

# Tight Bounds on 3-Neighbor Bootstrap Percolation

by

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We acknowledge with respect the Lekwungen peoples on whose traditional territory the university stands, and the Songhees, Esquimalt, and WSÁNEĆ peoples whose historical relationships with the land continue to this day.

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## ABSTRACT

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## ACKNOWLEDGEMENTS

## DEDICATION



# Chapter 1

## Introduction

1.1 Problem Overview

1.2 Literature Review

1.3 Outline of ...

# Chapter 2

## A Tight Bound on Grids of Size $\geq 7$

### 2.1 Introduction and Definitions

Let the ordered tuple  $(a, b, c)$  represent the  $a \times b \times c$  grid  $G$  where  $a \geq b \geq c$ . We refer to  $c$  as the “thickness” of  $G$ . For example, the tuple  $(5, 3, 3)$  represents a  $5 \times 3 \times 3$  grid of thickness 3. We refer to a tuple as “divisible”, or a “divisibility case”, if and only if  $ab + bc + ca \equiv 0 \pmod{3}$ . Observe that the divisibility cases are precisely those grids with integral lower bounds. The divisibility cases of thicknesses belonging to the three residue classes modulo 3 are illustrated in {Figure something}.

In the following lemmas, we use the notation  $(a, b, c) + (x, y, z) = (a + x, b + y, c + z)$  to represent respective increases of  $x$ ,  $y$ , and  $z$  to the side lengths  $a$ ,  $b$ , and  $c$  of  $G$ . We note the following:

**Remark 2.1.** By applying the recursion,  $(a, b, c) + (x, y, z)$  percolates at the lower bound when either:

1.  $(a, b, c), (a, y, z), (x, b, z), (x, y, c)$  all percolate at the lower bound, or;
2.  $(x, y, z), (x, b, c), (a, y, c), (a, b, z)$  all percolate at the lower bound.

We shall call a thickness “complete” if it can be shown that all divisibility cases in that thickness percolate at the lower bound. In this section, we demonstrate that thickness 5, thickness 6 and thickness 7 are all complete. As these belong to the residue classes 2, 0, and 1 modulo 3, respectively, we then use a recursive construction to show that all larger grids are also complete.

### 2.2 Completeness of Thickness 5

Leveraging {lemmas from earlier chapters yet to be written}, we show that all divisibility cases in thickness 5 percolate at the lower bound.

**Lemma 2.2.** *Thickness 5 is complete.*

*Proof.* Let  $(a, b, 2)$  represent an arbitrary (divisible) grid of thickness 2, and let  $x = a \pmod{6}$  and  $y = b \pmod{6}$ . By {some as of yet unwritten construction}, we have that  $(a, b, 2)$  percolates at the lower bound for all  $x, y \in \{0, 2, 3, 5\}$ , where  $x \neq y$ . We consider two constructions:  $(a, b, 2) + (6, 3, 3)$  and  $(a, b, 2) + (6, 6, 3)$ .

By item (1) of the remark, in order to show that  $(a, b, 2) + (6, 3, 3)$  percolates at the lower bound, it is sufficient to show that  $(a, b, 2), (a, 3, 3), (6, b, 3), (6, 3, 2)$  all percolate at the bound. By {more unwritten constructions}, this is true for all  $x, y \in \{0, 2, 3, 5\}$ , where  $x \neq y$  and at least one of  $\{a, b\} > 2$ . (Note that if  $a = 2$ , one of the tuples is  $(2, 3, 3)$ , which does not percolate at the lower bound; we accommodate for this by re-writing  $(a, b, 2) + (6, 3, 3)$  as  $(a, b, 2) + (3, 6, 3)$ .) The resulting tuple  $(a', b', 5)$  is a grid of thickness 5, with  $a'$  and  $b'$  in the same residue class modulo 6, and at least one of  $\{a', b'\} \geq 9$ . From {some figure representing the divisibility cases of thickness 5}, we see that the lower bound on  $a'$  and  $b'$  omits the following three grids:  $(5, 5, 5), (6, 6, 5)$  and  $(8, 8, 5)$ .

Applying an analogous argument to  $(a, b, 2) + (6, 6, 3)$ , we must demonstrate that  $(a, b, 2), (a, 6, 3), (6, b, 3), (6, 6, 2)$  all percolate at the lower bound. By {some other constructions}, we again find that this holds for all  $x, y \in \{0, 2, 3, 5\}$ , where  $x \neq y$  and  $a, b > 1$ . This gives all thickness 5 tuples  $(a', b', 5)$  with  $a'$  and  $b'$  in different residue classes modulo 6, where  $a', b' \geq 8$ .  $\square$

## 2.2.1 Intuition and Statement

## 2.2.2 Proof

## Chapter 3

# Chapter on the Next Thing