Q1)

Consider coverings $\{N_i\}_{i\in\mathbb{Z}}$ which we define $\varepsilon_i = 2^{-i}$ now assume that $|T| < \infty$ and $\operatorname{diam}(T) < \infty$ because in other way we can not discritisize the T. now we have:

$N_{Low} \Rightarrow$ the covering that $\varepsilon_{Low} \leq \min\limits_{s,t\in T} \frac{d(s,t)}{2} \Rightarrow$ all points in T are in this cover. l: $\max\limits_{l_i} \varepsilon_{l_i} \leq \min \frac{d(s,t)}{2}$ (1) $\Longrightarrow N_l$

and consider N_{High} which covers all set with only one point $\Longrightarrow$ $\varepsilon_H \geq \operatorname{diam}(T)$ h: $\min\limits_{l_i} l_i \geq \operatorname{diam}(T)$ (2)

So now mapping π_i is mapping from $T \to T_i$ we have.

$d(\pi_i(t), t) \leq \varepsilon_i \Longrightarrow t = \pi_i(t)$

so for every s,t in T:

$X_t = X_{\pi_L(t)}$ $X_s = X_{\pi_L(s)} \Rightarrow$ because L was each point covering. and $X_{\pi_h(t)} = X_{\pi_h(s)}$

so now by chaining we drive form below:

$$X_t - X_s = \left(X_{\pi_L(t)} - X_{\pi_{L-1}(t)} + X_{\pi_{L-1}(t)} - X_{\pi_{L-2}(t)} + \dots \right)$$
$$- \left(X_{\pi_L(s)} - X_{\pi_{L-1}(s)} + X_{\pi_{L-1}(s)} - X_{\pi_{L-2}(s)} + \dots \right)$$

$$= \sum_{i=h+1}^{L} \left(X_{\pi_i(t)} - X_{\pi_{i-1}(t)} \right) + \sum_{i=h+1}^{L} \left(X_{\pi_{i-1}}(s) - X_{\pi_i}(s) \right)$$

$$|X_t - X_s| = \Big| \underbrace{\sum_{i=h+1}^{L} \left(X_{\pi_i(t)} - X_{\pi_{i-1}}(t) \right)}_{\Omega_1} + \underbrace{\sum_{i=h+1}^{L} \left(X_{\pi_{i-1}}(s) - X_{\pi_i}(s) \right)}_{\Omega_2} \Big|$$

$$\leq |\Omega_1| + |\Omega_2|$$

because $X_t - X_s$ was subG Process with Param $d(t,s)$

we have : $X_{\pi_i(t)} - X_{\pi_{i-1}(t)} \sim \text{subG}\left(d^2(\pi_i(t), \pi_{i-1}(t)) \right)$

$\Longrightarrow$ with triangular ineq:

$d(a,b) \leq d(a,c) + d(c,b) \Longrightarrow$

$d(\pi_i(t), \pi_{i-1}(t)) \leq d(\pi_i(t), t) + d(t, \pi_{i-1}(t)) \leq \varepsilon_i + \varepsilon_{i-1}$

$\leq 2^{-i} + 2^{-i+1} = 3 \times 2^{-i} \Longrightarrow$

$X_{\pi_i(t)} - X_{\pi_{i-1}(t)} \sim \text{subG}\left((3 \times 2^{-i})^2 \right)$

$$\Longrightarrow \sup_{s,t \in \pi} |X_t - X_s| \leq \sum_{i=h+1}^{L} \sup |X_{\pi_i(t)} - X_{\pi_{i-1}(t)}| + \sum_{i=h+1}^{L} \sup |X_{\pi_{i-1}(s)} - X_{\pi_i(s)}|$$

and we know $\Longrightarrow \sup_{t \in T} |X_{\pi_i(t)} - X_{\pi_{i-1}}(t)| \leq \sup_{t' \in N_{i-1}, \tilde{t} \in N_i} |X_{\tilde{t}} - X_{t'}|$

for every $u_i \geq 0 \Rightarrow P\left[\sup |X_{\pi_i(t)} - X_{\pi_{i-1}(t)}| \geq u_i\right]$

$$\leq P\left[\sup |X_{\tilde{t}} - X_{t'}| > u_i\right] = P\left[\exists\, \tilde{t} \in N_i,\ t' \in N_{i-1}\ |X_{\tilde{t}} - X_{t'}| > u_i\right]$$

$$\leq |N_i|\,|N_{i-1}|\ \underbrace{P\left[|X_{\tilde{t}} - X_{t'}| > u_i\right]}_{\text{subG}} \leq$$

$$\Rightarrow \leq 2\exp\left(\frac{-u_i^2}{2(3\times 2^{-i})^2}\right)$$

$|N_i| > |N_{i-1}|$

$$\Rightarrow \quad \leq 2|N_i|^2 \exp\left(\frac{-u_i^2}{2(3\times 2^{-i})^2}\right) \Rightarrow$$

$$\leq \exp\left(\frac{\overbrace{4(3\times 2^{-i})^2}^{c}\log(|N_i|) - u_i^2}{2(\underbrace{3\times 2^{-i}}_{c'})^2}\right)$$

now set $u_i = \sqrt{c\log|N_i|} + c' z_i$

$$\Rightarrow P\left[\sup_{t\in T} |X_{\pi_i(t)} - X_{\pi_{i-1}(t)}| \geq u_i\right] \leq 2\exp\left(\frac{\cancel{c\log|N_i|} - \cancel{c\log|N_i|} - c' z_i^2}{2c'}\right)$$

$$\leq 2\exp\left(\frac{-u_i^2}{2}\right)$$ same happens with $X_{\pi_i(s)}$

now we desire that both same be less than

d_i where $d_i = 2C\sqrt{\lg N_i} + c' v_i$

$\Rightarrow$ we want $\forall t$ & s $\sup_{t \in T} |X_{\pi_i(t)} - X_{\pi_{i-1}(t)}| < d_i$

So we calculate the complement Probability:

$$P\Big[\exists t : k+1 \le t \le L,\ \sup_{t \in T} |X_{\pi_i(t)} - X_{\pi_{i-1}(t)}| \ge d_i$$
$$\exists s : \text{"} \quad \text{"} \quad \text{"} \quad

$$\sup_{s,t\in T} |X_t - X_s| \leq 2\sum \delta_i \lesssim \alpha \left(\sum_{i=h+1}^{L} 2^{-i}\sqrt{\log N_i} \right)$$

$$+ \alpha' \left(\sum 2^{-i}\right) g(H)$$

$$+ \alpha'' \sum 2^{-i} \sqrt{f(i,H)}$$

because $f(i,H)$ is bounded $\Longrightarrow$ $\alpha''' 2^{-H}$

$$\Longrightarrow \quad \lesssim \alpha \int_c^{+\infty} \sqrt{\log(N(T,d,\varepsilon))}\, d\varepsilon + \alpha''' \underbrace{2^{-H}}_{\lesssim \operatorname{diam}(T)}$$

$\alpha''' =$ some constant $\times$ u_i by definition of Σ

$$\Longrightarrow \quad 0 \lesssim \tilde{C} H \left[\int_0^{\infty} \sqrt{\log(N(T,d,\varepsilon))}\, d\varepsilon + u \operatorname{diam}(T) \right]$$

$\Longrightarrow$ with prob $1 - C'' \exp\left(-g(H)^2/2\right)$ above hold

set $C'' = 2$ redo calculations only $\tilde{C}$ will change $\boxed{\checkmark}$

Q2) From Q1 we know $\sup_{s,t\in T} |X_t - X_s|$ is a subG

from befor we know with probability at least

$1 - c'' \exp\left(-\frac{g^2(H)}{2}\right)$ we have

$$\sup_{\substack{s,t\in T \\ d(s,t)\le\delta}} \{X_t - X_s\} \le 2\sum \delta_i = 2\sum \left(2c\sqrt{\log N_i} + c\, v_i\right)$$

$$\le C\int_0^{2^{-H-1}} \sqrt{\log N(T,d,\varepsilon)}\, d\varepsilon + c'' \underbrace{2^{-H}}_{\text{diam}(T)}$$

because based on Q2 we have $\text{diam}(T) \le \delta$

$$\Rightarrow \sup_{\substack{s,t\in T \\ d(s,t)\le\delta}} \{X_t - X_s\} \le \underset{\text{new constant}}{C} \left[\int_0^{\delta} \sqrt{\log(N(T,d,\varepsilon)}\, \varepsilon + \delta u\right]$$

trick part here is how to define δ_i so we can have $-\frac{x^2}{\delta^2}$ in our final results.

define $x = \gamma u \delta \Rightarrow u = \frac{x}{\gamma\delta}$

$$P\left[\sup\{X_t - X_s\} \ge C\left(\sqrt{\log N}\, d\varepsilon + \gamma u \delta\right)\right] \le \tilde{\tilde{c}} \exp\left(\frac{-x^2}{\gamma^2\delta^2}\right)$$

$$\Rightarrow \bigcirc \le C e^{\left(\frac{-x^2}{\tilde{c}\delta^2}\right)}$$

Proof was inspired by theorem 5.29 vanhandel

Q3) (Problem 8.1D Vanhandel)

$\psi \to$ 1- Convex
2- $\psi(0) = \psi'(0) = 0$

we have $\log E\left[e^{\frac{\lambda\{x_t - x_s\}}{d(t,s)}}\right] \leq \psi(\lambda)$

From question 1 we have

+ L and H as ε_L and ε_H for covering sets with $\varepsilon_i = 2^{-i}$

$$E\left[\sup_{t\in T}\{X_t\}\right] = E\left[\sup\left\{\sum_{i=L}^{H+1} X_{\pi_i(t)} - X_{\pi_{i-1}(t)}\right\}\right]$$

$$\leq \sum_{i=L}^{H+1} \underbrace{E\left[\sup_{t\in T}\{X_{\pi_i(t)} - X_{\pi_{i-1}(t)}\}\right]}$$

$$E\left[\sup\{X_{\pi_i(t)} - X_{\pi_{i-1}(t)}\}\right] = E\left[\frac{c}{\lambda}\log\left\{\sup_{t\in T}\left\{\exp\left(\frac{\lambda(X_{\pi_i} - X_{\pi_{i-1}})}{c}\right)\right\}\right\}\right]$$

$$\leq \frac{c}{\lambda} E\left[\log\left(\sup_{\substack{t'\in N_i \\ \tilde{t}\in N_{i-1} \\ d(t,t')\leq c}}\left\{\exp\left(\frac{\lambda(X_{t'} - X_{\tilde{t}})}{d(t',\tilde{t})}\right)\right\}\right)\right]$$

union bound.

$$\leq \frac{c}{\lambda}\log\left(\sum_{\substack{t'\in N_i \\ \tilde{t}\in N_{i-1}}} e^{\psi(\lambda)}\right) \Rightarrow \frac{c}{\lambda}\log\left(|N_i|^2 e^{\psi}\right)$$

$$\Longrightarrow \frac{c}{\lambda}\left(2\log|N_i| + \psi(\lambda)\right)$$

$$2\log|N_i| \geq \sup_{\lambda>0}\left\{\frac{\lambda}{C}\,\frac{E\left[\sup_{t\in T}\{X_{\pi_i(t)}-X_{\pi_{i-1}(t)}\}\right]}{C}-\psi(\lambda)\right\}$$

$$\psi^*\left(\frac{E\left[\sup_{t\in T}\{X_{\pi_i(t)}-X_{\pi_{i-1}(t)}\}\right]}{C}\right) \quad *$$

(optimizing above)

chapter 3 and 5!

* this part is from vanhandel it has also used in previous Hw (Hw2).

to apply ψ^{*-1} we should see if it is increasing or not.

$$\psi^*(x) = \sup_{\lambda>0}\ \lambda x-\psi(\lambda) \geq \lambda x-\psi(\lambda)\Big|_{\lambda=0} = 0-0=0$$

$$\Rightarrow \psi^*(x)\geq 0 \Rightarrow \psi^{*-1} \text{ domain is positive numbers.}$$

$$2\log|N_i| = \psi^*(\psi^{*-1}(2\log|N_i|)) \geq \psi^*\left(\frac{E\sup[0]}{C}\right)$$

I

$$\Rightarrow E\left[\sup_{t\in T}\{X_t\}\right] \leq \sum_{i=H+1}^{L} E\left[\sup\{X_{\pi_i(t)}-X_{\pi_{i-1}}(t)\}\right]$$

$$\leq \tilde{C}\sum_{i=H+1}^{L} 2^{-i}\psi^{*-1}(2\log N_i) \leq C\int_0^\infty \psi^{*-1}(2\log(N(T,d,\varepsilon)))\,d\varepsilon$$

using Part I for us to claim that $\psi^{*-1}(2 \log N_i) \geq \frac{E \sup \ldots}{c}$

we need to find out sign of $\frac{d\psi}{d\lambda}$

$$\frac{\partial}{\partial \lambda}\left(\lambda x - \psi(\lambda)\right) = x - \psi'(\lambda) = 0 \Rightarrow \lambda^* = \psi^{-1}(x)\Big|_{x>0}$$

$$\psi^*(x) = x\lambda^* - \psi(\lambda^*) = x\,\psi^{-1}(x) - \psi(\psi^{-1}(x)) \Rightarrow$$

$$\psi'(x) = \psi^{-1}(x) + x\frac{d}{dx}(\psi^{-1}(x)) - \psi'(\psi^{-1}(x))\frac{d}{dx}\psi^{-1}(x) = \psi^{-1}(x)$$

$\Longrightarrow \psi'$ is increasing because of convexity of ψ

so because $\psi'(0) = 0 \Rightarrow \psi'(x) \geq 0$

$\Longrightarrow \psi^{*-1}$ is increasing $\Rightarrow$ ineq holds.

Q4)

a) we have to prove that $E \sup_{t \in T} \langle g, x \rangle$ is bounded where $g \sim N(0, I_{n \times n})$

$$E \sup_{t \in T} \frac{\sum e_{ti} g_i}{\sqrt{1 + \log(t)}} = E \sup_{t \in T} \left(\frac{g_t}{\sqrt{1 + \log t}} \right) \leq \sum \frac{E|g_i|}{\sqrt{1 + \log i}}$$

$$= \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} \frac{1}{\sqrt{1 + \log i}} \leq \infty \Rightarrow \text{this is bounded.}$$

or we can calculate this $E\left[\sup_{t \in T} \left\{ \frac{|g_t|}{\sqrt{1 + \log t}} \right\}\right]$

$$= \int_{u=0}^{\infty} P\left[\max_{t \in T} \left\{ \frac{|g_t|}{\sqrt{1 + \log t}} \right\} \geq u\right] du$$

$$\leq \int_{u=0}^{\infty} \underbrace{\sum_{t=1}^{n} P\left[|g_i| \geq u \sqrt{1 + \log t}\right]}_{C' e\left(-\frac{u^2}{2}\right)} dt + C_0$$

this is bounded too!

b) we know that $\frac{1}{\sqrt{1 + \log(\max(i,j))}} \leq d(t_i, t_j) \leq \frac{1}{\sqrt{\log \min(i,j) + 1}}$

So $\int_0^\infty \sqrt{N_\varepsilon} \geq \int_0^\infty \sqrt{D(r_2)} \geq \sum_i \int_{\frac{1}{\sqrt{1+1.j(\min(i,i+1))}}}^{\frac{1}{\sqrt{1+1.j\max(i,i+1)}}} \sqrt{i}$

$$= \sum_{i=1}^{n} \sqrt{i} \left(\frac{1}{1.j\, i} - \frac{1}{1.j(i+1)} \right)$$

if we $n \to \infty$ then this summation will diverge.

Q5) (theorem 5.31 Van handel):

like Question 1 we consider $\delta > 0$ and assume $E[X_t] = 0$

$\Rightarrow \{N_i\}_{i \in \mathbb{Z}}$ and $\varepsilon_i = 2^{-i}$ and $\varepsilon_L \Rightarrow 2^{-L}$ $\quad \varepsilon_L = \max_{l} 2^{-l} \leq \min_{s,t \in T} \frac{d(s,t)}{2}$

$\varepsilon_h \Rightarrow 2^{-H}$ $\quad \rightarrow \varepsilon_h = \min_{h} 2^{-h} \geq diam(T)$

$X_t = X_{\pi_L(t)}, \quad X_s = X_{\pi_L(s)}$

$X_{\pi_H(t)} = X_{\pi_H(s)}$

We have $\quad X_t = X_t - X_{\pi_L(t)} + \sum_{i=H+1}^{L} (X_{\pi_i(t)} - X_{\pi_{i-1}(t)}) + X_{\pi_H(t)}$

$d(\pi_i(t), \pi_{i-1}(t)) \leq d(\pi_i(t), t) + d(t, \pi_{i-1}(t)) \leq 2^{-i} + 2^{-i+1} \leq 3 \times 2^{-i}$

From lipschits $\Rightarrow \quad X_t - X_{\pi_L(t)} \leq C\, d(t, \pi_L(t)) \leq C 2\delta$

$$\Rightarrow \sup\{X_t\} \leq \sup_{t \in T} \{X_t - X_{\pi_L(t)}\} + \sum \sup\{X_{\pi_i(t)} - X_{\pi_{i-1}(t)}\} + \sup_{t \in T}\{X_{\pi_H(t)}\}$$

H was one point

covering for t covering point is the same!

$$\leq 2\delta C + \sum_{i=H+1}^{L} \sup_{\substack{t' \in N_i \\ \tilde{t} \in N_{i-1}}} \{X_{t'} - X_{\tilde{t}}\} + X_{\pi_H(t)}$$

$$E[\sup\{X_t\}] \leq 2\delta E[C] + \sum_{i=H+1}^{L} E[\sup\{X_{t'} - X_{\tilde{t}}\}] + E[X_{H(t)}]$$

$$= 2\delta E[c] + \sum_{i=H+1}^{L} E\Big[\sup_{\substack{t' \in N_i \\ \tilde{t} \in N_{i-1} \\ d(\tilde{t}, t') \leq 3\times 2^{-i}}} \{X_{t'} - X_{\tilde{t}}\}\Big] + \cancel{E[X_{H(e)}]}^{\,0}$$

because X_t is subg Process: for all $\lambda \geq 0$

$$E\Big[\sup_{\substack{t' \in N_i \\ \tilde{t} \in N_{i-1} \\ d \leq 3\times 2^{-i}}} \{X_{t'} - X_{\tilde{t}}\}\Big] = \frac{1}{\lambda} E\Big[\log\Big[\sup_{\substack{t' \in N_i \\ \tilde{t} \in N_{i-1} \\ d \leq 3\times 2^{-i}}} \{\exp(\lambda(X_{t'} - X_{\tilde{t}}))\}\Big]\Big]$$

union bound

$$\Longrightarrow \leq \frac{1}{\lambda} E\Big[\log\Big(\sum \ldots\Big)\Big] \leq \frac{1}{\lambda} \log\Big(\sum_{\substack{N_i \\ N_{i-1}}} E\big[e^{\lambda(X_t - X_{\tilde{t}})}\big]\Big)$$

$$\leq \frac{1}{\lambda} \log\Big(N_i^2 \exp\Big(\frac{\lambda^2 d^2}{2}\Big)\Big) \leq \frac{1}{\lambda} \log\Big(N_i^2 \exp\Big(\frac{\lambda^2 (3\times 2^{-i})^2}{2}\Big)\Big)$$

$$\leq \frac{2}{\lambda} \log(|N_i|) + \frac{\lambda^2 (3\times 2^{-i})^2}{2}$$

we desire to $E[\sup\{X_{t'} - X_{\tilde{t}}\}] \leq 6\times 2^{-i}\sqrt{\log N_i}$

$$\Longrightarrow \frac{2}{\lambda} \log N_i + \frac{\lambda^2 (3\times 2^{-i})^2}{2} \lesssim 6\times 2^{-i}\sqrt{\log N_i}$$

solve this inequality there would be $\lambda^* \Rightarrow$

set λ^* for the solution $\Rightarrow$

$$E[\sup[X_t]] \leq 2\delta E[C] + \sum_{i=H+1}^{L} 6\times 2^{-i}\sqrt{\log N_i}$$

$$\Rightarrow \leq 2\delta E[C] + 12\int_{\delta}^{\infty}\sqrt{\log(N,T,d,\varepsilon)}\,d\varepsilon$$

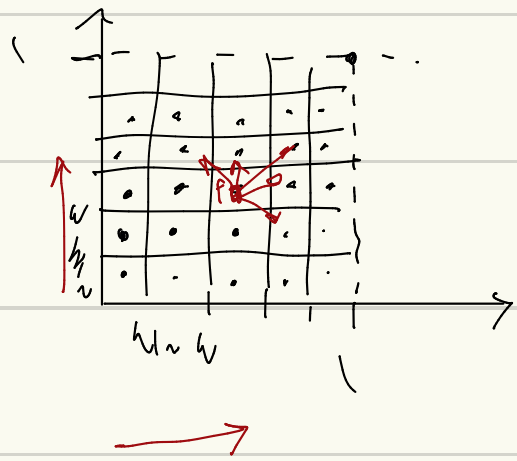
because results hold for all $\delta \geq 0$ then $\Rightarrow$

$$\leq \inf_{\delta\geq 0}\left\{2\delta E[C] + 12\int_{\delta}^{+\infty}\sqrt{\log(N(T,d,\varepsilon))}\,d\varepsilon\right\}$$

Q6) (Problem 5.11 van handell)

$$W_1(P_n, P) = \sup_{f:\ 1\text{-Lipshitz}} E_{P_n}[f] - E_P[f]$$

a)



moving directions are $\uparrow$ $\rightarrow$ so

for point P we have (moving up) (moving right)

states so

$$f(x) - (\text{nearest center point}) = (f(x) - \text{corner}) + (\text{corner} - \text{center}) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

function set of F_0 in I_{ij} compared to $I_{i-1,j-1}$ has only three states.

$I_{i-1,j-1} \pm \varepsilon$, $I_{i-1,j-1}$ $\Rightarrow$ 3

$$N(F_0, \|\cdot\|_\infty, \varepsilon) \le 3^{\lceil \frac{2}{\varepsilon} \rceil \lceil \frac{2}{\varepsilon} \rceil} \le 3^{(\frac{2}{\varepsilon}+1)^2} \le 3^{\frac{c}{\varepsilon^2}} = \exp\left(\frac{c'}{\varepsilon^2}\right)$$

b) as discussed in the class with hoffding lemma $X_f - X_g$ is subg with param $c\frac{\|f-g\|_\infty^2}{n}$ and X_f is subg with metric $d(f,g) = c\frac{\|f-g\|_\infty}{\sqrt{n}}$ ($\sqrt{n}$ is for normalization have)

$$\Rightarrow E[\sup X_f] \le c\int_0^\infty \sqrt{\log N}\, d\varepsilon = c'\int_0^\infty \sqrt{\log(N(F_0, \|\cdot\|_\infty, \tfrac{\varepsilon\sqrt{n}}{2}))}\, d\varepsilon$$

$$= \text{change of variable} = \frac{\tilde{c}}{\sqrt{n}}\int_0^\infty \sqrt{\log N} \le \frac{\bar{c}}{\sqrt{n}}\int_0^\infty \sqrt{\log e^{\frac{c'}{\varepsilon^2}}} = \frac{\tilde{\tilde{c}}}{\sqrt{n}}\int_0^\infty \frac{d\varepsilon}{\varepsilon} = \infty$$

c) extension from Q5 and part b

$$E[\sup X_f] = \inf_{\delta \geq 0} \left\{ 2\delta\sqrt{n} + 12 \int_\delta^\infty \sqrt{\log N(F_0, d, \varepsilon)}\, d\varepsilon \right\}$$

in $|x|$ square maximum distance is $\sqrt{2}$ $\Rightarrow$ for $\varepsilon > \sqrt{2}$ $N(F_0, \|\cdot\|_\infty, \varepsilon) = 1$

$\Rightarrow \log 1 = 0 \Rightarrow$ upper bound of integral is $\sqrt{2}$

$$= \inf_{\delta \geq 0} \left\{ 2\delta\sqrt{n} + \frac{c}{\sqrt{n}} \int_\delta^{\sqrt{2}} \sqrt{\log N} \right\} = \inf_{\delta \geq 0} \left\{ 2\delta\sqrt{n} + \frac{\tilde{c}}{\sqrt{n}} \int_\delta^{2} \frac{1}{\varepsilon}\, d\varepsilon \right\}$$

$$= \inf_{\delta \geq 0} \left\{ 2\delta\sqrt{n} + \frac{\tilde{c}}{\sqrt{n}} \log\left(\frac{\sqrt{2}}{\delta}\right) \right\}$$

$$\Rightarrow 2\sqrt{n} - \frac{\tilde{c}}{\sqrt{n}} \frac{1}{\delta} = 0 \Rightarrow \delta = \frac{\tilde{\tilde{c}}}{n}$$

$$\Rightarrow 2\frac{\tilde{\tilde{c}}}{n}\sqrt{n} + \frac{\tilde{c}}{\sqrt{n}} \log \frac{\sqrt{2}}{\frac{\tilde{\tilde{c}}}{n}} = 2\frac{\tilde{\tilde{c}}}{\sqrt{n}} + \frac{\text{some constant}}{\sqrt{n}} + \frac{\tilde{c}}{\sqrt{n}} \log n$$

$$\Rightarrow O\left(\frac{\log n}{\sqrt{n}}\right)$$

Q7) a) first note that (this Idea was with help of Marshal Pirgheibi)

* $w(aT) = |a|\, w(T)$ why? $\Rightarrow$

imagine $a \geq 0$ then only constant a would affect inside supremum part so $a\, w(\tau)$ (1)

if $a < 0$ then just use $-u$ intead of u and again we have. $\Rightarrow$ $-a\,(w(\tau))$ (2) $\Rightarrow$ (1)(2) $\Rightarrow$ $w(aT) = |a|\, w(T)$

* $w(S+T) \leq ?$ $\Rightarrow$

$$E[\sup\{(t+s)^T X\}] = \underbrace{E[\sup t^T X]}_{t} + \underbrace{E[\sup s^T X]}_{s}$$

$= w(S) + w(T)$ (this is because t and s can be chose independent (Y))

our Idea is to $w(T-T) = w(T) + w(-T) = 2w(T)$

$$w(T) = \frac{1}{2} w(T-T) \geq \frac{1}{2} E[\max\{(V_1-V_2)^T X, (V_2-V_1)^T X\}]$$

$$= \frac{1}{2} E[|(V_1-V_2)^T X|]$$

$\Rightarrow Y = (u-v)^T X \rightsquigarrow Y \sim N(0, \|V_1-V_2\|^2)$

$$E[Y] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma}} |y|\, e^{-z^2/2\sigma^2}\, dz = \sigma$$

$$\Rightarrow \quad w(T) \geq \frac{\sigma}{2}\sqrt{\frac{2}{\pi}} = \frac{\sigma}{\sqrt{2\pi}}$$

$$\sigma = \|v_1 - v_2\|_2$$

because for every v_1 and v_2 ineq holds we have

$$w(T) \geq \frac{1}{\sqrt{2\pi}} \sup_{u,v \in T} \{\|u-v\|_2\} = \frac{1}{\sqrt{2\pi}} \operatorname{diam}(T) \quad * \text{ I}$$

$$w_T = \frac{1}{2} w(T-T) = \frac{1}{2} E\left[\sup (v_1 - v_2)^T X\right]$$

$$\leq \frac{1}{2} E\left[\sup \{\|v_1 - v_2\|_2 \|X\|_2\}\right] = \frac{1}{2} \underbrace{E[\|X\|_2]}\, \overbrace{\sup\{\|u-v\|_2\}}^{\operatorname{diam}(T)}$$

$$E[\|X\|_2] = E\left[\sqrt{\|X\|_2^2}\right] = \sqrt{E\left[\sum_{i=1}^{n} X_i^2\right]}$$

$$E[X_i^2] = 1$$

$$\Rightarrow \sqrt{n} \quad \Rightarrow \quad \frac{1}{2}\sqrt{n} \operatorname{diam}(T) \quad \text{II}$$

$$\text{I}, \text{II} \Rightarrow \frac{\operatorname{diam}(T)}{\sqrt{2\pi}} \leq w(T) \leq \frac{1}{2}\sqrt{n} \operatorname{diam}(T)$$

b) $$w(AT) = E\left[\sup\{\langle g, At\rangle\}\right] = E\left[\sup\{(At)^T X\}\right]$$

$$= E\left[\sup\{t^T A^T X\}\right], \quad X \sim N(0, I_{n\times n}) \quad \text{I}$$

define $\Omega_t = t^T A^T X$

$\tilde{\Omega}_t = \langle \|A\|_{op}\, t, z\rangle$

$n, z \sim N(0, I)$

$$\Rightarrow \quad E[\Omega_t - \Omega_s] = E[(t-s)^T A^T X X^T A (t-s)] =$$

$$(t-s)\, A^T \underbrace{E[X X^T]}_{I} A (t-s) = (t-s)^T A^T A (t-s)$$

$$E[\tilde{\Omega}_t - \tilde{\Omega}_s] = \text{same process} \rightarrow \|A\|_{op}^2 \|t-s\|_2^2$$

from linear algebra

$$\Rightarrow \quad (t-s)^T A^T A (t-s) \leq (t-s)^T (t-s) \|A\|_{op}^2 = E[\tilde{\Omega}_t - \tilde{\Omega}_s]$$

$$\Rightarrow E[\Omega_t - \Omega_s] \leq E[\tilde{\Omega}_t - \tilde{\Omega}_s] \quad \text{II}$$

from I, II and vershynin section 7.2 we have

$$E[(X_t - X_s)^2] \leq E[(Y_t - Y_s)^2] \Longrightarrow$$

$$E[\sup_{t \in T}\{X_t\}] \leq E[\sup_{t \in T}\{Y_t\}] \rightarrow \text{in vershynin 7.2}$$

$$\downarrow$$

$$w(\Delta T) = E[\sup \Omega_t] \leq E[\sup \tilde{\Omega}_t] = \|A\|_{op}\, w(T)$$

c) $w(B_p^n) = E[\sup_{\|t\|_p \leq 1}\{t^T X\}]$

for all $p \geq 1 \rightsquigarrow B_p^n$ are in $[-1,1]^n \Rightarrow$

$$w = E\left[\sup\{t^T x\}\right] \leq E\left[\sup_{t \in [-1,1]^n}\{t^T x\}\right] = E\left[\max_{i \in n} |x_i|\right]$$

from class we know $\rightarrow$ $\max_{i \in n} |x_i| \leq \sqrt{2}\sqrt{\log n} = \sqrt{2 \log n}$

$$\Rightarrow E\left[\sqrt{2}\sqrt{\log n}\right] = w(B) \leq \sqrt{2 \log n}$$

Construct lagrangian function:

$$\mathcal{L}(t, \lambda) = -t^T x - \lambda(1 - \|t\|_p) = -\sum t_i x_i - \lambda\left(1 - \sum t_i^p\right)$$

$$\frac{\partial \mathcal{L}}{\partial t_i} = -x_i + \lambda p t_i^{p-1} = 0 \Rightarrow t_i = \left(\frac{x_i}{\lambda p}\right)^{\frac{1}{p-1}}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - \sum t_i^p = 0 \Rightarrow \sum_{i=1}^{n} \left(\frac{x_i}{\lambda p}\right)^{\frac{p}{p-1}} = 1 \Rightarrow \sum \left(\frac{x_i}{\lambda p}\right)^{\tilde{t}} = 1$$

call it $\tilde{p}$

$$\Rightarrow \lambda p = \left(\sum x_i^{\tilde{p}}\right)^{\frac{1}{\tilde{p}}}$$

$$\Rightarrow t_i^* = \left(\frac{x_i}{\left(\sum x_i^{\tilde{p}}\right)^{1/\tilde{p}}}\right)^{\frac{1}{p-1}} = x_i^{\frac{1}{p-1}} \left(\sum x_i^{\tilde{p}}\right)^{\frac{-1}{(p-1)\tilde{p}}} = -1/p$$

$$\tilde{p} = \frac{p}{\cancel{p-1}} \times \cancel{p-1} = p$$

$$\Rightarrow t^{*T} x = \sum t_i^* x_i = \left(\sum x^{\tilde{p}_i}\right)^{\frac{1}{\tilde{p}}}$$

$$w(B) = E\left[\left(\sum X_i^{\tilde{p}}\right)^{\frac{1}{\tilde{p}}}\right] \leq \left(\sum E[|X_i|^{\tilde{p}}]\right)^{\frac{1}{\tilde{p}}}$$

$$E[|X_i|^{\tilde{p}}] = \int_{+\infty}^{-\infty} \frac{1}{\sqrt{2\pi}} |X_i|^{\tilde{p}} e^{-x_i^2/2} \, dx_i = \frac{2}{\sqrt{2\pi}} \int_{0}^{+\infty} x_i^{\tilde{p}} e^{\frac{-x_i^2}{2}} dx_i$$

$$\frac{2}{\sqrt{2\pi}} \int_{0}^{+\infty} (2u)^{\frac{\tilde{p}-1}{2}} e^{-u} \, du = \underbrace{\frac{2}{\sqrt{2\pi}} 2^{\frac{\tilde{p}-1}{2}} \Gamma\left(\frac{\tilde{p}+1}{2}\right)}_{G(\tilde{p})}$$

$$\frac{-x_i^2}{2} = u \Rightarrow du = \frac{-2x_i}{2} dx_i = -x_i \, dx_i$$

from gamma function attributes:

$$G(\tilde{p}) \leq \frac{2}{\sqrt{2\pi}} 2^{\tilde{p}-1/2} \Gamma(\tilde{p})^{1/2} \Gamma(1)^{1/2}$$

From stirling approx we have. $\Gamma(z) \leq C\sqrt{2\pi(z-1)} \left(\frac{z-1}{e}\right)^{z-1}$

$$G(\tilde{p}) \leq \frac{2^{\tilde{p}+1/2}}{\sqrt{2\pi}} \left(C\sqrt{2\pi(\tilde{p}-1)} \left(\frac{\tilde{p}-1}{e}\right)^{\tilde{p}-1}\right)^{\frac{1}{2}}$$

$$w(B_p^n) \leq \left(\sum_{i=1}^{n} E[|X_i|^{\tilde{p}}]\right)^{\frac{1}{\tilde{p}}}$$

$$\leq \left(\sum_{i=1}^{n} G(\tilde{p})\right)^{\frac{1}{\tilde{p}}} \leq \tilde{C}\, n^{1/\tilde{p}} \sqrt{\tilde{p}}$$

$$\Longrightarrow \quad W(B) \leq \tilde{C}\, n^{1/\tilde{p}} \sqrt{\tilde{p}}$$

$$\Longrightarrow \quad w(B) \lesssim C \min\left(n^{\frac{1}{\tilde{p}}} \sqrt{\tilde{p}}\,,\ \sqrt{\log(n)} \right)$$