Tight Bounds on 3-Neighbor Bootstrap Percolation

by

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B.A.Sc., Quest University Canada, 2017

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

We acknowledge with respect the Lekwungen peoples on whose traditional territory the university stands, and the Songhees, Esquimalt, and W̱SÁNEĆ peoples whose historical relationships with the land continue to this day.

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ABSTRACT

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ACKNOWLEDGEMENTS

DEDICATION

Chapter 1
Introduction

1.1 Introduction and Definitions

Let the ordered tuple (a, b, c) represent the $a \times b \times c$ grid G where $a \geq b \geq c$. We refer to c as the "thickness" of G. For example, the tuple $(5, 3, 3)$ represents a $5 \times 3 \times 3$ grid of thickness 3. We refer to a tuple as "divisible", or a "divisibility case", if and only if $ab + bc + ca \equiv 0 \pmod{3}$. Observe that the divisibility cases are precisely those grids with integral lower bounds. The divisibility cases of thicknesses belonging to the three residue classes modulo 3 are illustrated in {Figure something}.

In the following lemmas, we use the notation $(a, b, c)+(x, y, z) = (a+x, b+y, c+z)$ to represent respective increases of x, y, and z to the side lengths a, b, and c of G. We note the following:

Remark 1.1. By applying the recursion, $(a, b, c) + (x, y, z)$ percolates at the lower bound when either:

1. $(a, b, c), (a, y, z), (x, b, z), (x, y, c)$ all percolate at the lower bound, or;
2. $(x, y, z), (x, b, c), (a, y, c), (a, b, z)$ all percolate at the lower bound.

We shall call a thickness "complete" if it can be shown that all divisibility cases in that thickness percolate at the lower bound. In this section, we demonstrate that thickness 5, thickness 6 and thickness 7 are all complete. As these belong to the residue classes 2, 0, and 1 modulo 3, respectively, we then use a recursive construction to show that all larger grids are also complete. [1]

1.1.1 Intuition and Statement

1.1.2 Proof

Chapter 2
A Tight Bound on Grids of Size ≥ 7

2.1 Introduction and Definitions

Let the ordered tuple (a, b, c) represent the $a \times b \times c$ grid G where $a \geq b \geq c$. We refer to c as the "thickness" of G. For example, the tuple $(5, 3, 3)$ represents a $5 \times 3 \times 3$ grid of thickness 3. We refer to a tuple as "divisible", or a "divisibility case", if and only if $ab + bc + ca \equiv 0 \pmod 3$. Observe that the divisibility cases are precisely those grids with integral lower bounds. The divisibility cases of thicknesses belonging to the three residue classes modulo 3 are illustrated in {Figure something}.

In the following lemmas, we use the notation $(a, b, c)+(x, y, z) = (a+x, b+y, c+z)$ to represent respective increases of x, y, and z to the side lengths a, b, and c of G. We note the following:

Remark 2.1. By applying the recursion, $(a, b, c) + (x, y, z)$ percolates at the lower bound when either:

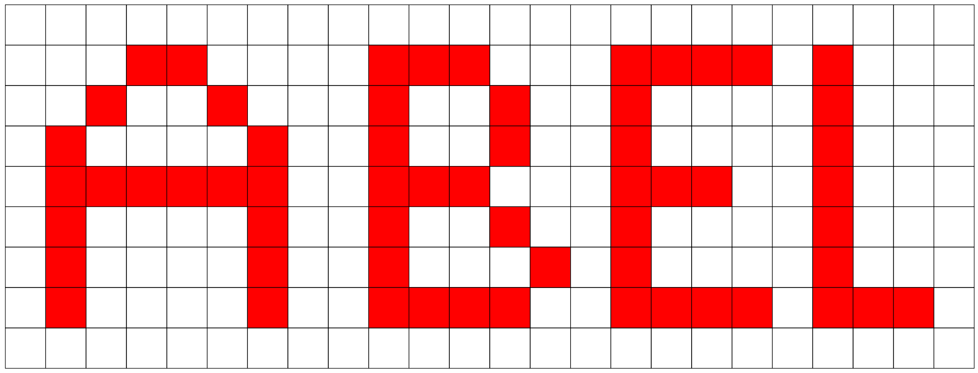
1. $(a, b, c), (a, y, z), (x, b, z), (x, y, c)$ all percolate at the lower bound, or;
2. $(x, y, z), (x, b, c), (a, y, c), (a, b, z)$ all percolate at the lower bound.

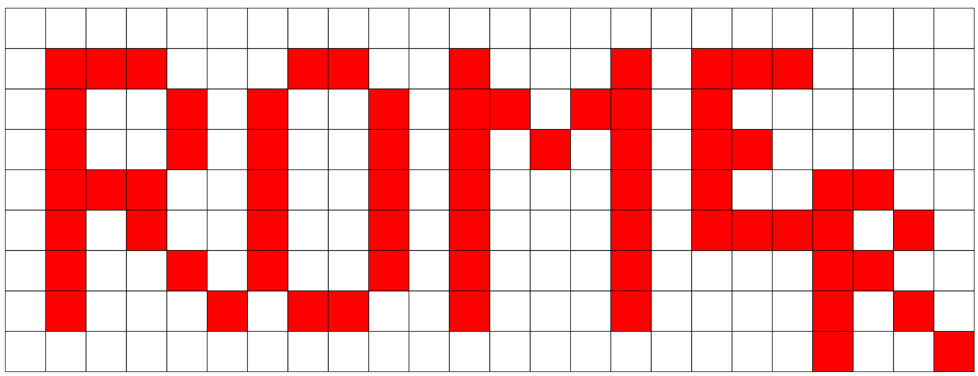
We shall call a thickness "complete" if it can be shown that all divisibility cases in that thickness percolate at the lower bound. In this section, we demonstrate that thickness 5, thickness 6 and thickness 7 are all complete. As these belong to the residue classes 2, 0, and 1 modulo 3, respectively, we then use a recursive construction to show that all larger grids are also complete.

2.2 Completeness of Thickness 5

Leveraging {lemmas from earlier chapters yet to be written}, we show that all divisibility cases in thickness 5 percolate at the lower bound.

Lemma 2.2. *Thickness 5 is complete.*





Proof. Let $(a, b, 2)$ represent an arbitrary (divisible) grid of thickness 2, and let $x = a$ (mod 6) and $y = b$ (mod 6). By {some as of yet unwritten construction}, we have that $(a, b, 2)$ percolates at the lower bound for all $x, y \in \{0, 2, 3, 5\}$, where $x \neq y$. We consider two constructions: $(a, b, 2) + (6, 3, 3)$ and $(a, b, 2) + (6, 6, 3)$.

By item (1) of the remark, in order to show that $(a, b, 2) + (6, 3, 3)$ percolates at the lower bound, it is sufficient to show that $(a, b, 2), (a, 3, 3), (6, b, 3), (6, 3, 2)$ all percolate at the bound. By {more unwritten constructions}, this is true for all $x, y \in \{0, 2, 3, 5\}$, where $x \neq y$ and at least one of $\{a, b\} > 2$. (Note that if $a = 2$, one of the tuples is $(2, 3, 3)$, which does not percolate at the lower bound; we accommodate for this by re-writing $(a, b, 2) + (6, 3, 3)$ as $(a, b, 2) + (3, 6, 3)$.) The resulting tuple $(a', b', 5)$ is a grid of thickness 5, with a' and b' in the same residue class modulo 6, and at least one of $\{a', b'\} \geq 9$. From {some figure representing the divisibility cases of thickness 5}, we see that the lower bound on a' and b' omits the following three grids: $(5, 5, 5), (6, 6, 5)$ and $(8, 8, 5)$.

Applying an analogous argument to $(a, b, 2) + (6, 6, 3)$, we must demonstrate that $(a, b, 2), (a, 6, 3), (6, b, 3), (6, 6, 2)$ all percolate at the lower bound. By {some other constructions}, we again find that this holds for all $x, y \in \{0, 2, 3, 5\}$, where $x \neq y$ and $a, b > 1$. This gives all thickness 5 tuples $(a', b', 5)$ with a' and b' in different residue classes modulo 6, where $a', b' \geq 8$.

SOMETHING IS DIFFERENT

□

2.2.1 Intuition and Statement

2.2.2 Proof

Bibliography

[1] A. P. Dove, J. R. Griggs, R. J. Kang, and J.-S. Sereni. Supersaturation in the boolean lattice. 2013.