Tight Bounds on 3-Neighbor Bootstrap Percolation

by

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We acknowledge with respect the Lekwungen peoples on whose traditional territory the university stands, and the Songhees, Esquimalt, and W̱SÁNEĆ peoples whose historical relationships with the land continue to this day.

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ABSTRACT

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2.1 Introduction and Definitions

Let the ordered tuple (a, b, c) represent the $a \times b \times c$ grid G where $a \geq b \geq c$. We refer to c as the "thickness" of G. For example, the tuple $(5, 3, 3)$ represents a $5 \times 3 \times 3$ grid of thickness 3. We refer to a tuple as "divisible", or a "divisibility case", if and only if $ab + bc + ca \equiv 0 \pmod 3$. Observe that the divisibility cases are precisely those grids with integral lower bounds. The divisibility cases of thicknesses belonging to the three residue classes modulo 3 are illustrated in {Figure something}.

In the following lemmas, we use the notation $(a, b, c) + (x, y, z) = (a+x, b+y, c+z)$ to represent respective increases of x, y, and z to the side lengths a, b, and c of G. We note the following:

Remark 2.1. By applying the recursion, $(a, b, c) + (x, y, z)$ percolates at the lower bound when either:

1. $(a, b, c), (a, y, z), (x, b, z), (x, y, c)$ all percolate at the lower bound, or;
2. $(x, y, z), (x, b, c), (a, y, c), (a, b, z)$ all percolate at the lower bound.

We shall call a thickness "complete" if it can be shown that all divisibility cases in that thickness percolate at the lower bound. In this section, we demonstrate that thickness 5, thickness 6 and thickness 7 are all complete. As these belong to the residue classes 2, 0, and 1 modulo 3, respectively, we then use a recursive construction to show that all larger grids are also complete.

2.2 Completeness of Thickness 5

Leveraging {lemmas from earlier chapters yet to be written}, we show that all divisibility cases in thickness 5 percolate at the lower bound.

Lemma 2.2. *Thickness 5 is complete.*

Proof. Let $(a, b, 2)$ represent an arbitrary (divisible) grid of thickness 2, and let $x = a \pmod 6$ and $y = b \pmod 6$. By {some as of yet unwritten construction}, we have that $(a, b, 2)$ percolates at the lower bound for all $x, y \in \{0, 2, 3, 5\}$, where $x \neq y$. We consider two constructions: $(a, b, 2) + (6, 3, 3)$ and $(a, b, 2) + (6, 6, 3)$.

By item (1) of the remark, in order to show that $(a, b, 2) + (6, 3, 3)$ percolates at the lower bound, it is sufficient to show that $(a, b, 2), (a, 3, 3), (6, b, 3), (6, 3, 2)$ all percolate at the bound. By {more unwritten constructions}, this is true for all $x, y \in \{0, 2, 3, 5\}$, where $x \neq y$ and at least one of $\{a, b\} > 2$. (Note that if $a = 2$, one of the tuples is $(2, 3, 3)$, which does not percolate at the lower bound; we accommodate for this by re-writing $(a, b, 2) + (6, 3, 3)$ as $(a, b, 2) + (3, 6, 3)$.) The resulting tuple $(a', b', 5)$ is a grid of thickness 5, with a' and b' in the same residue class modulo 6, and at least one of $\{a', b'\} \geq 9$. From {some figure representing the divisibility cases of thickness 5}, we see that the lower bound on a' and b' omits the following three grids: $(5, 5, 5), (6, 6, 5)$ and $(8, 8, 5)$.

Applying an analogous argument to $(a, b, 2) + (6, 6, 3)$, we must demonstrate that $(a, b, 2), (a, 6, 3), (6, b, 3), (6, 6, 2)$ all percolate at the lower bound. By {some other constructions}, we again find that this holds for all $x, y \in \{0, 2, 3, 5\}$, where $x \neq y$ and $a, b > 1$. This gives all thickness 5 tuples $(a', b', 5)$ with a' and b' in different residue classes modulo 6, where $a', b' \geq 8$. □

2.2.1 Intuition and Statement

2.2.2 Proof

Chapter 3
Chapter on the Next Thing