HDP Hw1:

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Q1) we want to prove

$$\inf_{k=0,1,2,\dots} \frac{E[|X|^k]}{\delta^k} \leq \inf_{\lambda>0} \frac{E[e^{\lambda X}]}{e^{\lambda\delta}}$$

$$e^{\lambda\delta} = 1 + (\lambda\delta) + \frac{(\lambda\delta)^2}{2!} + \dots = \sum_{i=0}^{\infty} (\lambda\delta)^i / i!$$

$$E[e^{\lambda X}] = E\left[1 + \lambda X + \lambda^2 X^2/2! + \lambda^3 X^3/3! \dots\right] = E\left[\sum_{i=0}^{\infty} (\lambda X)^i / i!\right] =$$

$$\sum_{i=0}^{\infty} E[(\lambda X)^i] \leq \sum_{i=0}^{\infty} \lambda^i E[|X|^i]$$

$$\forall_{\lambda>0} \quad \frac{E[e^{\lambda X}]}{E[e^{\lambda\delta}]} = \frac{E[e^{\lambda X}]}{e^{\lambda\delta}} = \frac{E\left[\sum_{k=0}^{\infty} (\lambda X)^i / i!\right]}{E\left[\sum_{k=0}^{\infty} (\lambda\delta)^i / i!\right]}$$

now set $E\left[\frac{(\lambda X)^i}{i!}\right] : t_i$

and set $E\left[(\lambda\delta)^i / i!\right] : l_i$

We want to prove that $\inf_k \frac{t_i}{l_i} \leq \frac{\sum t_i}{\sum l_i}$ $\forall\lambda$

Proof: assum $\inf_{k:\{0,\dots,k\}} \frac{t_i}{l_i} = c$ then we have $\forall_j \; c \leq \frac{t_j}{l_j}$

$$\Rightarrow \forall_j \; c l_j \leq t_j \Rightarrow \sum_{j=0}^{k} c l_j \leq \sum_{j=0}^{k} t_j$$

$$\Rightarrow c\left(\sum l_j\right) \leq \sum t_j \Rightarrow \boxed{c \leq \frac{\sum l_j}{\sum t_j}} \quad (\sum)$$

now by using I we have

$$\inf_{k=0,\dots,k} \frac{E[(\lambda X)^k / k!]}{E[(\lambda\delta)^k / k!]} \leq \frac{E[e^{\lambda X}]}{E[e^{\lambda\delta}]}$$

So now we just should prove that for every λ this is valid $\Rightarrow$

To do that we have to mention that for Proving Prior lemma (I) we didn't have any assumption of λ except for $\lambda > 0$ then we have.

$$\inf_{k=0,\dots,k} \frac{E[(\lambda x)^k/k!]}{E[(\lambda \gamma)^k/k!]} \leq \frac{E[e^{\lambda x}]}{E[e^{\lambda \gamma}]}$$ now assume

for $\lambda = \lambda^*$ λ^* = the λ that $\inf_{\lambda > 0} \frac{E[e^{\lambda x}]}{e^{\lambda \gamma}}$

so we would have λ^* in LHS of inequality

$$\Rightarrow \inf_k \frac{E[(\cancel{\lambda^*} x)^k/\cancel{k!}]}{E[(\cancel{\lambda^*} \gamma)^k/\cancel{k!}]} = \inf_k \frac{E[|x|^k]}{\gamma^k}$$ since $x \geq 0$

so LHS is independent of λ so

$$\Rightarrow \inf_{k \in \mathbb{Z}} \frac{E[|x|^k]}{\gamma^k} \leq \inf_{\lambda > 0} \frac{E[e^{\lambda x}]}{e^{\lambda \gamma}}$$ ✓

we conclude the Left side is better!

Q2)

a) $P[|X| \geq t] = F(-X) + 1 - F(X) = 1 - (F(X) - F(-X))$

we want to Proof $E[\Phi(|X|)] = \Phi(0) + \int_0^\infty \Phi'(t)\, \underbrace{P[|X| \geq t]}_{1 - F(x) + F(-x)}\, dt$

$$E[\Phi(X)] = \int_{-\infty}^{\infty} \Phi(|X|)\, d(F(X)) =$$

$d(1 - F(x)) = -d(F(x))$ so put (-1)

$$\int_0^{\infty} \Phi(|x|)\, \underbrace{d(F(-x))}_{-1} + \int_0^{\infty} \Phi(|x|)\, \underbrace{d(1 - F(x))}_{-1} =$$

$-\Phi(-\infty) \times (0) - (-\Phi(0)(1)) = \Phi(0)$

$$\int_0^{\infty} -\Phi(x)\, d(1 - F(x) + F(-x)) = \left[-\Phi(x)(1 - F(x) + F(-x))\right]_0^{\infty}$$

$$+ \int_0^{\infty} \phi'(x)(1 - F(x) + F(-x))\, d(x) = \boxed{\Phi(0) + \int_0^{\infty} \phi'(t)\, (P(|X| \geq t)\, dt}$$

b) $E[e^{X^2/6\sigma^2}] = 1 + \int_0^{\infty} \frac{2x}{3\sigma^2} e^{x^2/6\sigma^2} \times e^{-t^2/2\sigma^2} = \frac{2}{3\sigma^2} \int_0^{\infty} x e^{-t^2/3\sigma^2} dt + 1$

$$\frac{2}{3\sigma^2} \underbrace{\int_0^{\infty} x e^{-x^2/3\sigma^2} dx}_{\left[3\sigma^2 x - \frac{1}{2} x e^{-x^2/3\sigma^2}\right]_0^{\infty}} + 1 =$$

$\times \frac{2}{3\sigma^2} = \left[-e^{-t^2/3\sigma^2}\right]_0^{\infty} = 0 - (-1) = 1$

$\Longrightarrow \quad 1 + 1 = 2 \Longrightarrow E[e^{x^2/6\sigma^2}] \leq 2$

c) by definition $E[e^{\lambda x}] \leq \exp\left(\frac{\sigma^2 \lambda^2}{2}\right)$

and $P[x \geq t] \leq e^{-t^2/2\sigma^2}$

for large λ's: $E[e^{\lambda x}] \leq E[e^{\frac{a\lambda^2}{2} + \frac{x^2}{2a}}]$

$$e^{\frac{a\lambda^2}{2}} E[e^{\frac{x^2}{2a}}] \xrightarrow{a=3\sigma^2} e^{\frac{3\sigma^2\lambda^2}{2}} \underbrace{E[e^{x^2/6\sigma^2}]}_{=2}$$

$2 \times e^{\frac{3\sigma^2\lambda^2}{2}} \leq$ should be less than $e^{\frac{18\sigma^2\lambda^2}{2}}$

$$2e^{\frac{3\sigma^2\lambda^2}{2}} \leq e^{9\sigma^2\lambda^2} \Rightarrow \ln(2) + \frac{3\sigma^2\lambda^2}{2} \leq 9\sigma^2\lambda^2 \Rightarrow$$

$$\Rightarrow \ln(2) \leq \lambda^2\left(\frac{15}{2}\sigma^2\right) \Rightarrow \frac{2\ln(2)}{15\sigma^2} \leq \lambda^2 \Rightarrow$$

$$|\lambda| \geq \sqrt{\frac{2\ln(2)}{15}} \times \frac{1}{\sigma^2}$$

for small λ's:

param that will $f''(c)$: should be upper bound.

$$f(x) \leq f(0) + f'(0)x + \frac{f''(c)}{2!}x^2$$

$$E[e^{\lambda x}] \leq 1 + \lambda \cancel{E[x]}^{c} + E\left[\frac{\lambda^2 e^{\lambda c}}{2!} x^2\right]$$

Since $e^{\lambda x}$ is increasing with $x \Rightarrow e^{\lambda c} \leq e^{|\lambda x|}$

$$\Rightarrow E[e^{\lambda x}] \leq 1 + \lambda^2 E\left[\frac{e^{|\lambda x|}}{2!} x^2\right] = 1 + \frac{\lambda^2}{2} E[e^{|\lambda x|} x^2] \quad (\checkmark)$$

$$E[e^{\lambda x}] \leq 1 + \frac{\lambda^2}{2} E[x^2 e^{|tx|}] \leq 1 + \frac{\lambda^2}{2} E[x^2 e^{\frac{a\lambda^2}{2} + \frac{x^2}{2a}}]$$

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$$\leq 1+\frac{t^2}{2}e^{t^2a/2}\,E\left[x^2 e^{t^2/2\,a}\right] \leq 1+\frac{t^2}{2}a\,e^{\frac{t^2}{2}a}\,E\left[e^{x^2/a}\right]$$

$$\leq (1+t^2a)\,e^{\frac{t^2a}{2}}$$

$$a=6\sigma^2 \Rightarrow \quad \leq (1+6\sigma^2)\,e^{3\sigma^2t^2}$$

so now this bould be less than $e^{9\sigma^2\lambda^2}$

$$(1+6\sigma^2)\,e^{3\sigma^2\lambda^2} < e^{9\sigma^2\lambda^2} \Rightarrow$$

$$\ln(1+6\sigma^2)+3\sigma^2\lambda^2 < 9\sigma^2\lambda^2 \Rightarrow \ln(1+6\sigma^2) < 6\sigma^2\lambda^2$$

$$\Rightarrow |\lambda| \geq \frac{\sqrt{\ln(1+6\sigma^2)}}{\sqrt{6}\,\sigma}$$

d) we want to Prove $\forall k \in N \quad E[X^{2k}] \leq 2(2\sigma^2)^k k!$

$$X^{2k} \rightsquigarrow \leq 0 + \int_0^\infty 2k\, t^{2k-1} e^{-t^2/2\sigma^2} dt =$$

$$2k \int_0^\infty t^{2k-1} e^{-t^2/2\sigma^2} dt = 2k \int_0^\infty 2\sigma^2 \times \frac{1}{2\cancel{t}} \cancel{t^{2k-1}}\, t^{2k-2} e^{-w}$$

$$w = t^2/2\sigma^2 \Rightarrow dw = \frac{2t\, dt}{2\sigma^2} \Rightarrow dt = \frac{2\sigma^2}{2t} dw$$

$$= 2k \int_0^\infty \underbrace{\sigma^2 (t^2)^{k-1}}_{\sigma^2 (2\sigma^2 w)^{k-1}} e^{-w} dw = 2k\sigma^2 \int_0^\infty (2\sigma^2 w)^{k-1} e^{-w} dw$$

$$= \underbrace{2k\sigma^2\, 2^{k-1} (\sigma^2)^{k-1}}_{2^k (\sigma^2)^k} \int_0^\infty w^{k-1} e^{-w} dw = k\,(2\sigma^2)^k \int_0^\infty w^{k-1} e^{-w} dw$$

$$\int_0^\infty \underbrace{x^{a-1}}_{f} \underbrace{e^{-x}}_{g'} dx = \cancel{-e^{-x} x^{a-1} \Big|_0^\infty} + \int_0^\infty (a-1) x^{a-2} e^{-x} dx =$$

$$\ldots \quad (a-1)!\, e^{-x} \Big|_0^\infty = (a-1)!$$

$$\Longrightarrow E[X^{2k}] \leq k(k-1)!\,(2\sigma^2)^k \leq (2\sigma^2)^k k! \quad \text{d}$$

e) by definition: $E[e^{\lambda x}] \leq e^{\lambda^2 v^2/2} \quad \forall \lambda < \frac{1}{\alpha}$

$\text{subE}(v^2, \alpha)$

$$E\left[e^{\lambda(x^2 - E[x^2])}\right] = E\left[1 + \sum_{i=1}^{\infty} \frac{(\lambda(x^2 - E[x^2]))^i}{i!}\right]$$

$$= 1 + \sum_{i=1}^{\infty} \frac{\lambda^i E[(x^2 - E[x^2])^i]}{i!} =$$

for $i=1 \rightarrow E[x^2] - E[x^2] = 0 \Rightarrow i = 2 \rightarrow \infty$

$$= 1 + \sum_{i=2}^{\infty} \frac{\lambda^i E[(x^2 - E(x^2))^i]}{i!} \quad (I)$$

apply Jenson and set $-$ to $+$

$$I \leq 1 + \sum_{k=2}^{\infty} \frac{\lambda^k (E[x^2] + E[x^2])^k}{k!}$$

Jenson $\leq (E[x^2] + E[x^2])^k$

$2^k (E[x^2])^k \overset{\text{Jenson}}{\leq} 2^k E[|x|^{2k}]$

$$I \leq 1 + \sum_{k=2}^{\infty} \frac{\lambda^k 2^k E[|x|^{2k}]}{k!}$$

lemma:

X be $\text{subG}(\sigma^2)$ the we have

$$E[|x|^2] \leq (2\sigma^2)^{2/2} \cdot 2 \cdot \Gamma(2/2)$$

Proof: $E[|x|^k] = \int_0^{\infty} P(|x|^k > t)\, dt = \int_0^{\infty} P(|x| \geq t^{1/k})\, dt$

$$\leq 2 \int_0^{\infty} e^{-\frac{t^2/k}{2\sigma^2}}\, dt = (2\sigma^2)^{k/2}\, k\, \Gamma\left(\frac{k}{2}\right)$$

set $k = 2k \Rightarrow E[|x|^{2k}] \leq (2\sigma^2)^k \underbrace{k\,(k-1)!}_{k!}$

$$\Rightarrow I \leq 1 + \sum_{k=2}^{\infty} \frac{\lambda^k 2^k (2\sigma^2)^k \cancel{k!}}{\cancel{k!}} \leq 1 + \sum_{k=2}^{\infty} \lambda^k 4^k (\sigma^2)^k$$

$$\leq 1 + 2\sum_{k=2}^{\infty} (4\sigma^2\lambda)^k = 1 + 4\sigma^2\lambda^2 \times \frac{1}{1-4\sigma^2\lambda}$$

s.t. $4\sigma^2\lambda \leq 1 \Rightarrow \boxed{|\lambda| \leq \frac{1}{4\sigma^2}}$

for $|\lambda| \leq \frac{1}{4\sigma^2}$ we have.

$$\leq \exp\left(\frac{16\sigma^4\lambda^2}{1-4\sigma^2\lambda}\right) \leq \exp(32\sigma^4\lambda^2)$$

$$\Rightarrow X^2 \sim \mathrm{subE}(\overset{v}{\overset{\shortparallel}{8\sigma^2}}, \overset{\alpha}{\overset{\shortparallel}{8\sigma^2}})$$

$$1-4\sigma^2\lambda \geq \frac{1}{2} \Rightarrow \frac{1}{2} \geq 4\sigma^2\lambda \Rightarrow |\lambda| \leq \frac{1}{8\sigma^2}$$

$$\leq \exp(32\sigma^4\lambda^2)$$

Q3)

a) $E[e^{\lambda x}] \rightarrow$ apply Jenson

$$e^{\lambda E[x]} \leq E[e^{\lambda x}] \leq e^{\lambda^2\sigma^2/2 + \lambda\mu}$$

$$\Longrightarrow \lambda E[x] \leq \lambda\mu + \lambda^2\sigma^2/2 \Rightarrow$$

for $\lambda > 0 \Longrightarrow E[x] - \mu \leq \lambda\sigma^2/2$

for $\lambda < 0 \Longrightarrow E[x] - \mu \geq \lambda\sigma^2/2$

$$\Longrightarrow \frac{\delta\sigma^2}{2} \leq E[x] - \mu \leq \frac{\lambda\sigma^2}{2} \Longrightarrow$$

set $\delta \to 0^-$, $\lambda \to 0^+ \Longrightarrow 0 \leq E[x] - \mu \leq 0$

$$\Longrightarrow \boxed{E[x] = \mu}$$ [redacted]

b) $$E[e^{\lambda(x-\mu)}] = E\left[\sum \frac{((x-\mu)\lambda)^k}{k!}\right] =$$

$$1 + \cancel{\lambda E[x-\mu]}^{\,0} + E(x-\mu)^2 \frac{\lambda^2}{2} + o(\lambda^2) \Rightarrow$$

$$\underbrace{\frac{1}{\lambda^2} E[e^{\lambda(x-\mu)}]} = \frac{1}{\lambda^2} + E[(x-\mu)^2] \times \frac{1}{2} + o(1)$$

$$\frac{1}{\lambda^2} + E[(x-\mu)^2] \times \frac{1}{2} + o(1) \leq \frac{1}{\lambda^2} e^{\lambda^2\sigma^2/2} e^{\lambda\lambda}$$

$$\leq \frac{1}{\lambda^2}\left(1 + \lambda^2\sigma^2/2 + \frac{\lambda^4\sigma^4}{4} + o(\lambda^2)\right)$$

$$\Longrightarrow \cancel{\frac{1}{\lambda^2}} + E[(x-\mu)^2] \times \frac{1}{2} + \cancel{o(1)} \leq \cancel{\frac{1}{\lambda^2}} + \sigma^2/2 + \frac{\lambda^2\sigma^4}{4} + \cancel{o(1)}$$

$\Longrightarrow$ now set $\lambda \to 0 \Longrightarrow E[(x-\mu)^2] \times \frac{1}{2} \leq \frac{\sigma^2}{2} \Rightarrow$

$$E[(x-\mu)^2] \leq \sigma^2$$ [redacted]

c) $g(\lambda) = \lg E[e^{\lambda x}] - \frac{\lambda^2\sigma^2}{2} - \lambda\mu$

imagin Bernoulirandom variable with Probability of P

$var(X) = P(1-P)$

$E[X] = P$ $\qquad E[e^{\lambda x}] = Pe^{\lambda} + (1-P)$

$$Pe^{\lambda} + (1-P) \leq \exp\left(\lambda P + \frac{\sigma^2\lambda^2}{2}\right)$$

$g'(\lambda)$

$\exp(g(\lambda)) = Pe^{\lambda} + (1-P) - \frac{\lambda^2\sigma^2}{2} - \lambda P \Rightarrow$

$\sup_{\lambda} g'(\lambda) \rightarrow Pe^{\lambda} - \lambda\sigma^2 - P = 0 \Rightarrow$

$Pe^{\lambda} - \lambda\sigma^2 - P = 0 \Rightarrow \sigma^2 = \frac{Pe^{\lambda} - P}{\lambda}$

set $\lambda = 1 \Rightarrow \sigma^2 = Pe - P$

$\sigma^2 = Pe - P = P(e-1) =$

set $P = \frac{1}{2} \Rightarrow var(X) = \frac{1}{4}$ but above gives us

$P(e-1) = \frac{1}{2} \times (1.7) = .185 \Rightarrow var(X) \neq \sigma^2$

we can take another approach by using proxy function $\Rightarrow$ from stack ~~ex~~ du angle.

$$X \sim \begin{cases} 1-P & \rightarrow P \\ -P & \rightarrow 1-P \end{cases} \qquad E[X] = 0 \qquad E[X^2] = P(1-P)$$

$$E[e^{\lambda x}] = P \times e^{(1-P)\lambda} + (1-P)\, e^{-P\lambda} \leq e^{\frac{1}{2}\sigma^2\lambda^2 + P\lambda} \Rightarrow$$

$$\frac{2}{\lambda^2}\left(\lg(1-P+Pe^{\lambda}) - P\lambda\right) \leq \sigma^2 \Rightarrow$$

$$\sup_{\lambda} \frac{2}{\lambda^2}\left(1-P+Pe^{\lambda}) - P\lambda\right) \leq \sigma^2$$

at $\lambda = 1 \xrightarrow{P=1/4}$ we have

$\frac{1}{4} \times \frac{3}{4} = \frac{3}{16} \Rightarrow var(X) = .1875$

but $LHS = .12 \Rightarrow .12 \leq .18$. ✗ counter example ∎

Q4)

a) $\lg E[e^{\lambda(X_i - E[X_i])}]$ $\qquad Z = X_i - E[X_i] \Rightarrow |Z| \leq b$

$E[Z] = 0$

$$\lg\Big(E\big[\underbrace{1 + (X_i - E[X_i])\lambda + \frac{\lambda^2}{2}(X_i - E[X_i])^2 + \frac{\lambda^3}{3!}(X_i - E[X_i])^3 + \dots}_{I}\big]\Big)$$

$$I \leq 1 + \cancel{\lambda Z}^{\,0} + \frac{\lambda^2}{2}Z^2 + \sum_{k=3}^{\infty}\frac{(\lambda b)^k}{k!}$$

$$\Rightarrow E[I] = 1 + E(Z^2)\frac{\lambda^2}{2} + \sum_{k=3}^{\infty}\frac{E(\lambda b)^k}{k!} \leq 1 + \frac{\lambda^2\sigma^2}{2} + \sum_{k=3}^{\infty}\frac{E(\lambda b)^{k-2}\sigma^2}{k!}$$

$$\leq 1 + \lambda^2\sigma^2\underbrace{\left(\sum_{k=2}^{\infty}\frac{\lambda^{k-2}b^{k-2}}{k!}\right)} = 1 + \lambda^2\sigma^2\left(\sum_{k=2}^{\infty}\frac{\lambda^k b^k}{k!} \times \lambda^{-2}b^{-2}\right)$$

$$= 1 + \cancel{\lambda^2}\sigma^2\left(\left(\sum_{k=0}^{\infty}\frac{\lambda^k b^k}{k!} - \lambda b - 1\right)\cancel{\lambda^{-2}}b^{-2}\right) = 1 + \frac{\sigma^2}{b^2}\left(e^{\lambda b} - \lambda b - 1\right)$$

$$\Rightarrow 1 + \frac{\sigma^2}{b^2}\left(e^{\lambda b} - \lambda b - 1\right) \leq e^{\frac{\sigma^2}{b^2}\left(e^{\lambda b} - \lambda b - 1\right)} \quad \blacksquare$$

b) $P\left(\sum(X_i - EX_i) \geq n\gamma\right) \Rightarrow P\left(\exp\left(\lambda\sum(X_i - EX_i)\right) \geq \exp(n\lambda\gamma)\right)$

$\lambda > 0$

chernoff

$$\Rightarrow P \leq e^{-n\lambda\gamma}E\left[e^{\lambda\sum(X_i - EX_i)}\right] = e^{-n\lambda\gamma}\prod_{k=1}^{n}E\left[e^{\lambda(X_k - EX_k)}\right]$$

$$\Rightarrow \log P \leq -n\lambda\gamma + \sum \lg E\left[e^{\lambda(X_i - EX_i)}\right]$$

from a we have $\Rightarrow \lg P \leq -n\lambda\gamma + \sum \sigma_i^2\cancel{\lambda^2}\left[\frac{e^{\lambda b} - 1 - \lambda b}{\cancel{\lambda^2}b^2}\right]$

$$\Rightarrow \lg P \leq -n\lambda\gamma + \left(e^{\lambda b} - \lambda b - 1\right)\sum\sigma_i^2 = -n\lambda\gamma + \bigcirc\,\sigma^2 n$$

$$\Rightarrow \frac{-n\sigma^2}{b^2}\left(\frac{\lambda\gamma b^2}{\sigma^2} - e^{\lambda b} + \lambda b + 1\right) \Rightarrow$$

$$\log P \leq \inf_{\lambda} \left\{ \frac{-n\sigma^2}{b^2} \left(\frac{b\delta}{\sigma^2}\lambda b - e^{\lambda b} + \lambda b + 1 \right) \right\}$$

$$\Rightarrow \frac{\partial}{\partial \lambda} = \frac{-n\sigma^2}{b^2}\left(\frac{b^2\delta}{\sigma^2} - b e^{+\lambda b} + b \right) = 0 \Rightarrow \frac{b^2\delta}{\sigma^2} - b e^{\lambda b} + b = 0 \Rightarrow$$

$$e^{\lambda b} = \frac{b\delta}{\sigma^2} + 1 \Rightarrow \lambda b = \log\left(\frac{b\delta}{\sigma^2}+1\right) \Rightarrow$$ subtitude this in to above.

$$\log P \leq \frac{-n\sigma^2}{b^2}\left(\frac{b\delta}{\sigma^2}\log\left(\frac{b\delta}{\sigma^2}+1\right) - \frac{b\delta}{\sigma^2} - 1 + \left(\log\frac{b\delta}{\sigma^2}+1\right) + 1 \right)$$

$$\leq \frac{-n\sigma^2}{b^2}\underbrace{\left(\log\left(\frac{b\delta}{\sigma^2}+1\right)\left[\frac{b\delta}{\sigma^2}+1\right) - \frac{b\delta}{\sigma^2} \right]}_{h\left(\frac{b\delta}{\sigma^2}\right)} = \frac{-n\sigma^2}{b^2} h\left(\frac{b\delta}{\sigma^2}\right)$$

$$\Rightarrow P \leq \exp\left(\frac{-n\sigma^2}{b^2} h\left(\frac{b\delta}{\sigma^2}\right) \right)$$ ∎

$|z| < 3$

c) Bernstein: $e^z \leq 1 + z + \frac{z^2}{2}\left(\sum_{k=0}^{\infty} \frac{|z|^k}{3^k}\right) = 1 + z + \frac{z^2}{2} \times \left(\frac{1}{1-\frac{|z|}{3}}\right)$

$$\Rightarrow E\left[e^{\lambda x}\right] \leq 1 + 0 + \underbrace{\cancel{E\left[\frac{\lambda^2 x^2}{2}\right]}}_{\sigma^2\lambda^2/2}\left(\frac{1}{1-\frac{|\lambda| b}{3}}\right) \Rightarrow$$

$$E\left[e^{\lambda x}\right] \leq e^{\frac{\lambda^2\sigma^2}{2(1-\frac{\lambda b}{3})}}$$

$\frac{\lambda b}{3} < 1 \Rightarrow \lambda < \frac{3}{b}$

$$\Rightarrow \log P \leq -n\lambda\delta + \sum \frac{\lambda^2 \overbrace{\sigma_i^2}^{n\sigma^2}}{2(1-\frac{\lambda b}{3})} = \frac{-n\sigma^2}{b^2}\left(\frac{\lambda\delta b}{\sigma^2(1-\frac{\lambda b}{3})} - \frac{b^2\lambda^2 \overbrace{\sum \sigma_i^2}^{n\sigma^2}}{(1-\frac{\lambda b}{3})} \right)$$

a bunch of optimization we get

$$t = \frac{b\delta}{\sigma^2} : \quad \frac{-n\sigma^2}{b^2}\left(\frac{t^2}{2(1+t/3)}\right) \rightsquigarrow g(t) = \frac{t^2}{2(1+t/3)}$$

we want to proove $\frac{-n\sigma^2}{b^2} h(t) \leq \frac{-n\sigma^2}{b^2} g(t) \Rightarrow$

$\Rightarrow$ we want to proove $g(t) \leq h(t) \Rightarrow f(t) = h(t) - g(t)$

some calculation $\Rightarrow$

$\Rightarrow$

$$\left.\begin{array}{l} f(0)=0 \\ f'(0)=0 \\ f''(x)\geq 0 \\ f'(x)\geq 0 \end{array}\right\} \text{ so } \quad t>0 \Longrightarrow f(t)\geq 0 \Longrightarrow$$

$$h(t)-g(t)\geq 0 \Rightarrow h(t)\geq g(t)$$

Q5) assume without loss of generasity that $\sigma = 1$

let $Q = U \Lambda U^T \Rightarrow \langle X, QX \rangle = \sum_{i=1}^{n} \lambda_i X_i^2 = Z$

we know $X \sim N(0,1)$ then X^2 is subE $(2,4)$ therefore

we can conclude that $\lambda_i X_i^2$ is subE $(2\lambda_i, 4\lambda_i)$
which will lead us to Z is subE with

sum of σ_i and max of α

$$\Rightarrow v(v,\alpha) = \left(2\left(\sum_{i=1}^{n} \lambda_i^2\right)^{1/2} ; 4 \max \lambda_i\right) = \left(2\|Q\|_F, 4\|Q\|_2\right)$$

so we have $P(X^T Q X \geq tr[Q] + t) \leq \exp\left(-\min\left(\frac{t}{2\alpha}, \frac{t^2}{2v^2}\right)\right)$

$$\leq \exp\left(-\min\left(\frac{t}{8\|Q\|_2}, \frac{t^2}{8\|Q\|_F}\right)\right)$$

inspired from HDP github sols.

Some related assumption to the Problem.

$Z = X^T Q X$, $X = [X_1, \dots, X_n]^T$

$$E[Z] = \sum_{ij} E[Q_{ij} X_i X_j] = \sum Q_{ij} E[X_i X_j]$$

independence of X_i implies $\Rightarrow E[X_i X_j] = \begin{cases} i \neq j & 0 \\ i = j & E[X_i^2] \end{cases}$

$E[X_i^2] = var(X) + E[X]^2 = 1 + 0 = 1$

$$\sum Q_{ij} E[X_i X_j] = \sum Q_{ii} E[X_i^2] = Tr(Q)$$

$P(Z - E[Z] > t) \leq$

I saw another Proof in a Paper published 2013
Hanson – Wright inequalities and subg concentration.

Proof inspired by paper!

$$P(x^TQx - E[x^TQx] > t) = P\Big[\underbrace{\sum_{ij} Q_{ij}x_ix_j - \sum_i Q_{ii}E[x_i^2]}_{P_2} > t\Big]$$

$$\leq \underbrace{P\Big(\sum Q_{ii}(x_i^2 - E[x_i^2]) > t/2\Big)}_{P_1} + \underbrace{P\Big[\sum_{j\neq i} Q_{ij}x_ix_j > t/2\Big]}_{P_2}$$

note that $x_i^2 - E[x_i^2]$ are independent and zero-mean subE random variables:

$P_1 \leq \exp\left(-c\min\left(\frac{t^2}{\|Q\|_{Fr}}, \frac{t}{\|Q_2\|_2}\right)\right) \rightarrow$ from Vershynin remark 5.18 and lemma 5.14

now from Bernstein-type ineq (Vershynin - Proposition 5.16):

$$P_1 \leq \exp\left(-c\min\left(\frac{t^2}{\|Q\|_{Fr}}, \frac{t}{\|Q\|_2}\right)\right)$$

P_2 : $S = \sum_{i\neq j} Q_{ij}x_ix_j$ By chebyshev ineq we have:

$$P_2 : P(S > t/2) = P[\lambda S > \lambda t/2] \leq \exp\left(\frac{-\lambda t}{2}\right) E[e^{\lambda S}]$$

now we set $\delta_i \sim Bern(\frac{1}{2})$

$E[\delta_i(1-\delta_j)] = \frac{1}{4}$ for $i\neq j$ and zero for $i=j$

$S = 4E_\delta S_\delta$ where $S_\delta = \sum_{ij} \delta_i(1-\delta_j)Q_{ij}x_ix_j$

By applying Jenson we have

$E[e^{\lambda S}] \leq E_{x,\delta}[e^{4\lambda S_\delta}]$ for now consider set

$\Lambda_\delta = \{i \in [n] : \delta_i = 1\}$ $\quad S_\delta = \sum_{\substack{i\in\Lambda_\delta \\ j\in\Lambda_\delta^c}} Q_{ij}x_ix_j = \sum_{i\in C} x_j\left(\sum_{i\in\delta} Q_{ij}x_i\right)$

S_δ is Linear combination of mean-zero subg

now by vershynin lemma 5.9 $\Rightarrow \|S_g\| \leq C\sigma_g$

$$\sigma_g^2 = \sum_c \sum_s (Q_{ij} x_i)^2$$

from lemma 5.5 (vershynin) : $E[x_j]_{j\in c} \; E \exp(4\lambda S_g)]$

$$\leq \exp(c' \lambda^2 \sigma_g^2)$$

$$\Rightarrow E_{x_i} E[x_i]_j \; E[\exp(4\lambda S_g] \leq E_x \exp(c'\lambda^2 \sigma_g^2)$$

$$Z = \sum g_i (\sum Q_{ij} x_i) \sim N(0, \sigma_g^2)$$

$$g_i \sim N(0,1)$$

$$E_g[\exp(S_Z) = \exp(S^2 \sigma_g^2/2)$$

now set $S^2 = 2c'\lambda^2 \Rightarrow E_g = \begin{cases} E_{xg} \exp(C_1 \lambda Z) \\ C_1 = \sqrt{2c'} \end{cases}$

$$E_g \leq E_g[\exp(C_2 \lambda^2 \sum_j \sum_i (Q_{ij} g_i)^2]$$

$$E_g = [C_2 \lambda^2 \sum s_i^2 g_i^2] = \prod_i E_g \, e^{(C_2 \lambda^2 s_i^2 g_i^2)}$$

from section 4 of paper we have $P_2 \leq \exp\left(\frac{-\lambda t}{2} + C\lambda^2 |Q|_F\right)$

from optimizing over $\lambda \Rightarrow \lambda = \frac{t}{4|Q|_F}$

$$P_2 \leq \exp\left(-C\min\left(\frac{t^2}{|Q|_F}, \frac{t}{|Q|_2}\right)\right)$$

$$P_1 + P_2 \leq 2\exp(\bigcirc)$$

Q6)

a) $E[Y] = \underbrace{E[Y]\{Y \leq aE[Y]\}}_{\leq aE[Y]} + E[Y]\{Y \geq aE[Y]\}$

$(1-a)E[Y] \leq E[Y]\{Y \geq aE[Y]\}$

Cauchy-shawrz inequality:

$E[Y]\{Y \geq aE[Y]\}]^2 \leq E[Y^2] \cdot E[1^2_{\{Y \geq aE[Y]\}}]$

$E[Y^2] \cdot P(Y \geq aE[Y]) \Longrightarrow$

$((1-a)E[Y])^2 \leq E[Y^2] \; P(Y \geq aE[Y])$

$\Longrightarrow P(Y \geq aE[Y]) \geq \dfrac{(1-a)^2 E^2[Y]}{E[Y^2]}$ ✓

b) for special cases:

$n=1 \longrightarrow P(X > 1/2) \geq 3/32 \Longrightarrow P(X > 1/2) = 1/2$

$\frac{1}{2} \geq 3/32$ ✓

$n=2 \Longrightarrow P(X_1 + X_2 > \underbrace{\frac{\sqrt{2}}{2}}_{<1}) \Longrightarrow$

$\left.\begin{array}{l} 0+0 \\ 0+1 \\ 1+0 \\ 1+1 \end{array}\right\} \Omega$

$P(X_1 + X_2 > \frac{\sqrt{2}}{2}) = P(\Omega) = 3/4$

$3/4 \geq \frac{3}{32}$ ✓

for $n > 2$: $Y = \sum X_i$, $E[Y] = \frac{n}{2}$

$P(Y \geq aE[Y]) = P(Y \geq a\frac{n}{2})$: $a\frac{n}{2} = \frac{\sqrt{n}}{2} \Rightarrow a = \frac{\sqrt{n}}{n}$

and $E[\varphi^2] = var(\varphi) + E^2[\varphi] = \frac{n^2}{4} + \frac{n}{4} \Rightarrow \frac{n(n+1)}{4}$

$$\Rightarrow P\left(\sum X_i \geq \frac{\sqrt{n}}{2}\right) \geq \left(1 - \frac{\sqrt{n}}{n}\right)^2 \times \frac{n^2/4}{\frac{n(n+1)}{4}} = \frac{n+1-2\sqrt{n}}{n+1} \overset{?}{\geq} \frac{3}{32}$$

$$\Rightarrow 1 - 2\sqrt{n}/_{n+1} \geq 3/_{32} \Rightarrow \frac{29}{64} \geq \frac{\sqrt{n}}{n+1} \Rightarrow 29x^2 + 29 \geq 64x$$

$29x^2 - 64x + 29 \geq 0$ for all $n > 2$ which means $x^2 \geq 2$

$$4 \times 29 - \underset{\substack{\parallel \\ 14}}{64 \times \sqrt{2}} + 29 = \overset{8}{16} + 29 - \underbrace{64 - 32}_{-96} = > 0 \Rightarrow$$

for all $n > 2$ ✓ this is valid end of Proof!

Q7) let set function g a mapping from two sequence to number of LCS then we have $g(a_1 \dots, a_n, b_1, \dots b_n) \longrightarrow [0, 2n]$

now we can assume that g is function of $x_1 \dots x_{2n}$

$\Longrightarrow g(x_1 \dots x_{2n}) \Longrightarrow$ now imagine one of the x_i changes from zero to one or visaversa the it is trivial that

$$|g(x_1 \dots x_j' \dots x_n) - g(x_1 \dots x_j \dots x_n)| \leq 1 \quad (I)$$

it is because this change in sequence only can reduce the X_n by one. so statement (I) is valid.

now we can easily apply McDiarmid's inequality

$$\Longrightarrow X_n = g(a_1 \dots a_n, b_1 \dots, b_n) = subG\left(\frac{\sum c_i^2}{4}\right)$$

$\Longrightarrow$ from I we know $c_i = 1$

so we have $\frac{\sum c_i^2}{4} = \frac{\sum 1}{4} = \frac{2n}{4} = \frac{n}{2} \Rightarrow$

$X_n \sim subG\left(\frac{n}{2}\right)$

so $\mathbb{P}(|X_n - E[X_n]| \geq \delta) \leq 2e^{-\frac{\delta^2}{2n/2}} = 2\exp\left(\frac{-\delta^2}{n}\right)$ ✓

Q8)

a) first we would recall the theorm 2.3.1 of vershynin book.

if x_i are Bernoulli with p_i consider their sum $S_N = \sum x_i$ and set $\mu = E[S_N]$

then for any $t > \mu \Longrightarrow P[S_N > t] \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^t$

chernoff: $P[S_N > t] \leq e^{-\lambda t} \prod E[e^{\lambda x_i}]$ $\qquad 1+x \leq e^x$

$E[e^{\lambda x_i}] = e^{\lambda} p_i + (1-p_i) = 1 + (e^{\lambda}-1)p_i \leq \exp((e^{\lambda}-1)p_i)$

$\Longrightarrow \prod E[e^{\lambda x_i}] \leq \exp((e^{\lambda}-1)\sum p_i) \leq \exp((e^{\lambda}-1)\mu)$

subtitude into $\rightarrow \inf_{\lambda > 0} e^{-\lambda t} \exp((e^{\lambda}-1)\mu)$

$\lambda = \ln(t/\mu) \Longrightarrow$

$e^{-t\ln(t/\mu)} \exp((t/\mu - 1)\mu) \Rightarrow P[S_N > t] \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^t$

* $P(d_i > c\log n) \leq e^{-d}\left(\frac{ed}{c\log n}\right)^{c\log n}$ | $d = (n-1)p$ if we assume

$d = O(\log n) \Rightarrow$ $\qquad d \leq K\log n \Longrightarrow$ there exist an α that

$\frac{ed}{c\log n} < \alpha$ which for large n is valid we have $(\alpha < 1)$

$e^{-d}\left(\frac{ed}{c\log n}\right)^{c\log n} \leq \alpha^{c\log n} \Longrightarrow P(\forall i: d \leq c\log n) \geq 1 - \underbrace{n\alpha^{c\log n}}_{n \to \infty}$

$\Longrightarrow P(\forall i\ d \leq c\log n) \geq 1$ $\qquad$ c for $\alpha < 1$

$P[\text{for all } i: d_i \geq c\log n] \leq \sum P(d_i \geq c\log n) \leq n \times n^{-\alpha c}$

$\Longrightarrow P[\forall i\ d_i > c\log n] < n^{1-\alpha c}$ large c we have $\Longrightarrow 0$

$\Longrightarrow P = 0$

So we prooved that this is most likely that all vetices of G has degree of $O(n \lg n)$ ∎

b) using union bound, Prob that a node has degree at least k is $\binom{n-1}{k} p^k q^{n-1-k} \leq \binom{n-1}{k} p^k$

$Z \Rightarrow$ number of nodes with degree at least k

$$E[Z] \leq n \binom{n-1}{k} p^k \leq n \left(\frac{(n-1)e}{k}\right)^k p^k = n \left(\frac{de}{k}\right)^k$$

$$= \exp(\lg n - k \lg k + O(k))$$

for $k \lg k$ outgrow $\lg n$ we set $k = \frac{c \lg n}{\lg \lg n}$ $c > 2$

$$k \lg k = k \{ \lg c + \lg \lg n - \lg \lg \lg n \} = (1 + o(1)) k \lg \lg n \Rightarrow$$

$$= (1 + o(1)) c \lg n$$

$\Rightarrow$ $\lg n - k \lg k + O(k)$ goes to $-\infty$ with n

$\Rightarrow$ $\exp(\lg n - k \lg k + O(k)) = o(1)$

$\Rightarrow$ so this graph is most likely to have all degree $O\left(\frac{\lg n}{\lg \lg n}\right)$ ∎ (with help of mark exchange)