimed at readers comfortable with analytic philosophy and willing to engage with formalism. The MIT Press context suggests rigor is appropriate, but accessibility matters too; the primer chapter is crucial.

The risk with a project like this is vacuity—using tensors as a metaphor without real content. To avoid that, I'd suggest:

1. **Concrete examples throughout**: Every abstract claim should be illustrated with a moral case where the tensorial framing does work that scalar framing can't.
2. **Honest about limits**: Where does the analogy break down? What can't be captured?
3. **Engagement with existing pluralism literature**: Ross, Berlin, Raz, Chang—these thinkers have grappled with similar problems in different vocabularies.

Introduction: The Insufficiency of Scalar Ethics

The history of quantitative moral philosophy is largely a history of scalar thinking. From Bentham's felicific calculus to contemporary expected utility theory, the dominant impulse has been to assign single numerical values to the objects of moral concern—actions, outcomes, states of affairs, lives—and then to optimize, maximize, or satisfice with respect to these values. Even sophisticated pluralistic theories that acknowledge multiple values typically seek, at the moment of decision, to collapse this plurality into a single ordering or a single number. The question is always, in the end: *how much*?

This book argues that this scalar orientation, however natural and mathematically convenient, systematically distorts moral reality. The problems that have plagued quantitative ethics—the apparent incommensurability of values, the context-dependence of moral judgment, the perspectival character of ethical claims, the difficulty of aggregating welfare across persons while respecting their separateness—are not merely technical difficulties awaiting cleverer solutions. They are symptoms of a category error: the attempt to force inherently tensorial phenomena into scalar containers.

To see what this means, consider a parallel from physics. For centuries, physicists worked with scalar quantities like mass, temperature, and energy. These are numbers: they have magnitude but nothing more. The development of vector analysis in the nineteenth century marked a significant advance, allowing the representation of quantities with both magnitude and direction—velocity, force, electric field. But the deepest revolution came with the recognition that many physical quantities are neither scalars nor vectors but *tensors*: mathematical objects that encode how quantities transform when we change our coordinate system or frame of reference.

The stress at a point in a material, for instance, cannot be captured by a single number (a scalar) or even by an arrow (a vector). It requires specification of how force distributes across surfaces of different orientations—a rank-2 tensor with nine components in three dimensions. The curvature of spacetime in general relativity requires a rank-4 tensor. The insight is not merely that more numbers are needed, but that these numbers relate to each other in structured ways that encode transformation behavior—how the quantity appears from different perspectives, in different coordinate systems.

The central thesis of this book is that moral quantities exhibit analogous structure. Consider a simple case: the wrongness of a lie. A scalar approach assigns some number to this wrongness—perhaps derived from the harm caused, the violation of autonomy, or the breach of trust. But notice how the moral character of the lie transforms depending on the perspective from which it is evaluated: the liar's perspective (with access to intentions and perceived justifications), the deceived party's perspective (experiencing betrayal and damaged interests), a third party's perspective (observing violation of social norms), a retrospective perspective (knowing how things turned out). These are not merely different estimates of a single underlying magnitude. They are genuinely different moral aspects of the situation, irreducible to one another, yet systematically related through transformation rules.

Or consider the moral relationship between two persons, A and B. A may have obligations to B; B may have claims against A; both may share responsibilities to third parties that mediate their relationship; historical interactions may generate special duties; their positions in social structures may create role-based requirements. The moral content of "A's relation to B" is not a single quantity but a complex structure with multiple indices—agent, patient, context, normative framework, temporal frame—that transforms in specific ways when any of these indices are permuted or reinterpreted. This is precisely the signature of a tensor.

The Promissory Note

Before proceeding, I should be clear about the nature of the project. This book does not claim to have discovered that ethics *is* a branch of differential geometry, nor that moral truths can be read off from mathematical structures in the way that physical laws can (supposedly) be read off from the structure of spacetime. The relationship between the tensorial framework and moral reality is more subtle.

The claim is, first, *diagnostic*: that many persistent problems in moral philosophy arise from tacitly scalar assumptions that are poorly suited to the structure of ethical phenomena. When we try to compare values and find them incommensurable, we may be attempting an operation (scalar comparison) that is undefined for the quantities in question—like trying to ask whether a vector is greater or less than a different vector that points in an incomparable direction. When we try to aggregate welfare across persons and find ourselves unable to respect both total utility and the separateness of persons, we may be attempting to contract a tensor in a way that loses essential information. The tensorial framework helps us see *why* these problems arise and *what structure* a solution would need to have.

The claim is, second, *constructive*: that the mathematical resources of tensor analysis—transformation groups, invariants, metric structure, contraction operations—provide useful tools for articulating moral theories with greater precision and for identifying structural features that informal philosophical language tends to obscure. Just as the formalization of probability theory clarified problems of induction and decision, formalization via tensors may clarify problems of value pluralism, interpersonal comparison, and moral uncertainty.

The claim is, third, *heuristic*: that thinking tensorially suggests new questions, new distinctions, and new theoretical possibilities that might not arise within a purely scalar framework. What is the moral metric—the structure that determines distances and angles in moral space? What transformations leave moral claims invariant, and what does this invariance group tell us about the nature of moral objectivity? Are there tensorial identities—moral analogues to the Bianchi identities of differential geometry—that constrain the possible forms of ethical theory?

What the claim is *not* is reductionist. I do not think that moral reality is "nothing but" mathematical structure, or that the hard work of ethical reasoning can be replaced by calculation. The tensorial framework is a model, and like all models it abstracts and idealizes. But it is, I shall argue, a better model than the scalar frameworks that currently dominate—better in the sense of capturing more of the structure that matters while distorting less.

The Plan of the Book

The book proceeds in four parts. Part I establishes foundations: a mathematical primer on tensors for readers without physics backgrounds (Chapter 1), the philosophical motivations for thinking that ethics might be tensorial (Chapter 2), and a historical survey of proto-tensorial thinking in the moral philosophy tradition (Chapter 3).

Part II develops the formal framework. Chapter 4 introduces the moral manifold—the base space of points over which ethical tensors are defined, and considers what structure this space might have. Chapter 5 offers a taxonomy of ethical tensors by rank, from scalar quantities like undifferentiated welfare, through vector quantities like directed obligations, to higher-rank tensors encoding complex moral relationships. Chapter 6 addresses the moral metric: the structure that allows us to measure distances and angles in moral space, to say when two actions or outcomes are "close" or "far apart" in morally relevant respects. Chapter 7 examines transformation groups and the question of moral invariance—what remains constant when we shift perspectives, permute persons, or translate across time.

Part III turns to applications. Chapter 8 reconsiders the problem of value pluralism through a tensorial lens, arguing that incommensurability results from treating tensor components as competing scalars. Chapter 9 addresses the aggregation problem—how to combine welfare across persons while respecting their separateness—and shows how tensorial structure suggests alternatives to simple summation. Chapter 10 develops an account of moral responsibility as a tensor field, with indices for agent, action, outcome, affected party, and normative standard.

Part IV confronts objections and gestures toward future work. Chapter 11 addresses the formalism objection: the worry that this is all mathematical window-dressing with no real ethical content. Chapter 12 tackles the measurement problem: even if ethics is tensorial in structure, we cannot assign precise numerical values to moral quantities, so what use is the framework? Chapter 13 surveys open problems and speculative extensions.

A Note on Accessibility

This book is written for philosophers, not physicists or mathematicians. I assume familiarity with analytic philosophy and comfort with formal argumentation, but I do not assume prior knowledge of differential geometry or tensor analysis. Chapter 1 provides all the mathematical background needed to follow the arguments. Readers who want more rigor will find it in the appendices; readers who want to skip the mathematics and focus on the philosophical arguments will find that most chapters can be read with only qualitative understanding of the formal machinery.

My hope is that even readers skeptical of the entire project will find value in the reframings and distinctions the tensorial perspective affords. And perhaps some will be persuaded that the deep structure of moral thought has more in common with the geometry of spacetime than we have previously recognized.

Chapter 2: Why Ethics Might Be Tensorial

The Parable of the Old Man and His Horse

There is an ancient Chinese parable known as 塞翁失马 (*Sāi Wēng Shī Mǎ*)—"The Old Man at the Border Loses His Horse." It goes like this:

An old man living near the frontier lost his horse. His neighbors came to console him, but the old man said, "How do you know this isn't good fortune?"

Some months later, the horse returned, bringing with it a herd of fine wild horses. The neighbors came to congratulate him, but the old man said, "How do you know this isn't bad fortune?"

With so many horses, the old man's son took to riding. One day he fell and broke his leg. The neighbors came to console the old man, but he said, "How do you know this isn't good fortune?"

A year later, war came to the border. All the able-bodied young men were conscripted, and most died in battle. But the old man's son, with his broken leg, was spared.

The parable is usually read as a lesson in epistemic humility: we cannot know whether present events are good or bad because we cannot foresee their consequences. "Maybe" is the only honest answer.

But I want to suggest a different reading—one that reveals something structural about moral evaluation itself. The old man's "maybe" is not merely a confession of ignorance. It is a recognition that *scalar evaluation is the wrong tool for the job*.

When we say "losing the horse is bad," we are assigning a number—call it $S(x) = -1$—to the present state. When the horse returns with others, we revise: $S(x) = +3$. When the son breaks his leg, we revise again: $S(x) = -2$. And so on.

But notice what this scalar cannot represent:

1. **Which directions matter.** The loss of the horse is "bad" primarily along the *wealth* axis. It says nothing about the *health* axis, or the *family* axis, or the *political* axis (the son's eventual exemption from conscription). A scalar collapses all these dimensions into a single number, losing the information about *where* the badness lies.
2. **Where uncertainty concentrates.** The old man's uncertainty is not uniform. He is quite certain the horse is gone. What he is uncertain about is whether events will unfold along axes where the loss matters. Will famine come (making the lost horse

catastrophic)? Will the horse return (making the loss temporary)? Will war come (making his son's presence at home decisive)? The uncertainty has *shape*—it lies along some directions more than others.

3. **How evaluation changes along paths.** The moral status of "son breaks leg" depends on whether war is coming. The trajectory matters. Crossing from peacetime into wartime changes the evaluative landscape discontinuously—what was unambiguously bad (broken leg) becomes ambiguously fortunate (exemption from death). A scalar at a point cannot represent these *regime changes*.

The parable, I suggest, is pointing at a mathematical truth: **moral evaluation requires geometric structure that scalars cannot provide.**

The Insufficiency of Rank-0 Ethics

Let us be precise about what a scalar moral evaluation can and cannot do.

A *scalar* is a quantity fully specified by a single number. In ethics, scalar approaches assign a value—utility, welfare, goodness, rightness—to states of affairs, actions, or outcomes. The utilitarian calculus is scalar: it asks for the sum of pleasures minus pains, yielding a single number to be maximized. Cost-benefit analysis is scalar: all considerations reduce to a common currency. Even pluralistic theories that acknowledge multiple values typically seek, at the moment of decision, to collapse this plurality into a single ranking or a single number.

Formally, a scalar moral evaluation is a function:

$$S: \mathcal{M} \to \mathbb{R}$$

assigning to each point x in the moral space M a real number S(x). The defining feature of a scalar is *invariance*: under any coordinate transformation (any redescription of the situation), the value S(x) remains the same.

This invariance is both the strength and the weakness of scalar ethics. It is a strength because it promises objectivity: the goodness of a state should not depend on how we describe it. It is a weakness because it *loses information*: to achieve invariance, we must discard everything that varies with perspective.

The parable of the old man reveals three specific structural limitations:

Limitation 1: No Directional Information

A scalar S(x) tells us the magnitude of value at a point. It cannot tell us *which directions* in the moral space are responsible for that value, nor which directions would change the evaluation most dramatically.

Consider the moment the horse runs away. A scalar evaluation might say S = -1. But this number conceals the structure of the situation:

- Along the *wealth* dimension: strongly negative (valuable asset lost)
- Along the *labor* dimension: moderately negative (horse did farm work)
- Along the *health* dimension: neutral (no one is sick or injured)
- Along the *family* dimension: neutral (relationships unchanged)
- Along the *political* dimension: unknown (depends on future events)

A *vector* can represent this structure. Let v = (-0.8, -0.4, 0, 0, ?) be the "impact vector" of the horse's departure, with components along each morally relevant dimension. The scalar S = -1 is some contraction of this vector—perhaps its magnitude, or a weighted sum—but the vector itself contains information the scalar discards.

This matters because moral reasoning often requires knowing *which dimensions are engaged*. If a proposed remedy addresses the wrong dimension (say, offering emotional support when the problem is financial), it will be ineffective despite targeting the "badness." The vector structure tells us where to intervene; the scalar does not.

Limitation 2: Uncertainty Has Shape

The old man's "maybe" reflects uncertainty about the future. But his uncertainty is not uniform across all possibilities. It has *shape*: he is more uncertain about some developments than others, and—critically—his uncertainty is greatest along the dimensions that are most ethically decisive.

A scalar treatment of uncertainty adds error bars: S = -1 ± 0.5. This says the true value lies somewhere in the interval [-1.5, -0.5], but nothing about *why* we are uncertain or *where* the uncertainty matters.

A tensorial treatment represents uncertainty as a *covariance matrix* (or more generally, a rank-2 tensor) that encodes both the magnitude and the directional structure of our uncertainty:

$$\Sigma_{ij} = \langle (\delta m_i)(\delta m_j) \rangle$$

This tells us: uncertainty is large along axis i, small along axis j, and correlated between axes i and k.

In the parable, the old man's uncertainty might be:

- Small along the *current wealth* axis (the horse is definitely gone)
- Large along the *future wealth* axis (will more horses come?)
- Large along the *political* axis (will there be war?)
- Correlated between *political* and *son's welfare* (war affects conscription)

The crucial insight is that **uncertainty concentrated along ethically decisive directions matters more than uncertainty along irrelevant directions**. If the old man were uncertain about the color of next year's crops but certain about everything that affects his family's survival, the first uncertainty would be ethically negligible. But if he is uncertain precisely about war and conscription—the dimensions that determine whether the broken leg is a tragedy or a salvation—then his uncertainty is ethically maximal.

Scalar uncertainty ($S \pm \varepsilon$) cannot represent this. The covariance tensor Σ can.

Limitation 3: Paths Cross Boundaries

The most profound limitation of scalar evaluation is its inability to represent *trajectory-dependent* moral change, especially trajectories that cross *regime boundaries*.

In the parable, the evaluation of "son has a broken leg" depends on whether war comes. Before the declaration of war, a broken leg is unambiguously bad: pain, disability, inability to work. After war is declared, the evaluation bifurcates: for those without exemptions, conscription leads to probable death; for those with exemptions (including the son), survival is likely. The broken leg, unchanged in itself, has crossed a moral phase boundary.

This is not merely a matter of new information changing our estimate. It is a structural feature of the moral landscape: there exist *strata* (regimes, phases) within which smooth trade-offs apply, separated by *boundaries* where the rules change discontinuously.

A scalar function $S: M \to \mathbb{R}$, if it is continuous, cannot represent such discontinuities. It can represent gradual change—S increasing or decreasing smoothly—but not the sharp transitions that characterize moral thresholds: consent given vs. withheld, life vs. death, war vs. peace.

To represent regime boundaries, we need *stratified* spaces: spaces composed of smooth strata (within which scalar and vector calculus apply) joined along lower-dimensional boundaries (where discontinuities are permitted). And to represent how moral status evolves along paths that may cross these boundaries, we need *path-dependent* operations: parallel transport, holonomy, trajectory integrals.

What Tensorial Structure Provides

The three limitations point toward three geometric structures beyond scalars:

Limitation	Required Structure	Mathematical Object
No directional information	Vectors and covectors	∇S, O^μ, I_μ
Uncertainty has no shape	Covariance/correlation	Σ^{ij}, $G_{\mu\nu}$
No path-dependence	Stratification + transport	Strata, parallel transport, holonomy

Let us examine each.

Gradients and the Direction of Moral Change

If moral evaluation were purely scalar, there would be no meaningful sense of "direction" in moral space. But our actual moral reasoning is saturated with directional concepts: obligations *point* toward required states; interests *aim* at objects; responsibility *flows* from agents to patients; improvement *moves* toward better configurations.

These are not metaphors. They are descriptions of *vector* quantities—objects with both magnitude and direction.

Consider an obligation. "You ought to help your neighbor" is not merely a magnitude of oughtness. It specifies a *direction*: from the current state (neighbor unhelped) toward a required state (neighbor helped). The obligation can be stronger or weaker (magnitude), but it also has an orientation in the space of possible actions.

Formally, we can represent obligations as *vector fields* on the moral manifold:

$$O^\mu(x): \mathcal{M} \to T\mathcal{M}$$

At each point x in moral space, $O^\mu(x)$ is a tangent vector pointing in the direction of what is required.

Interests, conversely, can be represented as *covector fields*:

$$I_\mu(x): \mathcal{M} \to T^*\mathcal{M}$$

A covector (or 1-form) is a linear functional on vectors. The interest I_μ, applied to an obligation O^μ, yields a scalar: the *satisfaction* of interest I by obligation O.

$$S = I_\mu O^\mu$$

This is the fundamental formula of tensorial ethics: satisfaction is the contraction of obligations with interests. It is coordinate-invariant (a scalar), but it *arises from* vector quantities that carry directional information.

The gradient ∇S of the satisfaction function tells us: at this point in moral space, which direction increases satisfaction most rapidly? This is the direction of moral improvement—not a scalar claim ("things could be better") but a vector claim ("things could be better *in this specific way*").

The Metric Tensor and Moral Distance

To speak of directions, we need a way to compare them. To speak of distances, we need a way to measure them. In differential geometry, these functions are performed by the *metric tensor* $g_{\mu\nu}$.

The metric tensor is a rank-2 object that defines the inner product between vectors:

$$\langle u, v\rangle = g_{\mu\nu} u^\mu v^\nu$$

This allows us to say when two directions are orthogonal (their inner product is zero), when they are aligned (inner product is large and positive), and when they are opposed (inner product is negative). It also defines the length of vectors and the distance between points.

In moral space, the metric encodes *how we compare values*. Two values are orthogonal if trading off one against the other is undefined—there is no exchange rate between them. They are aligned if improving one tends to improve the other. They are opposed if they conflict.

The claim that some values are *incommensurable* is, in tensorial language, the claim that the moral metric is *degenerate* along certain directions—that there exist vectors v such that $g_{\mu\nu} v^\mu v^\nu = 0$, or that the metric blows up (becomes infinite) when we try to compare certain dimensions.

This is a structural claim, not a mystical one. It says that the geometry of moral space is not Euclidean—not all directions are comparable in the way that spatial directions are.

The Covariance Tensor and Structured Uncertainty

The old man's "maybe" reflects uncertainty that has directional structure. To represent this, we introduce a *covariance tensor*:

$$\Sigma^{ij} = \mathbb{E}[(\delta m^i)(\delta m^j)]$$

where δm^i is the deviation of the i-th moral dimension from its expected value.

This rank-2 tensor encodes:

- **Variance** along each axis: Σ^{ii} tells us how uncertain we are about dimension i
- **Covariance** between axes: Σ^{ij} ($i \neq j$) tells us whether uncertainty in dimension i correlates with uncertainty in dimension j
- **Principal directions**: the eigenvectors of Σ tell us the directions of maximum and minimum uncertainty

Ethically, what matters is the *alignment* between the uncertainty tensor and the gradient of moral value. If our uncertainty lies primarily along directions where the moral stakes are low, we can act confidently despite incomplete knowledge. If our uncertainty lies along the directions where moral stakes are highest, we should proceed with caution—or recognize, like the old man, that "maybe" is the honest answer.

The scalar quantity that captures this alignment is:

$$\sigma_S^2 = \Sigma^{ij} \frac{\partial S}{\partial m^i} \frac{\partial S}{\partial m^j}$$

This is the *variance of the moral evaluation* given structured uncertainty. It is large when uncertainty concentrates along morally decisive directions, and small when uncertainty lies along morally irrelevant directions.

Stratification and Moral Phase Transitions

The parable's most profound feature is the regime change brought by war. Before war is declared, the broken leg is bad. After war is declared, it becomes potentially good (exemption from death). This is not a gradual transition; it is a discontinuous jump at a boundary.

To represent such discontinuities, we need the apparatus of *stratified spaces*: spaces composed of smooth manifolds (strata) joined along boundaries where the smooth structure breaks down.

A stratified moral space M consists of:

1. **Strata** M_i: smooth manifolds of various dimensions, representing regimes within which ordinary calculus applies
2. **Boundary conditions**: rules for how strata are joined
3. **Discontinuous functions**: moral evaluations that are smooth on each stratum but may jump at boundaries

The boundary between "peacetime" and "wartime" is a moral stratum boundary. On either side, smooth trade-offs apply (more wealth is better, less pain is better). But crossing the boundary changes *which smooth trade-offs apply*. The rules are different.

This is why the old man cannot assign a stable scalar to his son's broken leg. The evaluation depends on which stratum the world occupies, and that is exactly what he is uncertain about.

The Parable Revisited

Let us return to the old man with the full apparatus of tensorial ethics.

Moment 1: The horse runs away.

The moral state is x_1. The impact lies primarily along the wealth dimension: obligation to provide for family is now harder to meet. The gradient ∇S points toward "recover the horse or find an alternative." The uncertainty tensor Σ is large along future-wealth and future-events axes—much could change.

A scalar evaluation says $S(x_1) \approx -1$. But this discards the directional structure. The old man, implicitly recognizing the tensor structure, says "maybe."

Moment 2: The horse returns with others.

The moral state is x_2. The impact is strongly positive along wealth. The gradient ∇S now points toward "maintain and increase this windfall." The uncertainty tensor remains large along future-events.

A scalar evaluation says $S(x_2) \approx +3$. The neighbors celebrate. The old man, still tracking the tensor structure, says "maybe."

Moment 3: The son breaks his leg.

The moral state is x_3. The impact is negative along health and capability. But now the uncertainty tensor Σ becomes crucial: there is high covariance between the *political* dimension (will there be war?) and the *welfare* dimension (will the son survive?).

Crucially, the son's condition now sits near a *stratum boundary*. If war comes, the moral evaluation of the broken leg will discontinuously shift. The gradient ∇S is undefined at the boundary—it points one way in peacetime, another way in wartime.

A scalar evaluation says $S(x_3) \approx -2$. The old man, sensing the proximity to regime change, says "maybe."

Moment 4: War comes; the son is spared.

The moral state crosses the boundary into wartime. The broken leg, unchanged in itself, is now on a different stratum. Its evaluation, relative to the counterfactual (able-bodied son conscripted and killed), is strongly positive.

The scalar is now $S(x_4) \approx +5$ or $+10$ (how do we quantify a life saved?). But this scalar conceals the path-dependence: the same physical state (broken leg) has different moral valence depending on which stratum it occupies.

Why "Maybe" Is Geometric, Not Merely Epistemic

The standard interpretation of the parable is epistemic: we should say "maybe" because we lack knowledge of the future. If only we knew whether war was coming, we could assign definite values.

But the tensorial interpretation suggests something deeper: "maybe" is the correct answer *even with perfect knowledge* when the evaluation structure is tensorial rather than scalar.

Suppose the old man had an oracle who told him exactly what would happen. Would he then assign a definite scalar to each moment?

Only if he were willing to commit to:

1. A fixed weighting of dimensions (how much does wealth matter vs. health vs. family?)
2. A fixed treatment of path-dependence (does the broken leg's value depend on the path through wartime, or just the final state?)
3. A specific contraction that collapses the tensor to a scalar

These choices are not determined by the facts. They are *perspective-dependent*—different agents, with different weights and different interests, will perform different contractions and arrive at different scalars.

The tensor is the invariant reality. The scalar is a projection, a shadow, a contraction that loses information. "Maybe" is what you say when you recognize that the tensor cannot be faithfully represented by any single scalar.

From Parable to Framework

The parable motivates the framework we will develop in subsequent chapters:

1. **Moral space has dimension** (Chapter 4). The space of morally relevant configurations is not a line (totally ordered by goodness) but a manifold of higher dimension, with independent axes for different values, agents, and considerations.
2. **Moral quantities are tensors** (Chapter 5). Obligations, interests, responsibilities, and evaluations are not scalars but tensors of various ranks, carrying directional and relational information that scalars discard.
3. **Moral space has a metric** (Chapter 6). The structure that allows comparison of values, measurement of moral distance, and identification of orthogonal (incommensurable) dimensions is a metric tensor $g_{\mu\nu}$.
4. **Moral space is stratified** (Chapter 4). The space is not uniformly smooth but divided into strata—regimes within which smooth trade-offs apply—separated by boundaries where rules change discontinuously.
5. **Moral transformation has structure** (Chapter 7). What happens to moral evaluations when we shift perspective, permute agents, or translate across contexts? The transformation behavior of tensors answers this question precisely.

The old man at the border, without the language of differential geometry, had the insight: moral reality is richer than any scalar can capture. What tensorial ethics provides is the mathematical apparatus to make that insight precise.

Anticipating Objections

The skeptical reader will have objections. Let me briefly anticipate two.

"Isn't this just saying ethics is complicated?"

No. "Complicated" suggests more of the same—more variables, more factors, more considerations to weigh. Tensorial structure is *different in kind*. A vector is not just a complicated scalar; it has properties (direction, transformation behavior) that scalars lack categorically. The claim is not that ethics has many dimensions (though it does) but that moral quantities *transform* in specific ways under change of perspective, and this transformation behavior is what tensors capture.

"We can't measure moral quantities precisely, so what use is this formalism?"

The same objection was raised against utility theory, and against the use of calculus in economics. The response is twofold. First, the framework's value lies in *structural* insights, not numerical precision. Knowing that two values are orthogonal (incommensurable) is useful even if we cannot measure their magnitudes exactly. Second, the framework identifies *what would need to be measured* to make ethical reasoning precise, even if current methods fall short. Physics progressed from qualitative insights ("force causes acceleration") to quantitative laws ($F = ma$) as measurement improved. Ethics might do the same.

These objections are serious enough to warrant dedicated chapters later in the book (Chapters 11 and 12). For now, the parable has done its work: it has shown that scalar ethics loses information that matters, and pointed toward the richer structures that might preserve it.

Conclusion

The old man's "maybe" is not resignation. It is recognition.

A scalar $S(x)$ can label the present, but cannot represent which directions are ethically decisive, whether uncertainty lies along those decisive directions, or how ethical status evolves along trajectories that cross stratification boundaries.

These requirements are naturally expressed by higher-order geometric objects:

- **Gradients** capture local fragility near moral phase transitions
- **Covariance tensors** encode uncertainty in morally relevant coordinates
- **Trajectory-level transport** captures path dependence

Without this higher-order structure, "maybe" must be bolted on as an ad hoc heuristic—a confession of ignorance—rather than emerging from the model as a structural feature of moral reality.

The parable, read tensorially, is not about the limits of human knowledge. It is about the geometry of ethical evaluation. The old man sees that the tensor cannot be contracted to a scalar without loss. His "maybe" is a holding operation—a refusal to project—until the full structure of the situation is revealed.

Tensorial ethics takes this insight and makes it mathematical. The chapters that follow develop the apparatus: the moral manifold, the tensor hierarchy, the metric, the transformations. But the core insight is here, in the parable: *ethics is not a number*. It is a geometric structure. And the first step in understanding that structure is recognizing what scalars cannot do.

The horse runs away. Good? Bad? Maybe.

The formalism agrees: the tensor is not yet fully contracted. Hold the projection. Watch the geometry unfold.

Chapter 3: Historical Precursors

Introduction: Tensors Before Tensors

The mathematical apparatus of tensor calculus was developed in the nineteenth and early twentieth centuries, reaching its canonical form in Einstein's general relativity. Moral philosophy, obviously, predates this development by millennia. Yet the *structural insights* that tensors formalize—transformation behavior, multi-dimensional interdependence, coordinate invariance, the distinction between intrinsic and perspectival properties—have appeared throughout the history of ethics in various guises.

This chapter traces a genealogy of proto-tensorial thinking in moral philosophy. The claim is not that Aristotle or Kant secretly knew tensor calculus, but that they grappled with phenomena that resist scalar treatment and developed conceptual tools that, in retrospect, capture aspects of tensorial structure. Reading these thinkers through a tensorial lens both illuminates their insights and shows that the framework developed in this book has deep roots in the philosophical tradition.

We proceed roughly chronologically, though the ordering also reflects increasing mathematical sophistication in the proto-tensorial concepts.

Aristotle: The Doctrine of the Mean as Context-Sensitive Calibration

Aristotle's *Nicomachean Ethics* presents virtue as a *mean* (μεσότης) between extremes of excess and deficiency. Courage lies between recklessness and cowardice; generosity between prodigality and miserliness; proper pride between vanity and undue humility. The virtuous person hits the mean "at the right times, with reference to the right objects, towards the right people, with the right motive, and in the right way" (1106b21).

This is emphatically not a scalar doctrine. Aristotle explicitly rejects the idea that virtue is a single quantity to be maximized:

"It is no easy task to find the middle… anyone can get angry—that is easy—or give or spend money; but to do this to the right person, to the right extent, at the right time, with the right motive, and in the right way, that is not for everyone, nor is it easy." (1109a26)

The mean is not a fixed point on a line but a *context-dependent calibration* across multiple dimensions. What counts as courage depends on the situation (battlefield vs. sickroom), the agent's role (soldier vs. physician), the stakes involved, and the alternatives available. The mean for one person in one situation may be quite different from the mean for another person in a different situation.

Tensorial Reading

In tensorial terms, Aristotle's mean can be understood as a *section* of a fiber bundle over the space of situations. At each point x in situation-space, there is a fiber of possible responses, and the virtuous response is determined by the local structure of the situation—not by a global, context-free rule.

More precisely, let S be the space of ethically relevant situations and let R be the space of possible responses. A *character trait* is a map $\sigma: S \to R$ assigning a response to each situation. The virtuous character trait σ^* is the one that, at each point, selects the response appropriate to that situation's specific configuration.

The "right time, right object, right person, right motive, right way" are *coordinates* on S. Virtue requires sensitivity to all of them. A scalar theory would say: "maximize courage" or "minimize cowardice." Aristotle says: the courageous response is a *function of the local coordinates*, not a global maximum.

This is proto-tensorial because it recognizes that ethical evaluation is:

1. **Multi-dimensional** (multiple "right X" conditions)
2. **Context-dependent** (the mean varies with situation)
3. **Not reducible to optimization of a single quantity**

What Aristotle lacks is the mathematical apparatus to describe how the mean *transforms* as we change coordinates—how the courageous response in one framing relates to the courageous response in another framing of the same situation. Tensor calculus provides exactly this.

The Doctrine of the Mean as a Metric Condition

There is another, deeper tensorial reading. The mean is defined relative to *us*—not the arithmetic mean of the extremes, but the mean "relative to us" (πρὸς ἡμᾶς). This suggests that the moral space has a *metric structure* that varies with the agent.

If we represent character traits as vectors in a space of dispositions, then "excess" and "deficiency" are directions away from the virtuous center. But what counts as excess depends on the metric: a step that is "too far" for one agent may be "not far enough" for another, because their metrics differ.

Formally, let $g_{\mu\nu}(a)$ be the metric tensor on disposition-space, parameterized by agent a. The mean for agent a is the point equidistant (under $g(a)$) from the extremes. Different agents, with different metrics, will locate the mean at different points.

This reading explains Aristotle's insistence that virtue cannot be taught by rule. Rules are coordinate-dependent; the mean is metric-dependent. Without knowing the agent's metric—their capacities, circumstances, history—one cannot specify the mean in advance.

Kant: The Categorical Imperative as an Invariance Condition

Kant's moral philosophy appears, at first glance, maximally anti-tensorial. The categorical imperative demands *universal* laws, applicable to all rational beings regardless of circumstance. "Act only according to that maxim whereby you can at the same time will that it should become a universal law" (Groundwork, 4:421). What could be more scalar than a single test applied uniformly to all actions?

But look again. The categorical imperative is not a command to maximize a quantity. It is a *constraint* on the form of permissible maxims: only those maxims that can be universalized without contradiction are morally permissible.

Tensorial Reading

In tensorial terms, the categorical imperative is an *invariance condition*. It asks: which maxims remain valid under a specific transformation—the transformation from "I, in my particular circumstances" to "any rational being in relevantly similar circumstances"?

Let T be the transformation that generalizes a maxim from first-personal to universal form. A maxim m is permissible if and only if:

$$T(m) = m$$

That is, the maxim is a *fixed point* of the universalization transformation. Maxims that change under T—that work for me but fail when universalized—are impermissible.

This is structurally identical to how physicists identify *scalars* (quantities invariant under coordinate transformations) and *tensors* (quantities that transform in specific lawful ways). Kant is asking: which moral claims are *invariant* under the transformation from particular to universal perspective?

The parallel is not superficial. In physics, the laws of nature must be the same in all reference frames—this is the principle of general covariance. In Kantian ethics, the laws of morality must be the same for all rational agents—this is the categorical imperative. Both are invariance conditions that constrain the form of legitimate laws.

The Kingdom of Ends as a Transformation Group

Kant's "kingdom of ends" deepens the tensorial reading. In the kingdom of ends, every rational being is both legislator (author of universal laws) and subject (bound by those laws). The moral community is defined by the *symmetry* between these roles.

In mathematical terms, the kingdom of ends is closed under the transformation that swaps legislator and subject. If a law L is valid, then the transformed law T(L)—where agent and patient are exchanged—must also be valid. This is a *symmetry condition* on the structure of moral laws.

Symmetry conditions of this form are the hallmark of tensor equations. Maxwell's equations are invariant under Lorentz transformations; Einstein's field equations are invariant under general coordinate transformations. Kant's moral laws are invariant under the permutation of rational agents.

What Kant identifies, without the mathematical language, is that *moral objectivity is transformation invariance*. A moral claim is objective not because it corresponds to some moral fact "out there," but because it remains valid under all admissible transformations of perspective.

Ross: Prima Facie Duties and the Problem of Tensor Combination

W.D. Ross's *The Right and the Good* (1930) introduced the concept of *prima facie duties*: duties that are binding unless overridden by stronger duties. We have prima facie duties of fidelity (keeping promises), reparation (making amends), gratitude, justice, beneficence, self-improvement, and non-maleficence.

These duties can conflict. A promise to meet a friend may conflict with an opportunity to prevent harm to a stranger. When they conflict, we must judge which duty is stronger *in this particular situation*—a judgment Ross calls the determination of our *actual duty*.

Tensorial Reading

Ross's prima facie duties are *components* of a moral vector. Each duty type corresponds to a dimension:

$$\mathbf{D} = (D_{fidelity}, D_{reparation}, D_{gratitude}, D_{justice}, D_{beneficence}, D_{improvement}, D_{nonmaleficence})$$

In any given situation, each component has some magnitude (possibly zero). The *actual duty* is some function of these components—but, crucially, not a simple sum.

Ross explicitly rejects the utilitarian move of reducing all duties to a single scalar (utility). He also rejects the idea that there is a fixed *priority ordering* among duty types. Instead, the determination of actual duty is a matter of *judgment* that weighs the components contextually.

In tensorial terms, Ross is grappling with the problem of *contraction*: how do we go from a multi-component vector to a single action-guiding prescription? His answer—that there is no mechanical rule, only trained judgment—reflects the fact that different situations call for different contraction operations.

The Interaction Problem

Ross's framework faces a difficulty: how do prima facie duties *interact*? If I have a strong duty of fidelity and a weak duty of beneficence, does the fidelity duty simply win? Or do they combine in some more complex way?

The tensorial framework suggests an answer. Duties are not merely magnitudes to be compared; they have *directions* in moral space. Two duties may be:

- **Aligned**: both point in the same direction (keeping a promise that also helps someone)
- **Orthogonal**: independent, neither reinforcing nor conflicting (a promise to one person, a beneficence opportunity involving another)
- **Opposed**: pointing in opposite directions (a promise that requires harming someone)

The combination of duties is then a *vector sum*, with the geometry determining how they add:

$$\mathbf{D}_{actual} = \sum_i \mathbf{D}_i$$

When duties are aligned, their magnitudes add. When orthogonal, they combine by the Pythagorean theorem. When opposed, they partially cancel.

This explains why strong orthogonal duties can coexist without conflict (I can keep my promise *and* help the stranger, if I have time for both), while even weak opposed duties create tension (breaking even a minor promise to prevent trivial harm still feels like a genuine moral loss).

Ross's "judgment" can now be understood as sensitivity to the *geometry* of the duty configuration in a particular case—something that resists codification in scalar terms but has definite structure.

Rawls: The Original Position as a Transformation-Invariant Framework

John Rawls's *A Theory of Justice* (1971) proposes that principles of justice are those that would be chosen by rational agents behind a "veil of ignorance"—not knowing their place in society, their natural talents, or their conception of the good. The original position is a thought experiment designed to identify principles that are fair because they are chosen without knowledge of how they will affect the chooser.

Tensorial Reading

The original position is a *symmetry condition*. By removing knowledge of particular position, it forces the choice of principles that are *invariant* under permutation of agents. If a principle benefits position A at the expense of position B, it cannot be chosen behind the veil, because the chooser might turn out to occupy position B.

Formally, let π be a permutation of social positions. A principle P is admissible in the original position if and only if:

$$\pi(P) = P \text{ for all permutations } \pi$$

This is precisely the condition for a *symmetric tensor*—a tensor that is unchanged under index permutation.

Rawls's two principles of justice can be understood as the *unique symmetric solution* (up to specification) to the problem of social cooperation. The first principle (equal basic liberties) is symmetric by construction: everyone gets the same liberties. The second principle (difference principle) permits inequalities only if they benefit the worst-off position—which is a *symmetric* condition because the worst-off position is defined relative to the structure of positions, not to any particular occupant.

The Metric of the Original Position

The original position also implicitly specifies a *metric* on social positions. The difference principle uses a *maximin* criterion: maximize the minimum position. This is equivalent to a metric in which distance is measured by the worst-off coordinate.

In tensorial terms, the Rawlsian metric is:

$$d(x,y) = \max_i |x_i - y_i|$$

This is the *supremum metric* (or L^∞ metric), which gives special weight to the worst-off dimension. Alternative metrics would yield different principles:

- The *utilitarian metric* (L^1): d(x,y) = Σ|x_i - y_i|, giving equal weight to all positions
- The *Euclidean metric* (L^2): d(x,y) = √(Σ(x_i - y_i)^2), weighting by squared deviations

Rawls's argument against utilitarianism can be read as an argument about *which metric* is appropriate for justice. The utilitarian metric permits sacrificing some positions for aggregate gain; the Rawlsian metric does not. This is a substantive geometric claim about the structure of fair social evaluation.

Sen and Nussbaum: Capabilities as a Basis for Moral Space

Amartya Sen and Martha Nussbaum developed the *capabilities approach* as an alternative to both utilitarian welfare measures and Rawlsian primary goods. The core idea is that what matters morally is not subjective well-being (utility) or objective resources (income, rights) but *capabilities*: the real freedoms people have to achieve "functionings" they have reason to value.

Sen identifies a plurality of capabilities: life, bodily health, bodily integrity, senses/imagination/thought, emotions, practical reason, affiliation, relation to other species, play, and control over one's environment (political and material). These are *irreducibly plural*—they cannot be reduced to a single scalar measure.

Tensorial Reading

The capabilities are *basis vectors* for moral space. Each capability defines an independent dimension along which a person's life can go well or badly. A person's overall situation is a *vector* in capability space:

$$\mathbf{c} = (c_{life}, c_{health}, c_{integrity}, c_{senses}, c_{emotions}, c_{reason}, c_{affiliation}, c_{nature}, c_{play}, c_{control})$$

This is explicitly multi-dimensional. Sen insists that capabilities cannot be aggregated into a single index without loss of essential information—precisely the claim that moral evaluation is tensorial, not scalar.

The Incompleteness Thesis

Sen argues that comparative judgments of capability sets are *incomplete*: we can often say that one situation is better than another along some dimensions and worse along others, without being able to say which is better overall. This incompleteness is not a failure of the theory but a feature of moral reality.

In tensorial terms, this is the claim that the moral metric is *degenerate* or *partial*. Not all vectors can be compared in length. Given two capability vectors c_1 and c_2, we may have:

- $c_1 > c_2$ along some dimensions
- $c_1 < c_2$ along other dimensions
- No basis for overall comparison

This is the tensorial signature of *incommensurability*. A scalar theory would force a comparison (by summing or by lexical priority); Sen's theory preserves the genuine incompleteness of the moral situation.

Nussbaum's Threshold and Stratum Structure

Nussbaum modifies the capabilities approach by introducing *thresholds*: minimum levels of each capability below which a life is not fully human. This introduces *stratum structure* into capability space.

Below the threshold, we are in a different moral regime—one where the imperative is to raise capabilities to the threshold level. Above the threshold, trade-offs and choices become permissible. The threshold is a *stratum boundary* separating regions with different moral rules.

This is proto-stratified geometry. Nussbaum's capability space is not a smooth manifold but a stratified space with distinguished hypersurfaces (the thresholds) where the moral rules change discontinuously.

Moral Uncertainty: Mixed States and Superposition

Recent work in moral philosophy has focused on *moral uncertainty*: what should we do when we are uncertain which moral theory is correct? If I am 60% confident in utilitarianism and 40% confident in deontology, and they recommend different actions, what should I choose?

Various approaches have been proposed: "my favorite theory" (act on whichever theory you find most plausible), "maximize expected moral value" (weight each theory's recommendation by your credence in it), and more sophisticated methods that account for intertheoretic comparisons.

Tensorial Reading

The structure of moral uncertainty is strikingly similar to *quantum superposition*. An agent under moral uncertainty is not in a definite moral state but in a *superposition* of moral states, weighted by credence.

Let |U⟩ be the state "utilitarianism is correct" and |D⟩ be the state "deontology is correct." An agent with 60% credence in utilitarianism is in the state:

$$|\psi\rangle = \sqrt{0.6}|U\rangle + \sqrt{0.4}|D\rangle$$

This is a vector in a *theory space*, not a scalar. The agent's moral situation cannot be captured by a single number (overall credence) but requires specification of the full vector.

When the agent acts, the superposition "collapses" to a definite choice—but the choice reflects the full vector structure. Expected value maximization is one way to perform this collapse (a specific *contraction* operation); other approaches perform different contractions.

Moral Hedging as Covariance

The sophisticated treatment of moral uncertainty involves *hedging*: choosing actions that are reasonably good under multiple theories, even if not optimal under any. This is analogous to *portfolio diversification* in finance—choosing investments that reduce variance across states of the world.

In tensorial terms, hedging is sensitivity to the *covariance structure* of moral uncertainty. If my uncertainty is concentrated in ways that some actions are robust to, I can act confidently. If my uncertainty lies along the dimensions that differentiate the recommended actions, I should hedge.

Let Σ be the covariance matrix of my moral beliefs (encoding correlations between credences in different theories). Let Δa = (a_U - a_D) be the vector of differences between what each theory recommends for action a. The "risk" of action a is:

$$\sigma_a^2 = \Delta_a^T \Sigma \Delta_a$$

Actions with low σ^2 are robust to moral uncertainty; actions with high σ^2 are risky bets on particular theories being correct.

This is the moral analogue of the portfolio variance formula in finance—and it requires the full tensor structure of uncertainty, not just scalar credences.

Synthesis: What the Precursors Share

Across two and a half millennia, these thinkers share a recognition that moral reality resists scalar reduction:

Thinker	**Insight**	**Tensorial Structure**
Aristotle	The mean is context-dependent	Evaluation as section of fiber bundle; agent-relative metric
Kant	Morality requires universalizability	Permissible maxims as transformation invariants
Ross	Duties are plural and interacting	Duties as vector components; combination as vector sum
Rawls	Justice requires position-independence	Principles as symmetric tensors; metric choice determines theory
Sen/Nussbaum	Capabilities are irreducibly plural	Capabilities as basis vectors; incompleteness as metric degeneracy
Moral uncertainty	Credences have structure	Beliefs as state vectors; hedging from covariance structure

None of these thinkers used the language of tensors. But they all developed conceptual tools to handle phenomena that tensors formalize:

1. **Multi-dimensionality**: Moral evaluation involves multiple independent considerations that cannot be reduced to one.
2. **Transformation behavior**: What happens to moral claims when we shift perspective, permute agents, or change framing?
3. **Metric structure**: How do we compare values, measure moral distance, identify incommensurability?
4. **Context-dependence**: The same abstract principle yields different concrete prescriptions in different situations.

5. **Structured combination**: When moral considerations combine, they do so geometrically (with alignment, orthogonality, opposition), not arithmetically.

What the Tensorial Framework Adds

If these insights are already present in the tradition, what does the tensorial framework add?

Precision. The tradition offers metaphors and intuitions; the framework offers definitions and theorems. "Duties interact" becomes "duties combine as vectors under the moral metric." "The mean is relative to us" becomes "virtue is determined by an agent-parameterized metric on disposition-space." Precision enables analysis, criticism, and extension.

Unification. The tradition offers disparate insights from incompatible systems. The tensorial framework reveals common structure beneath surface disagreement. Aristotle's context-sensitivity and Kant's universalizability are not opposed but complementary: both constrain the transformation behavior of moral claims, in different ways.

New questions. The framework suggests questions the tradition did not ask. What is the *signature* of the moral metric—is it positive-definite (Euclidean), indefinite (Lorentzian), or degenerate? What are the *symmetries* of moral space—which transformations leave the structure invariant? What *curvature* does moral space have—how does parallel transport around a loop change moral vectors?

Computability. Finally, and most relevant to the application of this framework to artificial systems, tensorial ethics is *computable*. Tensors can be represented in computers; tensor operations can be implemented in algorithms; tensor equations can be solved numerically. The tradition offers wisdom; the framework offers implementation.

Conclusion: Tensors as the Language of Moral Structure

The history of moral philosophy is, in significant part, a struggle against the reductionism of scalar ethics. Each thinker we have examined recognized that moral reality has structure that scalars cannot capture, and developed tools to articulate that structure.

Tensorial ethics is not a break from this tradition but its continuation—and, in a sense, its completion. The conceptual tools developed by Aristotle, Kant, Ross, Rawls, Sen, Nussbaum, and theorists of moral uncertainty find their natural mathematical expression in the language of tensors, metrics, transformations, and stratified spaces.

This is not to say that the tradition was secretly doing mathematics. It is to say that mathematics, properly understood, is the science of structure—and the structures that matter for ethics are precisely those that tensor calculus was developed to describe: transformation behavior, multi-linear combination, coordinate invariance, metric geometry.

The tradition provides the insights. The framework provides the language. Together, they enable a moral philosophy that is both faithful to the complexity of ethical life and precise enough to be implemented, tested, and refined.

In the chapters that follow, we develop the framework in full. But we do so in the company of these predecessors, whose insights we are formalizing, not replacing.

Chapter 4: The Moral Manifold

Introduction: The Question of Base Space

Every tensor lives on a manifold. The stress tensor in materials science lives on the manifold of spatial points within a body. The metric tensor in general relativity lives on the manifold of spacetime events. The question for tensorial ethics is: *what is the manifold on which moral tensors are defined?*

This is not a technical detail. The choice of base space determines:

- What counts as a "point" in moral reasoning
- What transformations are admissible
- What it means for two moral situations to be "nearby" or "far apart"
- Where discontinuities and singularities can occur

This chapter develops the concept of the *moral manifold*—the space of morally relevant configurations over which ethical tensors are defined. We proceed carefully, distinguishing geometric structure from metaphysical commitment, and being explicit about what our framework can and cannot do.

4.1 What Are the Points?

The first question is fundamental: what are the "points" of moral space?

Several candidates present themselves:

Candidate 1: Possible Actions

On this view, the moral manifold M is the space of possible actions available to an agent. Each point $x \in M$ represents a complete specification of what the agent does—not just "help the neighbor" but a full description including manner, timing, motivation, and consequences.

Strengths: This connects directly to the practical question of ethics ("what should I do?"). The gradient of satisfaction, ∇S, points toward the best action.

Weaknesses: Actions are agent-relative. My action space differs from yours. This makes interpersonal comparison difficult and global structure unclear.

Candidate 2: States of Affairs

On this view, points are possible states of the world—complete descriptions of how things are or could be. Actions are then paths or vectors, not points: an action takes the world from one state to another.

Strengths: This is agent-neutral. All agents evaluate the same space. It connects to consequentialist intuitions: what matters is the state of the world, not who brings it about.

Weaknesses: The space is enormous—potentially infinite-dimensional. And it privileges states over processes, which deontologists will resist.

Candidate 3: Situations

A *situation* is richer than a state: it includes not just how things are, but the relationships between agents, their histories, their commitments, and the options available. A situation is something like "Alice has promised Bob X, Bob is in need of Y, Alice can provide Y but only by breaking the promise to Bob, Carol is watching."

Strengths: This captures the morally relevant features without requiring a full specification of the universe. It's closer to how moral reasoning actually works.

Weaknesses: What counts as "morally relevant" is theory-dependent. Different ethical theories may carve situation-space differently.

Candidate 4: Agent-Situation Pairs

On this view, a point is a pair (a, s): an agent a in a situation s. This allows agent-relative evaluations while maintaining a common framework.

Strengths: Captures partiality and impartiality as different operations on the space (holding a fixed vs. varying a). Connects to the agent-indexed tensors of Chapter 8.

Weaknesses: More complex structure; requires specifying how the agent index interacts with the situation index.

Our Choice: Structured Situations

For the purposes of this book, we take the moral manifold M to be a space of *structured situations*—specifications of:

1. The agents involved and their relationships
2. The options available to each agent
3. The morally relevant features of the context (needs, promises, histories, stakes)
4. The epistemic state (what is known, by whom)

This is deliberately ecumenical. A consequentialist can project onto states; a deontologist can focus on the structure of relationships and commitments; a virtue ethicist can attend to the character of the agents. The manifold M is the common ground; different theories correspond to different tensors, metrics, and contractions on M.

Definition 4.1 (Moral Manifold, Informal). *The moral manifold M is a topological space whose points represent structured moral situations—complete specifications of agents, relationships, options, and morally relevant context.*

We will make this more precise as needed, but the key point is that M is *not* identified with any single candidate above. It is a space rich enough to support all of them as substructures.

4.2 Coordinates and Transformations: A Discipline

The Challenge

In physics, coordinate transformations have a precise meaning: a change from one coordinate chart to another covering the same region of manifold. The transformation laws for tensors tell us how components change under such redescription.

In ethics, we speak loosely of "different perspectives," "different framings," "different descriptions." But not all of these are coordinate transformations in the geometric sense. Conflating them invites the criticism that tensorial ethics is vacuous—that by choosing the right "coordinate system," we can make any answer come out.

This section introduces discipline. We distinguish three types of transformation, only the first of which is a coordinate change in the strict sense.

Type 1: Coordinate Redescriptions (Chart Changes)

A *coordinate redescription* is a change in how we parameterize the same underlying situation. The situation itself is unchanged; only our labels differ.

Example: Describing a resource allocation in terms of "amount to Alice" vs. "amount to Bob" (where total is fixed, so $x_{Bob} = T - x_{Alice}$). These are different coordinates on the same one-dimensional manifold.

Example: Describing an action as "withholding treatment" vs. "allowing natural death." If these are genuinely synonymous—if they pick out exactly the same action in all morally relevant respects—then they are coordinate redescriptions.

The tensorial requirement: Any moral quantity that is a genuine feature of the situation must transform appropriately under coordinate redescriptions. Scalars are invariant; vectors transform by the Jacobian; and so on.

What this rules out: It rules out moral evaluations that depend on *mere labeling*. If "allowing to die" sounds better than "withholding treatment" but they denote the same action, a proper moral evaluation should not distinguish them. This is the ethical analogue of general covariance in physics.

Type 2: Perspective Shifts (Agent Transformations)

A *perspective shift* changes the evaluating agent while holding the situation fixed. This is *not* a coordinate change on M—it is a change in the index on agent-relative tensors.

Example: Evaluating the kidney allocation from the physician's perspective vs. a patient's family's perspective. The situation (who the patients are, what their conditions are) is the same. What changes is *who is evaluating*.

Tensorial treatment: Perspective shifts are transformations on the *agent index*, not on the manifold coordinates. The rank-2 tensor M_{ia} (evaluation of option i by agent a) transforms in the agent index when we change which perspectives we're considering.

What this clarifies: The claim "morality is objective" does not mean all perspectives must agree. It means there are perspective-invariant *structures*—perhaps certain constraints, or the shape of disagreement itself—that remain stable across perspective shifts.

Type 3: Theory Shifts (Structural Transformations)

A *theory shift* changes the mathematical structure we impose on M: the metric, the connection, the admissible transformations themselves.

Example: Switching from a utilitarian metric (all dimensions commensurable) to a lexicographic metric (some dimensions have priority). This is not a coordinate change; it is a change in the *geometry* of moral space.

Example: Switching from a theory that treats agent identity as morally irrelevant (impartialism) to one that treats it as relevant (partiality). This changes which transformations are admissible.

Tensorial treatment: Theory shifts are *not* symmetries—they change the structure. Different theories correspond to different choices of metric g, connection ∇, and admissibility constraints. The tensorial framework does not adjudicate between theories; it represents each theory precisely, allowing explicit comparison.

Summary: The Transformation Hierarchy

Type	What Changes	Status	Example
Coordinate redescription	Labels/parameterization	True symmetry; tensors must transform appropriately	"x_Alice" vs. "T - x_Bob"
Perspective shift	Evaluating agent	Index transformation on agent-relative tensors	Physician vs. family evaluation
Theory shift	Metric, connection, structure	Not a symmetry; changes the geometry	Utilitarian vs. lexicographic metric

The discipline: When we say "moral claims should be invariant under redescription," we mean Type 1 transformations. Types 2 and 3 are not symmetries in this sense—they are legitimate sources of variation that the framework makes explicit.

4.3 Local vs. Global Structure

Local Structure: The Tangent Space

At each point $x \in M$, there is a *tangent space* T_xM representing the infinitesimal directions one can move from x. In moral terms, the tangent space contains the *possible variations* in the situation: small changes in resource allocation, slight modifications of action, marginal increases in risk.

Moral vectors—obligations, interests, gradients of value—live in tangent spaces. The obligation at x is a vector $O(x) \in T_xM$ pointing in the direction one *ought* to move.

Local structure is what we can see from a single point and its immediate neighborhood. It includes:

- The dimension of the tangent space (how many independent directions of variation exist)
- The metric at that point (how we measure distances and angles between nearby situations)
- The satisfaction gradient (which direction improves moral status)

Global Structure: Topology and Connectedness

Global structure concerns the manifold as a whole:

- **Connectedness:** Can any two situations be reached from each other by a continuous path? Or are there disconnected components (perhaps corresponding to incommensurable forms of life)?
- **Compactness:** Are there "limits" to moral space, or does it extend indefinitely?
- **Holes and handles:** Are there non-trivial loops in moral space? Can circling back to the same situation yield a different moral evaluation (holonomy)?

The global structure of M is an empirical question about ethics, not a mathematical stipulation. If there are genuine *discontinuities* in moral evaluation—situations where small changes produce large jumps in moral status—these appear as features of M's topology.

Moral Curvature

In differential geometry, *curvature* measures how much a space deviates from flatness. A flat space is one where parallel lines stay parallel; a curved space is one where they converge or diverge.

Does moral space have curvature?

Consider: the obligation at x points in direction O(x). If we move to a nearby point y, the obligation there is O(y). In flat space, we can compare O(x) and O(y) directly—they live in "the same" vector space. In curved space, comparison requires *parallel transport*: moving O(x) along a path to y and seeing how it differs from O(y).

Moral curvature would mean: the way obligations vary across moral space is path-dependent. The obligation you arrive at by going $x \to z \to y$ might differ from the obligation you arrive at by going $x \to w \to y$, even if both paths end at the same point y.

Is this plausible? Consider a case: you promise Alice to meet her, then encounter Bob in need. Your obligation to Bob depends on whether you're currently in the "promise to Alice" context or not. The moral landscape is shaped by prior commitments in a way that makes obligation path-dependent.

I do not claim this conclusively establishes moral curvature. But it suggests curvature is at least conceivable, and the framework is prepared to represent it if it turns out to be real.

4.4 Boundaries and Singularities

Stratum Boundaries

The most important structural feature for ethics is not curvature but *stratification*: the division of M into regions (strata) where different rules apply.

Within a stratum, smooth trade-offs are possible. You can exchange a little more of X for a little less of Y, and the moral evaluation varies smoothly. But at stratum boundaries, something changes discontinuously:

- **Threshold effects:** Consent at 0.99 is different from consent at 1.0; the distinction between "insufficient" and "sufficient" consent is a boundary.
- **Categorical distinctions:** The difference between killing and letting die, between lying and remaining silent, between a 17-year-old and an 18-year-old—these are boundaries where legal and moral rules change.
- **Veto regions:** Certain options are simply forbidden—not just low-value, but excluded from consideration. The boundary of the forbidden region is a stratum boundary.

Definition 4.2 (Stratum Boundary). *A stratum boundary $B \subset M$ is a lower-dimensional submanifold where the moral evaluation function S, or the structure of the moral metric g, fails to be smooth.*

Crossing a stratum boundary can produce:

- Discontinuous jumps in S (from permissible to forbidden)
- Changes in the effective dimension (some options disappear)
- Changes in the applicable rules (different duties become operative)

Singularities and Dilemmas

A *singularity* in M is a point where the normal geometric structure breaks down. In physics, singularities are points of infinite density or undefined curvature. In ethics, they are *genuine dilemmas*: situations where the framework cannot deliver a determinate answer, not because of ignorance, but because of structural conflict.

Example (Sophie's Choice): A mother must choose which of her two children will be killed. There is no "right" answer. Any choice involves irreducible moral loss. The situation is *singular*: the satisfaction function S has no maximum, or has multiple maxima with incomparable moral residue.

Tensorial representation: A singular point is one where the metric g becomes degenerate ($\det(g) = 0$), or where the satisfaction gradient ∇S is undefined, or where multiple constraint surfaces intersect in pathological ways.

Singularities are not bugs in the framework—they are features. The framework *represents* genuine dilemmas as singular points, rather than forcing a false determinacy. This is an advance over scalar approaches, which must either deliver an answer (by arbitrary tie-breaking) or fall silent (by declaring the situation beyond their scope).

The Forbidden Region

A special kind of boundary is the *constraint boundary*: the edge of the region where options are permissible.

Definition 4.3 (Constraint Set). *The constraint set $C \subset M$ is the set of points (situations, actions) that are absolutely forbidden—moral non-starters regardless of other considerations.*

Within C, the satisfaction function $S = -\infty$ by convention. The boundary ∂C is where S transitions from finite values to negative infinity. This is a hard discontinuity.

Examples of constraint boundaries:

- Actions involving non-consensual harm to innocents
- Violations of absolute rights
- Discrimination on protected characteristics

The tensorial framework does not determine *what* goes in C—that is the task of normative ethics, democratic deliberation, and institutional governance. But it represents the *structure* of constraints precisely: as a region with hard boundaries, distinguished from regions of trade-offs.

4.5 An Example: The Manifold of a Medical Decision

Let us construct M explicitly for a simplified medical decision.

Situation: A physician must allocate a scarce treatment among three patients (A, B, C). Each patient has:

- A medical benefit score $\beta_i \in [0, 1]$
- A waiting time $w_i \in [0, \infty)$
- An age $a_i \in [0, 120]$
- A number of dependents $d_i \in \{0, 1, 2, \ldots\}$

The Manifold: The space of possible allocations is the 2-simplex: $\Delta^2 = (p_A, p_B, p_C): p_i \geq 0, \sum_i p_i = 1$

where p_i is the probability (or fraction) of treatment allocated to patient i.

Strata:

- *Interior* (dim = 2): Allocations where all patients have positive probability. Smooth trade-offs possible.
- *Edges* (dim = 1): Allocations between two patients, one excluded.
- *Vertices* (dim = 0): Deterministic allocations. These are the *actual decisions*; the interior represents deliberation space.

Constraint region: Suppose Patient C's allocation would involve discrimination (the decision is based on a protected characteristic). Then the region p_C > 0 is forbidden: $C = (p_A, p_B, p_C): p_C > 0$

The feasible region is the edge from vertex A to vertex B.

Metric: Different metrics on Δ^2 correspond to different ethical theories:

- *Euclidean metric:* All movements equally costly. Trading probability from A to B costs the same as from B to A.
- *Weighted metric:* Movement toward sicker patients is "easier" (lower moral cost) than movement away.
- *Lexicographic metric:* The allocation must first maximize benefit to the worst-off; only then consider secondary criteria.

Satisfaction function: S: $\Delta^2 \to \mathbb{R}$ assigns a moral score to each allocation. On the interior, S is smooth (we can improve by tilting probability toward more deserving patients). On the constraint boundary, S = -∞.

This simple example illustrates all the features of a moral manifold: points, strata, boundaries, metrics, and the interplay between local trade-offs and global constraints.

4.6 What Tensorial Structure Does and Does Not Provide

What It Provides

1. **Precision about structure.** The framework makes explicit what is often implicit: the dimensionality of moral space, the presence of constraints, the metric that governs

trade-offs. Disagreements can be localized: do we disagree about the metric, the constraint set, or the contraction?

2. **Representation of hard cases.** Singularities and stratum boundaries are *represented*, not wished away. The framework can say "this is a genuine dilemma" in a precise way.
3. **Theory comparison.** Different ethical theories become different geometric structures on the same underlying manifold. We can ask precisely how they differ and where they agree.
4. **Computability.** Manifolds, tensors, and constraint regions can be implemented. This enables machine ethics that is explicit about its assumptions.

What It Does Not Provide

1. **Metaethical grounding.** The framework does not answer: "Why be moral?" or "What makes moral claims true?" It represents moral structure; it does not ground normativity.
2. **Content determination.** The framework does not tell you what the constraint set C should be, or what metric g is correct. That remains the work of normative ethics, empirical moral psychology, and democratic deliberation.
3. **Motivation.** Knowing the structure of moral space does not, by itself, motivate moral action. The question of moral psychology—why we care about morality—is orthogonal to the question of moral structure.
4. **Resolution of all disagreement.** If two theories correspond to genuinely different metrics, the framework represents both precisely but does not choose between them. Pluralism at the level of metric choice remains.

The Modest Claim

Our claim is structural, not metaphysical:

Tensorial representation is a better structural model of moral phenomena than scalarization.

"Better" means: it captures more of what we want to say, loses less information, makes implicit assumptions explicit, and enables analysis that scalar frameworks cannot support.

This is a modeling claim, not a claim about what morality *really is* at the deepest metaphysical level. Whether moral tensors are "out there" in some robust realist sense, or are useful constructs for organizing moral thought, is a question the framework does not settle. It is compatible with realism, constructivism, and expressivism alike—each can use the framework while giving different accounts of what the tensors represent.

4.7 Conclusion: The Manifold as Common Ground

The moral manifold M is the base space over which all ethical tensors are defined. Its points are structured situations; its structure includes local tangent spaces, global topology, stratum boundaries, and singular points.

Different ethical theories correspond to different structures *on* M:

- Different metrics (utilitarian vs. egalitarian vs. lexicographic)
- Different constraint sets (what is absolutely forbidden)
- Different contractions (how multi-dimensional evaluation reduces to action)

But all theories share M as common ground. This makes disagreement tractable: we can ask whether two theories differ in their metrics, their constraints, or their contractions. We can identify where they agree (perhaps on the constraint set) and where they diverge (perhaps on the metric).

The manifold is not the whole of ethics. It is the *stage* on which ethics plays out—the space of possibilities that moral reasoning navigates. The next chapters develop the actors: the tensors of various ranks that live on M, the metric that measures distances between points, and the transformations that reveal what is invariant and what is perspective-dependent.

But without the manifold, there is nowhere for tensors to live. The moral manifold is the foundation.

Technical Appendix: Formal Definitions

Definition A.1 (Moral Manifold, Formal). *A moral manifold is a paracompact Hausdorff topological space M equipped with:*

1. *A stratification $\{M_i\}_{i \in I}$ into smooth manifolds of varying dimensions;*
2. *A partial order $\preceq$ on I satisfying the frontier condition: $M_i \cap cl(M_j) \neq \emptyset$ implies $i \preceq j$;*
3. *Whitney's condition (B) at all stratum boundaries.*

Definition A.2 (Coordinate Chart). *A coordinate chart on a stratum M_i is a homeomorphism $\phi: U \to \mathbb{R}^{\dim(M_i)}$ from an open set $U \subset M_i$ to Euclidean space. A coordinate transformation is a composition $\phi' \circ \phi^{-1}: \mathbb{R}^n \to \mathbb{R}^n$.*

Definition A.3 (Admissible Transformation). *An admissible transformation of the moral manifold is a homeomorphism $\psi: M \to M$ that preserves the stratification: $\psi(M_i) = M_{\sigma(i)}$ for some permutation σ of strata. Type 1 transformations (coordinate redescriptions) are admissible; Type 3 transformations (theory shifts) are not.*

Definition A.4 (Constraint Set). *A constraint set $C \subset M$ is a closed subset such that any satisfaction function $S: M \to \mathbb{R} \cup \{-\infty\}$ satisfies $S|_C = -\infty$. The boundary ∂C is a stratum boundary where S has a discontinuity.*

Definition A.5 (Moral Singularity). *A point $x \in M$ is a moral singularity if:*

1. *The metric tensor g_x is degenerate ($\det(g_x) = 0$); or*
2. *The satisfaction function S is not differentiable at x; or*
3. *Multiple constraint surfaces intersect at x creating a cone of forbidden directions.*

Singularities represent genuine dilemmas: points where the moral structure does not determine a unique best response.

The manifold is the ground. The tensors are the figures.

Before we can say what obligation points toward, we must know the space in which it points.

This is that space: M, the moral manifold, where ethics takes its shape.

Chapter 5: Ethical Tensors of Various Ranks

The previous chapters established the motivation for thinking tensorially about ethics and introduced the mathematical machinery of tensors in a general way. This chapter gets specific. What, concretely, are the ethical tensors? What moral quantities are scalars, what are vectors, and what require the richer structure of higher-rank tensors?

The answer I shall develop is that rank correlates, roughly, with complexity of moral structure—specifically, with the number of independently variable dimensions along which a moral quantity must be specified. A scalar moral quantity is fully determined once we fix its magnitude. A vector moral quantity requires both magnitude and direction in some moral space. A rank-2 tensor requires specification along two independent dimensions and encodes how the quantity transforms under changes in both. And so on.

But we should not expect a clean one-to-one mapping between familiar moral concepts and tensor ranks. A single concept like "obligation" or "responsibility" may be scalar in some contexts, vectorial in others, and fully tensorial in its complete articulation. The tensorial framework reveals this context-dependence as structure rather than ambiguity.

5.1 Rank-0: Moral Scalars

A scalar is a quantity that is fully specified by a single number and that remains unchanged under coordinate transformations. In physics, mass and electric charge are scalars: an electron has the same charge regardless of the reference frame from which we measure it.

What moral quantities have this structure? The natural candidates are *magnitudes of value* in the simplest cases: the raw quantity of pleasure or pain in a moment of experience, the degree of preference satisfaction in an outcome, the intensity of an interest. These are the atoms of utilitarian calculation—the inputs to the felicific calculus.

But even here, we must be careful. Bentham famously enumerated dimensions of pleasure: intensity, duration, certainty, propinquity, fecundity, purity, extent. These are not independent scalar quantities to be separately maximized; they interact in complex ways. Intensity and duration trade off against each other in how they contribute to overall value. Certainty and propinquity function as discount factors. Extent raises questions of aggregation across persons that cannot be resolved by treating it as another scalar dimension.

Nonetheless, let us grant that there is *some* level of description at which simple scalar magnitudes exist in ethics. A moment of pain has some intensity. A preference has some strength. A moral wrong has some degree of seriousness. If nothing else, ordinal rankings—

this action is worse than that one—require at least a scalar structure (technically, a total or partial order, which need not be numerical but shares the one-dimensional character of scalars).

We can formalize this. Let $\mathcal{M}$ be the moral manifold—the space of morally relevant situations, which we will examine more carefully in Chapter 4. A *moral scalar field* is a function:

$$\phi: \mathcal{M} \to \mathbb{R}$$

assigning to each point in moral space a real number. The defining property of a scalar is that under a coordinate transformation $x \mapsto x'$, the value at a point is unchanged:

$$\phi'(x') = \phi(x)$$

In the moral context, this means: if we describe the same situation using different coordinates—different concepts, different framings, different perspectives—the scalar quantity remains the same. The pain intensity of this experience, described in these terms or those terms, from this perspective or that perspective, is what it is.

This invariance condition immediately raises a question: *are* there moral quantities with this strong invariance property? The worry is that almost every moral quantity seems to vary with perspective, framing, or description. My pain looks different from inside and outside. The wrongness of an action depends on how we describe it (intentional harm vs. negligent harm vs. accidental harm). The value of an outcome depends on whose values we consult.

I suggest that the scarcity of genuine moral scalars is itself significant. It indicates that scalar ethics—ethics that attempts to reduce everything to invariant magnitudes—is working with too impoverished a toolkit. The tensorial framework starts from this recognition and builds upward.

5.2 Rank-1: Moral Vectors

A vector is a quantity with both magnitude and direction. In physics, velocity, force, and electric field are vectors. Formally, a vector V at a point transforms under coordinate changes according to:

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}$$

where we use the Einstein summation convention (summing over repeated indices). The key feature is that the components of a vector are not invariant—they change with coordinates—but they change in a specific, lawful way that preserves the geometric object the vector represents.

What moral quantities are vectors? I propose that *directed obligations* have vector structure. Consider the claim "A ought to help B." This is not merely a magnitude (how much help? how strongly required?) but a directed quantity: help must flow *from* A *to* B, not vice versa. The obligation has a source, a target, and an orientation in moral space.

To formalize this, we need a moral space in which direction makes sense. Let us posit a space $\mathcal{P}$ of persons or moral patients (we will be more careful about this in Chapter 4). An obligation vector might live in the tangent space of some moral manifold, pointing from one state of affairs toward another—from "A has not helped B" toward "A has helped B."

Alternatively, consider *interests* as vector quantities. An interest is not merely a magnitude of concern; it has a directedness, pointing toward the object of interest. My interest in my own welfare points toward states of the world in which I flourish. This interest can be stronger or weaker (magnitude) but also points in a specific direction through the space of possible outcomes.

The vector framework illuminates several features of moral reasoning:

Non-comparability of orthogonal interests. If two interests point in orthogonal directions in moral space, there is no natural way to compare their magnitudes. Asking whether my interest in autonomy is stronger or weaker than my interest in welfare may be like asking whether the eastward component of velocity is greater than the northward component—a question that has no answer because the quantities are not comparable along any single dimension. This is a tensorial gloss on value incommensurability.

Conflicting obligations as opposing vectors. When I have an obligation to A and an obligation to B that cannot both be satisfied, these may be represented as vectors pointing in incompatible directions. The "net obligation"—if such a thing exists—is the vector sum, which may be smaller in magnitude than either component and may point in a direction that corresponds to neither obligation fully.

Supererogation as vector extension. The distinction between obligation and supererogation might be captured by distinguishing between the obligatory component of a moral vector (the part required) and the full vector (which includes praiseworthy but non-required extension in the same direction).

Mathematically, let us construct a simple model. Suppose we have $n$morally relevant dimensions—perhaps welfare, autonomy, justice, and so on. A moral situation can be represented as a point $x \in \mathbb{R}^n$. An obligation can be represented as a vector $\mathbf{O} \in T_x\mathbb{R}^n \cong \mathbb{R}^n$pointing from the current state toward the obligatory state.

For two obligations $\mathbf{O}_1$and $\mathbf{O}_2$, we can define their sum $\mathbf{O}_1 + \mathbf{O}_2$using standard vector addition. The magnitude of the net obligation is:

$$|\mathbf{O}_1 + \mathbf{O}_2| = \sqrt{|\mathbf{O}_1|^2 + |\mathbf{O}_2|^2 + 2|\mathbf{O}_1||\mathbf{O}_2|\cos\theta}$$

where $\theta$is the angle between the obligations. When $\theta = 0$(perfectly aligned obligations), the magnitudes simply add. When $\theta = \pi$(directly opposing obligations), they subtract. When $\theta = \pi/2$(orthogonal obligations), the magnitude is $\sqrt{|\mathbf{O}_1|^2 + |\mathbf{O}_2|^2}$

—less than the sum but more than either alone.

This already captures something important: combining orthogonal obligations does not simply sum their demandingness. The geometry matters.

5.3 Rank-2: Moral Tensors and Relations

A rank-2 tensor has two indices and transforms as:

$$T'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} T^{\alpha\beta}$$

(for a type (2,0) tensor; other variance types transform accordingly). Such tensors naturally represent *relations* or *bilinear maps*. In physics, the stress tensor σ_{ij}gives the force per unit area in the i-direction on a surface with normal in the j-direction. The metric tensor $g_{\mu\nu}$defines inner products between vectors.

In ethics, rank-2 tensors naturally represent *dyadic moral relations* between two entities—persons, actions, values, or times. Consider:

Responsibility. The moral responsibility of agent $A$for outcome $O$is not a scalar property of Aalone, nor of Oalone, but a relation between them. We can represent this as R^{AO}—a component of a rank-2 tensor. Different agent-outcome pairs have different responsibility values, and these values transform in specific ways under redescription of agents or outcomes.

Comparative value. The claim that outcome $X$is better than outcome Ymight be represented not as a comparison of scalars (the "value" of Xminus the "value" of Y) but as a component C^{XY} of an antisymmetric rank-2 tensor over the space of outcomes. Antisymmetry captures the logic of comparison: $C^{XY} = -C^{YX}$.

Interpersonal welfare comparisons. The relationship between person A's welfare in state S and person B's welfare in state S' might be encoded in a rank-2 tensor over persons-and-states, capturing the structure of interpersonal comparison without presupposing that there is a common scalar "utility" that both possess.

Let me develop the responsibility example in more detail. Define a space $\mathcal{A}$ of agents and a space $\mathcal{O}$ of outcomes. The responsibility tensor R is a bilinear map:

$$R: \mathcal{A}^* \times \mathcal{O}^* \to \mathbb{R}$$

where $\mathcal{A}^*$ and $\mathcal{O}^*$ are the dual spaces. In components, R^{ab} gives the responsibility of agent a for outcome b.

This framework immediately suggests structural properties. The tensor R can be decomposed into symmetric and antisymmetric parts:

$$R^{ab} = R^{(ab)} + R^{[ab]}$$

where $R^{(ab)} = \frac{1}{2}(R^{ab} + R^{ba})$ and $R^{[ab]} = \frac{1}{2}(R^{ab} - R^{ba})$.

If the agent and outcome spaces are identified (both are, say, events in the world), then the symmetric part $R^{(ab)}$ represents *mutual* or *shared* responsibility between events a and b, while the antisymmetric part $R^{[ab]}$ represents *directed* or *asymmetric* responsibility (agent-to-patient structure).

For a complete responsibility tensor over n agents and m outcomes, we have $n \times m$ components. This captures the intuition that responsibility is not a simple property of agents but a complex web of relations. The question "How responsible is Alice?" has no answer; we must ask "How responsible is Alice for *what*?"

Contraction and scalar extraction. Given the full rank-2 tensor R^{ab}, we can extract scalar quantities through contraction. If we have a vector v_a over agents (perhaps representing "degree of agency" or "moral weight"), we can form:

$$R^b = R^{ab} v_a$$

This gives a vector over outcomes: the "total responsibility" for each outcome, weighted by agent properties. Further contraction with an outcome vector w_b yields a scalar:

$$r = R^{ab} v_a w_b$$

This might represent the total responsibility in a situation, given weights on agents and outcomes.

The crucial point is that this scalar is *derived* from the tensor, not fundamental. Different weighting schemes (different v_a and w_b) yield different scalars from the same underlying tensorial reality. Disputes about "how much responsibility" there is in a situation may reflect different implicit weighting schemes rather than different assessments of the same quantity.

5.4 Higher Ranks and Complex Moral Structure

The pattern continues to higher ranks. A rank-3 tensor has three indices and can represent ternary relations; rank-4 tensors represent quaternary relations; and so on. In physics, the Riemann curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$ is rank-4 and encodes the curvature of spacetime.

What moral quantities might require rank-3 or higher tensors? Consider:

Mediated obligation. The claim "A ought to help B because C has requested it" involves three parties. The full moral structure might be represented as a rank-3 tensor O^{ABC} over the space of persons, encoding how the obligation depends on all three parties and their relations.

Contextual value. The value of outcome X for person A in context C might be a component of a rank-3 tensor V^{XAC}, capturing how value depends on outcome, person, and evaluative context simultaneously.

Temporal moral relations. If we include time explicitly, many moral quantities gain an additional index. The claim "A's action at t_1 wrongs B at t_2" involves four indices: agent, patient, and two times. This is naturally a rank-4 tensor.

Let me work out the mediated obligation example. Suppose we have three persons: A (the potential obligee), B (the potential beneficiary), and C (the requestor or mediator). The full structure of the mediated obligation can be represented as:

$$O^{ABC} \in \mathcal{P}^* \otimes \mathcal{P}^* \otimes \mathcal{P}^*$$

where $\mathcal{P}$ is the space of persons. Different configurations have different values:

- O^{ABB}: A's obligation to B when B requests help for themselves (direct request)

- O^{ABC} with $C \neq B$: A's obligation to B when a third party C requests help for B (mediated request)
- O^{AAC}: A's obligation to themselves when C requests it (paternalism?)

The tensor structure allows us to ask: Is O^{ABB} stronger or weaker than O^{ABC}? Does the presence of a mediator increase or decrease the obligation? These are questions about the structure of the tensor, not about scalar magnitudes.

We can also examine symmetries. Is $O^{ABC} = O^{ACB}$? This would mean that A's obligation to B at C's request equals A's obligation to C at B's request—a kind of reciprocity or symmetry in mediated obligations. Empirically (i.e., in our moral judgments), this may or may not hold, and the tensorial framework gives us precise vocabulary to articulate the difference.

5.5 Mixed Variance and the Metric

So far I have written tensors with all upper indices (contravariant). In the full tensorial framework, we distinguish contravariant indices (upper) from covariant indices (lower), and the metric tensor $g_{\mu\nu}$ mediates between them.

This distinction has moral significance. Contravariant and covariant quantities transform inversely under coordinate changes. In physics, position is contravariant while gradient (rate of change with position) is covariant. The distinction captures the difference between "things that live at points" and "things that measure change across points."

In ethics, we might distinguish:

Contravariant moral quantities: values, interests, welfare states—things that are located at points in moral space.

Covariant moral quantities: obligations, duties, requirements—things that measure how moral status changes as we move through moral space.

The metric tensor $g_{\mu\nu}$ on the moral manifold would then define how these are related. The claim that "welfare grounds obligation" might be formalized as a relationship mediated by the metric: the covariant obligation vector O_μ is related to the contravariant welfare vector W^ν via:

$$O_\mu = g_{\mu\nu} W^\nu$$

This is speculative, but it illustrates how the tensorial framework opens up structural questions that are invisible in scalar ethics.

5.6 Tensor Fields and Moral Variation

In physics, we typically work with tensor *fields*—tensors defined at each point of a manifold, varying smoothly from point to point. The stress at one location in a material differs from the stress at another; the curvature of spacetime varies across the cosmos.

Moral tensors are likewise *fields* over the moral manifold. The responsibility tensor for one situation differs from the responsibility tensor for another. The obligation vector at one point in moral space points in a different direction than the obligation vector at another point.

This introduces the apparatus of differential geometry: covariant derivatives, parallel transport, curvature. If I move through moral space—changing circumstances, encountering new information, shifting my situation—how do my obligations change? The covariant derivative $\nabla_\mu O_\nu$ measures the rate of change of the obligation tensor as we move through moral space, accounting for the curvature of the space itself.

This is the subject of Chapter 4 (the moral manifold) and Chapter 6 (the moral metric), but I mention it here to emphasize that the tensors we have been discussing are not static objects. They are components of a rich geometric structure that varies across the space of moral situations.

5.7 An Extended Example: Distributive Justice

Let me work through an extended example to show how the tensorial framework applies to a substantive moral problem.

Consider the question of distributive justice: how should resources be distributed among persons? Classical approaches offer scalar answers: maximize total utility (utilitarianism), maximize the minimum share (maximin), equalize resources, equalize welfare, or various combinations.

A tensorial analysis begins differently. We first identify the relevant spaces:

- $\mathcal{P}$: the space of persons (with n persons, $\mathcal{P} \cong \mathbb{R}^n$ or a discrete set)
- $\mathcal{R}$: the space of resources or goods
- $\mathcal{S}$: the space of states of affairs (possible distributions)

A distribution is a tensor D^{ar} giving the amount of resource r held by person a. This is already rank-2, with one person-index and one resource-index.

Now consider the *moral evaluation* of a distribution. A scalar evaluation assigns a single number—the "justice" or "goodness" of the distribution. But this discards information. A tensorial evaluation might be:

J^{ab}:the degree to which the distribution is just with respect to persons a and b

This captures relational justice: the justice of how $a$and $b$stand relative to each other under the distribution. The full picture of justice is not a scalar but a rank-2 tensor over persons.

We can impose structure on this tensor. If justice is purely relational (only comparisons matter, not absolute levels), then J^{ab}should depend only on the difference or ratio of a's and b's holdings. If justice is symmetric in persons, then $J^{ab} = J^{ba}$. If justice satisfies transitivity, then certain consistency conditions on J^{ab}must hold.

The scalar "total justice" is then a contraction:

$$J = c_{ab}J^{ab}$$

where $c_{ab}$is a weighting tensor. Different theories of justice correspond to different choices of c_{ab}:

- Utilitarian: $c_{ab} = \delta^{ab}$(only diagonal terms, weight each person equally, sum their individual welfare)
- Egalitarian: c_{ab}weights off-diagonal terms heavily (emphasize how persons stand relative to each other)
- Prioritarian: c_{ab}varies with welfare level (weight worse-off individuals more heavily)

The tensorial framework reveals that these are not simply different scalar answers but different ways of extracting a scalar from a common underlying tensorial structure. They agree on the tensor; they disagree on the contraction.

Furthermore, the tensorial framework suggests that the scalar may not be the right object of evaluation at all. Perhaps justice is irreducibly tensorial—a matter of how persons stand relative to each other that cannot be fully captured by any single number.

5.8 Summary and Transition

This chapter has argued that moral quantities come in various tensor ranks:

- **Scalars (rank-0):** Simple magnitudes of value, though these are rarer than commonly assumed

- **Vectors (rank-1):** Directed moral quantities like obligations and interests
- **Rank-2 tensors:** Dyadic relations like responsibility, comparative value, interpersonal welfare comparisons
- **Higher-rank tensors:** Complex moral structures involving multiple parties, contexts, times

The tensorial framework does not tell us *which* moral quantities exist or *what values* they take. That remains the work of substantive moral theory. But it provides a structural vocabulary for articulating moral theories with greater precision, revealing hidden assumptions (about symmetry, weighting, and aggregation), and identifying when scalar approaches are forcing tensorial phenomena into ill-fitting containers.

The next chapter takes up the question of the *moral metric*: the structure that defines distances and angles in moral space, allowing us to measure how "far apart" two moral states are and whether two moral vectors point in the "same direction." This metric structure is essential for making the tensorial framework quantitatively tractable, and its specification involves deep normative choices that have been largely invisible in scalar ethics.

Technical Appendix to Chapter 5

For readers who want the mathematics stated more rigorously:

Definition 5.1. A *moral manifold* is a smooth manifold $\mathcal{M}$ equipped with additional structure (to be specified in Chapter 4) representing the space of morally relevant situations.

Definition 5.2. A *moral tensor field of type (p, q)* is a smooth section of the bundle $T_q^p\mathcal{M} = TM^{\otimes p} \otimes T^*M^{\otimes q}$, where TM is the tangent bundle and T^*M is the cotangent bundle.

Definition 5.3. Under a coordinate transformation $x^\mu \mapsto x'^\mu$, a type (p, q) tensor $T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}$ transforms as:

$$T'^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_p}}{\partial x^{\alpha_p}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_q}}{\partial x'^{\nu_q}} T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}$$

Definition 5.4. Given a type (p, q) tensor and a type (r, s) tensor, their *tensor product* is a type $(p + r, q + s)$ tensor defined by:

$$\left(S \otimes T\right)^{\mu_1 \dots \mu_p \rho_1 \dots \rho_r}{}_{\nu_1 \dots \nu_q \sigma_1 \dots \sigma_s} = S^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} T^{\rho_1 \dots \rho_r}{}_{\sigma_1 \dots \sigma_s}$$

Definition 5.5. *Contraction* reduces tensor rank by summing over a matched pair of upper and lower indices. For a type $(2,0)$ tensor $T^{\mu\nu}$ and a metric $g_{\mu\nu}$, the contraction is:

$$T^{\mu}{}_{\mu} = g_{\mu\nu} T^{\mu\nu} = \sum_{\mu} T^{\mu}{}_{\mu}$$

This yields a scalar from a rank-2 tensor.

Proposition 5.1. If obligations are represented as vectors O^{μ} and interests as covectors I_{μ}, then the "satisfaction" of interest I by obligation O is the scalar:

$$S = I_{\mu} O^{\mu}$$

This is invariant under coordinate transformations.

Proposition 5.2. For a responsibility tensor R^{AB} over agents and outcomes, the "total responsibility" for outcome B is:

$$R^{B} = \sum_{A} R^{AB} = R^{AB} \delta_{A}$$

where the sum is over all agents. This is a vector over outcomes.

Proposition 5.3. The decomposition of a rank-2 tensor into symmetric and antisymmetric parts,

$$T^{ab} = T^{(ab)} + T^{[ab]}$$

is invariant under coordinate transformations. If T^{ab} represents a moral relation, the symmetric part $T^{(ab)}$ represents the mutual or shared component, and the antisymmetric part $T^{[ab]}$ represents the directed or asymmetric component.

End of Chapter 5