Fast sinc-transform for reconstruction of non-uniformely sampled images

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Interpolating images using the sinc-interpolation formula

A continuous signal, s_c, is sampled at discrete points x_n, so than $s[x_n]$ is the sample values. To re-construct the continuous signal from the discrete samples the discrete values are convolved with the *sinc*-function (Oppenheim and Schafer, 2014; Shannon, 1948):

$$s(x) = \sum_{i=1}^{N} s[x_i] \cdot sinc\left(\frac{x - x_i}{T}\right) \tag{1}$$

where T is the sampling-interval (inverse of frequency) so that $x_n = nT$. This is known as the *Nyquist-Shannon*-interpolation formula. When the sample rate is sufficiently high, satisfying the Nyquist-criterion of at least two times the highest frequency in the signal, the signal can be perfectly reconstructed. Here *sinc* is the normalized sinc-function:

$$sinc(x) = \frac{sin(\pi x)}{\pi x} \tag{2}$$

The fast sinc transform

The Fourier transform of the sinc-transform is the Π-function (rectangle). The Fourier-transform of $sinc^2$ is the Λ-function (triangle). The *sinc*-transform is defined as:

$$Um = \sum_{n=1}^{N} q_n sinc(\mathbf{k}_n - \mathbf{v}_m) \tag{3}$$

The (discrete) convolution in eq. 3 can be calculated quickly using the *NUFFT* library since the convolution in x equals a multiplication in the k-domain:

1. Weight $s[x]$ if non-uniformely spaced.
2. Take forward Fourier transform of $s[x]$ to quadrature nodes (e.g. Gauss-Legendre), to get $S[k]$.
3. Integrate $S[k]$ from $[-1, 1]$.
4. Take inverse Fourier transform of $\int S[k]$

Using the fast sinc transform to reconstruct images from non-uniform images

The approximate (inverse or *adjoint*) Fourier transform (Greengard et al., 2006):

$$\rho(\mathbf{r}_m) \approx \sum_{n=1} Ns(n)e^{-2\pi i \mathbf{k}(n)\cdot\mathbf{r}_m} \cdot w_n \tag{4}$$

Convention

$$sinc(\mathbf{k}) = sinc(k_1) \cdot sinc(k_2) \cdot \ldots \tag{5}$$

Sinc-kernel weights

Optimal weights (Sinc-3 in (Choi and Munson, 1998)):

$$\frac{1}{w_n} = \sum_{m=1}^{N} sinc^2(\mathbf{k}(m) - \mathbf{k}(n)) \tag{6}$$

Jacobian weights

Another choice weights is the difference between samples, Jacobian-weighting, see Sinc-2 in (Choi and Munson, 1998), so that densely sampled regions are scaled down proportionally. For a single-variable scalar function $f(x')$:

$$\mathbf{J} = \frac{\partial x'}{\partial x_{uf}} \tag{7}$$

where x' is the non-uniform samples and x_{uf} is an equidistant monotonically increasing grid.

$$w_n = x_{n+1} - x_n \tag{8}$$

up to $n = N - 1$, and $w_N = w_{N-1}$.

Jacobian of a scalar function of two variables

The points are transformed from non-uniform sampling to uniform sampling:

$x' = x$

$$x' = x(i)$$
$$y' = \theta(y)$$

where x' and y' is the uniform grid coordinates. The Jacobian for the set of equations is:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x'}{\partial \phi(x)} & \frac{\partial x'}{\partial \theta(y)} \\ \frac{\partial y'}{\partial \phi(x)} & \frac{\partial y'}{\partial \theta(y)} \end{bmatrix} \tag{9}$$

NUFFT Type 3

The non-uniform fast Fourier transform (Barnett et al., 2019), NUFFT, type 3 (most general) computes sums of type:

The forward transform:

$$G_j = \sum_{p=1}^{P} g_p e^{-i\mathbf{k}_j \cdot \mathbf{x}_p} \tag{10}$$

or, the inverse (adjoint) transform:

$$g_p = \sum_{j=1}^{J} G_j e^{+i\mathbf{k}_j \cdot \mathbf{x}_p} \tag{11}$$

These use the same implementation, with only the sign of i changed.

Gauss-Legendre quadrature

Weights are found for nodes on interval $[-1, 1]$ (re-scale input to this interval), multiply by weights to numerically integrate. This is exact for a polynominal with degree less or equal to $2n-1$, where n is number of nodes.

$$\int_{-1}^{1} f(x) \approx \sum_{i=0}^{n} w_i \cdot f(x_i) \tag{12}$$

Interpolating from non-uniform points to a uniform grid

$s[x_i]$ is the N samples at non-uniform points $\mathbf{x}$. We wish to find $s'[x_p]$, where x_p is a regular grid and $max(|s'[x] - s[x]|)$ is minimized (*minimax*).

Scheme

1. $S_k = \mathcal{F}\{s(x_i)\}$, the discrete non-uniform Fourier transform of the samples.

References

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