Tight Bounds on 3-Neighbor Bootstrap Percolation

by

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B.A.Sc., Quest University Canada, 2017

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We acknowledge with respect the Lekwungen peoples on whose traditional territory the university stands, and the Songhees, Esquimalt, and W̱SÁNEĆ peoples whose historical relationships with the land continue to this day.

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ABSTRACT

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Chapter 1
Introduction

We shall begin this thesis with a puzzle. Consider the lattice depicted in figure **??**. We refer to the elements of this lattice as *cells*. Suppose we have the capacity to infect some cells with a disease, and that this disease will, over a period of time, propagate through uninfected cells of the lattice. Let uninfected cells become infected if they are exposed to at least two infected neighboring cells in the vertical and/or horizontal directions. We say that the initial infection is *lethal* if the entire lattice ultimately becomes infected. Here is the puzzle:

Question. *What is the fewest number of infected cells necessary to spawn a lethal infection?*

Before we present the solution (which is particularly elegant and satisfying, and the reader is encouraged to play around with this problem on their own and get a feel for the behavior), let us take a moment to examine some properties of infectious sets and attempt to characterize what attributes might correspond to lethality. It should not take too long to observe that if an initial infection is in some way "spread too thin," it will be unable to jump between infected areas, leading to gaps in infection, which we refer to as *immune regions*. The perimeter of the lattice is particularly susceptible to this possibility, as those vertices have fewer neighbors from whom they might be exposed. Heuristically, then, a lethal set must have the ability to effectively span the entire lattice, and must be particularly virulent along the perimeter.

At this point, we are able to make some educated guesses regarding the specific structure of sets that are likely to be lethal. In particular, we would like to consider the two starting infections illustrated in figure **??**. Notice that while figure **??** (b) has far fewer perimeter infections, both (a) and (b) manage to form continuous bands of infected cells that appear to span the entire lattice after one step. Indeed, this holds with our notion of immune regions (or lack thereof), and we see that both infections will continue to propagate outwards from these bands until all cells become infected.

It is clear from figure **??** that we may obtain lethal sets on the $n \times n$ lattice of size n by simply infecting the diagonal. What is less obvious is whether it is possible to improve upon this result. Perhaps the most natural first attempt is to remove an infection from one of the cells along the diagonal. However, this seems to form an immune region around the removed cell. After some experimentation, one begins to believe it impossible to simultaneously satisfy the heuristic that a starting infection must span the lattice, while also using fewer than n initial infections. The question therefore becomes: how do we prove it?

1.1 Formalizing these observations

Before we present the proof, let us take a moment to formally define the process described above and develop some useful notation. The study of such cellular infection spread in grids (and more generally in graphs) is known in the literature as *bootstrap percolation*, and was introduced in the 1970s by Chalupa et al. [**?**] as a simplified model for the behavior of ferromagnetic fields. In their original 1979 paper, the authors research the stable structure of probabilistically selected initial infections. While this differs from the problem posed in question 1, the rules for the spread of infection and its broad behavior remain the same. It is worth noting that a large portion of contemporary research on bootstrap problems is focused on questions of probabilistic nature; while these problems are certainly of merit and quite interesting, they do not fall within the scope of this thesis. Rather, we shall focus on those problems where we have specific control over the structure of the initial infections; in particular, we aim to determine the smallest lethal set on a variety of graph classes.

Let us define our problem in concrete terms. Let G be a graph and $A_0 \subseteq V(G)$ be a set of initially infected vertices. Iteratively, we infect those vertices of G with at least r infected neighbors. For all $t > 0$, let A_t be the set of infected vertices at time step t. We then have

$$A_t = A_{t-1} \cup \{v \in V(G) : |N_G(v) \cap A_{t-1} \geq r\},$$

where $N_G(v)$ is the set of vertices adjacent to v in G. We define the *closure* of A_0 under r-neighbor bootstrap percolation to be $[A_0] = \bigcup_{t=0}^{\infty} A_t$. We say that A_0 *percolates* or is *lethal* if $[A_0] = V(G)$. We note that under these rules, it is not possible for vertices to become uninfected.

In the context of question 1, $G = P_{10} \,\square\, P_{10}$, and our diagonal construction shows us that $|A_0| \leq n$.

The problem of bootstrap percolation was first introduced in the 1970s by Chalupa et al. [**?**] as a simplified model for the behavior of ferromagnetic fields. In their original paper, the authors describe bootstrap percolation as the stabilization of a probabilistically occupied lattice, where each occupied site must be adjacent to at least m occupied neighbors. A re-rendering of the examples given in the original 1979

r	Grids								
	$[a_1]$	$[a_1] \times [a_2]$	$[n]^2$	$[a_1] \times [a_2] \times [a_3]$	$[n]^3$	$\cdots$	$\prod_{i=1}^{d} [a_i]$	$[n]^d$	$[2]^d$
$r = 0$	0	0	0	0	0		0	0	0
$r = 1$	1	1	1	1	1		1	1	1
$r = 2$	$\lceil \frac{a_1 - 1}{2} \rceil + 1$	$\lceil \frac{a_1 + a_2 - 2}{2} \rceil + 1$	n	$\lceil \frac{a_1 + a_2 + a_3 - 3}{2} \rceil + 1$	$\left\lceil \frac{3(n-1)}{2} \right\rceil + 1$		$\left\lceil \frac{\sum_{i=1}^{d}(a_i - 1)}{2} \right\rceil + 1$	$\left\lceil \frac{d(n-1)}{2} \right\rceil + 1$	$\lceil \frac{d}{2} \rceil + 1$
$r = 3$	**???**	**???**	$\lceil \frac{n^2+2n+4}{3} \rceil^*$	**S.A. bound**	n^2		**???**	**???**	$\lceil \frac{d(d+3)}{6} \rceil$
$\vdots$						$\ddots$			
$r = d$	**???**	**???**	**???**	**???**	**???**		S.A. bound	n^{d-1}	**???**

Table 1.1: A summary of known bootstrap percolation results for grids and the torus, $r \in \{0, 1, 2, 3, d\}$.

paper is presented in figures 1 and 2.

In this original problem, the authors are interested in the structural patterns of these stable arrangements. Put differently, given a set of randomly distributed occupants on a d-dimensional lattice, what configuration can we expect these occupied sites to fall into, subject to the constraint that each occupied site is adjacent to at least m other occupants? In this construction, a probabilistically populated lattice is iteratively depopulated until it reaches a stable state.

Alternatively, we might consider the behavior of a population as it grows, instead of shrinks. In this model, it is useful to consider the population as harboring an infection that steadily spreads from site to site, subject to population density. We shall consider these infections to take place on a graph, with vertices representing members of our population (sites), and edges indicating adjacency. In formal terms: let G be a graph and $A_0 \subseteq V(G)$ be a set of initially infected vertices. Iteratively, at every time step, infect those vertices of G with at least r infected neighbors. For all $t > 0$, let A_t be the set of infected vertices at time step t. We then have

$$A_t = A_{t-1} \cup \{v \in V(G) : |N_G(v) \cap A_{t-1} \geq r\},$$

where $N_G(v)$ is the set of vertices adjacent to v in G. We define the *closure* of A_0 under r-neighbor bootstrap percolation to be $[A_0] = \bigcup_{t=0}^{\infty} A_t$. This is analogous to the stable states introduced in [**?**]. We say that A_0 *percolates* or is *lethal* if $[A_0] = V(G)$. We note that under these rules, it is not possible for vertices to become uninfected.

Perhaps the most natural extremal question regarding r-neighbor bootstrap percolation is that of determining the size of the smallest percolating set $A_0 \subseteq V(G)$, for a given graph G. We represent this quantity by $m(G, r)$. There has been a great deal of work done on establishing the value of $m(G, r)$ for various classes of graphs and values of r (see {citations, citations, etc., etc.}). These results are incompletely summarized in table 1.1, and a selection of particularly noteworthy proofs are presented in detail in the following section.

We end this introduction with the presentation of a delightful question about the

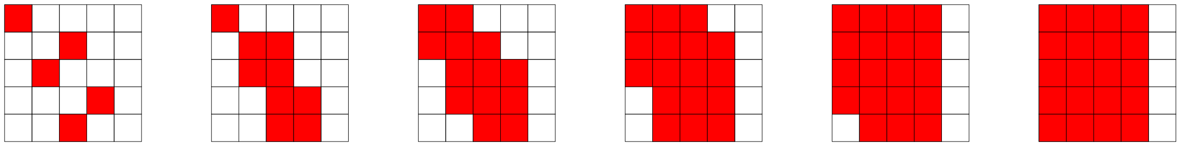

Figure 1.1: Percolation time-steps for an arbitrary initial infection on the 5×5 lattice, $r = 2$.

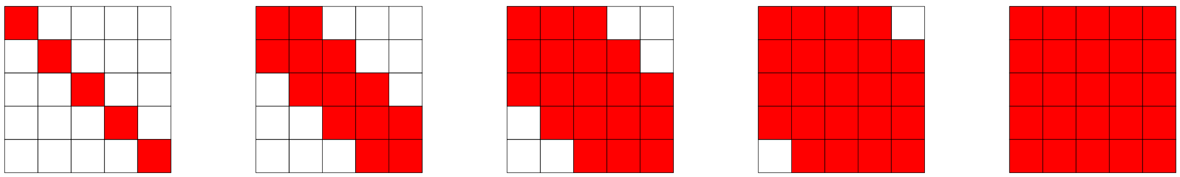

Figure 1.2: Percolation time-steps for a "spanning" initial infection on the 5×5 lattice, $r = 2$.

cardinality of 2-neighbor percolating sets on the two-dimensional lattice. The interested reader in encouraged to find a solution on their own; however, a proof of the result is presented at the beginning of the following section.

Question. *Let* (n, n) *represent the* $n \times n$ *lattice, given by* $G = P_n \times P_n$*. What is* $m(n, n, 2)$*?*

1.2 Minimum percolating sets and bounds

1.2.1 Foundations

Let us build some intuition for the behavior of percolating sets on the two-dimensional lattice. Figure 1.1 illustrates the percolation time-steps for an arbitrary initial infection on the graph $G = P_5 \times P_5$, where $r = 2$. (In general, we shall refer to the d-dimensional lattice with sides $a_1, \ldots, a_d$ as the d-tuple $(a_1, \ldots, a_d)$. The standard notation is $\prod_{i=1}^{d} a_i$.)

Note that this configuration fails to infect the entire grid; that is, the initial infection is not lethal. Heuristically, this appears to be a consequence of the fact that infected cells are unable to access the healthy cells in the rightmost column. We might, therefore, hypothesize that an initial infection must somehow "span" the entire lattice. A potential "spanning" construction is illustrated in figure 1.2. Observe that at each time-step, the infection spreads out laterally from the initial diagonal. It is a simple exercise to verify that this construction is lethal on all (n, n) grids for $r = 2$. We also note that a similar construction is lethal on all (n, m) grids for $r = 2$ (figure 1.3).

The (n, n) construction can be generalized to any dimension. Specifically, we have:

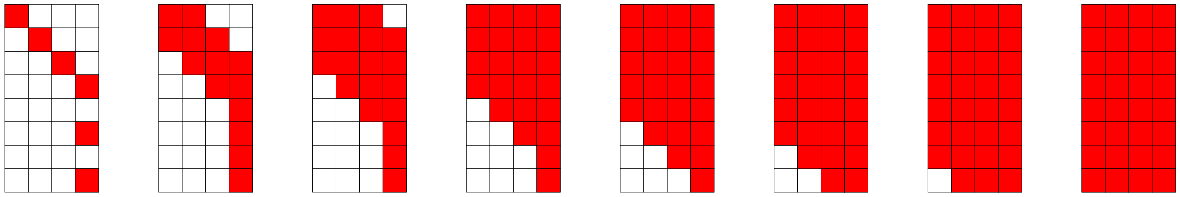

Figure 1.3: A lethal infection on the 8×4 lattice, $r = 2$.

Proposition 1.1. *For all $n, d \geq 1$,*

$$m([n]^d, d) \leq n^{d-1}.$$

This result has been known since at least Pete [?], although the particular constructions are difficult to render in general. The following proof (appearing in [?], but known in bootstrap percolation "folklore" for much longer) elegantly shows that this bound is tight.

Theorem 1.2. *For all $n, d \geq 1$,*

$$m([n]^d, d) = n^{d-1}.$$

Proof. The upper bound follows from proposition 1.1. The lower bound is given by a generalization of the famous "perimeter argument". Suppose $d = 2$ and consider an embedding of $G = [n]^d$ in the $n \times n$ grid. Let $A_0 \subseteq V(G)$ be a set of initially infected cells. We claim that the total perimeter of all infected regions in G is monotonically decreasing as a function of the time-step t. Consider an arbitrary healthy cell c. In order for c to become infected, at least two of its edges must abut infected cells. However, this implies that (upon infection of c) these edges are absorbed within the newly expanded infected region, thereby reducing the perimeter of infection by two. As c contains at most two un-absorbed edges, the perimeter of infection cannot increase.

Since a lethal set will infect the entire grid, and therefore have a final perimeter of infection of $4n$, it follows that the perimeter of infection of A_0 must be at least $4n$, and so $m([n]^2, 2) \geq n$.

The same argument generalizes nicely to higher dimensions. Simply observe that for a given hypercube cell to become infected, it must donate at least d of its $2d$ hyperplane faces to the infected region, thereby at most maintaining the current $(d-1)$-perimeter of infection. $\square$

We note that the perimeter argument extends directly to rectangular grids; however, the problem of obtaining tight constructions, should they exist, is largely unsolved and will be the main focus of this thesis. The following proposition, which we refer to informally as the *surface area bound* or *S.A. bound*, provides a lower bound on the size of lethal sets for d-dimensional rectangular grids where $r = d$.

Proposition 1.3. *For $d \geq 1$ and $a_1, \ldots, a_d \geq 1$,*

$$m(a_1, \ldots, a_d, d) \leq \frac{\sum_{i=1}^{d} \prod_{j \neq i} a_j}{d}.$$

Proof. Observe that the expression

$$\frac{\sum_{i=1}^{d} \prod_{j \neq i} a_j}{d}$$

is precisely the high-dimensional perimeter of the grid graph $(a_1, \ldots, a_d)$. The bound follows from the perimeter argument in theorem 1.2. □

In {some chapter of this thesis}, we will prove that this bound is tight in the case where $d = 3$ and $d_1, d_2, d_3 \geq 8$. We fully expect that this bound can be incrementally diminished; however, we feel that such small improvements do not at this time justify the effort required to obtain additional constructions.

In the remainder of this section, we shall present a number of additional bootstrap percolation results for different classes of grid graphs.

1.2.2 Additional results

Recall from the previous section that $m(n, n, 2) = n$, where the tight construction for the lower bound is given by a diagonal infection expanding laterally outwards. In a paper by Balogh and Bollobas [**?**], this result is generalized to all d-dimensional hypercubes $(a_1, \ldots, a_d)$, $a_i \geq 1$.

Theorem 1.4. *For $d \geq 1$ and $a_1, \ldots, a_d \geq 1$,*

$$m(a_1, \ldots, a_d, 2) = \left\lceil \frac{\sum_{i=1}^{d}(a_i - 1)}{2} \right\rceil + 1.$$

Proof. □

As suggested by table 1.1, general results become quite sparse for $r \notin \{2, d\}$. A nice result from Morrison and Noel resolves the question of $r = 3$ for hypercubes P_2^d of dimension $d \geq 3$.

Theorem 1.5. *For $d \geq 3$ and $a_1 = \cdots = a_d = 2$,*

$$m(a_1, \ldots, a_d, 3) = \left\lceil \frac{d(d+3)}{6} \right\rceil.$$

Proof. □

However, the issue of determining $m(a_1, \ldots, a_d, r)$ is largely unresolved. Furthermore, good lower bounds for $r \neq d$ are conspicuously absent.

Chapter 2
A Tight Bound on Grids of Size ≥ 5

2.1 Introduction and Definitions

Let the ordered tuple (a, b, c) represent the $a \times b \times c$ grid G where $a \geq b \geq c$. We refer to c as the "thickness" of G. For example, the tuple $(5, 3, 3)$ represents a $5 \times 3 \times 3$ grid of thickness 3. We refer to a tuple as "divisible", or a "divisibility case", if and only if $ab + bc + ca \equiv 0 \pmod 3$. Observe that the divisibility cases are precisely those grids with integral lower bounds. The divisibility cases of thicknesses belonging to the three residue classes modulo 3 are illustrated in {Figure something}.

In the following lemmas, we use the notation $(a, b, c) + (x, y, z) = (a + x, b + y, c + z)$ to represent respective increases of x, y, and z to the side lengths a, b, and c of G. We note the following:

Remark 2.1. By applying the recursion, $(a, b, c) + (x, y, z)$ percolates at the lower bound when either:

1. $(a, b, c), (a, y, z), (x, b, z), (x, y, c)$ all percolate at the lower bound, or;

2. $(x, y, z), (x, b, c), (a, y, c), (a, b, z)$ all percolate at the lower bound.

We shall call a thickness "complete" if it can be shown that all divisibility cases in that thickness percolate at the lower bound. In this section, we demonstrate that thickness 5, thickness 6 and thickness 7 are all complete. As these belong to the residue classes 2, 0, and 1 modulo 3, respectively, we then use a recursive construction to show that all larger grids are also complete.

2.2 Completeness of Thickness 5

Leveraging {lemmas from earlier chapters yet to be written}, we show that all divisibility cases in thickness 5 percolate at the lower bound.

NOTE: THE FOLLOWING LEMMAS HOLD ASSUMING WE HAVE A GENERAL CONSTRUCTION FOR $(2, 3, 3k)$ FOR ALL k.

Lemma 2.2. *All divisibility cases for grids of the form* $(k, 5, 5)$ *percolate at the lower bound.*

Proof. We consider grids obtained from $(5, 2, 2) + (a, 3, 3)$, for $a \equiv 0 \pmod 3$ and $a > 3$. By remark 2.1, it is sufficient to show that $(5, 2, 2), (5, 3, 3), (a, 2, 3), (a, 2, 3)$ are all perfect. By {a bunch of constructions}, each of these grids percolates at the lower bound for $a > 3$. We therefore obtain all grids of the form $(k, 5, 5)$, for $k > 8$. The only missing grids are $(5, 5, 5)$ and $(8, 5, 5)$, which we have by construction. This completes the proof. $\square$

Lemma 2.3. *All divisibility cases for grids of the form* $(k, 6, 5)$ *percolate at the lower bound.*

Proof. We consider grids obtained from $(6, 3, 2) + (a, 3, 3)$, for $a \equiv 0 \pmod 3$ and $a > 3$. By remark 2.1, it is sufficient to show that $(6, 3, 2), (6, 3, 3), (a, 3, 3), (a, 3, 2)$ are all perfect. By {a bunch of constructions}, each of these grids percolates at the lower bound for $a > 3$. We therefore obtain all grids of the form $(k, 6, 5)$, for $k > 8$. The only missing grid is $(6, 6, 5)$, which we have by construction. This completes the proof. $\square$

Lemma 2.4. *Thickness 5 is complete.*

Proof. Let $(a, b, 2)$ represent an arbitrary (divisible) grid of thickness 2, and let $x = a \pmod 6$ and $y = b \pmod 6$. By {some as of yet unwritten construction}, we have that $(a, b, 2)$ percolates at the lower bound for all $x, y \in \{0, 2, 3, 5\}$, where $x \neq y$. We consider two constructions: $(a, b, 2) + (6, 3, 3)$ and $(a, b, 2) + (6, 6, 3)$.

By item (1) of the remark, in order to show that $(a, b, 2)+(6, 3, 3)$ percolates at the lower bound, it is sufficient to show that $(a, b, 2), (a, 3, 3), (6, b, 3), (6, 3, 2)$ all percolate at the bound. By {more unwritten constructions}, this is true for all $x, y \in \{0, 2, 3, 5\}$, where $x \neq y$, $a, b > 1$, and at least one of $\{a, b\} > 2$. (Note that if $a = 2$, one of the tuples is $(2, 3, 3)$, which does not percolate at the lower bound; we accommodate for this by re-writing $(a, b, 2) + (6, 3, 3)$ as $(a, b, 2) + (3, 6, 3)$.) The resulting tuple $(a', b', 5)$ is a grid of thickness 5, with a' and b' in the same residue class modulo 6, $x, y \geq 8$, and at least one of $\{a', b'\} \geq 9$. From {some figure representing the divisibility cases of thickness 5}, we see that the lower bound on a' and b' omits all grids of the form $(5, 5, k)$ and $(5, 6, k)$, as well as the singular grid $(8, 8, 5)$.

Applying an analogous argument to $(a, b, 2) + (6, 6, 3)$, we must demonstrate that $(a, b, 2), (a, 6, 3), (6, b, 3), (6, 6, 2)$ all percolate at the lower bound. By {some other constructions}, we again find that this holds for all $x, y \in \{0, 2, 3, 5\}$, where $x \neq y$ and $a, b > 1$. This gives all thickness 5 tuples $(a', b', 5)$ with a' and b' in different residue classes modulo 6, where $a', b' \geq 8$.

Combining these results, we have completeness for all grids of thickness 5 except those of the form $(5, 5, k)$ and $(5, 6, k)$. By lemmas 2.2 and 2.3, these cases are also complete, and so thickness 5 is complete. This completes the proof. $\square$

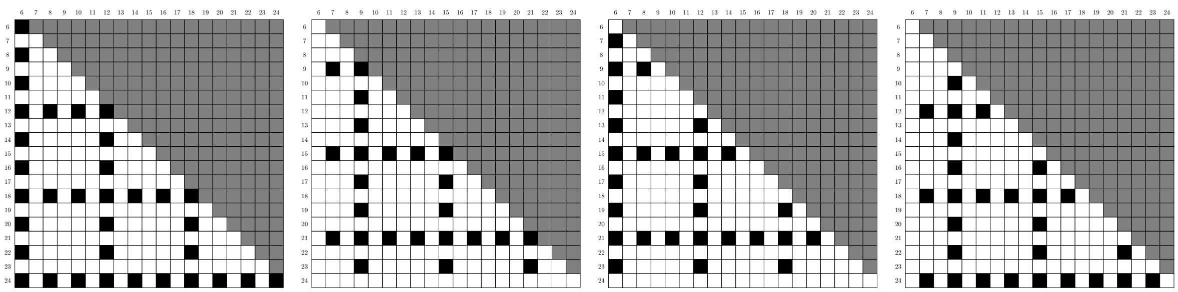

Figure 2.1: Thickness 6 grids with perfect percolating sets as obtained in lemma 2.5 (left), and divisibility cases of thickness 6 (right).

2.3 Completeness of Thickness 6

We shall show that all grids of thickness 6 can be obtained recursively from $(3, 3n, m)$, where $n, m \equiv 1 \pmod 2$ (see construction {something}), and one of $\{(3, 3, 3), (6, 3, 3), (6, 6, 3)\}$. As the following proof requires examining a number of cases, we separate thickness 6 into three categories, and show that each of these categories is complete.

Lemma 2.5. *All grids of the form $(a, b, 6)$, where $a \equiv 0 \pmod 6$ and $b \equiv 0 \pmod 2$, or $b \equiv 0 \pmod 6$ and $a \equiv 0 \pmod 2$, percolate at the lower bound.*

Proof. We consider $(3, 3n, m) + (3, 3, 3)$, for $n, m \equiv 1 \pmod 2$. Observe that the resulting grid is of the form $(6, 3k, l)$, where $k, l \equiv 0 \pmod 2$. This, in turn, is precisely the set of grids described above. By remark 2.1, we have that $(3, 3n, m) + (3, 3, 3)$ percolates if $(3, 3n, m), (3, 3, 3), (3, 3n, 3), (3, 3, m)$ all percolate. By construction {yet to be named}, all grids $(a, 3, 3)$ are perfect. Therefore, $(3, 3n, m) + (3, 3, 3)$ is perfect. This completes the proof. □

Lemma 2.6. *All grids of the form $(a, b, 6)$, where $a \equiv 3 \pmod 6$ and $b \equiv 1 \pmod 2$, or $b \equiv 3 \pmod 6$ and $a \equiv 1 \pmod 2$, percolate at the lower bound.*

Proof. Same argument as above, but add $(3, 6, 6)$. □

Lemma 2.7. *All remaining grids for thickness 6 work.*

Proof. Add $(3, 6, 3)$ or $(3, 3, 6)$. □

Lemma 2.8. *Thickness 6 is complete.*

Proof. Lemmas 2.5, 2.6, and 2.7 cover all divisibility cases for thickness 6. The result follows. □

2.4 Completeness of Thickness 7

We show that all divisibility cases for grids of thickness 7 percolate at the lower bound. Observe that divisibility cases for thickness 7 consist of grids of the form $(x, y, 7)$ for x, y in residue classes $\{0, 1, 3, 4\}$ modulo 6. We separate these divisibility cases into the following four categories and show that each category is complete:

1. $(x, y, 7)$ for $x, y \in \{1, 4\}$ and $x \equiv y \pmod 6$;
2. $(x, y, 7)$ for $x, y \in \{1, 4\}$ and $x \not\equiv y \pmod 6$;
3. $(x, y, 7)$ for $x, y \in \{0, 3\}$ and $x \equiv y \pmod 6$;
4. $(x, y, 7)$ for $x, y \in \{0, 3\}$ and $x \not\equiv y \pmod 6$.

Lemma 2.9. *All grids of the form $(x, y, 7)$ for $x, y \in \{1, 4\}$ and $x \equiv y \pmod 6$ are complete.*

Proof. Consider the construction $(a, b, 2) + (8, 5, 5)$ for $a, b \in \{2, 5\}$ and $a \not\equiv b \pmod 6$. Observe that this construction obtains all grids of the form described in (1) above. We show that the grids $(a, b, 2), (a, 5, 5), (8, b, 5), (8, 5, 2)$ are all complete. {The fact that these grids are complete follows from a number of constructions and the observation that thickness 5 is complete.} By remark 2.1, the construction $(a, b, 2) + (8, 5, 5)$ percolates at the lower bound. This completes the proof. □

Lemma 2.10. *All grids of the form $(x, y, 7)$ for $x, y \in \{1, 4\}$ and $x \not\equiv y \pmod 6$ are complete.*

Proof. Consider the construction $(a, b, 2) + (5, 5, 5)$ for $a, b \in \{2, 5\}$ and $a \not\equiv b \pmod 6$. The same argument as above shows that this construction is perfect. □

Lemma 2.11. *All grids of the form $(x, y, 7)$ for $x, y \in \{0, 3\}$ and $x \equiv y \pmod 6$ are complete.*

Proof. Consider the construction $(a, b, 2) + (5, 6, 9)$ for $a, b \in \{0, 3\}$ and $a \not\equiv b \pmod 6$. Same idea. □

Lemma 2.12. *All grids of the form $(x, y, 7)$ for $x, y \in \{0, 3\}$ and $x \not\equiv y \pmod 6$ are complete.*

Proof. Consider the construction $(a, b, 2) + (5, 6, 6)$ for $a, b \in \{0, 3\}$ and $a \not\equiv b \pmod 6$. Same idea. □

Lemma 2.13. *Thickness 7 is complete.*

Proof. By lemmas 2.9, 2.10, 2.11, and 2.12, all divisibility cases for thickness 7 percolate at the lower bound. □

2.5 Completeness of Grids of Size ≥ 5

We can get completeness in every residue class modulo 3 by simply considering the grids obtained from $(x, y, z) + (3, 3, 3)$. TA-DA!

Chapter 3
Constructions

3.1 Introduction

In this chapter, we present diagrammed proofs of lethal sets that percolate at the lower bound. The proofs are organized by the thickness of the grid. Many of the constructions in the following sections belong to infinite families of either optimal or perfect sets. In this case, we shall examine the grids by region, and observe that certain regions can be expanded to arbitrarily large sizes using mathematical induction.

We shall call a thickness *semi-complete* if all divisibility cases are optimal.

3.2 Useful lemmas and observations

We shall see that similar patterns and structures appear with some regularity in optimal sets. These structures always infect entire regions, and it will be helpful to recognize them within larger grids when they appear.

3.3 Thickness 2

Construction 3.1. *All* $(a, b, 2)$ *grids with* $a, b \in \{0, 3\} \pmod 6$ *and* $a \neq b \pmod 6$ *are perfect.*

Proof. We proceed by induction. For the base case, consider the grid $(6, 9, 2)$ (figure 3.2). $\square$

	1		1		1		1	
19	18	17		1		1		1
	5	16	15	14	13	12	1	
1	4	5		1		11		1
	1		1		1	12	13	14
1		1	6	7	8	13	14	15

21		21	22	23	24	25	26	27
20	19	20	21	22	23	24	25	26
1		17	18	19	20	21	22	23
	3	6	7	8	9	10	1	
1	2	3	4	1		1		1
	1		5		1		1	

Figure 3.1:

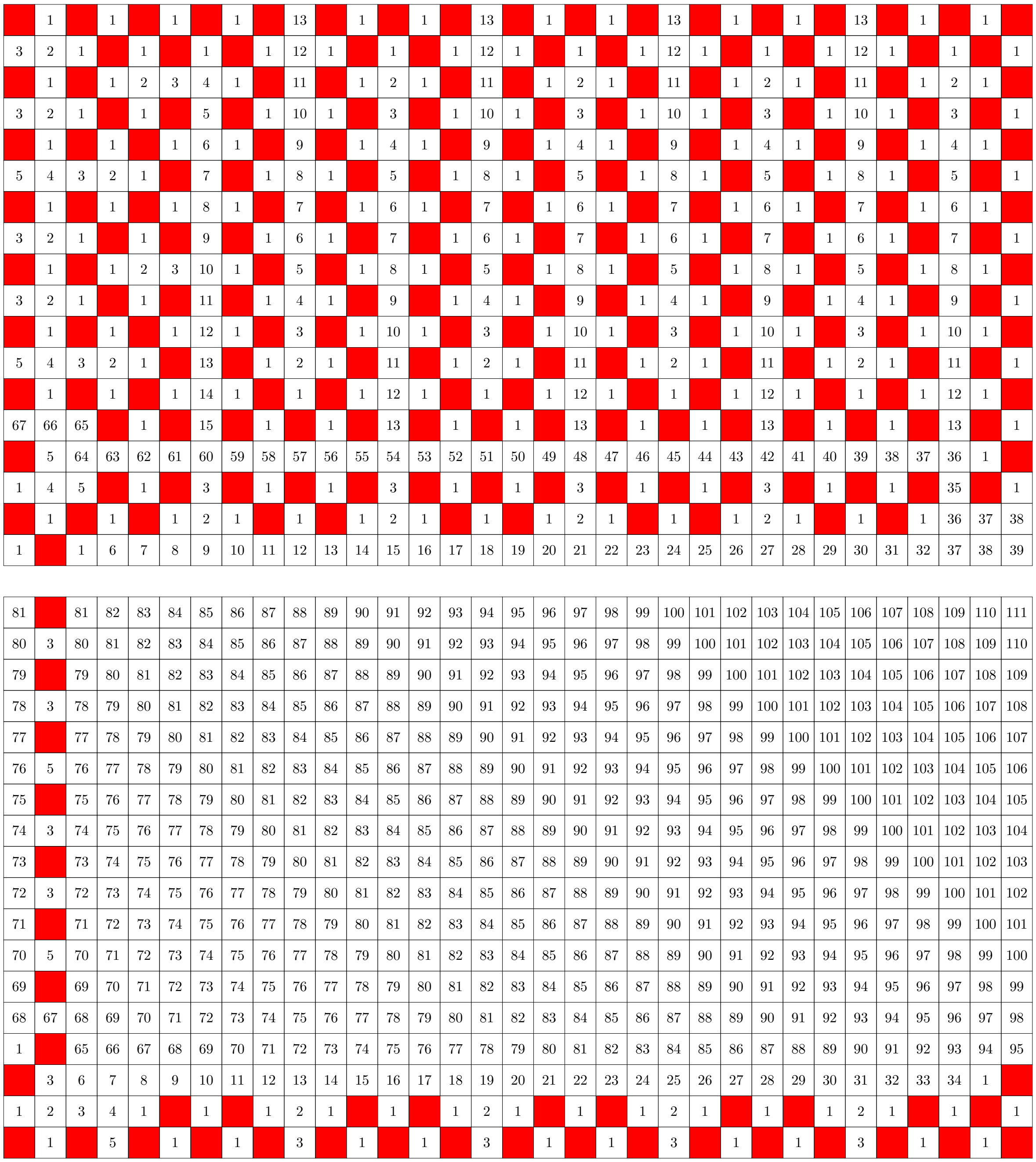
1 1 1 1 13 1 1 13 1 1 13 1 1 13 1 1
3 2 1 1 1 1 12 1 1 1 12 1 1 1 12 1 1 1 12 1 1 1
1 1 2 3 4 1 11 1 2 1 11 1 2 1 11 1 2 1 11 1 2 1
3 2 1 1 5 1 10 1 3 1 10 1 3 1 10 1 3 1 10 1 3 1
1 1 1 6 1 9 1 4 1 9 1 4 1 9 1 4 1 9 1 4 1
5 4 3 2 1 7 1 8 1 5 1 8 1 5 1 8 1 5 1 8 1 5 1
1 1 1 8 1 7 1 6 1 7 1 6 1 7 1 6 1 7 1 6 1
3 2 1 1 9 1 6 1 7 1 6 1 7 1 6 1 7 1 6 1 7 1
1 1 2 3 10 1 5 1 8 1 5 1 8 1 5 1 8 1 5 1 8 1
3 2 1 1 11 1 4 1 9 1 4 1 9 1 4 1 9 1 4 1 9 1
1 1 1 12 1 3 1 10 1 3 1 10 1 3 1 10 1 3 1 10 1
5 4 3 2 1 13 1 2 1 11 1 2 1 11 1 2 1 11 1 2 1 11 1
1 1 1 14 1 1 1 12 1 1 1 12 1 1 1 12 1 1 1 12 1
67 66 65 1 15 1 1 13 1 1 13 1 1 13 1 1 13 1
5 64 63 62 61 60 59 58 57 56 55 54 53 52 51 50 49 48 47 46 45 44 43 42 41 40 39 38 37 36 1
1 4 5 1 3 1 1 3 1 1 3 1 1 3 1 1 35 1
1 1 1 2 1 1 1 2 1 1 1 2 1 1 1 2 1 1 1 36 37 38
1 1 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 37 38 39
81 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110 111
80 3 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110
79 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109
78 3 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108
77 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107
76 5 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106
75 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105
74 3 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104
73 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103
72 3 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102
71 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101
70 5 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100
69 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99
68 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98
1 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95
3 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 1
1 2 3 4 1 1 1 2 1 1 1 2 1 1 1 2 1 1 1 2 1 1 1
1 5 1 1 3 1 1 3 1 1 3 1 1 3 1 1

Figure 3.2:

Bibliography

[1] A. P. Dove, J. R. Griggs, R. J. Kang, and J.-S. Sereni. Supersaturation in the boolean lattice. 2013.