

Tight Bounds on 3-Neighbor Bootstrap Percolation

by

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We acknowledge with respect the Lekwungen peoples on whose traditional territory
the university stands, and the Songhees, Esquimalt, and WSÁNEĆ peoples whose
historical relationships with the land continue to this day.

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ABSTRACT

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DEDICATION

Chapter 1

Introduction

The problem of bootstrap percolation was first introduced in the 1970s by Chalupa et al. [?] as a simplified model for the behavior of ferromagnetic fields. In their original paper, the authors describe bootstrap percolation as the stabilization of a probabilistically occupied lattice, where each occupied site must be adjacent to at least m occupied neighbors. A re-rendering of the examples given in the original 1979 paper is presented in figures 1 and 2.

In this original problem, the authors are interested in the structural patterns of these stable arrangements. Put differently, given a set of randomly distributed occupants on a d -dimensional lattice, what configuration can we expect these occupied sites to fall into, subject to the constraint that each occupied site is adjacent to at least m other occupants? In this construction, a probabilistically populated lattice is iteratively depopulated until it reaches a stable state.

Alternatively, we might consider the behavior of a population as it grows, instead of shrinks. In this model, it is useful to consider the population as harboring an infection that steadily spreads from site to site, subject to population density. We shall consider these infections to take place on a graph, with vertices representing members of our population (sites), and edges indicating adjacency. In formal terms: let G be a graph and $A_0 \subseteq V(G)$ be a set of initially infected vertices. Iteratively, at every time step, infect those vertices of G with at least r infected neighbors. For all $t > 0$, let A_t be the set of infected vertices at time step t . We then have

$$A_t = A_{t-1} \cup \{v \in V(G) : |N_G(v) \cap A_{t-1}| \geq r\},$$

where $N_G(v)$ is the set of vertices adjacent to v in G . We define the *closure* of A_0 under r -neighbor bootstrap percolation to be $[A_0] = \bigcup_{t=0}^{\infty} A_t$. This is analogous to the stable states introduced in [?]. We say that A_0 *percolates* or is *lethal* if $[A_0] = V(G)$. We note that under these rules, it is not possible for vertices to become uninfected.

Perhaps the most natural extremal question regarding r -neighbor bootstrap percolation is that of determining the size of the smallest percolating set $A_0 \subseteq V(G)$, for a

	Grids								
r	$[a_1]$	$[a_1] \times [a_2]$	$[n]^2$	$[a_1] \times [a_2] \times [a_3]$	$[n]^3$	\cdots	$\prod_{i=1}^d [a_i]$	$[n]^d$	$[2]^d$
$r = 0$	0	0	0	0	0		0	0	0
$r = 1$	1	1	1	1	1		1	1	1
$r = 2$	$\lceil \frac{a_1-1}{2} \rceil + 1$	$\lceil \frac{a_1+a_2-2}{2} \rceil + 1$	n	$\lceil \frac{a_1+a_2+a_3-3}{2} \rceil + 1$	$\lceil \frac{3(n-1)}{2} \rceil + 1$		$\lceil \frac{\sum_{i=1}^d (a_i-1)}{2} \rceil + 1$	$\lceil \frac{d(n-1)}{2} \rceil + 1$	$\lceil \frac{d}{2} \rceil + 1$
$r = 3$???	???	$\lceil \frac{n^2+2n+4}{3} \rceil^*$	S.A. bound	n^2		???	???	$\lceil \frac{d(d+3)}{6} \rceil$
\vdots						\ddots			
$r = d$???	???	???	???	???		S.A. bound	n^{d-1}	???

Table 1.1: A summary of known bootstrap percolation results for grids and the torus, $r \in \{0, 1, 2, 3, d\}$.

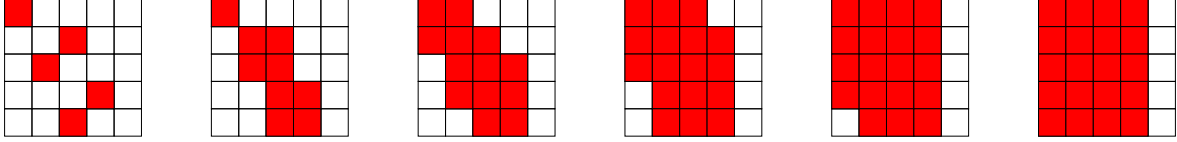


Figure 1.1: Percolation time-steps for an arbitrary initial infection on the 5×5 lattice, $r = 2$.

given graph G . We represent this quantity by $m(G, r)$. There has been a great deal of work done on establishing the value of $m(G, r)$ for various classes of graphs and values of r (see {citations, citations, etc., etc.}). These results are incompletely summarized in table 1.1, and a selection of particularly noteworthy proofs are presented in detail in the following section.

We end this introduction with the presentation of a delightful question about the cardinality of 2-neighbor percolating sets on the two-dimensional lattice. The interested reader is encouraged to find a solution on their own; however, a proof of the result is presented at the beginning of the following section.

Question. Let (n, n) represent the $n \times n$ lattice, given by $G = P_n \times P_n$. What is $m(n, n, 2)$?

1.1 Minimum percolating sets and bounds

1.1.1 Foundations

Let us build some intuition for the behavior of percolating sets on the two-dimensional lattice. Figure 1.1 illustrates the percolation time-steps for an arbitrary initial infection on the graph $G = P_5 \times P_5$, where $r = 2$. (In general, we shall refer to the d -dimensional lattice with sides a_1, \dots, a_d as the d -tuple (a_1, \dots, a_d) . The standard notation is $\prod_{i=1}^d a_i$.)

Note that this configuration fails to infect the entire grid; that is, the initial infection is not lethal. Heuristically, this appears to be a consequence of the fact that

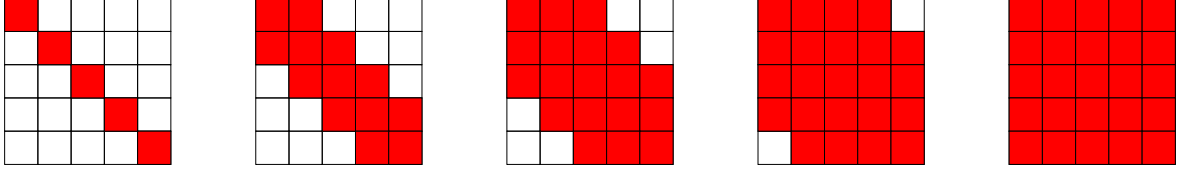


Figure 1.2: Percolation time-steps for a “spanning” initial infection on the 5×5 lattice, $r = 2$.

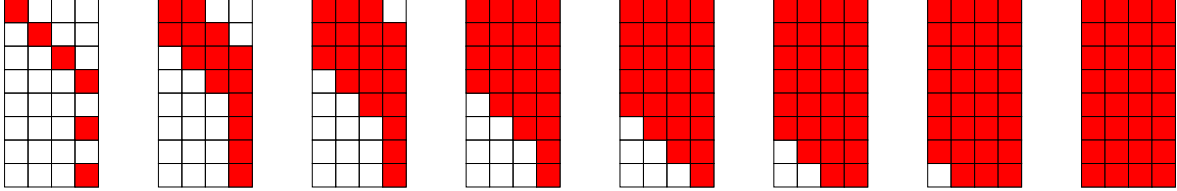


Figure 1.3: A lethal infection on the 8×4 lattice, $r = 2$.

infected cells are unable to access the healthy cells in the rightmost column. We might, therefore, hypothesize that an initial infection must somehow “span” the entire lattice. A potential “spanning” construction is illustrated in figure 1.2. Observe that at each time-step, the infection spreads out laterally from the initial diagonal. It is a simple exercise to verify that this construction is lethal on all (n, n) grids for $r = 2$. We also note that a similar construction is lethal on all (n, m) grids for $r = 2$ (figure 1.3).

The (n, n) construction can be generalized to any dimension. Specifically, we have:

Proposition 1.1. *For all $n, d \geq 1$,*

$$m([n]^d, d) \leq n^{d-1}.$$

This result has been known since at least Pete [?], although the particular constructions are difficult to render in general. The following proof (appearing in [?], but known in bootstrap percolation “folklore” for much longer) elegantly shows that this bound is tight.

Theorem 1.2. *For all $n, d \geq 1$,*

$$m([n]^d, d) = n^{d-1}.$$

Proof. The upper bound follows from proposition 1.1. The lower bound is given by a generalization of the famous “perimeter argument”. Suppose $d = 2$ and consider an embedding of $G = [n]^d$ in the $n \times n$ grid. Let $A_0 \subseteq V(G)$ be a set of initially infected cells. We claim that the total perimeter of all infected regions in G is monotonically decreasing as a function of the time-step t . Consider an arbitrary healthy cell c . In order for c to become infected, at least two of its edges must abut infected cells. However, this implies that (upon infection of c) these edges are absorbed within the

newly expanded infected region, thereby reducing the perimeter of infection by two. As c contains at most two un-absorbed edges, the perimeter of infection cannot increase.

Since a lethal set will infect the entire grid, and therefore have a final perimeter of infection of $4n$, it follows that the perimeter of infection of A_0 must be at least $4n$, and so $m([n]^2, 2) \geq n$.

The same argument generalizes nicely to higher dimensions. Simply observe that for a given hypercube cell to become infected, it must donate at least d of its $2d$ hyperplane faces to the infected region, thereby at most maintaining the current $(d-1)$ -perimeter of infection. \square

We note that the perimeter argument extends directly to rectangular grids; however, the problem of obtaining tight constructions, should they exist, is largely unsolved and will be the main focus of this thesis. The following proposition, which we refer to informally as the *surface area bound* or *S.A. bound*, provides a lower bound on the size of lethal sets for d -dimensional rectangular grids where $r = d$.

Proposition 1.3. *For $d \geq 1$ and $a_1, \dots, a_d \geq 1$,*

$$m(a_1, \dots, a_d, d) \leq \frac{\sum_{i=1}^d \prod_{j \neq i} a_j}{d}.$$

Proof. Observe that the expression

$$\frac{\sum_{i=1}^d \prod_{j \neq i} a_j}{d}$$

is precisely the high-dimensional perimeter of the grid graph (a_1, \dots, a_d) . The bound follows from the perimeter argument in theorem 1.2. \square

In {some chapter of this thesis}, we will prove that this bound is tight in the case where $d = 3$ and $d_1, d_2, d_3 \geq 8$. We fully expect that this bound can be incrementally diminished; however, we feel that such small improvements do not at this time justify the effort required to obtain additional constructions.

In the remainder of this section, we shall present a number of additional bootstrap percolation results for different classes of grid graphs.

1.1.2 Additional results

Recall from the previous section that $m(n, n, 2) = n$, where the tight construction for the lower bound is given by a diagonal infection expanding laterally outwards. In a paper by Balogh and Bollobas [?], this result is generalized to all d -dimensional hypercubes (a_1, \dots, a_d) , $a_i \geq 1$.

Theorem 1.4. For $d \geq 1$ and $a_1, \dots, a_d \geq 1$,

$$m(a_1, \dots, a_d, 2) = \left\lceil \frac{\sum_{i=1}^d (a_i - 1)}{2} \right\rceil + 1.$$

Proof. □

As suggested by table 1.1, general results become quite sparse for $r \notin \{2, d\}$. A nice result from Morrison and Noel resolves the question of $r = 3$ for hypercubes P_2^d of dimension $d \geq 3$.

Theorem 1.5. For $d \geq 3$ and $a_1 = \dots = a_d = 2$,

$$m(a_1, \dots, a_d, 3) = \left\lceil \frac{d(d+3)}{6} \right\rceil.$$

Proof. □

However, the issue of determining $m(a_1, \dots, a_d, r)$ is largely unresolved. Furthermore, good lower bounds for $r \neq d$ are conspicuously absent.

Chapter 2

A Tight Bound on Grids of Size ≥ 7

2.1 Introduction and Definitions

Let the ordered tuple (a, b, c) represent the $a \times b \times c$ grid G where $a \geq b \geq c$. We refer to c as the “thickness” of G . For example, the tuple $(5, 3, 3)$ represents a $5 \times 3 \times 3$ grid of thickness 3. We refer to a tuple as “divisible”, or a “divisibility case”, if and only if $ab + bc + ca \equiv 0 \pmod{3}$. Observe that the divisibility cases are precisely those grids with integral lower bounds. The divisibility cases of thicknesses belonging to the three residue classes modulo 3 are illustrated in {Figure something}.

In the following lemmas, we use the notation $(a, b, c) + (x, y, z) = (a+x, b+y, c+z)$ to represent respective increases of x , y , and z to the side lengths a , b , and c of G . We note the following:

Remark 2.1. By applying the recursion, $(a, b, c) + (x, y, z)$ percolates at the lower bound when either:

1. $(a, b, c), (a, y, z), (x, b, z), (x, y, c)$ all percolate at the lower bound, or;
2. $(x, y, z), (x, b, c), (a, y, c), (a, b, z)$ all percolate at the lower bound.

We shall call a thickness “complete” if it can be shown that all divisibility cases in that thickness percolate at the lower bound. In this section, we demonstrate that thickness 6, thickness 7 and thickness 8 are all complete. As these belong to the residue classes 0, 1, and 2 modulo 3, respectively, we then use a recursive construction to show that all larger grids are also complete.

2.2 Completeness of Thickness 6

We shall show that all grids of thickness 6 can be obtained recursively from $(3, 3n, m)$, where $n, m \equiv 1 \pmod{2}$ (see construction {something}), and one of $\{(3, 3, 3), (6, 3, 3), (6, 6, 3)\}$. As the following proof requires examining a number of cases, we separate thickness 6 into three categories, and show that each of these categories is complete.

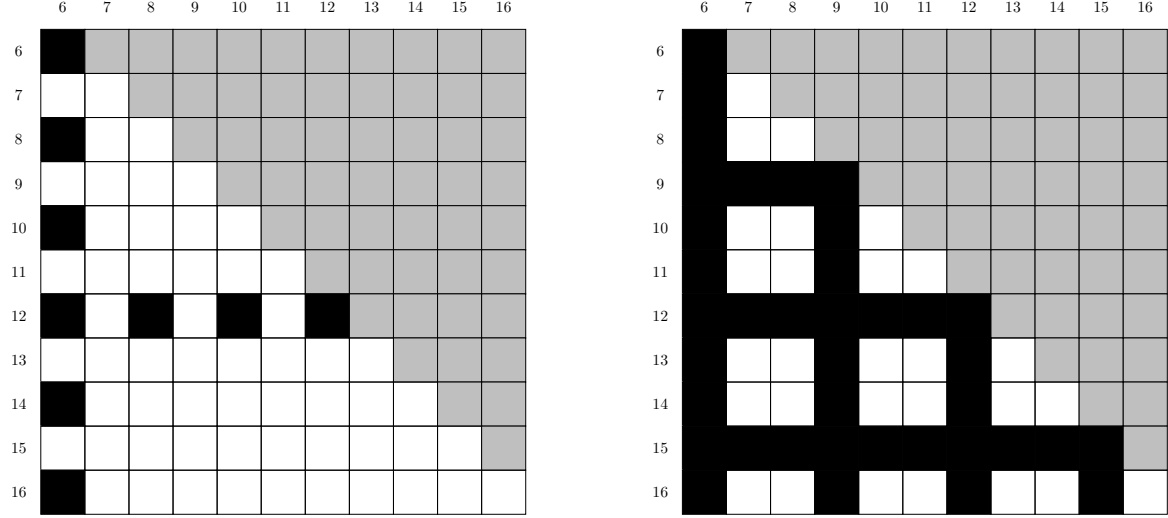


Figure 2.1: Thickness 6 grids with perfect percolating sets as obtained in lemma 2.2 (left), and divisibility cases of thickness 6 (right).

Lemma 2.2. *All grids of the form $(a, b, 6)$, where $a \equiv 0 \pmod{6}$ and $b \equiv 0 \pmod{2}$, or $b \equiv 0 \pmod{6}$ and $a \equiv 0 \pmod{2}$, percolate at the lower bound.*

2.3 Completeness of Thickness 7

2.4 Completeness of Thickness 5

Leveraging {lemmas from earlier chapters yet to be written}, we show that all divisibility cases in thickness 5 percolate at the lower bound.

Lemma 2.3. *Thickness 5 is complete.*

Proof. Let $(a, b, 2)$ represent an arbitrary (divisible) grid of thickness 2, and let $x = a \pmod{6}$ and $y = b \pmod{6}$. By {some as of yet unwritten construction}, we have that $(a, b, 2)$ percolates at the lower bound for all $x, y \in \{0, 2, 3, 5\}$, where $x \neq y$. We consider two constructions: $(a, b, 2) + (6, 3, 3)$ and $(a, b, 2) + (6, 6, 3)$.

By item (1) of the remark, in order to show that $(a, b, 2) + (6, 3, 3)$ percolates at the lower bound, it is sufficient to show that $(a, b, 2), (a, 3, 3), (6, b, 3), (6, 3, 2)$ all percolate at the bound. By {more unwritten constructions}, this is true for all $x, y \in \{0, 2, 3, 5\}$, where $x \neq y$ and at least one of $\{a, b\} > 2$. (Note that if $a = 2$, one of the tuples is $(2, 3, 3)$, which does not percolate at the lower bound; we accommodate for this by re-writing $(a, b, 2) + (6, 3, 3)$ as $(a, b, 2) + (3, 6, 3)$.) The resulting tuple $(a', b', 5)$ is a grid of thickness 5, with a' and b' in the same residue class modulo 6, and at least one of $\{a', b'\} \geq 9$. From {some figure representing the divisibility cases of thickness 5}, we

see that the lower bound on a' and b' omits the following three grids: $(5, 5, 5)$, $(6, 6, 5)$ and $(8, 8, 5)$.

Applying an analogous argument to $(a, b, 2) + (6, 6, 3)$, we must demonstrate that $(a, b, 2)$, $(a, 6, 3)$, $(6, b, 3)$, $(6, 6, 2)$ all percolate at the lower bound. By {some other constructions}, we again find that this holds for all $x, y \in \{0, 2, 3, 5\}$, where $x \neq y$ and $a, b > 1$. This gives all thickness 5 tuples $(a', b', 5)$ with a' and b' in different residue classes modulo 6, where $a', b' \geq 8$.

SOMETHING IS DIFFERENT

□

2.4.1 Intuition and Statement

2.4.2 Proof

Chapter 3

Constructions

3.1 Introduction

In this chapter, we present diagrammed proofs of lethal sets that percolate at the lower bound. The proofs are organized by the thickness of the grid. Many of the constructions in the following sections belong to infinite families of either optimal or perfect sets. In this case, we shall examine the grids by region, and observe that certain regions can be expanded to arbitrarily large sizes using mathematical induction.

We shall call a thickness *semi-complete* if all divisibility cases are optimal.

3.2 Thickness 2

Lemma 3.1. *All $(a, b, 2)$ grids with $a, b \in \{0, 3\} \pmod{6}$ and $a \not\equiv b \pmod{6}$ are perfect.*

Proof. We proceed by induction. For the base case, consider the grid $(6, 9, 2)$ (figure 3.1). □



Figure 3.1:

Bibliography

- [1] A. P. Dove, J. R. Griggs, R. J. Kang, and J.-S. Sereni. Supersaturation in the boolean lattice. 2013.