Tight Bounds on 3-Neighbor Bootstrap Percolation

by

Abel Emanuel Romer B.A.Sc., Quest University Canada, 2017

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We acknowledge with respect the Lekwungen peoples on whose traditional territory the university stands, and the Songhees, Esquimalt, and WSÁNEĆ peoples whose historical relationships with the land continue to this day.

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#### Supervisory Committee

Dr. Peter Dukes, Co-Supervisor (Department of Mathematics and Statistics)

Dr. Jonathan Noel, Co-Supervisor (Department of Mathematics and Statistics)

#### ABSTRACT

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#### ACKNOWLEDGEMENTS

#### **DEDICATION**

## Chapter 1 Introduction

- 1.1 Problem Overview
- 1.2 Literature Review
- 1.3 Outline of ...

### Chapter 2

### A Tight Bound on Grids of Size $\geq 7$

#### 2.1 Introduction and Definitions

Let the ordered tuple (a, b, c) represent the  $a \times b \times c$  grid G where  $a \ge b \ge c$ . We refer to c as the "thickness" of G. For example, the tuple (5, 3, 3) represents a  $5 \times 3 \times 3$  grid of thickness 3. We refer to a tuple as "divisible", or a "divisibility case", if and only if  $ab + bc + ca \equiv 0 \pmod{3}$ . Observe that the divisibility cases are precisely those grids with integral lower bounds. The divisibility cases of thicknesses belonging to the three residue classes modulo 3 are illustrated in {Figure something}.

In the following lemmas, we use the notation (a, b, c) + (x, y, z) = (a + x, b + y, c + z) to represent respective increases of x, y, and z to the side lengths a, b, and c of G. We note the following:

**Remark 2.1.** By applying the recursion, (a, b, c) + (x, y, z) percolates at the lower bound when either:

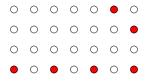
- 1. (a, b, c), (a, y, z), (x, b, z), (x, y, c) all percolate at the lower bound, or;
- 2. (x, y, z), (x, b, c), (a, y, c), (a, b, z) all percolate at the lower bound.

We shall call a thickness "complete" if it can be shown that all divisibility cases in that thickness percolate at the lower bound. In this section, we demonstrate that thickness 5, thickness 6 and thickness 7 are all complete. As these belong to the residue classes 2, 0, and 1 modulo 3, respectively, we then use a recursive construction to show that all larger grids are also complete.

#### 2.2 Completeness of Thickness 5

Leveraging {lemmas from earlier chapters yet to be written}, we show that all divisibility cases in thickness 5 percolate at the lower bound.

Lemma 2.2. Thickness 5 is complete.



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*Proof.* Let (a, b, 2) represent an arbitrary (divisible) grid of thickness 2, and let  $x = a \pmod{6}$  and  $y = b \pmod{6}$ . By {some as of yet unwritten construction}, we have that (a, b, 2) percolates at the lower bound for all  $x, y \in \{0, 2, 3, 5\}$ , where  $x \neq y$ . We consider two constructions: (a, b, 2) + (6, 3, 3) and (a, b, 2) + (6, 6, 3).

By item (1) of the remark, in order to show that (a, b, 2) + (6, 3, 3) percolates at the lower bound, it is sufficient to show that (a, b, 2), (a, 3, 3), (6, b, 3), (6, 3, 2) all percolate at the bound. By {more unwritten constructions}, this is true for all  $x, y \in \{0, 2, 3, 5\}$ , where  $x \neq y$  and at least one of  $\{a, b\} > 2$ . (Note that if a = 2, one of the tuples is (2, 3, 3), which does not percolate at the lower bound; we accommodate for this by re-writing (a, b, 2) + (6, 3, 3) as (a, b, 2) + (3, 6, 3).) The resulting tuple (a', b', 5) is a grid of thickness 5, with a' and b' in the same residue class modulo 6, and at least one of  $\{a', b'\} \geq 9$ . From {some figure representing the divisibility cases of thickness 5}, we see that the lower bound on a' and b' omits the following three grids: (5, 5, 5), (6, 6, 5) and (8, 8, 5).

Applying an analogous argument to (a, b, 2) + (6, 6, 3), we must demonstrate that (a, b, 2), (a, 6, 3), (6, b, 3), (6, 6, 2) all percolate at the lower bound. By {some other constructions}, we again find that this holds for all  $x, y \in \{0, 2, 3, 5\}$ , where  $x \neq y$  and a, b > 1. This gives all thickness 5 tuples (a', b', 5) with a' and b' in different residue classes modulo 6, where  $a', b' \geq 8$ .

SOMETHING IS DIFFERENT

#### 2.2.1 Intuition and Statement

#### 2.2.2 Proof

Chapter 3 Chapter on the Next Thing