سوال 5

آ

$$P[H \neq \hat{H}] \geq 1 - \frac{I(Y;S)+\ln 2}{\ln|S_K|} \leq \frac{1}{2} \Rightarrow \frac{I(Y;S)+\ln 2}{\ln|S_K|} \geq \frac{1}{2}$$

$$Y = X\theta^S + \varepsilon$$

$$I(Y;S) = h(Y) - h(Y|S) \overset{\text{when } S \text{ is fixed}}{=} h(Y) - h(\varepsilon)$$

$h(\varepsilon) = ?$ entropy of normal $= \frac{D}{2}(1 + \log_2 2\pi) + \frac{1}{2}\log_2|\Sigma|$

$\varepsilon = (\sigma^2)^D$

$$\Rightarrow h(\varepsilon) = \frac{D}{2}(\log_2 2 \times 2\pi \times \sigma^2)$$

$h(Y) = ?$ مطابق اصلی که به ما دادم در اینجا فرض می‌کنیم که توزیع گوسی $\max(h(Y))$ را می‌دهد.

ب

$$Y = XS + \varepsilon \Rightarrow \Sigma = E[(Y - E[Y])(Y - E[Y])^T]$$

$$E[Y] = E[XS + \varepsilon] = E[XS] + E[\varepsilon] \xrightarrow{x, S \text{ independent}} E[X]\cancel{E[S]}^{0} + \cancel{E[\varepsilon]}^{0} = 0$$

$$\Rightarrow \Sigma = E[YY^T] = E[(XS+\varepsilon)(S^TX^T + \varepsilon^T)] =$$

$$E[XSS^TX^T] + \cancel{E[XS\varepsilon^T]}^{0} + \cancel{E[\varepsilon S^TX^T]}^{0} + E[\varepsilon\varepsilon^T] =$$

a little abuse of notation $XS = X\theta^S$

$$\Rightarrow \Sigma = E[X\theta^S\theta^{S^T}X^T] + \underbrace{E[\varepsilon\varepsilon^T]}_{\sigma^2 I_n} \overset{x \text{ is determined}}{=} X\,E[\theta^S\theta^{S^T}]\,X^T + \sigma^2 I_n$$

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$(\theta^s \theta^{s^T})_{ij} \to \theta_i^s \theta_j^s \to E[\theta_i^s \theta_j^{s^T}] =$

if $i \neq j \Rightarrow$ independent $E[\theta_i^s]\, E[\theta_j^s] = 0$

$i = j \Rightarrow E[\theta_i^{s2}] =$ k index are $\neq 0$ and they are $-\theta_{min}$ or $+\theta_{min}$

and rest are $0 \Rightarrow E[\theta_i^{s2}] = \frac{\sum_k 1}{\sum_d 1} \theta_{min}^2$

$\Rightarrow E[\theta_i^{s2}] = \frac{k}{d} \theta_{min}^2$

$$\Rightarrow \Sigma = X E[\theta^s \theta^{s^T}] X^T + \sigma^2 I_n \Rightarrow$$

$$= X \frac{k}{d} \theta_{min}^2 X^T = \frac{k}{d} \theta_{min}^2 X X^T + \sigma^2 I_n$$

$\alpha =$

$$\Rightarrow \Sigma = \alpha \theta_{min}^2 X X^T + \sigma^2 I_n$$

$$h(Y|\theta) \leq h(Y_{gaus}) = \frac{n}{2} \log(\pi e) + \frac{1}{2} \log |\Sigma|$$

$|\Sigma| \Rightarrow ? \ \prod \lambda_i$

eigval

$$\sqrt[n]{\prod a_i} \leq \frac{\sum a_i}{n} \quad \text{تعمیم} \Leftarrow \sqrt{ab} \leq \frac{a+b}{2} \quad \text{نامساوی حسابی هندسی}$$

$$\Rightarrow \prod \lambda_i \leq \left(\frac{\sum \lambda_i}{n}\right)^n \Rightarrow \prod \lambda_i = |\Sigma| \leq \left(\frac{Tr(\Sigma)}{n}\right)^n$$

$$\Rightarrow Tr(\Sigma) = Tr\left(\frac{k}{d} \theta_{min}^2 X X^T + \sigma^2 I_n\right) \qquad |A|_F = \sqrt{Tr(AA^T)} \quad \text{از طرفی می‌دانیم}$$

$$Tr(\Sigma) = \frac{k}{d} \theta_{min}^2\, Tr(X X^T) + Tr(\sigma^2 I_n) = n\sigma^2 + \frac{k}{d} \theta_{min}^2 |X|_F^2$$

$$Tr(X X^T) \to |X|_F^2$$

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بنابراین

Because X is $n \times d$ then $D = n$ ← فرض سوال

$$h(Y) \leq \frac{D}{2}\log(4\pi) + \left(\frac{1}{n}\times\left(n\sigma^2 + \frac{k}{d}\theta_{min}^2 \|X\|_F^2\right)\right)^n$$

$$= \frac{D}{2}\log(4\pi) + \frac{1}{2}\cancel{n}\left[\sigma^2 + \frac{k}{d}\theta_{min}^2 \|n^{-\frac{1}{2}}X\|_F\right]^n =$$

$$\frac{D}{2}\log(4\pi) + \frac{1}{2}\times n \log_2\left(\sigma^2 + \frac{k}{d}\theta_{min}^2 \|n^{-\frac{1}{2}}X\|_F^2\right)$$

$$\Rightarrow h(Y;S) = h(Y) - \frac{D^n}{2}\log(4\pi) - \frac{n}{2}\log(\sigma^2) \Rightarrow$$

$$\frac{n}{2}\log_2\left[\sigma^2\left(1 + \frac{k}{d}\frac{\theta_{min}^2}{\sigma^2}\|n^{-\frac{1}{2}}X\|_F^2\right)\right] - \frac{n}{2}\log(\sigma^2) =$$

$$\cancel{\frac{n}{2}\log_2\sigma^2} + \frac{n}{2}\log_2\left(1 + \frac{k}{d}\frac{\theta_{min}^2}{\sigma^2}\|n^{-\frac{1}{2}}X\|_F^2\right) \cancel{- \frac{n}{2}\log\sigma^2}$$

$$\Rightarrow h(Y;S) \leq \frac{n}{2}\log_2\Big(1 + \underbrace{\frac{k}{d}\frac{\theta_{min}^2}{\sigma^2}\|n^{-\frac{1}{2}}X\|_F^2}_{\geq 1}\Big) \leq$$

$$\log_2(x+1) \leq x \rightsquigarrow x+1 \leq 2^x \quad (x \geq 1)$$

$x \geq 1$

$$h(Y;S) \leq \frac{n}{2}\left(1 + \boxed{\frac{k}{d}\frac{\theta_{min}^2}{\sigma^2}\|n^{-\frac{1}{2}}X\|_F^2}\right)$$

B ← این رو فعلا B بنامیم $\frac{nB}{2}$

$$\leq \frac{n}{2}B$$

$$\Rightarrow \frac{\frac{n}{2}B + \log 2}{\log|S_k|} \geq \frac{1}{2} \Rightarrow \frac{nB}{2} \geq \frac{1}{2}\log|S_k| - \log 2 \Rightarrow$$

$B \geq$

Q: $|S_k| \to$ (بر اساس) d بر $k \to \binom{d}{k}2^{k} \Rightarrow \log_2 \Rightarrow \log_2\binom{d}{k} + k\log_2 2$

$$\Rightarrow \frac{\frac{n}{2}\frac{k}{d}\frac{\theta_{min}^2}{\sigma^2}\|n^{-\frac{1}{2}}X\|_F^2 + \log_2 2}{\log\binom{d}{k}2^k} \geq \frac{1}{2} \Rightarrow \frac{n}{2}\frac{k}{d}\frac{\theta_{min}^2}{\sigma^2}\|n^{-\frac{1}{2}}X\|_F^2 \geq$$

$$\frac{1}{2}\log\binom{d}{k} + \frac{1}{2}k\log 2 - \log 2$$

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مهم نیست چون منفیه

$$\Rightarrow \frac{n}{\cancel{2}}\,\frac{k}{d}\,\frac{\theta_{min}^2}{\sigma^2}\,\|n^{-1/2}X\|_F^2 \geq \frac{\cancel{1}}{\cancel{2}}\log\left(\frac{d}{k}\right) + \frac{1}{2}k\log 2\;(-\log 2)$$

$$\longrightarrow n \geq \frac{\frac{d}{k}\log\left(\frac{d}{k}\right) + d\log 2}{\|n^{-1/2}X\|_F^2} \times \frac{\sigma^2}{\theta_{min}^2}$$

اگر از ترم $d\log 2$ به دلیل اینکه $k \ll d$ و $\log(\frac{d}{k})$ عدد بزرگی است صرف نظر شود داریم.

$$n \geq \frac{\frac{d}{k}\log\left(\frac{d}{k}\right)}{\|n^{-1/2}X\|_F^2} \times \frac{\sigma^2}{\theta_{min}^2}$$

$$\|X\|_F^2 = \sum_i \sum_j |X_{ij}|^2 = nd$$

$$\Rightarrow \frac{1}{n}\|X\|_F^2 = d \Rightarrow \|n^{-1/2}X\|_F^2 = d$$

$$\Rightarrow n \geq \frac{1}{k}\log\binom{d}{k} \times \underbrace{\left(\frac{\sigma}{\sigma_{min}}\right)^2}_{\beta^2} \geq \frac{1}{k}\log\binom{d}{k}\beta^2$$

هر چه مقدار β کمتر شود باید n کمتر شود $\Leftarrow$ σ_{min} زیادتر از σ باشد.

هر چه ورودی به اعوجاج (نویز) بیشتر باشد بهتر است.