سوال ۲

قسمت اول

$$Var(X) = Var(E(X|Y)) + E(Var[X|Y])$$

$$* \quad Var(E(X|Y)) = E(E^2(X|Y)) - E^2(E(X|Y))$$

می دانیم : $E(E(X|Y)) = E(X) \leftarrow$ چرا

$$\int_{-\infty}^{\infty} E(X|Y=y) f_Y dy =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx\, f_Y(y) dy = \int\int_{-\infty} x f_{X|Y}(x|y) f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)\, dx\, dy = \int_{-\infty}^{\infty} x f_X(x)\, dx = E(X) \quad \text{I}$$

$$\Longrightarrow \quad Var(E(X|Y)) = E(E^2(X|Y)) - E^2(X)$$

$$** \quad E(Var(X|Y)) = \underbrace{EE(X^2|Y)}_{\text{I} \to E(X^2)} - \underbrace{E(E(X|Y)^2)}_{} = E(X^2) - E([E(X|Y)]^2)$$

$$**, * \to \quad \cancel{E(E^2(X|Y))} - E^2(X) + E(X^2) \cancel{- E(E^2(X|Y))} = E(X^2) - E^2(X)$$

$$\Longrightarrow \quad Var(X) = E(X^2) - E^2(X)$$

$$Var(E(X|Y)) + E(Var[X|Y]) = E(X^2) - E^2(X) \quad \Big\} \Rightarrow$$

$$\Longrightarrow \quad Var(X) = Var(E(X|Y)) + E(Var[X|Y])$$

قسمت دوم

$$Cov(X,Y|Z) = E(XY|Z) - E[X|Z]\,E[Y|Z]$$

$$I = E([(X - E(X|Z))(Y - E(Y|Z))]|Z) = E([XY - E(X|Z)Y - E(Y|Z)X + E(X|Z)E(Y|Z)]|Z)$$

حال از مفهوم احتمال شرطی استفاده می کنیم $\leftarrow$

$$I = E(XY|Z) - \cancel{E(X|Z)E(Y|Z)} - E(X|Z)E(Y|Z) + \cancel{E(X|Z)E(Y|Z)}$$

$$I = E(XY|Z) - E(X|Z)E(Y|Z)$$

بنابراین

$$Cov(X,Y|Z) = E(XY|Z) - E(X|Z)E(Y|Z)$$

قسمت سوم:

$$Cov(X,Y) = E((X-\mu_x)(Y-\mu_y)) =$$

$$E(XY - \mu_x Y - \mu_y X + \mu_x \mu_y) = E(XY) - \mu_x E(Y) - \mu_y E(X) + \mu_x \mu_y$$

$$= E(XY) - \mu_x \mu_y - \cancel{\mu_x \mu_y} + \cancel{\mu_x \mu_y} = E(XY) - \mu_x \mu_y$$

$$E(X|Y) - E(X)E(Y) = Cov(X,Y) \rightarrow II$$

$$* \; E(Cov(X,Y|Z))$$

از قسمت دوم داریم که ← law of total expectation

$$E(E(XY|Z) - E(X|Z)E(Y|Z)) = E(XY) - E[E(X|Z)E(Y|Z)]$$

$$** \; Cov(E(X|Z), E(Y|Z)) = E[(E(X|Z) - \underbrace{E(E(X|Z))}_{E(X)})(E(Y|Z) - \underbrace{E(E(Y|Z))}_{E(Y)})] =$$

$$= E[(E(X|Z) - E(X))(E(Y|Z) - E(Y))] =$$

$$E[E(X|Z)E(Y|Z) - E(X|Z)E(Y) - E(Y|Z)E(X) + E(X)E(Y)] =$$

$$E[E(X|Z)E(Y|Z)] - E(Y)E[\underbrace{E(X|Z)}_{E(X)}] - \cancel{E(X)\underbrace{E(E(Y|Z))}_{E(Y)}} + \cancel{E(X)E(Y)} =$$

$$= E[E(X|Z)E(Y|Z)] - E(Y)E(X)$$

$$\Longrightarrow$$

$$*, ** \rightarrow E(Cov(X,Y|Z)] + Cov(E(X|Z), E(Y|Z)) =$$

$$E(XY) - \cancel{E(E(X|Z)E(Y|Z))} + \cancel{E(E(X|Z)E(Y|Z))} - E(Y)E(X) =$$

$$= E(XY) - E(X)E(Y)$$

$$\overset{II}{\Longrightarrow} \; \left.\begin{array}{l} Cov(X,Y) = E(XY) - E(X)E(Y) \\ Cov(E(X|Z), E(Y|Z)) + E(Cov(X,Y|Z)] = E(XY) - E(X)E(Y) \end{array}\right.$$

$$\Longrightarrow Cov(X,Y) = E(Cov(X,Y|Z)] + Cov(E(X|Z), E(Y|Z))$$