ME331B MIPSI: Bicycle Stability

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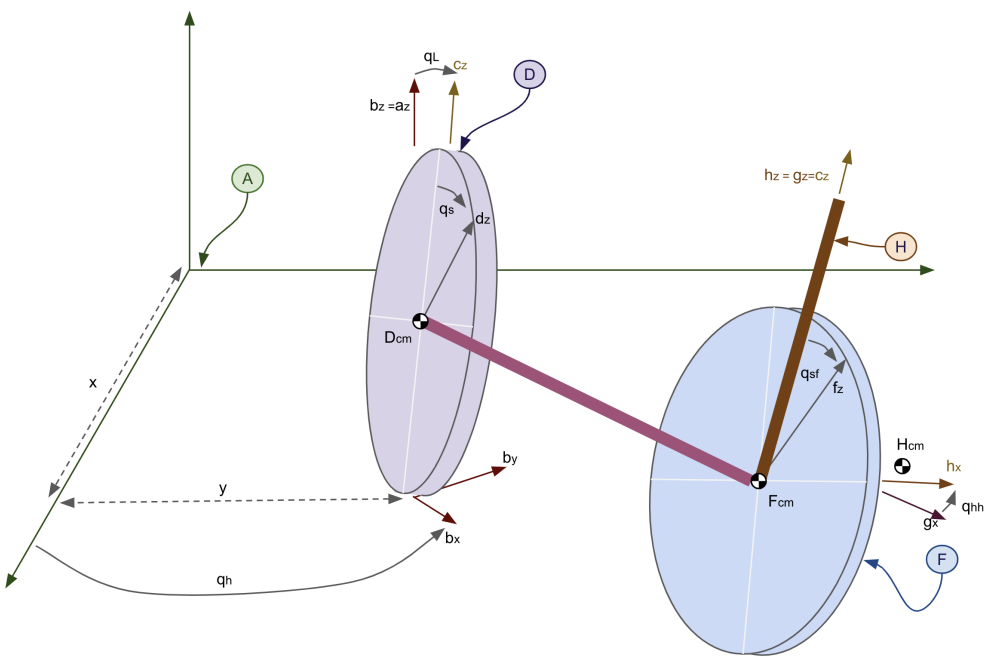
Question: What are a set of conditions that lead to a bike being stable without a rider? What simple changes make it unstable?



Answer: For a bike with a vertical head tube (traditionally considered unstable), having the handlebar assembly center of mass in front of the steering axis and below the front wheel rotation axis is a stable configuration. Moving the handlebar center of mass above the front wheel rotation axis leads to instability. Here is an animation of a stable bicycle configuration. Here is an animation of an unstable bicycle configuration.

1 Description of Model

Figure 1 shows a 3D mechanical model for a bicycle. The model consists of rigid bodies D and F, thin uniform disks representing the bike's wheels, and rigid body H modelling the handlebar and steering assembly. Disk D has three rotational degrees of freedom in Newtonian Frame A: heading, q_H, lean, q_L, and spin, q_S, with each angle corresponding to a simple body rotation between intermediate rigid frames B and C. The point of D in contact with A, D_A, is a distance $x\widehat{a}_x$ and $y\widehat{a}_y$ from point Ao. The center of mass of disk F is positioned a distance L_w from the center of mass of D, in the $\widehat{a}_x$ direction. The handlebar assembly origin, H_o, is coincident with D_{cm}, with $\widehat{h}_z$ equal to $\widehat{c}_z$. H is rotated by the steering angle, q_{HH} about $\widehat{h}_z$. Finally, the spin angle of the front wheel is q_{SF}.

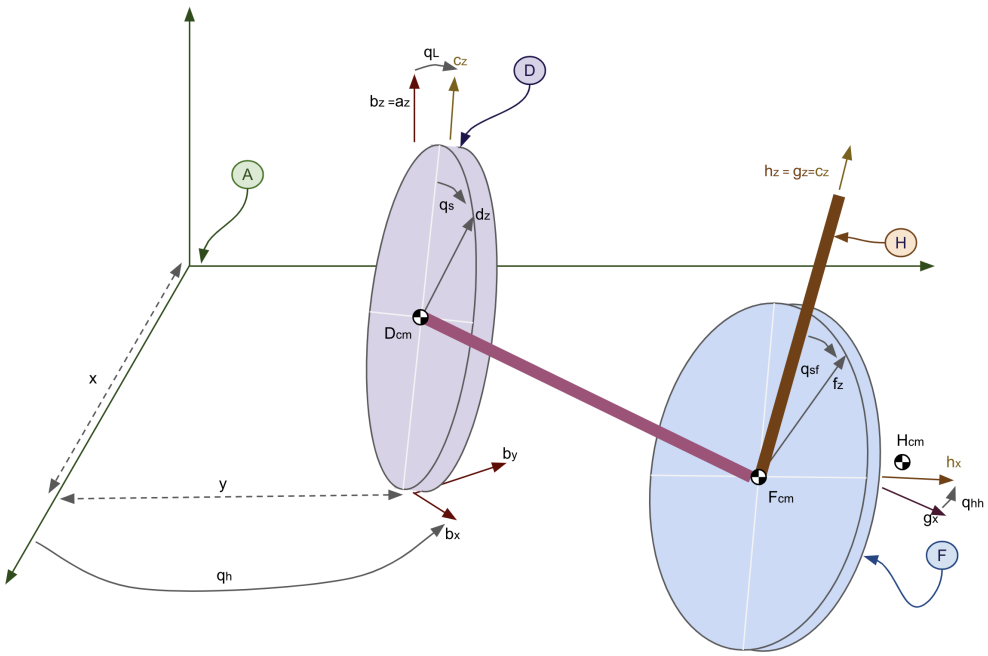


Figure 1: Bicycle model

2 Identifiers

Right-handed orthogonal unit vectors $\widehat{a}_x$, $\widehat{a}_y$, $\widehat{a}_z$; $\widehat{b}_x$, $\widehat{b}_y$, $\widehat{b}_z$; $\widehat{c}_x$, $\widehat{c}_y$, $\widehat{c}_z$; $\widehat{d}_x$, $\widehat{d}_y$, $\widehat{d}_z$; $\widehat{g}_x$, $\widehat{g}_y$, $\widehat{g}_z$; $\widehat{h}_x$, $\widehat{h}_y$, $\widehat{h}_z$; and $\widehat{f}_x$, $\widehat{f}_y$, $\widehat{f}_z$ are fixed in A,B,C,D,G,H and F respectively, with:

- $\widehat{a}_y$ horizontally right and $\widehat{a}_z$ vertically-upward
- $\widehat{b}_z = \widehat{n}_z$ and $\widehat{b}_x$ corresponding to the bike's heading in A
- $\widehat{c}_x = \widehat{b}_x$ and the angle between $\widehat{c}_z$ and $\widehat{b}_z$ corresponding to the bike's lean in A
- $\widehat{d}_y = \widehat{c}_y$ corresponding to D's spin axis
- $\widehat{g}_z = \widehat{c}_z$ and $\widehat{g}_x = \widehat{c}_x$
- $\widehat{h}_z = \widehat{g}_z$ corresponding to H's steering axis
- $\widehat{f}_y = \widehat{h}_y$ corresponding to F's spin axis

Quantity	Symbol	Type	Value
Mass of disk D or F	m	Constant	2 kg
Mass of handlebar H	m^H	Constant	1 kg
Radius of disk D or F	r	Constant	0.343 m
$\widehat{a}_x$ measure of F$_{cm}$'s position from D_{cm}	L_W	Constant	1 m
Position of H_{cm} relative to H_o in $\widehat{h}_x$ direction	x_{Hcm}	Constant	0.1 m
Position of H_{cm} relative to H_o in $\widehat{h}_z$ direction	z_{Hcm}	Constant	-0.1 m
Earth's gravitational acceleration	g	Constant	9.8 m/s^2
$\widehat{a}_x$ measure of D_A's position from A_o	x	Variable	–
$\widehat{a}_y$ measure of D_A's position from A_o	y	Variable	–
Angle from $\widehat{a}_x$to $\widehat{b}_x$ with $\widehat{a}_z$ sense	q_H	Variable	–
Angle from $\widehat{b}_z$to $\widehat{c}_z$ with $-\widehat{b}_x$ sense	q_L	Variable	–
Angle from $\widehat{c}_z$to $\widehat{d}_z$ with $\widehat{c}_y$ sense	q_S	Variable	–
Angle from $\widehat{g}_x$to $\widehat{h}_x$ with $\widehat{h}_z$ sense	q_{HH}	Variable	–
Angle from $\widehat{h}_z$to $\widehat{f}_z$ with $\widehat{h}_y$ sense	q_{SF}	Variable	–
$\widehat{c}_x$ measure of D's angular velocity in A	w_x	Variable	–
$\widehat{c}_y$ measure of D's angular velocity in A	w_y	Variable	–
$\widehat{c}_z$ measure of D's angular velocity in A	w_z	Variable	–
$\widehat{h}_y$ measure of F's angular velocity in A	w_{yF}	Variable	–
$\widehat{h}_z$ measure of F's angular velocity in A	w_{zF}	Variable	–

3 Analysis

3.1 Physics

The bike consists of three rigid bodies:

- rear wheel, **D**
- front wheel, **F**
- handlebars, **H**

The unconstrained system is represented by seven independent variables: D_A's $\widehat{a}_x$ position (x) and $\widehat{a}_y$ position (y) in A, heading (q_H), tilt (q_L), rear wheel spin (q_S), front wheel spin (q_{SF}), and steering (q_{HH}), where each angle is redefined with its corresponding angular velocity. With four motion constraints due to rolling of each wheel, the constrained system has $7 - 4 = three$ remaining degrees of freedom, which we represent as lean, rear wheel spin, and steering angle.

3.1.1 Mass Properties

The rear wheel and front wheel are modeled as thin uniform density disks. D_{cm} is located at the center of the rear wheel, ${}^{N_o}\vec{r}^{Dcm} = x\hat{a}_x + y\hat{a}_y + r\hat{c}_z$. F_{cm} is located at the center of the front wheel, ${}^{N_o}\vec{r}^{Fcm} = {}^{N_o}\vec{r}^{Dcm} + L_W\hat{c}_x - L_H\hat{h}_z$. Both wheels have moments of inertia about their center of mass $I_{xx} = \frac{1}{4}mr^2$, $I_{yy} = \frac{1}{2}mr^2$, $I_{zz} = \frac{1}{4}mr^2$ Where I_{xx}, I_{yy}, I_{zz} are about the $\hat{c}_x$, $\hat{c}_y$, and $\hat{c}_z$ directions for **D** and about the $\hat{h}_x$, $\hat{h}_y$, and $\hat{h}_z$ directions for **F**. H_{cm} is positioned relative to F_{cm}: ${}^{No}\vec{r}^{H_{cm}} = {}^{No}\vec{r}^{F_{cm}} + x_{H_{cm}}\hat{h}_x + z_{H_{cm}}\hat{h}_z$.

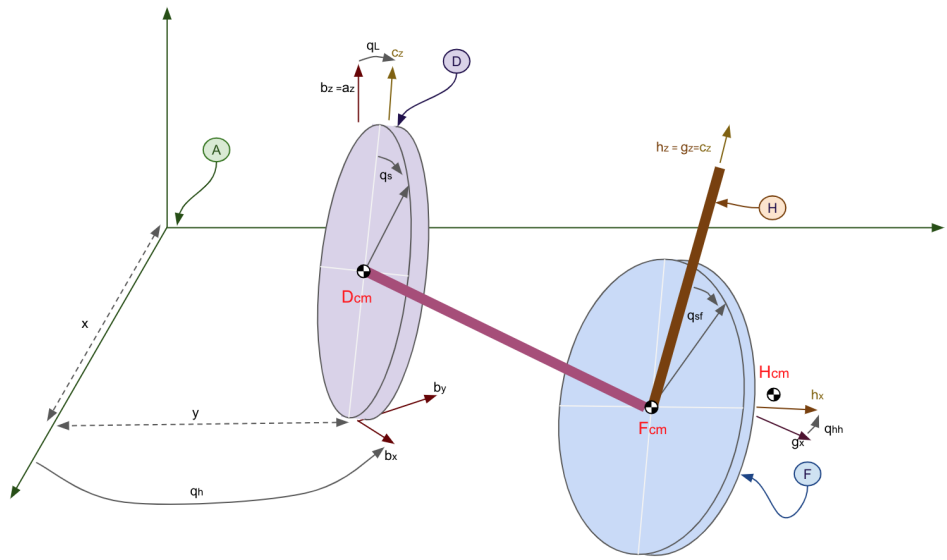


Figure 2: Bicycle mass properties

3.1.2 Forces

Sets of three forces interact with each wheel at the points of contact between the wheels and the ground. These forces are friction and the normal force. Each rigid body has a force due to gravity acting at its center of mass.

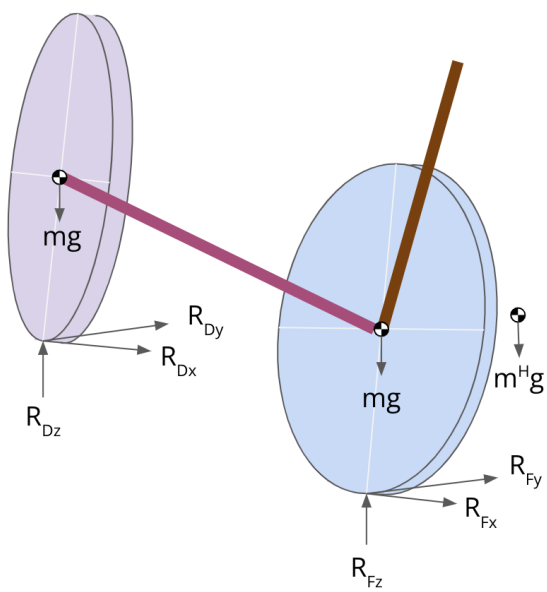


Figure 3: Bicycle free body diagram

3.2 Kinematics

Motion Constraints:
Both the rear wheel, **D**, and front wheel **F** are rolling. Therefore, there are four motion constraints:

$$^{A}\vec{v}^{D_A} \cdot \hat{b_x} \tag{1}$$

$$^{A}\vec{v}^{D_A} \cdot \hat{b_y} \tag{2}$$

$$^{A}\vec{v}^{F_A} \cdot \hat{b_x} \tag{3}$$

$$^{A}\vec{v}^{F_A} \cdot \hat{b_y} \tag{4}$$

Solving this system of equations for $\dot{x}$, $\dot{y}$, ω_z, and ω_{yF} enables the use of Kane's Embedded method in the following section.

3.3 Dynamics, Simplify & Solve

The dynamics of the system can be described using Kane's Embedded method where the motion constraints are embedded in Kane's equations. We selected three generalized speeds: ω_x, ω_y, and ω_{zH}. The equations are described by setting generalized forces equal to generalized effective forces.

$$\mathcal{F}_r = {}^{N}\mathcal{F}_r^S \tag{5}$$

Where the generalized force is defined as

$$\mathcal{F}_r^S = \vec{F}^S \cdot \frac{\partial^N \vec{v}^{B_p}}{\partial u_r} + \vec{M}^{S/B_p} \cdot \frac{\partial^N \vec{\omega}^B}{\partial u_r} \tag{6}$$

The forces acting at points D_N and F_N do not contribute to the generalized force because the ${}^{N}\vec{v}^{D_N} = \vec{0}$ and ${}^{N}\vec{v}^{F_N} = \vec{0}$ due to the rolling constraints. Therefore, for this bicycle model, the generalized force is

$$\mathcal{F}_r^S = -mg\hat{n}_z \cdot \frac{\partial^N \vec{v}^{D_{cm}}}{\partial u_r} - mg\hat{n}_z \cdot \frac{\partial^N \vec{v}^{F_{cm}}}{\partial u_r} - m^H g\hat{n}_z \cdot \frac{\partial^N \vec{v}^{H_{cm}}}{\partial u_r} \tag{7}$$

The generalized effective force for a rigid body is defined as

$${}^{N}\mathcal{F}_r^B = \frac{\partial^N \vec{v}^{B_p}}{\partial u_r} \cdot {}^{N}\vec{F}^B + \frac{\partial^N \vec{\omega}^B}{\partial u_r} \cdot {}^{N}\vec{M}^{B/B_p} \tag{8}$$

where ${}^{N}\vec{F}^B$ is the effective force

$${}^{N}\vec{F}^B = m^B\, {}^{N}\vec{a}^{B_{cm}} \tag{9}$$

and ${}^{N}\vec{M}^{B/B_p}$ is the effective moment of effective forces about B_p in N.

$${}^{N}\vec{M}^{B/B_p} = \frac{{}^{N}d\, {}^{N}\vec{H}^{B/B_p}}{dt} + ... \tag{10}$$

With the motion constraints embedded into Kane's dynamics equations, the resulting equations were solved and integrated.

Details of these calculations can be found in the MotionGenesis code in Appendix A.

4 Interpretation

Results are shown for a bicycle with three different locations for the handlebar center of mass, H_{cm}, relative to H_o / F_{cm}. In all three cases, H_{cm} is in front of the steering axis ($x_{H_{cm}} = 0.1$ m) and the bicycle starts with an initial velocity of 15 m/s in the $\hat{n}_x$ direction and an initial lean angle, $q_L = 10$ degrees. The first case (where H_{cm} is above the front wheel center of mass) shows an unstable system where the amplitude of the oscillations grows rapidly. The bike falls over (tilt/lean, q_L, approaches 90°), turns rapidly (heading, q_H, oscillations increase in amplitude), and the handlebars turn back and forth approaching their max rotation (steering, q_{HH}, approaches 90°). Here is an animation of the unstable bicycle configuration.

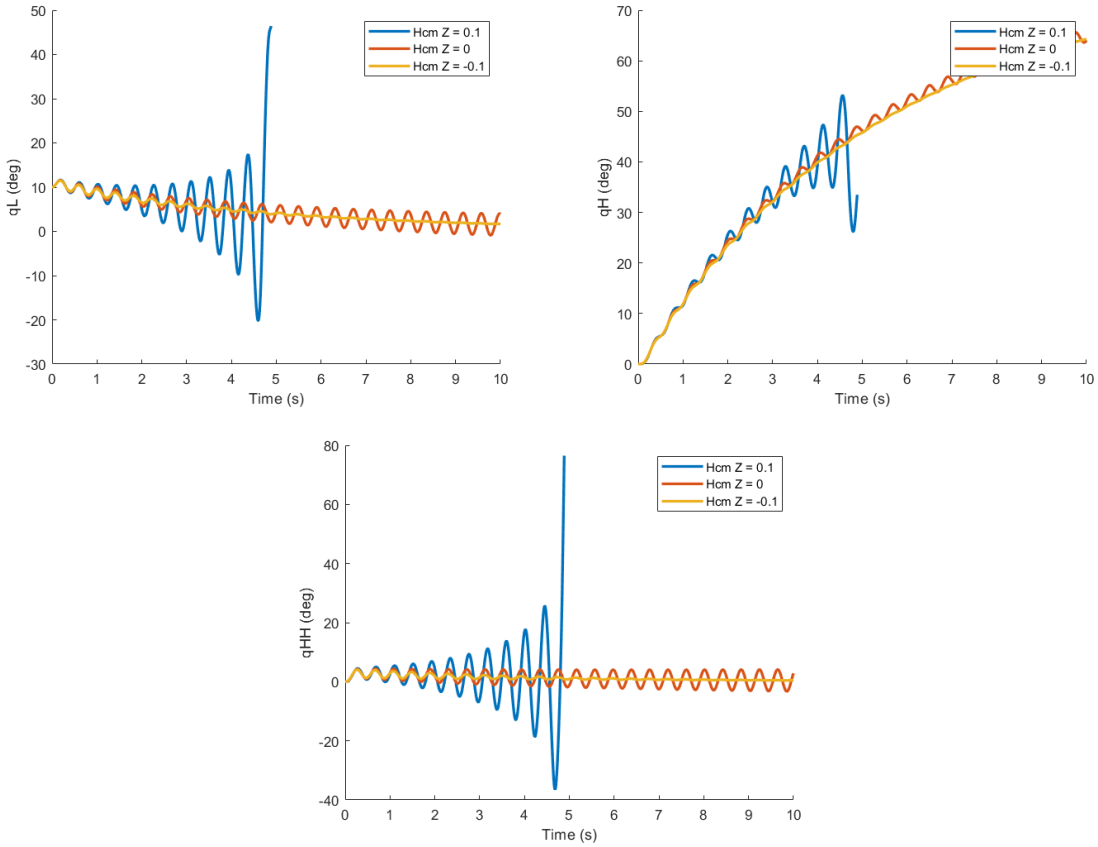


Figure 4: Bicycle response for $x_{H_{cm}} = 0.1$ m and varying $z_{H_{cm}}$ (0.1 m, 0 m, -0.1 m)

The second case (where H_{cm} is at the same $\widehat{h}_z$ position as the front wheel center of mass) also shows an unstable system, however the amplitude of the oscillations grows more slowly. Over a long enough period of time, the bike falls over, turns rapidly, and the handlebars approach their max rotation. However, since the amplitude of the oscillations remain small over a ten second time frame, a human user could correct the instability.

The final case (where H_{cm} is below the front wheel center of mass) shows a stable system where the amplitude of the oscillations approaches zero. The bike initially wobbles but remains upright (q_L). It turns to the left (q_H). And, the handlebars initially rotate back and forth before straightening out ($q_H H$). Here is an animation of the stable bicycle configuration.

A bicycle with a vertical steering tube and large enough initial speed is stable when the center of mass of the handlebars is in front of and below the wheel's center of mass.

Appendices

A Motion Genesis Code

```
% MotionGenesis file:  Bike_stable.txt
% Copyright (c) 2018 Motion Genesis LLC.  All rights reserved.
% Problem: Simulation of a bicycle.
% Authors: Leland Kirshen, Aaron Brown
%---------------------------------------------------------------------
SetAutoZee( ON )
NewtonianFrame  A                 % Flat road surface (possibly inclined)

% Rear wheel frames
RigidBody       D                 % Rolling disks, rear wheel
RigidFrame      B, C              % Intermediate heading and tilt reference frames for rear wheel (D)

% Rest of bike frames
RigidBody       F                 % Rolling disk, front wheel
RigidFrame      E                 % E frame of bike frame
RigidBody      H                 % H body of front handlebars
RigidFrame      G                 % Gz is front wheel steering column

% Contact points
Point           DA( D )           % Point of D in contact with A.
Point                     FA( F )           % point of F (front wheel) in contact with A
%---------------------------------------------------------------------
% Parameters
Variable  wx',  wy',  wz'         % Cxyz> measures of D's angular velocities in A
% Variable  wxF'                  % Hx> measure of F's angular velocity in A
Variable  wyF'                    % Hy> measure of F's angular velocity in A
Variable  wzH'                    % Hz> measure of H's angular velocity in A
Variable  x'',  y''               % Locates contact point DA from No.

Variable  qH',  qL',  qS'         % Heading angle, lean angle, spin angle.
Variable  qSF'
Variable  qHH'                    % steering angle from Ex> to Hx> w/ positive Ez> sense

%Variable  qP''
% Define qP as a constant for zero steering tube angle geometry
Constant qP = 0 deg
%---------------------------------------------------------------------
Constant  r = 0.343 meters        % Radius of disk (13.5 inches).
Constant  g = 9.8   m/s^2         % Earth's gravitational acceleration.
Constant  b = 0 N*m*s/rad         % Aerodynamic damping on disk.
Constant  LW = 1 m                % distance between wheels
Constant  LE = 1.02 m                % length of E frame
Constant HcmX = 0.1 m, HcmZ = -0.1 m % Position of Hcm relative to Ho in H frame
Constant lambda = 0 degrees % Angle of steeering column relative to vertical of bike
%Variable  LH                      % length of handle bars
Constant LH = 0 m % Length of handlebars, 0 when steering tube angle is 0 deg
%---------------------------------------------------------------------
D.SetMass( m = 2 kg )
D.SetInertia( DCm,  C,  J = 0.25*m*r^2,  I = 2*J,  J )
F.SetMass( m )
F.SetInertia( FCm,  H,  J,  I,  J )
H.SetMass( mH = 1 kg)
```

```
H.SetInertia(Hcm, 0,0,0) % Inertia not significant for now
%-----------------------------------------------------------------
%   Rotational kinematics for rear wheel
B.RotatePositiveZ( A, qH )
C.RotateNegativeX( B, qL )
D.RotatePositiveY( C, qS )

%   For efficient dynamics, make a change of variable for the rear wheel
wDA> = D.GetAngularVelocity( A )
ChangeVariablesD[1] = wx - Dot( wDA>, Cx> )
ChangeVariablesD[2] = wy - Dot( wDA>, Cy> )
ChangeVariablesD[3] = wz - Dot( wDA>, Cz> )
Solve( ChangeVariablesD = 0,  qH', qL', qS' )
D.SetAngularVelocity( A,  wx*Cx> + wy*Cy> + wz*Cz> )
%---------------------------------------------------------------
% Rotational kinematics for rest of bike
E.RotateNegativeY( C, qP ) % generally, E can have pitch relative to C frame

G.RotateNegativeY( C, lambda )

H.RotatePositiveZ(G, qHH, wzH ) % Rotation of front handlebars (heading)

F.RotatePositiveY( H, qSF, wyF ) % Spin of front wheel

%---------------------------------------------------------------
%   Translational kinematics.

Dcm.Translate( Ao,  x*Ax> + y*Ay> + r*Cz> )
Do.Translate(Dcm, 0>)
DA.Translate( Dcm,  -r*Cz> )
Bo.Translate(DA, 0>)
Eo.Translate( Dcm, 0> )
Go.Translate( Eo, LE*Ex> )
Ho.Translate( Go, 0> )
Fcm.Translate( Ho, -LH*Hz> )
Fo.Translate(Fcm, 0>)

% Location of FA only relevant for nonzero steering tube angle
projFxGround> = GetUnitVector(Cross(Fy>,Az>)) % CHECK ONCE MORE

unitFcmFA> = GetUnitVector(Cross(Fy>,projFxGround>)) % CHECK ONCE MORE

FA.Translate( Fcm, r*unitFcmFA> )

% Location of center of mass of H
Hcm.Translate(Ho, HcmX*Hx> + HcmZ*Hz>)


%-------------------------------------------------------------------
% Configuration Constraints - only necessary for nonzero pitch and steering tube angle
%Loop = Dot(FA.GetPosition(Ao),Az>)
%SolveDt(Loop=0, qP)
%SolveSetInputDt(Loop=0, qP = 30 deg)
%STOP

%---------------------------------------------------------------
```

```
%   Motion constraints (D rolls on A at point DA).
RollingConstraint[1] = Dot( v_DA_A>, Bx> )
RollingConstraint[2] = Dot( v_DA_A>, By> )
RollingConstraint[3] = Dot( v_FA_A>, Bx> )
RollingConstraint[4] = Dot( v_FA_A>, By> )
% ADDING ANOTHER ROLLING CONSTRAINT WILL OVERCONSTRAIN??
SolveDt( RollingConstraint = 0,  x', y', wyF, wz ) % constraint w corresponding to front wheel spin,
%-----------------------------------------------------------------------

% Interesting Output Quantities
zFA = Dot(FA.GetPosition(Ao),Az>)

xFA = Dot(FA.GetPosition(Ao),Ax>)
yFA = Dot(FA.GetPosition(Ao),Ay>)

xFo = Dot(Fo.GetPosition(Ao),Ax>)
yFo = Dot(Fo.GetPosition(Ao),Ay>)

%-------------------------------------------------------------
%   Add relevant contact and distance force/torques.
%   Note: Aerodynamic damping affects wx most, wy least.

System.AddForceGravity( -g*Az> )

%-----------------------------------------------------------
%   Form and simplify Kane's equations of motion.
SetGeneralizedSpeed( wx, wy, wzH )
KaneDynamics = System.GetDynamicsKane()
% FactorQuadratic( KaneDynamics,   wx, wy, wzH)
%-----------------------------------------------------------------------
%   Solve equations of motion for wx', wy', wz'.
%Solve( Dynamics = 0,  wx', wy', wz')
Solve( KaneDynamics = 0,   wx', wy',wzH')
%-----------------------------------------------------------------------
%   Integration parameters and initial values.
Input  tFinal = 10 sec,  tStep = 0.01 sec,  absError = 1.0E-08
Input  x = 0 m,  y = 0 m,  qH = 0 deg,  qL = 10 deg,  qS = 0 deg, qSF = 0 deg, qHH = 0 deg
Input  wx = 0 rad/sec,  wy = 15 rad/sec,  wzH = 0 rad/sec
%-----------------------------------------------------------------
%   List output quantities and solve ODEs.
Output  t sec,  x meters,  y meters,  qL degrees,  qH degrees, qHH degrees, zFA m
OutputPlot t sec, x meters, y meters, qL degrees, qH degrees, wy rad/s, wyF rad/s, qHH degrees, xFA m
%OutputPlot x meters, y meters
Animate(A,Ao,D )
Animate(A,Ao,F )
Animate(A,Ao,B )
Animate(A,Ao,H)
Animate(A,Ao,E)
Animate(A,Ao,Hcm)
ODE() BikeDynamics

%-----------------------------------------------------------------------
%Save MGBikeDynamics.html
%Quit
```