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سوال 2

الف

$P[\sum_{i=1}^{n}(Y_i-X_i) \geq nx] = \forall_{\theta\geq 0}\ P[\theta\sum(Y_i-X_i) \geq \theta nx] = \forall_{\theta\geq 0}\ P[e^{\theta\sum(Y_i-X_i)} \geq e^{\theta nx}]$

$\Rightarrow$ chernoff bound $P[\sum(Y_i-X_i) \geq nx] \leq \forall_{\theta\geq 0} \frac{E[e^{\theta\sum(Y_i-X_i)}]}{e^{\theta nx}} = e^{-\theta nx}E[e^{\theta(Y_i-X_i)}]^n$

$\Rightarrow P[\sum(Y_i-X_i) \geq nx] \leq \forall_{\theta\geq 0}\ e^{-n(\theta x - \log E[e^{\theta(Y_i-X_i)}])}$ $\cancel{\inf_\theta}$ $\Rightarrow$

$P[\sum(Y_i-X_i) \geq nx] \leq \inf_{\theta\geq 0}\{\exp(-n(\theta x - \log(E[e^{\theta Y_1}]E[e^{-\theta X_1}])))\} =$

$\inf_{\theta\geq 0}\{\exp(-n(\theta x - \log E[e^{\theta Y_1}] - \log E[e^{-\theta X_1}]))\} \Rightarrow$

$P[\sum(Y_i-X_i) \geq nx] \leq \exp(-n(\overbrace{\sup_{\theta\geq 0}\{\theta x - \psi_P(-\theta) - \psi_Q(\theta)\}}^{F(x)}))$

$\Rightarrow P[\sum(Y_i-X_i) \geq nx] \leq \exp(-nF(x))$ $\boxed{}$ ✓

ب

$F(x) = \sup_{\theta\geq 0}\{\theta x - \psi_P(-\theta) - \psi_Q(\theta)\} \Rightarrow F(0) = \sup_{\theta\geq 0}\{-\psi_P(-\theta) - \psi_Q(\theta)\}$

$\Rightarrow F(0) = \inf_{\theta\geq 0}\{\psi_P(-\theta) + \psi_Q(\theta)\} = \inf_{\theta\geq 0}\{\log E_P[e^{-\theta X_1}] + \log E_Q[e^{\theta Y_1}]\} =$

$= \inf_{\theta\geq 0}\{\log E_P[(\frac{p}{q})^{-\theta}] + \log E_Q[(\frac{p}{q})^{\theta}]\} = \inf_{\theta\geq 0}\{\log\sum_{x\in P}(\frac{q}{p})^{\theta}P(x) + \log\sum_{x\in Q}(\frac{p}{q})^{\theta}q(x)\} =$

تا این جا عبارت Σ را درنظر بگیرید.

$F(0) = \inf_{\theta\geq 0}\{\log(E_P[(\frac{p}{q})^{-\theta}]E_Q[(\frac{p}{q})^{\theta}])\} = \log\inf_{\theta\geq 0}\{E_P[(\frac{p}{q})^{-\theta}]E_Q[(\frac{p}{q})^{\theta}]\}$

$\Rightarrow \theta = \text{Arg}\inf_{\theta\geq 0} E_P[(\frac{p}{q})^{-\theta}]E_Q[(\frac{p}{q})^{\theta}] \xrightarrow{\text{مشتق}} -E_P[\log\frac{p}{q}(\frac{p}{q})^{-\theta}]E_Q[(\frac{p}{q})^{\theta}]$

$+E_q[\log\frac{p}{q}(\frac{p}{q})^{\theta}]E_p[(\frac{p}{q})^{-\theta}] = 0 \Longrightarrow$

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$$\frac{E_p\left[\log\frac{p}{q}\left(\frac{p}{q}\right)^{-\theta}\right]}{E_p\left[\left(\frac{p}{q}\right)^{-\theta}\right]} = \frac{E_q\left[\log\frac{p}{q}\left(\frac{p}{q}\right)^{\theta}\right]}{E_q\left[\left(\frac{p}{q}\right)^{\theta}\right]} \Rightarrow$$

$$\sum_{x\in p}\frac{\log\frac{p}{q}\, p^{1-\theta}q^{\theta}}{\sum_{x\in p} p^{1-\theta}q^{\theta}} = \sum_{x\in q}\frac{\log\frac{p}{q}\, p^{\theta}q^{1-\theta}}{\sum_{x\in q} p^{\theta}q^{1-\theta}}$$

از آنجا که $P \ll Q \Leftarrow \forall x\; Q(x)=0 \Rightarrow P(x)=0$

بنابراین سمت چپ را می‌توان بر روی Q برد چرا؟ ⇐ if $Q(x)=0 \Rightarrow P(x)=0 \to$ value inside sum is zero ✓

if $Q(x)\neq 0$ and $P(x)\neq 0 \to$ fine / if $Q(x)\neq 0$ and $P(x)=0 \Rightarrow$ value inside sum is zero ✓

$\underbrace{\log\frac{p}{q}}_{-\infty} \times p^{1-\theta}$ حاصل $0 \Rightarrow 0\times -\infty = 0$ مطلق

حال داریم

$$\sum_{x\in q}\frac{\log\frac{p}{q}\, p^{1-\theta}q^{\theta}}{\sum_{x\in q} p^{1-\theta}q^{\theta}} = \sum_{x\in q}\frac{\log\frac{p}{q}\, p^{\theta}q^{1-\theta}}{\sum_{q} p^{\theta}q^{1-\theta}}$$

$\theta=1/2 \Rightarrow$ LHS = RHS ✓

حالا باید بررسی کنیم آیا برای $\theta=$ مقدار $\inf\{\log E[e^{-\theta x}]+\log E[e^{\theta x}]\}$ هست یا نه $\theta=\frac{1}{2}$

از آنجا که MGF یک تابع convex است ⇐ $\Omega = \log MGF(-\theta) + \log MGF(\theta)$

⇐ چون دو تابع convex کانوکس است ⇐ اگر $\frac{\partial}{\partial\theta}\Omega\big|_{\theta=\frac{1}{2}}=0$ باشد،

⇐ $\theta=\frac{1}{2}$ همان نقطه inf است.

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حال داریم (I)

$$\log \sum_{x\in p} \left(\frac{p}{q}\right)^{-\frac{1}{2}} p(x) + \log \sum_{x\in q} \left(\frac{p}{q}\right)^{\frac{1}{2}} q(x) = \log \sum_{x\in p} p^{\frac{1}{2}} q^{\frac{1}{2}} + \log \sum_{x\in q} p^{\frac{1}{2}} q^{\frac{1}{2}}$$

$$= \log \sum_{x\in p} \sqrt{pq} + \log \sum_{x\in q} \sqrt{pq} \overset{P \ll Q}{\Longrightarrow} \log \sum_{x\in q} \sqrt{p(q)} + \log \sum_{x\in q} \sqrt{pq} = 2\log \sum_{x\in q} \sqrt{pq}$$

$$\Rightarrow F(0) = -2\log \sum_{x\in q} \sqrt{pq} = \boxed{-2\log B(P,Q)}$$

$$B(P,Q) = E_Q\left[\sqrt{\tfrac{P}{Q}}\right] q(x) = \sum_{q} \sqrt{\tfrac{p}{q}}\, q(x) = \sum_{q} \sqrt{pq}$$

$$\Rightarrow F(0) = -2\log B(P,Q) \Rightarrow F(0) = -\Psi_P(-1/2) - \Psi_Q(+1/2) = \alpha$$

(2)

$$F(x) = \sup_{\theta \geq 0} \{\theta x - \Psi_P(-\theta) - \Psi_Q(\theta)\} \Rightarrow$$

$$F(x) = \sup_{\theta \geq 0} (\theta x - \Psi_P(-\theta) - \Psi_Q(\theta)\} \geq F(x)\Big|_{\theta=\frac{1}{2}} = \frac{1}{2}x \overbrace{- \Psi_P(-1/2) - \Psi_Q(\theta)}^{\alpha}$$

$$\Rightarrow F(x) \geq \frac{x}{2} + \alpha$$

از قضیه الف داشتیم که

$$P\left[\sum [Y_i - X_i] \geq nx\right] \leq \exp(-n(F(x)))$$

$$\leq \exp\left(-n\left(\alpha + \frac{x}{2}\right)\right)$$