سوال ۴

$x \sim U(0,1)$ $\quad 0 \leq X \leq 1$ ابتدا CDF را حاسبه وسپس مشتق می‌گیریم:

a) $F_{\sqrt{x}}(a) = P(\sqrt{x} \leq a) = P(X \leq a^2) = F_x(a^2)$

$0 \leq X \leq 1 \rightarrow 0 \leq \sqrt{x} \leq 1$

$$F_{\sqrt{X}}(x) = F_X(x^2) = \begin{cases} 0 & x^2 > 1 \text{ or } x^2 < 0 \\ x^2 & 0 \leq x^2 \leq 1 \end{cases}$$

$$f_{\sqrt{x}}(x) = \frac{\partial F_{\sqrt{x}}(x)}{\partial x} = \begin{cases} 0 & x > 1 \\ 2x & 0 \leq x \leq 1 \end{cases} \rightarrow \begin{cases} \forall x: f_{\sqrt{x}}(x) \geq 0 \checkmark \\ \int_R 2x = \int_0^1 2x = 1 \checkmark \end{cases}$$

b) $F_{X^2}(a) = F(X^2 \leq a) = P(x \leq \sqrt{a})$ $\quad 0 \leq X \leq 1$, $0 \leq X^2 \leq 1$, $0 \leq a \leq 1$

$$F_{X^2}(x) = F_X(\sqrt{x}) = \begin{cases} 0 & x > 1 \text{ or } x < 0 \\ \sqrt{x} & 0 \leq x \leq 1 \end{cases}$$

$$f_{X^2}(x) = \frac{\partial F_{X^2}(x)}{\partial x} = \begin{cases} 0 & x > 1 \text{ or } x < 0 \\ \frac{1}{2} \cdot \frac{1}{\sqrt{x}} & 0 \leq x \leq 1 \end{cases} \rightarrow \forall x: f_{X^2}(x) \geq 0 \checkmark$$

بررسی $\int_{-\infty}^{\infty} f_{X^2}(x)\, dx = \int_0^1 \frac{1}{2} x^{-\frac{1}{2}}\, dx = 1 \checkmark$

بنابراین

$$f_{\sqrt{x}}(a) = \begin{cases} 0 & a > 1 \quad a < 0 \\ 2a & 0 \leq a \leq 1 \end{cases}$$

$$f_{X^2}(a) = \begin{cases} 0 & a > 1 \quad a < 0 \\ \frac{1}{2} \cdot \frac{1}{\sqrt{a}} & 0 \leq a \leq 1 \end{cases}$$