Fast sinc-transform for reconstruction of non-uniformely sampled images

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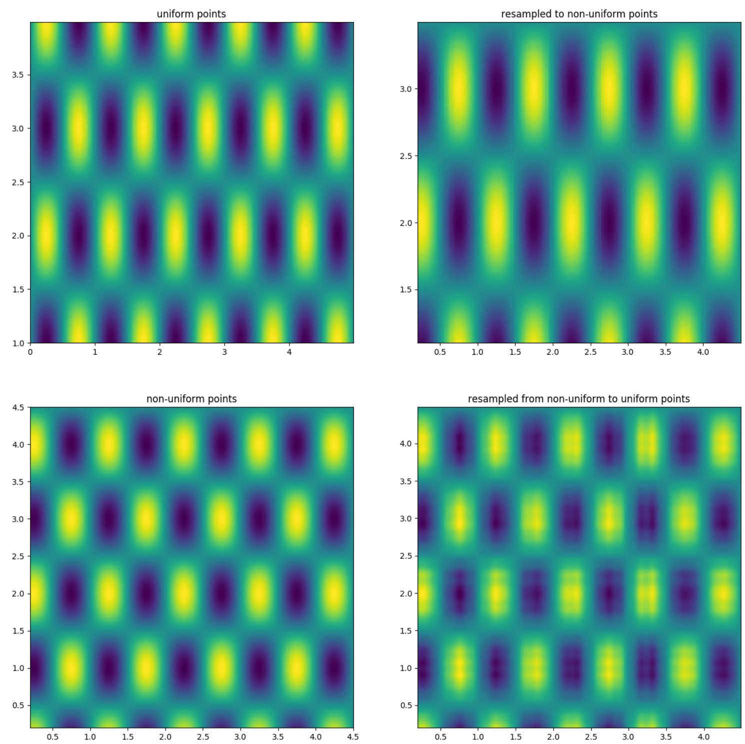

Figure 1: Sinc interpolation from non-uniform to uniform grids

Interpolating signals using the sinc-interpolation formula

A continuous signal, s_c, is sampled at discrete points x_n, so that $s[x_n]$ is the sample values. To re-construct the continuous signal from the discrete samples the discrete values are convolved with the *sinc*-function (Oppenheim and Schafer, 2014; Shannon, 1948):

$$s(x) = \sum_{i=1}^{N} s[x_i] \cdot sinc\left(\frac{x - x_i}{T}\right) \tag{1}$$

where T is the sampling-interval (inverse of frequency) so that $x_n = nT$. This is known as the *Nyquist-Shannon*-interpolation formula. When the sample rate is sufficiently high, satisfying the Nyquist-criterion of at least two times the highest frequency in the signal, the signal can be perfectly reconstructed. This stems from bandpassing the sampled signal. The perfect bandpass filter is a rectangle in the frequency-domain, the Fourier-transform of a rectangle is a *sinc*-function.

Here *sinc* is the normalized sinc-function:

$$sinc(x) = \frac{sin(\pi x)}{\pi x} \tag{2}$$

and

$$sinc^2(x) = \left(\frac{sin(\pi x)}{\pi x}\right)^2 \tag{3}$$

in higher dimensions:

$$sinc(\mathbf{x}) = sinc(x_1) \cdot sinc(x_2) \cdot ... \tag{4}$$

The fast sinc transform

The Fourier transform of the *sinc*-function is the Π-function (rectangle). The Fourier-transform of the $sinc^2$-function is the Λ-function (triangle).

The *sinc*-transform is defined as:

$$Um = \sum_{n=1}^{N} q_n sinc(\mathbf{k}_n - \mathbf{v}_m) \tag{5}$$

The (discrete) convolution in eq. 5 can be performed quickly using the *NUFFT* library since the convolution in x equals a multiplication in the k-domain, from (Greengard et al., 2006, sec. 2.):

1. Weight $s[x]$ according to sample spacing using e.g. the Sinc-3 scheme.
2. Take forward Fourier transform of $s[x]$ to quadrature nodes (e.g. Gauss-Legendre), to get $S[k]$ (re-projecting k_x to $[-1, 1]$).
3. The weights do not need to be scaled since the *sinc*-function is 1 for the frequency-band.
4. Integrate $S[k]$ from $[-1, 1]$ numerically.
5. Take inverse Fourier transform of $\int S[k]$

The Fourier transform on non-uniform samples

The approximate (inverse or *adjoint*) Fourier transform (Greengard et al., 2006):

$$\rho(\mathbf{r}_m) \approx \sum_{n=1}^{N} s(n) e^{-2\pi i \mathbf{k}(n) \cdot \mathbf{r}_m} \cdot w_n \tag{6}$$

The non-uniform fast Fourier transform (Barnett et al., 2019), NUFFT, type 3 (most general) computes sums of type:

The forward transform:

$$G_j = \sum_{p=1}^{P} g_p e^{-i \mathbf{k}_j \cdot \mathbf{x}_p} \tag{7}$$

or, the inverse (adjoint) transform:

$$g_p = \sum_{j=1}^{J} G_j e^{+i \mathbf{k}_j \cdot \mathbf{x}_p} \tag{8}$$

Sinc-kernel weights

Optimal weights (Sinc-3 in (Choi and Munson, 1998; Inati et al., 2005)):

$$\frac{1}{w_n} = \sum_{m=1}^{N} sinc^2(\mathbf{k}(m) - \mathbf{k}(n)) \tag{9}$$

These can be calculated quickly in a similar way as the sinc-transform (eq. 5). The only difference is that the quadrature weights are scaled with the triangle function (Λ).

Jacobian weights

Another choice weights is the difference between samples, Jacobian-weighting, see Sinc-2 in (Choi and Munson, 1998), so that densely sampled regions are scaled down proportionally. For a single-variable scalar function $f(x')$:

$$\mathbf{J} = \frac{\partial x'}{\partial x_{uf}} \tag{10}$$

where x' is the non-uniform samples and x_{uf} is an equidistant monotonically increasing grid.

$$w_n = x_{n+1} - x_n \tag{11}$$

up to $n = N - 1$, and $w_N = w_{N-1}$.

This is easy to approximate for 1d, but trickier to approximate for two or more.

Gauss-Legendre quadrature

Weights are found for nodes on interval $[-1, 1]$ (re-scale input to this interval), multiply by weights to numerically integrate. This is exact for a polynominal with degree less or equal to $2n - 1$, where n is number of nodes.

$$\int_{-1}^{1} f(x) \approx \sum_{i=0}^{n} w_i \cdot f(x_i) \tag{12}$$

The interpolation

Use the fast sinc transform in eq. 5 to evaluate the sinc-interpolation equation in eq. 1. The input samples can be weighted using the optimal weights (eq. 9), which is also calculated using the NUFFT.

References

Barnett, A. H., Magland, J., and Klinteberg, L. A. F. (**2019**). "A parallel nonuniform fast fourier transform library based on an "exponential of semicircle" kernel," SIAM Journal on Scientific Computing, **41**, C479–C504. doi:10.1137/18M120885X

Choi, H., and Munson, D. C. (**1998**). "Analysis and Design of Minimax-Optimal Interpolators," Greengard, L., Lee, J. Y., and Inati, S. (**2006**). "The fast sinc transform and image reconstruction from nonuniform samples in k-space," Communications in Applied Mathematics and Computational Science, **1**, 121–131. doi:10.2140/camcos.2006.1.121

Inati, S., Lee, J.-y., and Fleysher, L. (**2005**). "Fast, optimal weighting for image reconstruction from arbitrary k-space trajectories," Proc Intl Soc Mag . . . , **13**, 2297. Retrieved from http://cds.ismrm.org/ismrm-2005/Files/02297.pdf

Oppenheim, A. V., and Schafer, R. W. (**2014**). *Discrete-Time Signal Processing*, 3rd ed.

Shannon, C. E. (**1948**). "A Mathematical Theory of Communication," Bell System Technical Journal, **27**, 623–656. doi:10.1002/j.1538-7305.1948.tb00917.x